Speed and Accuracy Comparison of Noncentral Chi-Square Distribution Methods for Option Pricing and Hedging under the CEV Model*

Manuela Larguinho

Department of Mathematics, ISCAC

José Carlos Dias[†]

Finance Research Center (UNIDE/ISCTE) and Department of Finance, ISCAC

Carlos A. Braumann

Centro de Investigação em Matemática e Aplicações, Universidade de Évora

*The authors thank the participants of the 6th International Conference of the Portuguese Finance Network for helpful comments. Dias gratefully acknowledges the financial support from the Fundação para a Ciência e Tecnologia (FCT) grant number PTDC/EGE-ECO/099255/2008 and Braumann acknowledges the financial support from the Centro de Investigação em Matemática e Aplicações (CIMA) financed by the FCT.

[†]Corresponding author: Department of Finance, ISCAC, Quinta Agrícola, Bencanta, 3040-316 Coimbra, Portugal. Tel: +351 239 802185. Fax: +351 239 445445. E-mail: jdias@iscac.pt.

Speed and Accuracy Comparison of Noncentral Chi-Square Distribution Methods for Option Pricing and Hedging under the CEV Model

Abstract

Pricing options and evaluating greeks under the constant elasticity of variance (CEV) model require the computation of the noncentral chi-square distribution function. In this article, we compare the performance in terms of accuracy and computational time of alternative methods for computing such probability distributions against an externally tested benchmark. In addition, we present closed-form solutions for computing greek measures under the CEV option pricing model for both $\beta < 2$ and $\beta > 2$, thus being able to accommodate direct leverage effects as well as inverse leverage effects that are frequently observed in the options markets.

1. Introduction

Every option pricing model has to make a key assumption regarding the "right" distribution to be used when discounting the option's expected payoff. This expectation is typically computed by integrating the payoff function over a risk-neutral density function. Under the lognormal models of Black and Scholes (1973) and Merton (1973) (BSM model) it is assumed that the underlying asset price follows a geometric Brownian motion. Yet this prediction has been convincingly rejected in the finance literature. For instance, it is well documented—see, for example, Jackwerth and Rubinstein (1996)—that the lognormal assumption is unable to accommodate the negative skewness and the high kurtosis that are usually implicit in empirical asset return distributions.

The constant elasticity of variance (CEV) model of Cox (1975) is consistent with two well-known facts that have found empirical support in the literature: the existence of a negative correlation between stock returns and realized volatility (leverage effect), as observed, for instance, in Bekaert and Wu (2000); and the inverse relation between the implied volatility and the strike price of an option contract (implied volatility skew)—see, for example, Dennis and Mayhew (2002). More importantly, being a "local volatility" model, the CEV diffusion is consistent with a "complete market" setup and, therefore, allows the hedging of short option positions only through the underlying asset.

Computing option prices under the CEV model typically involves the use of the so-called complementary noncentral chi-square distribution function. There exists an extensive literature devoted to the efficient computation of this distribution function, with several alternative representations available (see, for instance, Farebrother (1987), Posten (1989), Schroder (1989), Ding (1992), Knüsel and Bablok (1996), Benton and Krishnamoorthy (2003), and Dyrting (2004)). The complementary noncentral chi-square distribution function can also be computed using a method based on series of incomplete gamma functions. For certain ranges of parameter values, some of the alternative representations available are more computationally efficient than the series of incomplete gamma functions. Moreover, for some parameter configurations the use of analytic approximations (e.g., Sankaran (1963), Fraser et al. (1998), and Penev and Raykov (2000)) may be preferable.

The main purpose of this article is to provide comparative results in terms of accuracy and computation time of existing alternative algorithms for computing the noncentral chi-square distribution function to be used for option pricing and hedging under the CEV model. A similar study has been conducted by Ağca and Chance (2003) to price compound options and min-max options whose computation requires approximations of the bivariate normal probability.

All tested methods are generally accurate over a wide range of parameters that are frequently needed for pricing options, though they all present relevant differences in terms of running times. The iterative procedure of Ding (1992) is the most efficient in terms of computation time needed for determining option prices under the CEV assumption. As expected, the analytic approximations run quickly but have an accuracy that varies significantly over the considered parameter space. Option pricing under the CEV assumption is computationally expensive especially when β is close to two, volatility is low, or the time to maturity is small in the CEV formulae. For these cases, a two-part strategy may be designed using the Ding (1992) method for small to moderate values of $2y$ and $2x$, and then using an approximation method based on Penev and Raykov (2000) for large values of $2y$ and $2x$.

Even though our numerical analysis focus on CEV European-type options, our results are also of interest for some options contracts with early exercise features and/or exotic payoffs. For instance, the valuation of plain-vanilla American options under the optimal stopping approach as proposed by Nunes (2009) requires an explicit solution of its European counterpart option contract and knowledge of the transition density function of the underlying price process. Thus, an efficient method in terms of accuracy and computation time for pricing European-type options should be similarly efficient for valuing plain-vanilla American options within this framework and under the CEV diffusion. The same line of reasoning applies when valuing both European and American (double) barrier options using the CEV assumption within the general multifactor pricing model offered by Nunes and Dias (2010).

The theoretical contribution of this paper is the derivation of closed-form solutions for computing greeks of European-type options under the CEV model that to our knowledge are not known in the finance literature. These new formulae are important for practitioners since closed-form solutions, when available, are generally preferable to simulation methods because of their computational speed advantage.

The structure of the paper is organized as follows. Section 2 outlines the noncentral chi-square distribution and presents different methods for computing it. Section 3 briefly reviews the CEV option pricing formulae expressed in terms of the noncentral chi-square distribution for valuing European-type options. Section 4 compares the alternative methods in terms of speed and accuracy. Section 5 gives some concluding remarks.

2. Alternative Methods for Computing the Noncentral Chi-Square Distribution

2.1. The Noncentral Chi-Square Distribution

If $Z_1, Z_2, ..., Z_v$ are independent unit normal random variables, and $\delta_1, \delta_2, ..., \delta_v$ are constants, then

$$Y = \sum_{j=1}^{v} (Z_j + \delta_j)^2 \tag{1}$$

is the noncentral chi-square distribution with v degrees of freedom and noncentrality parameter $\lambda = \sum_{j=1}^{v} \delta_j^2$, and is denoted as ${\chi'_v}^2(\lambda)$. When $\delta_j = 0$ for all j, then Y is distributed as the central chi-square distribution with v degrees of freedom, and is denoted as χ_v^2.

Hereafter, $p_{{\chi'_v}^2(\lambda)}(w) = p(w; v, \lambda)$ is the probability density function of a noncentral chi-square distribution ${\chi'_v}^2(\lambda)$, and $p_{\chi_v^2}(w) = p(w; v, 0)$ is the probability density function of a central chi-square distribution χ_v^2. Likewise, $P[{\chi'_v}^2(\lambda) \le w] = F(w; v, \lambda)$ is the cumulative distribution function of ${\chi'_v}^2(\lambda)$, and $P[\chi_v^2 \le w] = F(w; v, 0)$ is the cumulative distribution function of χ_v^2. The complementary distribution functions of $\chi_v'^2(\lambda)$ and χ_v^2 are denoted as $Q(w; v, \lambda)$ and $Q(w; v, 0)$, respectively.

The cumulative distribution function of $\chi_v'^2(\lambda)$ is given by (see, for instance, Johnson et al. (1995, Equation 29.2)):

$$
\begin{aligned}
P[\chi_v'^2(\lambda) \leq w] &= F(w; v, \lambda) \\
&= e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!\, 2^{v/2+j}\, \Gamma(v/2+j)} \int_0^w y^{v/2+j-1}\, e^{-y/2}\, dy, \quad w > 0,
\end{aligned} \tag{2}
$$

while $F(w; v, \lambda) = 0$ for $w < 0$. Alternatively, it is possible to express $F(w; v, \lambda)$, for $w > 0$, as a weighted sum of central chi-square probabilities with weights equal to the probabilities of a Poisson distribution with expected value $\lambda/2$. This is (see, for instance, Johnson et al. (1995, Equation 29.3), or Abramowitz and Stegun (1972, Equation 26.4.25)),

$$
\begin{aligned}
F(w; v, \lambda) &= \sum_{j=0}^{\infty} \left(\frac{(\lambda/2)^j}{j!}\, e^{-\lambda/2} \right) P[\chi_{v+2j}^2 \leq w] \\
&= \sum_{j=0}^{\infty} \left(\frac{(\lambda/2)^j}{j!}\, e^{-\lambda/2} \right) F(w; v+2j, 0),
\end{aligned} \tag{3}
$$

where the central chi-square probability function $F(w; v+2j, 0)$ is given by Abramowitz and Stegun (1972, Equation 26.4.1).

The complementary distribution function of $\chi_v'^2(\lambda)$ is

$$
\begin{aligned}
Q(w; v, \lambda) &= 1 - F(w; v, \lambda) \\
&= \sum_{j=0}^{\infty} \left(\frac{(\lambda/2)^j}{j!}\, e^{-\lambda/2} \right) Q(w; v+2j, 0),
\end{aligned} \tag{4}
$$

where the complementary central chi-square probability function $Q(w; v + 2j, 0)$ is given by Abramowitz and Stegun (1972, Equation 26.4.2).

The probability density function of ${\chi'_v}^2(\lambda)$ can, similarly, be expressed as a mixture of central chi-square probability density functions (see, for instance, Johnson et al. (1995, Equation 29.4)):

$$
\begin{aligned}
p_{{\chi'_v}^2(\lambda)}(w) &= p(w; v, \lambda) \\
&= \sum_{j=0}^{\infty} \left(\frac{(\lambda/2)^j}{j!} e^{-\lambda/2} \right) p(w; v + 2j, 0) \\
&= \frac{e^{-(\lambda+w)/2}}{2^{v/2}} \sum_{j=0}^{\infty} \left(\frac{\lambda}{4}\right)^j \frac{w^{v/2+j-1}}{j!\,\Gamma(v/2+j)} \\
&= \frac{1}{2} e^{-(\lambda+w)/2} \left(\frac{w}{\lambda}\right)^{(v-2)/4} I_{(v-2)/2}(\sqrt{\lambda w}), \quad w > 0,
\end{aligned}
\tag{5}
$$

where $I_q(\cdot)$ is the modified Bessel function of the first kind of order q, as defined by Abramowitz and Stegun (1972, Equation 9.6.10):

$$
I_q(z) = \left(\frac{z}{2}\right)^q \sum_{j=0}^{\infty} \frac{(z^2/4)^j}{j!\,\Gamma(q+j+1)}. \tag{6}
$$

Using equation (5) we may also express the functions $F(w; v, \lambda)$ and $Q(w; v, \lambda)$ as integral representations:

$$
F(w; v, \lambda) = \int_0^w \frac{1}{2} e^{-(\lambda+u)/2} \left(\frac{u}{\lambda}\right)^{(v-2)/4} I_{(v-2)/2}(\sqrt{\lambda u})\, du, \tag{7}
$$

$$
Q(w; v, \lambda) = \int_w^\infty \frac{1}{2} e^{-(\lambda+u)/2} \left(\frac{u}{\lambda}\right)^{(v-2)/4} I_{(v-2)/2}(\sqrt{\lambda u})\, du. \tag{8}
$$

2.2. The Gamma Series Method

It is well-known that the functions $F(w; v+2n, 0)$ and $Q(w; v+2n, 0)$ are related to the so-called incomplete gamma functions (see, for instance, Abramowitz and Stegun (1972, Equation 26.4.19)). Hence, we may express noncentral chi-square distribution functions (3) and (4) using series of incomplete gamma functions as follows:

$$F(w; v, \lambda) = \sum_{i=0}^{\infty} \frac{(\lambda/2)^i \, e^{-\lambda/2}}{i!} \, \frac{\gamma(v/2+i, w/2)}{\Gamma(v/2+i)}, \tag{9}$$

$$Q(w; v, \lambda) = \sum_{i=0}^{\infty} \frac{(\lambda/2)^i \, e^{-\lambda/2}}{i!} \, \frac{\Gamma(v/2+i, w/2)}{\Gamma(v/2+i)}, \tag{10}$$

with $\gamma(m, t)$ and $\Gamma(m, t)$ being, respectively, the incomplete gamma function and the complementary incomplete gamma function as defined by Abramowitz and Stegun (1972, Equations 6.5.2 and 6.5.3), and where $\Gamma(m)$ is the Euler gamma function, as defined by Abramowitz and Stegun (1972, Equation 6.1.1).[1]

The gamma series method has been applied by Fraser et al. (1998) as a benchmark for computing exact probabilities to be compared with several alternative methods for approximating the noncentral chi-square distribution function, and by Dyrting (2004) for computing the noncentral chi-square distribution function to be used under Cox et al. (1985b) diffusion processes. Carr and Linetsky (2006) also use the gamma series approach but for computing option prices under a jump-to-default CEV framework.

While this method is accurate over a wide range of parameters, the number of terms that must be summed increases with the noncentrality parameter λ. To avoid the infinite sum of the series we use the stopping rule as proposed by Knüsel and Bablok (1996) which allows the specification of a given error tolerance by the user.

[1] The incomplete gamma functions $\gamma(m, t)$ and $\Gamma(m, t)$, and the Euler gamma function $\Gamma(m)$ are all available in the *Mathematica* software package as built-in functions with the call Gamma[m,0,t], Gamma[m,t] and Gamma[t], respectively.

There have been several alternative proposals for evaluating expressions (9) and (10)—see, for instance, Farebrother (1987), Posten (1989), Schroder (1989), Ding (1992), Knüsel and Bablok (1996), Benton and Krishnamoorthy (2003), and Dyrting (2004)—all of which involve partial summation of the series. For certain ranges of parameter values, some of the alternative representations available are more computationally efficient than the series of incomplete gamma functions. Hence, it is important to evaluate the speed and accuracy of each method for computing the noncentral chi-square distribution as well as for option pricing and hedging purposes.

For the numerical analysis of this article we will concentrate the discussion on Schroder (1989) and Ding (1992) methods since both are commonly used in the finance literature. The algorithm provided by Schroder (1989) has been subsequently used by Davydov and Linetsky (2001). The popular book on derivatives of Hull (2008) suggests the use of the Ding (1992) procedure. We will also use the suggested approach of Benton and Krishnamoorthy (2003), since it is argued by the authors that their algorithm is more computationally efficient than the one suggested by Ding (1992).

2.3. Analytic Approximations

The cumulative distribution function of the noncentral chi-square distribution with degrees of freedom $v > 0$ and noncentrality parameter $\lambda \geq 0$ is usually expressed as an infinite weighted sum of central chi-square cumulative distribution functions. For numerical evaluation purposes this infinite sum is being approximated by a finite sum. For large values of the noncentrality parameter, the sum converges slowly. To overcome this issue, a number of approximations have been proposed in the literature. A comparison of early approximation methods is given in Johnson et al. (1995, chapter 29).

In this article, we will consider the approximation method of Sankaran (1963) which is well-known in the finance literature due to Schroder (1989) who recommends its use for large values of w and λ. In addition, two more recent approximations, namely Fraser et al. (1998) and Penev and Raykov (2000), will be considered also since both of them are commonly referenced by the statistic literature as accurate methods for approximating the noncentral chi-square distribution.

3. The CEV Option Pricing Model

The CEV call option pricing formula for valuing European options has been initially expressed in terms of the standard complementary gamma distribution function by Cox (1975) for $\beta < 2$, and by Emanuel and MacBeth (1982) for $\beta > 2$. Schroder (1989) has subsequently extended the CEV model by expressing the corresponding formulae in terms of the noncentral chi-square distribution as[2]

$$c_t := \begin{cases} S_t\, e^{-q\tau}\, Q\Big(2y; 2 + \frac{2}{2-\beta}, 2x\Big) - X\, e^{-r\tau}\Big[1 - Q\Big(2x; \frac{2}{2-\beta}, 2y\Big)\Big] & \Leftarrow \quad \beta < 2 \\ \\ S_t\, e^{-q\tau}\, Q\Big(2x; \frac{2}{\beta-2}, 2y\Big) - X\, e^{-r\tau}\Big[1 - Q\Big(2y; 2 + \frac{2}{\beta-2}, 2x\Big)\Big] & \Leftarrow \quad \beta > 2 \end{cases}, \tag{11}$$

with X being the strike price of the option, $Q(w; v, \lambda)$ being the complementary distribution function of a noncentral chi-square law with v degrees of freedom and noncentrality parameter λ, and where

$$k = \frac{2(r-q)}{\delta^2(2-\beta)[e^{(r-q)(2-\beta)\tau} - 1]}, \tag{12a}$$

$$x = k\, S_t^{2-\beta}\, e^{(r-q)(2-\beta)\tau}, \tag{12b}$$

$$y = kX^{2-\beta}, \tag{12c}$$

$$\delta^2 = \sigma_0^2\, S_0^{2-\beta}, \tag{12d}$$

$$\tau = T - t. \tag{12e}$$

[2]If one applies the put-call parity then the corresponding CEV put option pricing formula is obtained.

4. Computational Results

This section aims to present computational comparisons of the alternative methods of computing the noncentral chi-square distribution function for pricing and hedging European options under the CEV diffusion. Similarly to the study conducted by Ağca and Chance (2003), we examine CEV option pricing models using alternative combinations of input values over a wide range parameter space. The value of the assets will be $S_0 = 100$. The striking price of each option contract can assume values of $X = \{90, 95, 100, 105, 110\}$, which means that we are considering options with a moneyness factor of $m = \{0.90, 0.95, 1.00, 1.05, 1.10\}$. We let the volatility of the underlying asset to be $\sigma = \{0.10, 0.25, 0.40\}$. We use alternative times to expiration of $\tau = \{0.25, 0.50, 1.00, 3.00\}$. We let the risk-free interest rate to be $r = \{0.10, 0.05\}$, and the dividend yield to be $q = \{0.03, 0.00\}$. The β parameter is assumed to have the following values: $\beta = \{5, 3, 1, 0, -2, -4, -6\}$. These combinations generate a set of 3,360 probability distributions and 1,680 unique options for each type of CEV option.

All the calculations in this article were made using *Mathematica* 7.0 running on a Pentium IV (2.53 GhZ) personal computer. Option prices and greeks are computed using each of the alternative algorithms for approximating the complementary noncentral chi-square distribution. We have truncated all the series with an error tolerance of 1E−10. All values are rounded to four decimal places. In order to understand the computational speed of the alternative algorithms, we have computed the CPU times for all the algorithms using the function Timing[.] available in *Mathematica*. Since the CPU time for a single evaluation is very small, we have computed the CPU time for multiple computations.

4.1. Benchmark Selection

The noncentral chi-square distribution function $F(w; v, \lambda)$ as well as its complementary function $Q(w; v, \lambda)$ require values for w, v, and λ. For option pricing and hedging under the CEV model both w and λ can assume values of $2x$ or $2y$. Table 1 shows the maximum, minimum, and mean values for $2x$, $2y$, and v under the designed parameter space.

[Please insert Table 1 about here.]

To compare methods, in terms of speed and accuracy, for computing noncentral chi-square probabilities for pricing and hedging under the CEV model we need to choose a benchmark. An obvious candidate for a benchmark is to use the noncentral chi-square distribution $F(w; v, \lambda)$ and its complementary function $Q(w; v, \lambda)$ expressed as gamma series as given by equations (9) and (10), respectively. For instance, Fraser et al. (1998) uses the gamma series method as a benchmark for computing exact probabilities to be compared with several alternative methods for approximating the noncentral chi-square distribution function.

Alternatively, we can employ a standard numerical integration method for computing equations (7) and (8) or use a routine from an external source, such as *Matlab* or *R*, for computing noncentral chi-square probabilities.

Based on the results of Table 1 and to have a high degree of confidence in our results, we let the parameters w, v, and λ vary over a wide range of possible values. Thus, we let w and λ vary from 0.01 to 2,000.01 in increments of 20. We also let run v from 0.20 to 4.00 in increments of 0.20. These combinations of parameters produce 204,020 probabilities. For the benchmark selection we focus on the computation of the noncentral chi-square distribution $F(w; v, \lambda)$.

Table 2 compares the gamma series method (GS) for computing the noncentral chi-square distribution function $F(w; v, \lambda)$ based on equation (9), with a pre-defined error tolerance of 1E$-$10, against four external benchmarks based on the *Mathematica* built-in function (with the call CDF[NoncentralChiSquareDistribution[v,λ],w]), the integral representation method based on equation (7) and using the NIntegrate[.] function available in *Mathematica*, the *Matlab* built-in-function (with the call ncx2cdf(w,v,λ)), and the *R* built-in-function (with the call pchisq(w,v,λ)).

[Please insert Table 2 about here.]

Two test statistics obtained from computing these noncentral chi-square probabilities are shown. The first statistic, MaxAE, is the maximum absolute error, while the second, RMSE, is the root mean squared error. Ağca and Chance (2003) have adopted a similar procedure for choosing a benchmark to compute the bivariate normal probability for option pricing and hedging. The results show that the MaxAE and the RMSE are higher for the comparison between the GS vs CDF of *Mathematica* and the GS vs Integral Representation, though the number k_1 is small in relative terms (in both cases, it represents only about 0.03% of the 204,020 computed probabilities). However, the number k_2 is slightly higher for the CDF of *Mathematica*[3] and much higher for the Integral Representation method.

The results comparing the GS vs CDF of *Matlab* and GS vs CDF of *R* show that the corresponding differences are smaller. The number of $k_1 = 1,024$ (approximately 0.50% of the total) for the CDF of *R* is justified by the fact that when *R* computes noncentral chi-square probabilities whose value is very close to 1 it returns, by default, a value of 1. However, as in the gamma series method, we have not obtained any probability value greater than 1 ($k_2 = 0$) under the selected wide parameter space either in *Matlab* or *R*. In summary, we may conclude that the gamma series method is an appropriate choice for our benchmark.

4.2. Noncentral Chi-Square Distribution Using Alternative Methods

Now we want to evaluate the differences in approximations of noncentral chi-square probabilities $F(w; v, \lambda)$ for the iterative procedures of Schroder (1989) (S89), Ding (1992) (D92) and Benton and Krishnamoorthy (2003) (BK03), and the analytic approximations of Sankaran (1963) (S63), Fraser et al. (1998) (FWW98) and Penev and Raykov (2000)[4] compared against the benchmark based on the gamma series approach.

[3]This means that care must be taken if one wants to use the CDF built-in-function of *Mathematica* for computing the noncentral chi-square distribution function.

[4]For the analytic method of Penev and Raykov (2000) we have considered both the second order Wiener germ approximation (PR00a) and the improved first order Wiener germ approximation (PR00b).

Table 3 reports such comparison results using the following set of parameters: $S_0 = 100$, $X = \{90, 95, 100, 105, 110\}$, $\sigma = \{0.10, 0.25, 0.40\}$, $\tau = \{0.25, 0.50, 1.00, 3.00\}$, $r = \{0.10, 0.05\}$, $q = \{0.03, 0.00\}$, and $\beta = \{5, 3, 1, 0, -2, -4, -6\}$. The second rightmost column of the table reports the CPU time for computing 1,000 times the 3,360 probabilities (Panel A)[5], for computing 1,000 times the 80 probabilities when $2x > 1,000$ and/or $2y > 1,000$ (Panel B) and 1,000 times the 160 probabilities when $2x > 600$ and/or $2y > 600$ (Panel C), and for computing 1,000 times the 3,360 probabilities but using a combined two-part strategy, COMB1 (COMB2), based on D92 when $2x \leq 1,000\,(2x \leq 600)$ and/or $2y \leq 1,000\,(2y \leq 600)$ and on PR00a otherwise (Panel D). The MaxAE, MaxRE, RMSE, MeanAE, and k_1 denote, respectively, the maximum absolute error, the maximum relative error, the root mean squared error, the mean absolute error, and the number of times the absolute difference between the two methods exceeds 1E−07.[6]

[Please insert Table 3 about here.]

The iterative procedures based on S89, D92, and BK03 methods are accurate for determining noncentral chi-square probabilities that are needed for computing option prices. However, the differences in terms of computation time plays a key role for the tradeoff between speed and accuracy. The computational results show that the iterative procedure of D92 is the most efficient in terms of running time. Even though the BK03 method is more accurate than the S89 and D92 procedures the corresponding computational expenses do not compensate the improvement in terms of accuracy. Moreover, while Benton and Krishnamoorthy (2003) argue that their algorithm performs best when compared with the Ding (1992) procedure, we do not found such superiority at least for a typical parameter space to be used for option pricing and hedging purposes under the CEV diffusion.

As expected, the analytic approximations run quickly but have an accuracy that varies significantly over the considered parameter space. Thus, for small to moderate values of $2y$

[5] The CPU time for the gamma series method is 21,494.52 seconds.

[6] In order to compute the statistics of Panel A in Table 3 for the Penev and Raykov (2000) method we have excluded 3 probabilities whose values were indeterminate.

and $2x$ none of the approximation methods should be used and the preference is to use the D92 method.

It is well-known that the running time needed for computing the noncentral chi-square distribution $F(w; v, \lambda)$ and its complementary distribution function $Q(w; v, \lambda)$ increases when w and λ are large. Option pricing under the CEV assumption is computationally expensive especially when β is close to two, volatility is low, or the time to maturity is small in the CEV formulae. For this reason, Schroder (1989) has suggested a two-part strategy for computing the noncentral chi-square distribution where for small to moderate values of w and λ the iterative procedure is used, otherwise the distribution is evaluated using the analytic approximation of Sankaran (1963).

However, results from panels B and C of Table 3 indicate that the analytic approximation method of PR00a performs best in terms of accuracy for large values of $2y$ and $2x$. For these cases, a two-part strategy may be designed using the D92 method for small to moderate values of $2y$ and $2x$, and then using an approximation method based on PR00a for large values of $2y$ and $2x$ since the computational expenses will diminish substantially, as presented in panel D of Table 3. To achieve a higher value of accuracy, our preference is to use COMB1, though the use of other cut-off point, as the one used in COMB2, is also a viable alternative.

4.3. Option Pricing under the CEV Model

Even thought we have already analyzed the speed and accuracy of alternative methods of computing the noncentral chi-square distribution at the statistic level, it is also relevant to understand how quickly and accurate are those competing methods for pricing and hedging purposes under the CEV model. We will concentrate our analysis on call options, but the same line of reasoning applies also for put options.

Table 4 values the differences in call option prices under the CEV assumption using the iterative procedures of Schroder (1989) (S89), Ding (1992) (D92) and Benton and Krishnamoorthy (2003) (BK03), and the analytic approximations of Sankaran (1963) (S63), Fraser et al. (1998) (FWW98) and Penev and Raykov (2000) (PR00a and PR00b) compared against the benchmark based on the gamma series approach using the same parameter set as in Table 3. The second rightmost column of the table reports the CPU time for computing 1,000 times the 1,680 call option prices (Panel A)[7], for computing 1,000 times the 40 call option prices when $2x > 1,000$ and/or $2y > 1,000$ (Panel B) and 1,000 times the 80 call option prices when $2x > 600$ and/or $2y > 600$ (Panel C), and for computing 1,000 times the 1,680 call option prices but using a combined two-part strategy, COMB1 (COMB2), based on D92 when $2x \leq 1,000\,(2x \leq 600)$ and/or $2y \leq 1,000\,(2y \leq 600)$ and on PR00a otherwise (Panel D). The MaxAE, MaxRE, RMSE, MeanAE, and k_3 denote, respectively, the maximum absolute error, the maximum relative error, the root mean squared error, the mean absolute error, and the number of times the absolute difference between the two methods exceeds \$0.01.[8]

[Please insert Table 4 about here.]

The results of Table 4 highlight that the iterative procedures of S89, D92 and BK03 are all accurate for computing options prices under the CEV assumption, though the iterative procedure of D92 is still the most efficient in terms of computation time needed for determining option prices.

Again, the analytic approximations run quickly but have an unsatisfactory accuracy when all possible values of $2x$ and $2y$ are considered since all generate a high k_3. However, for large values of $2x$ and $2y$ all approximation methods returns a value of $k_3 = 0$ (see panels B and C of Table 4). In summary, for small to moderate values of $2x$ and $2y$ the iterative procedure of D92 is the most efficient in terms of computation time needed for determining option prices under the CEV assumption whereas the analytic approximation method of PR00a performs best for large values of $2y$ and $2x$.

[7]The CPU time for the gamma series method is 21,531.63 seconds.

[8]Once again, we have excluded 3 call option prices in order to compute the statistics of Panel A in Table 3 for the Penev and Raykov (2000) method, since the corresponding probabilities were indeterminate.

4.4. Greeks under the CEV Model

Information about derivatives of options (commonly known as greeks) is of practical and theoretical importance. In addition to pricing an option, a dealer of the financial services industry must also be able to hedge it. Thus, a practitioner needs to have knowledge regarding the sensitivity measures of derivative securities for designing hedging strategies to reduce the risk of a given security or a portfolio of securities, when closing the position is not viable or desirable. Greeks also enjoy many other multiple applications such as for market risk measurement, profit and loss attribution, model risk assessment and optimal contract design, and to imply out parameters from market prices.

For European-type options on dividend paying assets under the lognormal assumption, closed-form expressions for *delta* ($\partial/\partial S$), *gamma* ($\partial^2/\partial S^2$), *vega* ($\partial/\partial\sigma$), *theta* ($\partial/\partial t$), *rho* ($\partial/\partial r$), and *phi* (or *rho-q*, ($\partial/\partial q$)) are well documented in the literature (e.g., Hull (2008, Chapter 17)). Pelsser and Vorst (1994) discuss the computation of these greeks under the binomial option pricing model of Cox et al. (1979). Chung and Shackleton (2002) show that the so-called binomial Black-Scholes method advocated by Broadie and Detemple (1996) is not only useful for pricing, but also for computing greeks via numerical differentiation since it does not suffer from the problem of discreteness. Chung and Shackleton (2005) examine convergence problems when calculating vegas by comparing different alternative improvements to the traditional binomial method. Hull and White (1987) compare the relative performance of different hedging schemes available to a financial institution when it writes non-exchange-traded currency options. Garman (1992) introduces three more partial derivatives for derivative instruments, namely the *speed* ($\partial^3/\partial S^3$), the *charm* ($\partial^2/\partial S\partial t$), and the *colour* ($\partial^3/\partial S^2\partial t$). Many other greeks of options are discussed in Haug (2006).

Derivative information of option prices are also important at a theoretical level. For instance, Breeden and Litzenberger (1978) show that the second derivative with respect to the strike price ($\partial^2/\partial X^2$) can be interpreted as a state price density. Carr (2001) shows how delta, gamma, speed and other higher-order derivatives of an option's price with respect to the initial price of the underlying asset can be viewed as an expectation, through an appropriate change of measure, of the corresponding derivative at the terminal date. Bergman et al. (1996) derive a general theoretical expression for delta when volatility is a function of stock price and time. Grundy and Wiener (1999) derive theoretical and empirical bounds on deltas under the same volatility setting.

In this article we have derived closed-form solutions for delta, gamma, vega, theta, and rho under the CEV option pricing model for both $\beta < 2$ and $\beta > 2$ that, to our knowledge, have not been published in the literature.[9] Based on these new closed-form solutions, we should also consider how different methods for computing the complementary noncentral chi-square distribution affect the computation of greeks.

Table 5 shows results for deltas (Δ), gammas (Γ), vegas ($\mathcal{V}$), thetas (Θ), and rhos (ρ) for European-style standard call and put options under the CEV assumption for different specifications of the option parameters. The last five lines of the table report the CPU times for computing 1,000 times the greeks of the twenty one option contracts using our closed-form solutions based on the gamma series method (CPU time 1), on the iterative procedures of Schroder (1989), Ding (1992), and Benton and Krishnamoorthy (2003) (CPU time 2-4, respectively), and via elementary differentiation of the gamma series method through *Mathematica* with $nmax = 200$ (CPU time 5).

[Please insert Table 5 about here.]

Several points are noteworthy from Table 5. Inspection of results highlights that for at-the-money options we can imagine that a mirror has been placed at the parameter $\beta = 2$, which reflects similar values for some of the sensitivity meaures (e.g., vega, theta, and rho). Moreover, while symbolic algebra programs such as *Mathematica* or *Maple* can derive such sensitivity measures[10], these new closed-form solutions for determining greeks under the

[9] Even tough their closed-form solutions are omitted here due to constraints of space, they are available upon request.

[10] For instance, Shaw (1998) shows how to derive greeks under the geometric Brownian motion assumption via elementary differenciation using *Mathematica*. A similar symbolic algebra procedure can be used to derive any other arbitrary greek under alternative stochastic processes.

CEV model are important at least for three reasons. Firstly, as stated by Carr (2001), the derivation of greeks through symbolic algebra programs cannot replace an intuitive understanding of the role, genesis, and relationships between all the various greeks. Secondly, the computation time needed for computing analytic greeks will diminish substantially, which is extremely relevant when one needs to design hedging strategies through time. For example, while options under the CEV model have nonzero gammas and vegas, these two greek measures are not affected by the complementary noncentral chi-square distribution. Thus, the small computational expense needed for computing gammas and vegas is especially notable. For the other greeks (i.e., delta, theta, and rho) the D92 method is again the most efficient in terms of computation time. Lastly, the existence of analytical solutions allows that they can be coded in any desired computer language such as *Matlab*, *Fortran*, *R*, or *C*.

5. Conclusions

In this article, we compare the performance of alternative algorithms for computing the noncentral chi-square distribution function in terms of accuracy and computation time for evaluating option prices and greeks under the CEV model. We find that the gamma series method and the iterative procedures of Schroder (1989), Ding (1992), and Benton and Krishnamoorthy (2003) are all accurate over a wide range of parameters, though presenting significative speed computation differences. For small to moderate values of $2y$ and $2x$ the Ding (1992) algorithm is the most efficient in terms of computation time needed for determining option prices under the CEV assumption. However, for large values of $2y$ and $2x$ this method is more computationally expensive. For these cases, a two-part strategy may be designed using the Ding (1992) method for small to moderate values of $2y$ and $2x$, and then using an approximation method based on Penev and Raykov (2000) for large values of $2y$ and $2x$. Finally, we present closed-form solutions for computing greek measures under the CEV option pricing model for both $\beta < 2$ and $\beta > 2$, thus being able to accommodate direct leverage effects as well as inverse leverage effects that are frequently observed in the options markets.

Table 1: Maximum, minimum, and mean values for $2x$, $2y$, and v.

Parameter	Maximum	Minimum	Mean
$2x$	1,620.0833	0.1639	105.8837
$2y$	1,800.0926	0.0134	105.5586
v	4.0000	0.2500	1.9643

Table 2: Benchmark selection.

Methods	MaxAE	RMSE	k_1	k_2
GS vs CDF of *Mathematica*	3.07E−05	2.08E−07	55	2,500
GS vs Integral Representation	3.07E−05	2.17E−07	69	39,366
GS vs CDF of *Matlab*	7.68E−11	3.29E−11	0	0
GS vs CDF of *R*	1.35E−06	4.06E−08	1,024	0

This table compares the gamma series method (GS) for computing the noncentral chi-square distribution function $F(w; v, \lambda)$ based on equation (9), with a pre-defined error tolerance of 1E−10, against four external benchmarks based on the *Mathematica* built-in function (with the call CDF[NoncentralChiSquareDistribution[v,λ],w]), the integral representation method based on equation (7) and using the NIntegrate[.] function available in *Mathematica*, the *Matlab* built-in-function (with the call ncx2cdf(w,v,λ)), and the *R* built-in-function (with the call pchisq(w,v,λ)). The MaxAE, RMSE, k_1, and k_2 denote, respectively, the maximum absolute error, the root mean squared error, the number of times the absolute difference between the two methods exceeds 1E−07, and the number of times a computed probability is greater than 1.

Table 3: Differences in approximations of noncentral chi-square probabilities $F(w; v, \lambda)$ for each method compared against a benchmark based on the gamma series approach.

Methods	MaxAE	MaxRE	RMSE	MeanAE	CPU time	k_1
Panel A: For all possible values of $2x$ and $2y$						
S89	4.43E−10	1.17E−07	8.93E−11	4.61E−11	6,826.10	0
D92	9.71E−11	3.70E−08	5.70E−11	5.05E−11	6,137.68	0
BK03	2.47E−11	8.52E−10	3.00E−12	1.51E−12	48,668.51	0
S63	5.41E−02	2.87E+00	6.78E−03	2.84E−03	384.73	3,350
FWW98	1.74E−01	1.86E+00	1.54E−02	6.01E−03	318.54	3,161
PR00a	6.31E−01	1.00E+00	7.28E−02	1.79E−02	1,155.94	3,104
PR00b	6.31E−01	7.82E−01	7.20E−02	1.78E−02	1,115.42	3,312
Panel B: For values of $2x > 1,000$ and/or $2y > 1,000$						
S63	1.04E−06	4.23E−05	5.28E−07	4.62E−07	9.06	76
FWW98	1.55E−06	1.25E−05	6.97E−07	5.19E−07	7.36	66
PR00a	1.04E−07	1.98E−07	1.85E−08	8.57E−09	26.68	1
PR00b	3.42E−07	1.86E−06	1.54E−07	1.27E−07	25.72	43
Panel C: For values of $2x > 600$ and/or $2y > 600$						
S63	2.99E−06	7.15E−05	1.23E−06	9.37E−07	17.63	153
FWW98	4.57E−06	3.13E−05	1.70E−06	1.19E−06	14.54	146
PR00a	5.95E−05	1.19E−04	4.70E−06	4.06E−07	53.66	13
PR00b	5.95E−05	1.19E−04	4.72E−06	6.75E−07	52.01	117
Panel D: For all possible values of $2x$ and $2y$ but using a combined two-part strategy						
COMB1	1.04E−07	1.98E−07	2.86E−09	2.53E−10	4,633.53	1
COMB2	5.95E−05	1.19E−04	1.03E−06	1.94E−08	3,847.12	13

This table values the differences in approximations of noncentral chi-square probabilities $F(w; v, \lambda)$ for the iterative procedures of Schroder (1989) (S89), Ding (1992) (D92) and Benton and Krishnamoorthy (2003) (BK03), and the analytic approximations of Sankaran (1963) (S63), Fraser et al. (1998) (FWW98) and Penev and Raykov (2000) (PR00a and PR00b) compared against a benchmark based on the gamma series approach. The second rightmost column of the table reports the CPU time for computing 1,000 times the 3,360 probabilities (Panel A), for computing 1,000 times the 80 probabilities when $2x > 1,000$ and/or $2y > 1,000$ (Panel B) and 1,000 times the 160 probabilities when $2x > 600$ and/or $2y > 600$ (Panel C), and for computing 1,000 times the 3,360 probabilities but using a combined two-part strategy, COMB1 (COMB2), based on D92 when $2x \leq 1,000\,(2x \leq 600)$ and/or $2y \leq 1,000\,(2y \leq 600)$ and on PR00a otherwise (Panel D). The MaxAE, MaxRE, RMSE, MeanAE, and k_1 denote, respectively, the maximum absolute error, the maximum relative error, the root mean squared error, the mean absolute error, and the number of times the absolute difference between the two methods exceeds 1E−07. Parameters used in the calculations: $S_0 = 100$, $X = \{90, 95, 100, 105, 110\}$, $\sigma = \{0.10, 0.25, 0.40\}$, $\tau = \{0.25, 0.50, 1.00, 3.00\}$, $r = \{0.10, 0.05\}$, $q = \{0.03, 0.00\}$, and $\beta = \{5, 3, 1, 0, -2, -4, -6\}$.

Table 4: Differences in call option prices using each alternative method for computing the noncentral chi-square distribution compared against a benchmark based on the gamma series approach.

Methods	MaxAE	MaxRE	RMSE	MeanAE	CPU time	k_3
Panel A: For all possible values of $2x$ and $2y$						
S89	8.62E−08	1.29E−06	1.74E−08	8.94E−09	7,104.16	0
D92	1.96E−08	5.82E−07	1.07E−08	9.66E−09	6,499.81	0
BK03	2.10E−09	7.64E−08	4.74E−10	2.86E−10	47,904.63	0
S63	5.69E+00	9.34E−01	9.74E−01	4.04E−01	364.64	1,228
FWW98	1.34E+01	5.32E−01	1.86E+00	7.34E−01	282.81	1,223
PR00a	5.04E+01	1.03E+01	9.02E+00	3.16E+00	1,126.19	939
PR00b	4.75E+01	5.42E+00	8.87E+00	3.09E+00	1,086.92	1,052
Panel B: For values of $2x > 1,000$ and/or $2y > 1,000$						
S63	2.01E−04	1.37E−03	1.04E−04	9.18E−05	8.03	0
FWW98	2.05E−04	5.82E−04	1.24E−04	1.03E−04	6.38	0
PR00a	4.58E−06	2.05E−06	8.53E−07	2.59E−07	25.44	0
PR00b	4.90E−05	1.45E−04	3.03E−05	2.53E−05	24.85	0
Panel C: For values of $2x > 600$ and/or $2y > 600$						
S63	5.92E−04	1.37E−03	2.37E−04	1.81E−04	15.91	0
FWW98	6.09E−04	7.34E−04	3.03E−04	2.37E−04	12.75	0
PR00a	5.98E−03	2.10E−03	6.68E−04	7.73E−05	51.61	0
PR00b	6.06E−03	2.13E−03	6.83E−04	1.34E−04	49.42	0
Panel D: For all possible values of $2x$ and $2y$ but using a combined two-part strategy						
COMB1	4.58E−06	2.05E−06	1.32E−07	1.54E−08	4,606.05	0
COMB2	5.98E−03	2.10E−03	1.46E−04	3.69E−06	3,828.66	0

This table values the differences in call option prices under the CEV assumption using the iterative procedures of Schroder (1989) (S89), Ding (1992) (D92) and Benton and Krishnamoorthy (2003) (BK03), and the analytic approximations of Sankaran (1963) (S63), Fraser et al. (1998) (FWW98) and Penev and Raykov (2000) (PR00a and PR00b) compared against a benchmark based on the gamma series approach. The second rightmost column of the table reports the CPU time for computing 1,000 times the 1,680 call option prices (Panel A), for computing 1,000 times the 40 call option prices when $2x > 1,000$ and/or $2y > 1,000$ (Panel B) and 1,000 times the 80 call option prices when $2x > 600$ and/or $2y > 600$ (Panel C), and for computing 1,000 times the 1,680 call option prices but using a combined two-part strategy, COMB1 (COMB2), based on D92 when $2x \leq 1,000\,(2x \leq 600)$ and/or $2y \leq 1,000\,(2y \leq 600)$ and on PR00a otherwise (Panel D). The MaxAE, MaxRE, RMSE, MeanAE, and k_3 denote, respectively, the maximum absolute error, the maximum relative error, the root mean squared error, the mean absolute error, and the number of times the absolute difference between the two methods exceeds \$0.01. Parameters used in the calculations: $S_0 = 100$, $X = \{90, 95, 100, 105, 110\}$, $\sigma = \{0.10, 0.25, 0.40\}$, $\tau = \{0.25, 0.50, 1.00, 3.00\}$, $r = \{0.10, 0.05\}$, $q = \{0.03, 0.00\}$, and $\beta = \{5, 3, 1, 0, -2, -4, -6\}$.

Table 5: Greeks for European-style standard call and put options under the CEV assumption.

X	β	European standard call options					European standard put options				
		Delta	Gamma	Vega	Theta	Rho	Delta	Gamma	Vega	Theta	Rho
95	5	0.7912	0.0161	21.7198	-11.7066	31.3830	-0.2088	0.0161	21.7198	-2.6699	-13.8004
95	3	0.7616	0.0174	22.3075	-11.8030	31.1304	-0.2384	0.0174	22.3075	-2.7663	-14.0530
95	1	0.7293	0.0189	23.0771	-11.9410	30.8583	-0.2707	0.0189	23.0771	-2.9043	-14.3251
95	0	0.7118	0.0198	23.5433	-12.0286	30.7141	-0.2882	0.0198	23.5433	-2.9920	-14.4693
95	-2	0.6735	0.0218	24.6998	-12.2559	30.4045	-0.3265	0.0218	24.6998	-3.2192	-14.7789
95	-4	0.6286	0.0244	26.3712	-12.6035	30.0534	-0.3714	0.0244	26.3712	-3.5668	-15.1300
95	-6	0.5743	0.0277	28.5840	-13.0736	29.6381	-0.4257	0.0277	28.5840	-4.0369	-15.5453
100	5	0.6938	0.0197	26.5816	-12.1366	27.4562	-0.3062	0.0197	26.5816	-2.6243	-20.1053
100	3	0.6611	0.0206	26.3580	-12.0791	27.4483	-0.3389	0.0206	26.3580	-2.5669	-20.1132
100	1	0.6282	0.0216	26.3580	-12.0791	27.4483	-0.3718	0.0216	26.3580	-2.5669	-20.1132
100	0	0.6113	0.0222	26.4399	-12.1002	27.4513	-0.3887	0.0222	26.4399	-2.5879	-20.1102
100	-2	0.5763	0.0236	26.7924	-12.1907	27.4628	-0.4237	0.0236	26.7924	-2.6784	-20.0986
100	-4	0.5380	0.0255	27.5108	-12.3736	27.4796	-0.4620	0.0255	27.5108	-2.8613	-20.0818
100	-6	0.4946	0.0278	28.6855	-12.6709	27.4975	-0.5054	0.0278	28.6855	-3.1586	-20.0640
105	5	0.5944	0.0218	29.4169	-12.0217	23.3376	-0.4056	0.0218	29.4169	-2.0338	-26.6020
105	3	0.5561	0.0222	28.4533	-11.7794	23.3304	-0.4439	0.0222	28.4533	-1.7915	-26.6092
105	1	0.5202	0.0228	27.7716	-11.6182	23.3764	-0.4798	0.0228	27.7716	-1.6303	-26.5632
105	0	0.5028	0.0231	27.5180	-11.5632	23.4182	-0.4972	0.0231	27.5180	-1.5752	-26.5213
105	-2	0.4686	0.0240	27.1780	-11.5023	23.5387	-0.5314	0.0240	27.1780	-1.5144	-26.4008
105	-4	0.4344	0.0251	27.0996	-11.5166	23.7086	-0.5656	0.0251	27.0996	-1.5287	-26.2309
105	-6	0.3988	0.0265	27.3464	-11.6229	23.9313	-0.6012	0.0265	27.3464	-1.6350	-26.0082
CPU time 1		61.43	5.15	3.04	115.33	57.46	61.64	5.15	3.04	115.04	57.80
CPU time 2		28.97	5.15	3.04	55.88	30.56	29.28	5.15	3.04	55.99	30.76
CPU time 3		26.44	5.15	3.04	51.64	27.78	26.52	5.15	3.04	51.08	28.38
CPU time 4		298.82	5.15	3.04	591.63	293.47	295.51	5.15	3.04	584.94	293.30
CPU time 5		1,170.49	2,860.75	2,386.21	3,193.73	3,085.76	1,168.17	2,860.75	2,386.21	3,192.33	3,086.65

This table values deltas (Δ), gammas (Γ), vegas ($\mathcal{V}$), thetas (Θ), and rhos (ρ) for European-style standard call and put options under the CEV assumption for different specifications of the option parameters. The last five lines of the table report the CPU times for computing 1,000 times the greeks of the twenty one option contracts using the closed-form solutions presented in the appendix based on the gamma series method (CPU time 1), on the iterative procedures of Schroder (1989), Ding (1992), and Benton and Krishnamoorthy (2003) (CPU time 2-4, respectively), and via elementary differentiation of the gamma series method through *Mathematica* with $nmax = 200$ (CPU time 5). Parameters used in the calculations: $S_0 = 100$, $\sigma_0 = \sigma(S_0) = 0.25$, $r = 0.10$, $q = 0$, and $\tau = 0.50$.

References

Abramowitz, Milton and Irene A. Stegun, 1972, *Handbook of Mathematical Functions* (Dover, New York).

Ağca, Senay and Don M. Chance, 2003, Speed and Accuracy Comparison of Bivariate Normal Distribution Approximations for Option Pricing, *Journal of Computational Finance* 6, 61–96.

Bekaert, Geert and Guojun Wu, 2000, Asymetric Volatility and Risk in Equity Markets, *Review of Financial Studies* 13, 1–42.

Benton, Denise and K. Krishnamoorthy, 2003, Computing Discrete Mixtures of Continuous Distributions: Noncentral Chisquare, Noncentral t and the Distribution of the Square of the Sample Multiple Correlation Coefficient, *Computational Statistics and Data Analysis* 43, 249–267.

Bergman, Yaacov Z., Bruce D. Grundy, and Zvi Wiener, 1996, General Properties of Option Prices, *Journal of Finance* 51, 1573–1610.

Black, Fischer and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–654.

Breeden, Douglas T. and Robert H. Litzenberger, 1978, Prices of State-Contingent Claims Implicit in Option Prices, *Journal of Business* 51, 621–651.

Broadie, Mark and Jerome Detemple, 1996, American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods, *Review of Financial Studies* 9, 1211–1250.

Carr, Peter, 2001, Deriving Derivatives of Derivative Securities, *Journal of Computational Finance* 4, 5–29.

Carr, Peter and Vadim Linetsky, 2006, A Jump to Default Extended CEV Model: An Application of Bessel Processes, *Finance and Stochastics* 10, 303–330.

Chung, San-Lin and Mark Shackleton, 2002, The Binomial Black-Scholes Model and the Greeks, *Journal of Futures Markets* 22, 143–153.

Chung, San-Lin and Mark Shackleton, 2005, On the Errors and Comparison of Vega Estimation Methods, *Journal of Futures Markets* 25, 21–38.

Cox, John C., 1975, Notes on Option Pricing I: Constant Elasticity of Variance Diffusions. Working Paper, Stanford University.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1985b, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–408.

Cox, John C., Stephen A. Ross, and Mark Rubinstein, 1979, Option Pricing: A Simplified Approach, *Journal of Financial Economics* 7, 229–263.

Davydov, Dmitry and Vadim Linetsky, 2001, Pricing and Hedging Path-Dependent Options under the CEV Process, *Management Science* 47, 949–965.

Dennis, Patrick and Stewart Mayhew, 2002, Risk-Neutral Skewness: Evidence from Stock Options, *Journal of Financial and Quantitative Analysis* 37, 471–493.

Ding, Cherng G., 1992, Algorithm AS 275: Computing the Non-Central χ^2 Distribution Function, *Applied Statistics* 41, 478–482.

Dyrting, S., 2004, Evaluating the Noncentral Chi-Square Distribution for the Cox-Ingersoll-Ross Process, *Computational Economics* 24, 35–50.

Emanuel, David C. and James D. MacBeth, 1982, Further Results on the Constant Elasticity of Variance Call Option Pricing Model, *Journal of Financial and Quantitative Analysis* 17, 533–554.

Farebrother, R. W., 1987, Algorithm AS 231: The Distribution of a Noncentral χ^2 Variable with Nonnegative Degrees of Freedom, *Applied Statistics* 36, 402–405.

Fraser, D. A. S., J. Wu, and A. C. M. Wong, 1998, An Approximation for the Noncentral Chi-Squared Distribution, *Communications in Statistics - Simulation and Computation* 27, 275–287.

Garman, Mark, 1992, Charm School, *Risk* 5, 53–56.

Grundy, Bruce D. and Zvi Wiener, 1999, The Analysis of Deltas, State Prices and VaR: A New Approach. Working Paper, Wharton School, University of Pennsylvania.

Haug, Espen Gaarder, 2006, *The Complete Guide to Option Pricing Formulas*, 2nd ed. (McGraw-Hill, New York, NY).

Hull, John and Alan White, 1987, Hedging the Risks from Writing Foreign Currency Options, *Journal of International Money and Finance* 6, 131–152.

Hull, John C., 2008, *Options, Futures, and Other Derivatives*, 7th ed. (Prentice Hall, Upper Saddle River, NJ).

Jackwerth, Jens Carsten and Mark Rubinstein, 1996, Recovering Probability Distributions from Option Prices, *Journal of Finance* 51, 1611–1631.

Johnson, Norman L., Samuel Kotz, and N. Balakrishnan, 1995, *Continuous Univariate Distributions*, Vol. 2, 2nd ed. (John Wiley & Sons, New York, NY).

Knüsel, L. and B. Bablok, 1996, Computation of the Noncentral Gamma Distribution, *SIAM Journal of Scientific Computing* 17, 1224–1231.

Merton, Robert C., 1973, Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Nunes, João Pedro, 2009, Pricing American Options under the Constant Elasticity of Variance Model and Subject to Bankruptcy, *Journal of Financial and Quantitative Analysis* 44, 1231–1263.

Nunes, João Pedro and José Carlos Dias, 2010, Double Barrier Options Valuation under Multifactor Pricing Models. Working Paper, Finance Research Center (FRC/ISCTE) and ISCTE Business School.

Pelsser, Antoon and Ton Vorst, 1994, The Binomial Model and the Greeks, *Journal of Derivatives* 1, 45–49.

Penev, Spiridon and Tenko Raykov, 2000, A Wiener Germ Approximation of the Noncentral Chi Square Distribution and of its Quantiles, *Computational Statistics* 15, 219–228.

Posten, Harry O., 1989, An Effective Algorithm for the Noncentral Chi-Squared Distribution Function, *American Statistician* 43, 261–263.

Sankaran, Munuswamy, 1963, Approximations to the Non-Central Chi-Square Distribution, *Biometrika* 50, 199–204.

Schroder, Mark, 1989, Computing the Constant Elasticity of Variance Option Pricing Formula, *Journal of Finance* 44, 211–219.

Shaw, William, 1998, *Modelling Financial Derivatives with Mathematica* (Cambridge University Press, Cambridge, UK).