CHAOTIC SYNCHRONIZATION OF UNIMODAL AND BIMODAL MAPS

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We derive a threshold value for the coupling strength in terms of the growth number, to achieve synchronization of two coupled piecewise linear m-modal maps, with $m = 1$ and $m = 2$, for the unidirectional and for the bidirectional coupling. This gives us information about the synchronization of unimodal and bimodal maps. An application to the bidirectional coupling of two identical chaotic Duffing equations is given.

1. Introduction

The analysis of synchronization phenomena of dynamical systems started in the 17th century with the finding of Huygens that two very weakly coupled pendulum clocks become synchronized in phase. Recently, the search for synchronization has evolved to chaotic systems. Two or more, identical, separated, chaotic systems starting from slightly different initial conditions would evolve in time, with completely different behaviour, but if they are coupled, we may see that after some time they exhibit exactly the same behaviour. Synchronization is a process wherein two or more systems adjust a given property of their motion to a common behaviour, due to coupling or forcing. Various types of synchronization have been studied. This includes complete synchronization (CS), phase synchronization (PS), lag synchronization (LS) generalized synchronization (GS), anticipated synchronization (AS), and so on [2]. The coupled systems might be identical or different, the coupling might be unidirectional, (master-slave or drive-response), or bidirectional (mutual coupling) and the driving force might be deterministic or stochastic.

In [4], A. Kenfack studied the linear stability of the coupled double-well Duffing oscillators projected on a Poincaré section and observed numerically the bifurcations and chaotic behaviour of the system, when the parameters change. In [5], Kyprianidis *et al.* observed numerically the synchronization of two identical single-well Duffing oscillators as a function of the coupling parameter.

In this work we investigate the unidirectional and bidirectional synchronization of two identical unimodal and bimodal maps. We obtain, analytically, the value of the coupling parameter for which the complete synchronization is achieved. Then, we apply these results to the study of the chaotic synchronization of two identical bidirectionally coupled double-well Duffing oscillators. We discuss the synchronization in terms of symbolic dynamics. Symbolic dynamics is a fundamental tool available to describe complicated time evolution of a chaotic dynamical system. Instead of representing a trajectory by infinite sequences of numbers, one uses the alternation of symbols.

2. Main results

Consider the coupling of two identical maps $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$. To be able to say if the two systems are synchronized we must look to the difference

$$z_n = y_n - x_n$$

and see if this difference converges to zero, as $n \to \infty$.

Denoting by k the coupling parameter, if the coupling is unidirectional

$$\begin{cases} x_{n+1} = f(x_n) \\ y_{n+1} = f(y_n) + k\,[f(x_n) - f(y_n)] \end{cases}$$

then

$$z_{n+1} = (1-k)\,[f(y_n) - f(x_n)]\,. \tag{1}$$

If the coupling is bidirectional

$$\begin{cases} x_{n+1} = f(x_n) - k\,[f(x_n) - f(y_n)] \\ y_{n+1} = f(y_n) + k\,[f(x_n) - f(y_n)] \end{cases}$$

then

$$z_{n+1} = (1-2k)\,[f(y_n) - f(x_n)]\,. \tag{2}$$

These two systems are said in complete synchronization if there is an identity between the trajectories of the two systems. In [10] and [11] it is establish that this kind of synchronization can be achieved provided that all the Lyapunov exponents are negative.

2.1. *Synchronization of unimodal maps*

Consider the tent map f_s: $[0,1] \to [0,1]$ defined by

$$f_s(x) = \begin{cases} sx - s + 2, & \text{if } 0 \leqslant x < 1 - \frac{1}{s} \\ s - sx & \text{if } 1 - \frac{1}{s} \leqslant x \leqslant 1 \end{cases} .$$

Recall that any piecewise monotonic map of positive entropy and growth number s is topologically semi-conjugated to a piecewise linear map with slope $\pm s$ everywhere, see [9].

This map can be written as

$$f_s(x) = sx - s + 2 + 2\theta\,(s - sx - 1)\,, \tag{3}$$

with

$$\theta(x) = \begin{cases} 0 & \text{if } 0 \leqslant x < 1 - \frac{1}{s} \\ 1 & \text{if } 1 - \frac{1}{s} \leqslant x \leqslant 1 \end{cases} .$$

Definition 2.1. Let $S(n) = S_1 S_2 ... S_n$ be a symbolic sequence, using symbols S_i belonging to some alphabet A. Define a distance between two sequences

$$S_x(p) = S_{x_1} S_{x_2} ... S_{x_p} \quad \text{and} \quad S_y(q) = S_{y_1} S_{y_2} ... S_{y_q}\,,$$

by $d(S_x, S_y) = e^{-n}$, where $n = \min\{n \geqslant 1 : S_x(n) \neq S_y(n)\}$.

Compare with [3].

Theorem 2.1. *Let $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ be two identical coupled systems, with f given by (3) and $1 < s \leqslant 2$. Let h be the topological entropy of (3) ($h = \log s$) and $k \in [0,1]$ the coupling parameter. If*

$$\exists n \in \mathbb{N} : d\left(\theta(y_{n+j}), \theta(x_{n+j})\right) \leqslant e^{-n}, \ \forall j \geqslant 0,$$

then,

(i) the unidirectional coupled systems (1) are synchronized if $k > \frac{e^h - 1}{e^h}$.
(ii) the bidirectional coupled systems (2) are synchronized if $k > \frac{e^h - 1}{2e^h}$.

Proof. Attending to (1) and (3),

$$z_{n+1} = (1-k)\left[sy_n - s + 2 + 2\theta_{y_n}(s - sy_n - 1) - sx_n + s - 2 - 2\theta_{x_n}(s - sx_n - 1)\right].$$

If $\theta_{y_{n+j}} = \theta_{x_{n+j}}$, $\forall j \geqslant 0$, then $z_{n+1} = (1-k)(1+2\theta)\, s z_n$.

It follows that, $z_{n+m} = \left[(1-k)(1-2\theta)\, s\right]^m z_n$.

Thus, if $\theta = 0$, then $z_{n+m} = \left[(1-k)\, s\right]^m z_n$ and if $\theta = 1$, $z_{n+m} = \left[(1-k)(-s)\right]^m z_n$. In both cases, $z_{n+m} = r^m z_n$, with $|r| = (1-k)\, s$.

So, letting $m \to \infty$, we have $\lim\limits_{m\to\infty} r^m z_n = 0$, iff $|r| < 1$, i.e., $|(1-k)\, s| < 1 \Rightarrow k > \frac{e^h - 1}{e^h}$, for $k \in [0,1]$, as desired.

Attending to (2) and (3),

$$z_{n+1} = (1-2k)\left[sy_n - s + 2 + 2\theta_{y_n}(s - sy_n - 1) - sx_n + s - 2 - 2\theta_{x_n}(s - sx_n - 1)\right].$$

If $\theta_{y_{n+j}} = \theta_{x_{n+j}}, \forall j \geqslant 0$, then $z_{n+1} = (1-2k)(1+2\theta)\, s z_n$.

It follows that, $z_{n+m} = [(1-2k)(1-2\theta)\, s]^m \, z_n$.

Thus, if $\theta = 0$, then $z_{n+m} = [(1-2k)\, s]^m \, z_n$ and if $\theta = 1$, $z_{n+m} = [(1-2k)(-s)]^m \, z_n$. In both cases, $z_{n+m} = r^m z_n$, with $|r| = (1-2k)\, s$.

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2.2. *Synchronization of bimodal maps*

Consider the bimodal piecewise linear map $f_{s,r}\colon [0,1] \to [0,1]$, with slopes $\pm s$, and $s > 1$, defined by

$$f_{s,r}(x) = \begin{cases} -sx+1 & \text{if } 0 \leqslant x < c_1 \\ sx+r-1 & \text{if } c_1 \leqslant x < c_2 \\ -sx+s & \text{if } c_2 \leqslant x \leqslant 1 \end{cases}$$

with $r = \frac{3+s}{2} - s\,(c_1+c_2)$ and critical points $c_1 = \frac{2-r}{2s}$ and $c_2 = \frac{1+s-r}{2s}$, see [8]. Recall that any transitive bimodal map is semi-conjugated to such a map.

This map can be written as

$$f_{s,r}(x) = -sx+1+\theta_{c_1}(2sx+r-2)+\theta_{c_2}(-2sx+s-r+1), \tag{4}$$

with

$$\theta_{c_i}(x) = \begin{cases} 0, \text{ if } 0 \leqslant x < c_i \\ 1, \text{ if } c_i \leqslant x \leqslant 1 \end{cases} \qquad (i=1,2).$$

In this case, we may define $\theta(x) = \theta_{c_1}(x) + \theta_{c_2}(x)$, *i.e.*,

$$\theta(x) = \begin{cases} 0 & \text{if } 0 \leqslant x < c_1 \\ 1 & \text{if } c_1 \leqslant x < c_2 \\ 2 & \text{if } c_2 \leqslant x \leqslant 1 \end{cases}.$$

Note the similarity of the meaning of the symbols $\{0,1,2\}$ with the usual alphabet $\{L,M,R\}$ in the symbolic dynamics, see [9] and [6].

Theorem 2.2. *Let $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ be two identical coupled systems, with f given by (4) and $1 < s \leqslant 2$. Let h be the topological entropy of (4), $(h = \log s)$ and $k \in [0,1]$ the coupling parameter. If*

$$\exists n \in \mathbb{N} : d\left(\theta(y_{n+j}), \theta(x_{n+j})\right) \leqslant e^{-n}, \ \forall j \geqslant 0,$$

then,

(i) the unidirectional coupled systems (1) are synchronized if $k > \frac{3e^h-1}{3e^h}$.

(ii) the bidirectional coupled systems (2) are synchronized if $k > \frac{3e^h-1}{6e^h}$.

Proof. Attending to (1) and (4)

$$\begin{aligned} z_{n+1} = \quad & (1-k)\Big\{\Big[-sy_n + 1 + \theta_{c_{1_{y_n}}}(2sy_n + r - 2) + \theta_{c_{2_{y_n}}}(-2sy_n + s - r + 1)\Big] \\ & - \Big[-sx_n + 1 + \theta_{c_{1_{x_n}}}(2sx_n + r - 2) + \theta_{c_{2_{x_n}}}(-2sx_n + s - r + 1)\Big]\Big\}. \end{aligned}$$

If $\theta_{c_{1_{y_n}}} = \theta_{c_{1_{x_n}}} = \theta_{c_1}$ and $\theta_{c_{2_{y_n}}} = \theta_{c_{2_{x_n}}} = \theta_{c_2}$, then

$$z_{n+1} = (1-k)\left(-1 + 2\left(\theta_{c_1} - \theta_{c_2}\right)\right) s z_n.$$

It follows that $z_{n+m} = \left[(1-k)\left(-1 + 2\left(\theta_{c_1} - \theta_{c_2}\right)\right) s\right]^m z_n$.
Denoting $r = (1-k)\left(-1 + 2\left(\theta_{c_1} - \theta_{c_2}\right)\right) s$, we have $z_{n+m} = r^m z_n$.
Thus, if $\theta_{c_1} - \theta_{c_2} = 0$ or 1, then $|r| = (1-k)\, s < 1 \Rightarrow k > \frac{s-1}{s}$.
If $\theta_{c_1} - \theta_{c_2} = -1$, then $|r| = 3\,(1-k)\, s < 1 \Rightarrow k > \frac{3s-1}{3s}$.
So, as $\frac{3s-1}{3s} > \frac{s-1}{s}$, to have synchronization it suffices that $k > \frac{3e^h-1}{3e^h}$.
Attending to (2) and (4)

$$\begin{aligned} z_{n+1} = \quad & (1-2k)\Big\{\Big[-sy_n + 1 + \theta_{c_{1_{y_n}}}(2sy_n + r - 2) + \theta_{c_{2_{y_n}}}(-2sy_n + s - r + 1)\Big] \\ & - \Big[-sx_n + 1 + \theta_{c_{1_{x_n}}}(2sx_n + r - 2) + \theta_{c_{2_{x_n}}}(-2sx_n + s - r + 1)\Big]\Big\}. \end{aligned}$$

If $\theta_{c_{1_{y_n}}} = \theta_{c_{1_{x_n}}} = \theta_{c_1}$ and $\theta_{c_{2_{y_n}}} = \theta_{c_{2_{x_n}}} = \theta_{c_2}$, then

$$z_{n+1} = (1-2k)\left(-1 + 2\left(\theta_{c_1} - \theta_{c_2}\right)\right) s z_n.$$

It follows that $z_{n+m} = \left[(1-2k)\left(-1 + 2\left(\theta_{c_1} - \theta_{c_2}\right)\right) s\right]^m z_n$.
Denoting $r = (1-2k)\left(-1 + 2\left(\theta_{c_1} - \theta_{c_2}\right)\right) s$, we have $z_{n+m} = r^m z_n$.
Thus, if $\theta_{c_1} - \theta_{c_2} = 0$ or 1, then $|r| = (1-2k)\, s < 1 \Rightarrow k > \frac{s-1}{2s}$.
If $\theta_{c_1} - \theta_{c_2} = -1$, then $|r| = 3\,(1-2k)\, s < 1 \Rightarrow k > \frac{3s-1}{6s}$.
So, as $\frac{3s-1}{3s} > \frac{s-1}{s}$, to have synchronization it suffices that $k > \frac{3e^h-1}{6e^h}$. □

3. An example: coupled Duffing oscillators

Consider two identical bidirectionally coupled Duffing oscillators, see [4] and references therein.

$$\begin{cases} x''(t) = x(t) - x^3(t) - \alpha x'(t) + k\left[y(t) - x(t)\right] + \beta\; Cos(wt) \\ y''(t) = y(t) - y^3(t) - \alpha y'(t) - k\left[y(t) - x(t)\right] + \beta\; Cos(wt) \end{cases} \tag{5}$$

where k is the coupling parameter. A basic tool is to do an appropriate Poincaré section. In our case, we did a section defined by $y = 0$, since it is transversal to the flow, it contains all fixed points and captures most of the interesting dynamics. We consider parameter values for which each uncoupled ($k = 0$) oscillator exhibits a chaotic behaviour, so if they synchronize, that will be a chaotic synchronization. In a previous work we have found in the parameter plane (α, β), a region $\mathcal{U}$ where the first return Poincaré map behaves like a unimodal map and a region $\mathcal{B}$ where the first return Poincaré map behaves like a bimodal map. We choose, for example,

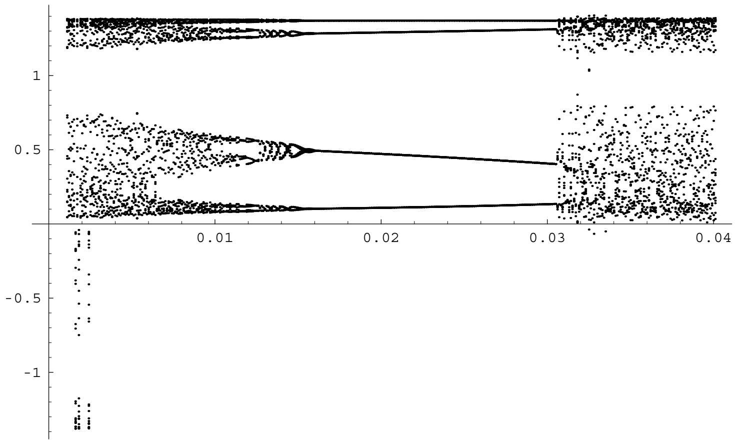

Figure 1. Bifurcation diagram for $k \in [0.001, 0.04]$

$w = 1.18$, $x_0 = 0.5$, $x_0' = -0.3$, $y_0 = 0.9$, $y_0' = -0.2$ and $\alpha = 0.4$, $\beta = 0.3578$, for the unimodal case and $\alpha = 0.5$, $\beta = 0.719$, for the bimodal case.

In Fig.1 the bifurcation diagram for the bidirectional coupled system (5) with $\alpha = 0.4$, $\beta = 0.3578$ and the coupling parameter $k \in [0.001, 0.04]$ shows several kinds of regions. In next section we will compute the topological entropy in some points of this regions.

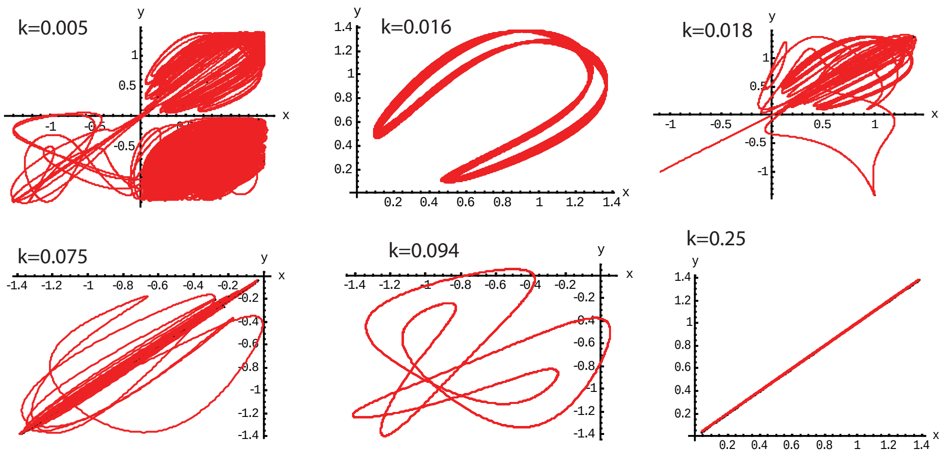

Figure 2. Evolution of x versus y for the bidirectional coupled Duffing oscillators, for some values of k, in the unimodal case ($\alpha = 0.4$, $\beta = 0.3578$).

Numerically we can also see the evolution of the difference $z = y - x$ with k.

The synchronization will occur when $x = y$. See some examples in Fig.2 for the unimodal case. Although not shown in this figure, the pictures for $k > 0.214\ldots$ are the same as for $k = 0.25$. Notice that, these pictures confirms numerically the theoretical results given by theorem 2.1. For $\alpha = 0.4$ and $\beta = 0.3578$ which correspond to $h = 0.2406\ldots$, the synchronization occurs for $k > 0.214\ldots$.

4. Symbolic Dynamics

As the value of k grows, the number of initial equal symbols in the x and y symbolic sequences, grows also. This can be expressed by the distance defined above and it is a numerical evidence that the two systems will be synchronized.

	k	n
$S_x : RLRRRLRLRRRLRRRLRRRLRLRLRRRLRL$	0.00601	
$S_y : RLRRRLRRRLRRRLRRRLRRRLRRRLRRRL$		7
$S_x : RLRRRRRLRRRLRRRLRRRLRRRLRRRLRL$	0.05	
$S_y : RLRRRLRLRLRLRRRLRRRLRLRRRLRRRL$		5
$S_x : RLRRRLRLRLRLRRRLRRRLRLRLRLRLRL$	0.06	
$S_y : RLRRRLRLRRRLRRRLRRRLRLRLRLRLRL$		9
$S_x : RLRRRLRLRLRRRLRRRLRRRLRRRLRRRL$	0.064	
$S_y : RLRRRLRLRLRLRLRRRLRRRLRLRRRLRR$		11
$S_x : RLRRRLRLRRRLRRRLRLRLRLRRRLRRRL$	0.065	
$S_y : RLRRRLRLRRRLRRRLRRRLRLRLRLRLRL$		17
$S_x : RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL$	0.07	
$S_y : RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL$		30
$S_x : RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL$	0.08	
$S_y : RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL$		30

Using techniques from Symbolic Dynamics, see [9] and [7], we compute the topological entropy h_{top} for some values of the coupling parameter k.

k	S_x	$D(t)$	h_{top}
0	$(CRLRRR)^\infty$	$\frac{(-1+t)\left[\left(-1+t^2\right)+t^4\right]}{1-t^{12}}$	$0.24061\ldots$
0.00601	$(CRLRRRLRLR)^\infty$	$\frac{(-1+t)\left[(-1+t^2)(1-t^4)+t^8\right]}{1-t^{20}}$	$0.20701\ldots$
0.03	$(CRLRRRLRRRLRRRLR)^\infty$	$\frac{(-1+t)\left[\left(-1+t^2\right)(1-t^4+t^8-t^{12}\right]}{1-t^{32}}$	0
0.1	$(CRLRRRLRLR)^\infty$	$\frac{(-1+t)\left[(-1+t^2)(1-t^4)+t^8\right]}{1-t^{20}}$	$0.20701\ldots$

Considering the return map for the first equation of system (5), with $\alpha = 0.4$ and $\beta = 0.3578$ (unimodal case), we obtain for several values of k, the kneading sequences S_x and the kneading determinants $D(t)$. Notice the correspondence of these values for the topological entropy with the evolution of k in the bifurcation diagram, see Fig.1. We have verified that the topological entropy for several values of k larger than $k \approx 0.032$ remains constant, but positive. Meanwhile we find values, of the k parameter, where the topological entropy is zero, that is, where there is chaos-destroying synchronization, see [12].

5. Conclusions

When doing Poincaré sections with $y = 0$, we obtained regions $\mathcal{U}$ and $\mathcal{B}$ where the Poincaré map behaves like a unimodal and bimodal map respectively. By a result from Milnor and Thurston [9] and Parry we know that every m-modal map F with growth rate s is topologically semi-conjugated to a $m+1$ piecewise linear map f defined on the interval $[0, 1]$, with slope $\pm s$ everywhere and $h_{top}(F) = h_{top}(f) = \log s$. So, the study and conclusions about synchronization of piecewise linear unimodal and bimodal maps, expressed in theorems 2.1 and 2.2, can be applied to understand the behaviour of more general maps.

From the previous theorems we may also verify that the unimodal map synchronizes faster than the bimodal map and that the bidirectional synchronization occurs at half the value of the coupling parameter for the unidirectional case, as mentioned by Belykh *et al* [1].

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