Fully nonlinear fourth order equations with functional boundary conditions

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Abstract

The aim of this paper consists in to give sufficient conditions to ensure the existence and location of the solutions of a nonlinear fully fourth order equation with functional boundary conditions.

The arguments make use of the upper and lower solutions method, a ϕ – laplacian operator and a fixed point theorem. An application of the beam theory to a nonlinear continuous model of the human spine allows to estimate its deformation under some loading forces.

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1 Introduction

In this paper it will be provided sufficient conditions to obtain existence and location results for the boundary value problem composed by the equation

$$-\left(\phi\left(u'''(x)\right)\right)' = f(x, u(x), u'(x), u''(x), u'''(x)), \text{ for a. e. } x \in I \equiv [a, b], \quad (1)$$

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with $\phi : \mathbb{R} \to \mathbb{R}$ an increasing and continuous function such that $\phi(0) = 0$, $f : I \times \mathbb{R}^4 \to \mathbb{R}$ a Carathéodory function and the functional boundary conditions

$$\begin{cases} u(b) &= A, \quad u'(b) = B, \qquad A, B \in \mathbb{R} \,, \\ 0 &= L_1(u, u', u'', u''(a), u'''(a)), \\ 0 &= L_2(u, u', u'', u''(b), u'''(b)), \end{cases} \tag{2}$$

where L_1 and L_2 are two continuous functions verifying some monotone conditions, to be precise.

The so-called beam equations have been studied in several works but, as far as we know, this type of fourth order fully nonlinear equation were never considered with such general functional boundary conditions. Therefore, the results contained in [9, 12, 13, 14, 17, 18] are improved.

Functional conditions (2) can be applied to several types of boundary value problems, such as, separated, multi-point or integro-differential problems. In this point of view, some cases of the four point boundary value problems of [5, 20] are generalized, more precisely when $b = d = 0$. The paper [6] is also improved, not only by the functional dependence on the boundary conditions but also by the more general definitions assumed for lower and upper solutions, allowing that, for example, both have the same sign.

The arguments make use of fixed point theory, lower and upper solutions method and some techniques suggested by [2, 4, 19] for second order and [1, 3] for third order nonlinear boundary value problems. The location part provided by these type of results is particularly useful, for instance, to prove the existence of positive, or negative, solutions (if the lower function is non-negative or the upper one non-positive) or to prove the multiplicity of solutions (if it is obtained the existence of solution in disjoint branches). Moreover, it gives some bounds on the solution and its derivatives, which are important tools in some models, as it is illustrated forward, for a continuous human spine model.

The second section contains some definitions and preliminary results, namely an *a priori* estimation on u''' obtained from a Nagumo-type growth assumption. Third section provides an auxiliary problem, with unique solution, and the main result: an existence and location theorem. In the last section it is referred a nonlinear model, that generalizes the classical beam equation presented in [16], used to study the deformation of the human spine under some loading conditions.

The results contained in this paper still hold if the explicit boundary data in (2) are replaced by

$$u(a) = A, \qquad u'(a) = B$$

or

$$u(a) = A, \qquad u'(b) = B$$

or

$$u(b) = A, \qquad u'(a) = B$$

under small and obvious modifications.

2 Definitions and *a priori* estimation

In this work it is considered a Carathéodory function $f : I \times \mathbb{R}^4 \to \mathbb{R}$, i. e., it satisfies the following conditions

(i) For each $x \in \mathbb{R}^4$ the function $f(\cdot, x)$ is measurable on I;

(ii) For a. e. $t \in I$ the function $f(t, \cdot)$ is continuous on $\mathbb{R}^4$;

(iii) For each compact set $K \subset \mathbb{R}^4$ there is a function $m_K \in L^1(I)$ such that $|f(t,x)| \le m_K(t)$ for a. e. $t \in I$ and all $x \in K$.

The functional part of the boundary conditions are defined by two continuous functions verifying the following monotonicity assumptions:

(H_1) $L_1 : C(I) \times C(I) \times C(I) \times \mathbb{R}^2 \to \mathbb{R}$ *is nondecreasing on the first, third and fifth variables and nonincreasing on the second one.*

(H_2) $L_2 : C(I) \times C(I) \times C(I) \times \mathbb{R}^2 \to \mathbb{R}$ *is nondecreasing on the first and third variables and nonincreasing on the second and fifth ones.*

Now we introduce the concept of lower and upper solutions of problem (1) – (2).

Definition 1 *A function $\alpha \in C^3(I)$, such that $\phi \circ \alpha''' \in AC(I)$, is a lower solution for problem (1)-(2) if it satisfies*

$$-\left(\phi\left(\alpha'''(x)\right)\right)' \le f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)), \text{ for a. e. } x \in I \tag{3}$$

together with

$$\begin{cases} \alpha(b) & \le & A, \quad \alpha'(b) \ge B, \\ 0 & \le & L_1(\alpha, \alpha', \alpha'', \alpha''(a), \alpha'''(a)), \\ 0 & \le & L_2(\alpha, \alpha', \alpha'', \alpha''(b), \alpha'''(b)). \end{cases}$$

A function $\beta \in C^3(I)$, such that $\phi \circ \beta''' \in AC(I)$ is an upper solution for problem (1)-(2) if

$$-\left(\phi\left(\beta'''(x)\right)\right)' \ge f(x, \beta(x), \beta'(x), \beta''(x), \beta'''(x)), \text{ for a. e. } x \in I \tag{4}$$

and

$$\begin{cases} \beta(b) & \ge & A, \quad \beta'(b) \le B, \\ 0 & \ge & L_1(\beta, \beta', \beta'', \beta''(a), \beta'''(a)), \\ 0 & \ge & L_2(\beta, \beta', \beta'', \beta''(b), \beta'''(b)). \end{cases} \tag{5}$$

We say that u is a solution of problem (1) – (2) if it is both a lower and an upper solution.

To deduce existence results we will use the following variation of Schauder fixed point theorem given in [10, Theorem 4.4.6]:

Theorem 2 *Let S be a bounded, closed, non-empty, convex subset of a normed space X and $F : S \to S$ a compact operator. Then F has a fixed point.*

To construct the set S and the compact operator F it is necessary to impose to the nonlinear part of (1) some growth restriction defined by a Nagumo-type condition. This condition is used in the study of boundary value problems and, despite they are not a necessary condition to deduce existence results [8], they cannot be avoid in general [7]. The imposed condition is the following

Definition 3 *Given $\gamma_j, \Gamma_j \in C(I)$ such that $\gamma_j(x) \leq \Gamma_j(x)$, for all $x \in I$ and $j = 0, 1, 2,$ consider the set*

$$E = \left\{(x, y_0, y_1, y_2, y_3) \in I \times \mathbb{R}^4 : \gamma_j(x) \leq y_j \leq \Gamma_j(x),\ \forall x \in I,\ j = 0, 1, 2\right\}.$$

A function $f : I \times \mathbb{R}^4 \to \mathbb{R}$ is said to satisfy a Nagumo-type condition in E if there exists $h_E \in C([0, +\infty), (0, +\infty))$, such that

$$|f(x, y_0, y_1, y_2, y_3)| \leq h_E(|y_3|),\ \text{for a. e. } (x, y_0, y_1, y_2, y_3) \in E, \tag{6}$$

and

$$\min\left\{\int_{\phi(\eta)}^{\phi(+\infty)} \frac{\left|\phi^{-1}(s)\right|^{\frac{p-1}{p}}}{h_E(\phi^{-1}(s))} ds\ ,\ \int_{\phi(-\infty)}^{\phi(-\eta)} \frac{\left|\phi^{-1}(s)\right|^{\frac{p-1}{p}}}{h_E(\phi^{-1}(s))} ds\right\} > \mu^{\frac{p-1}{p}} (b-a)^{\frac{1}{p}}, \tag{7}$$

where

$$\eta := \max\left\{\frac{\Gamma_2(b) - \gamma_2(a)}{b-a}, \frac{\Gamma_2(a) - \gamma_2(b)}{b-a}\right\}, \tag{8}$$

and

$$\mu := \max_{x \in I} \Gamma_2(x) - \min_{x \in I} \gamma_2(x). \tag{9}$$

Remark 4 *As $\phi(\mathbb{R}) = \mathbb{R}$ is not assumed then $\phi(\pm\infty)$ can be finite.*

Next result provides an *a priori* bound for the third derivative of solutions of equation (1).

Lemma 5 *Consider $\gamma_j, \Gamma_j \in C(I)$, $j = 0, 1, 2,$ such that*

$$\gamma_j(x) \leq \Gamma_j(x),\ \forall\, x \in I,$$

and let $f : E \to \mathbb{R}$ be a Carathéodory function satisfying a Nagumo-type condition in E. Then there exists $N > 0$ (depending only on γ_2, Γ_2 and h_E) such that for every solution u of (1) satisfying

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x),\ i = 0, 1, 2, \tag{10}$$

verifies

$$\|u'''\|_\infty < N.$$

Proof. Consider $N > 0$ large enough such that $N > \eta$ and

$$\min \left\{ \int_{\phi(\eta)}^{\phi(N)} \frac{\left|\phi^{-1}(s)\right|^{\frac{p-1}{p}}}{h_E(\phi^{-1}(s))} ds \; , \; \int_{\phi(-N)}^{\phi(-\eta)} \frac{\left|\phi^{-1}(s)\right|^{\frac{p-1}{p}}}{h_E(\phi^{-1}(s))} ds \right\} > \mu^{\frac{p-1}{p}} (b-a)^{\frac{1}{p}} \; , \tag{11}$$

with η given in (8) and μ defined in (9).

Let u be a solution of (1) such that (10) holds. Therefore there is $x_0 \in (a, b)$ such that

$$u'''(x_0) = \frac{u''(b) - u''(a)}{b - a}$$

and

$$-N < -\eta \leq \frac{\gamma_2(b) - \Gamma_2(a)}{b - a} \leq u'''(x_0) \leq \frac{\Gamma_2(b) - \gamma_2(a)}{b - a} \leq \eta < N.$$

If $|u'''(x)| < N$ for every $x \in I$ the proof is finished. If not, assume that there is $x \in I$ such that $u'''(x) > N$ or $u'''(x) < -N$. In the first case, suppose that there exists $[x_1, x_2] \subset [a, b]$ such that $u'''(x_2) = \max\{0, u'''(x_0)\}$ and $u'''(x_2) \leq u'''(x) \leq u'''(x_1) = N$ for every $x \in [x_1, x_2]$.

As f verifies the Nagumo condition then, by (6),

$$\left|(\phi(u'''(x)))'\right| = |f(x, u(x), u'(x), u''(x), u'''(x))| \leq h_E(|u'''(x)|), \; \forall x \in [x_1, x_2]$$

and, by (7), the following contradiction with (11) is obtained

$$\begin{aligned}
\int_{\phi(\eta)}^{\phi(N)} \frac{\left|\phi^{-1}(s)\right|^{\frac{p-1}{p}}}{h_E(|\phi^{-1}(s)|)} ds &\leq \int_{\phi(u'''(x_2))}^{\phi\left(u'''(x_1)\right)} \frac{\left|\phi^{-1}(s)\right|^{\frac{p-1}{p}}}{h_E(|\phi^{-1}(s)|)} ds \\
&= \int_{x_2}^{x_1} \frac{|u'''(x)|^{\frac{p-1}{p}}}{h_E(|u'''(x)|)} (\phi(u'''(x)))' \; dx \\
&\leq \int_{x_1}^{x_2} \frac{|f(x, u(x), u'(x), u''(x), u'''(x))|}{h_E(|u'''(x)|)} |u'''(x)|^{\frac{p-1}{p}} dx \\
&\leq (x_2 - x_1)^{\frac{1}{p}} [u''(x_2) - u''(x_1)]^{\frac{p-1}{p}} \\
&\leq (b-a)^{\frac{1}{p}} \; \mu^{\frac{p-1}{p}}.
\end{aligned}$$

For $[x_1, x_2] \subset [a, b]$ such that $u'''(x_1) = \max\{0, u'''(x_0)\}$ and $u'''(x_1) \leq u'''(x) \leq u'''(x_2) = N$ the arguments are similar and the same happen for the second case. ■

3 Existence and location results

The first result to be presented in this section ensures the existence and uniqueness of the solution of a suitable related problem of (1) – (2).

Lemma 6 *Consider $\varphi : \mathbb{R} \to \mathbb{R}$ an increasing homeomorphism such that $\varphi(0) = 0$ and $\varphi(\mathbb{R}) = \mathbb{R}$, $g : I \to \mathbb{R}$ a L^1 – function and $A,\ B,\ h_1,\ h_2 \in \mathbb{R}$. Then the problem*

$$\left\{ \begin{array}{c} -\left(\varphi\left(u'''(x)\right)\right)' = g(x), \quad \text{for a. e. } x \in I, \\ u(b) = A, \quad u'(b) = B, \\ u''(a) = h_1\ ,\ \ u''(b) = h_2, \end{array} \right. \tag{12}$$

has a unique solution given by the following expression

$$u(x) = A - B\ \ (b - x) + \int_x^b \int_s^b v\,(r)\ \ dr\ ds,$$

with

$$v\,(x) := h_1 + \int_a^x \varphi^{-1}\left(\tau_v - \int_a^s g\,(r)\ \ dr\right) ds$$

and $\tau_v \in \mathbb{R}$ the unique solution of the equation

$$h_2 - h_1 = \int_a^b \varphi^{-1}\left(\tau_v - \int_a^s g\,(r)\ \ dr\right) ds. \tag{13}$$

Proof. By the change of variable $u''(x) = v(x)$ it is obtained the Dirichlet problem

$$-\left(\varphi\left(v'(x)\right)\right)' = g(x), \quad \text{for a. e. } x \in I, \quad v(a) = h_1\ ,\ \ v(b) = h_2. \tag{14}$$

Then, for some $\tau \in \mathbb{R}$, the following identities hold

$$v'(x) = \varphi^{-1}\left(\tau - \int_a^x g\,(r)\ \ dr\right),$$

$$v\,(x) = h_1 + \int_a^x \varphi^{-1}\left(\tau - \int_a^s g\,(r)\ \ dr\right) ds. \tag{15}$$

Now, define

$$h(\tau) = h_1 + \int_a^b \varphi^{-1}\left(\tau - \int_a^s g\,(r)\ \ dr\right) ds.$$

From the properties of function φ, we have that function h is continuous, strictly increasing and satisfies $h(\mathbb{R}) = \mathbb{R}$. As consequence equation (13) has a unique solution τ_v. Integrating (15) twice,

$$u(x) = A - B\ \ (b - x) + \int_x^b \int_s^b v\,(r)\ \ dr\ ds$$

is the unique solution of problem (12). ∎

Before proving the existence result, we enunciate the following lemma given in [19, lemma 2].

Lemma 7 *For $z, w \in C(I)$ such that $z(x) \leq w(x)$, for every $x \in I$, define*

$$p(x,u) = max\{z, min\{u,w\}\}.$$

Then, for each $u \in C^1(I)$ the next two properties hold:

(a) $\dfrac{d}{dx}p(x,u(x))$ *exists for a.e. $x \in I$.*

(b) If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$ then

$$\frac{d}{dx}p(x,u_m(x)) \to \frac{d}{dx}p(x,u(x)) \text{ for a.e. } x \in I.$$

Theorem 8 *Assume that there is α a lower solution and β an upper solution of problem (1)-(2) such that $\alpha''(x) \leq \beta''(x)$, for every $x \in I$. Suppose that assumptions (H_1) and (H_2) hold and let $f : I \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying a Nagumo-type condition in*

$$E_* = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3) \in I \times \mathbb{R}^4 : \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), \ i = 0, 2, \\ \beta'(x) \leq y_1 \leq \alpha'(x) \end{array} \right\}$$

and, for $(x, y_2, y_3) \in I \times \mathbb{R}^2$ fixed, $\alpha(x) \leq y_0 \leq \beta(x)$, $\beta'(x) \leq y_1 \leq \alpha'(x)$,

$$f(x, \alpha(x), \alpha'(x), y_2, y_3) \leq f(x, y_0, y_1, y_2, y_3) \leq f(x, \beta(x), \beta'(x), y_2, y_3). \tag{16}$$

Then problem (1)-(2) has a solution u such that

$$\alpha(x) \leq u(x) \leq \beta(x), \ \beta'(x) \leq u'(x) \leq \alpha'(x), \ \alpha''(x) \leq u''(x) \leq \beta''(x), \ \forall\, x \in I.$$

Remark 9 *Note that $\alpha'' \leq \beta''$ coupled with the definition of lower and upper solutions imply that $\beta' \leq \alpha'$ and $\alpha \leq \beta$.*

Proof. Define the continuous truncations, for $i = 0, 2$,

$$\delta_i(x, y_i) = \begin{cases} \beta^{(i)}(t) & \text{if} \quad y_i > \beta^{(i)}(x), \\ y_i & \text{if} \quad \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), \\ \alpha^{(i)}(x) & \text{if} \quad y_i < \alpha^{(i)}(x). \end{cases} \tag{17}$$

and

$$\delta_1(x, y_1) = \begin{cases} \beta'(t) & \text{if} \quad y_1 < \beta'(x), \\ y_1 & \text{if} \quad \beta'(x) \leq y_1 \leq \alpha'(x), \\ \alpha'(x) & \text{if} \quad y_1 > \alpha'(x). \end{cases} \tag{18}$$

Consider

$$N > \max\{\eta, \|\alpha'''\|_\infty, \|\beta'''\|_\infty\}$$

satisfying condition (11), and define the homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(y) = \begin{cases} \phi(y) & \text{if} \quad |y| \leq N, \\ \frac{\phi(N)-\phi(-N)}{2N}\, y + \frac{\phi(N)+\phi(-N)}{2} & \text{if} \quad |y| > N. \end{cases}$$

Let now

$$q(y)=\max\{-N,\min\{y,\ N\}\} \tag{19}$$

and consider the modified problem composed by the differential equation

$$-\left(\varphi\left(u'''(x)\right)\right)'=f\left(x,\delta_0\left(x,u\right),\delta_1\left(x,u'\right),\delta_2\left(x,u''\right),q(\frac{d}{d\,t}\delta_2\left(x,u''\right))\right)\equiv F_u(x) \tag{20}$$

with the boundary conditions

$$\begin{aligned} u(b) &= A, \quad u'(b)=B, \\ u''(a) &= \delta_2\left(a,u''(a)+L_1(u,u',u'',u''(a),u'''(a))\right)\equiv\kappa_1(u), \\ u''(b) &= \delta_2(b,u''(b)+L_2(u,u',u'',u''(b),u'''(b)))\equiv\kappa_2(u). \end{aligned} \tag{21}$$

We say that $u\in C^3(I)$ such that $\phi\circ u'''\in AC(I)$ is a solution of problem (20)-(21) if it satisfies the previous five equalities. We remark that, from Lemma 7 and the definition of q, the right hand side of the equation (20) is a L^1- function.

Step 1: *Every solution of problem (20)-(21) verifies*

$$\alpha\left(x\right)\ \leq\ u\left(x\right)\leq\beta\left(x\right), \tag{22}$$

$$\beta'\left(x\right)\ \leq\ u'\left(x\right)\leq\alpha'\left(x\right), \tag{23}$$

$$\alpha''\left(x\right)\ \leq\ u''\left(x\right)\leq\beta''\left(x\right) \tag{24}$$

$$-N\ \leq\ u'''\left(x\right)\leq N \tag{25}$$

Let u be a solution of (20)-(21). Assume, by contradiction, that (24) does not hold. Define

$$\max_{x\in I}\left(u-\beta\right)''\left(x\right):=\left(u-\beta\right)''\left(x_0\right)>0.$$

As, by (21), $u''(a)\leq\beta''(a)$ and $u''(b)\leq\beta''(b)$ then $x_0\in(a,b)$, $u'''(x_0)=\beta'''(x_0)$ and $u''>\beta''$ on $[x_0,x_0+r)$ for some $r>0$ such that $u''(x_0+r)=\beta''(x_0+r)$. Moreover $u'''\left(x_0\right)=\beta'''\left(x_0\right)$.

So, by using (4), (16), (17), (19) and the choice of N, we arrive at the following inequality on (x_0,x_0+r):

$$\begin{aligned} -\left(\varphi\left(u'''(x)\right)\right)' &= f\left(x,\delta_0\left(x,u\right),\delta_1\left(x,u'\right),\delta_2\left(x,u''\right),q(\frac{d}{d\,t}\delta_2\left(x,u''\right))\right) \\ &= f\left(x,\delta_0\left(x,u\right),\delta_1\left(x,u'\right),\beta''\left(x\right),\beta'''\left(x\right)\right) \\ &\leq f\left(x,\beta\left(x\right),\beta'\left(x\right),\beta''\left(x\right),\beta'''\left(x\right)\right) \\ &\leq -\left(\phi\left(\beta'''(x)\right)\right)' \\ &= -\left(\varphi\left(\beta'''(x)\right)\right)'. \end{aligned}$$

This property implies that $u'''\geq\beta'''$ on (x_0,x_0+r), which contradicts the definition of $r>0$.

So $u''\left(x\right)\leq\beta''\left(x\right)$ for every $x\in I$ and by similar arguments it can be proved that $\alpha''\left(x\right)\leq u''\left(x\right)$ in I.

Integrating (24), conditions (23) and (22) are easily obtained by the boundary conditions of lower and upper solutions and (21).

At last, applying Lemma 5 with $\gamma_j(x) = \alpha^{(j)}(x)$, $\Gamma_j(x) = \beta^{(j)}(x)$, for $j = 0, 2$, $\gamma_1(x) = \beta'(x)$ and $\Gamma_1(x) = \alpha'(x)$, then condition (25) holds.

Step 2: *Problem (20)-(21) has a solution $u_1(x)$.*

Let $u \in C^3(I)$ be fixed. From Lemma 6 it is clear that the solutions of problem (20)-(21) are the fixed points of the operator

$$\mathcal{T}\, u(x) = A - B\ (b - x) + \int_x^b \int_s^b v_u\,(r)\ dr\ ds, \tag{26}$$

with

$$v_u\,(x) := \kappa_1(u) + \int_a^x \varphi^{-1}\left(\tau_u - \int_a^s F_u\,(r)\ dr\right) ds$$

and $\tau_u \in \mathbb{R}$ the unique solution of the equation

$$\kappa_2(u) - \kappa_1(u) = \int_a^b \varphi^{-1}\left(\tau_u - \int_a^s F_u\,(r)\ dr\right) ds. \tag{27}$$

Note that there exists $\Psi \in L^1(I)$ such that

$$\mid F_u(s) \mid \leq \Psi(s) \text{ for a. e. } s \in I \text{ and for all } u \in C^3(I),$$

and then, since $\kappa_2(u) - \kappa_1(u)$ is bounded in $C^3(I)$, there exists $L > 0$ such that

$$\mid \tau_u \mid \leq L \text{ for all } u \in C^3(I) \tag{28}$$

So, we conclude that operator $\mathcal{T}(C^3(I))$ is bounded in $C^3(I)$.

To verify that operator $\mathcal{T}$ is compact in $C^3(I)$ we follow the arguments given in [4, Theorem 2.1].

The proof follows as a direct application of Theorem 2.

Step 3: *$u_1(x)$ is a solution of problem (1)-(2).*

This function $u_1(x)$ will be a solution of (1)-(2) if it verifies,

$$\begin{aligned} \alpha''\,(a) &\leq u''(a) + L_1(u, u', u'', u''(a), u'''(a)) \leq \beta''\,(a) \quad (29)\\ \alpha''\,(b) &\leq u''(b) + L_2(u, u', u'', u''(b), u'''(b)) \leq \beta''\,(a)\,. \quad (30) \end{aligned}$$

Suppose, by contradiction, that

$$u''(a) + L_1(u, u', u'', u''(a), u'''(a)) > \beta''\,(a)\,.$$

Thus

$$u''(a) = \delta_2\,(a, u''(a) + L_1(u, u', u'', u''(a), u'''(a))) = \beta''\,(a)$$

by (24), $u'''\,(a) = u'''\,(a^+) \leq \beta'''\,(a^+) = \beta'''\,(a)$ and, by (H_1), (22), (23), (24) and (5), it is obtained the contradiction

$$\begin{aligned} u''\,(a) + L_1(u, u', u'', u''(a), u'''(a)) &= \beta''\,(a) + L_1(u, u', u'', \beta''(a), u'''(a)) \\ &\leq \beta''\,(a) + L_1(u, u', u'', \beta''(a), \beta'''(a)) \\ &\leq \beta''\,(a) + L_1(\beta, \beta', \beta'', \beta''(a), \beta'''(a)) \\ &\leq \beta''\,(a)\,. \end{aligned}$$

By analogous technique it can be proved that $\alpha''(a) \leq u''(a)+L_1(u,u',u'',u''(a),u'''(a))$ and (29) hold.

Inequality (30) is proved in the same way. ■

Now, given $A,\ B \in \mathbb{R}$ and $q_1,\ q_2 \geq 0$, defining

$$\begin{aligned} L_1(u,v,w,x,y) &= -p_1\,x + q_1\,y + C, \\ L_2(u,v,w,x,y) &= -p_2\,x - q_2\,y + D, \end{aligned}$$

we obtain, as a corollary of the previous result, the following one

Corollary 10 *Suppose that there exist* $\alpha,\ \beta \in C^3(I)$*, such that* $\phi\circ\alpha''',\ \phi\circ\beta''' \in AC(I)$*, satisfying inequalities (3) and (4) respectively. Moreover, if* $\alpha'' \leq \beta''$ *on* I *and the following inequalities hold:*

$$\begin{gathered} \alpha(b) \leq A \leq \beta(b), \\ \alpha'(b) \geq B \geq \beta'(b), \\ p_1\alpha''(a) - q_1\alpha'''(a) \leq C \leq p_1\beta''(a) - q_1\beta'''(a), \\ p_2\alpha''(b) + q_2\alpha'''(b) \leq D \leq p_2\beta''(b) + q_2\beta'''(b) \end{gathered}$$

and function f *satisfies the hypotheses imposed in Theorem 8, then problem composed by (1) together with the initial - Separated boundary conditions*

$$\begin{aligned} u(b) &= A, \\ u'(b) &= B, \\ p_1u''(a) - q_1u'''(a) &= C, \\ p_2u''(b) + q_2u'''(b) &= D. \end{aligned}$$

has a solution u *such that*

$$\alpha\,(x) \leq u\,(x) \leq \beta\,(x)\,,\ \ \beta'\,(x) \leq u'\,(x) \leq \alpha'\,(x)\,,\ \ \alpha''\,(x) \leq u''\,(x) \leq \beta''\,(x)\,,\ \ \forall\, x \in I.$$

Remark 11 *Note that in this case, on the contrary to the usual situations, we do not impose to the real constants* p_1 *and* q_1 *to be nonnegative.*

Another particular case is given by the multi-point boundary value conditions

$$u(b) = A, \quad u'(b) = B, \quad u''(a) = \sum_{i=1}^{m-2} a_i\,u''(\tau_i), \quad u''(b) = \sum_{i=1}^{n-2} b_i\,u''(\eta_i), \quad (31)$$

where $a < \tau_1 < \tau_2 < \cdots < \tau_{m-2} < b$, $a < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < b$ and a_i, $b_j \geq 0$ for all $i \in \{1,\ldots,m-2\}$ and $j \in \{1,\ldots,n-2\}$.

Defining

$$\begin{aligned} L_1(u,v,w,x,y) &= \sum_{i=1}^{m-2} a_i\,w(\tau_i) - x, \\ L_2(u,v,w,x,y) &= \sum_{i=1}^{n-2} b_i\,w(\eta_i) - x \end{aligned}$$

we arrive at the following result

Corollary 12 *Suppose that there exist $\alpha,\ \beta \in C^3(I)$, such that $\phi \circ \alpha'''$, $\phi \circ \beta''' \in AC(I)$, satisfying inequalities (3) and (4) respectively. Moreover if $\alpha'' \leq \beta''$ on I and the following inequalities hold:*

$$\begin{aligned} \alpha(b) \leq A &\leq \beta(b), \\ \alpha'(b) \geq B &\geq \beta'(b), \\ \alpha''(a) \leq \sum_{i=1}^{m-2} a_i\, \alpha''(\tau_i) &\leq \sum_{i=1}^{m-2} a_i\, \beta''(\tau_i) \leq \beta''(a), \\ \alpha''(b) \leq \sum_{i=1}^{n-2} b_i\, \alpha''(\eta_i) &\leq \sum_{i=1}^{n-2} b_i\, \beta''(\eta_i) \leq \beta''(b) \end{aligned}$$

and function f satisfies the hypotheses imposed in Theorem 8, then problem (1), (31) has a solution u such that

$$\alpha\,(x) \leq u\,(x) \leq \beta\,(x)\,,\ \beta'\,(x) \leq u'\,(x) \leq \alpha'\,(x)\,,\ \alpha''\,(x) \leq u''\,(x) \leq \beta''\,(x)\,,\ \forall\, x \in I.$$

4 Continuous model for the human spine deformation

The mechanical properties of the human spine have been studied by several authors (see [16] and the references therein) by a continuous beam model. Of course this type of models cannot describe certain local features of the spine. However they are useful to analyze the overall deformation of the spine under various loading conditions, such as, aircraft ejections and vehicle crash situations.

Furthermore the cantilever beam model correlate reasonably well some characteristics or certain form of scoliosis (see [15], its references and Figure 2). In fact the total lateral displacement, $y(x)$, of the beam-column, with length L, is expressed as the sum of the initial lateral displacement, $y_0(x)$, and the lateral displacement due to the axial and transverse loads, $y_1(x)$, i.e.,

$$y(x) = y_0(x) + y_1(x).$$

This displacement $y_1(x)$ is modelled [16] by the differential equation

$$EI\ y_1^{(4)} + P\ y_1'' = Q - P\ y_0''$$

where EI is the flexural rigidity of the beam-column, P the axial load and Q the transverse load. As it is well known in the elasticity theory, this shearing force Q is related with the third derivative of the displacement (see, for instance, [11]). Therefore we consider the nonlinear equation

$$EI\ y_1^{(4)} + P\ y_1'' = q(y_1''') - P\ y_0'' \tag{32}$$

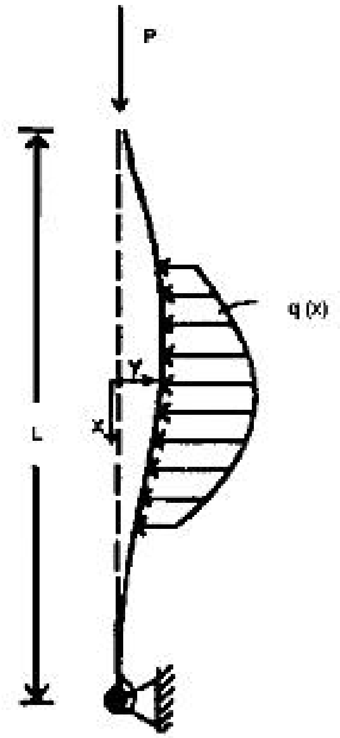

Figure 1: A continuous model for a beam-spine system

with q some continuous function (Figure 1).

The boundary conditions for the unknown lateral displacement $y_1(x)$, considered here, model a cantilever beam-column on the lower end whose curvature attains its maximal value on the left endpoint of the interval $J := [-L/2, L/2]$:

$$\left\{ \begin{array}{rcl} y_1\left(\dfrac{L}{2}\right) & = & 0, \\ y_1'\left(\dfrac{L}{2}\right) & = & 0, \\ y_1''\left(-\dfrac{L}{2}\right) & = & \max\limits_{x \in J} \{y_1''(x)\}, \\ y_1''\left(\dfrac{L}{2}\right) & = & 0. \end{array} \right. \tag{33}$$

By defining in this case

$$\phi(z) \equiv z,$$

$$f(x, z_1, z_2, z_3, z_4) = -\frac{P}{EI}\, z_3 + \frac{q(z_4) - P\, y_0''(x)}{EI},$$

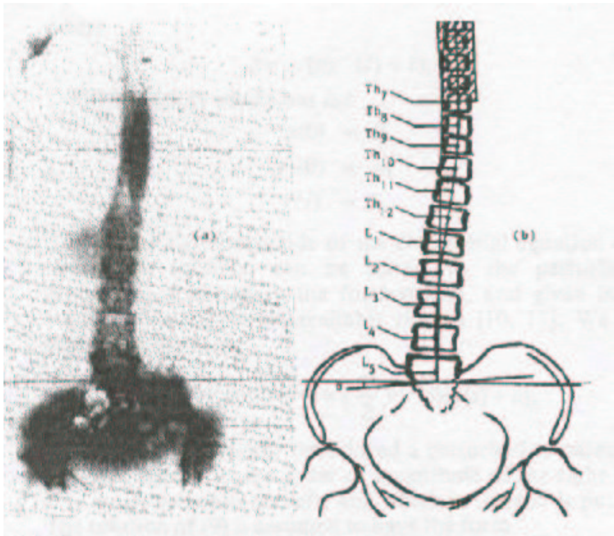

Figure 2: (a) Frontal X-ray picture of the flexible waist part Th7-Th12, L1-L5 of the spine of a patient with lumbar scoliosis, caused by a difference in leg length of 0.5 cm and standing in an upright muscle-relaxed position. (b) Extracted contour picture of (a).

$$L_1(z_1, z_2, z_3, z_4, z_5) = -z_4 + \max_{x \in J} \{z_3(x)\},$$

and

$$L_2(z_1, z_2, z_3, z_4, z_5) = -z_4$$

it is immediate to verify that problem (32) – (33) is a particular case of problem (1) – (2).

Assuming that the initial lateral displacement $y_0 \in W^{2,1}(J)$ then if the transverse force q, the axial force P, constant EI and the beam length L verify the following relations for $x \in J$:

$$\min_{t \in [-3aL/4, aL/4]} q(t) \geq a \left(-EI + P\frac{x^2}{2} - PL\frac{x}{4} - P\frac{L^2}{10} \right) + P\, y_0''(x), \tag{34}$$

$$\max_{t \in [-2bL, -bL]} q(t) \leq b \left(-EI + P\frac{x^2}{2} - PL\frac{3x}{2} + PL^2 \right) + P\, y_0''(x), \tag{35}$$

for some $b \geq 2a/25 \geq 0$, one can verify that the functions

$$\alpha\,(x) = \frac{a}{24}x^4 - \frac{a}{24}Lx^3 - \frac{a}{20}L^2x^2 + aL^3x - \frac{a}{2}L^4$$

and

$$\beta\,(x) = \frac{b}{24}x^4 - \frac{b}{4}Lx^3 + \frac{b}{2}L^2x^2 - 2bL^3x + bL^4$$

are, respectively, lower and upper solutions of problem (32) – (33). As f verifies a Nagumo-type condition in every set

$$E=\left\{(x,z)\in\left[-\frac{L}{2},\frac{L}{2}\right]\times\mathbb{R}:\frac{a}{2}x^2-\frac{a}{4}Lx-\frac{a}{10}L^2\leq z\leq\frac{b}{2}x^2-\frac{3}{2}bLx+b\ L^2\right\}$$

then, by Theorem 8, there is a solution of problem (32)-(33) such that the

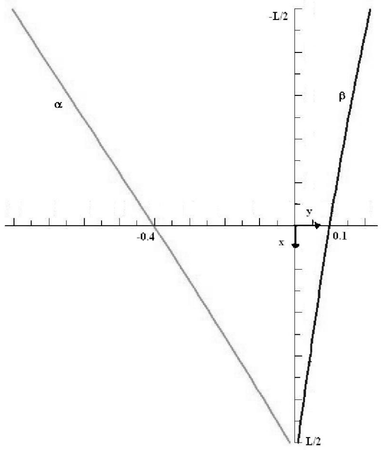

Figure 3: For $a=0.8, b=0.1, L=EI=P=1$ there is a solution in the strip bounded by lower and upper solutions of problem (32)-(33)

lateral displacement due to the axial and transverse loads verifies

$$\frac{a}{24}x^4-\frac{a}{24}Lx^3-\frac{a}{20}L^2x^2+aL^3x-\frac{a}{2}L^4\leq y_1(x)\leq\frac{b}{24}x^4-\frac{b}{4}Lx^3+\frac{b}{2}L^2x^2-2bL^3x+bL^4,$$

$$\frac{b}{6}x^3-\frac{3b}{4}Lx^2+b\ L^2x-2\ b\ L^3\leq y_1'(x)\leq\frac{a}{6}x^3-\frac{a}{8}Lx^2-\frac{a}{10}L^2x+aL^3$$

and

$$\frac{a}{2}x^2-\frac{a}{4}Lx-\frac{a}{10}L^2\leq y_1''(x)\leq\frac{b}{2}x^2-\frac{3}{2}bLx+b\ L^2$$

for every $x\in J$.

We note that small values of the transverse force q allow small values on a,b and, therefore, on the lateral displacement of the spine y_1.

As example, it can be mentioned that for $a=0.8, b=0.1, L=EI=P=1$, provided function q satisfies properties (34) and (35), there is a solution $y_1^*(x)$

of problem (32)-(33) such that

$$\frac{x^4}{30}-\frac{x^3}{30}-\frac{x^2}{25}+\frac{4x}{5}-\frac{2}{5}\leq y_1^*(x)\leq\frac{x^4}{240}-\frac{x^3}{40}+\frac{x^2}{20}-\frac{x}{5}+\frac{1}{10}$$

for $x\in J$, as it is illustrated in Figure 3.

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