DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2013

Website: www.aimSciences.org

pp. **217–226**

THE ROLE OF LOWER AND UPPER SOLUTIONS IN THE GENERALIZATION OF LIDSTONE PROBLEMS

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Abstract. In this the authors consider the nonlinear fully equation

$$u^{(iv)}(x) + f(x, u(x), u'(x), u''(x), u'''(x)) = 0$$

for $x \in [0,1]$, where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous functions, coupled with the Lidstone boundary conditions,

$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

They discuss how different definitions of lower and upper solutions can generalize existence and location results for boundary value problems with Lidstone boundary data. In addition, they replace the usual bilateral Nagumo condition by a one-sided condition, allowing the nonlinearity to be unbounded. An example will show that this unilateral condition generalizes the usual one and stress the potentialities of the new definitions.

1. **Introduction.** Fourth order differential equations are often called beam equations due to their relevance in beam theory, namely in the study of the bending of an elastic beam. In this work we consider the fully nonlinear equation

$$u^{(iv)}(x) + f(x, u(x), u'(x), u''(x), u'''(x)) = 0 \tag{1}$$

for $x \in [0,1]$, where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function, with the boundary conditions

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{2}$$

known as Lidstone boundary conditions. They appear in several physics and engineering situations such as simply supported beams ([6, 7]) and suspension bridges ([1, 8]). Different boundary conditions, meaning different types of support at

2010 *Mathematics Subject Classification.* Primary: 34B15; Secondary: 47H11.

Key words and phrases. Lidstone problems, one-sided Nagumo condition, non ordered lower and upper solutions.

the endpoints, are considered in the literature. As examples one can refer to [6, 9, 11, 12].

In this work we apply a more general Nagumo-type assumption: an unilateral condition. From this point of view, the results in the literature for problem (1)-(2) are improved because the nonlinearity can be unbounded from above or from below, following arguments suggested by [3, 4].

It is pointed out that, for Lidstone problems, where there is no information about the odd derivatives on the boundary, the replacement of the bilateral condition by an unilateral one is not trivial. It requires a new type of *a priori* bound and a more elaborate auxiliary problem in the proof of the main results. The last section contains an example where it can be seen how the one-sided Nagumo condition generalizes the usual bilateral one.

Moreover, the absence of data in the first derivative in the boundary conditions requires that the lower and upper solutions are defined as a pair of functions (see Definition 2.4) where condition (iii) could not be removed. This fact restricts the set of admissible functions as lower and upper solutions.

In Section 4, the above difficulty is overcome by introducing some adequate auxiliary functions and new definitions (see Definition 4.1). Moreover, the existence and location result still holds in the presence of not necessarily ordered lower and upper solutions and the corresponding first derivatives. More precisely, condition (11) is replaced by (23), where the "well order" is only required for the second derivatives.

2. **Definitions and auxiliary results.** In this section some auxiliary results and definitions, essential to the proof of main results, are presented.

We consider a one-sided Nagumo-type condition, meaning that the function f is only limited either from above, (3), or from below, (4). Therefore two different lemmas can be obtained, depending on the condition assumed on the nonlinearity f.

The one-sided Nagumo-type condition to be used and the consequent *a priori* estimation are as follows:

Definition 2.1. Given a subset $E \subset [0,1] \times \mathbb{R}^4$, a continuous function $f : E \to \mathbb{R}$ is said to satisfy a one-sided Nagumo-type condition in E if there exists a real continuous function $h_E : \mathbb{R}_0^+ \to [k, +\infty[$, for some $k > 0$, such that

$$f(x, y_0, y_1, y_2, y_3) \le h_E(|y_3|), \ \forall (x, y_0, y_1, y_2, y_3) \in E \tag{3}$$

or

$$f(x, y_0, y_1, y_2, y_3) \ge -h_E(|y_3|), \ \forall (x, y_0, y_1, y_2, y_3) \in E, \tag{4}$$

with

$$\int_0^{+\infty} \frac{t}{h_E(t)} dt = +\infty. \tag{5}$$

Lemma 2.2. *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function, verifying Nagumo-type conditions (3) and (5) in*

$$E = \left\{ (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \gamma_i(x) \le y_i \le \Gamma_i(x), \ i = 0,1,2 \right\}, \tag{6}$$

where $\gamma_i(x)$ and $\Gamma_i(x)$ are continuous functions such that, for $i = 0,1,2$, $\gamma_i(x) \le \Gamma_i(x)$, for every $x \in [0,1]$. Then for every $\rho > 0$ there is $R > 0$ such that every solution $u(x)$ of equation (1) satisying

$$u'''(0) \ge -\rho \ , \ u'''(1) \le \rho \tag{7}$$

and

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x), \ \forall x \in [0,1], \tag{8}$$

for $i = 0, 1, 2,$ *satisfies*

$$\|u'''\|_\infty < R.$$

Proof. The proof follows the method suggested in [4]. □

If the function f satisfies (4) the following lemma still holds:

Lemma 2.3. *Let* $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *be a continuous function, verifying Nagumo-type conditions (4) and (5) in* E *given by (6).*

Then for every $\rho > 0$ *there is* $R > 0$ *such that every solution* $u(x)$ *of equation (1) satisfying*

$$u'''(0) \leq \rho \ , \ u'''(1) \geq -\rho, \tag{9}$$

and

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x), \ \forall x \in [0,1], \tag{10}$$

for $i = 0, 1, 2,$ *satisfies*

$$\|u'''\| < R.$$

Remark 1. Observe that R depends only on the functions h_E, γ_2 and Γ_2 and not on the boundary conditions.

The functions used as lower and upper solutions are defined as a pair:

Definition 2.4. The functions $\alpha, \beta \in C^4(]0,1[) \cap C^2([0,1])$ verifying

$$\alpha^{(i)}(x) \leq \beta^{(i)}(x), \ i = 0, 1, 2, \ \forall x \in [0,1], \tag{11}$$

define a pair of lower and upper solutions of problem (1)-(2) if the following conditions are satisfied:

(i) $\alpha^{(iv)}(x) + f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)) \geq 0,$

$\beta^{(iv)}(x) + f(x, \beta(x), \beta'(x), \beta''(x), \beta'''(x)) \leq 0;$

(ii) $\alpha(0) \leq 0, \quad \alpha''(0) \leq 0, \quad \alpha''(1) \leq 0,$

$\beta(0) \geq 0, \quad \beta''(0) \geq 0, \quad \beta''(1) \geq 0;$

(iii) $\alpha'(0) - \beta'(0) \leq \min\{\beta(0) - \beta(1), \ \alpha(1) - \alpha(0)\}.$

As it was shown in [10], condition (iii) can not be removed for this type of definition. However if the minimum in (iii) is non-positive then assumption (11) can be replaced by $\alpha''(x) \leq \beta''(x)$ for every $x \in [0,1]$, as the other inequalities are obtained from integration.

3. **Existence and location result.**

A general existence and location result, where the nonlinear part can be unbounded from above or from below, is the following.

Theorem 3.1. *Suppose that there is a pair of lower and upper solutions of the problem (1)-(2),* $\alpha(x)$ *and* $\beta(x)$*, respectively. Let* $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *be a continuous function satisfying the one-sided Nagumo conditions (3) and (5) in*

$$E_* = \left\{(x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), \ i = 0, 1, 2\right\}$$

and

$$f(x, \alpha, \alpha', y_2, y_3) \leq f(x, y_0, y_1, y_2, y_3) \leq f(x, \beta, \beta', y_2, y_3), \tag{12}$$

for

$$\alpha(x) \leq y_0 \leq \beta(x), \ \alpha'(x) \leq y_1 \leq \beta'(x)$$

and for fixed $(x, y_2, y_3) \in [0,1] \times \mathbb{R}^2$. *Then the problem (1)-(2) has at least one solution* $u(x) \in C^4([0,1])$, *satisfying*

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \ \textit{for } i = 0,1,2, \ \forall x \in [0,1].$$

Proof. Consider the continuous truncations δ_i given by

$$\delta_i(x, y_i) = \begin{cases} \alpha^{(i)}(x), & \text{if} \quad y_i < \alpha^{(i)}(x), \\ y_i, & \text{if} \quad \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), \\ \beta^{(i)}(x), & \text{if} \quad y_i > \beta^{(i)}(x), \end{cases}, \tag{13}$$

for $i = 0,1,2$. For $\lambda \in [0,1]$, consider the homotopic equation

$$\begin{aligned} u^{(iv)}(x) &= u''(x) \\ &- \lambda \left[f(x, \delta_0(x,u), \delta_1(x,u'), \delta_2(x,u''), u''') + \delta_2(x,u'') \right] \end{aligned} \tag{14}$$

and the boundary conditions

$$\begin{gathered} u(0) = u(1) = 0, \\ (1-\lambda) u'''(0) = \lambda |u''(0)|, \\ (1-\lambda) u'''(1) = -\lambda |u''(1)|. \end{gathered} \tag{15}$$

Let $r_2 > 0$ be large enough such that, for every $x \in [0,1]$,

$$-r_2 < \alpha''(x) \leq \beta''(x) < r_2, \tag{16}$$

$$r_2 - f(x, \beta(x), \beta'(x), \beta''(x), 0) - \beta''(x) > 0, \tag{17}$$

$$r_2 + f(x, \alpha(x), \alpha'(x), \alpha''(x), 0) + \alpha''(x) > 0. \tag{18}$$

The proof follows similar steps to the proof of the main result in [2], therefore only the key points of the arguments are presented:

- *Every solution* $u(x)$ *of the problem (14)-(15) satisfies*

  $$|u''(x)| < r_2, \ \ |u'(x)| < r_1, \ \ |u(x)| < r_1, \ \forall x \in [0,1],$$

  with $r_1 := r_2 + u'(0)$ *independently of* $\lambda \in [0,1]$.

As for interior points, the technique is identical to [2] we give a proof for only the boundary points. Assume, by contradiction, that

$$\max_{x \in [0,1]} u''(x) := u''(0) \geq r_2 > 0.$$

Then for $\lambda \in]0,1]$, it is obtained

$$0 \geq (1-\lambda) u'''(0) = \lambda u''(0) \geq \lambda r_2 > 0.$$

For $\lambda = 0$, $u'''(0) = 0$. Therefore $u^{(iv)}(0) \leq 0$ and this case is identical to that of the interior points.

If

$$\max_{x \in [0,1]} u''(x) := u''(1) \geq r_2,$$

for $\lambda \in]0,1]$, the contradiction is similar:

$$0 \leq (1-\lambda) u'''(1) = -\lambda |u''(1)| \leq -\lambda r_2 < 0.$$

The case $\lambda = 0$, implies $u'''(1) = 0$ and $u^{(iv)}(1) \geq 0$ and the contradiction is obtained by the same technique as for the interior points.

The case $u''(x) \leq -r_2$ is analogous and so

$$|u''(x)| < r_2, \ \forall x \in [0,1], \forall \lambda \in [0,1].$$

Integrating in $[0,x]$, $u'(x) - u'(0) = \int_0^x u''(s)ds < r_2$, and

$$|u'(x)| < r_2 + u'(0), \ \forall x \in [0,1], \forall \lambda \in [0,1].$$

By integration, we obtain

$$|u'(x)| < r_1, |u(x)| < r_1, \forall x \in [0,1].$$

- *There is $R > 0$ such that, every solution $u(x)$ of the problem (14)-(15) satisfies*
$$|u'''(x)| < R, \ \forall x \in [0,1],$$
independently of $\lambda \in [0,1]$.
- *Problem (14)-(15) has at least a solution $u_1(x)$ for $\lambda = 1$.*

The existence of at least one solution $u_1(x)$ for problem (14)-(15) is obtained with the operators $\mathcal{L} : C^4([0,1]) \subset C^3([0,1]) \to C([0,1]) \times \mathbb{R}^4$ given by

$$\mathcal{L}u = \left(u^{(iv)} - u'', u(0), u(1), u'''(0), u'''(1)\right),$$

$\mathcal{N}_\lambda : C^3([0,1]) \to C([0,1]) \times \mathbb{R}^4$ by

$$\mathcal{N}_\lambda = \left(\begin{array}{c} \lambda\left[-f(x, \delta_0(x,u), \delta_1(x,u'), \delta_2(x,u''), u'''(x)) - \delta_2(x,u'')\right], \\ 0, 0, \lambda\left[u'''(0) + |u''(0)|\right], \lambda\left[u'''(1) - |u''(1)|\right] \end{array} \right)$$

and $\mathcal{T}_\lambda : \left(C^4([0,1]), \mathbb{R}\right) \to \left(C^4([0,1]), \mathbb{R}\right)$ by

$$\mathcal{T}_\lambda(u) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u).$$

The function $u_1(x)$ will be a solution of the initial problem (1)-(2) if it satisfies

$$\alpha^{(i)}(x) \leq u_1^{(i)}(x) \leq \beta^{(i)}(x), i = 0, 1, 2, \ \forall x \in [0,1].$$

Suppose, by contradiction, that there is $x \in [0,1]$ such that $\alpha''(x) > u_1''(x)$ and define

$$\min_{x \in [0,1]} [u_1''(x) - \alpha''(x)] := u_1''(x_1) - \alpha''(x_1) < 0.$$

If $x_1 \in]0,1[$, then $u_1'''(x_1) = \alpha'''(x_1)$ and $u^{(iv)}(x_1) \geq \alpha^{(iv)}(x_1)$.

By Definition 2.4 and (12) we obtain the contradiction:

$$\begin{aligned} \alpha^{(iv)}(x_1) &\leq u_1^{(iv)}(x_1) \\ &= -f(x_1, \delta_0(x_1,u), \delta_1(x_1,u'), \alpha''(x_1), \alpha'''(x_1)) \\ &\quad + u''(x_1) - \alpha''(x_1) \\ &< -f(x_1, \alpha(x_1), \alpha'(x_1), \alpha''(x_1), \alpha'''(x_1)) \leq \alpha^{(iv)}(x_1). \end{aligned}$$

If $x_1 = 0$ or $x_1 = 1$ the contradiction is trivial by Definition 2.4 (ii).

Therefore, $\alpha''(x) \leq u_1''(x)$ for every $x \in [0,1]$. In a similar way, it can be proved that $u_1''(x) \leq \beta''(x)$, and so

$$\alpha''(x) \leq u_1''(x) \leq \beta''(x), \quad \text{for every } x \in [0,1]. \tag{19}$$

By (2),

$$\begin{aligned} 0 &= \int_0^1 u_1'(x)\,dx = \int_0^1 \left(u_1'(0) + \int_0^x u_1''(s)\,ds\right)\,dx \\ &= u_1'(0) + \int_0^1 \int_0^x u_1''(s)\,ds\,dx, \end{aligned}$$

so

$$u_1'(0) = -\int_0^1 \int_0^x u_1''(s)\,dsdx. \tag{20}$$

By this technique

$$\int_0^1 \int_0^x \alpha''(s)\,ds\;dx = \alpha(1) - \alpha(0) - \alpha'(0)$$

and, by Definition 2.4 (iii), (19) and (20),

$$\begin{aligned} -\beta'(0) &\leq \alpha(1) - \alpha(0) - \alpha'(0) = \int_0^1 \int_0^x \alpha''(s)\,ds\;dx \\ &\leq \int_0^1 \int_0^x u_1''(s)\,ds\;dx = -u_1'(0)\,. \end{aligned}$$

Therefore $u_1'(0) \leq \beta'(0)$ and, by integration of (16), one obtains

$$u_1'(x) - u_1'(0) = \int_0^x u_1''(s)\,ds \leq \int_0^x \beta''(s)\,ds = \beta'(x) - \beta'(0)$$

and

$$u_1'(x) \leq \beta'(x) - \beta'(0) + u_1'(0) \leq \beta'(x), \forall x \in [0,1].$$

The relation $\alpha'(x) \leq u_1'(x)$, for every $x \in [0,1]$, can be proved by similar arguments. Then

$$\alpha'(x) \leq u_1'(x) \leq \beta'(x)\,, \forall x \in [0,1]\,.$$

By Definition 2.4 (ii)

$$\begin{aligned} \alpha(x) &\leq \int_0^x \alpha'(s)\,ds \leq \int_0^x u_1'(s)\,ds = u_1(x) \\ &\leq \int_0^x \beta'(s)\,ds = \beta(x) - \beta(0) \leq \beta(x). \end{aligned}$$

Therefore, $u_1(x)$ is a solution for problem (1)-(2). □

Remark 2. Theorem 3.1 still holds if condition (3) is replaced by (4) and conditions (15) are replaced by

$$\begin{gathered} u(0) = u(1) = 0, \\ (1-\lambda)\,u'''(0) = -\lambda\,|u''(0)|\,, \\ (1-\lambda)\,u'''(1) = \lambda\,|u''(1)|\,. \end{gathered}$$

4. **Generalized lower and upper solutions.** When looking at the definition of lower and upper solutions one can wonder about its impact and importance in the existence and location results presented in the previous sections.

It is immediate that they provide a very graphical information about some qualitative properties of the solution, but one can ask how deep is their influence in the final results, for instance, in Definition 2.4 is it possible to relax condition (11) and condition iii)? How does this change affect the final result?

With this thought in mind, we consider the following definitions for lower and upper solutions:

Definition 4.1. Functions $\alpha, \beta \in C^4\left(]0,1[\right) \cap C^2\left([0,1]\right)$ are a pair of lower and upper solutions of (1)-(2) if the following conditions are satisfied:

(i) $\alpha^{(iv)}(x) + f(x, \alpha_0(x), \alpha_1(x), \alpha''(x), \alpha'''(x)) \geq 0,$

where

$$\begin{aligned} \alpha_0(x) &= \textstyle\int_0^x \alpha_1(s)\,ds, \\ \alpha_1(x) &= \alpha'(x) - \alpha'(0) - \textstyle\int_0^1 \int_0^x |\beta''(s)|\,dsdx; \end{aligned} \tag{21}$$

(ii) $\alpha''(0) \le 0,\ \ \alpha''(1) \le 0;$

(iii) $\beta^{(iv)}(x) + f(x, \beta_0(x), \beta_1(x), \beta''(x), \beta'''(x)) \le 0,$

where

$$\begin{aligned} \beta_0(x) &= \textstyle\int_0^x \beta_1(s)\,ds, \\ \beta_1(x) &= \beta'(x) - \beta'(0) + \textstyle\int_0^1 \int_0^x |\alpha''(s)|\,dsdx; \end{aligned} \tag{22}$$

(iv) $\beta''(0) \ge 0,\ \ \beta''(1) \ge 0.$

Now, the main existence and location result becomes:

Theorem 4.2. *Suppose that there is a pair of lower and upper solutions of the problem (1)-(2), $\alpha(x)$ and $\beta(x)$, respectively satisfying*

$$\alpha''(x) \le \beta''(x),\ \ \forall x \in [0,1]. \tag{23}$$

Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying the one-sided Nagumo conditions (3), or (4), and (5) in

$$E_* = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \alpha_0(x) \le y_0 \le \beta_0(x), \\ \alpha_1(x) \le y_1 \le \beta_1(x),\ \ \alpha''(x) \le y_2 \le \beta''(x) \end{array} \right\}$$

and

$$f(x, \alpha_0, \alpha_1, y_2, y_3) \le f(x, y_0, y_1, y_2, y_3) \le f(x, \beta_0, \beta_1, y_2, y_3), \tag{24}$$

for

$$\alpha_0(x) \le y_0 \le \beta_0(x), \alpha_1(x) \le y_1 \le \beta_1(x)$$

and for fixed $(x, y_2, y_3) \in [0,1] \times \mathbb{R}^2$.

Then the problem (1)-(2) has at least one solution $u(x) \in C^4([0,1])$, satisfying

$$\alpha_i(x) \le u^{(i)}(x) \le \beta_i(x),\ \ \textit{for } i = 0,1,\ \ \forall x \in [0,1],$$

and

$$\alpha''(x) \le u''(x) \le \beta''(x),\ \ \forall x \in [0,1].$$

Proof. The arguments are similar to the proof of Theorem 3.1. So we only prove that the solution $u_1(x)$ of the modified problem will be a solution of the initial problem (1)-(2). For that it is sufficient to show that

$$\alpha''(x) \le u''(x) \le \beta''(x) \tag{25}$$

and

$$\alpha_i(x) \le u^{(i)}(x) \le \beta_i(x),\ \ \text{for } i = 0,1, \tag{26}$$

for every $x \in [0,1]$.

The inequalities (25) can be proved as in Theorem 3.1.

By integration,

$$u'(x) - u'(0) = \int_0^x u''(s)\,ds \le \int_0^x \beta''(s)\,ds = \beta'(x) - \beta'(0),$$

so

$$u'(x) \le \beta'(x) - \beta'(0) + u'(0). \tag{27}$$

Furthermore, by (2)

$$0=\int_0^1 u'(x)\,dx \le u'(0)+\int_0^1\int_0^x \beta''(s)\,dsdx.$$

Hence,

$$u'(0)\ge -\int_0^1\int_0^x \beta''(s)\,dsdx$$

and in a similar way $u'(0)\le -\int_0^1\int_0^x \alpha''(s)\,dsdx$.

Applying this in (27)

$$\begin{aligned} u'(x) &\le \beta'(x)-\beta'(0)-\int_0^1\int_0^x \alpha''(s)\,dsdx \\ &\le \beta'(x)-\beta'(0)+\int_0^1\int_0^x |\alpha''(s)|\,dsdx=\beta_1(x). \end{aligned}$$

Using the same arguments it can be proved that

$$\alpha_1(x)\le u'(x)\le \beta_1(x),\ \forall x\in[0,1].$$

Integrating the previous inequality

$$\alpha_0(x)=\int_0^x \alpha_1(s)\le u(x)\le \int_0^x \beta_1(s)\,ds=\beta_0(x),\ \forall x\in[0,1].$$

□

As one can notice the inclusion of the auxiliary functions α_0,β_0 and α_1,β_1 allows not only the use of non-ordered lower and upper solutions, increasing the range of admissible lower and upper solutions for the problem (1)-(2), but also overrun the order relation between the first derivatives, where there is no information.

5. **Example.** The next example illustrates a set of lower and upper solutions that were not covered by Definition 2.4 and Theorem 3.1 but are now included in Definition 4.1 and Theorem 4.2. In this example, lower and upper solutions are not ordered and condition (iii) from Definition 2.4 is eliminated, a case that was not possible by Definition 2.4 and Theorem 3.1.

Example 5.1. *For $x\in[0,1]$ consider the differential equation*

$$u^{(iv)}(x)+e^{u(x)}+\arctan(u'(x))-(u''(x))^3-|u'''(x)|^k=0, \tag{28}$$

with $k\in[0,2]$, along with the boundary conditions (2).

The functions $\alpha,\beta:\mathbb{R}\to\mathbb{R}$ given by

$$\begin{array}{c}\alpha(x)=-x^2+\frac{1}{2},\\ \beta(x)=x^2-\frac{1}{2},\end{array}$$

are lower and upper solutions, respectively, of problem (28),(2) satisfying (23) with the auxiliary functions given by Definition 4.1

$$\begin{array}{c}\alpha_0(x)=-x^2-x,\\ \alpha_1(x)=-2x-1,\end{array}$$

and

$$\begin{array}{c}\beta_0(x)=x^2+x,\\ \beta_1(x)=2x+1.\end{array}$$

The function

$$f(x,y_0,y_1,y_2,y_3)=e^{y_0}+\arctan(y_1)-(y_2)^3-|y_3|^k \tag{29}$$

is continuous, satisfies conditions (3)and (5) in

$$E=\left\{\begin{array}{c}(x,y_0,y_1,y_2,y_3)\in[a,b]\times\mathbb{R}^5:\alpha_i\le y_i\le\beta_i,\ i=0,1\\ \alpha''\le y_2\le\beta''\end{array}\right\}.$$

and satisfies (24). By Theorem 4.2 there is a solution $u(x)$ *of problem (28),(2), such that*

$$\begin{array}{c}-x^2-x\le u(x)\le x^2+x,\\ -2x-1\le u'(x)\le 2x+1,\\ -2\le u''(x)\le 2,\end{array}$$

Notice that the nonlinearity f *given by (29) does not satisfy the two-sided Nagumo type conditions and, therefore,* [10] *can not be applied to (28)-(2). In fact, suppose by contradiction that there are a set* E *and a positive function* φ *such that* $|f(x,y_0,y_1,y_2,y_3)|\le\varphi(|y_3|)$ *in* E *and*

$$\int_0^{+\infty}\frac{s}{\varphi(s)}=+\infty.$$

Consider, in particular, that

$$f(x,y_0,y_1,y_2,y_3)\le\varphi(|y_3|),\qquad\forall(x,y_0,y_1,y_2,y_3)\in E,$$

and $(0,0,0,y_3)\in E$*. So, for* $x\in[0,1]$, $y_0=0$, $y_1=0$, $y_2=0$, *and* $y_3\in\mathbb{R}^+$,

$$f(x,0,0,0,y_3)=1+|y_3|^k\le\varphi(|y_3|).$$

As

$$\int_0^{+\infty}\frac{s}{1+s^k}ds,$$

is finite, the following contradiction is obtained:

$$+\infty>\int_0^{+\infty}\frac{s}{1+s^k}ds\ge\int_0^{+\infty}\frac{s}{\varphi(s)}ds=+\infty.$$

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Received September 2012; revised February 2013.

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