AUTHOR QUERY FORM

ELSEVIER	**Journal:** Journal of Geometry and Physics **Article Number:** 1725	**Please e-mail or fax your responses and any corrections to:** **E-mail: corrections.esch@elsevier.river-valley.com** **Fax: +44 1392 285879**

Dear Author,

Please check your proof carefully and mark all corrections at the appropriate place in the proof (e.g., by using on-screen annotation in the PDF file) or compile them in a separate list.

For correction or revision of any artwork, please consult http://www.elsevier.com/artworkinstructions.

Any queries or remarks that have arisen during the processing of your manuscript are listed below and highlighted by flags in the proof. Click on the 'Q' link to go to the location in the proof.

Location in article	**Query / remark click on the Q link to go** **Please insert your reply or correction at the corresponding line in the proof**
Q1	Color statement has been added to the caption(s) of Figs. 7 and 8. Please check.
Q2	Uncited references: This section comprises references that occur in the reference list but not in the body of the text. Please cite each reference in the text or, alternatively, delete it. Any reference not dealt with will be retained in this section.

Thank you for your assistance.

GEOPHY: 1725 Model 3G pp. 1–13 (col. fig: NIL)

ARTICLE IN PRESS

Journal of Geometry and Physics xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Geodesic length spectrum on compact Riemann surfaces

Clara Grácio [a,*], J. Sousa Ramos [b]

[a] Department of Mathematics, Universidade de Évora, Rua Romão Ramalho, 59, 7000-585 Évora, Portugal

[b] Department of Mathematics, Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

ARTICLE INFO

Article history:
Received 26 February 2009
Received in revised form 14 June 2010
Accepted 19 June 2010
Available online xxxx

MSC:
37B10
34C15
37E05
37B40

Keywords:
Geodesic length spectrum
Hyperbolic geometry
Fundamental group

ABSTRACT

In this paper we use techniques linking combinatorial structures (symbolic dynamics) and algebraic–geometric structures to study the variation of the geodesic length spectrum, with the Fenchel–Nielsen coordinates, which parametrize the surface of genus $\tau = 2$. We explicitly compute length spectra, for all closed orientable hyperbolic genus two surfaces, identifying the exponential growth rate and the first terms of a growth series.

1. Introduction

The various relations between hyperbolic geometry and number theory have prosed a large number of interesting problems. Much of work carried out to resolve them involves interplay between these two fields and, in particular, the use of algebra and number theory to answer topological and geometric questions regrading low dimensional hyperbolic manifolds. The geodesic flow on a surface of constant negative curvature was the first non-trivial example of symbolic dynamics. It is known that the dynamical problem of counting periodic orbits is equivalent to the (group) problem of counting conjugacy classes in the fundamental group of the surface and that the limit set of this group can be coded as a subshift of finite type (see [1]). Several peculiarities are associated with this problem: the presence of an infinite algebra of Hermitian operators commuting with the Laplacian (Hecke operators) and an exponential growth in the number of lengths of periodic orbits (closed geodesics). The metric and geometric structure of surfaces can be analyzed using the closed geodesic spectrum and the Laplace–Beltrami operator spectrum. Obtaining these spectra is not easy; however, what is more difficult is describing their dependence with the parameters which determine the metric and geometric structure of the surface. We study these spectra dependence of these using a boundary map when a Riemann surface M of genus 2, thus with negative curvature, is considered.

The present paper is part of a program aimed at providing an understanding of Laplacian spectrum and geodesic length spectrum behavior, of the compact Riemannian manifold endowed with a metric of constant curvature of -1. We use techniques linking combinatorial structures (symbolic dynamics) and algebraic–geometric structures. Given a compact

* Corresponding author. Tel.: +351 266745370; fax: +351 266745393.
E-mail addresses: mgracio@uevora.pt (C. Grácio), sramos@math.ist.utl.pt (J.S. Ramos).

0393-0440/$ – see front matter © 2010 Published by Elsevier B.V.
doi:10.1016/j.geomphys.2010.06.006

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006

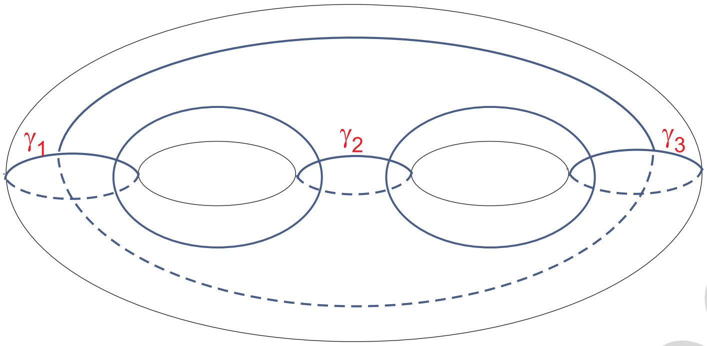

Fig. 1. Compact Riemann surface of genus 2 and the geodesics $\gamma_1, \gamma_2, \gamma_3$.

Riemannian manifold M, the Laplace–Beltrami operator Δ defined on functions on M is an elliptic operator with discrete spectrum

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty.$$

For surfaces of genus $\tau = 2$, in a previous paper, we computed the geodesic length spectrum of M (lengths of closed geodesics) (see [2,3]),

$$0 < \ell_1 \leq \ell_2 \leq \cdots \leq \ell_k \leq \cdots \to \infty.$$

These surfaces were obtained by gluing together pairs of pants with no twists in the boundary components. This corresponds to Riemann surfaces in a Teichmüller space for which the Fenchel–Nielsen coordinates are of the form $(\ell(\gamma_1), \ell(\gamma_2), \ell(\gamma_3), 0, 0, 0)$, where $\ell(\gamma_i)$, with $i = 1, 2, 3$, are the lengths of three geodesics on the surface (see Fig. 1) (see [4,5]). We will denote ℓ_i as length $\ell(\gamma_i)$ with $i = 1, 2, 3$. The special nature of genus 2 has made it more accessible for providing more detailed results. The set of equivalence classes of hyperbolic metrics (or equivalently complex structures) under orientation preserving diffeomorphisms on M forms the moduli space $\mathcal{M}$ of compact Riemann surfaces of genus τ. It is represented by a quotient space $M = H^2/G$ of the upper half-plane H^2 by a Fuchsian group G which is isomorphic to the fundamental group of M. The discrete group G is identified with the corresponding system of generators. A fundamental domain $\mathcal{F}$ is defined. The method is to decompose the Riemann surface into a set of 2 pairs of pants using three simple closed geodesics. Then the Fenchel–Nielsen coordinates are defined by using geodesic length functions of the three simple closed geodesics, γ_i and twist angles σ_i, along these geodesics, with $i = 1, 2, 3$. With explicit constructions and side pairing transformations (see [3]), we define the Fuchsian group G associated with the closed Riemann surface of genus 2 (see Fig. 3).

This paper is organized as follows. In Section 2 we introduce a geometric description of the surface, the universal covering space, the Fenchel–Nielsen parameters, a more familiar way of parametrizing the Teichmüller space, and an explicit geometrical construction of the fundamental domain. In Section 3 we compute the generators of the fundamental group and describe the construction of the boundary map and the associated Markov matrix. Finally, in Section 4, we obtain explicit values for the geodesic length spectrum and analyze its behavior with the variation of the Fenchel–Nielsen coordinates identifying, in particular, the exponential growth rate and the first terms of the growth series.

The preparation of this paper was overshadowed by the death of professor Sousa Ramos in January, 2007. Our original intention was to write it jointly: most of the main ideas in this paper were worked out jointly with him, and I have done my best to complete the paper. I grieve the loss of my professor and my friend, and dedicate this paper to him.

2. Geometric description and Fenchel–Nielsen parameters

Throughout this paper, all surfaces are closed Riemann surfaces of genus 2, and all references to lengths, distances, etc. are to be understood in terms of hyperbolic geometry. There are many interesting spaces associated with a surface $\mathcal{M}$. In order to studying dynamical systems on $\mathcal{M}$, one considers the group of automorphisms of $\mathcal{M}$ itself; for studying geometry, geometers are interested in Teichmuller space, which is the space of hyperbolic structures. We are interested in both approaches and the relations between them.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be compact orientable Riemann surfaces. $\mathcal{M}_1$ and $\mathcal{M}_2$ are called equivalent if there exists a conformal diffeomorphism $k : \mathcal{M}_1 \to \mathcal{M}_2$. Now let $\mathcal{M}_0$ be a Riemann surface of given topological type and let $k_i : \mathcal{M}_0 \to \mathcal{M}_i, i = 1, 2$ be diffeomorphisms. We denote by $[k]$ the homotopic class of a diffeomorphism. $(\mathcal{M}_1, [k_1])$ and $(\mathcal{M}_2, [k_2])$ are equivalent, if $k_2 \circ k_1^{-1} : \mathcal{M}_1 \to \mathcal{M}_2$, is homotopic to a conformal diffeomorphism. The space of equivalence classes is termed Teichmuller space $\mathcal{T}$. Then a point in $\mathcal{T}$ can be regarded as being an equivalence class of orientation preserving homeomorphisms h of the H^2. Two such homeomorphisms are equivalent if the corresponding representations are equivalent; two such representations, A and B are equivalent if there is an element $S \in PSL(2, \mathbb{R})$ so that $SAS^{-1} = B$. We then know that the space of metrics of constant curvature can be shown to be homeomorphic to $\mathbb{R}^6$ (see [6]). When we choose the rule of decomposition (the manner of gluing) and the lengths of closed geodesics we decide the decomposition. The set of lengths

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006

of all geodesics used in the decomposition into pants and the set of the so-called twisting angles used to glue the pieces provides a way of realizing this homeomorphism.

Given a surface $\mathcal{M}$ of negative curvature and genus $\tau = 2$, the universal covering surface of M is given by the hyperbolic plane which can be represented by the Poincaré disc, $D^2 = \{z \in \mathbb{C} : |z| < 1\}$, with metric $ds^2 = \frac{dz.dz}{(1-|z|^2)^2}$ or the upper half-plane, $H^2 = \{z = x + iy : y > 0\}$, with metric $ds^2 = \frac{dz.dz}{y^2}$. In both realizations, the isometry group is generated by the linear fractional transformations $h(z) = \frac{az+b}{cz+d}$. In the half-plane H^2, the matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$ belong to $SL_2(\mathbb{R})$, the *real unimodular group*.

In this paper, H^2 is the universal covering space of $\mathcal{M}$ and the fundamental group G, is a subgroup of $SL_2(\mathbb{R})$.

$\mathcal{M}$ can be decomposed into a union of two "pairs of pants" (surfaces of genus zero with three boundary circles) joined along 3 closed geodesics. The complex structure of a pair of pants P is uniquely determined by the hyperbolic lengths of the ordered boundary components of P.

A *chain* on a surface $\mathcal{M}$ is a set of four simple closed non-dividing geodesics, labelled $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, where γ_2 intersects γ_1 exactly once; γ_3 intersects γ_2 exactly once and is disjoint from γ_1; γ_4 intersects γ_3 exactly once and is disjoint from both γ_1 and γ_2.

Given the chain $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, it is easy to see that there are unique simple closed geodesics γ_5 and γ_6 so that γ_5 intersects γ_4 exactly once and is disjoint from γ_1, γ_2 and γ_3; and γ_6 intersects both γ_5 and γ_1 exactly once and is disjoint from the other γ_i. This set of six geodesics is known as a *geodesic necklace* (see [5]). If one cuts a surface $\mathcal{M}$ along the geodesics of a chain, one obtains a simply connected subsurface. It follows that we can find elements A, B, C, D of $\pi_1(\mathcal{M})$, that these elements generate $\pi_1(\mathcal{M})$, so that the shortest geodesic in the free homotopic class of loops corresponds to, respectively, A, B, C, D, is $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. There are several possible choices for these elements; we use the Maskit's option which yields the defining relation: $ABDA^{-1}C^{-1}D^{-1}CB^{-1} = 1$ (see [6]).

Let us construct our particular set of generators, A_0, B_0, C_0, D_0. These generators are normalized so that C_0 has its repelling fixed point at 0 and its attracting fixed point at ∞; the attracting fixed point of A_0 is positive and less than 1; and the product of the fixed points of A_0 is equal to 1.

In the non-deformed surface $\mathcal{M}_0 = H^2/G_0$ the group G_0 is a subgroup of the $(2, 4, 6)$-triangle group. One can use the fact that G_0 is a subgroup of the $(2, 4, 6)$-triangle group to calculate the corresponding multipliers or traces for $g_1, \ldots, g_6$ and we can write explicit matrices $A_0, \ldots, F_0 \in SL(2, \mathbb{R})$.

We set

$$A_0 = \begin{pmatrix} 2-2\sqrt{3} & 3 \\ -3 & 2+2\sqrt{3} \end{pmatrix}, \quad B_0 = \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 2+\sqrt{3} & 0 \\ 0 & 2-\sqrt{3} \end{pmatrix},$$

$$D_0 = \begin{pmatrix} 2 & -3-2\sqrt{3} \\ 3-2\sqrt{3} & 2 \end{pmatrix}, \quad E_0^{-1} = C_0 A_0 \quad \text{and} \quad F_0^{-1} = B_0 D_0.$$

It was proven in [5], that the group G_0, generated by $A_0, \ldots, D_0$, is appropriately normalized, discrete, purely hyperbolic and represents our surface $\mathcal{M}_0$, as described above. Therefore the axes of $A_0, \ldots, F_0$ split $\mathcal{M}_0$ into four right angle equilateral hexagons. Since the equilateral hexagon with all right angles is unique, it follows that our group and generators are as desired.

Once we have defined G_0, then we define the *normalized deformation space* $\mathcal{D}$ as the space of representations $\varphi: G_0 \to PSL(2, \mathbb{R})$; the image group $G = \varphi(G_0)$ is discrete, with $\mathcal{M} = H^2/G$ a closed Riemann surface of genus 2. Also here, the product of the fixed points of $A = \varphi(A_0)$ is equal to 1, with the attracting fixed point positive and smaller than the repelling fixed point; the repelling fixed point of $C = \varphi(C_0)$ is at 0; and the attracting fixed point of C is at ∞. The point of intersection of C_0 with the common orthogonal between A_0, and C_0 lies at the point i, of the imaginary axis. The normalizations given in the definition of $\mathcal{D}$ make clear that there is a well-defined real-analytic diffeomorphism between the Teichmüller space $\mathcal{T}$ and the *normalized deformation space* $\mathcal{D}$ (see [5]).

Definition 1. Let $\gamma_1, \gamma_2, \gamma_3$ be the oriented decomposition curves, ℓ_j, be the lengths of γ_j and θ_j, $j = 1, \ldots, 3$ the twist angles used to glue the pieces, respectively. The system $\{\ell_j, \theta_j\}_{j=1}^{3}$ is called the coordinates of Fenchel–Nielsen.

Using this decomposition, in order to obtain a geometric image and to study the dynamical properties, we construct a fundamental domain $\mathcal{F}$. For each fundamental domain, the fundamental group G is now generated by the side pairing transformations g_i (and their inverses) that we considered when we chose the side identifications. In this case the group G, as subgroup of $SL_2(\mathbb{R})$, is represented by the generators $G = \langle g_1, \ldots, g_6 \rangle$.

Definition 2. An open set $\mathcal{F}$ of the upper half-plane H^2 is a fundamental domain for G if $\mathcal{F}$ satisfies the following conditions:

(i) $g(\mathcal{F}) \cap \{\mathcal{F}\} = \varnothing$ for every $g \in G$ with $g \neq \text{id}$.
(ii) If $\overline{\mathcal{F}}$ is the closure of $\mathcal{F}$ in H^2 then $H^2 = \bigcup_{g \in \mathcal{F}} g(\overline{\mathcal{F}})$.
(iii) The relative boundary $\partial\mathcal{F}$ of $\mathcal{F}$ in H^2 has measure zero with respect to the two-dimensional Lebesgue measure.

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006

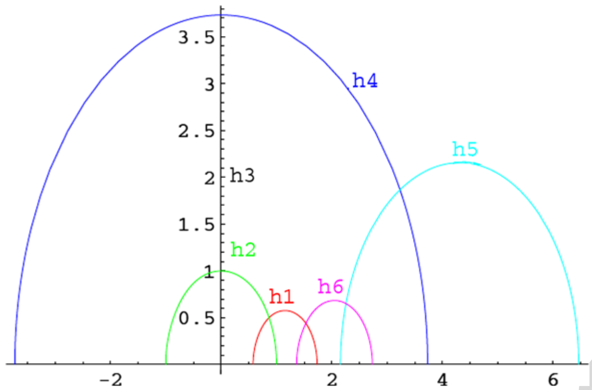

Fig. 2. The hexagon H_1.

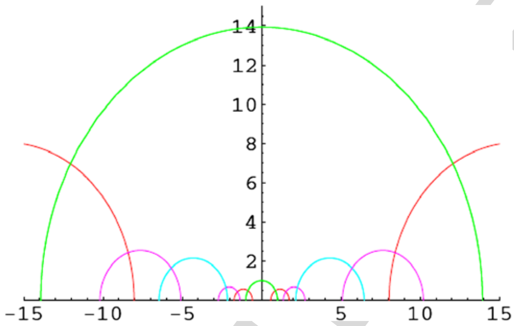

Fig. 3. Hyperbolic plane and the fundamental domain for twist angles are all equal to zero.

These conditions indicate that the Riemann surface $\mathcal{M} = H^2/G$ is considered as $\overline{\mathcal{F}}$ with points on $\partial\mathcal{F}$ identified under the covering group G. Using the hyperbolic geometry it is possible to determine explicit formulas for the generators. Images of $\mathcal{F}$ under G provide a tiling (tessellation) of H^2, where each image is a "tile" of the universal covering surface of $\mathcal{M}$. To explicitly describe the construction of the fundamental domain, we consider a chain on the surface $\mathcal{M}$, like the one in $\mathcal{M}_0$, (see Fig. 1). When we cut the surface $\mathcal{M}$ along these geodesics, we divide it into four equilateral hexagons. These geodesics are the shortest ones in the free homotopic class of loops corresponding to some elements h_i $(i = 1, \dots, 6)$ of $\pi_1(\mathcal{M})$, the fundamental group of $\mathcal{M}$. We have the hexagon H_1 whose sides s_i are the arcs of γ_i and these arcs are contained in the axes of the hyperbolic transformations h_i $(i = 1, \dots, 6)$. Their translation length in the positive direction along these axes is $2l_i$, where l_i denotes the length of $\gamma_i = \ell(\gamma_i)$. We select, as a reference, a geodesic segment γ, axis of h, (see Fig. 2) which is the common orthogonal between the axes of h_1 and h_3. Remark that if h_2 is orthogonal to h_1 and h_3 then $h_2 = h$. Let us term P the intersection point between h and h_1 and P_2 the intersection point between h_2 and h_1. The other parameters are the gluing angles. Then we consider the parameter σ determined by the distance between the intersection of h with h_1 and the intersection of h_2 with h_1. If $h_2 = h$ then l_5 is equal to zero. The other two parameters τ and ρ are determined by the angles θ_2 and θ_3 between h_2, h_1 and h_1, h_3, respectively. So $\ell_1 = \ell(\gamma_1)$, $\ell_2 = \ell(\gamma_2)$, $\ell_3 = \ell(\gamma_3)$, $\ell_4 = \ell(\gamma_4)$, $\sigma = |P - P_2|$, $\tau = \operatorname{arc\,tanh}(\cos(\theta_2))$ and $\rho = \operatorname{arc\,tanh}(\cos(\theta_3))$. Using the normalization selected, we obtain the hexagon H_1 represented in Fig. 2.

Let h be $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $c \neq 0$. The reflection through an axis h is represented by the transformation r given by

$$r(z) = \left(\frac{1}{\sqrt{a + d^2 - 4}}\right) \frac{(a-d)z + 2b}{2cz + (d-a)}.$$

The reflection with respect to the axis of h_4 sends H_1 to another right-angled hexagon H_2. Finally the reflection with respect to the imaginary axis (symmetry) sends H_1 and H_2 to the hexagons H_3 and H_4. Thus, we construct the fundamental domain $\mathcal{F} = H_1 \cup H_2 \cup H_3 \cup H_4$. If the twist angles (for our purposes we denote the twist angles $\sigma = \sigma_1$, $\tau = \sigma_2$ and $\rho = \sigma_3$) are zero, $\sigma_1 = \sigma_2 = \sigma_3 = 0$, then the fundamental domain is a right-angled polygon (see Fig. 3).

With the variation of Fenchel–Nielsen parameters we obtain different fundamental domains. This construction depends on the choice of the original geodesics γ_i, $i = 1, \dots, 4$ and the manner of gluing. For our computation purposes, we require the fundamental domain to be a dodecagon. We have to select either the convenient parameters $\{\ell_i, \theta_i\}_{i=1}^{3}$, or, the convenient geodesic chain $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, in order to obtain a dodecagon. Its sides are obtained by the intersection of the geodesic axes, and as a consequence; they are geodesic segments. We term vertex the single point which is at the intersection between

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006

two consecutive sides. The circular arc that contains a side s_i intersects the real axis at two points p_i and q_i. The sides are denoted $s_1, \ldots, s_{12}$ counting counterclockwise from zero. It is known (see [7]) that if $\mathcal{F}$ is any locally finite fundamental domain for a Fuchsian group G, then $\{g \in G : g(\overline{\mathcal{F}}) \cap \overline{\mathcal{F}} = \phi\}$ generates G.

Let $\mathcal{M} = H^2/G$ be our compact surface of genus 2. The fundamental domain $\mathcal{F}$ is a bounded fundamental polygon whose boundary $\partial\mathcal{F}$ consists of the 12 geodesic segments $s_1, \ldots, s_{12}$. Each side s_i of $\mathcal{F}$ is identified with s_j, by an element $g \in G$ and so each $g \in G$ produces a unique side s, namely, $s = \overline{\mathcal{F}} \cap g(\overline{\mathcal{F}})$. There is a bijection between the set of the sides of $\mathcal{F}$ and the set of elements g in G for which $\overline{\mathcal{F}} \cap g(\overline{\mathcal{F}})$ is a side of $\mathcal{F}$. We construct a map from the set of the sides of $\mathcal{F}$ onto itself $g : s_i \to s_j$ where s_i is identified with s_j. This is called a *side pairing* of $\mathcal{F}$ and the *side pairing* elements of G generate G. In this construction we select the side rule for the pairing

$$\begin{aligned} &s_1 \to s_7, \quad s_2 \to s_{12}, \quad s_3 \to s_5, \\ &s_4 \to s_{10}, \quad s_6 \to s_8, \quad s_9 \to s_{11}. \end{aligned} \tag{2.1}$$

Making this selection we explicitly calculate formulas for the side pairing transformations $g_1, \ldots, g_{12}$ and we explicitly obtain the generators $g_i = g_i(\ell_1, \ell_2, \ell_3, \sigma_1, \sigma_2, \sigma_3)$, $i = 1, \ldots, 12$, where $\ell_1, \ell_2, \ell_3, \sigma_1, \sigma_2, \sigma_3$ are the F–N coordinates. By using the linear fractional transformations defined above it is possible to obtain the boundary map: $f_G : \partial\mathcal{F} \to \partial\mathcal{F}$, defined by a piecewise linear fractional transformation in the partition $P = \{I_i = [p_i, p_{i+1}), i = 1, \ldots, 11, [p_{12}, p_1)\}$, which is orbit equivalent to the action of the fundamental group on $\partial\mathcal{F}$.

3. Boundary map

In this section, we describe in detail the construction of the boundary map and its domain. These constructions and results will be essential for the determination of geodesic length spectrum.

With the option (see the side rule (2.1)) we can, explicitly, calculate formulas for the side pairing transformations

$$g_1, \ldots, g_6, g_7 = g_1^{-1}, \ldots, g_{12} = g_6^{-1}.$$

This means that

$$s_7 = g_1(s_1), \ldots, s_9 = g_6(s_{11}), \qquad s_1 = g_7(s_7), \ldots, s_{11} = g_{12}(s_9).$$

Let

$$g_i(z) = (a_i z + b_i)/(c_i z + d_i), \quad \text{for } g_i(s_j) = s_k,$$

$$\text{with} \begin{cases} r_i = (q_i - p_i)/2, \\ c_i = 1/(r_j r_k)^{1/2}, \\ b_i = (a_i d_i - 1)/c_i. \end{cases}$$

Then we solve the system of equations

$$\begin{cases} (a_i p_j + b_i)/(c_i p_j + d_i) = q_k, \\ (a_i q_j + b_i)/(c_i q_j + d_i) = p_k \end{cases}$$

to determine $\{a_i, d_i\}$.

Thus we explicitly obtain the generators

$$g_i = g_i(\ell_1, \ell_2, \ell_3, \sigma_1, \sigma_2, \sigma_3), \quad i = 1, \ldots, 12.$$

We called G^0 the set of these generators.

The *isometric circle*, $I(g)$, is defined as the circle that is the complete locus of points in the neighborhood of whose lengths are unaltered in magnitude by the linear fractional transformation, $g - I(g) = \{z : |cz + d| = 1\}$. As a transformation $g \in G$ is defined by $g(z) = \frac{az+b}{cz+d}$ then $g'(z) = (cz+d)^{-2}$. Then g is expansive in the interior of $I(g)$ and contractive at the exterior. Let us consider fundamental domain $\mathcal{F}$ with sides $s_1, \ldots, s_{12}$ always labelled in an counterclockwise direction where $g(s_i) = g_i$ are the corresponding elements of G. We denote the intersection points with the boundary by $p_1, q_1, p_2, q_2, \ldots, p_{12}, q_{12}$. One way to construct the domain $\mathcal{F}$, is to take the region outside the isometric circles for each element $g \in G$ (see [8]). A vertex v_i, $i = 1, \ldots, 12$ of the dodecagon $\mathcal{F}$ is the intersection of s_{i-1} and s_i (see Fig. 4). We call $A(v_i)$ the set of all geodesic arcs that pass v_i (for instance, if $i = 1$, $A(v_1)$ is the set of brown arcs on Fig. 4) and $A = \bigcup_{i=1}^{12} A(v_i)$. The points of intersection between these arcs and the Poincaré upper half-plane boundary are denoted by $M(v_i)$. The point of $M(v_i)$, immediately before q_{i+1} is termed T_i and that immediately before p_{i-1}, is called S_i. Note that T_i precedes but it is not equal to S_{i+1}. We obtain 12 regions (which are 12 generators) involving each point v_i defined by the geodesic arcs of $M(v_i)$ (see Fig. 4).

Theorem 3. *The Poincaré upper half-plane boundary is divided, by the set $M = \bigcup_{i=1}^{12} M(v_i)$ into intervals.*

Proof. Let be an arbitrary point, P, of the Poincaré upper half-plane boundary; then $P \in [p_i, p_{i-1})$ for some $i = 1, \ldots, 12$. If $P \in [p_i, p_{i+1})$, the domain of the generator g_i, we consider the point $g_i(P)$. With the previous construction, P will belong to the set $M(v_i)$ or to the set $M(v_{i+1})$. But $P \in M(v_i)$ means that $P \in \{p_i, q_i, \ldots, T_i\}$. So the g_i image, of the arc $\widehat{v_i P}(\in A(v_i))$ will be a

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006

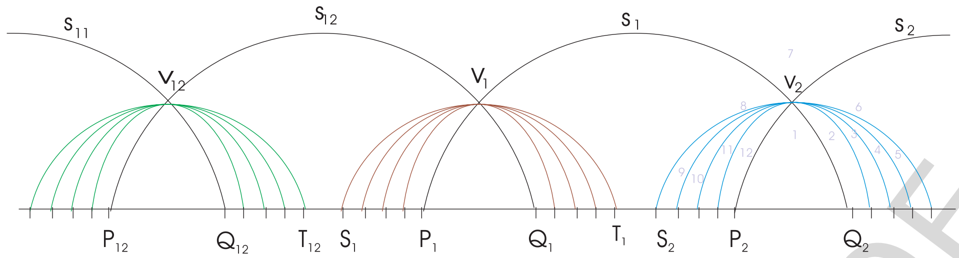

Fig. 4. Poincaré upper half-plane (pavimentation).

geodesic arc of $A = \bigcup_{i=1}^{12} A(v_i)$. As $g_i(v_i) = v_j, j = 1, \ldots, 12$ is a F vertex, then $g_i(P) \in g_i(M(v_i))$. We can state that $g_i(P) \in M$ because $M = \bigcup_{i=1}^{12} M(v_i)$. If $P \in M(v_{i+1})$ we can apply similar reasoning to the above, so that $g_i(M) \subset M, \forall_{\{i=1,\ldots,12\}}$. We can conclude that the Poincaré upper half-plane boundary its divided, by the set $M = \bigcup_{i=1}^{12} M(v_i)$, into intervals. □

To proceed from here we require the boundary map to have properties enabling a study of the dynamics of the geodesic flow, namely the codification of periodic orbits.

Definition 4. A map f is a Markov map for the partition $\mathcal{P}$ if f satisfies: (i) piecewise smoothness; (ii) local invertibility; (iii) (Markov property) each $f(J_j)$ is a union of intervals of the partition $\mathcal{P}$.

Unfortunately, in general, $f_G(p_i) \notin \{p_i\}_{i=1}^{12}$ so we need to refine the partition Q. For our purposes, it is important to ensure that our boundary map, f_G, is a Markov map, which implies the fulfilling of the Markov condition. However, we can introduce the lateral limits $p_i^{\pm}$, of the discontinuity points p_i, thus refining, the partition.

Definition 5. Let $\mathcal{P} = \{J_j\}_{j=1}^{N}$ be the Markov (finite or infinite) partition introduced through the itineraries of the lateral limits, $p_i^{\pm}$. Let be

$$\mathcal{W} = \left\{ \lim_{\epsilon\to 0} f_G^k(p_i - \epsilon), \lim_{\epsilon\to 0} f_G^k(p_i + \epsilon) \right\}_{i=1}^{12}, \quad \forall k \in \mathbb{N}.$$

The sets $\{J_j\}_{j=1}^{N}$ are subintervals defined by the points of partition $\mathcal{P}$.

With this partition, the map f_G is a Markov map. This follow, naturally from the construction of the fundamental group and the fundamental domain of the surface.

Theorem 6. *The boundary map f_G is a Markov map for the partition $\mathcal{P}$. Moreover, $\mathcal{P}$ is finite.*

Proof. The fundamental group G of the surface M is generated by the hyperbolic transformations, g_i, which were determined explicitly the beginning of this section. This is a Fuchsian group, which is purely hyperbolic, since M is compact. If m_i is the number of elements of the set $M(v_i)$, then $1 \le m_i \le \infty$. The fact that G is purely hyperbolic, ensures there are no parabolic points in $\partial\mathcal{F}$. Then $m_i < \infty$, allows us to conclude that the set $\mathrm{M} = \bigcup_{i=1}^{12} M(v_i)$ and partition $\mathcal{P}$ are finite. However if $G = \langle g_i, i = 1, \ldots, 12\rangle$ then it is a finitely generated group with $\mathrm{M} = H^2/G$ a genus 2 closed Riemann surface. Therefore G it is a finitely generated Fuchsian group of first kind. Finally, we can conclude that f_G is a Markov map (see [9]) and partition $\mathcal{P}$ is finite. □

Proposition 7. *The intervals $[p_i, q_i)$, $i = 1, \ldots, 12$, defined above, are not empty.*

Proof. Since we have constructed a fundamental domain for a Fuchsian group G that is a geodesic polygon with a finite number of sides identified with the side pairing rules, we know that partition $\mathcal{P}$ is finite and that parabolic vertices do not exist. Considering the definition of the *isometric circle* we have $\left|f_G'(x)\right|$ bounded by 1 in all intervals of the Markov partition $\mathcal{P}$, (with the exception of those that in the form $[p_i, q_i)$). According to the proof set out above, there are no parabolic points, so the points p_i and q_i do not coincide. Therefore the intervals $[p_i, q_i)$ are not empty. □

Corollary 8. *The map f_G satisfies:*

$$\exists n > 0, \ (n = 2), \quad \inf_{x\in(0,1)} \left|(f_G^n)'(x)\right| > \xi > 1.$$

Proof. This proof is a natural consequence of the previous proposition. □

In order to analyze the geodesic length spectrum with the parameters of F–N it is necessary to investigate if this partition, $\mathcal{P}$, that we have constructed, depends on these coordinates, that is, if it remains stable under conditions of deformation. This question has already been resolved (see [3]).

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006

Theorem 9. *Partition $\mathcal{P}$, introduced by the family of boundary map $f_G = f_G(\ell_1, \ell_2, \ell_3, \sigma_1, \sigma_2, \sigma_3)$, is stable under conditions of deformation, i.e., it does not depend on the variation of the Fenchel–Nielsen coordinates $(\ell_1, \ell_2, \ell_3, \sigma_1, \sigma_2, \sigma_3)$.*

The boundary map is represented by (see [2]),

$$f_G : \bigcup_{i=1,\dots,12} I_i \to \bigcup_{i=1,\dots,12} I_i,$$
$$x \to f_G(x)_{|I_i} = g_i(x), \quad i = 1, \dots, 12.$$

Let A_{f_G} be the matrix given by:

$$a_{ij} = \begin{cases} 1 & \text{if } J_j \subset f_G(J_i), \\ 0 & \text{otherwise} \end{cases}$$

which is the Markov matrix associated with f_G.

We can define the Markov subshift $(\Sigma_{A_{f_G}}, \sigma)$, where $\Sigma_{A_{f_G}}$ is the space of admissible symbolic sequences in the alphabet $\Sigma = \{1, \dots, 12\}$, corresponding to the Markov partition of $\partial\mathcal{F}$ and Markov matrix A_{f_G} and σ is the usual shift map.

4. Length spectrum

Although, the notion of closed geodesic lengths spectrum allows us to find the position of an element of the Teichmüller space $\mathcal{T}$, it does not indicate that the determined length corresponds to a determined closed geodesic.

This spectrum contains the lengths of all geodesics. While not all this information is necessary, we are able to identify which geodesic has a given length. Gelfand presented this conjecture (see [10]). To support it, he proved that there exists a continuous deformation of compact Riemann surfaces that does not modify the spectrum. Later, Mckean (see [11]) showed that the number of non-isometric compact Riemann surfaces, which have the same spectrum, is finite. Finally, Wolpert (see [12]) proved that surfaces for which this number is different from 1 are contained in a submanifold $\mathcal{V}_\tau$ (genus τ) of $\mathcal{T}$, with low dimension. This result proves the Gelfand conjecture for general cases. Simultaneously, Vignerás (see [13]) showed that the submanifold $\mathcal{V}_g$ is not empty. However, nothing is known about the explicit determination of the finite part of the spectrum that determines the total spectrum (which identifies a Riemann surface). The problem is how to explicitly determine this spectrum and to study its behavior in accordance with the variation of Fenchel–Nielsen coordinates. The identification, enumeration and codification of orbits use symbolic dynamics through constructions that involve the geometry of the surface and the algebraic structure of its fundamental group G. The action of the fundamental group on the boundary of the Poincaré upper half-plane (or Poincaré disc boundary) is shown to be orbit equivalent to the Markov map, f_G, that we have defined, and codification is obtained by the expansion of the boundary points. For this codification, successive ideas and constructions were produced by Nielsen, Hedlund (see [14]), Artin et al. (see [15]), Bowen and Series (see [9]).

We have determined the fundamental group G (finitely generated), associated with the fundamental domain $\mathcal{F}$ (of a negative curvature compact surface M) and the limit set of G is identified with a subshift of finite type (see [1]). A point x of the limit set of G belongs to the Poincaré upper half-plane boundary that is the real axis.

Definition 10. Let x_{i_0} be an element of the limit set of G. As it belongs to one of the intervals I_{i_0} of the Markov partition $\mathcal{P}$, the image under $f_G(x_{i_0}) = g_{i_0}(x_{i_0}) = x_{i_1}$, is another boundary point x_{i_1}. The point x_{i_1} belongs to the intervals I_{i_1} so $f_G(x_{i_1}) = g_{i_2}(x_{i_1}) = x_{i_2}$. We repeat this process successively and obtain a sequence

$$x \rightleftharpoons \cdots g_{i_2} g_{i_1} \quad \text{where } g_{i_2}, g_{i_1} \in G^0$$

which is called the expansion (f_G-expansion) of the boundary point x.

Definition 11. To each point x of the limit set of G we associate a f_G-expansion, that is the infinite word associated with the point x.

Proposition 12. *To each point x it is possible to associate only one word and the admissibility of this word is determined by the Markov matrix A_G.*

Proof. This matrix identifies the possible transitions between states in the associated subshift of finite type, (Σ_{A_G}, σ). Thus the occurrences in the limit set are given by the admissibility in subshift of finite type therefore by A_G. □

Previous results enable us to determine explicitly the words associated with each boundary point. In particular, they allow us to study its variation with F–N parameters. In the following tables we write some f_G-expansion (words) for different values of F–N parameters. The f_G-expansion is dependent on the selection of Fenchel–Nielsen coordinates.

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006

Example 13. In the first, regular case, which is of notable importance, as we shall see, the value of all F–N length coordinates is $\ell_1 = \ell_2 = \ell_3 = \log\left(2 + \sqrt{3}\right)$ and the twist angles $\sigma_1, \sigma_2, \sigma_3$ are zero.

$$x = 1.5 \leftrightarrow 6, 12, 3, 8, 7, 3, 3, 7, 2, 3, 6, 12, 3, 8$$
$$x = 2.0 \leftrightarrow 11, 7, 11, 6, 10, 5, 11, 12, 3, 12, 3, 3, 10, 6$$
$$x = 2.5 \leftrightarrow 3, 6, 10, 8, 12, 11, 10, 6, 4, 7, 4, 3, 6, 10.$$

Example 14. In the second case the value of all F–N length coordinates is $\ell_1 = \log(2 + \sqrt{3}) + 0.3$, $\ell_2 = \ell_3 = \log(2 + \sqrt{3})$ and the twist angles $\sigma_1, \sigma_2, \sigma_3$ are zero.

$$x = 1.5 \leftrightarrow 6, 10, 7, 2, 10, 2, 7, 10, 12, 3, 4, 7, 7, 7$$
$$x = 2.0 \leftrightarrow 12, 6, 10, 7, 9, 12, 5, 4, 8, 7, 4, 1, 5, 4$$
$$x = 2.5 \leftrightarrow 4, 11, 1, 8, 10, 7, 6, 10, 6, 10, 6, 9, 7, 10.$$

The elements of G can be considered as deformation classes of closed curves on M, relative to a fixed base point (see Section 2). A identification between deformation classes and conjugated classes exists: two elements g and g' belong to the same deformation classes if and only if $g' = k^{-1}gk$ for some $k \in G$. Simpler deformation classes occur when there are closed curves without self-intersections. In terms of f_G-expansions, this means that an element it cannot follow its inverse. Let us term this a *reduced* or *simple word*.

Definition 15. We define the f_G-expansion of a geodesic γ by taking the f_G-expansions of its endpoints γ_- and γ_+.

$$\gamma \leftrightarrows \gamma_-.\gamma_+ \leftrightarrows \cdots g_{i_2}g_{i_1}.h_{i_1}h_{i_2}\cdots \quad \text{where } g_{i_k}, h_{i_k}, \in G^0,\ k = 1, 2, 3, \ldots.$$

We obtain the representation (codification) of a geodesic γ in H^2 (or D^2) by juxtaposing the f_G*-expansions* of their endpoints γ_- and γ_+. A sequence $g_1g_2\cdots g_n (g_i \in G^0)$ is admissible if $\cup_{r=1}^{n}(f_G)^{-r}([g_i^{-1}]) \neq \varnothing$.
In order to proceed, we now need to introduce more terminology and definitions.

Definition 16. Given $g \in G$ (and $\gamma \in$ M) we define its word length $|g|$ as the smallest number of elements of G required in a presentation of g i.e. $|g| = \inf\{n : g = g_1 \cdots g_n \text{ with } g_1, \ldots, g_n \in G^0\}$; for a closed geodesic γ and associated class $[h]$ in G, we denote $|\gamma| = \inf\{|g| : g \in G \text{ e } [g] = [h]\}$, which is the word length of γ; the geometric length of γ is given by $\ell(\gamma) = \int_\gamma m(z)\,|dz|$ and is dependent on the metric of the surface.

Note 1. By convention, we set $|e| = 0$, where e is the identity element in G.

Let us call the geodesic γ joining γ_- to γ_+ as admissible if the sequence $(\gamma_-)^{-1} \bullet \gamma_+$ is admissible (see the previous proposition).
The above definitions are well defined because there is a known result (see [16]) that establishes a bijection between closed geodesics on M and (non-trivial) conjugacy classes $[g]$ for G.

Theorem 17. *Admissible geodesics are conjugated under G if and only if the corresponding sequences are shift equivalent.*

Proof. Let us consider admissible geodesic conjugated for G, which are in the same conjugated class $[g_\gamma]$. Should be recalled that the boundary map, f_G, is orbit equivalent to the action of the fundamental group G in boundary $\partial\mathcal{F}$. Then the f_G-expansions of their endpoints are the same, therefore the sequences are equivalent when (shift transformation $\sigma(x)_i = x_{i+1}$ is considered) they belong to the same σ-orbit. □

Using previous results, we can enumerate closed geodesics in M through the enumeration of the conjugacy classes of group G. With most finite exceptions, there is a bijection between the primitive closed geodesics γ on M (of length $\ell(\gamma)$ and word length $|\gamma|$) and the primitive k-periodic orbits, $\mathcal{O}(x) = \{x, f(x), f^2(x), \ldots, f^k(x)\}$. It should be noted that a closed geodesic is called primitive if it is not a repetition of a closed geodesic of strictly smaller length and, similarly, a closed orbit is called primitive if it is not the iterate of a closed orbit of a strictly smaller period. We establish equivalences between concepts and mathematical spaces that illustrate, in a sufficiently strong form, this geometric–algebraic linking, translated into symbolic dynamics, between the geometry of the surface and the algebraic structure of fundamental group G. The different results and considerations given above can be summarized by the following schema.

periodic points for f_G	$\longleftrightarrow$	admissible sequences for σ
$\updownarrow$		$\updownarrow$
primitive closed geodesics on $\mathcal{M}$	$\longleftrightarrow$	conjugated classes for G

This means that running through all the primitive admissible sequences relative to the Markov subshift corresponds to considering all the primitive periodic orbits in the flow. Moreover, this correspondence is equivalent to the consideration of

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006

all primitive closed geodesics, γ, on the surface and, consequently, it is sufficient for identifying all conjugated classes $[g]$ with word length $|g|$, in G. Now, we need to identify the conjugacy classes. Fortunately, this is possible by using traces of the elements of the group (see [7]). These elements are viewed as matrices in $SL_2(\mathbb{R})$ and the identification of the matrices g and $-g$ in $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm 1\}$ can be understood automatically. Thus, $\mathrm{tr}(g)$ denotes the corresponding matrix–trace. Let us consider two elements g and h of the group G. Their conjugacy classes are equal if and only if the squares of their traces are also equal. That is,

$$[g] = [h] \Leftrightarrow \mathrm{tr}^2(g) = \mathrm{tr}^2(h).$$

For different values of F–N parameters, we obtain different conjugacy classes, so identification is dependent on coordinates of F–N. We compute the conjugacy classes for different selections of word length and F–N coordinates.

Case 1. In the first (regular case), where the word length is $|g| = 1$, with options $\ell_1 = \ell_2 = \ell_3 = \log(2+\sqrt{3})$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$, only one conjugacy class exists.

g_i	g_1	$\cdots$	g_{12}
$\mathrm{tr}^2(g_i)$	16	$\cdots$	16

Case 2. In this case we have the same word length, $|g| = 1$. However, with the options $\ell_1 = \ell_2 = \log(2+\sqrt{3})$, $\ell_3 = 1.7$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$ there are 4 conjugacy classes.

g_i	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
$\simeq\mathrm{tr}^2(g_i)$	31.9975	16	49.2274	7.1839	49.2274	16	31.9975	16	49.2274	7.1839	49.2274	16

Case 3. In the third (regular case), where we have word length $|g| = 2$, with options $\ell_1 = \ell_2 = \ell_3 = \log(2+\sqrt{3})$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$, we have 9 conjugacy classes.

$g_1.g_i$	$\mathrm{tr}^2(g_1.g_i)$	$g_2.g_i$	$\mathrm{tr}^2(g_2.g_i)$	$g_3.g_i$	$\mathrm{tr}^2(g_3.g_i)$
$g_1.g_1$	196	$g_2.g_1$	64	$g_3.g_1$	400
$g_1.g_2$	64	$g_2.g_2$	196	$g_3.g_2$	676
$g_1.g_3$	400	$g_2.g_3$	676	$g_3.g_3$	196
$g_1.g_4$	64	$g_2.g_4$	400	$g_3.g_4$	64
$g_1.g_5$	16	$g_2.g_5$	100	$g_3.g_5$	4
$g_1.g_6$	64	$g_2.g_6$	2500	$g_3.g_6$	100
$g_1.g_7$	4	$g_2.g_7$	64	$g_3.g_7$	16
$g_1.g_8$	64	$g_2.g_8$	1156	$g_3.g_8$	676
$g_1.g_9$	16	$g_2.g_9$	676	$g_3.g_9$	1156
$g_1.g_{10}$	64	$g_2.g_{10}$	16	$g_3.g_{10}$	64
$g_1.g_{11}$	400	$g_2.g_{11}$	100	$g_3.g_{11}$	2500
$g_1.g_{12}$	64	$g_2.g_{12}$	4	$g_3.g_{12}$	100

The main purpose of this section is to provide an understanding of the behavior of the geodesic length spectrum of M, that is, the set of lengths of closed geodesics of M, with F–N coordinates. The determination of the geodesic length spectrum is thus equivalent to describing the set of traces of non-conjugate hyperbolic elements in the Fuchsian group. We can use results obtained by Beardon (see [7]) allowing for the determination of the length of the geodesic by using the traces of corresponding elements in the fundamental group.

We obtain the length spectrum of the closed geodesics by computing

$$\ell(g) = 2\cosh^{-1}[\mathrm{tr}(g)/2].$$

Case 4. In the regular case, with options $\ell_1 = \ell_2 = \ell_3 = \log(2+\sqrt{3})$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$, where we have word length, $|g| = 1$, there is just one distinct element of the geodesic length spectrum that is, $\ell(g) \simeq 2.63392$.

Case 5. It should be recalled that for the regular case $\mathrm{tr}(g_i) = 4$, with $i = 1, \ldots, 12$. So $\ell(g_i) = 2\cosh^{-1}[\mathrm{tr}(4)/2] \simeq 2.63392$

g_i	$[\mathrm{tr}(g_i)]^2$	$\ell(g_i)$
g_1	16	2.63392...
...	...	...
g_{12}	16	2.63392...

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006

Case 6. For options $\ell_1 = \ell_2 = \log(2+\sqrt{3})$, $\ell_3 = 1.7$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$, where we have word length, $|g| = 1$, there are 4 distinct elements of the geodesic length spectrum (see following table).

g_i	g_1	g_2	g_3	g_4	g_5	g_6
$\simeq[\mathrm{tr}(g_i)]^2$	31.9975	16	49.2274	7.18392	49.2274	16
$\simeq\ell(g_i)$	3.4	2.63392	3.85452	1.60608	3.85452	2.63392
g_i	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
$\simeq[\mathrm{tr}(g_i)]^2$	31.9975	16	49.2274	7.18392	49.2274	16
$\simeq\ell(g_i)$	3.4	2.63392	3.85452	1.60608	3.85452	2.63392

We study the behavior of the length spectrum where we consider word length, $|g| = 2$.

Case 7. For the regular case where we have word length, $|g| = 2$, we have 9 distinct values for the length spectrum, while it should be recalled that for this case we have 9 distinct conjugacy classes:

$g_1.g_i$	$\mathrm{tr}^2(g_1.g_i)$	$\simeq\ell(g_1g_i)$	$g_2.g_i$	$\mathrm{tr}^2(g_2.g_i)$	$\simeq\ell(g_2g_i)$	$g_3.g_i$	$\mathrm{tr}^2(g_3.g_i)$	$\simeq\ell(g_3g_i)$
$g_1.g_1$	196	5.26783	$g_2.g_1$	64	4.12687	$g_3.g_1$	400	5.98645
$g_1.g_2$	64	4.12687	$g_2.g_2$	196	5.26783	$g_3.g_2$	676	6.51323
$g_1.g_3$	400	5.98645	$g_2.g_3$	676	6.51323	$g_3.g_3$	196	5.26783
$g_1.g_4$	64	4.12687	$g_2.g_4$	400	5.98645	$g_3.g_4$	64	4.12687
$g_1.g_5$	16	2.63392	$g_2.g_5$	100	4.58486	$g_3.g_5$	4	0
$g_1.g_6$	64	4.12687	$g_2.g_6$	2500	7.82325	$g_3.g_6$	100	4.58486
$g_1.g_7$	4	0	$g_2.g_7$	64	4.12687	$g_3.g_7$	16	2.63392
$g_1.g_8$	64	4.12687	$g_2.g_8$	1156	7.05099	$g_3.g_8$	676	6.51323
$g_1.g_9$	16	2.63392	$g_2.g_9$	676	6.51323	$g_3.g_9$	1156	7.05099
$g_1.g_{10}$	64	4.12687	$g_2.g_{10}$	16	2.63392	$g_3.g_{10}$	64	4.12687
$g_1.g_{11}$	400	5.98645	$g_2.g_{11}$	100	4.58486	$g_3.g_{11}$	2500	7.82325
$g_1.g_{12}$	64	4.12687	$g_2.g_{12}$	4	0	$g_3.g_{12}$	100	4.58486

Case 8. For option $\ell_1 = \ell_2 = \log(2+\sqrt{3})$, $\ell_3 = 1.7$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$, where we have word length, $|g| = 2$, we have 16 distinct values for the length spectrum, which should be recalled that for this case we have 16 distinct conjugacy classes:

$g_1.g_i$	$\mathrm{tr}^2(g_1.g_i)$	$\ell(g_1g_i)$	$g_2.g_i$	$\mathrm{tr}^2(g_2.g_i)$	$\ell(g_2g_i)$	$g_3.g_i$	$\mathrm{tr}^2(g_3.g_i)$	$\ell(g_3g_i)$
$g_1.g_1$	899.848	6.8	$g_2.g_1$	127.99	4.83614	$g_3.g_1$	1908.65	7.5531
$g_1.g_2$	127.99	4.83614	$g_2.g_2$	196	5.26783	$g_3.g_2$	1724.46	7.45151
$g_1.g_3$	1908.65	7.5531	$g_2.g_3$	1724.46	7.45151	$g_3.g_3$	2230.41	7.70905
$g_1.g_4$	57.4668	4.01546	$g_2.g_4$	198.859	5.28246	$g_3.g_4$	88.4112	4.45898
$g_1.g_5$	16	2.63392	$g_2.g_5$	181.22	5.18858	$g_3.g_5$	4	0
$g_1.g_6$	127.99	4.83614	$g_2.g_6$	9602.51	9.16957	$g_3.g_6$	181.22	5.18858
$g_1.g_7$	4	0	$g_2.g_7$	127.99	4.83614	$g_3.g_7$	16	2.63392
$g_1.g_8$	127.99	4.83614	$g_2.g_8$	6722.76	8.81296	$g_3.g_8$	1724.46	7.45151
$g_1.g_9$	16	2.63392	$g_2.g_9$	1724.46	7.45151	$g_3.g_9$	1156	7.05099
$g_1.g_{10}$	57.4668	4.01546	$g_2.g_{10}$	11.4285	2.63392	$g_3.g_{10}$	88.4112	4.45898
$g_1.g_{11}$	1908.65	7.5531	$g_2.g_{11}$	181.22	5.18858	$g_3.g_{11}$	6926.78	8.84286
$g_1.g_{12}$	127.99	4.83614	$g_2.g_{12}$	4	0	$g_3.g_{12}$	181.22	5.18858

However, our aim is to carry out a more complete (analysis rather than that of just two cases) of this length spectrum with F–N coordinates, order to provide an understanding of how the geodesic length spectrum behaves under the conditions of deformation of the surface. It should be recalled that these coordinates are a system of global coordinates in the Teichmüller space $\mathcal{T}$. In Fig. 5 we study the geodesic length spectrum for word length $|g| = 1$. It can be clearly observed that the only case in which there is one element of the geodesic length spectrum, is the regular case ($\ell_1 = \ell_2 = \ell_3 = \log(2+\sqrt{3})$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$). This case is represented by the blue line. For the other cases, there are 4 distinct elements of the geodesic spectrum.

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006

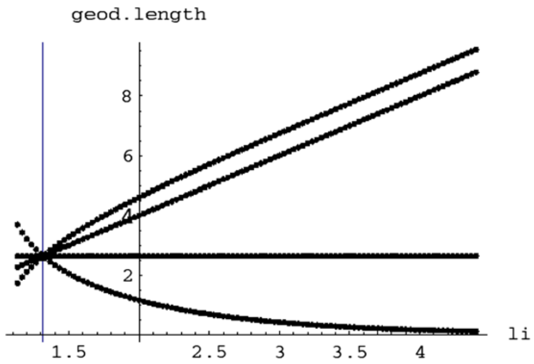

Fig. 5. The length spectrum with F–N coordinates ℓ_1, ℓ_2, ℓ_3 variation.

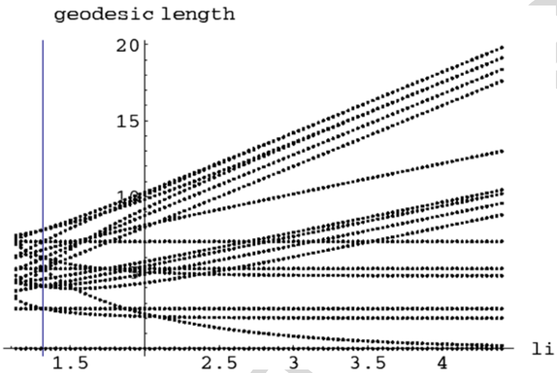

Fig. 6. The maximum number of distinct elements, for each value of ℓ_i, of the length spectrum is 17. For the regular case $\ell_i = \log(2 + \sqrt{3})$ there are 9 distinct elements.

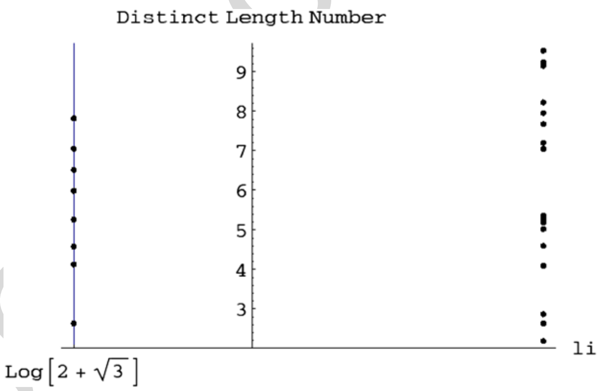

Fig. 7. For the regular case $\ell_i = \log(2+\sqrt{3}) \simeq 1.31696$ there are 8 distinct elements (blue line) and for $\ell_i = 1.7$ there are 17 elements. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

In Figs. 6 and 7 we study the geodesic length spectrum for word length $|g| = 2$. The maximum number of distinct elements of the length spectrum is 17 and in the regular case we have 8 distinct elements. This case is represented by the blue line.

For word length $|g| = 3$ (see Fig. 8) the maximum number of distinct elements of the length spectrum is 46 and in the regular case we have 24 distinct elements.

We can systematize in a table the number of distinct elements of the length spectrum when we select the value of the regular case for the coordinates of F–N ($\ell_i = \log(2 + \sqrt{3})$) or when we choose another value ($\ell_i \neq \log(2 + \sqrt{3})$) for F–N coordinates. As, can be observed in the table, this value depends on the word length considered.

Definition 18. Let G be a finitely presented group, with generating set S, such that S is closed under inverses. The growth series of G is the formal sum: $\Psi(G, S)(x) = \sum_{i=0}^{\infty} a_n x^n$ where a_n is the number of words in G of minimal word length n in

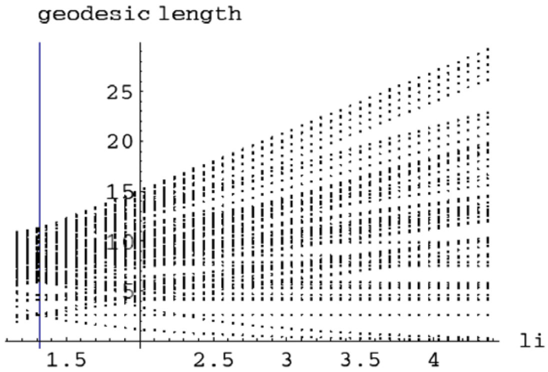

Fig. 8. For the regular case $\ell_i = \log(2+\sqrt{3}) \simeq 1.31696$ there are 24 distinct elements (blue line) and for $\ell_i = 1.7$ there are 46 elements. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 4.1
Number of distinct elements of the length spectrum.

Word length $(g) = n$	$\ell_i = \log(2+\sqrt{3})$	$\ell_i \neq \log(2+\sqrt{3})$
1	1	4
2	8	17
3	24	46
4	143	410
5	633	2623

the generating set S. The growth rate of G is given by

$$\lambda(G, S) = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}.$$

The growth series $\Psi(G, S)$ is rational, for example, when G is hyperbolic, automatic, or a Coxeter group (see for example [17], for more details). In the automatic case, we can obtain $\Psi(G, S)$ as $\lambda(f(G, S))$ where $f(G, S)$ is the characteristic polynomial of an associated matrix.

Theorem 19. (1) *Let $\mathcal{M}$ be a closed, non-deformed, surface with genus $\tau = 2$ (regular case). The exponential growth rate $\lambda = \lambda(G, S) = 4$ and the first terms of the growth series are:*

$$\xi(z) = \sum_{g\in G} z^{|g|} = z + z^2 + z^3 + z^4 + z^5 + z^6 + \cdots$$

$$\xi(z) = \sum_{n=0}^{\infty} a_n z^n = z + 8z^2 + 24z^3 + 143z^4 + 633z^5 + \cdots$$

and the radius of convergence of $\xi(z)$ is $\frac{1}{4}$.

(2) *Let $\mathcal{M}$ be any closed, deformed surface with genus $\tau = 2$ (non-regular case). The exponential growth rate $\lambda = \lambda(G, S) = 5$ and the first terms of the growth series are:*

$$\xi(z) = \sum_{g\in G} z^{|g|} = z + z^2 + z^3 + z^4 + z^5 + z^6 + \cdots,$$

$$\xi(z) = \sum_{n=0}^{\infty} a_n z^n = 4z + 17z^2 + 46z^3 + 410z^4 + 2623z^5 + \cdots$$

and the radius of convergence of $\xi(z)$ is $\frac{1}{5}$.

Proof. (1) Let us begin by recalling the notion of growth series for finitely generated groups (see [18]). Let G be a finitely generated group and let G^0 be a finite generating set. We are in this case so we can define the growth series $\xi(z) = \xi(G, S, z)$ by $\xi(z) = \sum_{g\in G} z^{|g|} = \sum_{n=0}^{\infty} a_n z^n$ where $a_n = \#\{g \in G : |g| = n\}$, is the number of words in G of minimal word length n in the generating set S. With the previous constructions, G is a finitely generated group with a finite generating set $S = \{g_1, \ldots, g_6, g_7 = g_1^{-1}, \ldots, g_{12} = g_6^{-1}\}$ (see section above). So the radius of convergence of $\xi(z)$ is given by $\frac{1}{\lambda} = \frac{1}{4}$, where $\lambda = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 4$ (see Table 4.1) and the number $\lambda = \lambda(G, S)$ is the exponential growth rate of the pair (G, S).

(2) Let us apply the same arguments as used in the previous demonstration. □

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006

Uncited references

[19], [20] and [21]. Q2

Acknowledgement

The authors are grateful to Carlos Correia Ramos for his many helpful comments and observations and we are pleased to notice his careful reading of this paper.

References

[1] T. Bedford, M. Keane, C. Series, Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces, Oxford Science Publications, 1991.
[2] C. Grácio, J. Sousa Ramos, Symbolic dynamics and hyperbolic groups, Grazer Math. Ber. 339 (1999) 195–206.
[3] C. Grácio, J. Sousa Ramos, Boundary maps and Fenchel–Nielsen coordinates, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13 (7) (2003) 1949–1958.
[4] A. Dupuy, P. Buser, M. Cibils, A. Kunzle, F. Steiner, Hyperbolic octagons and Teichmüller spaces in genus 2, J. Math. Phys. (2005).
[5] B. Maskit, New parameters for Fuchsian groups of genus 2, Proc. Amer. Math. Soc. 127 (12) (1999) 3643–3652.
[6] Y. Imayoshi, M. Taniguchi, An Introduction to Teichmüller Spaces, Springer-Verlag, Tokyo, 1992.
[7] A. Beardon, The Geometry of Discrete Groups, Springer, Berlin, 1983.
[8] L. Ford, Automorphic Functions, McGraw-Hill, New York, 1929.
[9] R. Bowen, C. Series, Markov Maps Associated with Fucshian Groups, vol. 50, I.H.E.S. Publications, 1979, pp. 153–170.
[10] I. Gelfand, 1962/63, Automorphic functions and the theory of representations, in: Proc. Int. Cong. Math., Stockholm, 1962, Almqvist & Wilsell, 1963, pp. 74–85.
[11] H. McKean, Selberg's trace formula as applied to a compact Riemann surface, Comm. Pure Appl. Math. XXV (1972) 225–246.
[12] S. Wolpert, The Fenchel–Nielsen deformation, Ann. of Math. 115 (1982) 501–528.
[13] M.-F. Vignéras, Variétés riemaniennes isospectrales et non isométriques, Ann. of Math. 112 (1980) 21–32.
[14] G. Hedlund, On the metrical transitivity of the geodesics on closed surfaces of constant negative curvature, Ann. of Math. 57 (1934) 787–808.
[15] M. Morse, A one-to-one representation of geodesics on a surface of negative curvature, Trans. Amer. Math. Soc. 22 (1921) 33–51.
[16] D. Gurarie, Symmetries and Laplacians, North-Holland, 1992.
[17] E. Hironaka, Special classes of algebraic integers in low-dimensional topology, in: Proceedings of RIMS Conference on Number Theory and Geometry, 2005.
[18] R. Sharp, Relative growth series in some hyperbolic groups, Math. Ann. 1 (1998) 125–132.
[19] M. Katz, S. Sabourau, Hyperelliptic surfaces are Loewner, Proc. Amer. Math. Soc. 134 (2006) 1189–1195.
[20] D. Lind, B. Marcus, An Introduction to Symbolic Dynamics and Coding, Cambridge University Press Publications, 1995.
[21] P. Schmutz, Riemann surfaces with shortest geodesic of maximal length, Geom. Funct. Anal. 3 (1993) 564–631.

Please cite this article in press as: C. Grácio, J.S. Ramos, Geodesic length spectrum on compact Riemann surfaces, Journal of Geometry and Physics (2010), doi:10.1016/j.geomphys.2010.06.006