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Rigidity and flexibility for surface groups

Clara Grácio* and José Sousa Ramos†

Abstract

The aim of this work is the flexibility of the hyperbolic surfaces. The results are about flexibility and geometrical boundedness. Bers are stated the universal property for all hyperbolic surface of finite area where introduced the constant of boundedness. We determine this constant, using symbolic dynamics.

1 Introduction and preliminaries

A central problem in topology is determining when two manifolds are the same, that is, homeomorphic or diffeomorphic. In this context rigidity (or flexibility) theorems are important, there are theorems about when a fairly weak equivalence between two manifolds (usually a homotopy equivalence) implies the existence of a stronger equivalence (a homeomorphism, diffeomorphism or isometry). So any kind of rigidity result is that an a priori mild condition of some sort forces unexpectedly strong one.

Mostow's rigidity theorem is a deep fundamental theorem in this theory. This theorem states that whenever M and N are two hyperbolic n-manifolds $(n > 2)$ of finite volume, with the same fundamental group, then they are isometric. This remarkable result links together concepts from topology, geometry and group theory. Of particular importance to topologists is the case where $\mathcal{M}$ and N are n-manifolds $(n > 2)$ of constant negative curvature with isomorphic fundamental groups. Mostow's theorem applies and they are isometric, by an isometry inducing the given isomorphism of fundamental groups. N-manifolds of constant negative curvature $(n > 2)$ are extremely rigid.

In the 2-dimensional case, any manifold of genus at least 2 has a hyperbolic structure. In fact there are many hyperbolic structures on any such manifold $\mathcal{M}$, each such structure corresponds to a point in Teichmüller space $\mathcal{T}$, which describes the geometry on various pieces making up $\mathcal{M}$. Mostow's theorem does not apply in this case.

What happens with $n = 2$?

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Any Riemann surface $\mathcal{M}$ (by the uniformatization theorem) has as its universal covering the Riemann sphere $\widehat{\mathbb{C}}$ or the complex plane $\mathbb{C}$ or the upper half-plane H. For our case, negative constant surface with genus g ($\geqslant 2$), the universal covering is H.

We can impose a hyperbolic structure on $\mathcal{M}$, so we can regard $\mathcal{M}$ as a complete 2-dimensional hyperbolic manifold. Hyperbolic structure and complex structure are in a one-to-one correspondence and are both represented by Fuchsian groups.

Every such surface is represented by a quotient space H^2/G of the upper half-plane H^2 by a fuchsian group G which is isomorphic to a fundamental group of $\mathcal{M}$. The discrete group G is identified with the corresponding system of generators. A fundamental domain $\mathcal{F}$ is defined. The method is to decompose Riemann surface into a set of 2 pairs of pants by simple closed geodesics. Then the Fenchel-Nielsen coordinates are defined by geodesic length functions of three simple closed geodesics and twist angles along these geodesics.

Here we use a real-analytic embedding of the Teichmüller space $\mathcal{T}$ of closed Riemann surfaces of genus 2 onto an explicitly defined region $\mathcal{R} \subset \mathbb{R}^6$ (see [8]). The parameters are explicitly defined in terms of the underlying hyperbolic geometry. The parameters are elementary functions of lengths of simple closed geodesics, angles and distances between simple closed geodesics. The embedding is accomplished by writing down four matrices in $PSL(2,\mathbb{R})$, where the entries in these matrices are explicit algebraic functions of the parameters. With explicit constructions and side pairing transformations (see [8]) we define the fuchsian group G representing the closed Riemann surface of genus 2.

One of rich research subjects who deal with the length spectrum is the systole of Riemann surfaces. In 1972 Marcel Berger (see [2]) defined a metric invariant that captures the size of k-dimensional homology of a Riemannian manifold, this invariant came to be called the k-dimensional systole. He asked if these invariants could be constrained by the volume. He constructs metrics, inspired by M. Gromov's 1993 example (see [3]). Now, this subject is developed in several directions. One of them enhances a long tradition of putting concepts of Euclidean geometry of numbers into the context of hyperbolic geometry. Quite recently it turned out with this concept of systole, this started with Schmutz, 82. The systoles provide perfect analogues to classical problems of lattice sphere packings, an important fact is that the function systole which associates to every surface the length of its systole is equivariant with a mapping class group.

We consider Riemann surfaces with hyperbolic metric. A systole is a closed geodesic of shortest length. Length-of-systole thus defines a function on Teichmüller space. One problem is to find the maximal value of the systole for Riemann surfaces of a given genus $g \geqslant 2$. But this Riemann surface is decomposed into pairs of pants, by cutting the surface along $3g-3$ simple closed non-intersecting geodesic curves and it is possible, always, to choose, these curves, in such a way that their hyperbolic lengths are bounded. Lipman Bers has shown that there exists a constant $B(g)$ depending only on g such that $\mathcal{M}$ has a decomposition into pairs of pants with curves of length minor than $B(g)$ (see [4]). This result is called Bers' theorem. Unfortunately Bers argument can not be used to get good estimates for the constant. This subject is related with the attainment of a maximal value for systoles. Bounds for the lengths of shortest closed curves have been studied for quite a long time.

James Hebda proved that $length \leq \sqrt{2\chi}$ where $\chi = area(\mathcal{M})$ (see [10]), Peter Buser has computed the numerical estimate $(6g-4)\cosh^{-1}(2\pi(g-1))$ (see [6]) and Bavard has obtained another one with the expression $\cosh(\frac{length}{4}) \leq \left(2\sin\frac{(g+1)\pi}{12g}\right)^{-1}$. Notice that this inequality for $g = 2$ is equivalent to $length \leq 2\log(1+\sqrt{2}+\sqrt{2+2\sqrt{2}})$, (see [1], [9]).

The main result of this paper is to obtain a sharper estimate for Bers'constant for a Riemann surface of genus $g = 2$.

More precisely, we prove, in section 4:

Theorem 1 *Let $\mathcal{M}$ be a closed surface with genus $g = 2$. Then the Bers' constant, $B(g)$, is $2\cosh^{-1}(2)$. Thus the length $l(\gamma)$ of every systole, γ, verify the inequality $l(\gamma) \leq B(g)$.*

2 Hyperbolic Surfaces-Properties

We can list several properties from the hyperbolic surfaces, but we only detach some, that are important for our subject like finiteness, finite area and the aim of this work, the flexibility.

An hyperbolic surface $\mathcal{M}$ is called algebraically finite if the fundamental group $\pi_1(\mathcal{M})$ is finitely generated, is called topologically finite when $\mathcal{M}$ is homeomorphic to the interior of a compact surface possibly with boundary and geometrically finite when the Nielsen kernel of $\mathcal{M}$ has finite hyperbolic area. For the 2-dimensional case this three finiteness conditions are equivalent.

The hyperbolic area of a closed surface is a topological invariant. When the genus of a closed surface $\mathcal{M}$ is g ($\geqslant 2$) the hyperbolic area $\chi(\mathcal{M})$ is given by the Gauss-Bonnet formula for the case of negative curvature -1. Then $\chi(\mathcal{M}) = 4\pi(g-1)$. For genus $g = 2$, the hyperbolic is constant, $\chi(\mathcal{M}) = 4\pi$.

Many distinct complex structures (or hyperbolic structures) can be introduced on a surface. The classical problem called *moduli problem* asked how may complex structures can be exist on a closed surface. Other classical result states that the Teichmüller space of a closed Riemann surface of genus g ($\geqslant 2$) is homeomorphic to $\mathbb{R}^{6g-6}$. But this super-abundance of hyperbolic structures on two-dimensions manifolds is not generally matched in higher dimensions, in contrast with the 3-dimensional (and higher dimensions) rigidity we have the flexibility of a hyperbolic surface.

How work this flexibility? Is the aim of this work.

Throughout this paper, all surfaces are closed Riemann surfaces of genus (the number of the handles) 2, all fuchsian groups are purely hyperbolic, and all references to lengths, distances, etc. are to be understood in terms of hyperbolic geometry. Given a surface $\mathcal{M}$ of negative curvature and genus $g = 2$ the universal covering surface of $\mathcal{M}$ is given by the hyperbolic plane which can be represented by the Poincaré disk, $D^2 = \{z \in \mathbb{C} : |z| < 1\}$, with metric

$$ds^2 = \frac{dzdz}{(1-|z|^2)^2}$$

or upper half-plane, $H^2 \; = \{z = x + iy : y < 0\}$, with metric

$$ds^2 = \frac{dzdz}{y^2}$$

In both realizations, the isometry group is made of the linear fractional transformations

$$h(z) = \frac{a\,z + b}{c\,z + d}$$

In the half-plane H^2, the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a\,d - b\,c \, = 1$$

belong to $SL_2(\mathbb{R})$, the *real unimodular group.*

In this work H^2 is the universal covering space of M, the fundamental group, G, is a subgroup of $SL_2(\mathbf{R})$. M can be decomposed into a union of two "pairs of pants" joined along 3 closed geodesics. The complex structure of a pair of pants P is uniquely determined by the hyperbolic lengths of the ordered boundary components of P.

It is well known that one can identify $\mathcal{T}$ with the (quasiconformal) deformation space of the fuchsian group G_0, within the space of fuchsian groups. We will construct our particular set of generators, A_0, B_0, C_0, D_0. These generators will be normalized so that C_0 has its repelling fixed point at 0, and its attracting fixed point at ∞; the attracting fixed point of A_0 is positive and less than 1; and the product of the fixed points of A_0 is equal to 1.

Sometimes we will use the same symbol to denote a orientation-preserving homeomorphisms h of the $H^2 \to H^2$, and the matrix A that represents them in $PSL(2, \mathbb{R})$. A point in $\mathcal{T}$ can be regarded as being an equivalence class of orientation-preserving homeomorphisms h of the H^2. Two such homeomorphisms are equivalent if the corresponding representations are equivalent; two such representations, A and B are equivalent if there is an element $S \in PSL(2, \mathbb{R})$ so that $SAS^{-1} = B$.

It is a classic result that the space of metrics of constant curvature can be shown to be homeomorphic to $\mathbb{R}^6$

When we choose the rule of the decomposition (the way of gluing) and the lengths of closed geodesics we decide the decomposition. The set of lengths of all geodesics used in the decomposition into pants and the set of so-called twisting angles used to glue the pieces is a way of realizing this homeomorphism.

A chain on a surface M is a set of four simple closed non-dividing geodesics, labelled $\gamma_1, \ldots, \gamma_4$, where γ_2 intersects γ_1 exactly once; γ_3 intersects γ_2 exactly once and is disjoint from γ_1; γ_4 intersects γ_3 exactly once and is disjoint from both γ_1 and γ_2. We assume throughout that these geodesics are directed so that, in terms of the homology intersection number, $\gamma_i \times \gamma_{i+1} = +1$.

Given the chain $\gamma_1, \ldots, \gamma_4$ it is easy to see that there are unique simple closed geodesics γ_5 and γ_6 so that γ_5 intersects γ_4 exactly once and is disjoint from γ_1, γ_2

and γ_3; and γ_6 intersects both γ_5 and γ_1 exactly once and is disjoint from the other γ_i. As above, we can assume that these geodesics are also directed so that, using cyclic ordering, $\gamma_i \times \gamma_{i+1} = +1$. This set of six geodesics is called a geodesic necklace (see [11]).

If one cuts the surface M_0 along the geodesics of a chain, one is left with a simply connected subsurface. It follows that we can find elements A_0, B_0, C_0, D_0 of $\pi_1(M_0)$, so that these elements generate $\pi_1(M_0)$, and so that the shortest geodesic in the free homotopy class of loops corresponding to, respectively, A_0, B_0, C_0, D_0, is, respectively, γ_1, γ_2, γ_3, γ_4. There are several possible choices for these elements; we make Maskit's choice, this yields the one defining relation:

$$A_0B_0D_0A_0^{-1}C_0^{-1}D_0^{-1}C_0B_0^{-1} = 1.$$

Then, G_0 is appropriately normalized, discrete, purely hyperbolic, and represents some surface of genus 2, which we could then take to be our base surface.

We first observe that C_0 has its attracting fixed point at ∞, and its repelling fixed point at 0. We also easily observe that the fixed points of A_0 are both positive, with product equal to 1, and that the attracting fixed point is smaller than the repelling fixed point. We also observe that $A_0, \ldots, D_0$ all have the same trace equal to 4 and it is easy to compute the trace of E_0 and F_0, and observe that it is equal to -4, and their axes are either disjoint or meet at right angles to form the hexagon H_1. Thus we obtain the discrete purely hyperbolic group G_0, representing a closed Riemann surface of genus 2.

In the not deformed surface $M_0 = H^2/G_0$ the group G_0 is a subgroup of the $(2, 4, 6)$-triangle group. One could use the fact that G_0 is a subgroup of the $(2, 4, 6)$-triangle group to calculate the corresponding multipliers or traces for $u_1, \ldots, u_6$ and we can write explicit matrices $A_0, \ldots, F_0 \in SL(2, R)$. We set

Proposition 2 *Let M_0 the not deformed surface and G_0 the fundamental group. The corresponding matrices are given by:*

$$A_0 = \begin{pmatrix} 2-2\sqrt{3} & 3 \\ -3 & 2+2\sqrt{3} \end{pmatrix} \quad B_0 = \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}$$
$$C_0 = \begin{pmatrix} 2+\sqrt{3} & 0 \\ 0 & 2-\sqrt{3} \end{pmatrix} \quad D_0 = \begin{pmatrix} 2 & -3-2\sqrt{3} \\ 3-2\sqrt{3} & 2 \end{pmatrix}$$
$$E_0^{-1} = C_0A_0 \qquad F_0^{-1} = B_0D_0$$

After doing the above computations, we see that the axes of $A_0, \ldots, F_0$ divide M_0 into four right angle equilateral hexagons. Since the equilateral hexagon with all right angles is unique, it follows that our group and generators are as desired.

Once we have defined G_0, then we define the *normalized deformation space* $\mathcal{D}$ as the space of representations $\varphi\colon G_0 \to PSL(2, \mathbb{R})$; the image group $G = \varphi(G_0)$ is discrete, with $M = H^2/G$ a closed Riemann surface of genus 2. Also here, the product of the fixed points of $A = \varphi(A_0)$ is equal to 1, with the attracting fixed point positive and smaller than the repelling fixed point; the repelling fixed point of $C = \varphi(C_0)$ is at 0; and the attracting fixed point of C is at ∞. The point of intersection of C_0 with the common orthogonal between A_0 and C_0 lies at the point i, of the imaginary axe. The normalizations given in the definition of $\mathcal{D}$ make it clear that there is a well defined real-analytic diffeomorphism between the Teichmüller space $\mathcal{T}$ and the *normalized deformation space* $\mathcal{D}$ (see [11]).

Definition 3 *Let $\gamma_1, .., \gamma_3$ the oriented decomposition curves, the functions l_j, and θ_j , $j = 1, ..., 3$ are the length of γ_j, and the twist angle used to glue the pieces respectively. The system $\{l_j, \theta_j\}$ are called the coordinates of Fenchel-Nielsen.*

With this decomposition, in order to obtain a geometric image and to study the dynamical proprieties, we construct a fundamental domain $\mathcal{F}$. For each fundamental domain, the fundamental group G is now generated for the side pairing transformations u_i (and their inverses), that considered when had chosen the side identifications. If the region is compact, the generators are hyperbolic transformation. For this case the group G as subgroup of $SL_2(\mathbb{R})$, is represented by the generators u_i

$$G =< u_1, .., u_6 > .$$

Definition 4 *An open set $\mathcal{F}$ of the upper half-plane H^2 is a fundamental domain for G if $\mathcal{F}$ satisfies the following conditions:*

i) $u(\mathcal{F}) \cap \mathcal{F} = \phi$ for every $u \in G$ with $u \neq id$.

ii) If $\overline{\mathcal{F}}$ is the closure of in H^2, then :

$$H^2 = \bigcup_{u \in \mathcal{F}} u(\overline{\mathcal{F}})$$

iii) The relative boundary $\partial\mathcal{F}$ of $\mathcal{F}$ in H^2 has measure zero with respect to the two-dimensional Lebesgue measure.

These conditions tell us that the Riemann surface $M = H^2/G$ is considered as $\overline{\mathcal{F}}$ with points on $\partial\mathcal{F}$ identified under the covering group G. With the hyperbolic geometry it is possible to determine explicit formulas for the generators.

The images of $\mathcal{F}$ under G provide a tiling (tessellation) of H^2 each image is a "tile" of the universal covering surface of M. To explicit construction of the fundamental domain we consider a chain on the surface M, see Figure 1. When we cut the surface M along these geodesics then we divide it into four equilateral hexagons. The sides are obtained by the intersection of the axis, they are geodesics segments.

These geodesics are the shortest geodesics in the free homotopy class of loops corresponding to some elements h_i $(i = 1, ..., 6)$ of $\pi_1(M)$, the fundamental group of M. We have the hexagon H_1 whose sides s_i are the arcs of γ_i and these arcs are contained in the axes of the hyperbolic transformations h_i $(i = 1, ..., 6)$. Their translation length in the positive direction along these axis is $2l_i$ where l_i denote the length of $\gamma_i = l(\gamma_i)$. They are four of the parameters. The other parameters are the gluing angles. So:

$$\begin{aligned}&c_1 = l(\gamma_1),\ c_2 = l(\gamma_2),\ c_3 = l(\gamma_3),\\ &c_4 = l(\gamma_4), \quad \sigma = |P - P_2|\ ,\\ &\tau = arc\tanh(\cos(\theta_2))\ ,\\ &\rho = arc\tanh(\cos(\theta_3))\end{aligned}$$

But c_4 is determined by the others parameters, so with this parametrization, each point t of the Teichmüller space $\mathcal{T}$ is $t = t(c_1, c_2, c_3, \sigma, \tau, \rho)$. This construction is dependent from de choice of the original geodesics γ_i, the chain, thus the dependence from the parameters $c_i = l(\gamma_i)$.

The sides are labelled $s_1, \ \ , s_{12}$ reading counterclockwise from the zero. It is known (see [Beardon, 1983]) that if $\mathcal{F}$ are any locally finite fundamental domain for a Fuchsian group G, then

$$\{u \in G : u(\overline{\mathcal{F}}) \cap \overline{\mathcal{F}} = \phi\}$$

generates G.

Let $M = H^2/G$ our compact surface of genus $g = 2$. The fundamental domain its a bounded fundamental polygon whose boundary ∂F consists of the 12 geodesics segments $s_1, ..., s_{12}$.

Each side s_i of F is identified with s_j , by an element $u \in G$ and so each $u \in G$ produces a unique side s , namely, $s = \overline{F} \cap u(\overline{F})$ There is a bijection between the set of the sides of F and the set of elements u in G for which $\overline{F} \cap u(\overline{F})$ is a side of F.

We construct a map from the set of the sides of F onto itself , $u : s_i \rightarrow s_j$ where .s_i is identified with s_j. This is called a *side-pairing* of F.The *side-pairing* elements of G generate G.

In this construction we choose the side rule for the pairing

$$\begin{aligned} s_1 &\rightarrow s_7,\ s_2 \rightarrow s_{12},\ s_3 \rightarrow s_5,\\ s_4 &\rightarrow s_{10},\ s_6 \rightarrow s_8,\ s_9 \rightarrow s_{11}\end{aligned}$$

With this choice we explicitly calculate formulas for the side pairing transformations $u_1, ..., u_6, u_7 = u_1^{-1}, ..., u_{12} = u_6^{-1}$. This mean that $s_7 = u_1(s_1)$,...,$s_9 = u_6(s_{11})$, $s_1 = u_7(s_7)$,...,$s_{11} = u_{12}(s_9)$, thus we obtain explicitly the generators $u_i = u_i(c_1, c_2, c_3, \sigma, \tau, \rho)$, $i = 1, ..., 12$.

3 Conjugacy Classes and Length Spectrum

Until now we had made an explicit geometrical description of the surface. All construction are generic for any choice of a surface of constant negative curvature and genus 2. So we can obtain a symbolic dynamics for the geodesic flow on these surfaces that involves the geometry and the structure of its fundamental group.

With the linear fractional transformations defined above it is possible to obtain the boundary map: $f_G : \partial\mathcal{F} \rightarrow \ \partial\mathcal{F}$, defined by piecewise linear fractional transformations in the partition $\mathcal{P} = \{ I_i = [p_i, p_{i+1}), i = 1, ..., 11, [p_{12}, p_1)\}$, which is orbit equivalent to the action of the fundamental group G on $\partial\mathcal{F}$. The boundary map is represented by

$$f_G : \bigcup_{i=1,...,12} I_i \to \bigcup_{i=1,...,12} I_i$$

$$f_G(x)_{|I_i} = u_i(x), \quad i = 1, ..., 12$$

We are able to define a map that codifies the expansion of boundary points of $\mathcal{F}$. And we determine the Markov matrix A_G associated to G.

Let be A_G the matrix

$$a_{ij} = \begin{cases} 1 & \text{if} \quad J_j \subset f_G(J_i) \\ 0 & \text{otherwise} \end{cases}$$

The fundamental group is isomorphic to G (a Fuchsian group), where the homotopic classes correspond to conjugacy classes $[u]$ of hyperbolic elements u in G.

Proposition 5 [Beardon] *(see [5]) Let be u and h elements of G. Then $[u] = [h] \Leftrightarrow tr^2(u) = tr^2(h)$.*

Proposition 6 *For each word length k, the number of conjugacy classes not depending of the Fenchel-Nielsen coordinates, depending only of the genus of the surface.*

Proof. We have introduced the Markov partition for the Bowen-Series boundary map associated with the fundamental group G (see [7]). Then we defined the 24×24 Markov matrix $[a_{ij}]$ associated to G. Each element u of G is a combination of the generators corresponding a matrix product. With the trace of the matrix and the Beardon's proposition it is possible to identify the conjugacy classes, $[u] = [h] \Leftrightarrow tr^2(u) = tr^2(h)$ and this number depending only on the genus of the surface. □

We show the computation of conjugacy class for some choice of parameters.

First we choose $\alpha = \beta = \gamma = \log(2 + \sqrt{3})$ and $\sigma = \tau = \rho = 0$, we obtain just one distinct conjugation class.

$$\begin{array}{ll} tr(u_1) = 4 & tr^2(u_1) = 16 \\ tr(u_2) = 4 & tr^2(u_2) = 16 \\ tr(u_3) = -4 & tr^2(u_3) = 16 \\ tr(u_4) = -4 & tr^2(u_4) = 16 \\ tr(u_5) = 4 & tr^2(u_5) = 16 \\ tr(u_6) = 4 & tr^2(u_6) = 16 \\ tr(u_7) = 4 & tr^2(u_7) = 16 \\ tr(u_8) = 4 & tr^2(u_8) = 16 \\ tr(u_9) = 4 & tr^2(u_9) = 16 \\ tr(u_{10}) = 4 & tr^2(u_{10}) = 16 \\ tr(u_{11}) = -4 & tr^2(u_{11}) = 16 \\ tr(u_{12}) = -4 & tr^2(u_{12}) = 16 \end{array} \tag{1}$$

Secondly, we choose $\alpha = \beta = \log(2+\sqrt{3})$, $\gamma = 1.7$ and $\sigma = \tau = \rho = 0$, we obtain 4 distinct conjugation classes. In that if it follows we will call $c_1 = \alpha, c_2 = \beta, c_3 = \gamma$.

$$\begin{array}{ll} tr(u_1) = -4 & tr^2(u_1) = 16 \\ tr(u_2) = 5.65663 & tr^2(u_2) = 31.9975 \\ tr(u_3) = -8.96989 & tr^2(u_3) = 80.459 \\ tr(u_4) = 2.89726 & tr^2(u_4) = 8.39412 \\ tr(u_5) = -8.96989 & tr^2(u_5) = 80.459 \\ tr(u_6) = 5.65663 & tr^2(u_6) = 31.9975 \\ tr(u_7) = 4 & tr^2(u_7) = 16 \\ tr(u_8) = 5.65663 & tr^2(u_8) = 31.9975 \\ tr(u_9) = 8.96989 & tr^2(u_9) = 80.459 \\ tr(u_{10}) = 2.89726 & tr^2(u_{10}) = 8.39412 \\ tr(u_{11}) = 8.96989 & tr^2(u_{11}) = 80.459 \\ tr(u_{12}) = 5.65663 & tr^2(u_{12}) = 31.9975 \end{array} \tag{2}$$

It is known that there are the correspondence between the closed geodesics of the surface and the conjugacy classes of the group so with the list above we identify each closed geodesic. We obtain the length spectrum of the closed geodesics by computing :

$$l(u) = 2\cosh^{-1}[\frac{tr(u)}{2}] \tag{3}$$

We show the correponding length spectrum

$$\begin{array}{l} l_{11} = 5.26783 \\ l_{12} = 4.12687 \\ l_{13} = 5.98645 \\ l_{15} = 2.63392 \\ l_{17} = 0 \\ l_{23} = 6.51323 \\ l_{25} = 4.58486 \\ l_{26} = 7.82325 \\ l_{28} = 7.05099 \end{array} \tag{4}$$

Definition 7 *Let be a geodesic chain $\gamma_1, \ldots, \gamma_4$ where the four geodesics have equal length and the twist parameters are zero. We call regular domain of the genus $g = 2$ the closed Riemann surface $\mathcal{M}$ defined by this Fenchel-Nielsen coordinates choice.*

We can observe the behavior of the length spectrum of the geodesics with the variation of the Fenchel-Nielsen parameters (c_i, $i = 1, 2, 3$). This variation is limited by the possible values of the parameters, that is, the values for which we get one dodecagon. All lengths are inferior to the value of the regular case ($l_{reg} = c_i = \log(2+\sqrt{3})$). As we see is not by chance that case is identified to this.

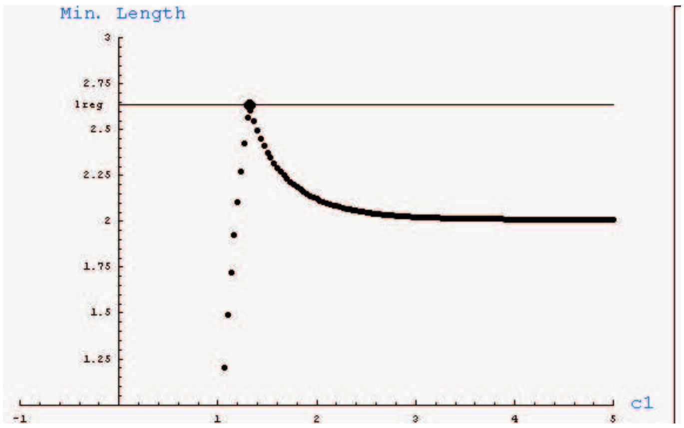

Figure 1: Length spectrum

4 Flexibility and Boundedness

Returning to our original question, we note that a closed surface of genus at least 2 has uncountable many hyperbolic structures up to homotopy relative to the boundary, and these may be parametrized by Fenchel-Nielsen coordinates in Teichmüller space, so they have the remarkable property, known as flexibility.

In spite of this flexibility, there is the following property introduced by Bers, that state the existence of constants, who guarantees a certain boundedness of hyperbolic structures under deformation of a hyperbolic surface.

Theorem 8 (Bers)*: Let $\mathcal{M}$ a hyperbolic surface of finite area. Then the length of the shortest closed geodesic on $\mathcal{M}$ is bounded from above by a constant c_0, depending only on the Euler characteristic of $\mathcal{M}$, or equivalently, on the area of $\mathcal{M}$.*

We remark that for a genus $g = 2$ surface $\mathcal{M}$, the hyperbolic area is $\chi(M) = 4\pi$, by the Gauss-Bonnet theorem.

The aim of this section is determine explicitly this constant. First we need to be clear this constant.

Remark 9 *It is possible to compute that when we have the regular case $c_i = l(\gamma_i) = \log(2+\sqrt{3})$, $i = 1, ..., 4$ and $\sigma = \tau = \rho = 0$, (see [8]).*

Definition 10 *We define the Bers constant by:*

$$B(g) = \max_{\{t \in T(\mathcal{M})\}} \{l(\gamma_0(t))\} \tag{5}$$

It is the maximum of the shortest closed geodesic length on M where $t = t(c_1, c_2, c_3, \sigma, \tau, \rho)$ is a point of the Teichmüller space $\mathcal{T}$.

We are able to introduced our main result.

Theorem 11 *Let $\mathcal{M}$ be a closed surface with genus $g = 2$.Then the Bers' constant, $B(g)$, is $2\cosh^{-1}(2)$. Thus the length $l(\gamma)$ of every systole, γ verify the inequality $l(\gamma) \leq B(g)$.*

Proof. Let c_i, $i = 1, 2, 3$ the Fenchel-Nielsen parameters. Each choice of these parameters (we can choose between values where the fundamental domain is a dodecagon) determine a surface. Then we can choose a fundamental domain and to obtain explicitly the generators $u_i = u_i(c_1, c_2, c_3, \sigma, \tau, \rho)$, $i = 1, ..., 12$ of the fundamental group G. But for each closed geodesic γ of the surface there are a correspondent conjugacy class $[u]$ of the group G. This conjugacy class is identified by its trace $tr(u)$. Then we can to determine its length: $l(u) = 2\cosh^{-1}[\frac{tr(u)}{2}]$. Thus we obtain the length spectrum of the closed geodesics when one varies the Fenchel-Nielsen parameters. For every possible value of these parameters and for each length of word the shortest closed geodesic of maximal length is bounded by the length geodesic that determine the regular case. If the Fenchel-Nielsen coordinates are lesser or bigger than $c_i = l(\gamma_i) = \log(2 + \sqrt{3})$, $i = 1, ..., 4$ then the length of shortest closed geodesic of maximal length, as if it can see in the previous graphs, is lesser than $B(g)$. □

We finish this article with the attainment it our main objective: the Bers constant is the length exact value for the regular case, $B(g) = 2\cosh^{-1}(2)$.

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Clara Grácio
Departamento de Matemática
Universidade de Évora
Rua Romão Ramalho,59
P - 7000-671 Évora, Portugal
e-mail: mgracio@uevora.pt

José Sousa Ramos
Departamento de Matemática
Instituto Superior Técnico
Av. Rovisco Pais, 1
P - 1049-001, Lisboa, Portugal
e-mail: sramos@math.ist.utl.pt