ITERATION THEORY (ECIT '04)
W. Förg-Rob, L. Gardini, D. Gronau,
L. Reich, J. Smítal (Eds.)
Grazer Math. Ber., ISSN 1016–7692
Bericht Nr. **350** (2006), 122-134

Spectrum of the Laplacian on hyperbolic surfaces

Clara Grácio and José Sousa Ramos

Abstract

Our main tool is a method for studying how the hyperbolic metric on a Riemann surface behaves under deformation of the surface. We study the variation of the first eigenvalue of the Laplacian and the conductance of the dynamical system, with the Fenchel-Nielsen coordinates, that parameterizes the surface.

1 Introduction

The present paper is part of a program to understand the behavior of the spectrum, in particular of the first eigenvalue $\lambda_1(M)$ of the Laplacian of a compact Riemannian manifold M, endowed with a metric of constant curvature -1. So a fundamental goal of our program is computing λ_1 under variations throughout moduli space. We use techniques link between combinatorial structures (symbolic dynamics) and algebraic-geometric structures. When there are graph theoretic analogous of these notions and results, we pass back and forth between the geometric and graph models. Given a compact Riemannian manifold M, the Laplace-Beltrami operator Δ on functions on M is an elliptic operator with discrete spectrum

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty.$$

The eigenvalues which are less than 1/4 its call *small eigenvalues* in particular, 0 is taken to be a small eigenvalue (see [17]). Whereas in the case of surfaces of genus zero and one the explicit computation is possible, this is not the case for surfaces of higher genus.

For surfaces of genus $\tau = 2$ we computed the geodesic length spectrum of M (lengths of closed geodesics)(see [8], [9])

$$0 < \ell_1 \leq \ell_2 \leq \cdots \leq \ell_k \leq \cdots \to \infty.$$

Mathematics Subject Classification 2000: Primary 37B10, 39B12, Secondary 37E05 .

These surfaces were obtained by gluing together pairs of pants with no twists on the boundary components. This corresponds to Riemann surfaces in Teichmüller space for which the Fenchel-Nielsen coordinates are of the form $(\ell(\gamma_1), \ell(\gamma_2), \ell(\gamma_3),$ 0, 0, 0), where $\ell(\gamma_i)$, with $i = 1, 2, 3$, are the lengths of 3 geodesics on the surface, see Figure 1. We will denote for ℓ_i the length $\ell(\gamma_i)$ with $i = 1, 2, 3$. The special

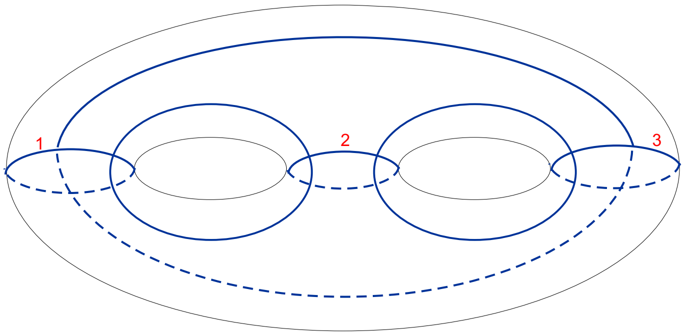

Figure 1: Compact Riemann surface of genus 2 and the geodesics γ_1, γ_2, γ_3.

nature of genus 2 has made it more accessible to produce more detailed results. There are several different ways to describe a closed Riemann surface of genus 2: representations as a hyperbolic manifold, an algebraic curve, a Fuchsian group, periodic matrices, the Fenchel-Nielsen (F-N) coordinates, etc. In this paper we use the F-N coordinates (see [14], [19]). They consist of the lengths and twists of $3\tau - 3$ disjoint simple closed geodesics. This space of coordinates is homeomorphic to the Teichmüller space $\mathcal{T}$ and the Teichmüller modular group acts on any such space of F-N coordinates as a group of algebraic diffeomorphisms. A general reference for this is [13]. The set of equivalence classes of hyperbolic metrics (or equivalently complex structures) under orientation preserving diffeomorphisms on M forms the moduli space $\mathcal{M}$ of compact Riemann surfaces of genus τ. It is represented by a quotient space $M = H^2/\Gamma$ of the upper half-plane H^2 by a Fuchsian group Γ which is isomorphic to the fundamental group of M. The discrete group Γ is identified with the corresponding system of generators. A fundamental domain $\mathcal{F}$ is defined. The method is to decompose Riemann surface into a set of 2 pairs of pants by simple closed geodesics. Then the Fenchel-Nielsen coordinates are defined by geodesic length functions of three simple closed geodesics, γ_i and twist angles σ_i, along these geodesics, with $i = 1, 2, 3$. With explicit constructions and side pairing transformations (see [9]), we define the Fuchsian group Γ representing the closed Riemann surface of genus 2, see Figure 2.

One approach to the Laplacian spectrum is made trought the Selberg trace for-

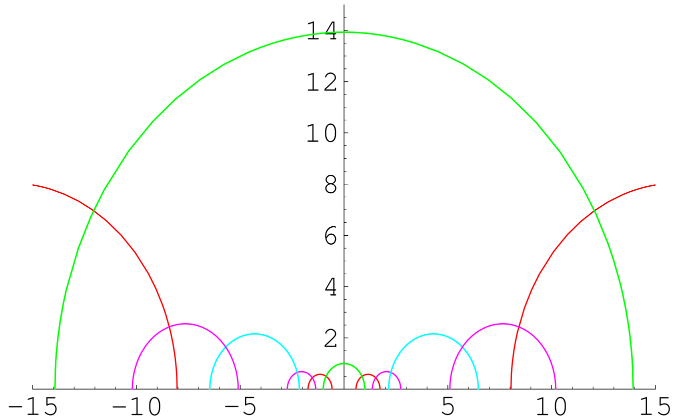

Figure 2: Hyperbolic plane and the fundamental domain.

mula. This trace formula has been of great interest to mathematicians for almost 50 years. It was discovered by Selberg in 1965, (see [16]), who also defined the Selberg zeta function, by analogy with the Riemann zeta function, to be a product over prime geodesics in a compact Riemann surface. An analogue of the Riemann hypothesis is provable for the Selberg zeta function. The trace formula shows that there is a relation between the length spectrum of these prime geodesics and the spectrum of the Laplace operator on the surface.

$$
\begin{aligned}
Tr(e^{t\Delta_M}) &= \sum_{k=0}^{\infty} e^{-t\lambda_k} \\
&= Area(M)\frac{e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}}\int_0^{+\infty}\frac{be^{-\frac{b^2}{4t}}}{sh\frac{1}{2}b}db + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{\ell(p)}\frac{\ell(p)}{sh\frac{1}{2}\ell(p^n)}\frac{e^{-\frac{t}{4}}}{(4\pi t)}e^{\frac{-(\ell(p^n))^2}{4t}}.
\end{aligned}
$$

In this case the geodesic length spectrum exactly determines the eigenvalues of the Laplacian, (see [20]).

Here we study the variation of the first eigenvalue $\lambda_1(M)$ of a compact Riemann surface M of genus 2 with the F-N coordinates.

2 Isoperimetric constant and Laplacian on hyperbolic surfaces and graphs

Let G be a discrete group. A mean is a linear functional $\mu : L^\infty(G) \to \mathbb{R}$ on the space of bounded real-valued functions such that $\mu(1) = 1$ and $u \geq 0 \Rightarrow \mu(u) \geq 0$. A group is amenable iff it admit a G-invariant mean, for the action $(g.u)(x) = u(xg)$. The free group $G =< r, s >$ is not amenable and the fundamental group of a closed surface of genus $\tau \geq 2$ is also nonamenable. The Cayley graph $\mathcal{G}$ of a finitely generated group G is a graph whose set of the vertices $V(\mathcal{G})$ represent the elements of the group G and whose edges connect elements differing by a generator. If $U \subset V$ is a subset of the Cayley graph we define its boundary ∂U to be the vertices connected to, but not lying in U. The isoperimetric constant, Cheeger constant or conductance $\varphi(\mathcal{G})$ of the graph $\mathcal{G}$ is given by

$$\varphi(\mathcal{G}) = \inf_U \frac{|\partial U|}{\min(|U|, |V - U|)}$$

where the infimum is over all finite sets U and $|X|$ is the number of vertices in X.

Theorem 1 (Folner) *Let G be a finitely generated group. Then G is amenable iff the isoperimetric constant of its Cayley graph is null.*

The Laplacian on functions is defined by

$$\Delta u = - * d * du.$$

The least eigenvalues of the Laplacian can be defined by minimizing the Ritz-Rayleigh quotient

$$\lambda_0(M) = \inf \frac{\int |\nabla u|^2}{\int |u|^2}.$$

On H with the metric $ds^2 = (dx^2 + dy^2)/y^2$ of constant curvature -1, we have

$$\Delta u = - * d * du = -y^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}).$$

It can be shown that $\lambda_0(H) = 1/4$. Given a closed Riemannian manifold M (compact without boundary) its Cheeger constant is defined by

$$h(M) = \inf_X \frac{area(X)}{\min(vol(A), vol(B))}$$

where the infimum is over all compact separating hypersurfaces $X \subset M$, and where $M - X = A \cup B$. If $vol(M) = \infty$ then the Cheeger constant reduces to the isoperimetric constant

$$h(M) = \inf_{A} \frac{area(\partial A)}{vol(A)}$$

where the inf is over all compact submanifolds A.

A celebrated inequality (see [5]) relates the first non-trivial eigenvalue of a compact manifold to an isoperimetric constant, the Cheeger constant.

Theorem 2 (Cheeger) *Let M be a closed Riemannian manifold we denote by $\lambda_1(M)$ the first nontrivial eigenvalue of M, if $vol(M)$ is finite, and by $\lambda_0(M)$ the bottom of the spectrum if $vol(M)$ is infinite. Then*

$$\lambda_i(M) \geq \frac{1}{4} h(M)^2$$

where $i = 0$ or 1.

This inequality is remarkable for its universal character.

Theorem 3 *For a closed hyperbolic surface M_τ of fixed genus τ, the $\lambda_1(M)$ is small iff M has a collection of disjoint simple geodesics $\gamma_1, \gamma_2, ..., \gamma_k$, $k \leq 3\tau - 3$, such that the length of $C = \cup_{i=1}^{k} \gamma_i$ is small and C separates M.*

This theorem with an added hypothesis on curvatures establish an upper bound for the first eigenvalue and was proved by Buser (1984) (see [4]).

Theorem 4 *Suppose that M is a smooth Riemannian manifold with curvature $Ricc(S) \geq -c$. Then there are constants c_1 and c_2 depending on c so that $\lambda_1 \leq c_1 h + c_2 h^2$.*

There are graph theoretic analogous of notions and results given above. A problem which appears difficult from one point of view may be more easy from the other point of view. Thus we return to graphs. In the analogous way one defines the Laplacian Δ of a graph. On a regular graph, like the Cayley graph $\mathcal{G}$ of a group, the degree d of its vertices $x \in V$ is a constant. We define the combinatorial Laplacian for functions $u : V \to \mathbb{R}$ by

$$(\Delta u)(x) = u(x) - \frac{1}{d} \sum_{y \sim x} u(y)$$

the sum is over the d vertices y adjacent to x.

Then, the Cheeger's inequality becomes (see [1], [2]).

Theorem 5 *Let $\mathcal{G}$ a regular graph of degree d. Then smallest non-zero eigenvalue of the Laplacian*

$$\lambda_i(\mathcal{G}) \geq \frac{1}{2d^2} h(\mathcal{G})^2,$$

where $i = 0$, if $|\mathcal{G}| = \infty$, and $i = 1$ if $|\mathcal{G}| < \infty$.

Theorem 6 *A finitely generated group G is amenable iff the smallest non-zero eigenvalue of the Laplacian $\lambda_0(\mathcal{G}) = 0$, where $\mathcal{G}$ is the Cayley graph of the group G.*

In the next section we extend for discrete dynamical systems the notion of the isoperimetric constant or conductance (see [6], [7]). In true, conductance distinguishes isospectral Riemann surfaces.

Theorem 7 (Brooks) *There exist two isospectral Riemann surfaces M_1 and M_2, such that M_1 is isospetral to M_2, but $h(M_1) \neq h(M_2)$.*

3 The notion of conductance of a discrete dynamical system

Let be (I, f) a discrete dynamical system defined by the iterates of a map f on the interval I. We associate a Markov partition and a transition matrix as usual, which is representable by a non-regular, oriented graph (digraph) $\mathcal{G}_f$. The edges E of $\mathcal{G}_f$ are now ordered pairs of vertices, defined by the adjacency matrix $A_f = (a_{ij})$.

Definition 1 *Let $A_f = (a_{ij})_{i,j=1}^n$ be the adjacency matrix associated to (I, f) and $\mathcal{G}_f$ the Markov graph. Define the diagonal matrix $D_f = (d_{ij})_{i,j=1}^n$, putting in the diagonal d_{ii} the number of edges that is incident (in and out) in the vertex i (loops contribute with 2). We call the matrix*

$$\Delta_f = D_f - (A_f + A_f^T)$$

the Laplacian matrix of the graph $\mathcal{G}_f$, where we designate by A^T the transpose matrix of A.

Here also, the smallest non-zero eigenvalue of the Laplacian is closely related with the conductance of the system, (see [7]). This result can be proved by symbolic dynamic methods.

Let be again H^2/Γ a compact surface of genus $\tau = 2$. A possible fundamental domain is a bounded fundamental polygon $\mathcal{F}$ whose boundary $\partial\mathcal{F}$ consists of the 12 geodesics segments $s_1, ..., s_{12}$, (see [8], [9] and Figure 2).

We construct a map from the set of the sides of $\mathcal{F}$ onto itself, $g : s_i \longrightarrow s_j$ where s_i is identified with s_j. This is called a *side-pairing* of $\mathcal{F}$. The *side-pairing* elements of Γ generate Γ. In this construction we choose the side rule for the pairing

$$s_1 \longrightarrow s_7,\ \ s_2 \longrightarrow s_{12},\ \ s_3 \longrightarrow s_5,$$
$$s_4 \longrightarrow s_{10},\ \ s_6 \longrightarrow s_8,\ \ s_9 \longrightarrow s_{11}.$$

With this choice we explicitly calculate formulas for the side pairing transformations $g_1, ..., g_{12}$. We obtain explicitly the generators $g_i = g_i(\ell_1,\ \ell_2,\ \ell_3,\ \sigma_1,\ \sigma_2,\ \sigma_3)$, $i = 1, ..., 12$, where $\ell_1, \ell_2, \ell_3, \sigma_1, \sigma_2, \sigma_3$ are the F-N coordinates. With the linear fractional transformations defined above it is possible to obtain the boundary map: $f_\Gamma : \partial\mathcal{F} \rightarrow\ \partial\mathcal{F}$, defined by piecewise linear fractional transformations in the partition $P = \{\, I_i = [p_i, p_{i+1}),\ i = 1, ..., 11,\ [p_{12}, p_1)\}$, which is orbit equivalent to the action of the fundamental group Γ on $\partial\mathcal{F}$. The boundary map is represented by

$$f_\Gamma : \bigcup_{i=1,...,12} I_i \rightarrow \bigcup_{i=1,...,12} I_i,$$

$$f_\Gamma(x)_{|I_i} = \ g_i(x), \quad i = 1, ..., 12.$$

We determine the Markov matrix A_{f_Γ} associated to f_Γ. Let be A_{f_Γ} the matrix

$$a_{ij} = \begin{cases} 1 & \text{if} \quad J_j \subset f_\Gamma(J_i) \\ 0 & \text{otherwise} \end{cases}$$

(see [8]). We obtained the length spectrum of the closed geodesics by computing

$$\ell(g) = 2 \quad \cosh^{-1}[tr(g)/2].$$

We can associate to the matrix A_{f_Γ} a stochastic matrix S and an invariant measure (the measure of the Parry) (S, π). We get thus what we call random walk where $S_{uv} = a_{uv} z_v/(\beta z_u)$ denotes the probability of moving from vertex u to v, β is the spectral radius and z the right eigenvector of A_{f_Γ}. Clearly, $S_{uv} > 0$ only if (u, v) is an edge and $\sum_v S_{uv} = 1$. The Perron-Frobenius theorem states that an irreducible matrix with non-negative entries has a unique (left and right) eigenvector with all entries positive. Let π denote the left eigenvector of S. We will call π the Perron vector of S. If $\mathcal{G}$ is strongly connected and aperiodic, the random walk converges to the stationary distribution, the Perron vector.

Now, with this Markov measure (S, π) we compute the conductance of the discrete dynamical system with stationary distribution π. The isoperimetric constant or conductance $\varphi(\mathcal{G})$ of the graph $\mathcal{G}$ is given by

$$\varphi(\mathcal{G}) = \min_{\substack{\varnothing \neq U \subset V \\ |U| \leq 1/2}} \frac{\sum\limits_{i \in U,\ j \in \overline{U}} \pi_i S_{ij}}{\sum\limits_{i \in U} \pi_i}$$

where the infimum is over all finite sets U and $|X|$ is the number of vertices in X.

In the context of the geometry there exists the following property, that states the existence of constants, who guarantees a certain limitation of hyperbolic structures under deformation of the hyperbolic surface.

We remember that eigenvalues which are less than $1/4$ are called *small eigenvalues,* in particular, 0 is taken to be a small eigenvalue (see [17]).

Theorem 8 *A compact hyperbolic surface M of genus $\tau = 2$ has at most two small eigenvalues of the Laplacian (see [17]).*

With the construction and computation above we can study the variation of the conductance (see Figure 3) and the first eigenvalue of the Laplacian (see Figure 4) with the Fenchel-Nielsen coordinates. We note that F-N coordinates are global coordinates so, define each surface.

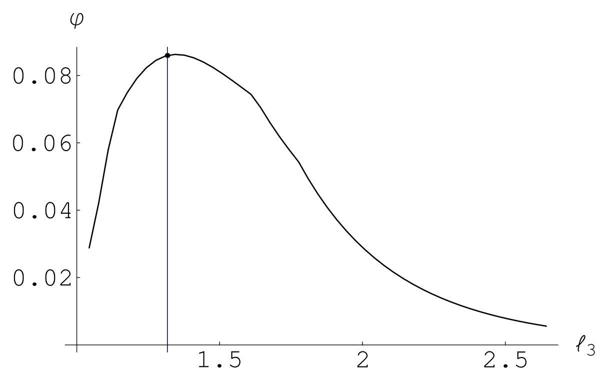

Figure 3: Variation of conductance φ with the Fenchel-Nielsen coordinate ℓ_3. For regular case $\ell_3 = \ell_0 = Log(2 + \sqrt{3}) = 1.31696...$

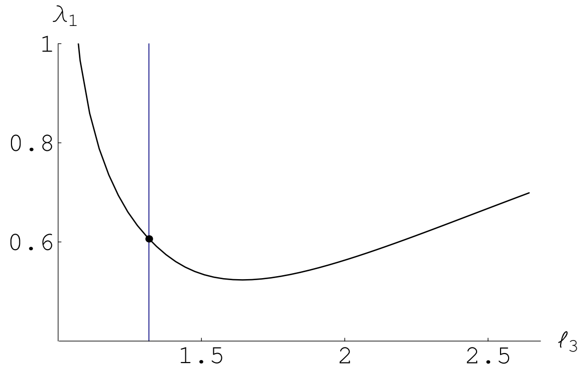

Figure 4: Variation of first eigenvalue λ_1 of the Laplacian with the Fenchel-Nielsen coordinate ℓ_3. For regular case $\ell_3 = \ell_0 = Log(2+\sqrt{3}) = 1.31696....$

Definition 2 *Let be a geodesic chain $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ where the four geodesics have equal length, ℓ_0, and the twist parameters are zero. We obtain the figure that we designate regular fundamental domain of the closed Riemann surface M (genus $\tau = 2$), for this Fenchel-Nielsen coordinates choice (see [14]).*

With the last construction it is possible to establish a upper bound for the conductance of the Laplacian.

Finally the main result, we denote by $\varphi(\ell_0)$ the conductance in the regular case, where $\ell_0 = Log(2+\sqrt{3})$ (see [9]) then we have.

Theorem 9 *Let M be a closed Riemannian manifold and let ℓ_i arbitrary Fenchel-Nielsen coordinates. Then the conductance $\varphi(\ell_i) \leq \varphi(\ell_0)$.*

The proof can be obtained following with analytic arguments the algorithms we had used to compute the conductance $\varphi(\ell_i)$ as a function of the F-N coordinates ℓ_i with $i = 1, 2, 3$.

At the Figure 3 it is possible to observe that the conductance has its maximum, exactly, when we consider the regular case, i.e., when Fenchel-Nielsen coordinates $\ell_i = \ell_0$ with $i = 1, 2, 3$.

Thus the Cheeger constant $h(\mathcal{G}_{f_\Gamma})$ and the conductances $\varphi(\ell_i)$ are maximum on M_t, where $t = t(\ell_1, \ell_2, \ell_3, \sigma_1, \sigma_2, \sigma_3)$ is a point of the Teichmüller space $\mathcal{T}$, when $t = t(\ell_0, \ell_0, \ell_0, 0, 0, 0)$(see Figure 3).

We also had studied the variation of the first eigenvalue $\lambda_1(\Delta_{f_\Gamma})$ of a Riemann surface (the smallest non-zero eigenvalue of the Laplacian) with $t \in \mathcal{T}$ and its relationship with the notion of conductance $\varphi(f_\Gamma)$ of the dynamical system defined by $f_\Gamma : \partial\mathcal{F} \to \partial\mathcal{F}$.

For each value of parameter of F-N coordinate, ℓ_i, we compute explicitly that

$$\lambda_1(\Delta_{f_\Gamma}) \geq \frac{1}{4}\varphi(f_\Gamma)^2$$

i.e. the conductance verifies the Cheeger's inequality (Cheeger's Theorem) (see Figure 5).

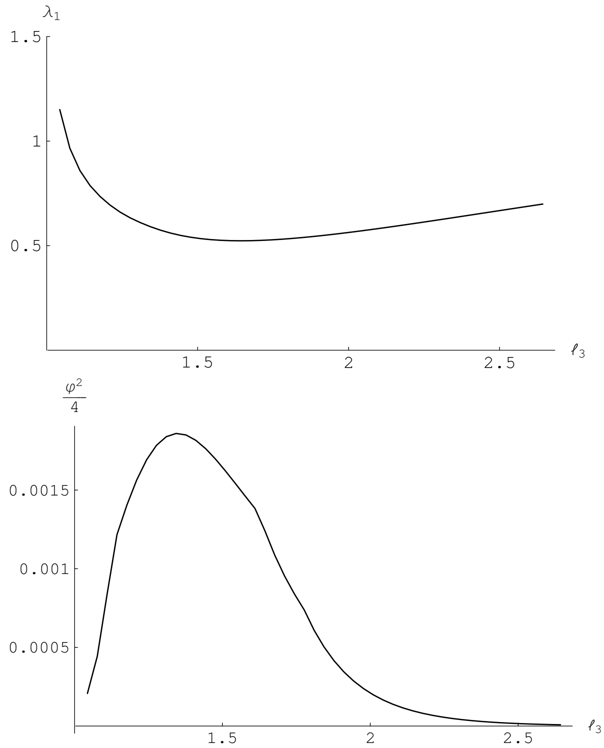

Figure 5: Variation of $\varphi(f_\Gamma)^2/4$ and the first non-zero eigenvalue $\lambda_1(\Delta_{f_\Gamma})$ of the Laplacian with the Fenchel-Nielsen coordinate ℓ_3

4 Systolic ratios

From a classical point of view the hyperelliptic surfaces are the most simple Riemann surfaces. They can be denned by an algebraic curve $y^2 = F(x)$ where $F(x)$ is a polynomial of degree $2\tau+1$ or $2\tau+2$ with distinct roots (τ is the genus of the surface). Hyperelliptic surfaces of genus τ are characterized by the fact that the number of different Weierstrass points is minimal, namely $2\tau+2$ (the fixed points of the hyperelliptic involution), while on the other hand, the weight of each Weierstrass point is maximal, namely $\frac{1}{2}\tau(\tau-1)$.

For us two results about surfaces (see [18]) are important:

Theorem 10 *A closed surface M of genus $\tau \geq 2$ is hyperelliptic if and only if M contains $2\tau-2$ different simple closed geodesics which all intersect in the same point and mutually intersect in no other point.*

Theorem 11 *All closed surfaces of genus 2 are hyperelliptic.*

For the next definition we denote by $sys\pi_1(M,m)$ the least length of a noncontractible loop of M. We define the systolic ratio SR of (M,m) as

$$SR(M,m) = \frac{sys\pi_1(M,m)^2}{vol(M,m)}, \tag{1}$$

and the optimal systolic ratio of M as

$$SR(M) = \sup_m \; SR(M,m), \tag{2}$$

where m runs over the space of all metrics, (see [12]).

The optimal systolic ratio of a genus 2 surface is unknown, but it satisfies the Loewner inequality $SR(M) \leq 2/\sqrt{3}$, the best available upper bound for the optimal systolic ratio of an arbitrary genus two surface, (see [11]). But the latter ratio is known for the Klein bottle in addition to the torus and real projective plane. Note that averaging a conformal metric by the hyperelliptic involution improves the systolic ratio of the metric.

Systolic geometry has recently seen a period of great growth, (see [10], [12]). Thus a surface is Loewner if $SR(M) \leq 2/\sqrt{3}$, and in [11] has recently been show that the genus 2 surface is Loewner.

Like some isoperimetric inequalities on manifolds can be generalized to graphs the same occurs for the systolic ratios. Let $(\mathcal{G},w)$ be a weighted graph. The *volume* of $(\mathcal{G},w)$ denoted by $\mathrm{Vol}(\mathcal{G},w)$, is the sum of the weight of its edges

$$\mathrm{Vol}(\mathcal{G},w) = \sum_{e\in E} w(e).$$

The *systole* of $(\mathcal{G},w)$ (or *girth*) is defined as

$$\mathrm{sys}(\mathcal{G},w) = \inf\{\ell_w(\gamma) \mid \gamma \text{ non trivial cycle of } \mathcal{G}\},$$

where the length of a cycle γ, noted $\ell_w(\gamma)$, is the sum of the weights of its edges. With this we can define the optimal systolic ratio of $\mathcal{G}$ as

$$SR(\mathcal{G}) = \sup_w \frac{\text{sys}(\mathcal{G}, w)^2}{\text{Vol}(\mathcal{G}, w)}.$$

where the supremum is taken over all the weight functions on the graph $\mathcal{G}$.

With the topological Markov chains, $(\Sigma_{A_\Gamma}, \sigma_{A_\Gamma})$, or subshift of finite type, associated to the 24×24 matrix A_Γ, and with the corresponding weighted matrix $Q_\Gamma(\ell_1, \ell_2, \ell_3)$ introduced in [8] and [9], we compute the *systole* of $(\mathcal{G}, w)$ and the optimal systolic ratio of $\mathcal{G}$, $SR(\mathcal{G})$. The set Σ_{A_Γ} can be identified with the space of bi-infinite paths of an oriented graph $\mathcal{G}_{A_\Gamma}$ whose vertices lie in $\mathbb{Z}_{24}$ and edges are the pairs (i, j) of vertices such that $a_{i,j} = 1$. We define the systole, girth or minimal period of $(\Sigma_{A_\Gamma}, \sigma_{A_\Gamma})$, as the smallest period of a periodic point of the dynamical system (S^1, f_Γ). It coincides with the shortest length of an oriented cycle of $\mathcal{G}_{A_\Gamma}$. Thus we compute explicitly this geometric quantities and its variation with the Fenchel-Nielsen coordinates.

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Clara Grácio
Department of Mathematics, Universidade de Évora,
Rua Romão Ramalho, 59, 7000-671 Évora, Portugal
e-mail: mgracio@uevora.pt

José Sousa Ramos
Department of Mathematics, Instituto Superior Técnico
Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal
e-mail: sramos@math.ist.utl.pt