INSTITUTO
SUPERIOR
TÉCNICO

UNIVERSIDADE TÉCNICA DE LISBOA

INSTITUTO SUPERIOR TÉCNICO

Modeling and Performance Evaluation of Mobile Ad Hoc Networks

Gonçalo João Costa Jacinto

Thesis prepared to obtain the PhD Degree in Mathematics

Supervisor: Doctor António Manuel Pacheco Pires
Co-Supervisor: Doctor Nelson Gomes Rodrigues Antunes

Jury

Chairman: Chairman of the IST Scientific Board
Members of the Committee: Doctor Rui Jorge Morais Tomaz Valadas
Doctor António Manuel Pacheco Pires
Doctor Luísa da Conceição dos Santos do Canto e Castro de Loura
Doctor Nelson Gomes Rodrigues Antunes
Doctor Cláudia Rita Ribeiro Coelho Nunes Philippart
Doctor Christine Fricker

February 2011

Modelação e Avaliação de Desempenho de Redes Móveis Ad Hoc

Nome: Gonçalo João Costa Jacinto

Doutoramento em: Matemática

Orientador: Professor Doutor António Manuel Pacheco Pires

Co-orientador: Professor Doutor Nelson Gomes Rodrigues Antunes

Resumo: As redes de telecomunicações móveis *ad hoc* são constituídas por nós que se organizam de forma autonóma e sem qualquer infraestutura, sendo uma das mais promissoras modernizações das actuais redes de telecomunicações sem fios. A mobilidade e a possibilidade de comunicação por rotas com múltiplos passos torna a topologia destas redes dinâmica e imprevisível, sendo necessário desenvolver modelos que descrevam a conectividade e a dinâmica dessas rotas.

A investigação inicia-se com o estudo da conectividade para redes unidimensionais e bidimensionais. É derivada a distribuição de probabilidade do número de passos duma rota quando a distribuição espacial dos nós provém de um processo de Poisson ou, utilizando o método de aleatorização de Poisson, quando um número fixo de nós está uniformemente distribuído numa dada região. Resultados numéricos ilustram o comportamento da distribuição de probabilidade do número de passos duma rota.

De seguida é desenvolvido um modelo para caracterizar a dinâmica das rotas através de um processo de Markov determinístico por troços. A distribuição e o tempo médio de duração das rotas são derivados, sendo estes resultados obtidos através de um sistema de equações integro-diferenciais. Um método recursivo é proposto para sua computação. Resultados numéricos ilustram o cálculo destas medidas, os quais são comparados com os obtidos quando se assumem rotas com passos independentes.

Palavras-chave: Redes de telecomunicações móveis ad hoc, distribuição do número de passos, aleatorização de Poisson, mobilidade, fiabilidade de rotas com múltiplos passos, processo de Markov determinístico por troços.

Modeling and Performance Evaluation of Mobile Ad Hoc Networks

Abstract: Mobile ad hoc networks are characterized by having nodes that are self-organized and cooperative without any kind of infrastructure, being the most promising upgrade of the current telecommunication systems. The mobility and multihop capability of these networks allows the network topology to change rapidly and unpredictably, turning necessary the development of appropriate models to describe the multihop connectivity and the dynamic of multihop paths.

The research carried on in this dissertation starts by addressing the multihop connectivity for one-dimensional and two-dimensional ad hoc networks. The hop count probability distributions are derived when the underlying node spatial distribution is drawn from a Poisson process and, by using a Poisson randomization technique, when a fixed number of relay nodes are uniformly distributed in a region of interest. Numerical results illustrate the computation of the hop count probabilities.

We then present an analytical framework to characterize the random behavior of a multihop path by means of a piecewise deterministic Markov process. The mean path duration and the path persistence metrics are obtained as the unique solution of a system of integro-differential equations, and a recursive scheme for their computation is provided. Numerical results are presented to illustrate the computation of the metrics and to compare the associated results with independent link approximation results.

Keywords: Mobile ad hoc networks, hop count distribution, Poisson randomization, mobility, multihop path reliability, piecewise deterministic Markov process.

Acknowledgments

It has been an enormous pleasure to work with Prof. António Pacheco and Prof. Nelson Antunes. My huge gratitude for their endless support and for all that they have transmitted to me.

To Prof. António Pacheco for the possibility to learn with his outstanding knowledge and rigour. Thanks for all his support and guidance. Thanks for transmitting all is rigour and humanity. He will be a reference in my future as scientific researcher.

To Prof. Nelson Antunes for his support, guidance and friendship. I had the fortune to work in a daily basis with him, and his interest, dedication, enthusiasm and encouragement was contagious. A special thanks to him by making it possible for me to work at the Algarve University facilities.

I would like to thank my parents for the education and principles they have granted me. Their support and incentive was always important. Thanks to my brothers for their support.

A special thanks to my friends, for their friendship and to be there whenever I needed. They know who they are and I'll never forget them.

In the end, but the most important of all, to Ana, to her endless patience, support, friendship and by having always a word of understanding. She knows everything I owe her, and the importance she had. My huge gratitude to her.

This work was partially supported by: Centro de Investigação em Matemática e Aplicações da Universidade de Évora (CIMA-UE).

Contents

List of Figures

Glossary

3G	the third generation of telecommunications systems
4G	the fourth generation of telecommunications systems
GSM	Global System of Mobile communications
CR	Compass Routing protocol
NR	Nearest distance Routing protocol
FR	Furthest distance Routing protocol
LAN	Local Area Network
MANET	Mobile Ad Hoc NETwork
MFR	Most Forward within Radius routing protocol
ODE	Ordinary Differential Equation
PDMP	Piecewise Deterministic Markov Process
UMTS	Universal Mobile Telecommunications Systems

Chapter 1

Introduction

The aim of this dissertation is to contribute to the modeling and performance evaluation of mobile ad hoc networks (MANETs), a type of network where the mobile nodes are connected by wireless links and where nodes, despite being free to move independently, are cooperative and self-organized in random topologies without any kind of supporting infrastructure or centralized administration. Each node can communicate directly with another node within its radio coverage, or with other nodes outside its radio coverage by using intermediate nodes to *relay* or forward traffic in a multihop fashion. Our research will focus on the probabilistic analysis of the connectivity problems that arise in one-dimensional and two-dimensional MANETs and on the modeling of the dynamics of multihop paths in such networks.

In this chapter, we start with a short overview of this kind of networks and some motivation for the study of MANETs. A description of the problems and the main contributions in literature are given and the main methodologies used in the dissertation are presented.

1.1 Mobile ad hoc networks

In this section we give a short overview of the history and the importance that MANETs have gained in the past few years. For additional reading see, e.g., Toh (2001) and Aggelou (2005), which provide a comprehensive introduction to a wide range of MANET's concepts, models, and technologies.

Wireless communications has grown exponentially in recent years. Second generation of telecommunication systems, such as the Global System for Mobile communications (GSM), enable voice traffic and are undoubtedly the largest technology worldwide. As the number of users using Internet connections and requiring multimedia applications has increased, a new generation of telecommunications has risen: the third generation (3G) mobile communication systems, also known as Universal Mobile Telecommunication Systems (UMTS). The combination of several evolving and emerging access technologies into a common platform will be the key characteristic of 4G systems. However, the high data rates envisioned will decrease the quality of services and, to guarantee them, it is necessary that the density of base stations (that provide users their communication capabilities) increases, which originates higher deployment costs.

With these factors in mind, the integration of MANETs multihop capability into wireless networks is perhaps the most promising architectural upgrade to envisage area coverage without significant additional infrastructure cost. The salient features of this breed of networks is that they can operate in different propagation and networking conditions, are self organized, and are rapidly deployable.

This type of networks has significant advantages when compared with traditional communication networks, since they have increased mobility and flexibility and can be brought up and torn down in a very short time. The absence of fixed infrastructure reduces the economical costs and, having short hop communications, the radio emission levels can be kept at a low level, increasing the spectrum reuse and permitting that unlicensed bands can be allocated. Additionally, a MANET may be connected to the edges of a fixed, wired internet, or other type of network, expanding the current networks.

The set of MANET applications is diverse, ranging from networks where nodes are static and constrained by power sources to large scale, mobile, highly dynamic networks. Their rapid deployment and low configuration profile make them suitable to be used in emergency scenarios, in cases where the existing fixed and wireless communications are destroyed, allowing rescue agencies to rapidly create a communication platform. These networks are also appropriate to be used in military scenarios, in remote regions, since the low node power decreases the probability of detection and can be deployed instantaneously and without any kind of infrastructure. In fact, military applications were one of the propulsors of MANET's development.

For commercial communications, they are suited to be used in local area networks (LAN) events, like congresses, meetings, and forums, where participants, in an ad hoc manner and with their laptops, have instant network formations in addition to file and information sharing without the presence of fixed stations or system infrastructures.

For vehicular networks, several communication systems are being developed. They arise in safety and information applications, permitting the vehicles to transmit and receive information about traffic, news, weather, etc.

Recently, the Swedish company TerraNet has implemented a system for personal communications in regions without landline connections or mobile network coverage, such that users with mobile phones equipped with their technology will form their own network and communicate directly with each other. Projects backed by Terranet were recently launched in Tanzania and Ecuador. This company argues that with their systems the era of free communications has arrived.

At last but not the least, applications of ad hoc networks extend to sensor networks. These are characterized by having sensor nodes (that could be static or mobile) who react to some variable of interest and collect data to be transmitted to a collector node, usually by a multihop path. The sensors are used to collect data related to temperature, humidity, noise, pollution, animal movements, etc, and have a high range of applications in life and environmental sciences.

1.2 Motivation

Recent advances in mobile equipment technologies and wireless communications gave rise to a new kind of networks, the MANETs, which are receiving a lot of attention mainly due to the high range of their applications (cf., e.g., Toh (2001) and Basagni et al. (2004)). These networks are characterized by being autonomous collections of mobile users, with small, powerfull, and versatile equipments (nodes), that communicate over relatively bandwidth constrained wireless links and are responsible to relay traffic towards its destination.

In MANETs, nodes can dynamically form a network in a self-organized manner without the need of an existing fixed infrastructure. Nodes are expected to act cooperatively in order to route traffic and adapt the network to the highly dynamic status of its links and mobility patterns. The dynamic behavior of MANETs, whose network topology may change rapidly and unpredictably, gives to mathematical models, specially the ones that include a strong stochastic component, an important role in evaluating the performance of MANETs.

The multihop capability is one of the key features of MANETs. When the source and destination nodes are at a distance greater than the transmission range, the communication between them is made via multiple hops, using the neighbor nodes to redirect the traffic towards the destination node. The multihop path is determined by the routing protocol and influences the performance of the network (cf., e.g., Mauve et al. (2001)). The functionality of the network critically depends on its connectivity properties, being necessary to develop models that focus on the multihop connectivity between a pair of nodes, both for one-dimensional and two-dimensional networks. These properties depend on the number of nodes, their transmission ranges, the spatial distribution resulting from the mobility pattern, and the routing protocol. The most common routing protocols for one-dimensional networks are the furthest distance routing protocol (FR) and the nearest distance routing protocol (NR), where the relay node chosen is at the furthest distance or at the nearest distance from the emissor node towards the destination node, respectively.

One of the most important metrics to evaluate the performance of the routing protocols is the hop count (cf. Kuo and Liao (2007)), defined as the number of relay nodes of a multihop path between the source and the destination nodes. Obtaining the hop count distribution for one-dimensional networks will not only be important for the design of routing protocols, but it will also bring new insights to the connectivity probability in one-dimensional networks. Despite of that, there are very few analytical studies on the distribution of the number of hops in a multihop path.

In one-dimensional ad hoc networks, the existing studies are focused on the connectivity probability, i.e., the probability that there exists a path between the source and destination nodes regardless of the number of hops, when relay nodes are uniformly distributed between the source and the destination nodes, (cf., e.g., Desai and Manjunath (2002), and Ghasemi and Nader-Esfahani (2006)). The study of the hop count distribution for one-dimensional networks has applications, e.g., in vehicular ad hoc networks built along a road in a city environment, in ad hoc networks along an attack route in battlefields, and in other scenarios where nodes are approximately in a line. This is the case for dense two dimensional ad hoc networks where routes are approximately straight line segments (Hyytia et al. 2005).

For two-dimensional ad hoc networks, additionally to the variables considered in one-dimensional networks, nodes are randomly distributed in the plane and the choice of the next relay node of the multihop path becomes even more decisive in the performance of the routing protocols. The most used routing protocols for two dimensional scenarios are the position-based routing protocols that use the geographical position of nodes to make routing decisions. These protocols require that a node knows its own geographical position and the geographical position of the destination node. The similarity between the position-based routing protocols is that all of them guarantee that the hop progress is made towards the destination node.

Generally, there are different strategies a node can use to decide to which neighbor a given transmission should be forwarded. In addition to the furthest and the nearest distance routing protocols also used in one-dimensional networks, one intuitive strategy is to forward the packet to the node that makes the most forward progress towards the destination node. This strategy is known as the Most Forward within Radius (MFR) and tries to minimize the number of hops a packet has to traverse in order to reach the destination node. Node that in one-dimensional ad hoc networks the MFR and the FR protocols coincide. Another used strategy is the compass routing, which selects the neighbor node closest to the straight line between the source and destination nodes. The compass routing (CR) tries to minimize the spatial distance a packet travels. For a detailed description of the most common routing protocols see, e.g., Basagni et al. (2004) and Mauve et al. (2001).

Due to the complexity involved, the hop count distribution in two dimensional networks is hard to derive and, by this reason, the existing studies in the plane use single link models (cf., eg., Srinivasa and Haenggi (2010) and Vural and Ekici (2005)) or approximation results (cf., eg., Dulman et al. (2006) and Kuo and Liao (2007)). Based on the lack of analytical results on the hop count distribution for two-dimensional ad hoc networks, the investigation to be carried out in this dissertation should address the distribution of the number of hops for two-dimensional ad hoc networks. To select a multihop path, an efficient routing model should be proposed and its performance compared with the most common routing protocols.

Once a multihop path is active, node mobility causes frequent failure and activation of new links, affecting the performance of a MANET (cf, e.g., McDonald and Znati (1999) and Bai et al. (2004)). The derivation of path metrics that characterize the reliability of a multihop path can be used in the design and performance evaluation of routing protocols. Therefore, the development of models that integrate the mobility and the connectivity demands of MANETs are essential for a better understanding of the complex behavior of these networks.

As far as we know, an exact analysis of the reliability of a multihop path appears to be unavailable in the literature and most of the analytical studies that focus on link stability extend the analysis to multihop paths assuming independent link failures, that is, links behave independently of each other and the path duration is given by the minimum of the durations of the links of the multihop path (cf., e.g., Han et al. (2006), McDonald and Znati (1999) and Xu et al. (2007)). However, as a link shares a common node with each of its neighboor links, this introduces dependences on the mobility of the shared nodes, which may be extended to other links in the case of group mobility models where the mobility of nodes is correlated. The exception to the assumption of independent time failures is the study of La and Han (2007), where using Palm theory and assuming that the number of hops is large, proves that the distribution of path duration can be well approximated by an exponential distribution.

Aware of the necessity of exact analytical models, the investigation to be carried in this dissertation should develop a model for the multihop path dynamics allowing the derivation of important path metrics, like the mean path duration and path persistence (defined as the probability that a path is continuously active until some time instant). Our main motivation is to derive a new mathematical framework to model the random behavior of a multihop path, integrating both connectivity and mobility requirements of MANETs. This mathematical framework should permit to consider that the duration of the links of a multihop path are not independent, providing new insights on the multihop path dynamics.

1.3 Objectives

An analysis of the literature on MANETs made us realize that the distribution of the hop count, which is one of the most important metrics to evaluate the performance of routing protocols (c.f. Kuo and Liao (2007)), has not received a lot of attention, even in one-dimensional networks. The literature on the subject considers the underlying node spatial distribution as being either drawn from a Poisson process or that a fixed and known number of nodes are uniformly distributed in the region of interest.

For one-dimensional networks, most of the studies focus on the connectivity probability between a source node and a destination node, when relay nodes are uniformly distributed between the source and destination nodes (cf., e.g., Desai and Manjunath (2002) and Ghasemi and Nader-Esfahani (2006)). For two-dimensional networks, the only analytical results for the hop count distribution are the ones derived by Dulman et al. (2006) and Kuo and Liao (2007) and both of these studies consider some simplifying assumptions to cope with the mathematical complexity. The existing studies use different routing protocols and there is no exact analytical study to evaluate their performance.

In this line, we propose in this dissertation:

- To derive the hop count probability distribution in one-dimensional and two dimensional scenarios, assuming that the underlying node distribution is drawn from a Poisson process or that a fixed number of relay nodes are uniformly distributed in that region.
- To evaluate the performance of routing protocols for two-dimensional networks by means of the hop count probability distribution.

After a multihop path is built, its evolution highly determines the performance of the network in the short term. The dynamics of a multihop path in an ad hoc network call for a systematic formulation of the geometrical relations governing the complex random movement of the nodes of the path, with the state of its links limited by the transmission range of each relay node. We have come to realize that, due to the complexity of an exact analysis of these dynamic, the existing studies consider simulation analysis (cf. Bai et al. (2004)) or simplifying assumptions to model the multihop path dynamics (cf. Han et al. (2006), McDonald and Znati (1999) and, Xu et al. (2007)). Almost all studies consider that the duration of links of the multihop path are independent from each other, permitting to focus the analysis on a single link. The multihop path duration is then obtained as the minimum of the durations of the single links. However, as shown by Han et al. (2006), these assumptions are only valid when there exists a high number of links, which in fact is not common in MANETs. To address this problem La and Han (2007) study the distribution of path durations considering that links are dependent, by proving some mixing conditions in order to use Palm's theorem (cf. La and Han, 2007, Section V) and by assuming that the number of hops is large. Our main objective is to consider a model where the joint dynamics of the links is taken into account. To address this problem we propose:

- To model the multihop path dynamic by a mathematical framework that integrates the mobility and connectivity aspects.
- To obtain path based metrics from the multihop path model, like the mean path duration and the path persistence, by considering that links that share a common node do not behave independently.

1.4 Claim of contributions

The contributions within this thesis are twofold: connectivity results for one-dimensional and two-dimensional networks, where the hop count distribution is derived; and the dynamics of a multihop path, where a model for the multihop path dynamics is proposed and important metrics of network performance are obtained.

Concerning the research on the connectivity of mobile ad hoc networks, we propose a randomization technique to derive the hop count probability distribution in one-dimensional and in a two-dimensional region, when a fixed and known number of relay nodes are uniformly distributed in some region of interest. Analogous results are derived when the relay nodes are randomly distributed according to a Poisson process.

In one-dimensional networks, the multihop path selected is the one that selects the node with the most forward progress within radius, and we prove that the connectivity probability derived by Ghasemi and Nader-Esfahani (2006) can be decomposed as the sum of the probability masses at each possible value of the hop count.

In two-dimensional networks, the multihop path is selected by using the FR and NR protocols, and the performance of both protocols is compared. To guarantee an efficient progress towards the destination node, we propose a novel propagation model which we call the dynamic propagation model. This model establishes that each node transmits within a routing region defined by an angular section of a circular disk with radius equal to the transmission range and oriented to the destination node, such that the angular span depends of the distance to the destination node.

Our analysis allow us to derive the exact hop count probability for any number of hops and, as far as we know, these are the first exact analytical results for the hop count probability distribution. Moreover, our dynamic propagation model is also a generalization of the model proposed by Srinivasa and Haenggi (2010).

Our main contributions for the connectivity in one-dimensional and two-dimensional ad hoc networks can be summarized as follows:

- We derive the hop count probability distribution for one-dimensional and two-dimensional networks, when the underlying node spatial distribution is drawn from a Poisson process and, by using a Poisson randomization technique, when a fixed number of relay nodes are uniformly distributed in a region of interest.

- We derive the hop count distribution in one-dimensional networks for the case where the selected path provides the greatest forward progress towards the destination.

- We extend the connectivity probability presented by Ghasemi and Nader-Esfahani (2006) for one-dimensional networks through its decomposition as the sum of the probabilities for each possible value of the hop count.
- To derive the hop count probability distribution for two-dimensional networks, we propose a novel propagation model, called the dynamic propagation model, to guarantee an efficient progress towards the destination node.
- We derive the hop count probability distribution for two-dimensional networks, in the cases where the multihop path selects the closest and the furthest relay node within the routing region of the emissor node.

Concerning the research on the multihop path dynamics, we propose to model the random behavior of multihop paths by a Piecewise Deterministic Markov Process (PDMP). A PDMP is a Markov process that follows deterministic trajectories between random time instants. The usage of a PDMP to model a multihop path arises naturally, since PDMPs are a mixture of deterministic motion and random events, just like the multihop path dynamics. Moreover, the PDMP description of a multihop path allow us to derive related performance measures.

Our main contributions on the multihop path dynamics for mobile ad hoc networks can be summarized as follows:

- We propose a new mathematical framework to model the multihop path dynamics by a PDMP, integrating both mobility and connectivity requirements of MANETs.
- We derive the mean path duration and path persistence through a system of integro-differential equations.

- We propose a recursive scheme that transforms the system of integro-differential equations in a system of first order ordinary differential equations that can be numerically computed.
- We evaluate numerically the effects of the independent link assumption and the connectivity and mobility parameters in the path metrics.

1.5 Overview of the dissertation

We now describe the contents of the remaining chapters of this dissertation, that are based partially in Antunes, Jacinto and Pacheco (2008, 2010a, 2010b).

In Chapter 2 we derive the minimum hop count probability distribution in one-dimensional networks, assuming that the location of the source and destination nodes are known and there are a fixed number of relay nodes uniformly distributed between them. We also obtain the joint probability density function of relay node locations in the multihop path and conclude that the probability that the source and destination nodes are connected can be obtained by summing the probability masses for each possible value of the minimum hop count. Numerical results illustrate the effect of the number of nodes and of the transmission range in the minimum hop count probability.

In Chapter 3 we derive the hop count probability distribution in two-dimensional networks using the dynamic propagation model and two routing protocols, the furthest distance and the nearest distance routing protocols. We consider that the location of the source and destination nodes are known and the underlying node spatial distribution is assumed to be either drawn from a Poisson process or that a known and fixed number of relay nodes are uniformly distributed in a region of interest. Moreover, we obtain numerical results to evaluate the model performance and to compare the routing protocols.

In Chapter 4 we propose an analytical framework to fully describe the random behavior of a multihop path in ad hoc networks and obtain path based metrics for computing its reliability. We derive formulas for two performance evaluation metrics: the mean path duration and the path persistence. Finally, we apply our framework to compute numerical results for these metrics and compare them with those obtained when assuming that the links of the multihop path behave independently.

Finally, in Chapter 5 some conclusions are drawn and some ideas for future work are presented.

Chapter 2

Connectivity in one-dimensional ad hoc networks

2.1 Introduction

In this chapter we derive the probability distribution of the minimum hop count in one-dimensional ad hoc networks when a fixed and known number of relay nodes are uniformly distributed between the source and the destination nodes. We show that the connectivity probability, defined as the probability that there exists a multihop path between the source and the destination nodes regardless of the number of hops, can be obtained from its decomposition as the sum of the probabilities of the minimum hop count assuming each of its possible values.

Several studies of connectivity between two nodes and for the entire network in one-dimensional ad hoc networks have appeared in the literature. One of the first studies deriving the connectivity probability for a finite set of nodes uniformly distributed in a finite interval was presented by Desai and Manjunath (2002). Foh and Lee (2004) and Foh et al. (2005) derived closed form approximation formulas for the connectivity between two nodes when relay nodes are uniformly and non-uniformly distributed, respectively.

Ghasemi and Nader-Esfahani (2006) obtained the connectivity probability in an one-dimensional ad hoc network when relay nodes are uniformly distributed between the source and the destination nodes. Other issues of interest that have been investigated are the number of nodes and the transmission range to ensure that the network is connected (see, e.g., Santi and Blough (2003) and Bettstetter (2002)). Assuming a Poisson process for node locations, Mathar and Mattfeldt (1996) studied the optimal transmission ranges that maximize the expected hop progress of a packet, and Dulman et al. (2006) and Vural and Ekici (2005) investigated the distribution of the node distance.

We consider an one dimensional ad hoc network where the source and destination nodes are located at the edge of the network and a fixed number of relay nodes are uniformly distributed between them. This spatial distribution of nodes arises when nodes move according to a Random Walk or a Random Direction mobility model over the region of interest (Camp et al. 2002). A relay node in the path is selected by the Most Forward within Radius (MFR) routing protocol (see, e.g., Takagi and Kleinrock (1984)). Starting from the source node, each successive relay node in the MFR path is selected so that it provides the greatest forward progress toward the destination node within the fixed transmission range. Therefore, the minimum hop count is given by the number of hops in the MFR path. We show that the connectivity probability derived in Ghasemi and Nader-Esfahani (2006) can be obtained from its decomposition as the sum of the probability masses for each possible value of the minimum hop count, providing new insights on the connectivity probability.

To derive the minimum hop count distribution and the joint density location of relay nodes for the case when a fixed number of relay nodes are uniformly distributed between the source and the destination nodes, we use a poissonification technique (Domb 1952) that randomizes the number of relay nodes by assuming that they are distributed according to a Poisson process, followed by conditioning on the number of nodes. Thus, our results also apply to the case when relay nodes are distributed according to a Poisson process.

The outline of this chapter is described as follows. In Section 2.2 we describe the model and define the minimum hop count. In Section 2.3 we obtain a recursive formula that in each step updates the density location of the next relay node that will forward the packet. We then present the main result of the chapter, where, using a Poisson randomization method, we obtain the distribution of the number of nodes in the minimum hop path. In Section 2.4 we present results for network dimensioning, mainly we obtain the critical value of the number of nodes and of the transmission range that guarantee a certain connectivity probability with the minimum hop count. Finally, in Section 2.5 we present some numerical results, illustrating the influence of the number of nodes and the transmission range in the minimum hop count.

2.2 Model description

We consider a multihop ad hoc network with a source node, a destination node, and N relay nodes between them. Let X_i, $i = 1, 2, \ldots, N$, denote the location of relay node i, and $X_0 = 0$ and $X_{N+1} = L$ denote the location of the source and destination nodes, respectively. We assume that $\{X_i\}_{i=1}^N$ are independent and uniformly distributed random variables on $(0, L)$ and let $X_{(i)}$, $i = 0, 1, \ldots, N+1$, denote the location of the ith node from the origin, i.e.,

$$0 = X_{(0)} < X_{(1)} < \ldots < X_{(N+1)} = L.$$

Given a fixed transmission range $0 < R < L$ equal for all nodes, nodes i and j are connected with no hops if $|X_i - X_j| < R$. If the source and destination nodes are connected, i.e., $X_{(i+1)} - X_{(i)} < R$, $i = 0, 1, \ldots, N$, the MFR multihop path between them is given by $Y^M = (Y_1, Y_2, \ldots, Y_M)$ where

$$Y_1 = \max\{X_{(i)} : X_{(i)} < R\},\ Y_2 = \max\{X_{(i)} : X_{(i)} < Y_1 + R\}, \ldots,$$
$$Y_M = \max\{X_{(i)} : X_{(i)} < Y_{M-1} + R\}$$

with $Y_{M-1} + R < X_{(N+1)} < Y_M + R$ (see Figure 2.1), and M is called the minimum hop count.

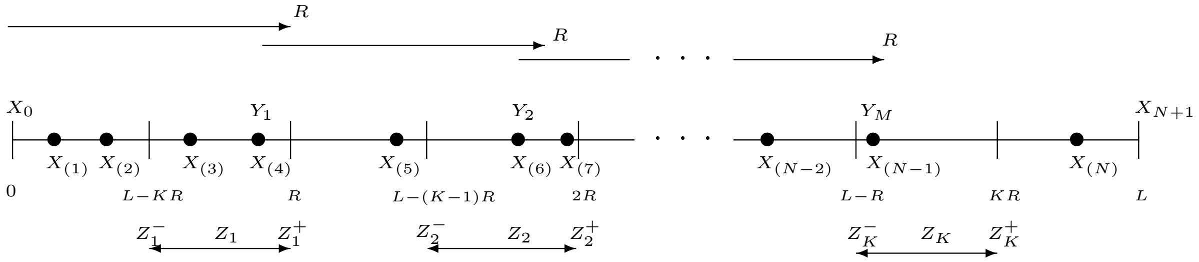

Figure 2.1: Minimum hop count.

The minimum number of hops that may be needed to connect the source and destination nodes is denoted by

$$K = \left\lfloor \frac{L}{R} \right\rfloor \tag{2.1}$$

To simplify the exposition, we define K zones between the source and destination nodes denoted by $Z_1, Z_2, \ldots, Z_K$, with zone i starting at location $Z_i^- = L - R(K+1-i)$ and ending at $Z_i^+ = iR$, with length $Z = (K+1)R - L$. From Figure 2.1, the minimum hop count is K if there exist relay nodes in the zones $Z_1, \ldots, Z_K$ in such way that

$$Z_1^- < Y_1 < Z_1^+, \quad Z_2^- < Y_2 < Y_1 + R, \ \ldots, \quad Z_K^- < Y_K < Y_{K-1} + R.$$

On the other hand, the maximum number of hops that may be needed to connect the source and destination nodes is $2K$, which occurs if the zones Z_i, $i = 1, 2, \ldots, K$, are empty, the first relay node in the MFR multihop path is before Z_1^-, and between each Z_i^+ and Z_{i+1}^- there exists two relay nodes in the multihop path with the last relay node after Z_K^+, such that

$$Y_1 < Z_1^-, \quad Z_1^+ < Y_2 < Y_1 + R < Y_3 < Z_2^-, \quad Z_2^+ < Y_4 < Y_3 + R < Y_5 < Z_3^-, \ \ldots,$$
$$Z_{K-1}^+ < Y_{2K-2} < Y_{2K-3} + R < Y_{2K-1} < Z_K^-, \quad Z_K^+ < Y_{2K} < Y_{2K-1} + R.$$

2.3 Connectivity probability with the minimum hop path

In this section we obtain the connectivity probability with minimum hop count. We first present a recursive formula to obtain that probability, but it suffers from combinatorial problems turning it hard to evaluate. For that reason we then use a Poisson randomization method that provides a simple closed formula for the same probability.

2.3.1 Recursive formula

The probability density function of the location of the first hop in the MFR path being t and the order of the associated relay node being i is

$$f^{(i)}(t|N,R,L) = \frac{i}{L}\binom{N}{i}\left(\frac{t}{L}\right)^{i-1}\left(1-\frac{R}{L}\right)^{N-i}, \qquad 0<t<R,\ i=1,\dots,N.$$

This results from the marginal density location of $X_{(i)}$ (see, e.g., Kulkarni (1995), page 209, Eq. (5.78)) taking into account that exactly $N-i$ ordered nodes have to be located on (R, L).

Given the location t of the first hop and the order i of the associated relay node, the locations of the last $N-i$ ordered relay nodes are independent and uniformly distributed on $[R, L)$. In addition, the probability that the minimum hop count is m given that the location and the order of the first hop are t and i, respectively, is equal to the probability that the minimum hop count is $m-1$ between a pair of nodes located at the endpoints of interval $[R, L]$ where: the source is at R and has transmission range t; and, $N-i$ relay nodes, each with transmission range R, are uniformly distributed on $[R, L)$. This fact can be used to derive a recursive formula to compute the probability distribution of the minimum hop count, as it will be explained.

We let $p(m|n, r, l)$ denote the probability that the minimum hop count is m when the distance between the source and destination nodes is l, the transmission range of the source is r, and there are n relay nodes, each with transmission range R, uniformly placed between the source and destination nodes. Then,

$$p(m|n,r,l) = \begin{cases} \int_0^r \sum_{i=1}^n f^{(i)}(t|n,r,l)p(m-1|n-i,t+R-r,l-r)dt & r<l, k\leq m\leq \min(2k,n), \\ 1 & r\geq l, m=0, \\ 0 & \text{otherwise}, \end{cases} \tag{2.2}$$

where $k = 1 + \lfloor (l-r)/R \rfloor$, and the probability that the minimum hop count is m is then given by $p(m|N,R,L)$.

2.3.2 Poisson randomization method

The recursive formula given by equation (2.2) suffers from combinatorial problems and its computation is hard even for moderate values of m and n. In this section we show that the use of Poisson randomization, an exact method originally proposed by Domb (1952), allows the derivation of an elegant formula without the mentioned drawbacks.

Our use of Poisson randomization consists in randomizing the distribution of relay nodes, and as a consequence of the parameter N, by assuming that relay nodes are distributed according to a Poisson process with a fixed rate, say λ. This assumption decouples the relay nodes in the sense that the number of relay nodes located in disjoint intervals are independent. Moreover, by conditioning on the number of nodes that lie on $(0,L)$, the assumption that the relay nodes are uniformly distributed on $(0,L)$ pops up. As a result of these facts, the probability that the minimum hop count (M) is equal to m can be explicitly derived with little effort. Moreover, by viewing it as an instance of the total probability law formula in the form

$$\mathbb{P}(M=m) = \sum_{n=m}^{\infty} e^{-\lambda L}\frac{(\lambda L)^n}{n!}\mathbb{P}(M=m|N=n), \tag{2.3}$$

we can identify the probability that the minimum hop count is equal to m when the number of relay nodes is n, $P(M = m|N = n)$, $n \in \mathbb{N}$. It turns out that this leads to a more efficient way to compute the minimum hop count distribution for a fixed number of relay nodes than the one described through (2.2).

The next result characterizes the exact probability distribution of the minimum hop count given the number of relay nodes between the source and destination nodes, which is a consequence of the successful use of the Poisson randomization method.

Theorem 2.1. *Given that the relay nodes are uniformly distributed on* $(0, L)$*, the probability distribution of the minimum hop count is given by*

$$\mathbb{P}(M = m|N = n) = \int_{S_m} \frac{n!}{(n-m)!L^n}(L - mR + y_m)^{n-m} dy_m dy_{m-1} \dots dy_1, \tag{2.4}$$

with $K \le m \le \min\{2K, n\}$*,* $y_0 = 0$ *and* $y_{m+1} = L$*, where* K *is given by (2.1) and*

$$S_m = \{(y_1, y_2, \dots, y_m) \in (0, L)^m : y_1 < R,\ y_{j-2} + R \le y_j < y_{j-1} + R,\ 2 \le j \le m+1\}.$$

Proof. The first relay node is located at y_1 if an only if there is a relay node at y_1 and there are no relay nodes between y_1 and R, with $y_1 < R$. Because the probability that there are no relay nodes between y_1 and R is $e^{-\lambda(R-y_1)}$, the density function of the location of the first relay node y_1 is given by

$$\lambda e^{-\lambda(R-y_1)}, \qquad 0 < y_1 < R.$$

The second relay node must be the furthest one from the first relay node within its transmission range. Given that the location of the first relay node is y_1, the density function of the location of the second relay node at y_2 is

$$\lambda e^{-\lambda(y_1+R-y_2)}, \qquad R < y_2 < y_1 + R.$$

Proceeding in the same manner, given that the location of the $(m-1)$-th relay node is y_{m-1}, the density function of the location of the m-th relay node is

$$\lambda e^{-\lambda(y_{m-1}+R-y_m)}, \qquad y_{m-2} + R < y_m < y_{m-1} + R.$$

Note that the locations of the m relay nodes belong to S_m, where S_m is the set of the possible locations of the m relay nodes in order to the origin node (located at position 0) is connected to the destination node (located at position L) through those m relay nodes

$$S_m = \{(y_1, y_2, \ldots, y_m) \in (0, L)^m : y_1 < R,\ y_{j-2} + R \leq y_j < y_{j-1} + R,\ 2 \leq j \leq m\},$$

where $y_0 = 0$ and $y_{m+1} = L$.

Putting all things together, the density of the locations of the consecutive m relay nodes being $y_1, y_2, \ldots, y_m$ is given by

$$\lambda e^{-\lambda(R-y_1)} \prod_{i=1}^{m-1} \lambda e^{-\lambda(y_i+R-y_{i+1})}, \qquad (y_1, y_2, \ldots, y_m) \in S_m. \tag{2.5}$$

Thus, integrating (2.5) over the set S_m we obtain the probability that the minimum hop count is m, when the relay nodes are randomly distributed according to a Poisson process

$$\mathbb{P}(M = m) = \int_{S_m} \lambda e^{-\lambda(R-y_1)} \prod_{i=1}^{m-1} \lambda e^{-\lambda(y_i+R-y_{i+1})} dy_m dy_{m-1} \ldots dy_1. \tag{2.6}$$

Multiplying equation (2.6) by $e^{\lambda L}$, we obtain

$$\begin{aligned} e^{\lambda L}\mathbb{P}(M = m) &= e^{\lambda L} \int_{S_m} \lambda^m e^{-\lambda(mR-y_m)} dy_m dy_{m-1} \ldots dy_1 \\ &= \int_{S_m} \lambda^m \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (L - mR + y_m)^n dy_m dy_{m-1} \ldots dy_1 \\ &= \sum_{n=m}^{\infty} \frac{(\lambda L)^n}{n!} \frac{n!}{(n-m)! L^n} \int_{S_m} (L - mR + y_m)^{n-m} dy_m dy_{m-1} \ldots dy_1 \end{aligned}$$

where the change between the sum and the integral follows by the dominated convergence theorem. On the other hand, conditioning on the value of N, which is Poisson distributed with mean λL, the total probability law produces

$$e^{\lambda L}\mathbb{P}(M = m) = \sum_{n=m}^{\infty} \mathbb{P}(M = m | N = n) \frac{(\lambda L)^n}{n!}.$$

Since the coefficients of $(\lambda L)^n/n!$ in the previous two expressions for $e^{\lambda L}\mathbb{P}(M = m)$ must match, the result follows. ❑

We remark that the integrand function in (2.4)

$$\frac{n!}{(n-m)!L^n}(L-mR+y_m)^{n-m}$$

is the density function of the location of the relay nodes in the MFR path (Y^M). The next results show that, for some values of M, we can find a closed form formula for the connectivity probability with the minimum hop count.

Corollary 2.1. *The probability that the minimum hop count is equal to K is given by*

$$\mathbb{P}(M=K|N=n)=\sum_{i=K}^{n}\binom{n}{i}\left(\frac{Z}{L}\right)^{i}\left(1-\frac{Z}{L}\right)^{n-i} \tag{2.7}$$

for $K=\lfloor L/R\rfloor\leq n$ and $Z=(K+1)R-L$.

Proof. The probability with minimum hop count being equal to K is obtained by integrating (2.5) over set S_K, where by observing Figure 2.1 we can conclude that the set S_K must be defined as

$$S_K=\{(y_1,y_2,\dots,y_m)\in(0,L)^m:Z_1^-<y_1<Z_1^+,Z_2^-<y_2<y_1+R,\dots,Z_k^-<y_k<y_{K-1}+R\}.$$

Note that this set is in fact the same as the set S_m with $m=K$. For the first relay node in the set S_m, if $y_1<Z_1^-$ then $y_K+R<L$ and the multihop path with exactly K relay nodes cannot be established. Then, in fact, we must have $Z_1^-<y_1<Z_1^+$, where $Z_1^+=R$. By the same arguments, the condition $R\leq y_2<y_1+R$ of the set S_m must be in fact $Z_2^-\leq y_2<y_1+R$ of the set S_K, since if $y_2<Z_2^-=L-R(K-1)$ the path cannot be established with exactly K relay nodes because $y_K+R<L$. The same argument applies to the derivation of the remain conditions defining the set S_K.

From Theorem 2.1, we have

$$\begin{aligned}\mathbb{P}(M=K|N=n)&=\frac{n!}{(n-K)!L^n}\int_{Z_1^-}^{Z_1^+}\int_{Z_2^-}^{y_1+R}\dots\int_{Z_K^-}^{y_{K-1}+R}(L-KR+y_K)^{n-K}dy_Kdy_{K-1}\dots dy_1\\&=\frac{n!}{(n-K)!L^n}\int_0^Z\int_0^{s_1}\dots\int_0^{s_{K-1}}(L-Z+s_K)^{n-K}ds_Kds_{K-1}\dots ds_1\end{aligned}$$

where the last equality comes by the change of variables $s_i = y_i - Z_i^-$, $i = 1, 2, \ldots, K$. Using the binomial theorem, we obtain

$$\mathbb{P}(M = K|N = n) = \int_0^Z \int_0^{s_1} \ldots \int_0^{s_{K-1}} \sum_{m=0}^{n-K} \binom{n-K}{m} \frac{n!}{(n-K)!} \frac{s_K^m (L-Z)^{n-K-m}}{L^n} ds_K ds_{K-1} \ldots ds_1.$$

Integrating in order to the variables s_i, $i = 1, 2, \ldots, K$, after some algebra we obtain

$$\begin{aligned}\mathbb{P}(M = K|N = n) &= \sum_{m=0}^{n-K} \frac{n!}{(n-m-K)!(m+K)!} \left(\frac{L-Z}{L}\right)^{n-m-K} \left(\frac{Z}{L}\right)^{m+K} \\ &= \sum_{i=K}^{n} \frac{n!}{(n-i)!i!} \left(\frac{L-Z}{L}\right)^{n-i} \left(\frac{Z}{L}\right)^{i}.\end{aligned}$$

❑

Equation (2.7) is the survival function at $K - 1$ of the binomial distribution with parameters N and Z/L. Since the survival function of the binomial distribution is stochastically increasing in the number of trials, when the success probability is kept fixed, the probability that the minimum hop count is equal to K is increasing in N.

The probability that the minimum hop count is $K+1$ can be obtained in a similar way. Closed form expressions for other values of M are out of reach because of the complexity involved.

Corollary 2.2. *The probability that the minimum hop count is equal to $K + 1$ is*

$$\mathbb{P}(M = K+1|N = n) = \sum_{i=K+1}^{n} \binom{n}{i} \left(\frac{L-R-Z}{L}\right)^{n-i}$$
$$\left[\sum_{j=0}^{K-1} \binom{i}{j} \left(\left(\frac{Z}{L}\right)^{j} \left(\frac{R}{L}\right)^{i-j} - (i-j)\left(\frac{R-Z}{L}\right)\left(\frac{Z}{L}\right)^{i-1}\right) - \left(\frac{Z}{L}\right)^{i} (1 + i(K-1))\right] \tag{2.8}$$

for $K = \lfloor L/R \rfloor < n$ and $Z = (K+1)R - L$.

Proof. From Theorem 2.1, the probability that the minimum hop count is equal to $K+1$ is

$$\mathbb{P}(M = K+1|N = n) = \frac{n!}{(n-(K+1))!L^n} \int_{S_{K+1}} (L - (K+1)R + y_{K+1})^{n-(K+1)} \mathrm{d}y_{K+1} \ldots \, \mathrm{d}y_1$$

where S_{K+1} is the set of possible locations of the $K+1$ relay nodes such that the connectivity between the source and the destination nodes is done in exactly $K+1$ hops, that can be written as the union of 2 disjoint sets. The first set S^1_{K+1}, is the set of the locations of the $K+1$ relay nodes when one of the zones $Z_j, 1 \leq j \leq K$ (see Figure 2.1) does not have relay nodes belonging to the multihop path Y^M. Thus, $S^1_{K+1} = \bigcup_{j=1}^{K} S^{1,j}_{K+1}$, where $S^{1,j}_{K+1}$ stands for the set of node locations when the j-th intersection zone has no relay nodes

$$\begin{aligned} S^{1,j}_{K+1} = \big\{ &(y_1, ..., y_{K+1}) \in (0,L)^{K+1} : Z_1^- < y_1 < Z_1^+, Z_i^- < y_i < y_{i-1} + R, 2 \leq i \leq j-1 \\ Z^+_{j-1} &< y_j < Z_j^-, Z_j^+ < y_{j+1} < y_j + R, \text{ and } Z^-_{i-1} < y_i < y_{i-1} + R, j+2 \leq i \leq K+1 \big\}, \end{aligned}$$

where we recall that $Z_i^- = L - R(K+1-i)$ and $Z_i^+ = iR$.

The second set S^2_{K+1}, is the set of the locations of the $K+1$ relay nodes when one of the remaining zones $Z_j, 2 \leq j \leq K-1$, has two relay nodes. Thus $S^2_{K+1} = \bigcup_{j=2}^{K-1} S^{2,j}_{K+1}$, where $S^{2,j}_{K+1}$ stands for the set of node locations when the j-th intersection zone has two relay nodes

$$\begin{aligned} S^{2,j}_{K+1} = \big\{ (y_1, ..., y_{K+1}) \in (0,L)^{K+1} &: Z_1^- < y_1 < Z_1^+, Z_i^- < y_i < y_{i-1} + R, 2 \leq i \leq j, \\ &y_{j-1} + R < y_{j+1} < Z_j^+, y_j + R < y_{j+2} < y_{j+1} + R; \\ &\text{and } Z^-_{i-1} < y_i < y_{i-1} + R, j+3 \leq i \leq K+1 \big\}. \end{aligned}$$

Integrating in order to the set $S^{1,j}_{K+1}$, $1 \leq j \leq K$ we obtain

$$\begin{aligned}
&\frac{n!}{(n-(K+1))!L^n} \int_{S^{1,j}_{K+1}} (L-(K+1)R+y_{K+1})^{n-(K+1)} \, \mathrm{d}y_{K+1} \ldots \, \mathrm{d}y_1 = \\
=&\frac{n!}{(n-K-1)!L^n} \int_{Z_1^-}^{Z_1^+} \int_{Z_2^-}^{y_1+R} \cdots \int_{Z_{j-1}^-}^{y_{j-2}+R} \int_{Z_{j-1}^+}^{Z_j^-} \int_{Z_j^+}^{y_j+R} \int_{Z_{j+1}^-}^{y_{j+1}+R} \cdots \\
&\qquad \cdots \int_{Z_K^-}^{y_K+R} (L-(K+1)R+y_{K+1})^{n-K-1} \, \mathrm{d}y_{K+1} \ldots \, \mathrm{d}y_1 \\
=&\frac{n!}{(n-K-1)!L^n} \int_0^Z \int_0^{s_1'} \cdots \int_0^{s_{j-2}'} \int_Z^R \int_Z^{s_j} \int_0^{s_{j+1}} \cdots \\
&\qquad \cdots \int_0^{s_K} (L-Z-R+s_{K+1})^{n-K-1} \, \mathrm{d}s_{K+1} \ldots \, \mathrm{d}s_j \, \mathrm{d}s_{j-1}' \ldots \, \mathrm{d}s_1'
\end{aligned}$$

where the last equality results by the change of variables $s_i' = y_i - Z_i^-$, $i = 1, \ldots, j-1$, and $s_i = y_i - Z_{i-1}^-$, $i = j, \ldots, K+1$. Using the binomial theorem and integrating in order to the variables s_i, s_i', and after some algebra we obtain

$$\begin{aligned}
&\frac{n!}{(n-(K+1))!L^n} \int_{S^1_{K+1}} (L-(K+1)R+y_{K+1})^{n-(K+1)} \mathrm{d}y_{K+1} \ldots \, \mathrm{d}y_1 = \\
&= \sum_{m=0}^{n-K-1} \binom{n}{m+K+1} \frac{1}{L^n} (L-Z-R)^{n-m-K-1} \\
&\qquad \int_0^Z \int_0^{s_1'} \cdots \int_0^{s_{j-1}'} \int_Z^R \int_Z^{s_j} \int_0^{s_{j+1}} \cdots \int_0^{s_K} s_{K+1}^m \, \mathrm{d}s_{K+1} \ldots \, \mathrm{d}s_j \, \mathrm{d}s_{j-1}' \ldots \, \mathrm{d}s_1' \\
&= \sum_{i=K+1}^{n} \binom{n}{i} \frac{1}{L^n} (L-Z-R)^{n-i} \binom{i}{j-1} \Big(Z^{j-1} \big(R^{i-j+1} - Z^{i-j+1} \big) - (i-j+1)(R-Z)Z^{i-1} \Big) .
\end{aligned} \tag{2.9}$$

For the set set $S^{2,j}_{K+1}, 2 \leq j \leq K-1$, using the change of variables $s_i' = y_i - Z_i^-$,

$i = 1, \ldots, j$ and $s_i = y_i - Z_{i-1}^-$, $i = j+1, \ldots, K+1$, we obtain

$$
\begin{aligned}
&\frac{n!}{(n-(K+1))!L^n} \int_{S_{K+1}^{2,j}} (L-(K+1)R+y_{K+1})^{n-(K+1)} \, \mathrm{d}y_{K+1} \ldots \, \mathrm{d}y_1 = \\
=&\frac{n!}{(n-K-1)!L^n} \int_{Z_1^-}^{Z_1^+} \int_{Z_2^-}^{y_1+R} \ldots \int_{Z_j^-}^{y_{j-1}+R} \int_{y_{j-1}+R}^{Z_j^+} \int_{y_j+R}^{y_{j+1}+R} \int_{Z_{j+2}^-}^{y_{j+2}+R} \ldots \\
&\qquad \ldots \int_{Z_K}^{y_K+R} (L-(K+1)R+y_{K+1})^{n-K-1} \, \mathrm{d}y_{K+1} \ldots \, \mathrm{d}y_1
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
&= \sum_{m=0}^{n-K-1} \binom{n}{m+K+1} \frac{1}{L^n} (L-Z-R)^{n-m-K-1} \\
&\qquad \int_0^Z \int_0^{s_1'} \ldots \int_0^{s_{j-1}'} \int_{s_{j-1}'}^Z \int_{s_j'}^{s_{j+1}} \int_0^{s_{j+2}} \ldots \int_0^{s_K} s_{K+1}^m \, \mathrm{d}s_{K+1} \ldots \, \mathrm{d}s_j \, \mathrm{d}s_{j-1}' \ldots \, \mathrm{d}s_1' \\
&= \sum_{i=K+1}^{n} \binom{n}{i} \frac{1}{L^n} (L-Z-R)^{n-i} Z^i \left(\binom{i}{j} - i \right).
\end{aligned}
\tag{2.11}
$$

Summing (2.9) in order to j, $1 \leq j \leq K$, the integration in order to the set $S_{K+1}^1 = \bigcup_{j=1}^K S_{K+1}^{1,j}$ is derived. In the same way, summing (2.10) in order to j, $2 \leq j \leq K-1$, the integration in order to the set $S_{K+1}^2 = \bigcup_{j=2}^{K-1} S_{K+1}^{2,j}$ is derived. Joining both results, (2.8) follows after some algebra. ❏

Since the set of all locations of relay nodes, where the source and destination nodes are connected, can be partitioned in the sets of locations of nodes associated to each possible value of the minimum number of hops necessary to connect the source and destination nodes, the connectivity probability P_c may be computed through the relation

$$
P_c = \sum_{m=K}^{\min(2K,n)} \mathbb{P}(M = m | N = n). \tag{2.12}
$$

2.4 Network dimensioning

In this section, we obtain the critical number of relay nodes and the critical transmission range that guarantee a desired connectivity probability with the minimum number of hops. These results can be used in network dimensioning.

Since we have derived a closed formula for the minimum value of the hop count, if we invert that formula in order to the number of nodes n or the transmission range R, we obtain a value for the number of nodes or a value for the transmission range that guarantee a given hop count probability (and also a given connectivity probability because this probability is greater or equal to the hop count probability with K hops). The cases where the number of nodes is either too high or too low, originating a minimum hop count probability close to 1 or 0, respectively, have no interest for the analysis. However, for moderate values of the number of nodes, knowing the values of n and R that guarantee a desired minimum hop count probability can be of interest for applications.

Given that there are n relay nodes uniformly distributed between the source and destination nodes, the connectivity probability with the minimum number of hops K is given by

$$\begin{aligned} P\left(M=K|N=n\right) &= \sum_{i=K}^{n}\binom{n}{i}\left(\frac{Z}{L}\right)^{i}\left(1-\frac{Z}{L}\right)^{n-i} \\ &= 1-F_{B\left(n,\frac{Z}{L}\right)}(K-1). \end{aligned} \tag{2.13}$$

where the right hand side of (2.13) is the survival function of a binomial distribution with parameters N and Z/L.

A possible approximation of the binomial distribution by the normal distribution in (2.13) gives, with p denoting the desired connectivity probability with the minimum number of hops,

$$p \simeq 1-\Phi\left(\frac{K-N\frac{Z}{L}}{\sqrt{N\frac{Z}{L}\left(1-\frac{Z}{L}\right)}}\right). \tag{2.14}$$

which can be written as

$$\Phi^{-1}(1-p)=\frac{K-N\frac{Z}{L}}{\sqrt{N\frac{Z}{L}\left(1-\frac{Z}{L}\right)}}. \tag{2.15}$$

where Φ^{-1} is the inverse of the cumulative distribution function of the standard normal distribution. Note that in order to obtain an upper bound for N, in (2.14) we haven't made the usual approximation of the binomial distribution by the normal distribution, and we consider this approximation valid for the rule of thumb $N > 5$ and $1/\sqrt{N}(\sqrt{(L-Z)/Z} - \sqrt{Z/(L-Z)}) < 0.3$ presented in Box et al. (1978, p. 130). Letting $A = \Phi^{-1}(1-p)$ and solving equation (2.15) in order to N we obtain

$$N = \left\lceil \frac{2KL + A^2(L-Z) - A\sqrt{4KL(L-Z) + A^2(L-Z)^2}}{2Z} \right\rceil \tag{2.16}$$

the critical number of relay nodes, that is, the minimum number of nodes that guarantee a connectivity probability with K hops with at least probability p, for a given value of R (note that $Z = (K+1)R - L$).

Obtaining the critical value of the transmission range is a bit more involved. We can see in Figure 2.2 a representation of the connectivity probability and the hop count probability with K hops, for $N = 20$, as a function of the transmission range R. We can observe that for each possible value of K, obtained for each $R \in (L/(K+1), L/K]$, the connectivity probability with K hops has an increasing behavior within each interval $R \in (L/(K+1), L/K]$.

Given that the source and destination nodes are at a distance L from each other and that K is the minimum number of hops, we can find the critical (i.e. smallest) value of the transmission range R, with $R \in (L/(K+1), L/K]$ that originates a connectivity probability with K hops of at least p, if there exists one. This is done using exactly the same approach used to obtain the critical value of the number of relay nodes and so we will just present numerical results for the critical value of the transmission range.

2.5 Numerical results

In this section, we evaluate the effect of the number of nodes and the transmission range in the distribution of the minimum hop count. We scale the parameters with respect to the distance between the source and destination nodes. Thus we set $L = 1$ and $1/(K+1) < R \leq 1/K$, so that $0 < Z \leq 1/K$. The most interesting cases are when K is

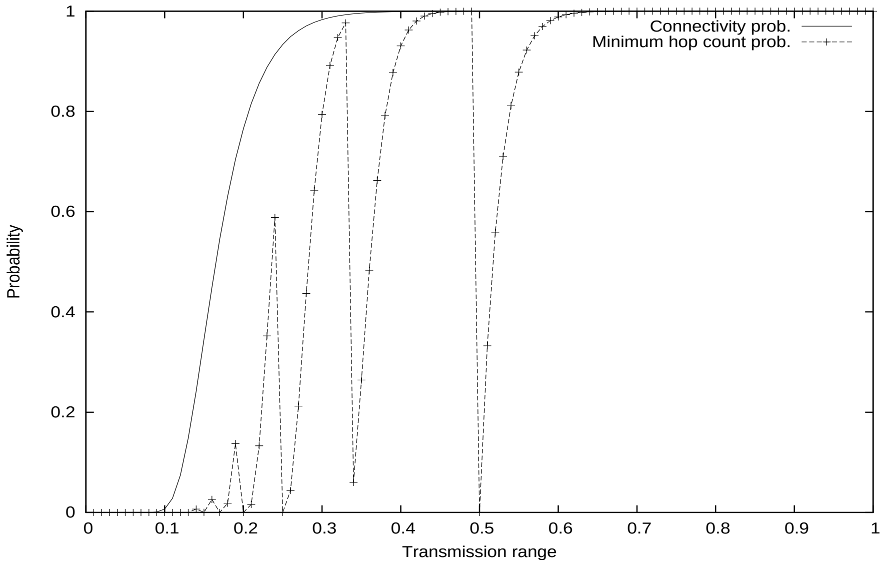

Figure 2.2: Connectivity probability and hop count probability with K hops as a function of R, with $N = 20$.

small since as the number of hops increases a multihop path may break frequently due to mobility of nodes (Jiang and Rao 2005).

In Figure 2.3 we can observe the connectivity probability decomposed in terms of the possible values of the hop count. Considering $R = 0.32$ ($K = 3$, $Z = 0.28$), the minimum hop count goes from $K = 3$ to $2K = 6$. Each value of the hop count probability is denoted in Figure 2.3 by $3, 4$ and $5+$ (for values greater or equal to 5) hops. We can observe that the hop count with $K = 3$ has the highest hop count probability, giving values close to the connectivity probability when the number of nodes is large (around 30 or more). The sum of the probabilities for the two smallest values of the hop count ($K = 3$ and $K = 4$) is close to the connectivity probability, indicating that the connectivity probability can be approximated by the sum of the hop count probabilities for the two smallest values of K. As expected, with the increase of the number of relay nodes, the probability that the minimum hop count is 4 starts to decrease and goes to zero, turning the hop count probability with K relay nodes the only non negligible probability.

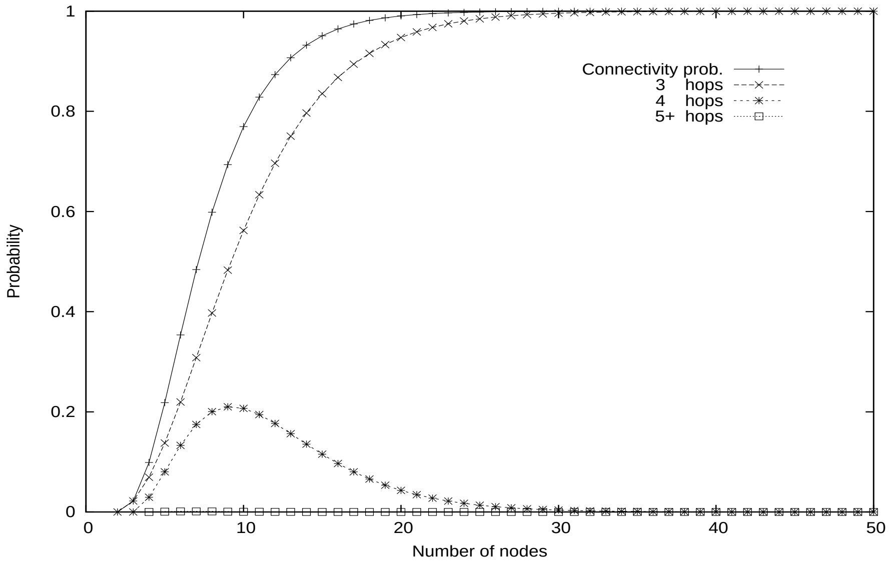

Figure 2.3: Minimum hop count/connectivity probability ($L = 1$, $R = 0.32$).

In Figure 2.4, we consider the connectivity probability and values of the probability function of the minimum hop count between the source and destination nodes when $R = 0.23$ ($K = 4$, $Z = 0.15$) for different values of the number of nodes N. Here, the connectivity probability is given by the sum of the probabilities of the minimum hop count being 4, 5, 6, and 7 or more (denoted by 7+). As can be seen from the Figure 2.4, this probability is approximately equal to the sum of the probabilities of the minimum hop count being 4, 5, and 6. Again, the connectivity probability can be approximated by the sum of the probabilities for the two smallest values of K, 4 and 5, when the number of relay nodes is large.

As the length of the intersections zones (Z) decreases, due to the increase of K or the value of the transmission range (R), and gets close to zero, the connectivity probability in more than $K + 1$ hops stops being negligible. However, as the number of relay nodes increases this probability starts to decrease, converging to zero.

In Figure 2.5, we depict jointly the probabilities that the minimum hop count is equal to K and $K + 1$, as a function of R, for $N = 20$. We can see that, in this setting, K

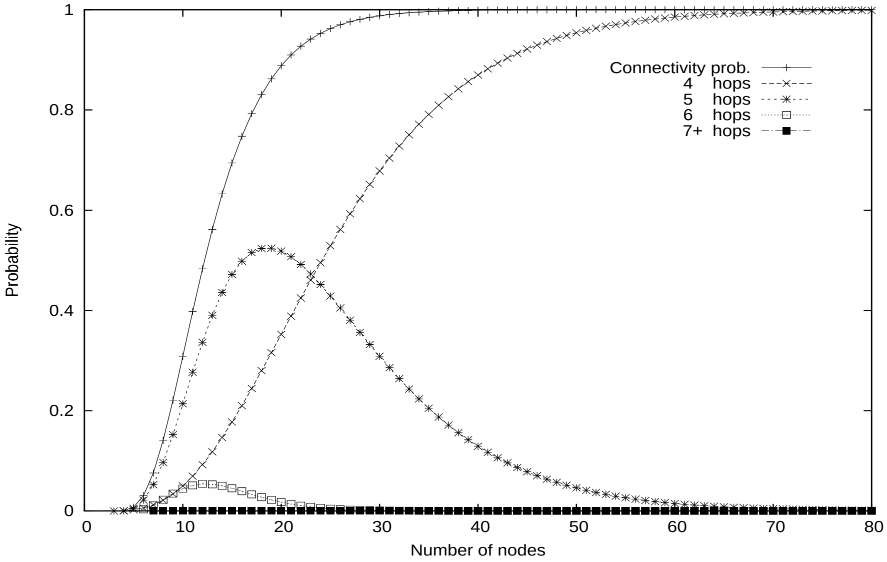

Figure 2.4: Minimum hop count/connectivity probability ($L = 1$, $R = 0.23$).

takes values 2, 3, 4, 5 in the interval of the transmission range. In an interval for R of the form $(1/(K + 1), 1/K]$, the probability that the minimum hop count is K increases with R, since the size of the intersection zones (Z) also increases as R increases.

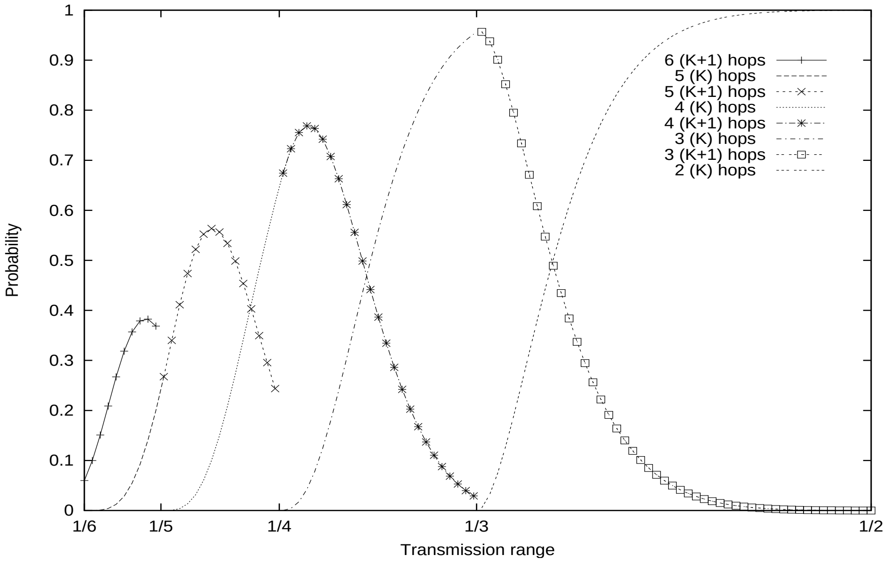

Figure 2.5: Probability of the minimum hop count being K and $K + 1$ ($L = 1$, $N = 20$).

We now present some numerical results for the critical values of N and R. In Figure 2.6 we can observe the critical values of the number of relay nodes, N, needed to obtain a connectivity probability with K hops of at least p, where the hop count is equal to $K = 2, 3$ and $Z = (K+1)R - L = 0.30$. These results are obtained using (2.16). We can conclude that as p increases also the necessary number of nodes increases, as expected. For example, to obtain a connectivity probability with K hops of at least 0.95 we will need a minimum of 16 relay nodes if $K = 2$, and a minimum of 19 relay nodes if $K = 3$, when $Z = 0.3$.

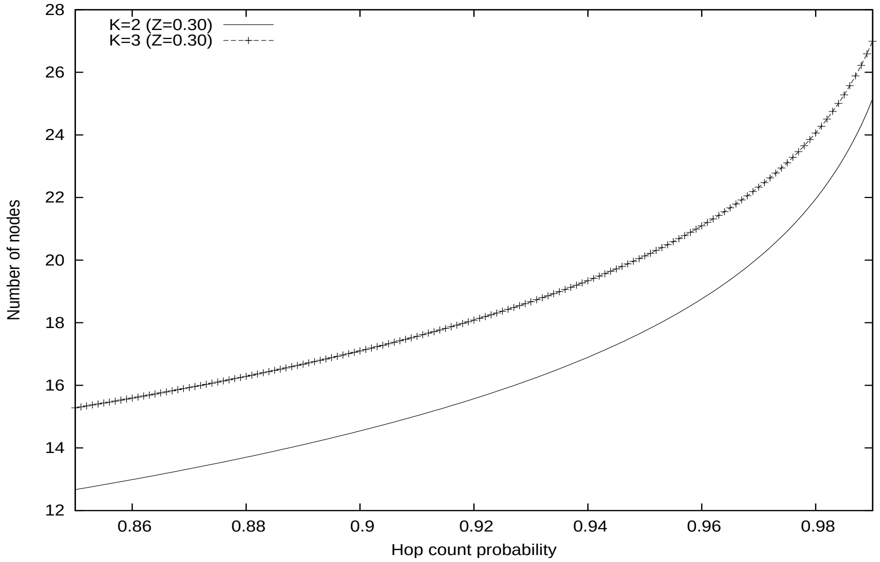

Figure 2.6: Critical value of the number of relay nodes N as a function of p.

In Figure 2.7 we can observe the relation between R and N for a hop count probability of $p = 0.95$ and for $K = 2, 3$ hops. We can observe that as the transmission range increases, the number of relay nodes needed to connect the source and destination nodes decreases.

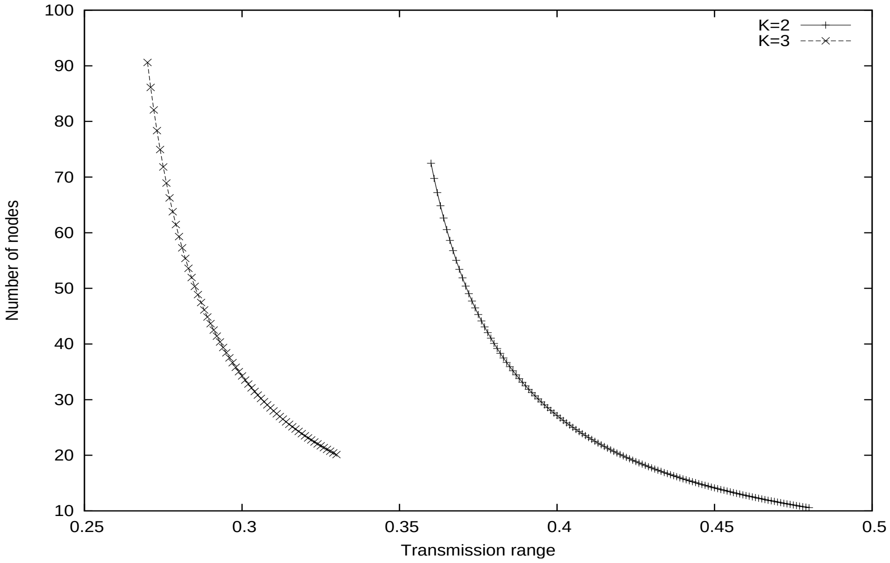

Figure 2.7: Relation between R and N for $p = 0.95$

Chapter 3

Connectivity in two-dimensional ad hoc networks

3.1 Introduction

In this chapter we derive the exact probability distribution of the hop count in a two-dimensional ad hoc network in which: either a fixed number of relay nodes are uniformly distributed between the source and the destination nodes, or the relay nodes are distributed according to a Poisson process. We recall that in Chapter 2 we derived the hop count distribution for a one-dimensional ad hoc network. In this chapter we will generalize the results of Chapter 2 for a two-dimensional ad-hoc network.

When the source and destination nodes are at a distance greater than the transmission range, the communication between them is made via a multiple hop path determined by the routing protocol (cf., e.g., Mauve et al. (2001)). Among the existing position-based routing protocols for two dimensional networks we have described the most used: the MFR (most forward within radius routing), the NR (nearest distance routing), the FR (furthest distance routing) and CR (compass routing) protocol. The similarity between these protocols is that all of them guarantee that the hop progress is made towards the destination node.

As stated by Kuo and Liao (2007) and the references therein, one of the most important metrics to evaluate the performance of routing protocols is the number of hops of the multihop path. Derivation of the hop count distribution in a two-dimensional scenario must take into account, among other factors: the node spatial distribution, the transmission range and the routing protocol. The interaction of these characteristics turns the derivation of the hop count distribution a difficult task. This is the reason why, despite its importance, there are few analytical studies on the subject and most of them address only single link models (Haenggi (2005), Srinivasa and Haenggi (2010) and Vural and Ekici (2005)) and/or approximation results (Dulman et al. (2006) and Kuo and Liao (2007)).

Haenggi (2005), assuming that relay nodes are distributed according to a Poisson process, derived the distribution of the distance from the source to the furthest neighbor node within transmission range. The analysis was extended in Srinivasa and Haenggi (2010) to a model where a finite number of relay nodes is uniformly distributed in a region of interest. Also assuming a Poisson process for the node locations, Vural and Ekici (2005) derived an approximation of the distribution of the distance to the furthest neighbor node within transmission range.

As far as we know, only two papers focus their analysis in more than a single link. Dulman et al. (2006) derived an approximation for the relationship between the number of hops and the distance between the source and the destination nodes, and Kuo and Liao (2007) derived an approximation for the probability of existence of a multihop path between the source and destination nodes.

Other issues of interest that have been analyzed in the literature are the expected hop progress and the expected hop distance. The expected hop progress from a relay node is defined as the expected value of the length of the hop, measured along the axis between the relay node and the destination node, whereas the expected hop distance is defined as the expected distance between two consecutive relay nodes of the multihop path. Assuming that relay nodes are randomly distributed by a Poisson process, Hou and Li (1986) compared the expected hop progress of a single link for the FR and NR routing protocols.

In this chapter, we assume that the source and destination nodes are fixed at a known distance. In addition, the underlying node distribution, in a given region of interest, is either assumed as a stationary Poisson process or that a known and fixed number of relay nodes are uniformly distributed in that region. To obtain the multihop path, we propose a propagation model where the routing region of each relay node is defined by a given angular span and a radius equal to the transmission range. Our model is a generalization of the model proposed by Srinivasa and Haenggi (2010) in the sense that we consider a variable angular span instead of a fixed angular span. This is accomplished by considering that the angular span depends on the distance between the relay and destination nodes. We call this model the dynamic propagation model.

Using the dynamic propagation model, our results permit to derive the exact hop count probability distribution with an arbitrary number of hops for a multihop path selected by the FR and NR routing protocols. The hop count distribution is derived when relay nodes are distributed according to a Poisson process and, by resorting to the Poissonification technique, we derive the hop count distribution when a finite number of relay nodes is uniformly distributed in an area of interest. As far as we know, these are the first exact analytical results for the hop count distribution with an arbitrary number of hops in a two-dimensional scenario. From the hop count probability distribution, we numerically compute the expected hop progress and the expected hop distance to evaluate the performance of the routing protocols.

The outline of this chapter is the following. In Section 3.2 we describe the dynamic propagation model. In Section 3.3 we derive the density location of relay nodes and the hop count distribution for the dynamic propagation model with the nearest and the furthest distance routing protocols, when the relay nodes are randomly distributed according to a

Poisson process. In Section 3.4 we derive analogous results assuming that a fixed number of relay nodes are uniformly distributed in a region of interest. In Section 3.5 we present some numerical results to compare both routing protocols and to evaluate the performance of the dynamic propagation model. Finally, in Section 3.6 we present the proofs of some auxiliary results used in the former sections.

3.2 Model description

We consider a multihop ad hoc network with the source node fixed at the origin and the destination node fixed at a distance L from the source node. A multihop path with m hops is defined as an existing path from the source to the destination node using exactly m relay nodes. Denote by $X_i, 1 \leq i \leq m$, the location of the relay node i of a multihop path, with these nodes ordered according to their distance to the origin, and let $X_0 = (0, 0)$ and $X_{m+1} = (L, 0)$ denote the locations of the source and destination nodes, respectively. Note that, without loss of generality, we have assumed that the destination node is located in the x-axis. Given a fixed transmission range R, $0 < R < L$, equal for all nodes, nodes i and j are connected with no hops if $\|X_i - X_j\| < R$.

We assume that the locations of the source node, the destination node, and all relay nodes of the multihop path belong to a compact set $\Omega \subset \mathbb{R}^2$. The set Ω is defined by an isosceles triangle with one vertice at the origin $(0, 0)$ with associated angle $\phi_0 = 2 \arctan (R/L)$, and the height of the triangle lies on the horizontal axis and is equal to $L + R$. For the underlying node distribution, we assume that either a fixed and known number of nodes are uniformly distributed in Ω, or a random number of nodes are distributed in Ω according to a stationary Poisson process.

3.2.1 Dynamic propagation model

For efficient routing progress towards the destination, we consider that each relay node transmits within a routing region limited by the transmission radius R and an angular span oriented to the destination node. The angular span ϕ_i of relay node i is chosen in a dynamic way, being dependent on the location X_i of the relay node. The choice of the angular span is such that it originates a triangle with vertices at points (L, R), $(L, -R)$ and X_i (see Figure 3.1), decreasing when the relay node gets further away from the destination node and increasing when it gets closer to the destination node.

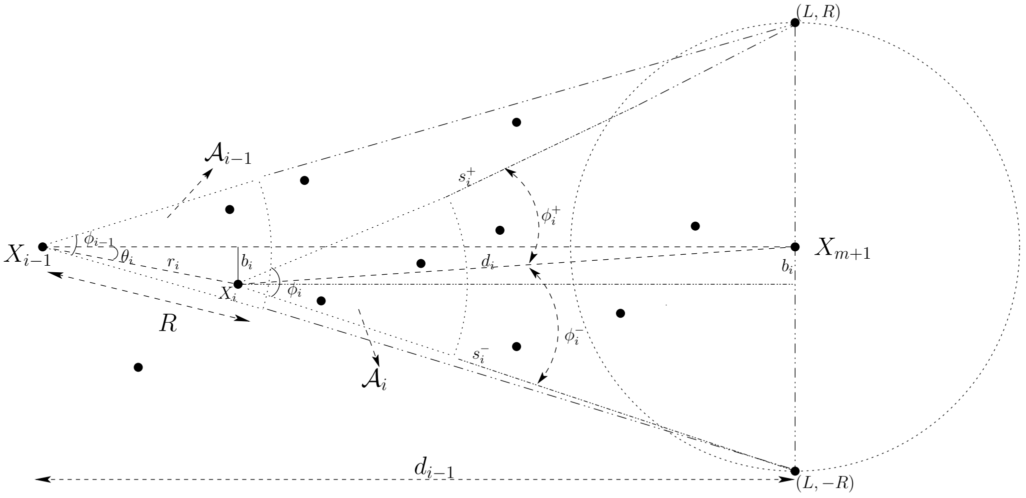

Figure 3.1: Routing regions and angular spans of relay nodes $i-1$ and i.

The polar coordinates of the location of the relay node i relative to the location of relay node $i-1$ are denoted by (r_i, θ_i), $-\pi \leq \theta_i \leq \pi$. Given (r_i, θ_i) and the distance from relay node $i-1$ to the destination node, d_{i-1}, the distance from relay node i to the destination node, d_i, is given by the function

$$d_i \equiv f(d_{i-1}, r_i, \theta_i) = \sqrt{(d_{i-1} - r_i \cos\theta_i)^2 + (r_i \sin\theta_i)^2}, \qquad 1 \leq i \leq m, \tag{3.1}$$

with $d_0 = L$. The angle ϕ_i of relay node i, can then be written as a function of d_{i-1} and (r_i, θ_i),

$$\phi_i \equiv \phi(d_{i-1}, r_i, \theta_i), \tag{3.2}$$

and is given by

$$\phi_i = \arcsin\left(\frac{R-b_i}{s_i^+}\right) + \arcsin\left(\frac{R+b_i}{s_i^-}\right), \tag{3.3}$$

where $b_i = r_i \sin\theta_i$, so that $|b_i|$ is the minimum distance between X_i and the axis that goes from X_{i-1} to X_{m+1}, and $s_i^{\pm} = \sqrt{(d_{i-1} - r_i\cos\theta_i)^2 + (R \mp b_i)^2}$ is the distance between X_i and $(L, \pm R)$; see Figure 3.1. Using geometric arguments, we can show that the angle ϕ_i^+ formed by the points (L, R), X_i and X_{m+1} is given by

$$\phi_i^+ = \arcsin\left(\frac{R-b_i}{s_i^+}\right) + \arcsin\left(\frac{b_i}{d_i}\right)$$

and the angle ϕ_i^- formed by the points $(L, -R)$, X_i and X_{m+1} is given by

$$\phi_i^- = \arcsin\left(\frac{R+b_i}{s_i^-}\right) - \arcsin\left(\frac{b_i}{d_i}\right).$$

Note that the angular span of relay node i is given by $\phi_i = \phi_i^+ + \phi_i^-$.

To describe the routing regions of each relay node, we make a translation and rotation of the plane to locate the origin of the new plane at the current emissor node (in this case at relay node i), with horizontal axis being the line drawn from the emissor node to the destination node. For a relay node i located at X_i, the routing region relative to X_i is denoted by $\mathcal{A}_i \equiv \mathcal{A}(X_i, X_{m+1}, \phi_i)$ and, at each hop, an angular slice of a circular disk with radius R and with area $\frac{\phi_i}{2}R^2$ is covered (see Figure 3.1). More precisely, the routing region of relay node i relative to X_i is

$$\mathcal{A}_i \equiv \mathcal{A}(X_i, X_{m+1}, \phi_i) = \left\{(r,\theta) : 0 < r < R, -\phi_i^- \leq \theta \leq \phi_i^+\right\}$$

Since the routing regions will be dynamic and oriented towards the destination node, we call this model the *dynamic propagation model*. Using this model, two routing protocols are analyzed: the furthest distance routing (FR) and the nearest distance routing (NR). With the FR protocol the transmission is made to the furthest neighbor node within the routing region, whereas in the NR protocol the transmission is made to the nearest neighbor node within the routing region. In Figure 3.2 we can observe the dynamic propagation model with the FR protocol for a path with 3 hops.

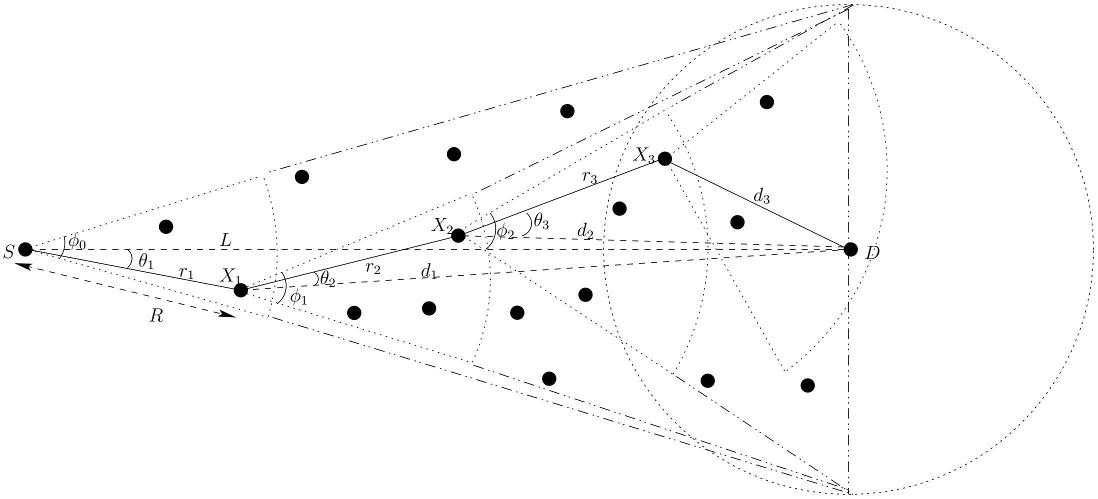

Figure 3.2: Dynamic propagation model with the FR protocol for a path with 3 hops.

The minimum number of hops needed to connect the source and the destination nodes for the FR and the NR routing protocols is

$$K = \left\lfloor \frac{L}{R} \right\rfloor, \tag{3.4}$$

which can be easily verified by putting each relay node $X_i = (x_i, y_i)$ on the source-destination axis (i.e., $y_i = 0$) and such that their abscissas are

$$Z_1^- < x_1 < Z_1^+, \quad Z_2^- < x_2 < x_1 + R, \; \ldots, \quad Z_K^- < x_K < x_{K-1} + R.$$

with $Z_i^- = L - R(K + 1 - i)$ and $Z_i^+ = iR$, just like in the unidimensional case (cf. Antunes, Jacinto, and Pacheco (2008)).

3.3 Hop count distribution with a random number of relay nodes

In this section we assume that relay nodes are randomly distributed in the plane according to a stationary Poisson point process with intensity λ. In this scenario, the number of relay nodes in disjoint sets are independent of each other and for any given set with area

B the number of relay nodes in the set has Poisson distribution with mean λB. For the dynamic propagation model with the furthest distance and the nearest distance routing protocols, we derive the density location of relay nodes for a multihop path with m hops and the distribution of the number of hops.

3.3.1 Dynamic propagation model with the nearest distance protocol

In this subsection we derive the hop count probability distribution for a multihop path that selects the relay node closer to the emissor node.

To derive the hop count probability distribution we need to obtain the area of the *vacant region* of each relay node of the multihop path. The vacant region $\mathcal{V}_i^{NR}$ of relay node i is defined to be the subset of the routing region of relay node i that has no relay nodes. That is, since the relay node selected is the closest one from the emissor node, the vacant region of relay node i is given by the set of points that are closer to i than relay node $i+1$, having an area $V_i^{NR} = \frac{\phi_i}{2} r_{i+1}^2$, see Figure 3.3.

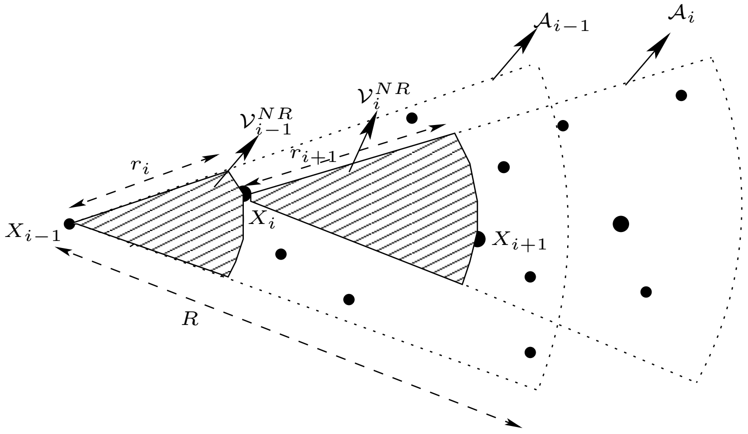

Figure 3.3: Routing regions and vacant regions of relay nodes $i-1$ and i.

Denote by $\mathbf{l}_m = (l_1, l_2, \ldots, l_m)$ the vector of relative locations of the m relay nodes, with $l_i = (r_i, \theta_i)$, and let $\mathrm{d}\mathbf{l}_m = \mathrm{d}r_m\, \mathrm{d}\theta_m\, \mathrm{d}\theta_{m-1}\, \mathrm{d}r_{m-1} \ldots \mathrm{d}r_1\, \mathrm{d}\theta_1$.

Theorem 3.1. *Given that relay nodes are distributed according to a Poisson process with rate λ, the probability distribution of the hop count for the dynamic propagation model with the nearest distance routing protocol is given by*

$$\mathbb{P}(M = m) = \int_{N_m} \lambda^m \mathrm{e}^{-\lambda \sum_{i=0}^{m-1} V_i^{NR}} \prod_{i=1}^{m} r_i \, \mathrm{d}\mathbf{l}_m \tag{3.5}$$

with $m \geq K$, where

$$N_m = \Big\{ \mathbf{l}_m : l_i = (r_i, \theta_i) \in \mathcal{A}_{i-1}, i = 1, 2, \ldots, m, d_m < R \leq d_{m-1} \Big\}. \tag{3.6}$$

Proof. We first derive the joint density location of the m relay nodes of the multihop path. For that, fix $(r_1, \theta_1) \in \mathcal{A}_0 = \big\{ (r_1', \theta_1') : 0 < r_1' < R, -\frac{\phi_0}{2} < \theta_1' < \frac{\phi_0}{2} \big\}$ and define

$$\mathcal{V}_0 = \left\{ (r_1', \theta_1') : 0 < r_1' < r_1, -\frac{\phi_0}{2} < \theta_1' < \frac{\phi_0}{2} \right\}$$
$$\mathcal{V}_0^{\epsilon} = \left\{ (r_1', \theta_1') : r_1 \leq r_1' < r_1 + \epsilon_1, \theta_1 \leq \theta_1' < \theta_1 + \epsilon_2 \right\}.$$

Denote by $N(A)$ the number of points of the Poisson process that lies in A. Then we have

$$\begin{aligned} P\left(N(\mathcal{V}_0) = 0, N(\mathcal{V}_0^{\epsilon}) > 0\right) &= P\left(N(\mathcal{V}_0) = 0\right) P\left(N(\mathcal{V}_0^{\epsilon}) > 0\right) \\ &= e^{-\lambda \frac{\phi_0}{2} r_1^2} \left(1 - P\left(N(\mathcal{V}_0^{\epsilon} = 0\right)\right) \\ &= e^{-\lambda \frac{\phi_0}{2} r_1^2} \left(1 - \exp\left(-\lambda \int_{r_1}^{r_1+\epsilon_1} \int_{\theta_1}^{\theta_1+\epsilon_2} r \mathrm{d}r \mathrm{d}\theta\right)\right) \\ &= e^{-\lambda \frac{\phi_0}{2} r_1^2} \lambda \int_{r_1}^{r_1+\epsilon_1} \int_{\theta_1}^{\theta_1+\epsilon_2} r \mathrm{d}r \mathrm{d}\theta + o(\epsilon_1 \epsilon_2). \end{aligned}$$

The density function of the location of the first relay node being at (r_1, θ_1) is given by

$$\begin{aligned} h(r_1, \theta_1) = \lim_{\epsilon_1, \epsilon_2 \to 0^+} \frac{P\left(N(\mathcal{V}_0) = 0, N(\mathcal{V}_0^{\epsilon}) > 0\right)}{\epsilon_1 \epsilon_2} = \\ = \lambda r_1 e^{-\lambda \frac{\phi_0}{2} r_1^2}. \end{aligned}$$

To derive the density location of the first two relay nodes, we make a rotation and translation of the plane in order to place the origin of the new plane at $(r_1 + \epsilon, \theta_1)$ with horizontal axis being the line drawn from $(r_1+\epsilon, \theta_1)$ to the destination node. The following three sets are defined relative to this new origin.

Fix $(r_2, \theta_2) \in \mathcal{A}_1 = \left\{(r_2', \theta_2') : 0 < r_2' < R, -\frac{\phi_1}{2} < \theta_2' < \frac{\phi_1}{2}\right\}$ and define

$$\mathcal{V}_1 = \left\{(r_2', \theta_2') : 0 < r_2' < r_2, -\frac{\phi_1}{2} < \theta_2' < \frac{\phi_2}{2}\right\}$$
$$\mathcal{V}_1^\delta = \{(r_2', \theta_2') : r_2 \le r_2' < r_2 + \delta_1, \theta_2 \le \theta_2' < \theta_2 + \delta_2\}.$$

Therefore,

$$\begin{aligned}
&P\left(N(\mathcal{V}_0) = 0, N(\mathcal{V}_0^\epsilon) > 0, N(\mathcal{V}_1) = 0, N(\mathcal{V}_1^\delta) > 0\right) = \\
&= P\left(N(\mathcal{V}_0) = 0\right) P\left(N(\mathcal{V}_0^\epsilon) > 0\right) P\left(N(\mathcal{V}_1) = 0\right) P\left(N(\mathcal{V}_1^\delta) > 0\right) \\
&= P(N(\mathcal{V}_0) = 0)\left(1 - P(N(\mathcal{V}_0^\epsilon) = 0)\right) P(N(\mathcal{V}_1) = 0)\left(1 - P(N(\mathcal{V}_1^\delta) = 0)\right) \\
&= e^{-\lambda\frac{\phi_0}{2}r_1^2}\left(\lambda\int_{r_1}^{r_1+\epsilon_1}\int_{\theta_1}^{\theta_1+\epsilon_2} r\mathrm{d}r\mathrm{d}\theta + o(\epsilon_1\epsilon_2)\right) e^{-\lambda\frac{\phi_1}{2}r_2^2}\left(\lambda\int_{r_2}^{r_2+\delta_1}\int_{\theta_2}^{\theta_2+\delta_2} r\mathrm{d}r\mathrm{d}\theta + o(\delta_1\delta_2)\right).
\end{aligned}$$

The density function of the location of the first two relay nodes is given by

$$\begin{aligned}
h(r_1, \theta_1, r_2, \theta_2) &= \lim_{\epsilon_1, \epsilon_2, \delta_1, \delta_2 \to 0^+} \frac{P\left(N(\mathcal{V}_0) = 0, N(\mathcal{V}_0^\epsilon) > 0, N(\mathcal{V}_1) = 0, N(\mathcal{V}_1^\delta) > 0\right)}{\epsilon_1\epsilon_2\delta_1\delta_2} = \\
&= \lambda^2 r_1 r_2 e^{-\lambda\frac{\phi_0}{2}r_1^2} e^{-\lambda\frac{\phi_1}{2}r_2^2}.
\end{aligned}$$

Proceeding in the same manner until the m-th relay node reaches the destination node we obtain the joint density location of the m relay nodes of the multihop path

$$h(\mathbf{l}_m) = \lambda^m \mathrm{e}^{-\lambda\sum_{i=1}^m \frac{\phi_{i-1}}{2}r_i^2} \prod_{i=1}^m r_i, \tag{3.7}$$

where the given node locations are in N_m and the last relay node is m because $d_m < R \le d_{m-1}$.

Thus, integrating (3.7) over the set N_m we obtain the probability that the hop count is m for the nearest distance routing protocol, when the relay nodes are randomly distributed according to a Poisson process

$$\mathbb{P}(M = m) = \int_{N_m} \lambda^m \mathrm{e}^{-\lambda\sum_{i=1}^m \frac{\phi_{i-1}}{2}r_i^2} \prod_{i=1}^m r_i \; \mathrm{d}\mathbf{l}_m \tag{3.8}$$

❏

Note that the function (3.7) is the joint density location of the relay nodes in the multihop path selected with the dynamic propagation model and the NR protocol.

3.3.2 Dynamic propagation model with the furthest distance protocol

In this subsection we characterize the exact probability distribution of the hop count for the dynamic propagation model that selects the relay node that is furthest away from the emissor within the routing region.

We denote by $\widehat{\mathcal{A}}_i$ the *admissible propagation region* of relay node i, defined as the possible relative locations of relay node $i+1$, (r_{i+1}, θ_{i+1}), given the location of relay node i. For different values of i, the sets $\widehat{\mathcal{A}}_i$ are disjoint. For a given relative location of relay node $i+1$, and since it is at the furthest distance from the relay node i within its routing region, the *vacant region* of relay node i, denoted by $\mathcal{V}_i^{FR}$, is given by the set of points of the routing region of relay node i that are at a distance greater than r_{i+1}. We denote by V_i^{FR} the area of the vacant region of relay node i and, for a multihop path with m hops, we will denote the sum of the areas of the disjoint vacant regions by

$$V^{FR}(\mathbf{l}_m) = \sum_{i=0}^{m-1} V_i^{FR}. \tag{3.9}$$

Note that, to simplify the notation, we write $\mathcal{V}_i^{FR}$ and V_i^{FR} instead of the more accurate $\mathcal{V}_i^{FR}(r_i, \theta_i, d_i, r_{i+1}, \theta_{i+1})$ and $V_i^{FR}(r_i, \theta_i, d_i, r_{i+1}, \theta_{i+1})$. Since the derivations of the areas of V_i^{FR} and $\widehat{\mathcal{A}}_i$ have a high level of details, being necessary to consider all possible relative locations of relay nodes i and $i+1$, we present the auxiliary lemmas that derive these results at the end of this chapter, in Section 3.6.

Theorem 3.2. *Given that relay nodes are distributed according to a Poisson process with rate λ, the probability distribution of the hop count for the dynamic propagation model* *with the furthest distance routing protocol is*

$$\mathbb{P}(M=m)=\int_{G_m}\lambda^m \mathrm{e}^{-\lambda V^{FR}(\mathbf{l}_m)}\prod_{i=1}^{m} r_i \, \mathrm{d}\mathbf{l}_m \tag{3.10}$$

with $m \geq K$*, where*

$$G_m=\Big\{\mathbf{l}_m : l_i=(r_i,\theta_i)\in \widehat{\mathcal{A}}_i, i=1,2,\ldots,m, d_m < R \leq d_{m-1}\Big\}. \tag{3.11}$$

Proof. Proceeding as in the proof of Theorem 3.1 we can prove that, in a Poisson random network with intensity λ, the density function of the location of the first relay node that is at furthest distance from the source, is given by

$$g(l_1)=\lambda r_1 \mathrm{e}^{-\lambda\frac{\phi_0}{2}(R^2-r_1^2)}=\lambda r_1 \mathrm{e}^{-\lambda V_0^{FR}},$$

with $l_1=(r_1,\theta_1)\in\mathcal{A}_0$.

The density location of the first two relay nodes is given by

$$g(l_1,l_2)=\lambda^2 r_1 r_2 e^{-\lambda(V_0^{FR}+V_1^{FR})}.$$

where V_1^{FR} is the area of the vacant region of relay node 1 and $(r_1,\theta_1)\in\mathcal{A}_0$, $(r_2,\theta_2)\in\widehat{\mathcal{A}}_1$, with V_1^{FR} and $\widehat{\mathcal{A}}_1$ being given by Lemma 3.2 of Section 3.6.

Proceeding in the same manner for the m relay nodes, the density location of the m relay nodes is

$$g(\mathbf{l}_m)=\lambda^m \mathrm{e}^{-\lambda V^{FR}(\mathbf{l}_m)}\prod_{i=1}^{m} r_i, \tag{3.12}$$

where the possible locations of the m given relay nodes are in G_m and the last relay node is m because $d_m < R \leq d_{m-1}$.

Thus, integrating (3.12) over the set G_m we obtain the probability that the hop count is m for the furthest distance routing protocol, when the relay nodes are randomly distributed according to a Poisson process

$$\mathbb{P}(M=m)=\int_{N_m}\lambda^m \mathrm{e}^{-\lambda\sum_{i=0}^{m-1}V_i^{NR}}\prod_{i=1}^{m} r_i \, \mathrm{d}\mathbf{l}_m \tag{3.13}$$

❏

Note that the function (3.12) is the density location of the relay nodes in the multi-hop path selected by the dynamic propagation model with the furthest distance routing protocol.

For a path with one hop, a simple closed formula can be obtained.

Theorem 3.3. *Given that relay nodes are distributed according to a Poisson process with rate* λ*, for* $R \leq L < 2R$*, the probability that the hop count is equal to 1 is given by*

$$P(M=1) = 1 - \mathrm{e}^{-\lambda A},$$

where

$$A = \begin{cases} \left(\frac{R^2}{2}\left(\phi_0 - a + \sin a\right) - h^2 \tan\left(\frac{\phi_0}{2}\right)\right), & R < L \leq \sqrt{3}R \\ \left(2R^2 \arccos\left(\frac{L}{2R}\right) - \frac{L}{2}\sqrt{4R^2 - L^2}\right), & \sqrt{3}R < L \leq 2R \end{cases} \tag{3.14}$$

where $a = 2\arccos\left(\frac{L-h}{R}\right)$*,* $h = L - \sqrt{R^2 - (b \sin\frac{\phi_0}{2})^2}$ *and* $b = L\cos\frac{\phi_0}{2} - \sqrt{R^2 - (L\sin\frac{\phi_0}{2})^2}$*.*

Proof. Denote by $B(x,a)$ a circumference with center at x and radius a. The area of intersection I between the transmission ranges of the source node S and the destination node D, $I = B(S,R) \cap B(D,R)$, is given by (cf., http://mathworld.wolfram.com/Circle-CircleIntersection.html)

$$2R^2 \arccos\left(\frac{L}{2R}\right) - \frac{L}{2}\sqrt{4R^2 - L^2}.$$

If the routing region is such that $\sqrt{3}R < L \leq 2R$, the intersection region I is completely inside the routing region of the source $\mathcal{A}_0$. Since relay nodes are distributed according to a Poisson point process, the probability that the hop count is equal to 1 is given by the probability that there is at least one node in the intersection region I, which is

$$1 - \exp\left(-\lambda\left(2R^2 \arccos\left(\frac{L}{2R}\right) - \frac{L}{2}\sqrt{4R^2 - L^2}\right)\right).$$

Case $R < L \leq \sqrt{3}R$, the routing region of the source cuts, above and below, the intersection region I (see Figure 3.4), and we proceed to find the area of $J = I \cap \mathcal{A}_0$.

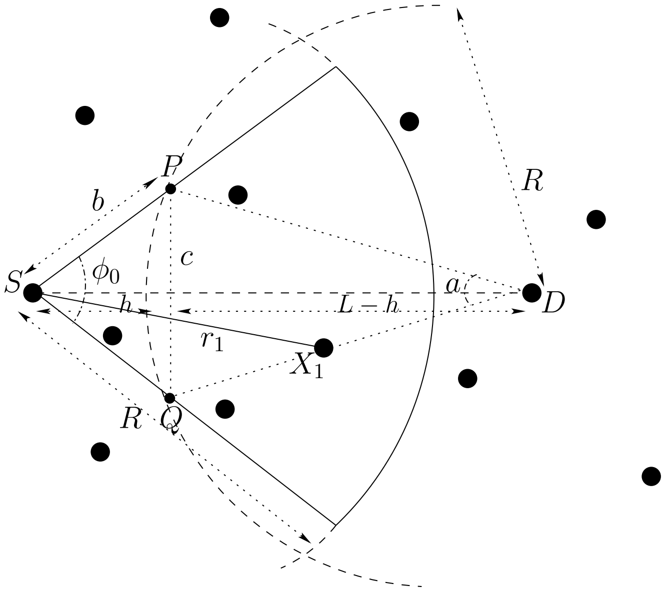

Figure 3.4: Intersection between $\mathcal{A}_0$, $B(S,R)$ and $B(D,R)$.

To obtain the area of J, we first need to obtain the area of the circular segment defined by the line between P and Q and the boundary of $B(D,R)$. The area of this circular segment is (cf. Polyanin and Manzhirov, 2006, p. 38, formula 3.1.3.1)

$$\frac{R^2}{2}\left(a - \sin a\right).$$

with the angle a given by $a = 2\arccos\left(\frac{L-h}{R}\right)$, where $L-h$ is the height of the triangle $\overline{PDQ}$, which is given by

$$L - h = \frac{1}{2}\sqrt{4R^2 - c^2},$$

where c is the length of the cord $\overline{PQ}$ which is also the base of the triangle $\overline{PSQ}$. By the law of sines (cf. Polyanin and Manzhirov, 2006, p. 44),

$$c = 2b\sin\frac{\phi_0}{2},$$

where b is the length of $\overline{SP} = \overline{SQ}$ and is obtained by the law of cosines (cf. Polyanin and

Manzhirov, 2006, p. 44), $R^2 = b^2 + L^2 - 2bL\cos\frac{\phi_0}{2}$, whose solution gives

$$b = L\cos\frac{\phi_0}{2} - \sqrt{R^2 - \left(L\sin\frac{\phi_0}{2}\right)^2}.$$

Knowing the area of the circular segment and the area of the triangle $\overline{PSQ}$ (which is $h^2\tan\frac{\phi_0}{2}$), we can obtain the area of the cone $\mathcal{B}$ defined by the region that is not intersected by $B(D,R)$ and is inside $\mathcal{A}_0$, which is

$$\mathrm{area}(\mathcal{B}) = h^2\tan\frac{\phi_0}{2} - \frac{R^2}{2}\left(a - \sin a\right)$$

Finally, knowing the area of $\mathcal{B}$ and the area of the routing region $\mathcal{A}_0$, we obtain the area of J,

$$\mathrm{area}(J) = \mathrm{area}(\mathcal{A}_0) - \mathrm{area}(\mathcal{B}) = \frac{R^2}{2}\left(\phi_0 + a - \sin a\right) - h^2\tan\frac{\phi_0}{2}.$$

Then, the probability that the hop count is equal to 1 is given by the probability that there is at least one node in J, which is

$$1 - \exp\left(-\lambda\left(\frac{R^2}{2}\left(\phi_0 + a - \sin a\right) - h^2\tan\frac{\phi_0}{2}\right)\right).$$

❑

3.4 Hop count distribution with a finite number of relay nodes

We now assume that a finite and known number N of relay nodes are uniformly distributed in Ω, in order to derive the hop count probability distribution for a multihop path selected by the dynamic propagation model with the FR and the NR protocols. The hop count probability distribution is obtained by using the Poisson randomization technique, used in Section 2.3.2, that consists in randomizing the number of relay nodes by assuming that relay nodes are distributed in Ω according to a Poisson process with rate λ.

3.4.1 Dynamic propagation model with the nearest distance protocol

In this subsection we derive the hop count probability distribution for a multihop path selected by the dynamic propagation model with the NR protocol when a fixed and known number of relay nodes are uniformly distributed on Ω. We denote by B the area of Ω

Theorem 3.4. *Given that there are n relay nodes uniformly distributed on Ω, the probability distribution of the hop count for a multihop path selected by the dynamic propagation model with the nearest distance routing protocol is given by*

$$\mathbb{P}(M=m|N=n)=\int_{N_m}\frac{n!}{(n-m)!}\frac{1}{B^m}\left(1-\frac{1}{B}\sum_{i=0}^{m-1}V_i^{NR}\right)^{n-m}\prod_{i=1}^{m}r_i\,\mathrm{d}\mathbf{l}_m \tag{3.15}$$

with $K\le m\le n$ and where

$$N_m=\Big\{\mathbf{l}_m:l_i=(r_i,\theta_i)\in\mathcal{A}_{i-1},i=1,2,\dots,m,d_m<R\le d_{m-1}\Big\}.$$

Proof. From (3.5) the probability that the hop count is m when relay nodes are distributed according to a Poisson process is

$$\mathbb{P}(M=m)=\int_{N_m}\lambda^m\mathrm{e}^{-\lambda\sum_{i=0}^{m-1}V_i^{NR}}\prod_{i=1}^{m}r_i\,\mathrm{d}\mathbf{l}_m \tag{3.16}$$

Multiplying equation (3.16) by $e^{\lambda B}$ where B is the area of Ω, we obtain

$$\begin{aligned}\mathrm{e}^{\lambda B}\mathbb{P}(M=m) &= \mathrm{e}^{\lambda B}\int_{N_m}\lambda^m\mathrm{e}^{-\lambda\sum_{i=0}^{m-1}V_i^{NR}}\prod_{i=1}^{m}r_i\,\mathrm{d}\mathbf{l}_m\\ &= \int_{N_m}\lambda^m\sum_{n=0}^{\infty}\frac{(\lambda B)^n}{n!}\left(1-\frac{1}{B}\sum_{i=0}^{m-1}V_i^{NR}\right)^{n}\prod_{i=1}^{m}r_i\,\mathrm{d}\mathbf{l}_m\\ &= \sum_{n=m}^{\infty}\frac{(\lambda B)^n}{n!}\int_{N_m}\frac{n!}{(n-m)!}\frac{1}{B^m}\left(1-\frac{1}{B}\sum_{i=0}^{m-1}V_i^{NR}\right)^{n-m}\prod_{i=1}^{m}r_i\,\mathrm{d}\mathbf{l}_m\end{aligned}$$

where the change between the sum and the integral follows by the dominated convergence theorem. On the other hand, conditioning on the value of N, which is Poisson distributed with mean λB, the total probability law produces

$$e^{\lambda B}\mathbb{P}(M=m)=\sum_{n=m}^{\infty}\mathbb{P}(M=m|N=n)\frac{(\lambda B)^n}{n!}.$$

Since the coefficients of $\frac{(\lambda B)^n}{n!}$ in the previous two expressions for $e^{\lambda B}\mathbb{P}(M=m)$ must match, the result follows.

❏

Note that the integrand function in (3.15),

$$\frac{n!}{(n-m)!}\frac{1}{B^m}\left(1-\frac{1}{B}\sum_{i=0}^{m-1}V_i^{NR}\right)^{n-m}\prod_{i=1}^{m}r_i$$

is the density location of the relay nodes when a finite and known number n of relay nodes are uniformly distributed on Ω with the multihop path selected by the dynamic propagation model with the NR protocol.

3.4.2 Dynamic propagation model with the furthest distance protocol

In this subsection we characterize the exact probability distribution of the hop count for the dynamic propagation model with the FR protocol, given the number of relay nodes in the region Ω. We also obtain a closed formula for the connectivity probability with 1 hop.

Theorem 3.5. *Given that there are n relay nodes uniformly distributed on Ω, the probability distribution of the hop count for a multihop path selected by the dynamic propagation model and the furthest distance routing protocol is given by*

$$\mathbb{P}(M=m|N=n)=\int_{G_m}\frac{n!}{(n-m)!}\frac{1}{B^m}\left(1-\frac{V^{FR}(\mathbf{l}_m)}{B}\right)^{n-m}\prod_{i=1}^{m}r_i\,\mathrm{d}\mathbf{l}_m \tag{3.17}$$

with $K\le m\le n$ and where

$$G_m=\Big\{\mathbf{l}_m : l_i=(r_i,\theta_i)\in\widehat{\mathcal{A}}_{i-1}, i=1,2,\ldots,m, d_m<R\le d_{m-1}\Big\}.$$

Proof. Assuming that the relay nodes are distributed on Ω by a Poisson process with rate λ, the probability that the hop count is m is given by (3.10),

$$\mathbb{P}(M = m) = \int_{G_m} \lambda^m \mathrm{e}^{-\lambda V^{FR}(\mathbf{l}_m)} \prod_{i=1}^{m} r_i \, \mathrm{d}\mathbf{l}_m. \tag{3.18}$$

Multiplying equation (3.18) by $e^{\lambda B}$, where B is the area of Ω, we obtain

$$\begin{aligned} e^{\lambda B}\mathbb{P}(M = m) &= e^{\lambda B} \int_{G_m} \lambda^m e^{-\lambda V^{FR}(\mathbf{l}_m)} \prod_{i=1}^{m} r_i \, \mathrm{d}\mathbf{l}_m \\ &= \int_{G_m} \lambda^m \sum_{n=0}^{\infty} \frac{(\lambda B)^n}{n!} \left(1 - \frac{V^{FR}(\mathbf{l}_m)}{B}\right)^n \prod_{i=1}^{m} r_i \, \mathrm{d}\mathbf{l}_m \\ &= \sum_{n=m}^{\infty} \frac{(\lambda B)^n}{n!} \int_{G_m} \frac{n!}{(n-m)!} \frac{1}{B^m} \left(1 - \frac{V^{FR}(\mathbf{l}_m)}{B}\right)^{n-m} \prod_{i=1}^{m} r_i \, \mathrm{d}\mathbf{l}_m \end{aligned}$$

where the change between the sum and the integral follows by the dominated convergence theorem. By the same arguments used in the proof of Theorem 3.4, the result follows.

❑

Note that the integrand function in (3.17),

$$\frac{n!}{(n-m)!} \frac{1}{B^m} \left(1 - \frac{V^{FR}(\mathbf{l}_m)}{B}\right)^{n-m} \prod_{i=1}^{m} r_i$$

is the density location of the relay nodes when a finite and known number n of relay nodes are uniformly distributed on Ω with the multihop path selected by the dynamic propagation model with the FR protocol.

The next result gives an exact closed formula for the probability that there is a path with 1 hop.

Theorem 3.6. *Given that there are n relay nodes uniformly distributed on Ω, for $R \leq L < 2R$, the probability that the hop count is equal to 1 for the FR protocol is given by the binomial distribution*

$$\mathbb{P}(M = 1 | N = n) = 1 - (1 - p)^n \tag{3.19}$$

where $p = A/B$ *is the probability that a relay node uniformly distributed on* Ω *is connected with both the source and destination nodes, and* A *is given by (3.14).*

Proof. The result follows directly using the same arguments as in Theorem 3.3 and noting that if there are n relay nodes uniformly distributed on Ω, vectors whose coordinates are the number of nodes in the sets of a finite partition of Ω have a multinomial distribution. Here p is the probability that a relay node uniformly distributed on Ω is connected with both the source and destination nodes. ❑

3.5 Numerical results

In this section we evaluate the performance of the dynamic propagation model and compare the hop count probability distributions for the FR and the NR protocols. We scale all parameters with respect to the distance between the source and destination nodes assuming that $L = 1$, having the set Ω an area $B = R(R+1)^2$. Therefore, depending on the value of R, for $1/(K+1) < R \leq 1/K$, $K \in \mathbb{N}$, we have multihop paths with a minimum number of hops equal to K. The results were obtained by numerical integration using a Monte Carlo algorithm. We only present the results when the number of relay nodes are uniformly distributed on Ω, since similar results are obtained when assuming that the node distribution on Ω follows a stationary Poisson process with rate λ equal to n/B, the mean number of nodes per unit area.

Figure 3.5 shows the connectivity probability with the minimum number of hops K, $K = 1, 2, 3$, with the FR protocol and for different values of the number of nodes, when the nodes are uniformly distributed on Ω. We can observe that when the number of nodes increases, the probability of reaching the destination node with the minimum number of hops increases and approaches the value 1. For the same number of relay nodes, the hop count probability decreases as K increases.

Figure 3.6 shows the connectivity probability with the minimum number of hops K,

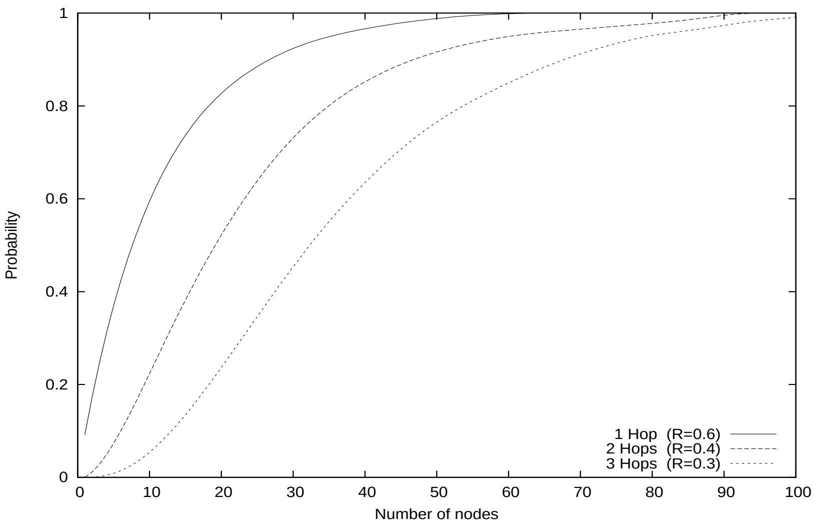

Figure 3.5: Connectivity probability with the minimum number of hops for the FR protocol.

$K = 1, 2, 3$, with the NR protocol and for different values of the number of nodes, when the nodes are uniformly distributed on Ω. We can observe that when the number of nodes increases, the NR protocol is ineffective because it cannot transmit with a high probability with the minimum number of hops. This protocol is only worth of consideration when there is a small number of nodes in the network. When the number of nodes increases, the minimum hop count probability decreases and approaches the value 0.

In Figure 3.7 we compare the hop count probability for the FR protocol, with the minimum number of hops, to different hop count probabilities for the NR protocol. We consider that $R = 0.4$, giving $K = 2$ for the FR protocol and $K = 2, 3, 4$ for the NR protocol. In this way, we compare the efficiency of the FR protocol using the minimum number of hops versus the efficiency of the NR protocol using longer hop paths ($K = 3, 4$). We can observe that, when there is a small number of nodes, the NR protocol with $K + 1 = 3$ hops produces probabilities similar to those obtained by the FR protocol. The probability of having a path with $K = 2, 3, 4$ hops for the NR protocol is always smaller or

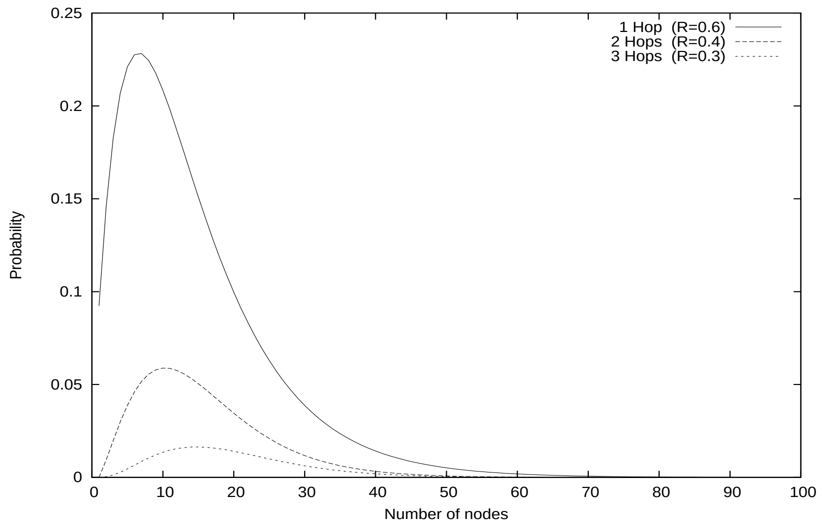

Figure 3.6: Connectivity probability with the minimum number of hops for the NR protocol.

equal to the probability of having a path with $K = 2$ hops for the FR protocol. Moreover, when the number of nodes increases all the probabilities ($K = 2, 3, 4$) for the NR protocol approach zero, whereas for the FR protocol the probability with $K = 2$ approaches 1.

In Figure 3.8 we compare the hop count probability for the FR protocol, with the minimum number of hops, to different hop count probabilities for the NR protocol. We consider that $R = 0.3$, giving $K = 3$ for the FR protocol and $K = 3, 4, 5, 6$ for the NR protocol. We observe again that when there is a small number of nodes, the NR protocol with $K + 1 = 4$ hops produces probabilities similar to those obtained by the FR protocol. Despite that all probabilities ($K = 3, 4, 5, 6$) on the NR protocol approach zero with the increase of the number of nodes, the probabilities obtained for paths with a large number of hops are generally higher than the ones obtained for paths with a smaller number of hops. Thus, we can conclude that the FR protocol with the minimum hop count outperforms the NR protocol with other number of hops, being more effective in finding a multihop path.

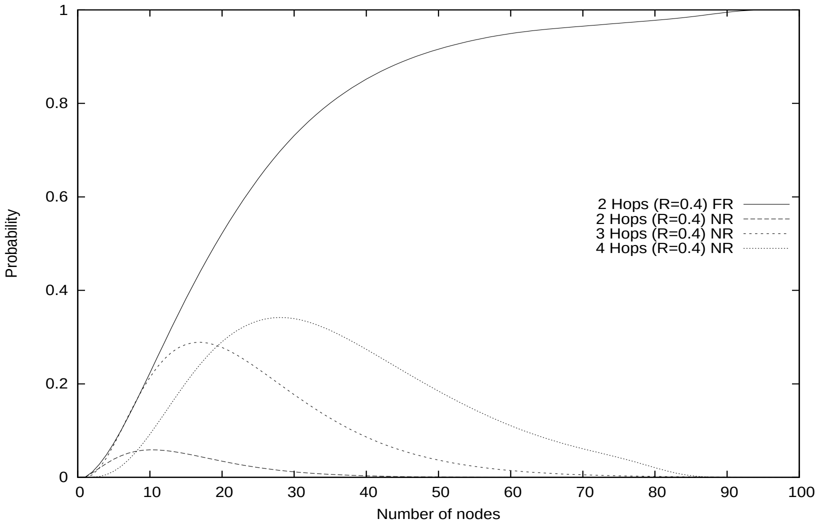

Figure 3.7: Probability that the hop count equals $K = 2$ for the FR protocol, versus the probability that the hop count equals $K = 2, 3, 4$ for the NR protocol.

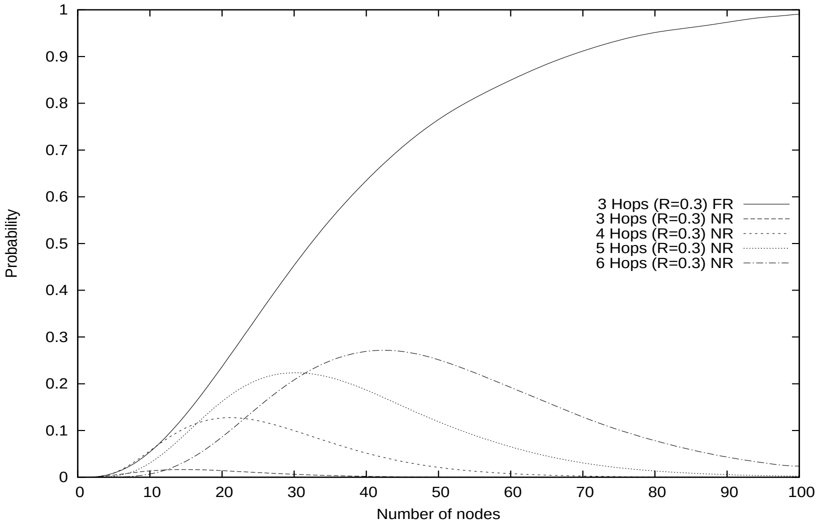

Figure 3.8: Probability that the hop count equals $K = 3$ for the FR protocol, versus the probability that the hop count equals $K = 3, 4, 5, 6$ for the NR protocol.

In Figure 3.9 we present the expected hop progress for the FR and NR protocols, for a multihop path with 2 hops ($R = 0.4$). These results were obtained numerically from the density location of relay nodes given by (3.18) and (3.15), respectively. We observe that for the FR protocol the expected hop progress increases with the number of nodes and approaches its maximum value 0.4 as the number of nodes increases, whereas for the NR protocol the expected hop progress decreases as the number of nodes increases. For the expected hop distance similar figures (not displayed) and similar conclusions were obtained.

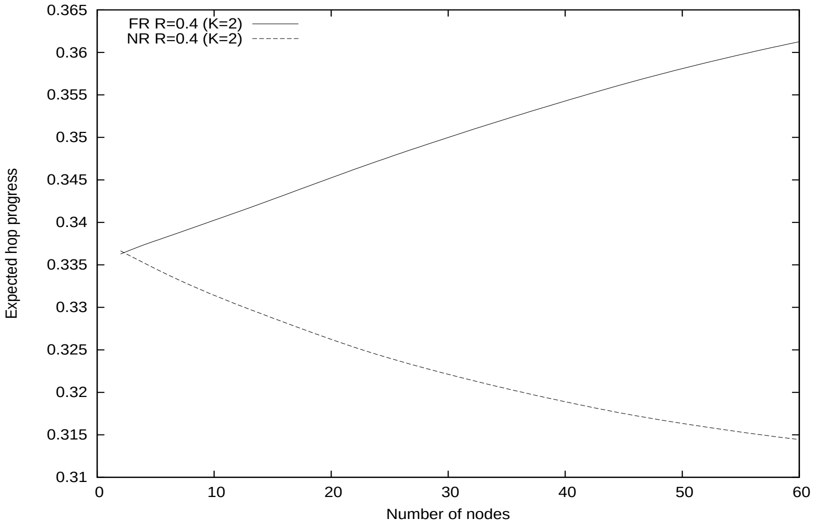

Figure 3.9: Expected hop progress for the FR and the NR protocols as a function of the number of nodes.

3.6 Auxiliary lemmas

In this section we derive the area of the *vacant region* of relay node i for the FR protocol, denoted by V_i^{FR}, and the *admissible propagation region* $\widehat{\mathcal{A}}_i$ of relay node i. Recall that the vacant region of relay node i is given by the set of points of the routing region of relay node i that are further away from relay node i than relay node $i+1$ is. Moreover, the admissible propagation region of relay node i is defined as the possible relative locations of relay node $i+1$, (r_{i+1}, θ_{i+1}), given the location of relay node i, that is, is the set of points of $\mathcal{A}_i$ that are not intersected by $\mathcal{A}_{i-1}$.

As before, we denote by $B(x,a)$ a circumference with center at x and radius a. In the following lemma we derive the coordinates of the intersection points between the circumferences $B(X_i, r)$ and $B(X_{i-1}, R)$, for $X_i \in B(X_{i-1}, R)$ and $r < R$. This lemma will provide the necessary results to prove the main result of this section, which enables us to derive the regions V_i^{FR} and $\widehat{\mathcal{A}}_i$ for each relay node i of the multihop path.

Lemma 3.1. *For $r < R$ and $X_i \in B(X_{i-1}, R)$, the polar coordinates relative to X_i of the intersection points between $B(X_i, r)$ and $B(X_{i-1}, R)$, are given by (r, g_θ^+) and (r, g_θ^-), where*

$$g_\theta^\pm \equiv g_\theta^\pm(r, r_i, \theta_i, d_i) = \theta_i - \gamma_i \pm \arccos\left(\frac{R^2 - r_i^2 - r^2}{2r_i r}\right), \tag{3.20}$$

with $\gamma_i = -\arcsin\left(\frac{r_i \sin\theta_i}{d_i}\right)$.

In the counterpart, the distance from X_i to the point of $B(X_{i-1}, R)$ that is at an angle θ from X_i is given by the function

$$g_r \equiv g_r(\theta, r_i, \theta_i, d_i) = -r_i \cos(\theta + \gamma_i - \theta_i) + \sqrt{R^2 - (r_i \sin(\theta + \gamma_i - \theta_i))^2}. \tag{3.21}$$

Proof. Relative to a cartesian coordinate system (x,y) with origin at X_{i-1}, and horizontal axis given by the line drawn from X_{i-1} to X_{m+1}, the equation of the circumference $B(X_{i-1}, R)$ is

$$x^2 + y^2 = R^2. \tag{3.22}$$

We rewrite equation (3.22) relative to relay node i, by making a translation and a rotation of the plane (x,y) in order to locate the origin of the new plane at X_i and the horizontal axis being the line drawn from X_i to X_{m+1}. The translation and rotation is obtained by the following transformation of the coordinate system (x,y) into $(\widehat{x}, \widehat{y})$ (see, e.g., Polyanin and Manzhirov, 2006, p. 80, eq. 4.1.2.3),

$$\begin{cases} x = \widehat{x}\cos\gamma_i - \widehat{y}\sin\gamma_i + r_i\cos\theta_i \\ y = \widehat{x}\sin\gamma_i + \widehat{y}\cos\gamma_i + r_i\sin\theta_i \end{cases}$$

where the new plane origin (x_i, y_i) (that corresponds to the cartesian coordinates of relay node i) is written by its polar coordinates relative to relay node $i-1$, $(r_i \cos\theta_i, r_i \sin\theta_i)$, and the angle of rotation (in the interval $(-\pi/2, \pi/2)$) is given by γ_i

$$\gamma_i = -\arcsin\left(\frac{r_i \sin\theta_i}{d_i}\right),$$

where (r_i, θ_i) are the polar coordinates of relay node i relative to relay node $i-1$.

Transforming equation (3.22) into the new coordinate system $(\widehat{x}, \widehat{y})$, we obtain

$$(\widehat{x}\cos\gamma_i - \widehat{y}\sin\gamma_i + r_i\cos\theta_i)^2 + (\widehat{x}\sin\gamma_i + \widehat{y}\cos\gamma_i + r_i\sin\theta_i)^2 = R^2 \tag{3.23}$$

and, by transforming into polar coordinates $(\widehat{x}, \widehat{y}) = (\widehat{r}\cos\widehat{\theta}, \widehat{r}\sin\widehat{\theta})$, after some algebra, equation (3.23), that describes the points of the circumference $B(X_{i-1}, R)$, can be written in polar coordinates in the new coordinate system and relative to the location of relay i by

$$\widehat{r}^2 + r_i^2 + 2r_i\widehat{r}\cos\left(\widehat{\theta} + \gamma_i - \theta_i\right) = R^2. \tag{3.24}$$

Solving (3.24) in order to $\widehat{r}$ we obtain the distance from X_i to the point of $B(X_{i-1}, R)$ that is at an angle $\widehat{\theta}$ from X_i,

$$g_r(\widehat{\theta}, r_i, \theta_i, d_i) = -r_i\cos(\widehat{\theta} + \gamma_i - \theta_i) + \sqrt{R^2 - (r_i\sin(\widehat{\theta} + \gamma_i - \theta_i))^2}.$$

For a given r, $0 < r < R$, the two intersection points between $B(X_{i-1}, R)$ and $B(X_i, r)$ are obtained by solving (3.24) in order to $\widehat{\theta}$. Then, relative to relay node i location and in the new coordinate system, the angles of the two points that are at a distance r from X_i are given by

$$g_\theta^\pm \equiv g_\theta^\pm(r, r_i, \theta_i, d_i) = \theta_i - \gamma_i \pm \arccos\left(\frac{R^2 - r_i^2 - r^2}{2r_i r}\right). \tag{3.25}$$

where the solution of arccos is obtained in its principal value, that is in the interval $(0, \pi)$ (cf. Polyanin and Manzhirov, 2006, p. 32). In Figure 3.10 we can observe a representation of the intersection points that are at a distance r_{i+1} from relay node i.

❏

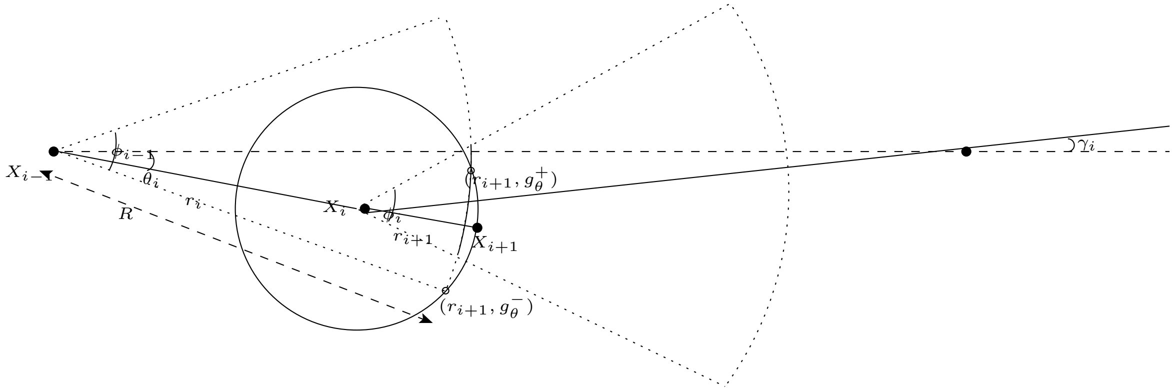

Figure 3.10: Intersection points, (r, g_θ^-) and (r, g_θ^+), between $B(X_{i-1}, R)$ and $B(X_i, r_{i+1})$.

In the next lemma we derive the vacant region V_i^{FR} and the admissible propagation region $\widehat{\mathcal{A}}_i$ for each relay node of the multihop path. We have to distinguish between two cases: when the vacant region is obtained just from $\widehat{\mathcal{A}}_i$, as represented in Figure 3.11(a), and when the vacant region is obtained from both $\widehat{\mathcal{A}}_i$ and $\mathcal{A}_{i-1}$, as represented in Figure 3.11(b).

Lemma 3.2. *Given that relay node i, located at X_i, is at a distance $d_i > R$ from the destination node, the set of possible locations (r, θ) of relay node $i+1$ relative to relay node i, is given by the admissible propagation region $\widehat{\mathcal{A}}_i$,*

$$\widehat{\mathcal{A}}_i \equiv \widehat{\mathcal{A}}_i(r_i, \theta_i, d_i) = \{(r, \theta) : r_{i+1}^{min} < r \leq R, \theta_{i+1}^{min} \leq \theta \leq \theta_{i+1}^{max}\}.$$

and for a given location (r_{i+1}, θ_{i+1}) of relay node $i+1$ relative to relay node i, the vacant region of relay node i is given by

$$\mathcal{V}_i^{FR} \equiv \mathcal{V}_i^{FR}(r_i, \theta_i, d_i, r_{i+1}, \theta_{i+1}) = \{(r, \theta) : r_{i+1} < r \leq R, \theta_{i+1}^{min} \leq \theta \leq \theta_{i+1}^{max}\}.$$

Here

$$r_{i+1}^{min} = \begin{cases} R - r_i & , sign(\theta_i)\phi_i^{sign(\theta_i)} \leq \theta_i - \gamma_i \\ g_r\left(\phi_i^{sign(\theta_i)}, r_i, \theta_i, d_i\right) & , sign(\theta_i)\phi_i^{sign(\theta_i)} > \theta_i - \gamma_i \end{cases} \tag{3.26}$$

when $\theta_{i-1} \neq 0$, and $r_{i+1}^{min} = R - r_i$ when $\theta_i = 0$, with $\theta_{i+1}^{min} = \max\left(g_\theta^-, -\phi_i^-\right)$, $\theta_{i+1}^{max} = \min\left(g_\theta^+, \phi_i^+\right)$, and $\gamma_i = -\arcsin\left(\frac{r_i \sin\theta_i}{d_i}\right)$. The functions $g_r\left(\theta, r_i, \theta_i, d_i\right)$ and $g_\theta^\pm$ are given in Lemma 3.1

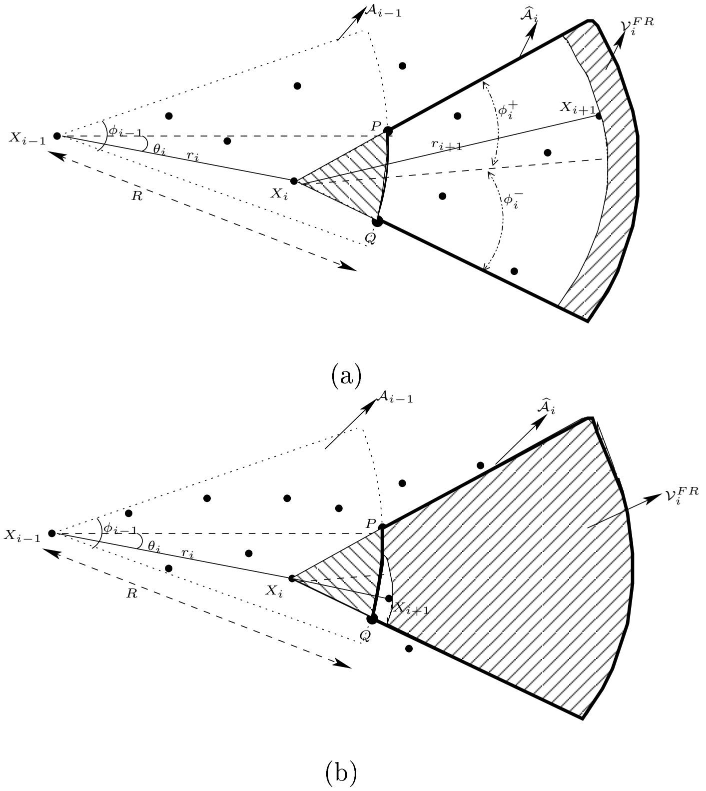

Figure 3.11: Admissible propagation region, $\widehat{\mathcal{A}}_i$, and vacant region, $\mathcal{V}_i^{FR}$, of relay node i, obtained from: (a) only $\widehat{\mathcal{A}}_i$; (b) both $\widehat{\mathcal{A}}_i$ and $\mathcal{A}_{i-1}$.

From the result above, we can obtain the area of the vacant region of relay node i, that is given by

$$V_i^{FR} \equiv V^{FR}(r_i, \theta_i, d_i, r_{i+1}, \theta_{i+1}) = \int_{r_{i+1}}^{R} \int_{\theta_{i+1}^{min}}^{\theta_{i+1}^{max}} r \, \mathrm{d}\theta \, \mathrm{d}r. \tag{3.27}$$

Proof. To distinguish between the two cases presented in Figure 3.11, we need to obtain the maximum and minimum distances from relay node i to the curve $\overline{PQ}$ (as shown in Figure 3.12), denoted by r_{i+1}^{max} and r_{i+1}^{min}, respectively. If the distance between relay node i and relay node $i+1$ is such that $r_{i+1} \geq r_{i+1}^{max}$, we are in case (a) of Figure 3.11, and if $r_{i+1}^{min} < r_{i+1} < r_{i+1}^{max}$ we are in case (b) of Figure 3.11.

If $\theta_i \leq 0$ (like in Figure 3.11), r_{i+1}^{max} is obtained by replacing θ by ϕ_i^+ in equation (3.21) of Lemma 3.1, and if $\theta_i > 0$, r_{i+1}^{max} is obtained by replacing θ by $-\phi_i^-$, that is

$$r_{i+1}^{max} = \begin{cases} g_r\left(\phi_i^+, r_i, \theta_i, d_i\right) & ,\theta_i \leq 0 \\ g_r\left(-\phi_i^-, r_i, \theta_i, d_i\right) & ,\theta_i > 0 \end{cases} . \tag{3.28}$$

Thus, for a given relative location of relay node $i+1$, (r_{i+1}, θ_{i+1}), such that $r_{i+1} > r_{i+1}^{max}$ (the case of Figure 3.11(a)), the vacant region of relay node i is given by

$$\mathcal{V}_i^{FR} \equiv \mathcal{V}_i^{FR}(r_i, \theta_i, d_i, r_{i+1}, \theta_{i+1}) = \{(r, \theta) : r_{i+1} < r \leq R, -\phi_i^- \leq \theta \leq \phi_i^+\}.$$

To derive the minimum distance from relay node i, first note that $\theta_i - \gamma_i$ is the relative angle between relay nodes $i-1$ and i after the plane rotation, where γ_i is the angle of the rotation (see Figure 3.12 and the proof of Lemma 3.1),

$$\gamma_i = -\arcsin\left(\frac{r_i \sin\theta_i}{d_i}\right).$$

By simple geometric arguments it can be shown that, case $\theta_i < 0$

$$r_{i+1}^{min} = \begin{cases} R - r_i & ,\phi_i^- \geq \gamma_i - \theta_i \\ g_r\left(\phi_i^-, r_i, \theta_i, d_i\right) & ,\phi_i^- < \gamma_i - \theta_i \end{cases} \tag{3.29}$$

and case $\theta_i > 0$,

$$r_{i+1}^{min} = \begin{cases} R - r_i & ,\phi_i^+ \leq \theta_i - \gamma_i \\ g_r\left(\phi_i^+, r_i, \theta_i, d_i\right) & ,\phi_i^+ > \theta_i - \gamma_i \end{cases} . \tag{3.30}$$

Putting all things together, we conclude that the minimum distance of relay node i to the curve $\overline{PQ}$ is given by

$$r_{i+1}^{min} = \begin{cases} R - r_i & ,sign(\theta_i)\phi_i^{sign(\theta_i)} \leq \theta_i - \gamma_i \\ g_r\left(\phi_i^{sign(\theta_i)}, r_i, \theta_i, d_i\right) & ,sign(\theta_i)\phi_i^{sign(\theta_i)} > \theta_i - \gamma_i \end{cases}$$

when $\theta_i \neq 0$, and given by $r_{i+1}^{min} = R - r_i$ when $\theta_i = 0$.

In Figure 3.12 we can observe a representation of the maximum and minimum distances from relay node i to $\overline{PQ}$.

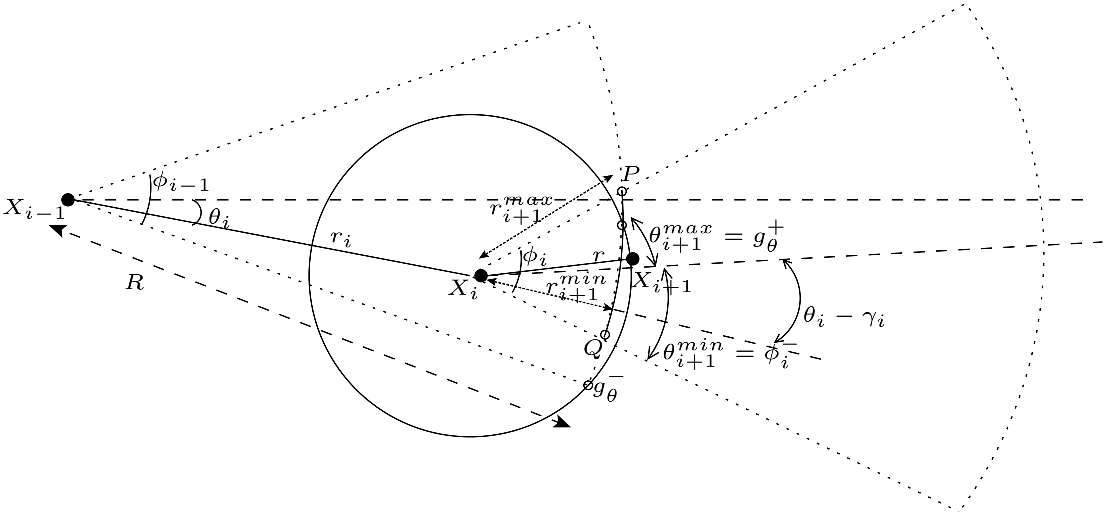

Figure 3.12: Representation of the parameters $r_{i+1}^{min}, r_{i+1}^{max}, \theta_{i+1}^{min}, \theta_{i+1}^{max}$.

To completely describe the regions $\mathcal{V}_i^{FR}$ and $\widehat{\mathcal{A}}_i$ for each possible value of r such that $r_{i+1}^{min} < r < r_{i+1}^{max}$, we just need to find the possible relative angles. In Lemma 3.1, we derive the coordinates of the intersection points between the circumferences $B(X_{i-1}, R)$ and $B(X_i, r)$, for $r < R$, which are given by (r, g_θ^+) and (r, g_θ^-), where $g_\theta^\pm$ are given by (3.25). Since we are only interested in the set of points located inside the routing region of relay node i, the angle between the consecutive relay nodes as to be restricted to the interval $(-\phi_i^-, \phi_i^+)$.

Consider first the case $\theta_i \leq 0$. If the solution g_θ^- given by equation (3.25) is smaller than $-\phi_i^-$ the point $(r, g_\theta^-) \notin \mathcal{A}_i$, being the minimum relative angle to relay node i given by $\theta_{i+1}^{min} = -\phi_i^-$ (see Figure 3.12), otherwise $\theta_{i+1}^{min} = g_\theta^-$. On the other hand, since $r_{i+1}^{min} < r < r_{i+1}^{max}$ and $\theta_i \leq 0$, the other solution of (3.25), g_θ^+, is such that $g_\theta^+ < \phi_i^+$, being the maximum relative angle given by $\theta_{i+1}^{max} = g_\theta^+$. Then, case $\theta_i \leq 0$, we have

$$\widehat{\mathcal{A}}_i = \Big\{(r, \theta) : r_{i+1}^{min} < r \leq r_{i+1}^{max}, \max\left(g_\theta^-, -\phi_i^-\right) \leq \theta \leq g_\theta^+\Big\}.$$

Then, for a given relative location of relay node $i+1$, (r_{i+1}, θ_{i+1}), such that $r_{i+1}^{min} <$

$r_{i+1} \leq r_{i+1}^{max}$, the vacant region of relay node i is given by

$$\mathcal{V}_i^{FR} = \{(r,\theta) : r_{i+1} < r \leq R,\ \max\left(g_\theta^-, -\phi_i^-\right) \leq \theta \leq g_\theta^+\}.$$

The arguments for the case $\theta_i > 0$ are analogous

$$\widehat{\mathcal{A}}_i = \left\{(r,\theta) :\ r_{i+1}^{min} < r \leq r_{i+1}^{max},\ g_\theta^- \leq \theta \leq \min\left(g_\theta^+, \phi_i^+\right)\right\}$$

and, for a given relative location of relay node $i+1$, (r_{i+1}, θ_{i+1}), such that $r_{i+1}^{min} < r_{i+1} \leq r_{i+1}^{max}$, the vacant region of relay node i is given

$$\mathcal{V}_i^{FR} = \{(r,\theta) : r_{i+1} < r \leq R,\ g_\theta^- \leq \theta \leq \min\left(g_\theta^+, \phi_i^+\right)\}.$$

Putting all things together, the possible locations of relay $i+1$ are given by the set

$$\widehat{\mathcal{A}}_i = \left\{(r,\theta) :\ r_{i+1}^{min} < r \leq R,\ \theta_{i+1}^{min} \leq \theta \leq \theta_{i+1}^{max}\right\},$$

where $\theta_{i+1}^{min} = \max\left(g_\theta^-, -\phi_i^-\right)$ and $\theta_{i+1}^{max} = \min\left(g_\theta^+, \phi_i^+\right)$. Note that, for the source node, since $(r_0, \theta_0) = (0,0)$, $\theta_1^{min} = -\frac{\phi_0}{2}$ and $\theta_1^{max} = \frac{\phi_0}{2}$, we have $\widehat{\mathcal{A}}_0 = \mathcal{A}_0$.

Given the location of relay node $i+1$, (r_{i+1}, θ_{i+1}), the vacant region of relay node i is given by

$$\mathcal{V}_i^{FR} \equiv \mathcal{V}_i^{FR}(r_i, \theta_i, d_i, r_{i+1}, \theta_{i+1}) = \{(r,\theta) : r_{i+1} < r \leq R, \theta_{i+1}^{min} \leq \theta \leq \theta_{i+1}^{max}\}.$$

❑

Chapter 4

Multihop path duration

4.1 Introduction

In the previous chapters we have studied the multihop connectivity between the source and the destination nodes, and have obtained the hop count distribution to compare and evaluate the performance of routing protocols. After a multihop path is built, its evolution highly determines the performance of the network.

Modeling the random movement of nodes in a multihop path plays an important role in examining the statistical properties of link and path reliability. In the general case, the dynamic of a multihop path in an ad hoc network requires a formulation of the geometric relations governing the complex dynamic of random movement of the nodes along the multihop path, with the state of its links limited by the transmission range of each relay node. In this chapter we propose a model for the multihop path dynamics. To evaluate the performance of the model, we derive two path metrics: the mean path duration and the path persistence - defined as the probability that the path is continuously in existence until time t_0 provided the path is set-up (or already active) at time 0.

One of the earliest analysis of mobility was done by McDonald and Znati (1999), who addressed the link and path availability, assuming independent links and that nodes move according to a variant of the random walk mobility model. The mobility model proposed models the evolution of the network topology. By characterizing the random movement of mobile nodes, analytical expressions are derived for the link availability (the probability that a link between two nodes is available at time t, given that the link exists at time t_0, $t_0 < t$). However, this analysis does not consider that disruptions in one of the links could exist between times t_0 and t.

Samar and Wicker (2004) investigated the behavior of the communication links of a node in a random mobility environment and derived analytical expressions to characterize the total link duration and link residual time (the link residual time and link duration are lengths of time of the remaining and the total duration of paths, respectively). They assumed a straight-line mobility such that nodes do not change their mobility behavior (direction and velocity) until the path is broken.

Xu et al. (2007) used a Markov chain mobility model to derive several analytical path metrics, under a straight-line mobility rather than a random direction mobility model. Assuming that links are independent, they derived the link persistence, link availability, link residual time, and link duration.

Due to the complexity involved, a common approach to study the properties of path durations in mobile ad hoc networks is using simulation. One of the first complete studies concerning the analysis of path durations under different mobility models and routing protocols based on simulations was done by Bai et al. (2004). These authors concluded that, for moderate and high velocities, the density function of the path duration for paths with 4 or more links can be approximated by an exponential distribution. They also concluded that the path duration is a good metric for the performance evaluation of routing protocols, and observed a relationship between the duration of the multihop paths and the mobility parameters.

Han et al. (2006) used Palm theory to prove analytically what Bai et al. (2004) had concluded by simulation. They proved that, when the link count is large, the distribution of path duration converges to an exponential distribution. Assuming that links are mutually independent, they provided a solution for the analysis of paths which is valid for routes with a large number of hops. However, the importance of short hop paths is reinforced by the fact that in order to reduce the effects of wireless retransmissions on the performance of the network, most of the relevant protocols in MANETs use the minimum hop count as the metric to select a route and, thus, the hop count in MANETs is small. La and Han (2007) relaxed the independent link assumption assumed by Han et al. (2006), proving that the dependence between links goes away asymptotically with increasing link count.

We propose an analytical framework to fully describe the random behavior of a multihop path in ad hoc networks and to obtain path based metrics for computing its reliability. This framework models the dynamic of a multihop path with any number of nodes, and takes into account the dependencies of the durations of the links of a multihop path. The path is characterized through a Piecewise Deterministic Markov Process (PDMP, see Davis (1993)) where, for simplicity, the mobility of each node along the path is given by the random walk model. The PDMP is known as the most general class of continuous-time Markov processes which includes both discrete and continuous processes, except diffusion processes. A PDMP is a Markov process that follows deterministic trajectories between event times that can be random (as for example in a Poisson-like fashion) or fixed (when the process hits the boundary of its state space). Other mobility models (c.f. Camp et al. (2002)) admit a PDMP description of a multihop path, such as group mobility models where, instead of independent mobility, mobile nodes have correlated mobility.

The model is characterized by a vector of phase states governed by an alternating Markov renewal process, and by a vector of phase attributes. The phase attributes of each mobile node describe its movement, mainly: its velocity, its direction, and the sojourn time in the current state. To completely characterize a PDMP, we need to describe the jump rate, the transition measure, and the flow of the process. The jump rate function is a measurable function on the state space of the process that describes the transition rate from each state of the process; the transition measure describes the transition probability between each pair of states of the process; and the flow of the process is a locally Lipschitz continuous function that describes the deterministic motion of the process between random jumps. For a detailed description of the characteristics of a PDMP, see Davis (1993, Section 24.8).

Using the PDMP model, the mean path duration and the path persistence are derived. We establish that these path metrics are obtained as functionals of the underlying process and are the unique solution of a set of integro-differential equations. Since direct methods to solve them are problematic, we introduce a recursive method by which numerical solutions of the metrics can be obtained. This is accomplished by transforming the set of integro-differential equations into a system of first order ordinary differential equations, that solved recursively converge to the desired path metrics. Finally, we apply our framework to compute numerical results for the metrics and compare them with those presented by Han et al. (2006) assuming independent links. Our work creatively applies the power of the PDMP formalism to study the reliability of multihop paths in ad hoc networks.

This chapter is organized as follows. In Section 4.2 we describe the multihop path model under a PDMP, describing the jump rate function, the transition measure, and the deterministic behavior of the process. Section 4.3 shows that the mean path duration and path persistence are the unique solution of a set of integro-differential equations and can be obtained as expectations of functionals of the PDMP. Section 4.4 gives a recursive method to apply in the computation of the path metrics. Numerical results are presented in Section 4.5 to show the adequacy of the recursive method and the effect of the mobility and connectivity parameters in the metrics.

4.2 Multihop path

We consider that a multihop path is set-up (or already active) at time 0 with $N-1$ links and extends from node 1 along nodes $2, 3, \ldots$, until it reaches node N. Each node in the path moves across the plane independently of other nodes according to a variation of the random walk mobility model (see, e.g., Camp et al. 2002) next described.

4.2.1 Random walk mobility model

A node alternates between two phases, *pause* (0) and *move* (1), with the phase process being an alternating Markov renewal process. If at a transition instant a node goes into phase i, the amount of time it stays in phase i is drawn independently of the past according to a continuous distribution function F_i with support on the set $\mathbb{R}^+$. We assume that the hazard rate function of F_i, denoted by $\lambda_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}F_i(t)/(1-F_i(t))$, is bounded on the positive reals. When the phase of a node changes to move, the node picks a mobility vector according to a distribution function H_M on an open set S_M. Choosing a mobility vector m corresponds to choosing independently a direction θ and a velocity v through $m = (v\cos\theta, v\sin\theta)$. The node travels from the current location in the direction and with the velocity drawn from the mobility vector during the entire phase duration, with distribution F_1. Once this time expires, independently of the past, the node pauses for a random time period with distribution F_0 before starting to move again.

4.2.2 Link duration

We consider a transmission range R equal for all nodes in the multihop path. Given two consecutive nodes in the path, $i-1$ and i with locations in the plane l^{i-1} and l^i, respectively, they can communicate if $\|l^{i-1} - l^i\| < R$.

In cellular networks, the characterization of the handoff metrics is based on the analysis of the movement of a node with respect to a fixed base station (Hong and Rappaport 1986).

In ad hoc networks, the link duration can be transformed into the handoff problem by considering the relative movement between the two nodes. Let p^j denote the phase of node j and m^j its mobility vector if $p^j = 1$ (i.e. the node is in the move phase). The relative location and relative mobility vector of node i with respect to node $i-1$ are defined, respectively, by

$$l_r^i = l^i - l^{i-1}, \qquad m_r^i = m^i - m^{i-1}$$

where m^i (resp. m^{i-1}) is omitted in the expression if $p^i = 0$ (resp. $p^{i-1} = 0$); and, if both nodes are in pause phases, $m_r^i = \mathbf{0}$ with $\mathbf{0} = (0,0)$. Let $x \to (\|x\|, \theta(x))$ denote the one-to-one correspondence between the cartesian coordinates in the plane and the polar coordinates on $\{\mathbf{0}\} \cup \mathbb{R}^+ \times [0, 2\pi)$ with $\mathbf{0}$ being the polar coordinates of the cartesian origin. From the relative mobility vector m_r^i, the direction and velocity of node i with respect to node $i-1$ are, respectively, $\theta(m_r^i)$ and $\|m_r^i\|$. In Figure 4.1, we show the relative motion of node i, within the transmission region $S_L = \{x \in \mathbb{R}^2 : \|x\| < R\}$, with respect to node $i-1$. After traveling a distance

$$Z(l_r^i, m_r^i) = \sqrt{R^2 - (\|l_r^i\| \sin\theta')^2} - \|l_r^i\| \cos\theta'$$

where $\theta' = |\theta(m_r^i) - \theta(l_r^i)|$, node i moves out of the range of node $i-1$. The duration of the link i is

$$d_{link}(l_r^i, m_r^i) = Z(l_r^i, m_r^i)/\|m_r^i\| \tag{4.1}$$

for $m_r^i \neq \mathbf{0}$. Case $m_r^i = \mathbf{0}$, the duration of the link is infinity and we set $d_{link}(l_r^i, m_r^i) = \infty$.

4.2.3 Multihop path model

To characterize the multihop path as a PDMP we need to incorporate, for each node in the path: ‘the phase’, ‘the elapsed time since the previous phase transition’, ‘the mobility vector’, and ‘the relative location with respect to the previous node’. Thus, we obtain a process

$$\mathbf{X} = (\mathbf{P}, \mathbf{A})$$

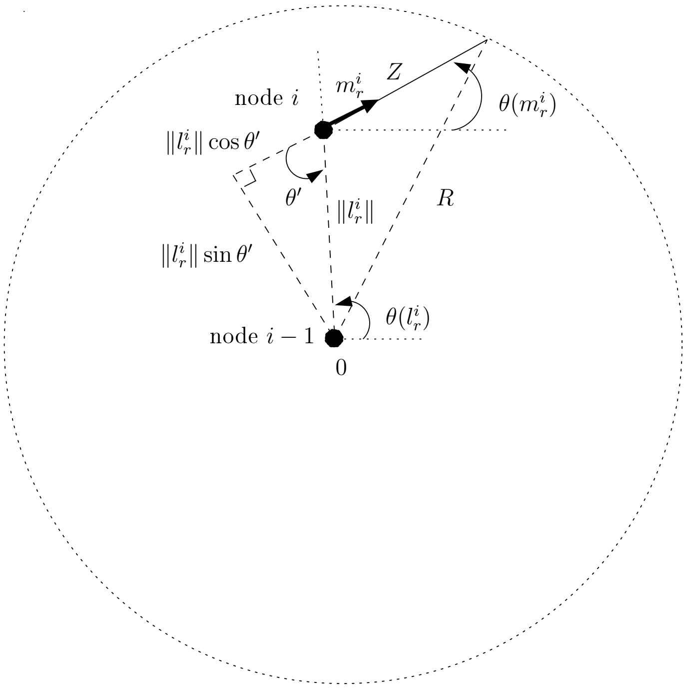

Figure 4.1: Link duration between two nodes with respect to the relative motion.

with $\mathbf{P} = (P^i)_{1\leq i\leq N}$, where P^i denotes the phase process of node i, and

$$\mathbf{A} = (\mathbf{E}, \mathbf{M}, \mathbf{L}_r)$$

denotes the joint attribute process where $\mathbf{E} = (E^i)_{1\leq i\leq N}$, $\mathbf{M} = (M^i)_{1\leq i\leq N}$ and $\mathbf{L}_r = (L_r^i)_{2\leq i\leq N}$ are defined as follows:

- The process E^i gives the elapsed time since the previous phase transition of node i.
- The process M^i is the mobility vector process of node i, such that $M^i(t)$ is the mobility vector of node i at time t if $P^i(t) = 1$ and is omitted if $P^i(t) = 0$.
- The process L_r^i is the relative location process of node i with respect to node $i-1$.

The process $\mathbf{X}$ is defined as having a completely deterministic motion between the random jumps of the process. To completely characterize a PDMP we need to describe its local characteristics: the *flow* ϕ, the *jump rate* function $\boldsymbol{\lambda}$, and the *transition measure* $\mathcal{Q}$.

Deterministic motion and state space

Constructing a PDMP as a path model requires a description of the deterministic trajectory of the process between random jumps along with the explicit definition of the boundary of its state space where jumps occur.

From the definition of $\mathbf{X}$, a state will be denoted by $\mathbf{x} = (\mathbf{p}, \mathbf{a})$, with vector of attributes $\mathbf{a} = (\mathbf{e}, \mathbf{m}, \mathbf{l}_r)$ and node phase vector $\mathbf{p} = (p^1, \ldots, p^N)$, with $\mathbf{e} = (e^1, \ldots, e^N)$ being the elapsed times of the N nodes in the current phases, $\mathbf{m} = (m^1, \ldots, m^N)$ including the mobility vectors of the nodes (having dimension N when all $p^j = 1$ and with m^j omitted if $p^j = 0$), and $\mathbf{l}_r = (l_r^2, \ldots, l_r^N)$ the relative locations of nodes $2, 3, \ldots, N$ with respect to nodes $1, 2, \ldots, N-1$, respectively.

From a state $\mathbf{x}$, the deterministic trajectory of $\mathbf{X}$ until the next jump time is characterized by $\phi(t, \mathbf{x}) = (\mathbf{p}, \phi_{\mathbf{p}}(t, \mathbf{a}))$ with

$$\phi_{\mathbf{p}}(t, \mathbf{a}) = (\mathbf{e} + t\mathbf{1}, \mathbf{m}, \mathbf{l}_r + t\mathbf{m}_r), \quad t \in \mathbb{R}$$

denoting the evolution of the vector of attributes $\mathbf{a}$ over time, where $\mathbf{1}$ denotes a vector of $1's$ with dimension N.

The function $\phi_{\mathbf{p}}(t, \mathbf{a})$ is the flow of a vector field $\mathcal{V}_{\mathbf{p}}$, defined on the open set

$$S_{\mathbf{p}} = (0, \infty)^N \times {S_M}^{\sum p_i} \times {S_L}^{N-1}$$

where we recall that S_M and S_L are open sets, whose time derivative at each point along the flow is the value of the vector field at that point, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_{\mathbf{p}}(t, \mathbf{a}) = \mathcal{V}_{\mathbf{p}}(\phi_{\mathbf{p}}(t, \mathbf{a})), \qquad \phi_{\mathbf{p}}(0, \mathbf{a}) = \mathbf{a}, \tag{4.2}$$

where $\mathcal{V}_{\mathbf{p}}(\phi_{\mathbf{p}}(t, \mathbf{a}))$ is the tangent vector at point $\phi_{\mathbf{p}}(t, \mathbf{a})$ given by

$$\mathcal{V}_{\mathbf{p}}(\phi_{\mathbf{p}}(t, \mathbf{a})) = (\mathbf{1}, \mathbf{0}, \mathbf{m}_r)$$

where each of the vector coordinates $\mathbf{1}, \mathbf{0}$ and $\mathbf{m}_r$ are given by $\mathbf{1} = \left(\frac{\mathrm{d}e^1(t)}{\mathrm{d}t}, \dots, \frac{\mathrm{d}e^N(t)}{\mathrm{d}t}\right)$, $\mathbf{0} = \left(\frac{\mathrm{d}m^1(t)}{\mathrm{d}t}, \dots, \frac{\mathrm{d}m^N(t)}{\mathrm{d}t}\right)$ and $\mathbf{m}_r = \left(\frac{\mathrm{d}l^2(t)}{\mathrm{d}t}, \dots, \frac{\mathrm{d}l^N(t)}{\mathrm{d}t}\right)$. For each $(t, \mathbf{a}) \in \mathbf{S}_\mathbf{p}$, since $\mathcal{V}_\mathbf{p}$ is a Lipschitz continuous function, there is a unique flow $\phi_\mathbf{p}(t, \mathbf{a})$ for $\mathcal{V}_\mathbf{p}$ passing through $\mathbf{a}$ at time 0 and satisfying (4.2).

Let $\partial S_\mathbf{p}$ denote the boundary of the set $S_\mathbf{p}$. Now, let

$$\partial^+ S_\mathbf{p} = \{\mathbf{a}' \in \partial S_\mathbf{p} : \mathbf{a}' = \phi_\mathbf{p}(t, \mathbf{a}) \text{ for some } \mathbf{a} \in S_\mathbf{p}, t > 0\}$$

denote the set of boundary points at which the multihop path process exits from $S_\mathbf{p}$, and

$$\partial^- S_\mathbf{p} = \{\mathbf{a}' \in \partial S_\mathbf{p} : \mathbf{a}' = \phi_\mathbf{p}(-t, \mathbf{a}) \text{ for some } \mathbf{a} \in S_\mathbf{p}, t > 0\}$$

denote the set of boundary points that take the process into $S_\mathbf{p}$. The disjoint union of the sets $\partial^+ S_\mathbf{p}$ is

$$B = \coprod_{\mathbf{p} \in \{0,1\}^N} \partial^+ S_\mathbf{p} = \{(\mathbf{p}, \mathbf{a}) : \mathbf{p} \in \{0,1\}^N, \mathbf{a} \in \partial^+ S_\mathbf{p}\},$$

which represents the set of states through which the multihop path disconnects. When the process hits a state in the boundary B, it means that the path breaks and $\mathbf{X}$ jumps to an absorbing state which we denote by Δ. Therefore, the state space of $\mathbf{X}$ becomes

$$S_\mathbf{X}^\Delta = S_\mathbf{X} \cup \{\Delta\},$$

where $S_\mathbf{X}$ denotes the disjoint union of the sets $S_\mathbf{p}^- = S_\mathbf{p} \cup \partial^- S_\mathbf{p}$,

$$S_\mathbf{X} = \coprod_{\mathbf{p} \in \{0,1\}^N} \mathbf{S}_\mathbf{p}^- = \{(\mathbf{p}, \mathbf{a}) : \mathbf{p} \in \{0,1\}^N, \mathbf{a} \in S_\mathbf{p}^-\}.$$

For $\mathbf{x} \in S_\mathbf{X}$, define $d_{path}(\mathbf{x})$ as the path duration (i.e. the time to hit a state in B) constrained to no phase transitions of the nodes taking place when starting from state $\mathbf{x}$,

$$\begin{aligned} d_{path}(\mathbf{x}) &= \inf\{t > 0 : \phi_\mathbf{p}(t, \mathbf{a}) \in B\} \\ &= \inf\{d_{link}(l_r^i, m_r^i) : i = 2, \dots, N\}, \end{aligned}$$

and recall that $d_{link}(l_r^i, m_r^i)$ is given by (4.1). This time is equal to infinity if all nodes are in the pause phase or all nodes have the same mobility vector.

Jump rate

The function $\boldsymbol{\lambda} : S_{\mathbf{X}}^{\Delta} \to \mathbb{R}_0^+$ characterizes the jump rate in each state of the process. For $\mathbf{x} \in S_{\mathbf{X}}$, the jump rate depends only on the phase and the time since the previous phase transition of each node, and is given by the sum of the hazard rate functions of the phase duration distributions, i.e.,

$$\boldsymbol{\lambda}(\mathbf{x}) = \sum_{i=1}^{N} \lambda_{p^i}(e^i)$$

while at the absorbing state $\boldsymbol{\lambda}(\Delta) = 0$.

Transition measure

Before introducing the transition measure, first some words about notation. For a vector $\mathbf{y} = (y^1, \ldots, y^N)$, let $[\mathbf{y}]_z^j$ denotes a vector that differs from $\mathbf{y}$ only on the component j, taking the value z on that component (i.e., the j-th component of the vector $[\mathbf{y}]_z^j$ is equal to z and the other components are equal to y^i, $i \neq j$). In addition, we let $\mathbf{y}_{\setminus j}$ denote the vector obtained from $\mathbf{y}$ by omitting y^j and retaining y^i for all $i \neq j$. The interpretation of $\mathbf{P}$ and $\mathbf{A}$ makes it clear that from any $\mathbf{x} = (\mathbf{p}, \mathbf{a}) \in S_{\mathbf{X}}$ it is only possible to jump to a state where a node changes its phase characteristics (phase, elapsed time in the phase, and mobility vector) and all the other values of the components remain the same, i.e., for some j, we make the transition

$$\mathbf{x} = (\mathbf{p}, \mathbf{e}, \mathbf{m}, \mathbf{l}_r) \longrightarrow \mathbf{x}^{(j)} = \begin{cases} ([\mathbf{p}]_0^j, [\mathbf{e}]_0^j, \mathbf{m}_{\setminus j}, \mathbf{l}_r) & p^j = 1 \\ ([\mathbf{p}]_1^j, [\mathbf{e}]_1^j, [\mathbf{m}]_m^j, \mathbf{l}_r) & p^j = 0 \end{cases}$$

with $m \in S_M$. The transition measure $Q : (S_{\mathbf{X}} \cup B) \times \mathcal{E} \to [0,1]$, with $\mathcal{E}$ denoting the event space of $S_{\mathbf{X}}$, is such that for $\mathbf{x} \in S_{\mathbf{X}}$, $Q(\mathbf{x}, \cdot)$ is a probability measure defined by

$$\begin{aligned} Q(\mathbf{x}, \{\mathbf{x}^{(j)}\}) &= \tfrac{\lambda_{p^j}(e^j)}{\boldsymbol{\lambda}(\mathbf{x})} & p^j = 1 \\ Q(\mathbf{x}, \mathbf{dx}^{(j)}) &= \tfrac{\lambda_{p^j}(e^j)}{\boldsymbol{\lambda}(\mathbf{x})} H_m(\mathrm{d}m) & p^j = 0 \end{aligned} \tag{4.3}$$

and for $\mathbf{x} \in B$ we have $Q(\mathbf{x}, \{\Delta\}) = 1$, where $\mathbf{dx}^{(j)} = \{\mathbf{p}\} \times \{\mathbf{e}\} \times \mathbf{d}^j\mathbf{m} \times \{\mathbf{l}_r\}$ and $\mathbf{d}^j\mathbf{m} = \{m^1\} \times \ldots \times \{\,\mathrm{d}m^j\} \times \ldots \times \{m^N\}$.

Motion of the multihop path process

Putting all things introduced above together, the evolution of $\mathbf{X}$ starting from state $\mathbf{x} \in S_{\mathbf{X}}$ can be constructed as follows. The survival function of the first jump time of the multihop path model, T_1, is defined by

$$G_{\mathbf{x}}(t) = \begin{cases} \exp\left(-\int_0^t \boldsymbol{\lambda}(\phi(s,\mathbf{x}))\mathrm{d}s\right) & t < d_{\text{path}}(\mathbf{x}) \\ 0 & t \geq d_{\text{path}}(\mathbf{x}) \end{cases} \tag{4.4}$$

and the state at an instant of time before the first jump is

$$\mathbf{X}(t) = \phi(t, \mathbf{x}), \qquad t < T_1.$$

If $T_1 < d_{\text{path}}(\mathbf{x})$, then one of the nodes in the path changes phase and mobility attributes at time T_1, and the next state of the multihop process, $\mathbf{X}(T_1)$, has distribution

$$Q(\phi(T_1, \mathbf{x}), \cdot)$$

given by (4.3). Otherwise, $T_1 = d_{\text{path}}(\mathbf{x})$ which means that the path breaks since the process hits a state in B and the next state, $\mathbf{X}(T_1)$, is Δ with probability 1; the process then stays in Δ forever since the jump rate out of Δ is zero. The process restarts from $\mathbf{X}(T_1)$ in a similar way if $T_1 < d_{\text{path}}(\mathbf{x})$, with survival function of the next inter jump time $T_2 - T_1$ given by $G_{\mathbf{X}(T_1)}$, and so on ....

We assume that at time $t = 0$ the state of a multihop path with N nodes is drawn according to some specified initial distribution; see Section 4.5 for an initialization rule.

Note that, as we have described a variant of the random walk mobility model, the transition measure $\mathcal{Q}$ and the jump rate function $\boldsymbol{\lambda}$ only depend on the vector phase $\mathbf{p}$ and the vector of the elapsed times $\mathbf{e}$. By adding a little more complexity on the notation, $\mathcal{Q}$ and $\boldsymbol{\lambda}$ could depend on the whole state vector $\mathbf{x}$, making possible to describe other mobility models, like group mobility models, where the mobility of the nodes are correlated. Thus, the framework proposed here has a huge potential in the description of a wide variety of mobility models used in mobile ad hoc networks, requiring only the adjustment of the transition measure $\mathcal{Q}$, the jump rate function $\boldsymbol{\lambda}$, and the deterministic motion ϕ.

Note also that, an inspection in the definition shows that technical problems will arise if mobility vectors can assume values from a closed set S_M. We can relax this assumption when the velocities or directions that describe the mobility vectors are discrete sets. This includes the case of one dimensional ad hoc network with directions on a straight line. All that needs to be done is to consider the discrete quantities (velocities or directions) in the first component of $\mathbf{X}$, along with the node phases, and to redefine the PDMP properly according to these changes. In the rest of the paper, we shall focus only on the assumptions defined in Subsection 4.2.1.

4.3 Path based metrics

In this section, we will express as expectations of functionals of the process $\mathbf{X}$ two main path metrics to characterize the reliability of paths: the mean path duration and the path persistence. Given that a path is set-up (or is already active) at time 0, the path duration refers to the amount of time the path remains available until one of its links fails for the first time. The path persistence at time t_0 is defined as the probability that the path duration is greater than t_0, provided the path is set-up (or already active) at time 0.

4.3.1 Mean path duration

Given the state of the multihop path process $\mathbf{x} \in S_{\mathbf{X}}^{\Delta}$, the mean path duration is denoted by

$$D(\mathbf{x}) = \mathbb{E}_{\mathbf{x}} \left[\int_0^{\infty} \mathbb{1}_{S_{\mathbf{X}}}(\mathbf{X}(s)) \mathrm{d}s \right] \tag{4.5}$$

where $\mathbb{1}_A$ is the indicator function of a set A.

Theorem 4.1. *The expected path duration $D(\mathbf{x})$ is a bounded function of $\mathbf{x} \in \mathbf{S}_{\mathbf{X}}$, and $\mathbb{E}_{\mathbf{x}}[D(\mathbf{X}(t))]$ goes to zero as t goes to infinity.*

Proof. For any multihop path process $(\mathbf{X}(t))$ with initial state $\mathbf{x} = (\mathbf{p}, \mathbf{a}) \in \mathbf{S_X}$, we have that

$$D(\mathbf{x}) \leq D(\mathbf{x}')$$

where $D(\mathbf{x}')$ denotes the expected path duration of a multihop path process $(\mathbf{X}'(t))$ with one hop whose initial state

$$\mathbf{x}' = (\mathbf{p}', \mathbf{a}')), \mathbf{p}' = (p^j, p^{j+1})$$

and

$$\mathbf{a}' = ((e^j, e^{j+1}), (m^j, m^{j+1}), l_r^{j+1}),$$

$j \in \{1, \ldots, n-1\}$ is given by the state of two consecutive nodes of $(\mathbf{X}(t))$ at time 0. If T_i', $i \geq 1$, are the jump times of the process $(\mathbf{X}'(t))$ then taking $T_0' = 0$ and $S_i' = T_{i+1}' - T_i'$, we can write

$$D(\mathbf{x}') = \mathbb{E}_{\mathbf{x}'}\left[\sum_{i=0}^{\infty} S_i' \mathbb{1}_{S_\mathbf{X}}(X'(T_i'))\right].$$

Given that at a jump time the path is not broken, the expected time between jumps is bounded by

$$\mathbb{E}_{\mathbf{x}'}[S_i' \mathbb{1}_{S_\mathbf{X}}(X'(T_i'))] \leq K \equiv 1/(2\min(\lambda_0^{\min}, \lambda_1^{\min}))$$

where we recall that we assume that the hazard rate functions of the times distributions in the phases are bounded such that $0 < \lambda_i^{\min} \leq \lambda_i(t) \leq \lambda_i^{\max}$. If at time T_{i-1}' the process is in a state $\mathbf{X}'(T_{i-1}')$ where the two nodes are in different phases then, $d_{path}(\mathbf{X}'(T_{i-1}'))$, the duration of the path constrained to no phase transitions of the nodes, is smaller or equal to $2R/v_{\min}$ (i.e., the maximum time it takes a node in move phase to cross the transmission range of a node in pause phase). Hence,

$$\mathbb{E}_{\mathbf{x}'}\left[\mathbb{1}_{S_\mathbf{X}}(\mathbf{X}'(T_i'))|\mathbf{X}'(T_{i-1}') \in S_{(0,1)}^- \cup S_{(1,0)}^-\right] \leq p \equiv 1 - \exp(-2R/v_{\min}(\lambda_0^{\max} + \lambda_1^{\max})).$$

Note that for a multihop path process $(\mathbf{X}'(t))$ with two nodes, the process alternates between states in which the two nodes are in different phases and states in which both nodes are in the same phase. Therefore, assuming that the path just breaks when the two nodes are in different phases, the probability that the path is alive after i jumps is bounded by

$$\mathbb{E}_{\mathbf{x}'}\left[\mathbb{1}_{S_{\mathbf{X}}}(X'(T_i'))\right] \leq p^{\lfloor i/2 \rfloor}$$

where $\lfloor i/2 \rfloor$ is the minimum number of jumps from a state where the two nodes are in different phases in i jumps of the process from any initial state $\mathbf{x}'$. Thus, putting all together, we have that

$$D(\mathbf{x}') \leq K \sum_{i=1}^{\infty} p^{\lfloor i/2 \rfloor} = \frac{K(1+p)}{1-p}$$

which proves the first statement of the theorem. From the above

$$\begin{aligned} \mathbb{E}_{\mathbf{x}}[D(\mathbf{X}(t))] &= \mathbb{E}_{\mathbf{x}}\left[D(\mathbf{X}(t))\mathbb{1}_{S_{\mathbf{X}}}(X'(t))\right] \\ &\leq D(\mathbf{x}')\mathbb{P}(X'(t) \in S_{\mathbf{X}}). \end{aligned}$$

Since $\lim_{t\to\infty} \mathbb{P}(X'(t) \in S_{\mathbf{X}}) = 0$, the theorem is proved. ❑

Let $f : S_{\mathbf{X}}^{\Delta} \cup B \to \mathbb{R}^+$ be a bounded measurable function and, for $\mathbf{x} \in B$, define $f(\mathbf{x}) \equiv \lim_{t\downarrow 0} f(\phi(-t,\mathbf{x}))$. Let $\mathcal{Q}$ be an operator mapping the set of bounded measurable functions from $S_{\mathbf{X}} \cup B$ into itself such that $\mathcal{Q}f$ is a function defined by

$$\mathcal{Q}f(\mathbf{x}) = \int_{S_{\mathbf{X}}^{\Delta}} f(\mathbf{y})Q(\mathbf{x},\mathbf{dy}) \tag{4.6}$$

and therefore for $\mathbf{x} \in S_{\mathbf{X}}$,

$$\mathcal{Q}f(\mathbf{x}) = \sum_{\{j:p^j=0\}} \int_{\{m:m\in S_M\}} Q(\mathbf{x},\mathbf{dx}^{(j)})f(\mathbf{x}^{(j)}) + \sum_{\{j:p^j=1\}} Q(\mathbf{x},\{\mathbf{x}^{(j)}\})f(\mathbf{x}^{(j)}),$$

and for $\mathbf{x} \in B$, $\mathcal{Q}f(\mathbf{x}) = f(\Delta)$.

For $\mathbf{x} \in S_{\mathbf{X}}$, the state of $\mathbf{X}$ after a short time t is, roughly speaking, $\phi(t,\mathbf{x})$ with probability $(1-\boldsymbol{\lambda}(\mathbf{x})t)$, while with probability $\boldsymbol{\lambda}(\mathbf{x})t$ the process jumps to another state $\mathbf{X}(t)$ with transition measure Q and all other events have probability $o(t)$. Thus, we have

$$\mathbb{E}_{\mathbf{x}}[f(\mathbf{X}(t))] = (1-\boldsymbol{\lambda}(\mathbf{x})t)f(\phi(t,\mathbf{x})) + \boldsymbol{\lambda}(\mathbf{x})t\mathcal{Q}f(\phi(t,\mathbf{x})) + o(t)$$

so that

$$\frac{1}{t}\mathbb{E}_{\mathbf{x}}[f(\mathbf{X}(t)) - f(\mathbf{x})] = \frac{1}{t}(f(\phi(t,\mathbf{x})) - f(\mathbf{x})) \qquad (4.7)$$
$$+ \boldsymbol{\lambda}(\mathbf{x})\left(\mathcal{Q}f(\phi(t,\mathbf{x})) - f(\phi(t,\mathbf{x}))\right) + o(1).$$

Let $\mathcal{A}$ denote an operator acting on the domain of bounded measurable real-valued functions on $S_{\mathbf{X}}^{\Delta} \cup B$ defined by

$$\mathcal{A}f(\mathbf{x}) = \lim_{t\to 0}\frac{1}{t}\mathbb{E}_{\mathbf{x}}[f(\mathbf{X}(t)) - f(\mathbf{x})].$$

In order to define the derivative of f with respect to the flow $\phi(t,\mathbf{x})$ in a rigorous way, we need to define its *phase function* by $f_{\mathbf{p}} : S_{\mathbf{p}}^{-} \to \mathbb{R}^{+}$ such that $f_{\mathbf{p}}(\mathbf{a}) = f(\mathbf{x})$. If for a state $\mathbf{x}$ the function $t \to f_{\mathbf{p}}(\phi_{\mathbf{p}}(t,\mathbf{a}))$ is differentiable almost everywhere on $[0, d_{path}(\mathbf{x})[$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{\mathbf{p}}(\phi_{\mathbf{p}}(t,\mathbf{a})) = \mathcal{V}_{\mathbf{p}} f_{\mathbf{p}}(\phi_{\mathbf{p}}(t,\mathbf{a})), \qquad \phi_{\mathbf{p}}(0,\mathbf{a}) = \mathbf{a} \qquad (4.8)$$

holds for almost all t, being equivalent to (4.2), and where $\mathcal{V}_{\mathbf{p}}$ is a vector field and $\phi_{\mathbf{p}}(t,\mathbf{a})$ is the unique integral curve of $\mathcal{V}_{\mathbf{p}}$ such that (4.8) is satisfied.

Therefore, making $t \to 0$ in equation (4.7), we obtain

$$\mathcal{A}f(\mathbf{x}) = \mathcal{V}f(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{x})(\mathcal{Q}f(\mathbf{x}) - f(\mathbf{x}))$$

for $\mathbf{x} \in S_{\mathbf{X}}$, where to simplify the notation we write $\mathcal{V}f(\mathbf{x})$ instead of the more accurate $\mathcal{V}_{\mathbf{p}} f_{\mathbf{p}}(\mathbf{a})$. Also, any reference to a function $t \to f(\phi(t,\mathbf{x}))$ should be read as $t \to f_{\mathbf{p}}(\phi_{\mathbf{p}}(t,\mathbf{a}))$.

The next result follows by theorems 32.2 and 32.10 of Davis (1993) conveniently applied to the expectation functional D of the PDMP $\mathbf{X}$ with finite time horizon and taking into account the specific boundary conditions.

Theorem 4.2. *For each* $\mathbf{x} \in S_{\mathbf{X}}$, $t \to D(\phi(t,\mathbf{x}))$ *is an absolutely continuous function on* $[0, d_{path}(\mathbf{x})[$ *and* D *is the unique bounded solution of the equations*

$$\mathcal{A}f(\mathbf{x}) = -1, \quad \mathbf{x} \in S_{\mathbf{X}}, \qquad (4.9)$$

and at a boundary state $\mathbf{x} \in B$*,* $f(\mathbf{x}) = f(\Delta) = 0$*.*

The result above states that, for $\mathbf{x} \in S_{\mathbf{X}}$, the function $t \to D(\phi(t, \mathbf{x}))$ is differentiable almost everywhere on $[0, d_{path}(\mathbf{x})[$ with derivative denoted by $\mathcal{V}D(\mathbf{x}')$ at $\mathbf{x}' = \phi(t, \mathbf{x})$, when it exists. The value of the derivative at such points is equal to

$$-1 - \boldsymbol{\lambda}(\mathbf{x}')(\mathcal{Q}D(\mathbf{x}') - D(\mathbf{x}')).$$

Since (4.9) includes an integral term, the equations are systems of integro-differential equations. Solving them provides a way of computing the mean path duration for different initial conditions.

Proof. By definition $D(\Delta) = 0$. We assume that $\mathbf{x} \in S_{\mathbf{X}}$ and $t \in]0, d_{path}(\mathbf{x})[$. By the strong Markov property, the function D satisfies

$$\begin{aligned} D(\mathbf{x}) &= \mathbb{E}_{\mathbf{x}}[(T_1 \wedge t) + D(\phi(T_1 \wedge t, \mathbf{x}))] \\ &= G_{\mathbf{x}}(t)(t + D(\phi(t, \mathbf{x}))) + \mathbb{E}_{\mathbf{x}}\left[(T_1 + \mathcal{Q}D(\phi(T_1, \mathbf{x})))\mathbb{1}_{]0,t]}(T_1)\right]. \end{aligned} \tag{4.10}$$

Using the density function of first phase transition time T_1 on $[0, t]$, given by $\lambda(\phi(s, \mathbf{x}))G_{\mathbf{x}}(s)$, the second term on the right hand side of (4.10) is

$$\begin{aligned} \mathbb{E}_{\mathbf{x}}\left[(T_1 + \mathcal{Q}D(\phi(T_1, \mathbf{x})))\mathbb{1}_{]0,t]}(T_1)\right] &= \int_0^t \lambda(\phi(s, \mathbf{x}))G_{\mathbf{x}}(s)\left(\int_0^s 1 \, \mathrm{d}v + \mathcal{Q}D(\phi(s, \mathbf{x}))\right) \mathrm{d}s \\ &= \int_0^t G_{\mathbf{x}}(s)(1 + \lambda(\phi(s, \mathbf{x}))\mathcal{Q}D(\phi(s, \mathbf{x}))) \, \mathrm{d}s - G_{\mathbf{x}}(t)t, \end{aligned}$$

where the last equality is obtained by changing the order of integration. Then (4.10) becomes

$$\begin{aligned} D(\phi(t, \mathbf{x})) =& G_{\mathbf{x}}(t)^{-1}\left(D(\mathbf{x}) - \int_0^t G_{\mathbf{x}}(s)\left(1 + \lambda(\phi(s, \mathbf{x}))\mathcal{Q}D(\phi(s, \mathbf{x}))\right) \mathrm{d}s\right) \qquad (4.11) \\ =& G_{\mathbf{x}}(t)^{-1}D(\mathbf{x}) - \int_0^t (1 + \lambda(\phi(s, \mathbf{x}))\mathcal{Q}D(\phi(s, \mathbf{x}))) \, \mathrm{d}s \\ & - \int_0^t G_{\mathbf{x}}(s)(1 + \lambda(\phi(s, \mathbf{x}))\mathcal{Q}D(\phi(s, \mathbf{x})))(G_{\mathbf{x}}(t)^{-1} - G_{\mathbf{x}}(s)^{-1}) \, \mathrm{d}s \end{aligned}$$

and, since

$$\int_s^t \lambda(\phi(v,\mathbf{x}))G_{\mathbf{x}}(v)^{-1}\,\mathrm{d}v = G_{\mathbf{x}}(t)^{-1} - G_{\mathbf{x}}(s)^{-1}, \qquad 0 \le s \le t, \tag{4.12}$$

we have

$$\begin{aligned}
D(\phi(t,\mathbf{x})) = &\left(\int_0^t \lambda(\phi(s,\mathbf{x}))G_{\mathbf{x}}(s)^{-1}\mathrm{d}s + 1\right) D(\mathbf{x}) - \int_0^t (1 + \lambda(\phi(s,\mathbf{x}))\mathcal{Q}D(\phi(s,\mathbf{x})))\,\mathrm{d}s \\
&- \int_0^t G_{\mathbf{x}}(s)(1 + \lambda(\phi(s,\mathbf{x}))\mathcal{Q}D(\phi(s,\mathbf{x}))) \int_s^t \lambda(\phi(v,\mathbf{x}))G_{\mathbf{x}}(v)^{-1}\,\mathrm{d}v\,\mathrm{d}s \\
=& D(\mathbf{x}) - \int_0^t (1 + \lambda(\phi(s,\mathbf{x}))\mathcal{Q}D(\phi(s,\mathbf{x})))\mathrm{d}s \\
&+ \int_0^t \lambda(\phi(v,\mathbf{x}))G_{\mathbf{x}}(v)^{-1} \left(D(\mathbf{x}) - \int_0^v G_{\mathbf{x}}(s)(1 + \lambda(\phi(s,\mathbf{x}))\mathcal{Q}D(\phi(s,\mathbf{x})))ds\right) \mathrm{d}v,
\end{aligned}$$

where the last equality is obtained by changing the order of integration. Then, by (4.11) we obtain

$$\begin{aligned}
D(\phi(t,\mathbf{x})) =& D(\mathbf{x}) - \int_0^t (1 + \lambda(\phi(s,\mathbf{x}))\mathcal{Q}D(\phi(s,\mathbf{x})))\mathrm{d}s \\
&+ \int_0^t \lambda(\phi(v,\mathbf{x}))G_{\mathbf{x}}(v)^{-1} \left(D(\mathbf{x}) - (D(\mathbf{x}) - G_{\mathbf{x}}(v)D(\phi(v,\mathbf{x})))\right) \mathrm{d}v \\
=& D(\mathbf{x}) - \int_0^t (1 + \lambda(\phi(s,\mathbf{x}))\mathcal{Q}D(\phi(s,\mathbf{x})))\mathrm{d}s + \int_0^t \lambda(\phi(s,\mathbf{x}))D(\phi(s,\mathbf{x}))\mathrm{d}s \\
=& D(\mathbf{x}) - \int_0^t (1 + \lambda(\phi(s,\mathbf{x}))\left(\mathcal{Q}D(\phi(s,\mathbf{x})) - D(\phi(s,\mathbf{x}))\right)\mathrm{d}s.
\end{aligned}$$

Thus, this last equation can be written as

$$D(\phi(t,\mathbf{x})) = D(\mathbf{x}) + \int_0^t g(s)\,\mathrm{d}s$$

where

$$g(s) = -1 - \boldsymbol{\lambda}(\phi(s,\mathbf{x}))(\mathcal{Q}D(\phi(s,\mathbf{x})) - D(\phi(s,\mathbf{x}))).$$

In Theorem 4.1 we have proven that D is a bounded function for all $\mathbf{x} \in S^{\Delta}_{\mathbf{X}}$ and $\mathbb{E}_{\mathbf{x}}[D(\mathbf{X}(t))]$ goes to zero as t tends to infinity. Then $\mathcal{Q}D$ is also bounded and $\int_0^t |g(s)|\,\mathrm{d}s$ is finite. Therefore, by the fundamental theorem of calculus $t \to D(\phi(t,\mathbf{x}))$ is absolutely continuous on $[0,t]$ with derivative $g(t)$ and equation (4.9) is satisfied. Now, let $\mathbf{x}' \in B$ and $t > 0$ such that the initial state is $\mathbf{x} = \phi(-t, \mathbf{x}')$. By the strong Markov property, we have

$$D(\mathbf{x}) = G_{\mathbf{x}}(t)(t + \mathcal{Q}D(\mathbf{x}')) + \int_0^t \boldsymbol{\lambda}(\phi(s,\mathbf{x}))G_{\mathbf{x}}(s)(s + \mathcal{Q}D(\phi(s,\mathbf{x})))\,\mathrm{d}s.$$

Thus, as $t \downarrow 0$ the function D possesses a limit as a boundary state is approached and $D(\mathbf{x}') = 0$.

Under the conditions that, for $\mathbf{x}{\in}S_{\mathbf{X}}$, $t \to f(\phi(t,\mathbf{x}))$ is an absolutely continuous function on $[0, d_{path}(\mathbf{x})[$ and f is a bounded function, from Davis (1993, Theorem 31.3, p. 83) conveniently applied, the process $(M^f(t))$ defined by

$$M^f(t) = f(\mathbf{X}(t)) - f(\mathbf{X}(0)) - \int_0^t \mathcal{A}f(\mathbf{X}(s))\mathrm{d}s, \qquad t \geq 0$$

is a martingale and $\mathcal{A}f(\mathbf{X}(s)) = \lim_{t\to 0} \frac{1}{t}\mathbb{E}_{\mathbf{X}(s)}[f(\mathbf{X}(s+t)) - f(\mathbf{X}(s)]$. Since $(M^f(t))$ is a martingale, then taking conditional expectation we have $\mathbb{E}_{\mathbf{x}}[M^f(t)] = M^f(0) = 0$, which implies that the Dynkin formula holds in the form

$$f(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}[f(\mathbf{X}(t))] - \mathbb{E}_{\mathbf{x}}\left[\int_0^t \mathcal{A}f(\mathbf{X}(s))\mathrm{d}s\right].$$

In view of (4.9) and $f(\Delta) = 0$, we have that $\mathcal{A}f(\mathbf{X}(s)) = -\mathbb{1}_{S_{\mathbf{X}}}(\mathbf{X}(s))$; therefore, if $\lim_{t\to\infty} \mathbb{E}_{\mathbf{x}}[f(\mathbf{X}(t))] = 0$, by the monotone converge theorem we get

$$f(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}\left[\int_0^\infty \mathbb{1}_{S_{\mathbf{X}}}(\mathbf{X}(s))\mathrm{d}s\right]$$

and the solution is unique. ❏

4.3.2 Path persistence

We can derive the probability that a path is continuously in existence until time $t_0 \in (0, t^*]$ (where $t^* > 0$ is fixed) provided that the path is alive at time 0 as the expectation of a functional of $\mathbf{X}$. For that, we need to include explicitly the time variation in the state of the PDMP $\mathbf{X}$. We denote the extended multihop path process by $\widehat{\mathbf{X}}$ with state space $S^{\Delta}_{\widehat{\mathbf{X}}} = S_{\widehat{\mathbf{X}}} \cup \{\Delta\}$ where

$$S_{\widehat{\mathbf{X}}} = S_{\mathbf{X}} \times (0, t^*],$$

and the set of boundary states is represented by

$$\widehat{B} = (B \times [0, t^*]) \cup \widehat{B}^0$$

with $\widehat{B}^0 = S_{\mathbf{X}} \times \{0\}$. From the definition of $\widehat{\mathbf{X}}$, a state in $S_{\widehat{\mathbf{X}}} \cup \widehat{B}$ is denoted by $\widehat{\mathbf{x}} = (\mathbf{x}, t_0)$ where t_0 is the time variation component and $\mathbf{x} = (\mathbf{p}, \mathbf{a})$ is defined as before. The deterministic evolution of the process from a state $\widehat{\mathbf{x}}$ is given by $\widehat{\phi}(t, \widehat{\mathbf{x}}) = (\mathbf{p}, \widehat{\phi}_{\mathbf{p}}(t, (\mathbf{a}, t_0)))$ with

$$\widehat{\phi}_{\mathbf{p}}(t, (\mathbf{a}, t_0)) = (\phi_{\mathbf{p}}(t, \mathbf{a}), t_0 - t), \qquad t \in \mathbb{R}.$$

Let $\widehat{d}_{path}(\widehat{\mathbf{x}})$ be the time to hit a state in $\widehat{B}$ constrained to no phase transition of nodes in the path when starting from state $\widehat{\mathbf{x}}$,

$$\begin{aligned}\widehat{d}_{\text{path}}(\widehat{\mathbf{x}}) &= \inf\{t > 0 : \widehat{\phi}_{\mathbf{p}}(t, (\mathbf{a}, t_0)) \in \partial^+ S_{\mathbf{p}} \times [0, t^*] \cup S^-_{\mathbf{p}} \times \{0\}\} \\ &= \inf\{d_{\text{path}}(\mathbf{x}), t_0\}.\end{aligned}$$

When the extended multihop path process hits the boundary $\widehat{B}$, it means that the path breaks or the time variation component has reached zero, and $\widehat{\mathbf{X}}$ jumps to the absorbing state Δ. The remaining characteristics of the PDMP $\widehat{\mathbf{X}}$ are the jump rate $\widehat{\boldsymbol{\lambda}} : S^{\Delta}_{\widehat{\mathbf{X}}} \to \mathbb{R}^+$ defined by

$$\widehat{\boldsymbol{\lambda}}(\widehat{\mathbf{x}}) = \boldsymbol{\lambda}(\mathbf{x}), \qquad \widehat{\mathbf{x}} \in S_{\widehat{\mathbf{X}}}$$

and $\widehat{\boldsymbol{\lambda}}(\Delta) = 0$; the transition measure $\widehat{Q} : (S_{\widehat{\mathbf{X}}} \cup \widehat{B}) \times \widehat{\mathcal{E}} \to [0, 1]$, where $\widehat{\mathcal{E}}$ denotes the event space of $S_{\widehat{\mathbf{X}}}$, is such that for $\widehat{\mathbf{x}} \in S_{\widehat{\mathbf{X}}}$ we have

$$\widehat{Q}(\widehat{\mathbf{x}}, \mathbf{dx}^{(j)} \times \{t_0\}) = Q(\mathbf{x}, \mathbf{dx}^{(j)})$$

where Q is defined in (4.3), and $\widehat{Q}(\widehat{\mathbf{x}}, \{\Delta\}) = 1$ for $\widehat{\mathbf{x}} \in \widehat{B}$. The extended process $\widehat{\mathbf{X}}$ has the same behavior as $\mathbf{X}$ until it hits a state in boundary $\widehat{B}$. Starting from state $\widehat{\mathbf{x}} \in S_{\widehat{\mathbf{X}}}$, the survival function of the first jump time $\widehat{T}_1$ of the process $\widehat{\mathbf{X}}$ is given by

$$\widehat{G}_{\widehat{\mathbf{x}}}(t) = \mathbb{1}_{[0,\widehat{d}_{\text{path}}(\widehat{\mathbf{x}}))}(t) G_{\mathbf{x}}(t), \qquad t \geq 0, \tag{4.13}$$

and

$$\widehat{\mathbf{X}}(t) = \widehat{\phi}(t, \widehat{\mathbf{x}}), \qquad t < \widehat{T}_1.$$

The next state $\widehat{\mathbf{X}}(\widehat{T}_1)$ has distribution $\widehat{Q}(\widehat{\phi}(\widehat{T}_1, \widehat{\mathbf{x}}), \cdot)$. If $\widehat{\mathbf{X}}(\widehat{T}_1) = \Delta$ the process stays in state Δ there forever since the jump rate out of Δ is zero. Otherwise, the process restarts from $\widehat{\mathbf{X}}(\widehat{T}_1)$ in a similar way if $\widehat{T}_1 < \widehat{d}_{\text{path}}(\widehat{\mathbf{x}})$ with survival function of the next inter jump time $\widehat{T}_2 - \widehat{T}_1$ given by $\widehat{G}_{\widehat{\mathbf{X}}(\widehat{T}_1)}$ and so on ....

The path persistence at time $t_0 \in]0, t^*]$ starting from $\widehat{\mathbf{x}} \in S_{\widehat{\mathbf{X}}}$ can be written as the expectation of a functional of $\widehat{\mathbf{X}}$ by

$$U(\widehat{\mathbf{x}}, t_0) = \mathbb{E}_{(\widehat{\mathbf{x}}, t_0)} \left[\mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{X}}(\widehat{T}_*)) \right]$$

where $\widehat{T}_* = \inf\{\widehat{T}_i : \widehat{X}(\widehat{T}_i) \in \widehat{B}\}$ is the time to reach the absorbing state Δ and, by convention, $U(\Delta) = 0$.

Let $f : S^{\Delta}_{\widehat{\mathbf{X}}} \cup \widehat{B} \to \mathbb{R}^+$ denote an arbitrary bounded measurable function and at the boundary state $\widehat{\mathbf{x}} \in \widehat{B}$ we define $f(\widehat{\mathbf{x}}) \equiv \lim_{t \downarrow 0} f(\widehat{\phi}(-t, \widehat{\mathbf{x}}))$. Let $\widehat{\mathcal{Q}}$ denote an operator mapping the set of bounded measurable functions on $S_{\widehat{\mathbf{X}}} \cup \widehat{B}$ into itself. The operation of $\widehat{\mathcal{Q}}$ on f is a function defined by $\widehat{\mathcal{Q}} f(\widehat{\mathbf{x}}) = \int_{S^{\Delta}_{\widehat{\mathbf{X}}}} f(\widehat{\mathbf{y}}) \widehat{Q}(\widehat{\mathbf{x}}, \mathbf{d}\widehat{\mathbf{y}})$ and, for $\widehat{\mathbf{x}} \in S_{\widehat{\mathbf{X}}}$, we have

$$\widehat{\mathcal{Q}} f(\widehat{\mathbf{x}}) = \sum_{\{j : p^j = 0\}} \int_{\{m : m \in S_M\}} Q(\mathbf{x}, \mathbf{dx}^{(j)}) f(\mathbf{x}^{(j)}, t_0) + \sum_{\{j : p^j = 1\}} Q(\mathbf{x}, \{\mathbf{x}^{(j)}\}) f(\mathbf{x}^{(j)}, t_0)$$

and, for $\widehat{\mathbf{x}} \in \widehat{B}$, $\widehat{\mathcal{Q}} f(\widehat{\mathbf{x}}) = f(\Delta)$. Using the same heuristic arguments as in Subsection 4.3.1, we obtain, for $\widehat{\mathbf{x}} \in S_{\widehat{\mathbf{X}}}$,

$$\frac{1}{t} \mathbb{E}_{\widehat{\mathbf{x}}}[f(\widehat{\mathbf{X}}(t)) - f(\widehat{\mathbf{x}})] = \frac{1}{t}(f(\widehat{\phi}(t, \widehat{\mathbf{x}})) - f(\widehat{\mathbf{x}})) + \boldsymbol{\lambda}(\mathbf{x}) \left(\widehat{\mathcal{Q}} f(\widehat{\phi}(t, \widehat{\mathbf{x}})) - f(\widehat{\phi}(t, \widehat{\mathbf{x}})) \right) + o(1). \tag{4.14}$$

In addition, let $\widehat{\mathcal{A}}$ be an operator acting on the domain of bounded measurable real-valued functions on $S^{\Delta}_{\widehat{\mathbf{X}}} \cup \widehat{B}$ such that $\widehat{\mathcal{A}}f$ is a function defined by

$$\widehat{\mathcal{A}}f(\widehat{\mathbf{x}}) \equiv \lim_{t \to 0} \frac{1}{t} \mathbb{E}_{\widehat{\mathbf{x}}}[f(\widehat{\mathbf{X}}(t)) - f(\widehat{\mathbf{x}})]. \tag{4.15}$$

For a function f define its phase function by $f_{\mathbf{p}} : S^{-}_{\mathbf{p}} \cup (0, t^*] \to \mathbb{R}^+$ such that $f_{\mathbf{p}}(\mathbf{a}, t_0) = f(\widehat{\mathbf{x}})$. If for a state $(\mathbf{x}, t_0)$ the function $t \to f_{\mathbf{p}}(\widehat{\phi}_{\mathbf{p}}(t, (\mathbf{a}, t_0)))$ is differentiable almost everywhere on $[0, \widehat{d}_{path}(\widehat{\mathbf{x}}))$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{\mathbf{p}}(\widehat{\phi}_{\mathbf{p}}(t, (\mathbf{a}, t_0))) = \widehat{\mathcal{V}}_{\mathbf{p}} f_{\mathbf{p}}(\widehat{\phi}_{\mathbf{p}}(t, (\mathbf{a}, t_0))), \qquad \widehat{\phi}_{\mathbf{p}}(0, (\mathbf{a}, t_0)) = (\mathbf{a}, t_0), \tag{4.16}$$

holds for almost all t, where $\widehat{\mathcal{V}}_{\mathbf{p}}$ is a vector field and $\widehat{\phi}_{\mathbf{p}}(0, (\mathbf{a}, t_0)) = (\mathbf{a}, t_0)$ is the unique integral curve of $\widehat{\mathcal{V}}_{\mathbf{p}}$ such that (4.16) is satisfied.

Therefore, letting t tend to zero, equation (4.14) becomes

$$\widehat{\mathcal{A}}f(\mathbf{x}, t_0) = \widehat{\mathcal{V}}f(\mathbf{x}, t_0) + \boldsymbol{\lambda}(\mathbf{x}) \left(\widehat{\mathcal{Q}}f(\mathbf{x}, t_0) - f(\mathbf{x}, t_0) \right)$$

where, with a slight abuse of notation, $\widehat{\mathcal{V}}f(\mathbf{x}, t_0)$ should be read as $\widehat{\mathcal{V}}f_{\mathbf{p}}(\mathbf{a}, t_0)$. Finally, we write from now on $t \to f(\widehat{\phi}(t, \widehat{\mathbf{x}}))$ in place of more cumbersome $t \to f_{\mathbf{p}}(\widehat{\phi}_{\mathbf{p}}(t, (\mathbf{a}, t_0)))$.

The next result shows that the augmentation of the state space of $\mathbf{X}$ allows to define U as the unique solution of a set of integro-differential equations. The impact of this transformation will become more relevant in Section 4.4.

Theorem 4.3. *For each* $\widehat{\mathbf{x}} \in S_{\widehat{\mathbf{X}}}$, $t \to U(\widehat{\phi}(t, \widehat{\mathbf{x}}))$ *is an absolutely continuous function on* $[0, \widehat{d}_{path}(\widehat{\mathbf{x}})[$ *and* U *is the unique solution of the equations*

$$\widehat{\mathcal{A}}f(\widehat{\mathbf{x}}) = 0, \qquad \widehat{\mathbf{x}} \in S_{\widehat{\mathbf{X}}}, \tag{4.17}$$

$f(\Delta) = 0$, *and at a boundary state* $\widehat{\mathbf{x}} \in \widehat{B}$, $f(\widehat{\mathbf{x}}) = \mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{x}})$.

Proof. By definition, $U(\Delta) = 0$. We assume that $\widehat{\mathbf{x}} = (\mathbf{x}, t_0) \in \mathbf{S}_{\mathbf{X}} \times]0, t^*]$ and $t \in]0, \widehat{d}_{path}(\widehat{\mathbf{x}})[$. By the strong Markov property, we have

$$U(\widehat{\mathbf{x}}) = \widehat{G}_{\widehat{\mathbf{x}}}(t) U(\widehat{\phi}(t, \widehat{\mathbf{x}})) + \mathbb{E}_{\widehat{\mathbf{x}}} \left[\widehat{\mathcal{Q}} U(\widehat{\phi}(\widehat{T}_1, \widehat{\mathbf{x}})) \mathbb{1}_{]0,t]}(\widehat{T}_1) \right].$$

Now, using the density function of the first transition $\widehat{T}_1$ on $[0,t]$, given by $\widehat{\lambda}(\widehat{\phi}(s,\widehat{\mathbf{x}}))\widehat{G}_{\widehat{\mathbf{x}}}(s)$ and proceding in a similar way as in the proof of Theorem 4.2, it follows that

$$U(\widehat{\phi}(t,\widehat{\mathbf{x}})) = U(\widehat{\mathbf{x}}) + \int_0^t \widehat{\lambda}(\widehat{\phi}(s,\widehat{\mathbf{x}}))(U(\widehat{\phi}(s,\widehat{\mathbf{x}})) - \widehat{\mathcal{Q}}U(\widehat{\phi}(s,\widehat{\mathbf{x}})))\,\mathrm{d}s. \tag{4.18}$$

Since $\widehat{\mathcal{Q}}U$ is clearly bounded (U is bounded) by the fundamental theorem of calculus $t \to U(\widehat{\phi}(t,\widehat{\mathbf{x}}))$ is an absolutely continuous function on $[0,t]$ and equation (4.17) is satisfied. Now fix $\widehat{\mathbf{x}}^{'} \in \widehat{B}$ and $t > 0$ such that the initial state is $\widehat{\mathbf{x}} = \widehat{\phi}(-t,\widehat{\mathbf{x}}')$. By the strong Markov property, we have

$$\begin{aligned} U(\widehat{\mathbf{x}}) &= \mathbb{E}\left[\mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{x}}')\mathbb{1}_{\{\widehat{T}_1=\widehat{d}_{path}(\widehat{\mathbf{x}})\}} + \widehat{\mathcal{Q}}U(\widehat{X}(\widehat{T}_1^-))\right] \\ &= \mathbb{P}(\widehat{T}_1 = \widehat{d}_{path}(\widehat{\mathbf{x}}))(\mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{x}}') + \widehat{\mathcal{Q}}U(\widehat{\mathbf{x}}^{'})) + \int_0^t \widehat{\lambda}(s,\widehat{\phi}(t,\widehat{\mathbf{x}}))\widehat{G}_{\widehat{\mathbf{x}}}(s)\widehat{\mathcal{Q}}U(\widehat{\phi}(s,\widehat{\mathbf{x}}))\,\mathrm{d}s. \end{aligned}$$

Letting $t \downarrow 0$, the function U has a limit as a boundary state is approached, and $U(\widehat{\mathbf{x}}') = \mathbf{I}_{\widehat{B}^0}(\widehat{\mathbf{x}}')$.

If $\widehat{\mathbf{x}} \in [0,t^*) \times S_{\widehat{\mathbf{X}}}$, $t \to f(\widehat{\phi}(t,\widehat{\mathbf{x}}))$ is an absolutely continuous function on $[0,\widehat{d}_{path}(\widehat{\mathbf{x}}))$ and f is bounded. From conditions from Davis (1993, Theorem 31.3, p. 83) conveniently applied, the process $(\widehat{M^f}(t))$ defined by

$$\widehat{M^f}(t) = f(\widehat{\mathbf{X}}(t)) - f(\widehat{\mathbf{X}}(0)) - \sum_{\{\widehat{T}_i \le t, \widehat{\mathbf{X}}(\widehat{T}_i^-) \in \widehat{B}\}} (\widehat{\mathcal{Q}}f(\widehat{\mathbf{X}}(\widehat{T}_i)) - f(\widehat{\mathbf{X}}(\widehat{T}_i^-))), \quad t \ge 0$$

is a martingale. Taking conditional expectations, $\mathbb{E}_{\widehat{\mathbf{x}}}(\widehat{M^f}(t)) = \widehat{M^f}(0) = 0$ and, therefore,

$$f(\widehat{\mathbf{x}}) = \mathbb{E}_{\widehat{\mathbf{x}}}\left[f\widehat{\mathbf{X}}(t))\right] - \mathbb{E}_{\widehat{\mathbf{x}}}\left[\sum_{\{\widehat{T}_i \le t, \widehat{\mathbf{X}}(\widehat{T}_i^-) \in \widehat{B}\}} (\widehat{\mathcal{Q}}f(\widehat{\mathbf{X}}(\widehat{T}_i)) - f(\widehat{\mathbf{X}}(\widehat{T}_i^-)))\right]. \tag{4.19}$$

In view of $f(\widehat{\mathbf{x}}) = \mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{x}})$ and $\widehat{\mathcal{Q}}f(\widehat{\mathbf{x}}) = 0$ for $\widehat{\mathbf{x}} \in \widehat{B}$, we have that $\widehat{\mathcal{Q}}f(\widehat{\mathbf{X}}(\widehat{T}_i)) - f(\widehat{\mathbf{X}}(\widehat{T}_i)) = -\mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{X}}(\widehat{T}_i))$. Since $\widehat{\mathbf{X}}(t) = \Delta$ for all $t \ge t^*$,

$$\lim_{t\to\infty} \mathbb{E}_{\widehat{\mathbf{x}}} f(\widehat{\mathbf{X}}(t)) = 0.$$

Letting $t \to \infty$ the solution of the equation (4.19) is unique and $f = U$. ❑

The mean path duration and the path persistence written as expectations of functionals of the multihop path process are the unique solutions of systems of integro-differential equations. However, any direct method to solve them is quite problematic and depends very much on the specific characterization of the multihop path process (number of nodes, deterministic motion, jump rate, transition measure). There are some recent studies (cf. Annunziato 2007, 2008) addressing the numerical solution of integro-differential equations, but they can be only applied when the number of initial states is small. In our case, we have several nodes in the path and each one as a large set of attributes and so these methodologies cannot be applied.

4.4 Recursive computations

In this section we present a recursive scheme which provides the basis for practical numerical techniques for computing the path metrics defined in Section 4.3. The basic idea is to obtain a simple system of first order ordinary differential equations and prove that this system converges to the original system of integro-differential equation.

4.4.1 Mean path duration

Let D^0 be a function such that $D^0(\mathbf{x}) = 0$, for all $\mathbf{x} \in S_{\mathbf{X}}^{\Delta}$, and let $\mathcal{O}$ be an operator mapping the set of bounded measurable functions on $S_{\mathbf{X}}^{\Delta} \cup B$ into itself. The action of the operator $\mathcal{O}$ on D^0 gives the function $D^1 \equiv \mathcal{O}D^0$ defined by

$$D^1(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}\left[\int_0^{T_1} \mathbb{1}_{S_{\mathbf{X}}}(X(s))\,\mathrm{d}s + D^0(\mathbf{X}(T_1))\right], \ \mathbf{x} \in S_{\mathbf{X}}^{\Delta}.$$

Iterating $k(\geq 1)$ times the operator $\mathcal{O}$ on D^0 results in the function $D^k \equiv \mathcal{O}^k D^0$ given by

$$\begin{aligned} D^k(\mathbf{x}) &= \mathcal{O}D^{k-1}(\mathbf{x}) \\ &= \mathbb{E}_{\mathbf{x}}\left[\int_0^{T_1} \mathbb{1}_{S_{\mathbf{X}}}(X(s))\,\mathrm{d}s + D^{k-1}(\mathbf{X}(T_1))\right] \end{aligned}$$

for $\mathbf{x} \in S_{\mathbf{X}}^{\Delta}$. The metric $D^k(\mathbf{x})$ denotes the mean path duration constrained to at most k jumps of the multihop process $\mathbf{X}$ starting from state $\mathbf{x}$. As the number of jumps increases we obtain in the limit the mean path duration $D(\mathbf{x})$ defined by (4.5).

Theorem 4.4. *For each* $\mathbf{x} \in S_{\mathbf{X}}$ *and* $k \geq 1$*,*

$$D^k(\mathbf{x}) = \mathbb{E}_{\mathbf{x}} \left[\int_0^{T_k} \mathbb{1}_{S_{\mathbf{X}}}(X(s))\, \mathrm{d}s + D^0(\mathbf{X}(T_k)) \right] \tag{4.20}$$

and

$$\lim_{k \to \infty} D^k(\mathbf{x}) = D(\mathbf{x}). \tag{4.21}$$

Proof. For $k = 1$, equation (4.20) follows from definition. Suppose that (4.20) holds for a fixed k, then by induction

$$\begin{aligned} D^{k+1}(\mathbf{x}) &= \mathcal{O}^k(\mathcal{O}D^0)(\mathbf{x}) \\ &= \mathbb{E}_{\mathbf{x}} \left[\int_0^{T_k} \mathbb{1}_{S_{\mathbf{X}}}(X(s))\, \mathrm{d}s + D^1(\mathbf{X}(T_k)) \right]. \end{aligned}$$

By the strong Markov property,

$$\begin{aligned} D^{k+1}(\mathbf{x}) &= \mathbb{E}_{\mathbf{x}} \left[\int_0^{T_k} \mathbb{1}_{S_{\mathbf{X}}}(X(s))\, \mathrm{d}s + D^1(\mathbf{X}(T_k)) \right] \\ &= \mathbb{E}_{\mathbf{x}} \Bigg[\int_0^{T_k} \mathbb{1}_{S_{\mathbf{X}}}(X(s))\, \mathrm{d}s \\ &\qquad\qquad + \mathbb{E}_{\mathbf{x}} \left[\int_{T_k}^{T_{k+1}} \mathbb{1}_{S_{\mathbf{X}}}(X(s))\, \mathrm{d}s + D^0(T_{k+1}) | \mathcal{F}_{T_k} \right] \Bigg] \\ &= \mathbb{E}_{\mathbf{x}} \left[\int_0^{T_{k+1}} \mathbb{1}_{S_{\mathbf{X}}}(X(s))\, \mathrm{d}s + D^0(T_{k+1}) \right], \end{aligned}$$

where $\mathcal{F}_{T_k}$ is the history of $\mathbf{X}$ until time T_k. Since $T_k \to \infty$ w.p.1 as $k \to \infty$, for all initial states $\mathbf{x} \in S_{\mathbf{X}}$, equation (4.21) follows by monotone convergence. ❑

If the path starts at $\mathbf{x} \in S_{\mathbf{X}}$, then after a short time t either there has been no jump, in which case $D^k(\mathbf{X}(t))$ is equal to $D^k(\phi(t, \mathbf{x}))$, with probability $(1 - \lambda(\mathbf{x})t + o(t))$, or one jump occurs with probability $\lambda(\mathbf{x})t + o(t)$, in which case $D^k(\mathbf{X}(t)) = \mathcal{Q}D^{k-1}(\phi(t,\mathbf{x}))$ (the probability of other events is $o(t)$). Thus,

$$\begin{aligned}\frac{1}{t}\mathbb{E}_{\mathbf{x}}[D^k(\mathbf{X}(t)) - D^k(\mathbf{x})] &= \frac{1}{t}(D^k(\phi(t,\mathbf{x})) - D^k(\mathbf{x})) \\ &\quad + \boldsymbol{\lambda}(\mathbf{x})\left(\mathcal{Q}D^{k-1}(\phi(t,\mathbf{x})) - D^k(\phi(t,\mathbf{x}))\right) + o(1),\end{aligned}$$

where the operator $\mathcal{Q}$ is defined by (4.6). From the definition of D^k and proceeding as in (4.7), we obtain, for $\mathbf{x} \in S_{\mathbf{X}}$,

$$\mathcal{A}D^k(\mathbf{x}) = \mathcal{V}D^k(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{x})\left(\mathcal{Q}D^{k-1}(\mathbf{x}) - D^k(\mathbf{x})\right)$$

where $\mathcal{V}D^k$ is the vector field $\mathcal{V}$, as defined in (4.8), and note that $\mathcal{Q}$ now acts on the externally given function D^{k-1}. To establish the next result, we use Proposition 32.20 in Davis (1993), conveniently applied to the PDMP $\mathbf{X}$.

Theorem 4.5. *Suppose that the function D^{k-1} is given. For each $\mathbf{x} \in S_{\mathbf{X}}$, $t \to D^k(\phi(t,\mathbf{x}))$ is an absolutely continuous function on $[0, d_{path}(\mathbf{x})[$ and D^k is the unique bounded solution of the equations*

$$\mathcal{V}f(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{x})\left(\mathcal{Q}D^{k-1}(\mathbf{x}) - f(\mathbf{x})\right) = -1,\ \mathbf{x} \in S_{\mathbf{X}}, \tag{4.22}$$

and at a boundary state $\mathbf{x} \in B$, $f(\mathbf{x}) = f(\Delta) = 0$.

The difference between equations (4.22) and (4.9) is that now the operator $\mathcal{Q}$ acts only on the externally given function D^{k-1}, which turn them into a system of first order ordinary differential equations. Combining this result with Theorem 4.4 provides a recursive way of computing the mean path duration D.

Proof. To show that D^k satisfies equation (4.22) and $t \to D^k(\phi(t,\mathbf{x}))$ is absolutely continuous, we may proceed along the same lines as in the proof of Theorem 4.2 with the difference that the operator $\mathcal{Q}$ now acts on the given function D^{k-1}. The same also holds to prove that D^k satisfies the boundary conditions.

Fix a state $\mathbf{x} \in S_{\mathbf{X}}$ such that $d_{path}(\mathbf{x}) = \infty$. If we write $y_{\mathbf{x}}(t) = f(\phi(t, \mathbf{x}))$, $a_{\mathbf{x}}(t) = \boldsymbol{\lambda}(\phi(t, \mathbf{x}))$ and $b_{\mathbf{x}}(t) = 1 + \boldsymbol{\lambda}(\phi(t, \mathbf{x}))\mathcal{Q}D^{k-1}(\phi(t, \mathbf{x}))$, then equation (4.22) is a linear scalar ordinary differential equation (ODE) with general coefficients

$$\frac{\mathrm{d}}{\mathrm{d}t} y_{\mathbf{x}}(t) = a_{\mathbf{x}}(t) y_{\mathbf{x}}(t) - b_{\mathbf{x}}(t), \qquad y_{\mathbf{x}}(0) = f(\mathbf{x}) \tag{4.23}$$

which, by Davis (1993, Formula 22.10, p. 55), has a unique solution along $\{\phi(t, \mathbf{x}), t < d_{path}(\mathbf{x})\}$ given by

$$y_{\mathbf{x}}(t) = \exp\left(\int_0^t a_{\mathbf{x}}(s)\,\mathrm{d}s\right) f(\mathbf{x}) - \int_0^t \exp\left(\int_s^t a_{\mathbf{x}}(u)\,\mathrm{d}u\right) b_{\mathbf{x}}(s)\,\mathrm{d}s.$$

Using the distribution of T_1 and noting that

$$\exp\left(\int_0^s -a_{\mathbf{x}}(u)\,\mathrm{d}u\right) = G_{\mathbf{x}}(s),$$

$D^k(\mathbf{x})$ can be expressed as

$$\begin{aligned}
D^k(\mathbf{x}) &= \mathbb{E}_{\mathbf{x}}\left[\int_0^{T_1} \mathbb{1}_{S_{\mathbf{X}}}(X(s))\,\mathrm{d}s + D^{k-1}(\mathbf{X}(T_1))\right] \\
&= \int_0^\infty \lambda(\phi(s, \mathbf{x}))G_{\mathbf{x}}(s)\left(\int_0^s \mathbb{1}_{S_{\mathbf{X}}}(\phi(s, \mathbf{x}))\,\mathrm{d}u + \mathcal{Q}D^{k-1}(\phi(s, \mathbf{x}))\right)\,\mathrm{d}s \\
&= \int_0^\infty\int_s^\infty \lambda(\phi(t, \mathbf{x}))G_{\mathbf{x}}(t)\,\mathrm{d}t\,\mathrm{d}s + \int_0^\infty \lambda(\phi(s, \mathbf{x}))G_{\mathbf{x}}(s)\mathcal{Q}D^{k-1}(\phi(s, \mathbf{x}))\,\mathrm{d}s \\
&= \int_0^\infty G_{\mathbf{x}}(s)\,\mathrm{d}s + \int_0^\infty \lambda(\phi(s, \mathbf{x}))G_{\mathbf{x}}(s)\mathcal{Q}D^{k-1}(\phi(s, \mathbf{x}))\,\mathrm{d}s \\
&= \int_0^\infty \exp\left(\int_0^s -a_{\mathbf{x}}(u)\,\mathrm{d}u\right) b_{\mathbf{x}}(s)\,\mathrm{d}s.
\end{aligned}$$

where the third equality is obtained by changing the order of integration.

Since $b_{\mathbf{x}}(t)$ is bounded and $a_{\mathbf{x}}(t) > 0$, by the variant-of-constants formula we can show that the unique solution for which $y_{\mathbf{x}}(t)$ is bounded is given by the initial condition $y_{\mathbf{x}}(0) = f(\mathbf{x}) = D^k(\mathbf{x})$ and therefore

$$y_{\mathbf{x}}(t) = \int_t^\infty \exp\left(\int_t^s -a_{\mathbf{x}}(s)\,\mathrm{d}s\right) b_{\mathbf{x}}(s)\,\mathrm{d}s$$

with $y_{\mathbf{x}}(t) = f(\phi(t, \mathbf{x})) = D^k(\phi(t, \mathbf{x}))$.

Suppose now that $\mathbf{x} \in S_{\mathbf{X}}$ such that $d_{path}(\mathbf{x}) < \infty$. Let $\mathbf{x}' = \phi(d_{path}(\mathbf{x}), \mathbf{x})$ and $t \in [-d_{path}(\mathbf{x}), 0]$. Then, equation (4.23) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} y_{\mathbf{x}'}(t) = a_{\mathbf{x}'}(t) y_{\mathbf{x}'}(t) - b_{\mathbf{x}'}(t), \qquad y_{\mathbf{x}'}(0) = f(\mathbf{x}').$$

Since the solution of the equation is unique and $f(\mathbf{x}') = D^k(\mathbf{x}') = 0$, by the boundary condition, then $y_{\mathbf{x}'}(t) = f(\phi(t, \mathbf{x}')) = D^k(\phi(t, \mathbf{x}'))$. This completes the proof. ❏

4.4.2 Path persistence

Define the function U^0 such that $U^0(\widehat{\mathbf{x}}) = 0$ for all $\widehat{\mathbf{x}} \in S^{\Delta}_{\widehat{\mathbf{X}}}$. Let $\widehat{\mathcal{O}}$ denote an operator acting on the domain of bounded measurable real valued functions on $S^{\Delta}_{\widehat{\mathbf{X}}} \cup \widehat{B}$ such that the operation of $\widehat{\mathcal{O}}$ on U^0 gives the function $U^1 \equiv \mathcal{O} U^0$ given by

$$U^1(\widehat{\mathbf{x}}) = \mathbb{E}_{\widehat{\mathbf{x}}} \left[\mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{X}}(\widehat{T}_1^-)) + U^0(\widehat{\mathbf{X}}(\widehat{T}_1)) \right], \qquad \widehat{\mathbf{x}} \in S^{\Delta}_{\widehat{\mathbf{X}}}.$$

Iterating successively $k(> 1)$ times the operator $\widehat{\mathcal{O}}$ on U^0 results in the function $U^k \equiv \widehat{\mathcal{O}}^k U^0$ given by

$$U^k(\widehat{\mathbf{x}}) = \widehat{\mathcal{O}} U^{k-1}(\widehat{\mathbf{x}}) = \mathbb{E}_{\widehat{\mathbf{x}}} \left[\mathbb{1}_{\widehat{B}^0} \widehat{\mathbf{X}}(\widehat{T}_1^-)) + U^{k-1}(\widehat{\mathbf{X}}(\widehat{T}_1)) \right],$$

for $\widehat{\mathbf{x}} \in S^{\Delta}_{\widehat{\mathbf{X}}}$. The metric $U^k(t_0, \mathbf{x})$, for $(t_0, \mathbf{x}) \in S_{\widehat{\mathbf{X}}}$, denotes the path persistence at time t_0 constrained to at most k jumps of the process $\widehat{\mathbf{X}}$ starting from state $\mathbf{x}$. As k tends to infinity, we obtain the expectation functional $U(\widehat{\mathbf{x}})$. By the strong Markov property and the definition of $\widehat{\mathbf{X}}$, we can state the following result.

Theorem 4.6. *For each $\widehat{\mathbf{x}} \in S_{\widehat{\mathbf{X}}}$ and $k \geq 1$,*

$$U^k(\widehat{\mathbf{x}}) = \mathbb{E}_{\widehat{\mathbf{x}}} \left[\sum_{i=1}^{k} \mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{X}}(\widehat{T}_i^-)) + \widehat{U}^0(\widehat{\mathbf{X}}(\widehat{T}_k)) \right]$$

and

$$\lim_{k \to \infty} U^k(\widehat{\mathbf{x}}) = U(\widehat{\mathbf{x}}).$$

Proof. The proof follows along the same arguments as in the proof of Theorem 4.4. Just note that for $k = 1$, $U^1 \equiv \mathcal{O}U^0$, and by induction

$$\begin{aligned} U^{k+1}(\widehat{\mathbf{x}}) &= \widehat{\mathcal{O}}^k(\widehat{\mathcal{O}}U^0)(\widehat{\mathbf{x}}) \\ &= \mathbb{E}_{\widehat{\mathbf{x}}}\left[\sum_{i=1}^{k} \mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{X}}(\widehat{T}_i^-)) + U^1(\widehat{\mathbf{X}}(\widehat{T}_k))\right] \end{aligned}$$

and, by the strong Markov property,

$$\begin{aligned} U^{k+1}(\widehat{\mathbf{x}}) &= \mathbb{E}_{\widehat{\mathbf{x}}}\left[\sum_{i=1}^{k} \mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{X}}(\widehat{T}_i^-)) + \mathbb{E}_{\widehat{\mathbf{x}}}\left[\mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{X}}(\widehat{T}_{k+1}^-)) + U^0(\widehat{\mathbf{X}}(\widehat{T}_{k+1}))|\widehat{\mathcal{F}}_{\widehat{T}_k}\right]\right] \\ &= \mathbb{E}_{\widehat{\mathbf{x}}}\left[\sum_{i=1}^{k+1} \mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{X}}(\widehat{T}_i^-)) + U^0(\widehat{\mathbf{X}}(\widehat{T}_{k+1}))\right] \end{aligned}$$

where $\widehat{\mathcal{F}}_{\widehat{T}_k}$ is the history of the process $\widehat{X}$ until time $\widehat{T}_k$. ❑

Using the definition of U^k and following the same steps as in (4.14), we have

$$\widehat{\mathcal{A}}U^k(\mathbf{x}, t_0) = \widehat{\mathcal{V}}U^k(\mathbf{x}, t_0) + \boldsymbol{\lambda}(\mathbf{x})\left(\widehat{\mathcal{Q}}U^{k-1}(\mathbf{x}, t_0) - U(\mathbf{x}, t_0)\right), \quad (\mathbf{x}, t_0) \in S_{\widehat{\mathbf{X}}}$$

where $\widehat{\mathcal{V}}U^k$ is the vector field applied to U^k and defined as in (4.16). Note that now $\widehat{\mathcal{Q}}$ acts on the function U^{k-1}. This gives rise to the following result whose proof is omitted, since it uses merely the arguments used in the proof of Theorem 4.5.

Theorem 4.7. *Suppose that the function U^{k-1} is as given before. Then, for each $\widehat{\mathbf{x}} \in S_{\widehat{\mathbf{X}}}$, $t \to U^k(\widehat{\phi}(t, \widehat{\mathbf{x}}))$ is an absolutely continuous function on $[0, \widehat{d}_{path}(\widehat{\mathbf{x}})[$ and U^k satisfies and is the unique solution of the equations*

$$\widehat{\mathcal{V}}f(\mathbf{x}, t_0) + \boldsymbol{\lambda}(\mathbf{x})\left(\widehat{\mathcal{Q}}U^{k-1}(\mathbf{x}, t_0) - f(\mathbf{x}, t_0)\right) = 0 \tag{4.24}$$

for $(\mathbf{x}, t_0) \in S_{\widehat{\mathbf{X}}}$, $f(\Delta) = 0$, and, at a boundary state $\widehat{\mathbf{x}} \in \widehat{B}$, $f(\widehat{\mathbf{x}}) = \mathbb{1}_{\widehat{B}^0}(\widehat{\mathbf{x}})$.

Proof. The proof follows along the same arguments used in the proof of Theorem 4.5 by letting $\widehat{b}_{\widehat{\mathbf{x}}}(t) = -\frac{\partial f}{\partial t_0}(\mathbf{x}, t_0) + \lambda(\widehat{\phi}(t, \widehat{\mathbf{x}}))\widehat{\mathcal{Q}}U^{k-1}(\widehat{\phi}(t, \widehat{\mathbf{x}}))$, $\widehat{y}_{\widehat{\mathbf{x}}}(t) = f(\widehat{\phi}(t, \widehat{\mathbf{x}})$ and $\widehat{a}_{\widehat{\mathbf{x}}}(t) = \lambda(\widehat{\phi}(t, \widehat{\mathbf{x}}))$

❑

Computing D^k (resp. U^k) requires only to solve ODEs instead of solving integro-differential equations. The results of these calculations are then used to compute the next iteration $k+1$. Since we are dealing with independent ODEs, they can be computed using parallel computation. The convergence of the solution depends on how large k needs to be before D^k (resp. U^k) is close to D (resp. U). The derivation of error bounds for these quantities is possible (as shown in the proof of Theorem 4.1), which also gives an estimate of number of iterations needed for convergence.

To exemplify the computation of the recursive equations, in the case where there are only two nodes, the set $\{(1,1),(1,0),(0,1),(0,0)\}$ contains the possible phases of the two nodes and, for a given initial state, we need to solve a system of 4 ordinary differential equations, one for each possible phase pair. For $\mathbf{p} = (1,1)$ and initial state

$$\mathbf{a} = (\mathbf{e}, \mathbf{m}, l_r^2) = ((e_1, e_2), (v_1 \cos\theta_1, v_1 \sin\theta_1, v_2 \cos\theta_2, v_2 \sin\theta_2), l_r^2),$$

the equation (4.22) for the mean path duration is given by

$$\begin{aligned}\frac{\partial}{\partial t} D^k_{(1,1)}(\mathbf{e}(t), \mathbf{m}, l_r^2) &= 2\lambda_1 D^k_{(1,1)}(\mathbf{e}(t), \mathbf{m}, l_r^2 + (v_2 - v_1)t) \\ &\quad - \lambda_1 D^{k-1}_{(1,0)}(\mathbf{e}(t), (v_1 \cos\theta_1, v_1 \sin\theta_1, 0, 0), l_r^2 + (v_2 - v_1)t) \\ &\quad - \lambda_1 D^{k-1}_{(0,1)}(\mathbf{e}(t), (0, 0, v_2 \cos\theta_2, v_2 \sin\theta_2), l_r^2 + (v_2 - v_1)t) - 1,\end{aligned}$$

with λ_0 (λ_1) being the mean duration of the pause (move) phase, $\mathbf{e}(t) = (e_1 + t, e_2 + t)$ being the sojourn time in phase after t units of time, and where the boundary condition is $D^k_{(1,1)}(\mathbf{e}(d_{path}(\mathbf{x})), \mathbf{m}, l_r^2 + (v_2 - v_1) d_{path}(\mathbf{x})) = 0$. The equations for the other initial phase states are obtained in an analogous way. The solution is obtained profiting from the fact that at step k the values of $D^{k-1}(\mathbf{x})$ for all states are known from previous calculations. The ordinary differential equation (4.24) for the path persistence can be obtained in a similar way.

Any direct implementation of these equations requires a discretization of the state space and solving at each grid point an independent ODE, providing the data for calculating the next iteration. Therefore, numerical software routines that implement the corresponding differential equations in a recursive way are needed for obtaining the final results. It is unrealistic to hope that numerical solutions are possible for a medium size number of links in a single workstation due to the great number of computer processing cycles and the need of storing large amounts of data. However, it is possible to solve the equations in the case of one or two links and in one dimensional ad hoc networks. It should be noted that, for numerical problems of this kind, the use of distributed computing for studying the reliability of multihops paths in a general scenario seems to be a *conditio sine qua non*. Such a development will be efficient if, in addition, we use techniques such as function interpolation and eliminate sample paths whose contribution is negligible. It seems that the equations presented in this section can be effectively solved for non trivial cases, however, more research in this direction is necessary.

4.5 Numerical results

In this section we illustrate an application of the preceding results of the chapter to study the differences between independent links. The scenario proposed is based in La and Han (2007) with pause times and restricted mobility direction of nodes, which intends to model a military scenario where vehicles moves in low velocities in a given direction. We consider that the phase durations are exponentially distributed with means of $30s$ and $120s$ in move and pause phase, respectively. The transmission range of a node is set up to $250m$. The mobility vector is obtained choosing a velocity (m/s) and direction of nodes uniformly distributed in $]10, 20[$ and $]-\pi/4, \pi/4[$, respectively. For a multihop path with N nodes, initially each node i ($2 \leq i \leq N$) is deployed inside node $i-1$'s radio coverage, with an angle uniformly distributed in the interval $]-\pi/4, \pi/4[$ and with a distance following a triangular distribution in the interval $(0, 250)$ with mode 62.5. If the initial phase is move, the mobility vector is chosen as done at a phase transition instant.

Figure 4.2 shows the results of the mean path duration after each iteration for different link count. The departure states of the multihop path were sampled according to the initial

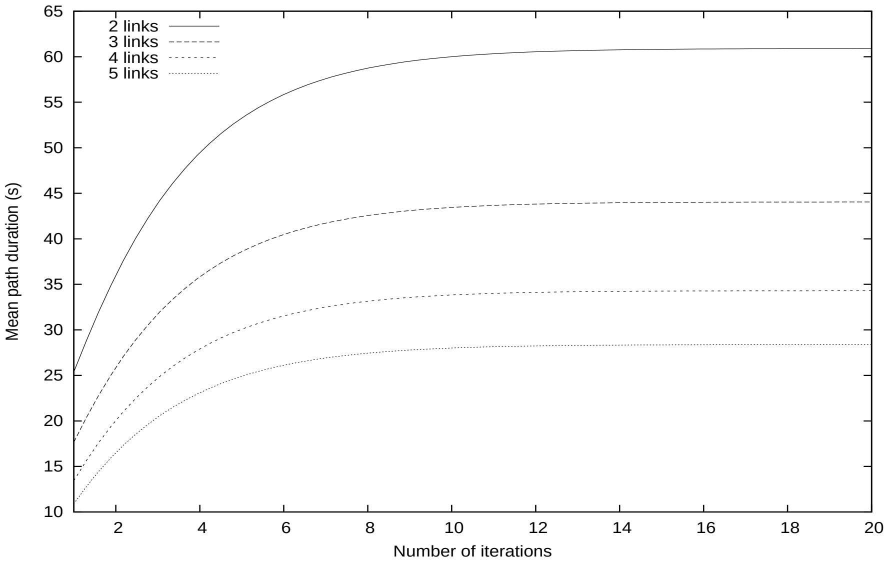

Figure 4.2: Mean path duration after each iteration.

distribution and their respective mean path durations were estimated in each iteration using Monte Carlo methods (in a single computer workstation). The results were averaged out in the final of each iteration. The difference between iterations gets smaller as the number of links increases since it is more likely that a path failure occurs after a small number of phase transitions. However, all curves have converged before iteration 20.

In Figure 4.3 we investigate the impact of neglecting the dependency between links in the mean path duration for different link count. Numerical routines were developed for independent links. Since each link is only dependent on its neighboor links, the difference between mean path durations reaches its maximum value for two links and decreases with the increase of link count. The percentage error from assuming independent links under this scenario could achieve values up to 30%.

Figure 4.4 depicts the path persistence for 2 and 5 links in the interval [0, 200]. The initial states of the multihop path were selected according to the initial distribution (described above) and the path persistences were estimated in each iteration through Monte Carlo methods. After convergence, the results were averaged out. The curves obtained

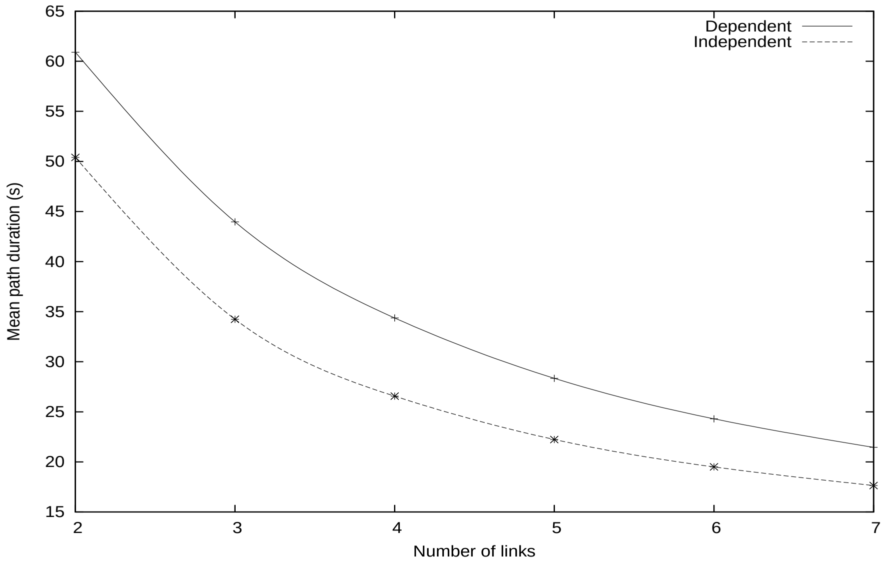

Figure 4.3: Mean path duration *vs* link count

have been plotted against the independent link assumption. The marginal probabilities of independent links were computed using the results of PDMP with one link. For 2-link paths the maximum difference between the curves is smaller than for 5-link paths but persists for a longer time since the path duration is stochastically decreasing in the number of links. As expected, for values of t_0 at the beginning and at the end of the interval the differences are small. Finally, the independent link failure assumption leads to underestimation of the path metrics in the scenario presented.

In Figure 4.5 we study the impact of mean time in pause phase on the mean duration of a 4-link path. The results are rather sensitive to the mean value of pause phase and getting an estimate for the mean path duration using the link independence assumption may in fact be a major problem when the inactive time of a node is large.

In spite of the goal of this model being to present an analytical framework to study path reliability, we highlight from our experiments that slower moving nodes along with shorter link distances conducts to more significant differences in the path metrics when compared with the corresponding independent link approximations.

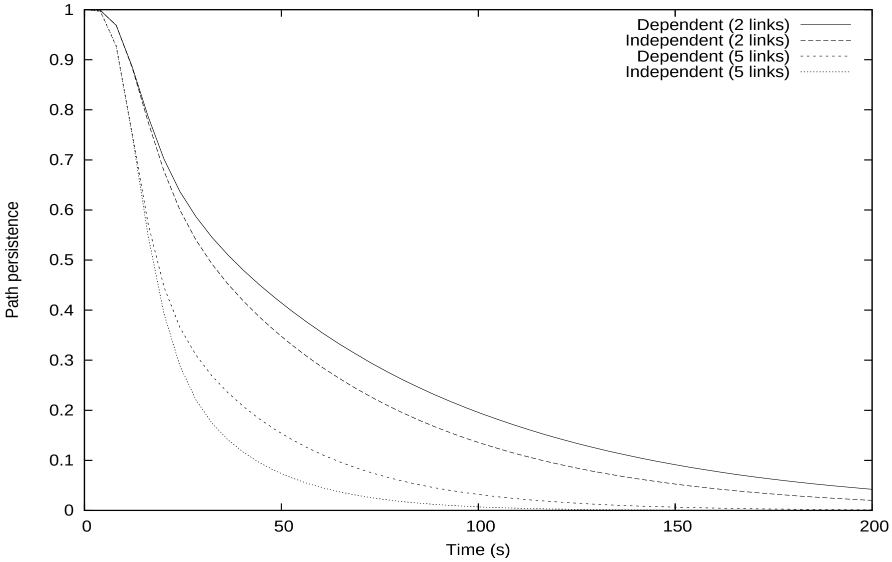

Figure 4.4: Path persistence *vs* link count

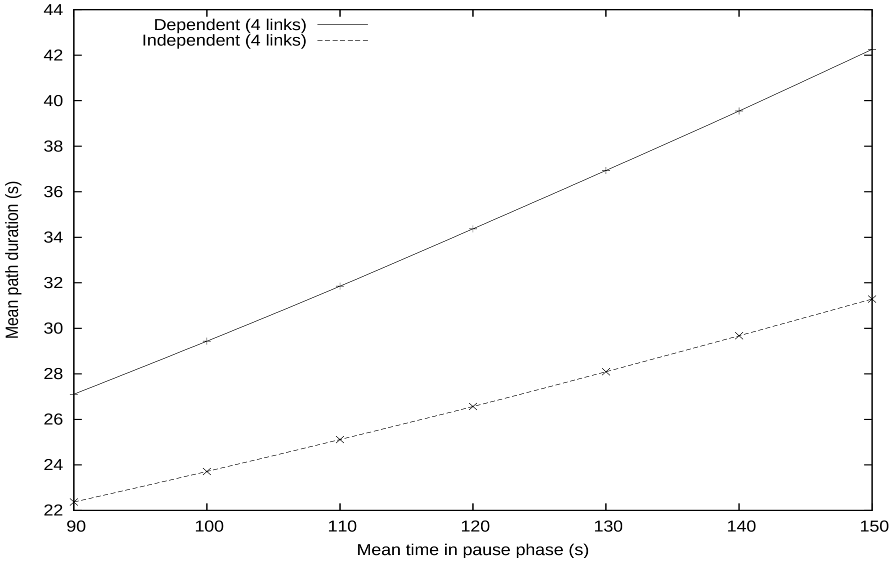

Figure 4.5: Mean path duration *vs* mean pause phase

Chapter 5

Concluding remarks

In this chapter we present a general overview of this dissertation, review the main contributions and provide some directions for future work.

5.1 General overview

Motivated by the absence of mathematical models to cope with one of the main characteristics of MANETs - the multihop ability - in this dissertation we have focused on probabilistic models for the existence of multihop paths and their dynamics.

The first two chapters after the introduction study the distribution of the number of hops of a multihop path between the source and destination nodes. Chapter 2 is dedicated to one-dimensional ad hoc networks, which have applications in vehicular networks and sensor networks, and Chapter 3 is dedicated to two-dimensional ad hoc networks, which have applications in commercial communications, emergency services, military networks, sensor networks, etc. Once a multihop path between two nodes is built, the duration of the multihop path depends on the mobility of the relay nodes. This originate a complex behavior that integrates both connectivity and mobility requirements of such networks. In Chapter 4 we have studied the multihop path dynamics by a mathematical framework to fully describe the random behavior of a multihop path in ad hoc networks.

5.2 Connectivity in one-dimensional MANETs

In Chapter 2 we have studied the connectivity for one-dimensional ad hoc networks, when a known number of relay nodes are uniformly distributed on a straight line between the source and the destination nodes. We then derived the hop count probability distribution from the density location of relay nodes in a multihop path selected with the most forward progress within radius. To obtain the hop count distribution, we used a Poissonification technique that randomizes the number of relay nodes by assuming that they are distributed according to a Poisson process. So, in fact, the results obtained can also be used in the case that relay nodes are randomly distributed by a Poisson process on the line.

The results derived in Chapter 2 can be summarized as follows:

- We have derived the joint density function of relay node locations and, from this density, we derived the hop count probability distribution when the multihop path selected provides the greatest forward progress towards the destination node.
- We have obtained the connectivity probability, that is, the probability that the source and the destination nodes are connected regardless the number of hops, by summing the probability masses for each possible value of the hop count, extending the results of Ghasemi and Nader-Esfahani (2006).
- We have obtained a closed formula for the hop count probabilities of the two smallest possible values of the number of hops. We proved that the hop count probability distribution with the minimum number of hops follows a binomial distribution.

We have also obtained the critical number of relay nodes that, for a given transmission range, guarantees a desired minimum hop count probability. The inverse problem was also analyzed, that is, given the number of relay nodes, the critical transmission range that guarantees a desired minimum hop count probability is obtained. These results can be used for network dimensioning tools since, as a function of the characteristics of the wireless devices, the number of relay nodes that guarantees a desired minimum hop count probability (and also of the connectivity probability) can be obtained.

In terms of practical conclusions for telecommunications engineering, we can make the following considerations:

- The connectivity probability can be well approximated by the sum of the probabilities for the two smallest values of the hop count when there is, at least, a moderate number of nodes.
- With the increase of the number of relay nodes, the minimum hop count probability distribution has the highest contribution for the connectivity probability, and tends to 1. On the other hand, the probabilities with other values of the hop count, decreases and tend to 0.
- The minimum hop count probability increases with the increase of the transmission range, when the number of hops is fixed.

5.3 Connectivity in two-dimensional MANETs

In Chapter 3 we focused on the connectivity in two-dimensional ad-hoc networks. We have assumed that the source and the destination nodes are fixed at a known distance and have considered that the underlying node spatial distribution is drawn from a Poisson process and, by using a Poisson randomization technique, that a fixed number of relay nodes are uniformly distributed in a region of interest. To find a multihop path, we proposed a novel propagation model which we have called the dynamic propagation model. The main characteristic of this model is that the routing region is defined by an angular section of a circular disk with radius equal to the transmission range and oriented to the destination node, with the angular section being dynamic since it depends on the distance to the destination node. We derived the hop count probability distribution when the multihop path chosen follows two routing protocols: the relay node chosen is at the furthest distance within the routing region or the relay node chosen is at the nearest distance within the routing region.

We can summarize the results obtained in Chapter 3 as follows:

- We derived the joint density function of relay nodes and, from this density, the hop count probability distribution when relay nodes are distributed according to a Poisson process. Using a Poissonification technique, analogous results were obtained when a finite number of relay nodes is uniformly distributed in a region of interest.
- The hop count probability distribution was derived when the multihop path is obtained from the dynamic propagation model for each of the two routing protocols considered. Our results extended the results of Srinivasa and Haenggi (2010) and Vural and Ekici (2005), by considering multihop paths instead of single link models.

The numerical results derived in our model allowed us to compare the routing protocols. For practical telecommunication networks we can make the following considerations:

- A novel dynamic propagation model was proposed, which guarantees an efficient progress towards the destination node while increasing the connectivity probability.
- With the increase of the number of relay nodes, the hop count probability distribution for the furthest distance routing protocol with the minimum number of hops increases and tends to 1, while the connectivity probabilities for other values of the minimum hop count, decreases and tend to 0.
- With the increase of the number of relay nodes, the hop count probabilities for the nearest distance routing protocol decrease and tend to 0.

- The furthest distance routing protocol outperforms the nearest distance routing protocol, originating, for the same number of hops, higher probabilities for the minimum number of hops and being more effective in finding a multihop path.
- The expected hop progress and the expected hop distance with the furthest (nearest) distance protocol increase (decrease) with the number of nodes.

5.4 Multihop path model

In Chapter 4 we have proposed a piecewise deterministic Markov process (PDMP) to model the dynamics of a multihop path. We considered that nodes move randomly along the plane according to a variation of a random walk mobility model (other mobility models can also be described with the PDMP, by adjusting the transition measure, the jump rate function and the deterministic behavior between jump epochs).

As a summary, for the results presented in Chapter 4, we can make the following remarks:

- We proposed a new mathematical framework to model the multihop path dynamics by a PDMP.
- We derived exact analytical results for two main path metrics: the mean path duration and path persistence. These metrics were obtained as the solution of a system of integro-differential equations.
- To complement the model with potential applications, a recursive scheme was given, permitting the numerical computation of the path metrics considered by transforming the system of integro-differential equations into a recursion of simple systems of ordinary differential equations.
- Our model permitted to extend the results presented by Han et al. (2006) and La and Han (2007), where the distribution of the path duration was obtained assuming

that the hop count is large. In Han et al. (2006) it is was also assumed that each link is independent of the remaining links of the multihop path.

We numerically compared the path metrics obtained with our model with the corresponding results assuming that the links are independent. We have assumed a military scenario where nodes have low velocities but restricted mobility. We can summarize our conclusions for this scenario as follows:

- The independent link assumption is only feasible when: the nodes have such a high mobility that leads to very small path durations; and, the number of links of the path is very high.
- The independent link assumption leads to underestimation of the path metrics. Our results showed that by assuming independent links the percentage error on the computation of mean path durations could achieve values up to 30%.
- In respect to the mobility parameters, we concluded that higher velocities reduce the mean path duration and the path persistence. Decreasing the mean duration of the pause phase also increase the node mobility, reducing the duration of the paths.
- As regards to the connectivity parameters, we conclude that higher transmission ranges increase the duration of the paths. We have also concluded that multihop paths with higher number of hops have smaller durations.

5.5 Future work

The models presented in this dissertation cover important lacks of the literature concerning the dynamics of multihop paths and the performance of routing protocols. However there is still a lot of work in this area that can be done in the future.

As regards the connectivity in two-dimensional networks, we intend to model other routing protocols and compare them with the ones used in this dissertation. Mainly, we intend to compare the furthest distance routing protocol used here with the most forward routing protocol (see, e.g., Takagi and Kleinrock (1984)), and with a greedy protocol that chooses the relay node that is closer to the destination (see, e.g., Dulman et al. (2006) and Kuo and Liao (2007)).

We also intend to include other connectivity issues, like fading and shadow environments (see, e.g., Haenggi (2008) and Patwari and Agrawal (2008)), where a link between two nodes is established not only as a function of the distance between nodes, but also as a function of the environment, turning the transmission range of each relay node into a random variable.

We also envisage to include an interference model along the multihop path (see, e.g., Dousse et al. (2005) and Srinivasa and Haenggi (2007)). In this case the links between nodes are not only affected by their physical distance, but also by the locations of other transmitting relay nodes, that create interferences and originate link failures.

As concerns the dynamic of multihop paths, we envisage to apply our framework to derive other path metrics Jiang and Rao (2005). Future work will also include other individual or group mobility models (see, e.g., Camp et al. (2002)), that admit a PDMP description. We have already concluded that random mobility and correlated mobility models can be described by a PDMP, just by changing the characteristics of the PDMP, mainly the jump rate function, the transition measure, and the deterministic behavior between jump epochs.

Bibliography

Aggelou, G. (2005). *Mobile Ad Hoc Networks: From Wireless LANs to 4G Networks.* New York, USA: McGraw Hill.

Annunziato, M. (2007). A finite difference method for piecewise deterministic processes with memory. *Mathematical Modelling and Analysis 12*(2), 157–179.

Annunziato, M. (2008). Analysis of upwind method for piecewise deterministic markov processes. *Computation Methods in Applied Mathematics 8*(1), 3–20.

Antunes, N., G. Jacinto, and A. Pacheco (2008). On the minimum hop count and connectivity in one-dimensional ad hoc wireless networks. *Telecommunication Systems 39*(2), 366–376.

Antunes, N., G. Jacinto, and A. Pacheco (2010a). An analytical framework to infer multihop path reliability in MANETs. In *SIGMETRICS '10: Proceedings of the ACM SIGMETRICS International Conference on Measurement and Modeling of Computer Systems*, New York, USA, pp. 323–332. ACM.

Antunes, N., G. Jacinto, and A. Pacheco (2010b). Hop count distribution in two-dimensional ad hoc wireless networks. *Technical Report*.

Bai, F., N. Sadagopan, B. Krishnamachari, and A. Helmy (2004). Modeling path duration distributions in MANETs and their impact on reactive routing protocols. *IEEE Journal on Selected Areas in Communications 22*(7), 1357–1373.

Basagni, S., M. Conti, and S. Giordano (2004). *Mobile Ad Hoc Networking.* New Jersey, USA: Wiley-IEEE Press.

Bettstetter, C. (2002). On the minimum node degree and connectivity of a wireless multihop network. In *Proceedings of the 3rd ACM International Symposium on Mobile Ad Hoc Networking & Computing*, pp. 80–91.

Box, G., W. Hunter, and S. Hunter (1978). *Statistics for Experimenters: An Introduction to Design, Data Analysis, and Model Building.* New York: Wiley.

Camp, T., J. Boleng, and V. Davies (2002). A survey of mobility models for ad hoc network research. *Wireless Communications and Mobile Computing 2*(5), 483–502.

Davis, M. (1993). *Markov Models and Optimization.* Monographs on Statistics and Applied Probability 49. London/New York: Chapman and Hall.

Desai, M. and D. Manjunath (2002). On the connectivity in finite ad hoc networks. *IEEE Journal Transactions on Communications 6*(10), 437–439.

Domb, C. (1952). On the use of a random parameter in combinatorial problems. *Proceedings Physical Society, Sec. A 65*(1), 305–309.

Dousse, O., F. Baccelli, and P. Thiran (2005). Impact of Interferences on Connectivity in Ad Hoc Networks. *IEEE/ACM Transactions on Networking 13*(2), 425–436.

Dulman, S., M. Rossi, P. Havinga, and M. Zorzi (2006). On the hop count statistics for randomly deployed wireless sensor networks. *International Journal of Sensor Network 1*(1/2), 89–102.

Foh, C. and B. Lee (2004). A closed form network connectivity formula one-dimensional MANETs. In *IEEE International Conference on Communications (ICC 2004)*, Volume 6, Paris (France), pp. 3739–3742.

Foh, C., G. Liu, B. S. Lee, B.-C. Seet., K.-J. Wong, and C. Fu (2005). Network connectivity of one-dimensional MANETs with random waypoint movement. *IEEE Journal Transactions on Communications 9*(1), 31–33.

Ghasemi, A. and S. Nader-Esfahani (2006). Exact probability of connectivity one-dimensional ad hoc wireless networks. *IEEE Journal Transactions on Communications 10*(4), 251–253.

Haenggi, M. (2005). On distances in uniformly random networks. *IEEE Transactions on Information Theory 51*(10), 3584–3586.

Haenggi, M. (2008). A geometric interpretation of fading in wireless networks: Theory and applications. *IEEE Transactions on Information Theory 54*(12), 11–25.

Han, Y., R. J. La, A. M. Makowski, and S. Lee (2006). Distribution of path durations in mobile ad-hoc networks: Palm's theorem to the rescue. *Computer Networks 50*(12), 1887–1900.

Hong, D. and S. Rappaport (1986). Traffic model and performance analysis for cellular mobile radio telephone systems with prioritized and nonprioritized handoff procedures. *IEEE Transactions on Vehicular Technology 35*(3), 77–92.

Hou, T.-C. and V. Li (1986). Transmission range control in multihop packet radio networks. *IEEE Transactions on Communications 34*(1), 38–44.

Hyytia, E., P. Lassila, P. Penttinen, and J. Roszik (2005). Traffic load in a dense wireless multihop network. In *PE-WASUN '05: Proceedings of the 2nd ACM International Workshop on Performance Evaluation of Wireless Ad Hoc, Sensor, and Ubiquitous Networks*, New York, USA, pp. 9–17. ACM Press.

Jiang, S. and J. Rao (2005). A prediction-based link availability estimation for routing metrics in MANETs. *IEEE/ACM Transactions on Networking 13*(6), 1302–1312.

Kulkarni, V. G. (1995). *Modeling and Analysis of Stochastic Systems.* London, UK: Chapman & Hall, Ltd.

Kuo, J.-C. and W. Liao (2007). Hop count distribution of multihop paths in wireless networks with arbitrary node density: Modeling and its applications. *IEEE Transactions on Vehicular Technology 56*(4), 2321–2331.

La, R. J. and Y. Han (2007). Distribution of path durations in mobile ad hoc networks and path selection. *IEEE/ACM Transactions on Networking 15*(5), 993–1006.

Mathar, R. and J. Mattfeldt (1996). Optimal transmission ranges for mobile communication in linear multihop packet radio networks. *Wireless Networks 2*(4), 329–342.

Mauve, M., A. Widmer, and H. Hartenstein (2001). A survey on position-based routing in mobile ad hoc networks. *IEEE Network 15*(6), 30–39.

McDonald, A. B. and T. Znati (1999). A path availability model for wireless ad-hoc networks. In *Proceedings of the 5th ACM International Symposium on Mobile Ad Hoc Networking & Computing*, Volume 1, New Orleans, LA, pp. 35–40.

Patwari, N. and P. Agrawal (2008). Effects of correlated shadowing: Connectivity, localization, and RF tomography. In *IPSN '08: Proceedings of the 7th International Conference on Information Processing in Sensor Networks*, Washington, DC, USA, pp. 82–93. IEEE Computer Society.

Polyanin, A. and A. Manzhirov (2006). *Handbook of Mathematics for Engineers and Scientists*. Boca Raton, Florida, USA: Chapman Hall/CRC Press.

Samar, P. and S. Wicker (2004). On the behavior of communications links of a node in a multi-hop mobile environment. In *Proceedings 5th ACM International Symppsium Mobile Ad Hoc Network Computing*, Tokyo, Japan, pp. 145–156.

Santi, P. and D. Blough (2003). The critical transmitting range for connectivity in sparse wireless ad hoc networks. *IEEE Transactions on Mobile Computing. 2*(1), 25–39.

Srinivasa, S. and M. Haenggi (2007). Modeling interference in finite uniformly random networks. In *International Workshop on Information Theory for Sensor Networks (WITS 2007)*, Santa Fe, NM.

Srinivasa, S. and M. Haenggi (2010). Distance distributions in finite uniformly random networks: Theory and applications. *IEEE Transactions on Vehicular Technol-*

ogy 59(2), 940–949.

Takagi, H. and L. Kleinrock (1984). Optimal transmission ranges for randomly distributed packet radio terminals. *IEEE Journal Transactions on Communications 32*(3), 246–257.

Toh, C. K. (2001). *Ad Hoc Wireless Networks: Protocols and Systems.* Upper Saddle River, New Jersey, USA: Prentice Hall PTR.

Vural, S. and E. Ekici (2005). Analysis of hop-distance relationship in spatially random sensor networks. In *MobiHoc '05: Proceedings of the 6th ACM International Symposium on Mobile Ad Hoc Networking and Computing*, New York, USA, pp. 320–331. ACM.

Xu, S., K. L. Blackmore, and H. M. Jones (2007). An analysis framework for mobility metrics in mobile ad hoc networks. *EURASIP Journal on Wireless Communications and Networking 1*(16), 26–26.