BOUNDARY MAPS AND FENCHEL-NIELSEN COORDINATES

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We consider a genus 2 surface, M, of constant negative curvature and we construct a 12-sided fundamental domain, where the sides are segments of the lifts of closed geodesics on M (which determines the Fenchel-Nielsen-Maskit coordinates). Then we study the linear fractional transformations of the side pairing of the fundamental domain. This construction gives rise to 24 distinct points on the boundary of the hyperbolic covering space. Their itineraries determine Markov partitions that we use to study the dependence of the Lyapunov exponent and length spectrum of the closed geodesics with the Fenchel-Nielsen coordinates.

1. Introduction

The metric and geometric structure on surfaces can be studied by the closed geodesics spectrum and the Laplace-Beltrami operator spectrum. To obtain these spectra is not easy but more difficult is to describe their dependence on the parameters which determine the metric and geometric structure of the surface. We study these spectra dependence through the boundary map dependence from these parameters, considering a Riemann surface M of genus 2, thus with negative curvature.

Every Riemann surface M is represented by a quotient space H^2/G of the upper half-plane H^2 by a fuchsian group G which is isomorphic to a fundamental group of M. The discrete group G is identified with the corresponding system of generators. A fundamental domain $\mathcal{F}$ is defined. The method is to decompose the Riemann surface into a set of 2 pairs of pants by simple closed geodesics. Then the Fenchel-Nielsen coordinates are defined by geodesic length functions of three simple closed geodesics and twist angles along these geodesics.

Here we use a real-analytic embedding of the Teichmüller space $\mathcal{T}$ of closed Riemann surfaces of genus 2 onto an explicitly defined region $\mathcal{R} \subset \mathbb{R}^6$ (see [Maskit, 1999]). The parameters are explicitly defined in terms of the underlying hyperbolic geometry. The parameters are elementary functions of lengths of simple closed geodesics, angles and distances between simple closed geodesics. The embedding is accomplished by writing down four matrices in $PSL(2,\mathbb{R})$, where the entries in these matrices are explicit algebraic functions of the parameters. Explicit constructions and side pairing transformations are given to define the fuchsian group G representing a closed Riemann surface of genus 2.

We start with a Riemann surface M_0, and a specific set of normalized generators, $A_0, B_0, C_0, D_0 \in PSL(2,\mathbb{R})$, for the fuchsian group G_0 representing $\pi_1(M_0)$. Then we can realize a point in $\mathcal{T}$ as a set of appropriate normalized generators $A, B, C, D \in PSL(2,\mathbb{R})$ for the fuchsian group G representing a deformation M of M_0 (see [Maskit, 1999]). We write the entries in the generators, $A, \ldots, D$, as elementary functions of six Fenchel-Nielsen-Maskit coordinates and we write down explicit formula.

After we define a Markov map on the boundary of the hyperbolic covering space. Then, we study the dynamical properties in the symbolic dynamics framework.

Our other results are related to dynamical quantities, in particular Lyapunov exponents, of boundary map with respect to the variation of the parameters, the Fenchel-Nielsen-Maskit coordinates. Then we verify that the Lyapunov exponents decrease when the parameters are going out of the corresponding to the regular fundamental domain.

2. Preliminaries, Definitions and Geometric Description

Throughout this paper, all surfaces are closed Riemann surfaces of genus 2, all fuchsian groups are purely hyperbolic, and all references to lengths, distances, etc. are to be understood in terms of hyperbolic geometry. Given a surface M of negative curvature and genus $g = 2$ the universal covering surface of M is given by the hyperbolic plane which can be represented by the Poincaré disk, $D^2 = \{z \in C : |z| < 1\}$, with metric

$$ds^2 = \frac{dz.dz}{(1-|z|^2)^2}$$

or upper half-plane, $H^2 = \{z = x + iy : y > 0\}$, with metric

$$ds^2 = \frac{dz.dz}{y^2}$$

In both realizations, the isometry group is made of the linear fractional transformations

$$h(z) = \frac{a\,z + b}{c\,z + d}$$

In the half-plane H^2, the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a\,d - b\,c = 1$$

belong to $SL_2(\mathbb{R})$, the *real unimodular group.*

In this work H^2 is the universal covering space of M, the fundamental group, G, is a subgroup of $SL_2(\mathbf{R})$. M can be decomposed into a union of two "pairs of pants" joined along 3 closed geodesics. The complex structure of a pair of pants P is uniquely determined by the hyperbolic lengths of the ordered boundary components of P.

A *chain* on a surface M is a set of four simple closed non-dividing geodesics, labelled $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, where γ_2 intersects γ_1 exactly once; γ_3 intersects γ_2 exactly once and is disjoint from γ_1; γ_4 intersects γ_3 exactly once and is disjoint from both γ_1 and γ_2. We assume throughout that these geodesics are directed so that, in terms of the homology intersection number, $\gamma_i \times \gamma_{i+1} = +1$.

Given the chain $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, it is easy to see that there are unique simple closed geodesics γ_5 and γ_6 so that γ_5 intersects γ_4 exactly once and is disjoint from γ_1, γ_2 and γ_3; and γ_6 intersects both γ_5 and γ_1 exactly once and is disjoint from the other γ_i. As above, we can assume that these geodesics are also directed so that, using cyclic ordering, $\gamma_i \times \gamma_{i+1} = +1$. This set of six geodesics is called a *geodesic necklace* (see [Maskit, 1999]).

If one cuts the surface M_0 along the geodesics of a chain, one obtains a simply connected subsurface. It follows that we can find elements A_0, B_0, C_0, D_0 of $\pi_1(M_0)$, so that these elements generate $\pi_1(M_0)$, and so that the shortest geodesic in the free homotopy class of loops corresponding to, respectively, A_0, B_0, C_0, D_0, is, respectively, γ_1, γ_2, γ_3, γ_4. There are several possible choices for these elements; we make Maskit's choice which yields to the defining relation:

$$A_0 B_0 D_0 A_0^{-1} C_0^{-1} D_0^{-1} C_0 B_0^{-1} = 1.$$

It is well known that one can identify $\mathcal{T}$ with the (quasiconformal) deformation space of the fuchsian group G_0, within the space of fuchsian groups (see [Imayoshi *et al.*, 1992]). We will construct our particular set of generators, A_0, B_0, C_0, D_0. These generators will be normalized so that C_0 has its repelling fixed point at 0, and its attracting fixed point at ∞; the attracting fixed point of A_0 is positive and less than 1; and the product of the fixed points of A_0 is equal to 1.

Sometimes we will use the same symbol to denote a orientation-preserving homeomorphisms h of the $H^2 \to H^2$, and the matrix A that represents them in $PSL(2,\mathbb{R})$. A point in $\mathcal{T}$ can be regarded as being an equivalence class of orientation-preserving homeomorphisms h of the H^2. Two such homeomorphisms are equivalent if the corresponding representations are equivalent; two such representations, A and B are equivalent if there is an element $S \in PSL(2,\mathbb{R})$ so that $SAS^{-1} = B$.

We know then that the space of metrics of constant curvature can be shown to be homeomorphic to $\mathbb{R}^6$ (see [Imayoshi *et al.*, 1992]).

When we choose the rule of the decomposition (the way of gluing) and the lengths of closed geodesics we decide the decomposition. The set of lengths of all geodesics used in the decomposition into pants and the set of so-called twisting angles used to glue the pieces is a way of realizing this homeomorphism.

In the not deformed surface $M_0 = H^2/G_0$ the group G_0 is a subgroup of the $(2,4,6)$-triangle group. One could use the fact that G_0 is a subgroup of the $(2,4,6)$-triangle group to calculate the corresponding multipliers or traces for $g_1, \dots, g_6$ and we can write explicit matrices $A_0, \dots, F_0 \in SL(2,\mathbb{R})$. We set

$$A_0 = \begin{pmatrix} 2-2\sqrt{3} & 3 \\ -3 & 2+2\sqrt{3} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 2+\sqrt{3} & 0 \\ 0 & 2-\sqrt{3} \end{pmatrix},$$

$$D_0 = \begin{pmatrix} 2 & -3-2\sqrt{3} \\ 3-2\sqrt{3} & 2 \end{pmatrix},$$

$$E_0^{-1} = C_0.A_0 \ , \qquad F_0^{-1} = B_0 D_0 \ .$$

We need to prove that the group G_0, generated by $A_0, \dots, D_0$, is appropriately normalized, discrete, purely hyperbolic and represents our surface M_0, as described above. We remark that it would suffice for our purposes to show that G_0 is appropriately normalized, discrete, purely hyperbolic, and represents some surface of genus 2, which we could then take to be our base surface.

We first observe that C_0 has its attracting fixed point at ∞, and its repelling fixed point at 0. We also easily observe that the fixed points of A_0 are both positive, with product equal to 1, and that the attracting fixed point is smaller than the repelling fixed point. We also observe that $A_0, \dots, D_0$ all have the same trace equal to 4 and it is easy to compute the trace of E_0 and F_0, and observe that it is equal to -4, and their axes are either disjoint or meet at right angles to form the hexagon H_1. Thus we obtain the discrete purely hyperbolic group G_0, representing a closed Riemann surface of genus 2.

After doing the above computations, we see that the axes of $A_0, \dots, F_0$ split M_0 into four right angle equilateral hexagons. Since the equilateral hexagon with all right angles is unique, it follows that our group and generators are as desired.

Once we have defined G_0, then we define the *normalized deformation space* $\mathcal{D}$ as the space of representations $\varphi\colon G_0 \to PSL(2,\mathbb{R})$; the image group $G = \varphi(G_0)$ is discrete, with $M = H^2/G$ a closed Riemann surface of genus 2. Also here, the product of the fixed points of $A = \varphi(A_0)$ is equal to 1, with the attracting fixed point positive and smaller than the repelling fixed point; the repelling fixed point of $C = \varphi(C_0)$ is at 0; and the attracting fixed point of C is at ∞. The point of intersection of C_0 with the common orthogonal between A_0 and C_0 lies at the point i, of the imaginary axe. The normalizations given in the definition of $\mathcal{D}$ make clear that there is a well defined real-analytic diffeomorphism between the Teichmüller space $\mathcal{T}$ and the *normalized deformation space* $\mathcal{D}$ (see [Maskit, 1999]).

Definition 2.1. Let $\gamma_1, \gamma_2, \gamma_3$ the oriented decomposition curves, the functions l_j, and θ_j , $j = 1, ..., 3$ are the length of γ_j, and the twist angle used to glue the pieces respectively. The system $\{l_j, \theta_j\}_{j=1}^3$is called the coordinates of Fenchel-Nielsen.

With this decomposition, in order to obtain a geometric image and to study the dynamical proprieties, we construct a fundamental domain $\mathcal{F}$. For each fundamental domain, the fundamental group G is now generated for the side pairing transformations g_i (and their inverses), that considered when had chosen the side identifications. If the region is compact, the generators are hyperbolic transformation. For this case the group G as subgroup of $SL_2(\mathbb{R})$, is represented by the generators g_i

$$G = < g_1, .., g_6 > \ .$$

Definition 2.2. An open set $\mathcal{F}$ of the upper half-plane H^2 is a fundamental domain for G if $\mathcal{F}$ satisfies the following conditions:

i) $g(\mathcal{F}) \cap \mathcal{F} = \phi$ for every $g \in G$ with $g \neq id$.

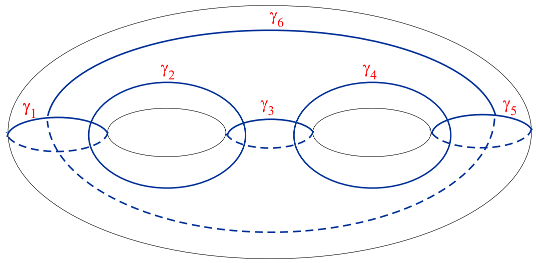

Fig. 1. A chain on the surface M

ii) If $\overline{\mathcal{F}}$ is the closure of in H^2, then :

$$H^2 = \bigcup_{g \in \mathcal{F}} g(\overline{\mathcal{F}})$$

iii) The relative boundary $\partial\mathcal{F}$ of $\mathcal{F}$ in H^2 has measure zero with respect to the two-dimensional Lebesgue measure.

These conditions tell us that the Riemann surface $M = H^2/G$ is considered as $\overline{\mathcal{F}}$ with points on $\partial\mathcal{F}$ identified under the covering group G. With the hyperbolic geometry is possible to determine explicit formulas for the generators.

The images of $\mathcal{F}$ under G provide a tiling (tessellation) of H^2 each image is a "tile" of the universal covering surface of M.

To explicit the construction of the fundamental domain we consider a chain on the surface M, like in M_0, see Fig. 1.

When we cut the surface M along these geodesics then we divide it into four equilateral hexagons. These geodesics are the shortest geodesics in the free homotopy class of loops corresponding to some elements h_i $(i = 1, ..., 6)$ of $\pi_1(M)$, the fundamental group of M. We have the hexagon H_1 whose sides s_i are the arcs of γ_i and these arcs are contained in the axes of the hyperbolic transformations h_i $(i = 1, ..., 6)$. Their translation length in the positive direction along these axis is $2\,l_i$ where l_i denote the length of $\gamma_i = l(\gamma_i)$. They are four of the parameters on this construction.

We choose, as reference a geodesic segment, γ, axis of h, which is the common orthogonal between the axes of h_1 and h_3. Remark that if h_2 is orthogonal to h_1 and h_3 then $h_2 = h$. We called P the intersection point between h and h_1 and P_2 the intersection point between h_2 and h_1.

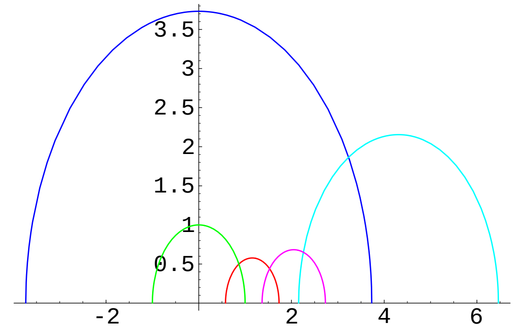

Fig. 2. Hexagon H_1

The other parameters are the gluing angles. So we consider the parameter σ determined by the distance between the intersection of h with h_1 and the intersection of h_2 with h_1. If $h_2 = h$ then l_5 is equal to zero. The other two parameters τ and ρ are determined by the angles θ_2 and θ_3 between h_2, h_1 and h_1, h_3, respectively. So

$$\begin{aligned} &l_1 = l(\gamma_1), \;\; l_2 = l(\gamma_2), \; l_3 = l(\gamma_3), \\ &l_4 = l(\gamma_4), \quad \sigma = |P - P_2| \;, \\ &\tau = arc\tanh\left(\cos(\theta_2)\right) \;, \\ &\rho = arc\tanh\left(\cos(\theta_3)\right) \end{aligned}$$

With the chosen normalization we obtain the hexagon H_1 represented in Fig. 2.

Let h, with $c \neq 0$, be

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The reflection through an axis h is represented by the transformation r

$$r(z) = \left(\frac{1}{\sqrt{a + d^2 - 4}}\right)\frac{(a-d)\,z + 2\,b}{2\,c\,z + (d-a)}.$$

The reflection with respect to the axis of h_4 sends H_1 to another right angled hexagon H_2, see Fig.3.

Finally the reflection with respect to the imaginary axis (symmetry) sends H_1 and H_2 to the hexagons H_3 and H_4. Thus, we had construct the fundamental domain: $\mathcal{F} = H_1 \cup H_2 \cup H_3 \cup H_4$. If the twist angles are zero so the fundamental domain are a right-angle polygon. That is $\sigma = \tau = \rho = 0$, see Fig. 4.

For $\sigma, \tau, \rho \neq 0$ we can see the twist angle parameters in the Fig. 5. The way of gluing is not the same.

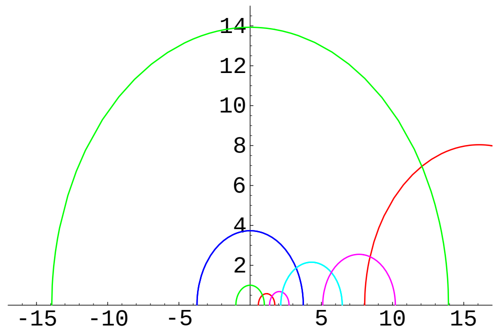

Fig. 3. Hexagon $H_1 \cup H_2$.

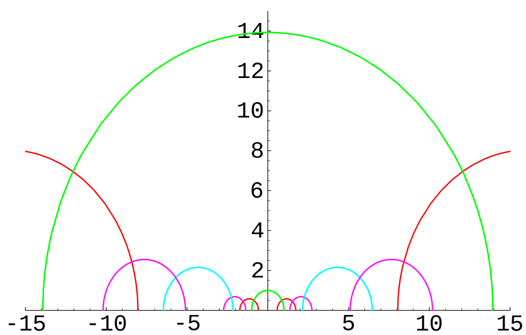

Fig. 4. Hexagon $H_1 \cup H_2 \cup H_3 \cup H_4$.

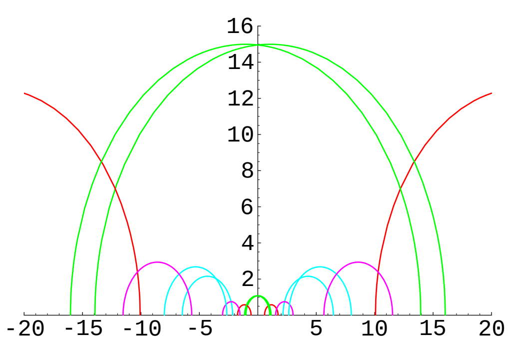

Fig. 5. Dodecagon $\sigma, \tau, \rho \neq 0$.

This construction depends on the choice of the original geodesics γ_i, $i = 1, ..., 4$. The chain is then dependent of the parameters $l_i = l(\gamma_i)$.

The sides are obtained by the intersection of the axis, they are geodesic segments. We called vertex the single point which is the intersection between two consecutive sides. The circular arc that contains a side s_i intersects the real axis in two points p_i and q_i. The sides are labelled $s_1, ..., s_{12}$ reading counterclockwise from the zero.

It is known (see [Beardon, 1983]) that if $\mathcal{F}$ is any locally finite fundamental domain for a Fuchsian group G, then

$$\{g \in G : g(\overline{\mathcal{F}}) \cap \overline{\mathcal{F}} = \phi\}$$

generates G.

Let $M = H^2/G$ our compact surface of genus 2. The fundamental domain $\mathcal{F}$ is a bounded fundamental polygon whose boundary $\partial\mathcal{F}$ consists of the 12 geodesics segments $s_1, ..., s_{12}$. Each side s_i of $\mathcal{F}$ is identified with s_j, by an element $g \in G$ and so each $g \in G$ produces a unique side s, namely, $s = \overline{\mathcal{F}} \cap g(\overline{\mathcal{F}})$. There is a bijection between the set of the sides of $\mathcal{F}$ and the set of elements g in G for which $\overline{\mathcal{F}} \cap g(\overline{\mathcal{F}})$ is a side of $\mathcal{F}$.

We construct a map from the set of the sides of $\mathcal{F}$ onto itself, $g : s_i \to s_j$ where s_i is identified with s_j. This is called a *side pairing* of $\mathcal{F}$. The *side pairing* elements of G generate G. In this construction we choose the side rule for the pairing

$$\begin{aligned} g_1 &: s_1 \to s_7, \quad g_2 : s_2 \to s_{12}, \quad g_3 : s_3 \to s_5, \\ g_4 &: s_4 \to s_{10}, \quad g_5 : s_6 \to s_8, \quad g_6 : s_9 \to s_{11} \end{aligned}$$

3. Deformations, Parameters and the Parameter Map

Let $\varphi : G_0 \to PSL(2, \mathbb{R})$ be a deformation in $\mathcal{D}$. We define $A, \ldots, F$ by $A = \varphi(A_0), \ldots, F = \varphi(F_0)$, and let $G = \varphi(G_0)$. Since φ can be realized by an orientation-preserving homeomorphism of the closed disc, the axes of $A, \ldots, F$ form a hexagon, H, and the axes of A and C do not meet, even on the circle at infinity. Also G is normalized so that the axis of C lies on the imaginary axis, with 0 as the repelling fixed point, so that the point of intersection of the axis of C with the common orthogonal to the axes of A and C lies at the point i (given the orientation, and given that the axis of

C lies on the imaginary axis, this is equivalent to saying that the product of the fixed points of A is equal to 1). We observe that H necessarily lies in the right half-plane. We let $A, \ldots, F$ be the sides of H, where A lies on the axis of A, etc. We orient each side so that its orientation agrees with that of the positive direction of the corresponding hyperbolic Möbius transformation.

The axes of $A, \ldots, F$ form a geodesic necklace on the underlying Riemann surface $M = H^2/G$. This necklace divides M into four hexagons, which we can also label as $H_1, \ldots, H_4$. In this section, we consider a general deformation $\varphi \in \mathcal{D}$; we set $(A, B, C, D) = (\varphi(A_0), \varphi(B_0), \varphi(C_0), \varphi(C_0))$. We define Maskit eight basic parameters, and we write down matrices (A, B, C, D), such that the entries in these matrices are particular functions of these parameters. In the same way define $E = A^{-1}C^{-1}$, and $F = D^{-1}B^{-1}$. Then, since φ is a deformation, the axes of $A, \ldots, F$ form a hexagon H_1, with sides $A, \ldots, F$, where A lies on the axis of A, etc. Let $G = \langle A, B, C, D \rangle$, and let $M = H^2/G$.

Maskit basic parameters are α, β, γ, δ, σ, τ, ρ and μ, defined as follows. Set $\alpha = l_1/2$, $\beta = l_2/2$, $\gamma = l_3/2$, $\delta = l_4/2$. Let L be the common orthogonal between the axes of A and C and ℓ its length define μ by $\coth\mu = \cosh\ell$. Let σ be the distance, measured in the positive direction along the axis of C, between L and the point where the axis of B crosses the axis of C. Let θ_2 be the angle inside H between the axes of B and C; and let θ_3 be the angle inside H^2 between the axes of C and D. Define τ and ρ by $\tanh\tau = \cos\theta_2$ and $\tanh\rho = -\cos\theta_3$.

We note that α, β, γ, δ and μ are necessarily positive. We define $A, \ldots, D$ as being matrices in $SL(2, \mathbb{R})$, with positive trace, representing, respectively, $A, \ldots, D$.

Maskit in [1999] introduced the following representations

$$A = \frac{1}{\sinh\mu}\begin{pmatrix} \sinh(\mu-\alpha) & \sinh\alpha \\ -\sinh\alpha & \sinh(\mu+\alpha) \end{pmatrix},$$

$$B = \frac{1}{\cosh\tau}\begin{pmatrix} \cosh(\tau+\beta) & e^{\sigma}\sinh\beta \\ e^{-\sigma}\sinh\beta & \cosh(\tau-\beta) \end{pmatrix},$$

$$C = \begin{pmatrix} e^{\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix},$$

$$D = \frac{1}{\cosh\rho}\begin{pmatrix} \cosh(\rho-\delta) & -e^{\sigma+\gamma}\sinh\delta \\ -e^{-\sigma-\gamma}\sinh\delta & \cosh(\rho+\delta) \end{pmatrix},$$

$$E^{-1} = CA, \qquad F^{-1} = BD.$$

Like the projection of the axes of $A_0, \ldots, F_0$, form a geodesic necklace on M_0, the projection of the axes of $A, \ldots, F$ form a geodesic necklace on M. The length of A is half the length of the closed geodesic formed by the projection of the axis of A; it follows that $trace(A) = 2\cosh\alpha$; $trace(B) = 2\cosh\beta$; $trace(C) = 2\cosh\gamma$ and $trace(D) = 2\cosh\delta$. We easily compute that the matrices above all have unit determinant.

Here we use the same normalization. It follows from the normalization that the common orthogonal L between the axes of A and C intersects the axis of C at the point i, with the attracting fixed point of A positive and smaller than the repelling fixed point; the repelling fixed point of A is at e^{μ} and the attracting fixed point of A is at $e^{-\mu}$. Maskit defined σ to be the distance, measured along the axis of C, in the positive direction, between the point of intersection with L, which has been normalized to be at the point i, and the point of intersection with the axis of B. Then the axis of B crosses the imaginary axis at the point ie^{σ}; the attracting fixed point of B is at $e^{\sigma+\tau}$ and the repelling fixed point is at $-e^{\sigma-\tau}$. Easy computations show that τ and θ_2 are related by $\tanh\tau = \cos\theta_2$.

We observed above that the distance, along the axis of C, between the point of intersection with the axis of B and the point of intersection with the axis of D must be half the translation length of C. Hence this point of intersection is the point $ie^{\sigma+\gamma}$, the repelling fixed point of D is at $e^{\sigma+\gamma+\rho}$, and the attracting fixed point is at $-e^{\sigma+\gamma-\rho}$. Then ρ and θ_3 are related by $\tanh\rho = -\cos\theta_3$.

Proposition 3.1. *(Maskit coordinates [Maskit, 1999]) The parameters,* $\sinh\alpha$, $\sinh\beta$, $\sinh\gamma$, $\sinh\delta$, $\sinh\mu$, $\sinh\sigma$, $\sinh\tau$, $\sinh\rho$, *depend algebraically on the point* $X \in \mathcal{D}$.

Observe that $2\cosh\alpha = trace(A)$; $2\cosh\beta = trace(B)$; $2\cosh\gamma = trace(C)$; $2\cosh\delta = trace(D)$; $2\cosh\mu$ is the sum of the fixed points of A; $e^{2\sigma}$ is the product of the fixed points of B; $2e^{\sigma}\sinh\tau$ is the sum of the fixed points of C; and $2e^{\sigma+\gamma}\sinh\rho$ is the sum of the fixed points of D.

We also remark that the entries in the matrices A, B, C, D are algebraic functions of the parameters, $\sinh\alpha$, $\sinh\beta$, $\sinh\gamma$, $\sinh\delta$, $\sinh\mu$, $\sinh\sigma$,

$\sinh\tau$ and $\sinh\rho$.

We will see below that $\sinh\mu$ and $\sinh\delta$ can be written as algebraic functions of the other parameters. Maskit defined the map $\psi\colon\mathcal{D}\to\mathbb{R}^6$, by $\psi(X) = (\sinh\alpha, \sinh\beta, \sinh\gamma, \sinh\sigma, \sinh\tau, \sinh\rho)$.

Now we assume that the matrices, $A, \ldots, D$, are defined by the formulas as functions of the eight parameters, $\alpha, \ldots, \rho$. We denote the corresponding Möbius transformations by $A, \ldots, D$. And we assume that there is a deformation $\varphi \in \mathcal{F}$, so that $(A, \ldots, D) = (\varphi(A_0), \ldots, V(D_0))$. We also explicitly assume that $\mu > 0$. Let $G = \varphi(G_0)$.

We have normalized C so that 0 is the repelling fixed point; this means that $\gamma > 0$. It also follows from our normalization that the attracting fixed point of B is positive; it follows that $\beta > 0$. We have normalized A so that its attracting fixed point lies between 0 and 1; this, together with our assumption that $\mu > 0$, implies $\alpha > 0$.

We state these three inequalities as Maskit first condition

$$\alpha > 0, \quad \beta > 0, \quad \gamma > 0. \tag{1}$$

For future use, we remark that it also follows from Maskit normalization that the attracting fixed point of D is negative; this implies that $\delta > 0$. Maskit [1999] determines the non-trivial inequalities

$$\coth\mu > \frac{1+\cosh\alpha\cosh\gamma}{\sinh\alpha\sinh\gamma} \tag{2}$$

$$\cosh(\rho+\sigma) < \cosh\gamma\cosh\mu - \coth\alpha\sinh\gamma\sinh\mu \tag{3}$$

and the following equalities

$$\cosh\mu = \coth\beta\cosh\sigma\cosh\tau + \sinh\sigma\sinh\tau, \quad \mu > 0 \tag{4}$$

$$\coth\delta = \frac{\cosh\gamma\cosh\mu - \coth\alpha\sinh\gamma\sinh\mu}{\cosh\sigma\cosh\rho} - \frac{\sinh\sigma\sinh\rho}{\cosh\sigma\cosh\rho} \tag{5}$$

Then Maskit has shown the following.

Proposition 3.2. *Let $\mathcal{R} \subset R^6$ be the region defined by the inequalities (1), (2) and (3), where μ is defined by equation (4) and δ is defined by equation (5). Then the image of ψ is contained in $\mathcal{R}$. $\mathcal{R}$ is equal to the image of ψ.*

We now assume that we have a point $(\alpha, \beta, \gamma, \sigma, \tau, \rho) \in \mathcal{R}$, defined by inequalities (1), (2) and (3); we assume that μ is defined by (4) and that δ is defined by (5). We write the matrices $A, \ldots, D$. We need to show that there is a $\varphi \in \mathcal{D}$, with $(A, \ldots, D) = (\varphi(A_0), \ldots, \varphi(D_0))$.

Finally, as above in the regular case, we obtain a purely hyperbolic discrete group, $G = \langle A, B, C, D\rangle$, representing a closed Riemann surface of genus 2.

Also, since the above construction uses combination theorems in exact analogy with their use in the construction of G_0, there is a topological deformation of G_0 onto G, where this deformation takes $(A_0, \ldots, D_0)$ onto $(A, \ldots, D)$. It follows from our normalization that this deformation is orientation-preserving. It is well known that an orientation-preserving topological deformation can be approximated by a quasiconformal one. The mapping $\chi\colon\mathcal{T} \to \mathcal{R}$ is a real-analytic embedding of the Teichmüller space of surfaces of genus 2 onto the region $\mathcal{R} \subset \mathbb{R}^6$

$$\mathcal{R} = \{\sinh\alpha, \sinh\beta, \sinh\gamma, \sinh\sigma, \sinh\tau, \sinh\rho\}.$$

As remarked above, $\sinh\alpha, \ldots, \sinh\rho$ are algebraic functions of the entries in the matrices.

With this choice we obtain the explicit formulas for the generators $h_1 = B$, $h_2 = A$, $h_3 = F$, $h_4 = E$, $h_5 = BD$, $h_6 = DF^{-1}$, $h_7 \equiv h_1^{-1}$, .., $h_{12} \equiv h_6^{-1}$, whose determine the axes.

We label the end points of the axes of h_i on $\partial\mathcal{F}$, p_i, q_i, $i = 1, ..., 12$, with $p_1 = -1$, $q_1 = 1$ and p_i occurring before q_i in the anti-clockwise order. These points are the intersections of circular arcs $C(h_i)$, axes of h_i, orthogonal to $\partial\mathcal{F}$.

With this choice we explicitly calculate formulas for the side pairing transformations $g_1, ..., g_6$, $g_7 = g_1^{-1}, ..., g_{12} = g_6^{-1}$. This mean that $s_7 = g_1(s_1)$, ..., $s_9 = g_6(s_{11})$, $s_1 = g_7(s_7)$, ..., $s_{11} = g_{12}(s_9)$. Let be

$$g_i(z) = (a_i z + b_i)/(c_i z + d_i),$$

for $g_i(s_j) = s_k$, with

$$\begin{aligned} r_i &= (q_i - p_i)/2, \\ c_i &= 1/(r_j r_k)^{1/2}, \\ b_i &= (a_i d_i - 1)/c_i, \end{aligned}$$

then we solve the system of equations

$$\begin{cases} (a_i p_j + b_i)/(c_i p_j + d_i) = q_k, \\ (a_i q_j + b_i)/(c_i q_j + d_i) = p_k \end{cases}$$

and we determine $\{a_i, d_i\}$, thus we obtain explicitly the generators $g_i = g_i(\alpha, \beta, \gamma, \sigma, \tau, \rho)$, $i = 1, ..., 12$.

Until now we had made an explicit geometrical description of the surface. All construction are generic for any choice of a surface of constant negative curvature and genus 2. So we can obtain a symbolic dynamics for the geodesic flow on these surfaces that involves the geometry and the structure of its fundamental group.

Bowen & Series in [1979] introduce a boundary map $f_G : \partial\mathcal{F} \to \partial\mathcal{F}$, defined by piecewise linear fractional transformations in the partition $\mathcal{P} = \{\, I_i = [p_i, p_{i+1}),\ i = 1, ..., 11, [p_{12}, p_1)\}$, which is orbit equivalent to the action of the fundamental group G on $\partial\mathcal{F}$, see Fig. 6. With the linear fractional transformations defined above it is possible to obtain the boundary map. The boundary map is represented by

$$f_G : \bigcup_{i=1,...,12} I_i \to \bigcup_{i=1,...,12} I_i$$

$$f_G(x)_{|I_i} = g_i(x), \quad i = 1, ..., 12$$

In general $f_G(p_i^{\pm}) \notin \{p_i\}_{i=1}^{12}$, then we need to refine the partition $\mathcal{P}$. Now, we consider the finite or infinite Markov partition $\mathcal{M} = \{J_j\}_{j=1}^{N}$ introduced by the itineraries of the lateral limits $p_i^{\pm}$ of the discontinuous points p_i. Let

$$W = \{ \lim_{\epsilon\to 0} f_G^k(p_i - \epsilon), \lim_{\epsilon\to 0} f_G^k(p_i + \epsilon)\}_{i=1}^{12}$$

for all $k \in \mathbb{N}$, where $\{J_j\}_{j=1}^{N}$ are the subintervals defined by the partition points W and f_G^k is the k iterate of f_G. With this set we obtain a Markov map for the partition $\mathcal{M}$.

Definition 3.3. A map f_G is a Markov map for $\mathcal{M}$ if f_G satisfies

i) piecewise smoothness,

ii) local invertibility,

iii) Markov property: each $f_G(J_j)$ is a union of intervals of the partition $\mathcal{M}$.

Now we can study the dependence of the boundary map with the parameters.

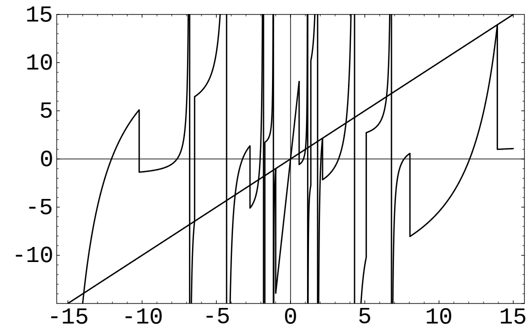

Fig. 6. Graph of $f_{\mathbf{G}}$.

We are able to define a map that codifies the expansion of boundary points of $\mathcal{F}$. And we determine the Markov matrix A_G associated to G.

The matrix A_G is given by

$$a_{ij} = \begin{cases} 1 & \text{if} \quad J_j \subset f_G(J_i) \\ 0 & \text{otherwise} \end{cases}$$

Now we can prove the following result.

Theorem 3.4. *The partition introduced by the family of boundary map $f_G(\alpha, \beta, \gamma, \delta, \sigma, \tau)$ through points in W is stable under deformation, i.e., it does not depend on the variation of the Fenchel-Nielsen-Maskit coordinates $(\alpha, \beta, \gamma, \delta, \sigma, \tau) \in \mathcal{R}$.*

Proof. It is known, in the regular case $(\alpha, \alpha, \alpha, \alpha, 0, 0)$, with $\alpha = arc\cosh(2)$, that the Markov transition matrix has 24×24 elements and is determined by the transition of the intervals J_j under f_G (see [Adler & Flato, 1991] and [Grácio & Sousa Ramos, 1999]). In this regular case $W = \{p_i, q_i\}_{i=0}^{11}$, where now we put in index, $i \bmod 12$, the Markov partition $\mathcal{M}$ determines the 24 intervals $J_{2i+1} = [p_{i+1}, q_i)$, $J_{2i+2} = [q_i, p_{i+2})$, and the transitions

$$\begin{aligned} g_i(p_i) &= q_{\pi(i)}, \\ g_i(q_{i-1}) &= q_{\pi(i)+1}, \\ g_i(p_{i+1}) &= q_{\pi(i)-1}, \end{aligned}$$

where $i = 0, ..., 11$ and π is defined by

$$\pi(i) = \begin{cases} (8 - i) \bmod 12 & \text{if } i \text{ odd} \\ (2 - i) \bmod 12 & \text{if } i \text{ even} \end{cases} .$$

Now we prove that, when we change the coordinates of Fenchel-Nielsen-Maskit this matrix remains constant. The entries in the matrices A, B, C, D (and

$E,F)$ are algebraic functions of the parameters, $\sinh(\alpha)$, $\sinh(\beta)$, $\sinh(\gamma)$, $\sinh(\delta)$, $\sinh(\sigma)$, $\sinh(\tau)$, as we saw. Thus their fixed points $\{p_i,q_i\}_{i=0}^{11}$ also depends algebraically in these Fenchel-Nielsen-Maskit coordinates. Finally the hyperbolic pairing linear fractional transformations determined by these fixed points also has a algebraic dependence in theses coordinates. One time we choose the pairing type, we have unique pairing transformations that transform the set $W=\{p_i,q_i\}_{i=0}^{11}$ into itself, according the same rules, then it determines the same Markov matrix.□

Corollary 3.5. *The topological entropy*

$$h_{top}(f_G(\alpha,\beta,\gamma,\delta,\sigma,\tau))=\log[\lambda_{\max}(A_G)],$$

where $\lambda_{\max}(A_G)$ *is the spectral radius of the matrix* A_G, *do not depend on the variation of the Fenchel-Nielsen-Maskit coordinates* $(\alpha,\beta,\gamma,\delta,\sigma,\tau)\in\mathcal{R}$ *(it is a topological invariant).*

For the closed Riemann surface of genus 2, $\lambda_{\max}(A_G)=6.97984....$ (see [Grácio & Sousa Ramos, 1999]).

Now, we could give explicit formula which show the dependence of others dynamical quantities of the boundary map $f_G(\alpha,\beta,\gamma,\delta,\sigma,\tau)$ with the parameters. When we change the Fenchel-Nielsen-Maskit coordinates we modify the metric structure of the surface and the quantities that depends of the metric. Thus we obtain the dependence the Lyapunov exponent with the Fenchel-Nielsen-Maskit coordinates. The Lyapunov exponent $\lambda(f_G(\alpha,\beta,\gamma,\delta,\sigma,\tau))$ (resp. Lyapunov multiplier $m(f_G(\alpha,\beta,\gamma,\delta,\sigma,\tau)))$ is given by

$$\lambda(f_G(\alpha,\beta,\gamma,\delta,\sigma,\tau),x)=\lim_{k\to\infty}1/k\,\log\left|(f_G^k)'(x)\right|$$

$$m(f_G(\alpha,\beta,\gamma,\delta,\sigma,\tau),x)=\lim_{k\to\infty}\left|(f_G^k)'(x)\right|^{1/k}$$

and the pressure $P(f_G(\alpha,\beta,\gamma,\delta,\sigma,\tau),\ s)$ is

$$\lim_{k\to\infty}1/k\,\log\sum_{f_G^k(x)=x}\exp(-s\left|(f_G^k)'(x)\right|)$$

Also we can define the zeta function (see [Pollicott & Rocha, 1997] and [Grácio & Sousa Ramos, 1999])

$$Z(t,s)\ =\ \det\left[I-t\,Q_{(-s\log|(f_G)'|)}\right]$$

$$=\ exp(-\sum_{k=1}^{\infty}\frac{t^k}{k}tr(Q^k_{(-s\log|(f_G)'(x)|)}))$$

where

$$tr\left(Q^k_{(-s\log|(f_G)'|)}\right)=\sum_{x\in Fix(f_G^k)}\frac{\exp\left(-s\,(f_G^k)'(x)\right)}{\left|1-(f_G^k)'(x)^{-1}\right|}.$$

Remark 3.6. This zeta function is a generalization of Selberg zeta function associated to the group G

$$\varsigma(s)=\prod_{k=0}^{\infty}\prod_{\gamma}\left(1-\exp\left[(s+k)\,\ell\left(\gamma\right)\right]\right)$$

$$\varsigma(s)=Z(t,s),$$

for $t=1$, where $\ell(\gamma)$ denotes the length of the closed geodesic γ.

Recalling that $\ell((\gamma)=2\,arc\cosh\left(tr(g)/2\right)$ and that there is a bijection between primitive geodesics on M and the conjugacy classes $[g]$ for G, we can enumerate the length spectrum $\ell(\gamma_i)$ of the primitive closed geodesics through periodic orbits of $f_G(\alpha,\beta,\gamma,\delta,\sigma,\tau)$. The allowed primitive words of the associated Markov shift determine elements $g_k=g_{i_1}g_{i_2}...g_{i_k}$ such that $\ell\left(\gamma_k\right)=2\,arc\cosh\left(tr(g_k)/2\right)=\log((f_G^k)'(\alpha,\beta,\gamma,\delta,\sigma,\tau))$. Then we obtain the zeta function in terms of the length spectrum

$$Z(t,s)=1+\sum_{k=1}^{\infty}z_k(s)\,t^k$$

where $z_k(s)$ is given by

$$\sum_{\substack{\gamma_1,\dots,\gamma_n\\|\gamma_1|+\dots+|\gamma_n|=k}}\frac{(-1)^n\quad\exp\left(-s\left(\ell(\gamma_1)+...+\ell(\gamma_n)\right)\right.}{(1-\exp\left(-\ell(\gamma_1)\right)...(1-\exp(-\ell(\gamma_n)))}$$

We finish illustrating the dependence for the Lyapunov multiplier with the parameter $\alpha=\ell_1\left(\gamma_1\right)/2$, see Fig.7.

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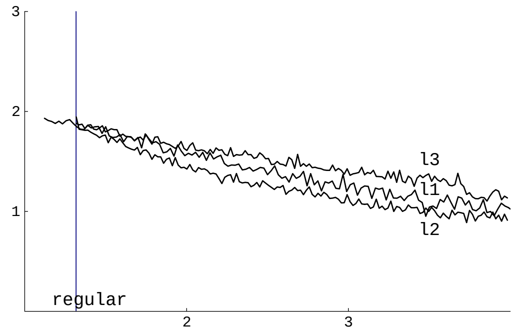

Fig. 7. Lyapunov expoent $(l_i), i = 1, 2, 3$.

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