Universidade de Évora - Instituto de Investigação e Formação Avançada

Programa de Doutoramento em Matemática

Tese de Doutoramento

Study of some varieties of Frobenius number

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Orientador(es) | José Carlos Rosales González
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For everything!

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For the ideas, proactivity, hymns, faith, and love. VIRTUOSITY and resilience.

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Abstract

Common behaviors in families of numerical semigroups led to the introduction of Frobenius varieties, pseudo-varieties and restricted varieties concepts. These facts allow us to build, arrange and give algorithms to compute its elements with a given genus, multiplicity and Frobenius number.

This thesis is devoted to the study of families of numerical semigroups fitting one of the three above concepts. Families of numerical semigroups covered are modularly equidistant, with concentration two, with fixed multiplicity and concentration, without consecutive small elements, coated with odd elements and with distances no admissible between gaps greater than its multiplicity.

Lastly, we study the Frobenius restricted variety of the numerical semigroups contained in a given one where we give formulas for the Frobenius and genus number restricted. Further, we generalize Bras-Amorós and Wilf's conjecture.

Keywords: Numerical semigroups, Frobenius number, varieties, pseudo-varieties, restricted varieties.

Resumo

Estudo de algumas variedades do número de Frobenius

Comportamentos comuns em famílias de semigrupos numéricos levaram à introdução de conceitos de variedades, pseudo-variedades e variedades restritas de Frobenius. Esses fatos nos permitem construir, organizar e fornecer algoritmos para computar seus elementos com um determinado gênero, multiplicidade e número de Frobenius.

Esta tese é dedicada ao estudo de famílias de semigrupos numéricos que se encaixam em um dos três conceitos acima. Famílias de semigrupos numéricos abordados são modularmente equidistantes, com concentração dois, com multiplicidade e concentração fixas, sem elementos consecutivos menores que o número de Frobenius, revestidos com elementos ímpares e com distâncias não admissíveis entre buracos maiores que sua multiplicidade.

Por fim, estudamos a variedade restrita de Frobenius dos semigrupos numéricos contidos num semigrupo dado onde damos fórmulas para os números de Frobenius e gênero restritos. Além disso, generalizamos as conjecturas de Bras-Amorós e Wilf.

Palavras-chave: Semigrupos numéricos, número de Frobenius, variedades, pseudo-variedades, variedades restritas.

Contents

Introduction

At the end of the 19th century, the German Frobenius[1] (1849-1917) raised the following question: given relative prime integers $a_1, a_2, \ldots, a_n$ find the largest positive integer that cannot be expressed as an integer conical combination[2] of these numbers.

This problem became known as the **(diophantine) Frobenius problem** (also referred to as the coin problem, postage stamp problem, or the Chicken McNuggets numbers) and is usually denoted by $g(a_1, \ldots, a_n)$ [**Ram05**].

In 1857, while investigating the **partition number function** which represents the number of possible partitions of a non-negative integer, Sylvester[3] (1814-1897) [**Syl57**] defined the function $d(m; a_1, \ldots, a_n)$, called the **denumerant**, as the number of non-negative integer representations of m by $a_1, \ldots, a_n$. In 1882, Sylvester [**Syl82**, page 134] solved the Frobenius problem for $n = 2$ getting $g(a_1, a_2) = a_1a_2 - a_1 - a_2$. Later, in 1884, in the *Educational Times* journal, Sylvester [**Syl84**] put the question of finding such a formula as a recreational problem.

Curtis showed that, in some sense, a search for a simple formula when $n = 3$ is impossible. Indeed, Curtis [**Cur90**] proved that in the case $n = 3$, and consequently in all cases $n \geq 3$, the Frobenius number cannot be given by closed formulas of a certain

[1]Ferdinand Georg Frobenius (1849-1917), was a German mathematician, best known for his contributions to the theory of elliptic functions, differential equations, number theory, and to group theory. Also, he was the first to give full proof for the Cayley-Hamilton theorem in linear algebra.

[2]An integer conical combination is a type of linear combination where the coefficients are non-negative integers.

[3]James Joseph Sylvester (1814-1897), was an English mathematician. He made fundamental contributions to matrix theory, invariant theory, number theory, partition theory, and combinatorics.

type (formulas which can be reduced to a finite set of certain polynomials) [**Ram05**, p. 35].

Recently, the Frobenius problem, or FP for short, has aroused great interest in the computational area due to its complexity.

The study of numerical semigroups is equivalent to that of nonnegative integer solutions to Diophantine equations of the form $a_1x_1 + \cdots + a_nx_n = b$ with positive integer coefficients reduced to the case $\gcd(a_1, \ldots, a_n) = 1$. For a thorough treatment of the subject see [**RGS09**] and [**RGS99**, chapter 10]. The book [**ADGS20**] covers some applications such as how to compute the minimal presentations, Betti elements, factorizations and divisibility of numerical semigroups.

The numerical semigroup membership problem, or NSMP, is the problem of determining if, given a certain positive integer t and a numerical semigroup S, the integer t is contained in S.

The numerical semigroup membership problem is NP-complete (that is, it is in NP and is hard for NP), as shown in [**PS98**]. This fact was used by Jorge Ramírez Alfonsín in 1996 [**Ram96**] to finally prove that the Frobenius problem is NP-hard (under Turing reductions). He proved that the NSMP can be Turing reduced to the FP. As the NSMP is NP-complete, he concluded that the FP is NP-hard [**OT21a**, p. 4].

The numerical semigroup membership problem can be trivially Turing reduced to the problem of finding the Sylvester denumerant. Thus, the Sylvester denumerant problem is in NP-hard [**Oss19**, p. 24].

Problem	Computational Complexity
Frobenius problem	NP-hard
NS membership problem	NP-complete
Sylvester denumerant	NP-hard

Taking advantage of this, work has been done in the field of code theory with numerical semigroups, see for instance [**BA13, DFGSL13**] and the references therein.

New works have gone even further by proposing quantum algorithms for these combinatoric invariants of numerical semigroups [**Oss19**, **OT21a**, **OT21b**].

In the present work we will study some families of numerical semigroups which present common behaviors strongly related to the Frobenius number. These common behaviors allow us to build and arrange the elements of families of numerical semigroups. We arrange them in a tree and give algorithms that allow the computation of its whole set with a given Frobenius number and other invariants.

We know that if S and T are numerical semigroups, then $S \cap T$ is a numerical semigroup. Furthermore, if S is a numerical semigroup different from $\mathbb{N}$, then $S \cup \{F(S)\}$ will be also [**RGS09**, Lemma 4.1].

In this way, one can define a **Frobenius variety** by families of numerical semigroups closed under finite intersections and the union with the Frobenius number [**Ros08b**].

We already know that the set of all numerical semigroups is a Frobenius variety. Families of numerical semigroups as **Arf**, **Saturated** and **with Toms decompositon** are also notable examples of varieties [**Ros08b**]. The intersection of Frobenius varieties is a Frobenius variety. So, for example, the set of **Arf numerical semigroups having a Toms decomposition** is a Frobenius variety.

However, there exist families of numerical semigroups that are not varieties, but with a very similar behavior. Thus the concept of **Frobenius pseudo-varieties** was introduced in [**RR15**] and it generalizes the concept of varieties. By the way, every variety is a pseudo-variety.

Unfortunately, this was not enough. Because we can still find significant families of numerical semigroups which are not pseudo-varieties. So the concept of R**-variety** (that is, Frobenius restricted variety) came on the scene and it generalizes the concept of pseudo-varieties [**RR18**]. Indeed, every pseudo-variety is a R-variety.

Families studied in chapters 2-7 fit into one of the three above concepts. For instance, a family of numerical semigroups without consecutive small elements is a variety, the family of numerical semigroups with concentration k is a pseudo-variety and the family of numerical semigroups contained in a given one is a R-variety.

If S is a numerical semigroup, then $\mathrm{m}(S) = \min(S\setminus\{0\})$, $\mathrm{F}(S) = \max(\mathbb{Z}\setminus S)$, $\mathrm{c}(S) = \mathrm{F}(S)+1$, $\mathrm{g}(S)$ the cardinality of $\mathbb{N}\setminus S$, $\mathrm{n}(S)$ the cardinality of the elements of S bellow the Frobenius number and $\mathrm{e}(S)$ the cardinality of the minimal system of generators of S are six important invariants of S known as **multiplicity**, **Frobenius number**, **conductor**, **genus**, **the number of small elements** and **embedding dimension** of S, respectively. See chapter 1 for more details.

In 1978, Wilf[4] conjectured that every numerical semigroup S satisfies

$$\mathrm{g}(S) \leq (\mathrm{e}(S)-1)\mathrm{n}(S).$$

Wilf question [**Wil78**] which is nowadays know as **Wilf's conjecture** was:

(a) Is it true that for a fixed k the fraction Ω/χ of omitted values is at most $1-(1/k)$?

In Wilf's notation χ stands for the conductor c, Ω for the genus and k for the embedded dimension. Thus

$$\boxed{g/c \leq 1-1/e}$$

which we can rearrange as

$$\frac{c}{g} \geq \frac{e}{e-1} = \frac{e-1+1}{e-1} = 1 + \frac{1}{e-1}$$

so

$$\frac{c}{g} - 1 \geq \frac{1}{e-1} \Leftrightarrow \frac{g}{c-g} \leq e-1 \Leftrightarrow g \leq (e-1)(c-g)$$

Hence

$$\boxed{\mathrm{g}(S) \leq (\mathrm{e}(S)-1)\mathrm{n}(S)}$$

[4]Herbert Saul Wilf (1931-2012), was an American mathematician, who specialized in combinatorics and graph theory.

This question is still widely open and it is one of the most important problems in numerical semigroup theory. Some families of numerical semigroups for which it is known that the conjecture is true are collected in [Del20] and [ADGS20, p. 11]. We summarize some results here in table 1.

1882	Sylvester [Syl82]	• $e(s) = 2$
1986	Fröberg *et al* [FGH86]	• $e(s) = 3$
2012	Sammartano [Sam12]	• $e(S) \geq m(S)/2$ • $m(S) \leq 8$
2012	Zhai [Zha13]	• the average of ns fulfilling this inequality tends to one when the genus goes to infinity
2015	Fromentin and Hivert [FH16]	• $g(S) \leq 60$
2015	Eliahou [Eli18]	• c(s) ≤ 3m(S)
2019	Bruns [BGSOW20] *et al*	• $m(s) \leq 18$
2019	Eliahou and Marín-Aragón[EMA21]	• $n(s) \leq 12$
2019	Eliahou [Eli20]	• $e(S) \geq m(S)/3$
2019	Delgado and Eliahou [Eli20]	• $g(S) \leq 80$ • $g(S) \leq 100$?

TABLE 1. Numerical semigroups satisfying Wilf's conjecture so far.

The result c(s) ≤ 3m(S) of Eliahou (2015), together with the work of Zhai (2012), imply that Wilf's conjecture is *asymptotically true* as $g \to \infty$. The case $e(S) \geq m(S)/3$ covers more than 99.999% of numerical semigroups of genus $g \leq 45$.

In chapter 3 we prove that numerical semigroups with concentration 2 and its generalization satisfy Wilf's conjecture. Coe-semigroups for which a particular condition is true also satisfy (see chapter 5). Furthermore, we generalize Wilf's conjecture for Frobenius R-variety of the numerical semigroups contained in a given one.

Many authors agree that mathematics is a science of pattern and order (see for example [Ste88, Dev96]) and, in the context of numerical semigroups Bras-Amorós, after computing the number n_g of numerical semigroups with genus up to 50 in 2008 [BA08], saw a beautiful behavior and conjectured

(i) $n_g \geq n_{g-1} + n_{g-2}$, for $g \geq 2$

(ii) $\lim_{g\to\infty} \frac{n_{g-1} + n_{g-2}}{n_g} = 1$

(iii) $\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

From (ii) and (iii) we have a Fibonacci-like property on the number of numerical semigroups of a given genus and that the associated quotient sequence approaches the golden ratio. Notice also that (iii) $\Rightarrow$ (ii). These items (2/3 of the conjecture) were proved by Alex Zhai while he was an undergraduate [**Zha13**][**Kap17**, p. 2].

However, it is still not known in general if for a fixed positive integer g there are more numerical semigroups with genus $g+1$ than numerical semigroups with genus g (the first item).

In other words, the sequence of the number n_g of numerical semigroups with genus g grows as fast as the Fibonacci numbers, but it is still not known whether it is nondecreasing.

Computations have been extended to $g \leq 67$ by Fromentin and Hivert [**FH16**] and Delgado, Garcia-Sánchez, and Morais have implemented a program to find the set of all numerical semigroups of genus g in the *NumericalSgps* package for the computer algebra system GAP [**DGSM20**, **GAP22**].

In chapter 7 we generalize Bras-Amorós conjecture for Frobenius restricted varieties of the numerical semigroups contained in a given one.

This work is organized as follows:

Chapter 1 - Numerical Semigroups Preliminaries. Here we present some basic definitions and known results related to numerical semigroups: notable elements, irreducible numerical semigroups and a collection of common behaviors in families of numerical semigroups. In the last one we will see the concepts and comparisons of (Frobenius) varieties, pseudo-varieties and R-varieties.

Chapter 2 - Modularly Equidistant numerical semigroups. In this chapter we study the so-called Modularly Equidistant numerical semigroups. We arrange this kind of numerical semigroups modulo a in a tree and give algorithms for computing its whole set with fixed multiplicity, genus and Frobenius number. We close this chapter by giving attention to those with maximal embedding dimension. Results are published in [**RBT21**].

Chapter 3 - Numerical semigroups with concentration 2 **and** k**.** This chapter is broken into two sections. The first is dedicated to the study of numerical semigroups of concentration 2. We give algorithms to calculate the whole set of this class of semigroups with given multiplicity, genus or Frobenius number. Separately, we prove that this class of semigroups verifies Wilf's conjecture. In the second section we study numerical semigroups with fixed multiplicity and concentration, $C_k[m]$. With the same approach, we give algorithms to calculate the whole set of this class of semigroups with a given genus or Frobenius number. In addition, we prove that if $S \in \mathsf{C}_k[m]$ with $k \leq \sqrt{\frac{m}{2}}$, then S verifies the Wilf's conjecture. The first section has already been published [**RBT22b**] and the second is submitted.

Chapter 4 - Numerical semigroups without consecutive small elements. In this chapter we study A-semigroups, that is, numerical semigroups which have no consecutive elements less than the Frobenius number. We will see that the set of all A-semigroups is a Frobenius variety and we give algorithms that allow the computation of its whole set with a given genus, multiplicity and Frobenius number. From this we examine interesting families of A-semigroups which are Frobenius varieties, pseudo-varieties and R-varieties. Results are published in [**RBT22c**].

Chapter 5 - Numerical semigroups coated with odd elements. A numerical semigroup S is coated with odd elements (Coe-semigroup), if $\{x-1, x+1\} \subseteq S$ for all odd element x in S. In this chapter, we will study this kind of numerical semigroups. In particular, we are interested in the study of the Frobenius number, genus and embedding dimension of a numerical semigroup of this type. Furthermore, we arrange the set of all Coe-semigroups in a rooted tree and give an algorithm to recursively obtain its elements. Also, Coe-semigroups satisfy Wilf's conjecture if a particular condition holds. The results of this chapter are submitted for publication.

Chapter 6 - Numerical semigroups with distances no admissible between gaps greater than its multiplicity. In this chapter we will study the sets of numerical semigroups with distances no admissible between gaps greater than its multiplicity, denoted by $\mathcal{P}(A)$, and the ones with fixed multiplicity, denoted by $\mathcal{P}(A, m)$. First, we order the elements of $\mathcal{P}(A)$ in a tree with root $\mathbb{N}$. Second, we found that a partition of the set $\mathcal{P}(A, m)$ is a Frobenius pseudo-variety and, the set $\mathcal{P}(A, m)$ is a finite tree wherein its vertices are Frobenius pseudo-varieties. The results of this chapter are submitted for publication.

Chapter 7 - Frobenius R-Variety of the Numerical Semigroups Contained in a Given One. Let Δ be a numerical semigroup and $\mathrm{R}(\Delta) = \{S \mid S \text{ is a numerical semigroup and } S \subseteq \Delta\}$. We prove that $\mathrm{R}(\Delta)$ is a Frobenius R-variety that can be arranged in a tree rooted in Δ. We introduce the concepts of Frobenius and genus number of S restricted to Δ (respectively $\mathrm{F}_\Delta(S)$ and $\mathrm{g}_\Delta(S)$). We give formulas for $\mathrm{F}_\Delta(S)$, $\mathrm{g}_\Delta(S)$ and generalizations of the Bras-Amorós's and Wilf's conjecture. Moreover, we will show that most of the results of irreducibility can be generalized to $\mathrm{R}(\Delta)$-irreducibility. Results are published in [**RBT22a**].

CHAPTER 1

Numerical Semigroups Preliminaries

In this chapter we present some basic definitions and known results, needed later in this work, related to the numerical semigroups. Some more specific definitions and known results may be presented locally when needed. Results from sections 1 and 2 were extracted from [**RGS99**, chapter 10], [**RGS09**] and [**ADGS20**]. Section 3 is a compilation of [**Ros08b**], [**RR15**] and [**RR18**].

1. Notable elements

We use $\mathbb{N}$ and $\mathbb{Z}$ to denote the set of nonnegative integers and the set of the integers, respectively.

A **semigroup** is a pair $(S, +)$, where S is a nonempty set and $+$ is a binary operation defined on S verifying the associative law, that is, for all $a, b, c \in S$ we have $a+(b+c) = (a + b) + c$. If there exists an element $t \in S$ such that $t + s = s + t = s$ for all $s \in S$ we say that $(S, +)$ is a **monoid** . This element is usually denoted by 0. In addition, S is a **commutative monoid** if for all $a, b \in S$, $a + b = b + a$. An example of a commutative monoid is $(\mathbb{N}, +)$. All semigroups and monoids considered in this work are commutative. A **submonoid** of a monoid S is a subset A of S such that $0 \in A$ and for every $a, b \in A$ we have that $a + b \in A$. Clearly, $\{0\}$ and $\mathbb{N}$ are submonoids of $\mathbb{N}$.

Given a nonempty subset A of a monoid S, the **monoid generated** by A is the least (with respect to set inclusion) submonoid of S containing A, which turns out to be the intersection of all submonoids of S containing A. It follows easily that

$$\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid a_i \in A, \lambda_i \in \mathbb{N} \text{ for all } i \in \{1, \ldots, n\}\}$$

The set A is a **system of generators** of S if $\langle A \rangle = S$, and we will say that S is generated by A. A monoid S is finitely generated if there exists a system of generators of S with finitely many elements. Moreover, we say that A is a **minimal system of generators** of S if no proper subset of A generates S. It is denoted by $A = \mathrm{msg}(S)$.

Given two monoids X and Y, a map $f : X \to Y$ is a **monoid homomorphism** if $f(a + b) = f(a) + f(b)$ for all $a, b \in X$ and $f(0) = 0$. We say that f is a **monoid isomorphism** if f is bijective.

A **numerical semigroup** is a submonoid of $(\mathbb{N}, +)$ such that the greatest common divisor of its elements is equal to one.

Lemma 1. [**RGS99**, p. 105] [**RGS09**, Lemma 2.1] *Let A be a nonempty subset of* $\mathbb{N}$. *Then $\langle A \rangle$ is a numerical semigroup if and only if* $\gcd(A) = 1$.

Let S be a submonoid of $\mathbb{N}$ and let $\mathbf{G}(S)$ be the subgroup of $\mathbb{Z}$ generated by S (that is, $\mathbf{G}(S) = \{s' - s \mid s', s \in S\}$). If $1 \in \mathbf{G}(S)$, then we say that S is a **numerical semigroup**.

Example 2. Let $S = 2\mathbb{N}$, the set of even nonnegative integers. Then, S is a submonoid of $\mathbb{N}$. The group spanned by S is $2\mathbb{Z}$; hence S is not a numerical semigroup.

The following result gives us alternative ways of defining a numerical semigroup.

Proposition 3. [**RGS99**, p. 105] *Let S be a submonoid of* $\mathbb{N}$. *The following conditions are equivalent:*

(1) S is a numerical semigroup,
(2) the group spanned by S is $\mathbb{Z}$,
(3) $\mathbb{N} \backslash S$ is finite.

Proof. (1) *implies* (2) $\mathbf{G}(S) = n\mathbb{Z}$ for some $n \in \mathbb{N}$. Since $\gcd(S) = 1$, the element n must be equal to 1.

(2) *implies* (3) Since $\mathbf{G}(S) = \mathbb{Z}$, we have that $1 \in \mathbf{G}(S) = \mathbb{Z}$ and by the definition of $\mathbf{G}(S)$, there exists s and s' in S such that $s' - s = 1$. Hence $s' = s + 1$ which implies that

$$\{s, s+1, 2s+2, 3s+3, \ldots, (s-1)s + (s-1)\} \subseteq S.$$

We show that for every $n \in \mathbb{N}$ such that $n \geq (s-1)s + (s-1)$, the element n belongs to S. Let q and r be elements of $\mathbb{N}$ such that $n = qs + r$, with $0 \leq r < s$. Since $n = qs + r \geq (s-1)s + (s-1)$, we have $q \geq s - 1 \geq r$. Thus $n = qs + r = (q-r)s + r(s+1) \in S$.

(3) *implies* (1) Assume that $\mathbb{N} \backslash S$ has finitely many elements. Then there exist $s \in S$ such that $s + 1 \in S$. This implies that $\gcd(S) = 1$. □

Proposition 4. [**RGS09**, Proposition 2.2] *Every nontrivial submonoid of* $\mathbb{N}$ *is isomorphic to a numerical semigroup.*

Proof. Let M be a nontrivial submonoid of $\mathbb{N}$ and let $d = \gcd(M)$. We know that $S = \left\langle \left\{ \frac{m}{d} \mid m \in M \right\} \right\rangle$ is a numerical semigroup [**RGS09**, Lemma 2.1]. The map

$$f \colon M \to S,\ f(m) = \frac{m}{d}$$

is clearly a monoid isomorphism (*i.e.*, a bijective monoid homomorphism). □

Example 5. The submonoid $\langle 3, 7 \rangle = \{0, 3, 6, 7, 9, 10, 12, \rightarrow\}$ is an example of a numerical semigroup. The arrow means that every integer larger than 12 is in the set.

Taking into account Proposition 3 it makes sense to consider the greatest integer not belonging to S. We call this element the **Frobenius number** of S and it is denoted by $\mathrm{F}(S)$. The number $c(S) = F(S) + 1$ is said to be the **conductor** of S.

The set $\mathbb{N} \backslash S$ will be denoted by $G(S)$ and we call it the set of **gaps** of S. Its cardinality is called the **genus** of S and is denoted by $g(S)$. The **small elements** (also known as sporadic or left elements) of S are those elements that are smaller than $F(S)$. Its set is denoted $N(S)$ and its cardinality $n(S)$.

Given $n \in S\backslash\{0\}$, the **Apéry set** (named so in honour of [**Apé46**]) of S with respect to n is defined by

$$\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$$

A proof for the following result can be seen in [**RGS09**, Lemma 2.4].

Lemma 6. [**RGS09**, Lemma 2.4] *Let S be a numerical semigroup and let n be a nonzero element of S. Then,* $\mathrm{Ap}(S, n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$*, where $w(i)$ is the least element of S congruent with i modulo n, for all* $i \in \{0, \ldots, n-1\}$.

So the Apéry set consists of the least element of S in each congruence class modulo n. Hence, $\#\mathrm{Ap}(S, n) = n$. Here and from now on # before a set will denote its cardinality.

Lemma 7. [**RGS09**, Lemma 2.6] *Let S be a numerical semigroup and let $n \in S\backslash\{0\}$. Then for all $s \in S$, there exists a unique $(k, w) \in \mathbb{N} \times \mathrm{Ap}(S, n)$ such that*

$$s = kn + w.$$

The set $\mathrm{Ap}(S, n)$ determines completely the semigroup S, since $S = \langle \mathrm{Ap}(S, n) \cup \{n\} \rangle$. Moreover, $\mathrm{Ap}(S, n)$ contains in general more information that an arbitrary set of generators of S.

Remark 8. If S is a numerical semigroup and $n \in S\backslash\{0\}$ then $\mathrm{Ap}(S, n) = \{w(0) = 0, w(1), \ldots, w(n-1)\}$. From Lemma 7 we have that an integer z is in S if and only if $z \geq w(z \bmod n)$.

Let A and B be subsets of integer numbers. We define $A + B := \{a + b \ : \ a \in A, \ b \in B\}$. Furthermore, from hereon $S^* := S\backslash\{0\}$ sometimes.

Lemma 9. [**RGS09**, Lemma 2.3] *Let S be a numerical semigroup. Then $S^*\backslash(S^* + S^*)$ is a system of generators of S. Furthermore, every system of generators of S contains $S^*\backslash(S^* + S^*)$.*

Lemma 9, which says that $S^*\backslash(S^* + S^*)$ is the minimal system of generators of S, combined with $S = \langle \mathrm{Ap}(S, n) \cup \{n\}\rangle$, for any $n \in S\backslash\{0\}$, results:

Theorem 10. [**RGS09**, Theorem 2.7] *Every numerical semigroup admits a unique minimal system of generators. This minimal system of generators is finite.*

From Proposition 4 and Theorem 10 we obtain the following consequence.

Corollary 11. [**RGS09**, Corollary 2.8] *Every submonoid of* $(\mathbb{N}, +)$ *has a unique minimal system of generators, which is finite.*

Let S be a numerical semigroup. The cardinality of the minimal system of generators of S is called **embedding dimension** of S, and is denoted by $\mathrm{e}(S)$. The smallest nonzero element of S is called the **multiplicity** of S and is denoted by $\mathrm{m}(S)$.

Lemma 12. [**RGS09**, Proposition 2.10] *Let S be a numerical semigroup. We have $e(S) \leq m(S)$.*

Let S be a numerical semigroup. We say that S has **maximal embedding dimension** if $\mathrm{e}(S) = m(S)$. We will call it a MED-semigroup for short.

In particular, we have the **ordinary numerical semigroups** (also known as **half-line**), and denoted by O_m, which are MED-semigroups. Indeed, if $S = \{0, m, \rightarrow\}$ is a numerical semigroup with multiplicity m. It is easy to check that a minimal system of generators for S is $\{m, m+1, \ldots, 2m-1\}$ and $\mathrm{Ap}(S, m) = \{0, m+1, \ldots, 2m-1\}$. Hence $e(S) = m(S) = m$.

The next result is due to Selmer [**Sel77**] and can be used to compute $\mathrm{F}(S)$ and $\mathrm{g}(S)$, from one of the Apéry sets of the numerical semigroup S.

Proposition 13. [**RGS09**, Proposition 2.12] *Let S be a numerical semigroup and let n be a nonzero element of S. Then*

(1) $\mathrm{F}(S) = \max(\mathrm{Ap}(S, n)) - n$*;*

(2) $\mathrm{g}(S) = \frac{1}{n}(\sum_{w \in \mathrm{Ap}(S,n)} w) - \frac{n-1}{2}$.

Example 14. [Syl82, page 134][Syl84] Let $S = \langle a, b\rangle$ be a numerical semigroup. We have

$$\mathrm{Ap}(S, a) = \{0, b, 2b, \ldots, (a-1)b\}.$$

(1) $\mathrm{F}(S) = (a-1)b - a = ab - a - b$.
(2) $\mathrm{g}(S) = \frac{1}{a}(b + 2b + \cdots + (a-1)b) - \frac{a-1}{2} = \frac{(a-1)(b-1)}{2} = \frac{F(S)+1}{2}$.

Let S be a numerical semigroup. Following the notation introduced in [RB02], we say that the **pseudo-Frobenius numbers** of S are the elements of the set

$$\mathrm{PF}(S) = \{x \in \mathbb{Z}\backslash S \mid x + s \in S \text{ for every } s \in S\backslash\{0\}\}$$

The cardinality of the previous set is an important invariant of S called the **type** of S denoted by $\mathrm{t}(S)$. From the definition it easily follows that $\mathrm{F}(S) \in \mathrm{PF}(S)$, in fact, is the maximum of this set.

Let $a, b \in \mathbb{Z}$. We define $\leq_S$ as follows: $a \leq_S b$ if $b - a \in S$. Clearly, $\leq_S$ is a (partial) **order relation** (reflexive, antisymmetric, and transitive). With this order relation $\mathbb{Z}$ becomes a **partially ordered set** (or poset) . The following result states that $\mathrm{PF}(S)$ are precisely the maximal gaps of S with respect to $\leq_S$.

Proposition 15. [ADGS20, Proposition 7] *Let S be a numerical semigroup. We have*

$$\mathrm{PF}(S) = \mathrm{Maximals}_{\leq_S}(\mathbb{Z}\backslash S).$$

We can also recover the pseudo-Frobenius elements in terms of the Apéry sets.

Proposition 16. [ADGS20, Proposition 8] *Let S be a numerical semigroup and let $n \in S^*$. Then*

$$\mathrm{PF}(S) = \{w - n \mid w \in \mathrm{Maximals}_{\leq_S} \mathrm{Ap}(S, n)\}$$

From the previous proposition, we obtain an upper bound for the type of a numerical semigroup.

Corollary 17. *Let S be a numerical semigroup other than $\mathbb{N}$, then*

$$t(S) \leq m(S) - 1$$

2. Irreducible numerical semigroups

One type of numerical semigroups which are among the most studied are the irreducible numerical semigroups for their relevance in ring theory. A numerical semigroup is **irreducible** if it cannot be expressed as an intersection of two numerical semigroups properly containing it. Irreducible numerical semigroups gather both **symmetric** and **pseudo-symmetric** numerical semigroups.

Lemma 18. [RGS09, Lemma 4.1] *Let S be a numerical semigroup other than $\mathbb{N}$. Then $S \cup \{F(S)\}$ is a numerical semigroup.*

Proof. The complement of $S \cup \{F(S)\}$ in $\mathbb{N}$ is finite, because $\mathbb{N} \backslash S$ is finite.

Take $a, b \in S \cup \{F(S)\}$. If any of them is $F(S)$, then $a + b \geq F(S)$ and thus $a + b \in S \cup \{F(S)\}$. If both a and b are in S, then $a + b \in S \subseteq S \cup \{F(S)\}$. As $0 \in S \cup \{F(S)\}$, this proves that $S \cup \{F(S)\}$ is a numerical semigroup. □

Lemma 19. [RGS09, Exercise 2.2] *Let S and T be numerical semigroups. Then $S \cap T$ is a numerical semigroup.*

Proof. Clearly, $S \cap T$ contains the zero element. Now, if $a, b \in S \cap T$, then $a, b \in S$ and $a, b \in T$, both of which are closed under addition. So $a+b$ lives in S and T, thereby $a + b \in S \cap T$.

Finally, as S and T are numerical semigroups the complements of S and T in $\mathbb{N}$ are finite, $\overline{S} = \mathbb{N} \backslash S$ and $\overline{T} = \mathbb{N} \backslash T$ respectively. By De Morgan's law of intersection we have $\overline{S} \cup \overline{T} = \overline{(S \cap T)}$, combined with the fact that the union of finite sets is finite leads us to the desired result. Namely $\overline{S \cap T} = \mathbb{N} \backslash (S \cap T)$ is finite, and this completes the proof. □

The next result shows that the irreducible numerical semigroups are maximal in the set of numerical semigroups with fixed Frobenius number.

Theorem 20. [RB03, Theorem 1] *The following conditions are equivalent:*

(1) S *is irreducible;*

(2) S *is maximal in the set of all numerical semigroups with Frobenius number* $\mathrm{F}(S)$*;*

(3) S *is maximal in the set of all numerical semigroups that do not contain* $\mathrm{F}(S)$.

A numerical semigroup S is **symmetric** (respectively, **pseudo-symmetric**) if it is irreducible and $\mathrm{F}(S)$ is odd (respectively, even).

Proposition 21. [RGS09, Proposition 4.4] *Let* S *be a numerical semigroup.*

(1) S *is symmetric if and only if* $\mathrm{F}(S)$ *is odd and* $x \in \mathbb{Z}\backslash S$ *implies* $\mathrm{F}(S) - x \in S$*;*

(2) S *is pseudo-symmetric if and only if* $\mathrm{F}(S)$ *is even and* $x \in \mathbb{Z}\backslash S$ *implies that either* $\mathrm{F}(S) - x \in S$ *or* $x = \frac{\mathrm{F}(S)}{2}$.

Sometimes the previous proposition is used as the definition of the concepts of symmetric and pseudo-symmetric numerical semigroups.

The maximality of irreducible numerical semigroups in the set of numerical semigroups with the same Frobenius number translates to minimality in terms of gaps.

Corollary 22. [RGS09, Corollary 4.5] *Let* S *be a numerical semigroup.*

(1) S *is symmetric if and only if* $\mathrm{g}(S) = \dfrac{\mathrm{F}(S)+1}{2}$.

(2) S *is pseudo-symmetric if and only if* $\mathrm{g}(S) = \dfrac{\mathrm{F}(S)+2}{2}$.

Hence irreducible numerical semigroups are those with the least possible genus once the Frobenius number is fixed.

Corollary 23. [RGS09, Corollary 4.7] *Every numerical semigroup with embedding dimension two is symmetric.*

A numerical semigroup is said to be **almost symmetric** if its genus is the arithmetic mean of its Frobenius number and its type. It is a class of semigroups that includes the symmetric and the pseudo-symmetric ones.

Corollary 24. [ADGS20, Corollary 10] *Every irreducible numerical semigroup is almost symmetric.*

Note that Wilf's conjecture holds for irreducible numerical semigroups [DM06, Proposition 2.2].

The next result reveals that we can decompose a numerical semigroup into irreducibles.

Proposition 25. [RGS09, Proposition 4.44] *Every numerical semigroup can be expressed as a finite intersection of irreducible numerical semigroups.*

3. Common behaviors in families of numerical semigroups

3.1. Trees. A **graph** $G = (V, E)$ consists of a set V and a collection E of ordered pairs (v, w) of distinct elements from V. Elements of V are called **vertices** and elements of E are called **edges**. A **path** of length n connecting the vertices u and v of G is a sequence of n distinct edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ with $v_0 = u$ and $v_n = v$.

A graph G is a **tree** if there exists a vertex r (known as the root of G) such that for every other vertex v of G, there exists a unique path connecting v and r. If (u, v) is an edge of a tree, then we say that u is a **child** of v. If there exists a path connecting the vertices u and v, then we say that u is a **descendant** of v.

A **binary tree** is a rooted tree in which every vertex has 0, 1 or 2 children. A vertex of a tree with no child is a **leaf**.

Definition 26 (The tree of the set of all numerical semigroups)**.** Let $\mathcal{S}$ be the set formed by all numerical semigroups. We denote by $G(\mathcal{S})$ the tree associated do $\mathcal{S}$. In this tree, the vertices are the elements of $\mathcal{S}$, (T, S) is an edge if $S = T \cup \{\mathrm{F}(T)\}$ and $\mathbb{N}$ is the root.

Proposition 27. *If S is a numerical semigroup, then the unique path connecting S with $\mathbb{N}$ is given by $C(S) = \{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n\}$ (the **chain** of numerical semigroups associated to S), where $S_0 = S$, $S_{i+1} = S_i \cup \{F(S_i)\}$, for all $i < n$, and $S_n = \mathbb{N}$.*

Example 28. $C(\langle 4,5,6,7\rangle) = \{\langle 4,5,6,7\rangle, \langle 3,4,5\rangle, \langle 2,3\rangle, \langle 1\rangle = \mathbb{N}\}$.

Note that if $S = T \cup \{\mathrm{F}(T)\}$, then $\mathrm{F}(T)$ becomes a minimal generator of S, which in addition is greater than the Frobenius number of S. Conversely, if we choose a minimal generator a of S, then $S\setminus\{a\}$ is a numerical semigroup, and if this generator is greater than the Frobenius number, then $\mathrm{F}(S\setminus\{a\}) = a$. With this information in mind, we have the following property.

Proposition 29. [**RGS09**, Proposition 7.1] *The children of $S \in \mathcal{S}$ are $S\setminus\{a_1\}, \ldots, S\setminus\{a_r\}$, where $a_1, \ldots, a_r$ are the minimal generators of S that are greater than* $\mathrm{F}(S)$.

This result allows us to construct recurrently the set of all numerical semigroups.

Example 30. The first levels (with respect to the genus) of $G(\mathcal{S})$, figure 1.

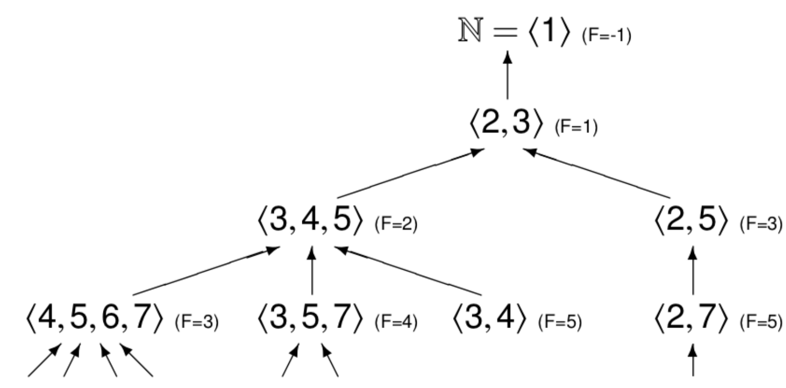


FIGURE 1. The tree of the set of all numerical semigroups

Observe that $\langle 3,4\rangle$ is a leaf: it has not got any child.

The purpose of this section is to define a structure that allows us to build and arrange the elements of families of numerical semigroups. To perform this task we will see definitions, monoids associated and the minimal system of generators with respect to varieties, pseudo-varieties and restricted varieties.

3.2. Analysis of the set of numerical semigroups. We have already seen that if S and T are numerical semigroups (with $S \neq \mathbb{N}$). Then $S \cup \{\mathrm{F}(S)\}$ and $S \cap T$ are numerical semigroups, Lemmas 18 and 19.

Lemma 31. [**Ros08a**, Lemma 1.7] *Let S be a numerical semigroup. Then $S\setminus\{a\}$ is a numerical semigroup if and only if $a \in \mathrm{msg}(S)$.*

Proposition 32. [**RGS09**, Propositon 7.1] *Let S and T be numerical semigroups. Then $S = T \cup \{\mathrm{F}(T)\}$ if and only if $T = S\setminus\{a\}$ for some $a \in \mathrm{msg}(S)$ such that $a > \mathrm{F}(S)$.*

Proposition 33. [**RGS09**, Propositon 7.4] *Let S and T be numerical semigroups such that $S = T \cup \{\mathrm{F}(T)\}$. Then $\mathrm{F}(S) < \mathrm{F}(T)$ and $\mathrm{g}(T) = \mathrm{g}(S) + 1$.*

These facts enable us to construct recursively all numerical semigroups with genus $g + 1$ from all numerical semigroups with genus g.

3.3. Frobenius varieties. For certain families of numerical semigroups we can observe a similar behavior to that described in the two previous subsections. This observation allows us to introduce the concept of (Frobenius) variety.

Definition 34. [**Ros08b**] A **Frobenius variety** is a nonempty family $\mathcal{V}$ of numerical semigroups fulfilling the following conditions:

(1) if $S, T \in \mathcal{V}$, then $S \cap T \in \mathcal{V}$;

(2) if $S \in \mathcal{V}$ and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{V}$.

Clearly the set of all numerical semigroups is a Frobenius variety. The chain associated to a numerical semigroup is also a Frobenius variety. Moreover, the set of oversemigroups of S is a Frobenius variety. Other notable examples of Frobenius varieties are:

Example 35. Families that are varieties:

- Arf numerical semigroups;
- Saturated numerical semigroups;
- Numerical semigroups having a Toms decomposition;
- Numerical semigroups defined by strongly admissible linear patterns.

Example 36. Families that are not varieties:

- Numerical semigroups with maximal embedding dimension and multiplicity m;
- Numerical semigroups defined by non-homogeneous patterns.

Remark 37. $\mathbb{N}$ belongs to every Frobenius variety.

Proposition 38. [**RGS09**, Proposition 7.19] *The intersection of Frobenius varieties is a Frobenius variety.*

Example 39. The set of Arf numerical semigroups having a Toms decomposition is a Frobenius variety.

3.4. $\mathcal{V}$-monoids and $\mathcal{V}$-systems of generators. Lemma 19 proves that the intersection of finitely many numerical semigroups is a numerical semigroup. However, nonfinite intersections of numerical semigroups are not in general numerical semigroups as it is shown in the following example. Nevertheless, they are always submonoids of $\mathbb{N}$.

Example 40. It can be easily seen that $\bigcap_{n\in\mathbb{N}} \langle n, n+1\rangle = \{0\}$.

Let $\mathcal{V}$ be a Frobenius variety. A submonoid M of $\mathbb{N}$ is a **$\mathcal{V}$-monoid** if it can be expressed as an intersection of elements in $\mathcal{V}$.

Lemma 41. [**Ros08b**, Lemma 10] *The intersection of $\mathcal{V}$-monoids is a $\mathcal{V}$-monoid.*

Let $A \subseteq \mathbb{N}$. The **$\mathcal{V}$-monoid generated** by A (denoted by $\mathcal{V}(A)$) is the intersection of all the $\mathcal{V}$-monoids containing A.

Lemma 42. [**Ros08b**, Lemma 11] $\mathcal{V}(A)$ *is the intersection of all elements of $\mathcal{V}$ containing A.*

If $M = \mathcal{V}(A)$, then A is a **$\mathcal{V}$-system of generators** of M.

If no proper subset of A is a $\mathcal{V}$-system of generators of M, then we say that A is a **minimal $\mathcal{V}$-system of generators** of M.

Theorem 43. [**Ros08b**, Corollary 19] *Every $\mathcal{V}$-monoid M has a unique minimal $\mathcal{V}$-system of generators, which in addition is finite ($A = \mathrm{msg}_{\mathcal{V}}(M)$).*

Lemma 44. [**Ros08b**, Proposition 24] *If M is a $\mathcal{V}$-monoid and $x \in M$, then the set $M \backslash \{x\}$ is a $\mathcal{V}$-monoid if and only if $x \in \mathrm{msg}_{\mathcal{V}}(M)$.*

3.5. The tree of a Frobenius variety. Let $\mathcal{V}$ be a Frobenius variety. Let $G(\mathcal{V})$ be the **tree** associated to $\mathcal{V}$. We have that the vertices are the elements of $\mathcal{V}$, (T, S) is an edge if $S = T \cup \{\mathrm{F}(T)\}$ and $\mathbb{N}$ is the root.

If $S \in \mathcal{V}$, then the unique path connecting S with $\mathbb{N}$ is $C(S) = \{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n\}$ (the **chain** of numerical semigroups associated to S), where $S_0 = S$, $S_{i+1} = S_i \cup \{F(S_i)\}$, for all $i < n$, and $S_n = \mathbb{N}$.

Theorem 45. [**Ros08b**, Theorem 27] *Let $\mathcal{V}$ be a Frobenius variety. The graph $G(\mathcal{V})$ is a tree with root equal to $\mathbb{N}$. Furthermore, the children of a vertex $S \in \mathcal{V}$ are $S \backslash \{a_1\}, \ldots, S \backslash \{a_r\}$ where $a_1, \ldots, a_r$ are the elements of* $\mathrm{msg}_{\mathcal{V}}(s)$ *which are greater than* $\mathrm{F}(S)$.

3.6. Frobenius pseudo-varieties. The common behavior of families of numerical semigroups led up to defining the Frobenius varieties. However, there exist families of numerical semigroups that are not varieties, but with a very similar behavior.

The family of numerical semigroups with maximal embedding dimension and multiplicity equal to m is not a variety.

The following definition generalizes the concept of varieties.

Definition 46. [**RR15**] A **Frobenius pseudo-variety** is a non-empty family $\mathcal{P}$ of numerical semigroups that fulfills the following conditions:

(1) $\mathcal{P}$ has a maximum element $\max(\mathcal{P})$ (with respect to the inclusion order);
(2) If $S, T \in \mathcal{P}$, then $S \cap T \in \mathcal{P}$;
(3) If $S \in \mathcal{P}$ and $S \neq \max(\mathcal{P})$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{P}$.

Example 47. Examples of pseudo-varieties:

- The set of all numerical semigroups with multiplicity m.
- The set of numerical semigroups with maximal embedding dimension and multiplicity m.
- The set of numerical semigroups admitting a strong admissible pattern and multiplicity m: m-varieties [**BAGSVO13**].

In general, the intersection of pseudo-varieties is not a pseudo-variety. For example, if S and T are different numerical semigroups, then $\mathcal{P}_1 = \{S\}$ and $\mathcal{P}_2 = \{T\}$ are pseudo-varieties, but $\mathcal{P}_1 \cap \mathcal{P}_2 = \{\emptyset\}$ is not.

3.7. Pseudo-varieties and varieties. From the definitions, it is clear that every variety is a pseudo-variety. However, there are pseudo-varieties that are not varieties. For example, if S is a numerical semigroup different from $\mathbb{N}$, then $\{S\}$ is a pseudo-variety but not a variety.

Proposition 48. [**RR15**, Propositon 2.1] *If $\mathcal{P}$ is a pseudo-variety, then $\mathcal{P}$ is a variety if and only if $\mathbb{N} \in \mathcal{P}$.*

Proposition 49. [**RR15**, Propositon 2.2] *If $\mathcal{P}$ is a family of numerical semigroups with maximum Δ, then $\mathcal{P}$ is a pseudo-variety if and only if $\mathcal{P} \cup C(\Delta)$ is a variety.*

Lemma 50. [**RR15**, Lemma 2.4] *If $\mathcal{P}$ is a pseudo-variety and $S \in \mathcal{P}$, then* $\max(\mathcal{P}) \in C(S)$.

Lemma 51. [**RR15**, Lemma 2.6] *If S_1, S_2, Δ are numerical semigroups such that $\Delta \in C(S_1) \cap C(S_2)$, then $\Delta \in C(S_1 \cap S_2)$.*

Theorem 52. [**RR15**, Theorem 2.7] *Let $\mathcal{V}$ be a variety and let $\Delta \in \mathcal{V}$. Then $\mathcal{D}(\mathcal{V}, \Delta) = \{S \in \mathcal{V} \mid \Delta \in C(S)\}$ is a pseudo-variety. Every pseudo-variety can be obtained in this way.*

Example 53. [**RR15**, Example 2.8] $D(\mathcal{S}, \{0, m, \rightarrow\}) = \{S \in \mathcal{S} \mid S \in \{0, m, \rightarrow\}\}$ is a pseudo-variety.

3.8. $\mathcal{P}$-monoids and $\mathcal{P}$-systems of generators. Let $\mathcal{P}$ be a pseudo-variety. A submonoid M of $\mathbb{N}$ is a **$\mathcal{P}$-monoid** if it can be expressed as an intersection of elements in $\mathcal{P}$.

Lemma 54. [**RR15**, Lemma 3.1] *The intersection of $\mathcal{P}$-monoids is a $\mathcal{P}$-monoid.*

Let $A \subseteq \max(\mathcal{P})$. The **$\mathcal{P}$-monoid generated** by A (denoted by $\mathcal{P}(A)$) is the intersection of all the $\mathcal{P}$-monoids containing A.

Lemma 55. [**RR15**, Lemma 3.2] *$\mathcal{P}(A)$ is the intersection of all elements of $\mathcal{P}$ containing A.*

If $M = \mathcal{P}(A)$, then A is a **$\mathcal{P}$-system of generators** of M.

If no proper subset of A is a $\mathcal{P}$-system of generators of M, then we say that A is a **minimal $\mathcal{P}$-system of generators** of M.

Theorem 56. [**RR15**, Corollary 3.9] *Every $\mathcal{P}$-monoid M has a unique minimal $\mathcal{P}$-system of generators, which in addition is finite ($A = \mathrm{msg}_{\mathcal{P}}(M)$).*

Lemma 57. [**RR15**, Lemma 4.2] *If M is a $\mathcal{P}$-monoid and $x \in M$, then the set $M \backslash \{x\}$ is a $\mathcal{P}$-monoid if and only if $x \in \mathrm{msg}_{\mathcal{P}}(M)$.*

3.9. The tree of a pseudo-variety. Let $\mathcal{P}$ be a pseudo-variety (with maximum $\max(\mathcal{P})$). Let $G(\mathcal{P})$ be the **tree** associated to $\mathcal{P}$. We have that the vertices are the elements of $\mathcal{P}$, (T, S) is an edge if $S = T \cup \{\mathrm{F}(T)\}$ and $\max(\mathcal{P})$ is the root.

If $S \in \mathcal{P}$, then the unique path connecting S with $\max(\mathcal{P})$ is the **chain** $C_{\mathcal{P}}(S) = \{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n\}$, where $S_0 = S$, $S_{i+1} = S_i \cup \{F(S_i)\}$, for all $i < n$, and $S_n = \max(\mathcal{P})$.

Theorem 58. [**RR15**, Theorem 4.3] *The children of $S \in \mathcal{P}$ are $S\backslash\{a_1\}, \ldots, S\backslash\{a_r\}$ where $a_1, \ldots, a_r$ are the elements of $\mathrm{msg}_{\mathcal{P}}(S)$ which are greater than $\mathrm{F}(S)$.*

A pseudo-variety is a subtree obtained from the tree of a variety when we take a vertex and all its descendants.

3.10. Frobenius restricted varieties. The concept of Frobenius restricted variety (R-variety for short) generalizes the concept of pseudo-variety and there exist significant families of numerical semigroups which are R-varieties but not pseudo-varieties (see [**RR18**, Example 2.3]).

If S and T are numerical semigroups such that $S \subsetneq T$, then $S \cup \{\max(T\backslash S)\}$ is another numerical semigroup [**RGS09**, Lemma 4.35].

$F_T(S) = \max(T\backslash S)$ is the **Frobenius number of S restricted to T**.

Definition 59. [**RR18**] A ***R*-variety** (that is, **Frobenius restricted variety**) is a non-empty family $\mathcal{R}$ of numerical semigroups that fulfills the following conditions:

(1) $\mathcal{R}$ has a maximum element $\max(\mathcal{R})$ (with respect to the inclusion order);
(2) If $S, T \in \mathcal{R}$, then $S \cap T \in \mathcal{R}$;
(3) If $S \in \mathcal{R}$ and $S \neq \max(\mathcal{R})$, then $S \cup \{F_{\max(\mathcal{R})}(S)\} \in \mathcal{R}$.

3.11. *R*-varieties, pseudo-varieties and varieties.

Proposition 60. [**RR18**, Proposition 2.2] *Every pseudo-variety is a R-variety.*

The converse of the above result is false (see [**RR18**, Example 2.3]).

Proposition 61. [**RR18**, Proposition 2.4] *If $\mathcal{R}$ is a R-variety, then $\mathcal{R}$ is a pseudo-variety if and only if* $\mathrm{F}(S) \in \max(\mathcal{R})$ *for all* $S \in \mathcal{R}$ *such that* $S \neq \max(\mathcal{R})$.

Corollary 62. [**RR18**, Corollary 2.5] *If $\mathcal{R}$ is a R-variety, then $\mathcal{R}$ is a variety if and only if* $\mathbb{N} \in \mathcal{R}$.

Theorem 63. [**RR18**, Theorem 2.9] *Let $\mathcal{V}$ be a variety and let T be a numerical semigroup. Then $\mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\}$ is a R-variety. Every R-variety is of this form.*

Corollary 64. [**RR18**, Crollary 2.11] *If $\mathcal{R}$ is a R-variety and U is a numerical semigroup, then $\mathcal{R}_U = \{S \cap U \mid S \in \mathcal{R}\}$ is a R-variety.*

Corollary 65. [**RR18**, Crollary 2.12] *Let $\mathcal{P}$ be a pseudo-variety and T be a numerical semigroup. Then $\mathcal{P}_T = \{S \cap T \mid S \in \mathcal{P}\}$ is a R-variety. Every R-variety is of this form.*

3.12. $\mathcal{R}$-monoids and $\mathcal{R}$-systems of generators. Let $\mathcal{R}$ be an R-variety. A submonoid M of $\mathbb{N}$ is a **$\mathcal{R}$-monoid** if it can be expressed as an intersection of elements in $\mathcal{R}$.

Lemma 66. [**RR18**, Lemma 3.1] *The intersection of $\mathcal{R}$-monoids is a $\mathcal{R}$-monoid.*

Let $A \subseteq \max(\mathcal{R})$. The **$\mathcal{R}$-monoid generated** by A (denoted by $\mathcal{R}(A)$) is the intersection of all the $\mathcal{R}$-monoids containing A.

Lemma 67. [**RR18**, Lemma 3.2] $\mathcal{R}(A)$ *is the intersection of all elements of $\mathcal{R}$ containing A.*

If $M = \mathcal{R}(A)$, then A is a **$\mathcal{R}$-system of generators** of M.

If no proper subset of A is a $\mathcal{R}$-system of generators of M, then we say that A is a **minimal $\mathcal{R}$-system of generators** of M.

Theorem 68. [**RR18**, Theorem 3.7] *Every $\mathcal{R}$-monoid M has a unique minimal $\mathcal{R}$-system of generators, which in addition is finite ($A = \mathrm{msg}_{\mathcal{R}}(M)$).*

Lemma 69. [**RR18**, Proposition 3.8] *If M is a $\mathcal{R}$-monoid and $x \in M$, then the set $M\backslash\{x\}$ is a $\mathcal{R}$-monoid if and only if $x \in \mathrm{msg}_{\mathcal{R}}(M)$.*

3.13. The tree of an R-variety. Let $\mathcal{R}$ be an R-variety (with maximum $\max(\mathcal{R})$). Let $G(\mathcal{R})$ be the **tree** associated to $\mathcal{R}$. We have that the vertices are the elements of $\mathcal{R}$, (T, S) is an edge if $S = T \cup \{F_{\max(\mathcal{R})}(T)\}$ and $\max(\mathcal{R})$ is the root.

If $S \in \mathcal{R}$, then the unique path connecting S with $\max(\mathcal{R})$ is the **chain** $C_{\mathcal{R}}(S) = \{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n\}$, where $S_0 = S$, $S_{i+1} = S_i \cup \{F_{\max(\mathcal{R})}(S_i)\}$, for all $i < n$, and $S_n = \max(\mathcal{R})$.

Theorem 70. [**RR18**, Theorem 4.4] *The children of $S \in \mathcal{R}$ are $S\backslash\{a_1\}, \ldots, S\backslash\{a_r\}$ where $a_1, \ldots, a_r$ are the elements of $\mathrm{msg}_{\mathcal{R}}(S)$ which are greater than $\mathrm{F}(S)$.*

There exists R-varieties that are not the set formed by all the descendants of an element belonging to a variety.

CHAPTER 2

Modularly Equidistant numerical semigroups

If S is a numerical semigroup and $s \in S$, we denote by $\mathrm{next}_S(s) = \min\{x \in S \mid s < x\}$. Let a be an integer greater than or equal to two. A numerical semigroup is equidistant modulo a if $\mathrm{next}_S(s) - s - 1$ is a multiple of a for every $s \in S$. In this chapter we give algorithms for computing the whole set of equidistant numerical semigroups modulo a with fixed multiplicity, genus and Frobenius number. Moreover, we will study this kind of semigroups with maximal embedding dimension. Results are published in [**RBT21**].

1. Definitions and preliminaries

Given a numerical semigroup S and $s \in S$, we denote by $\mathrm{next}_S(s) = \min\{x \in S \mid s < x\}$. For $a \in \mathbb{N}\backslash\{0, 1\}$, we say that S is an **equidistant numerical semigroup modulo** a if $\mathrm{next}_S(s) - s - 1$ is a multiple of a for every $s \in S$.

We denote by

$$\mathsf{E}(a) = \{S \mid S \text{ is equidistant numerical semigroup modulo } a\}.$$

Our aim in this chapter is to study of this kind of numerical semigroups.

This work is a generalization of the study of the parity numerical semigroups [**MR20a**]. Indeed a numerical semigroup S is parity if $s + \mathrm{next}_S(s)$ is odd for every $s \in S$. Clearly $s + \mathrm{next}_S(s)$ is odd if and only if $\mathrm{next}_S(s) - s - 1$ is a multiple of 2. Therefore, the parity numerical semigroups are equidistant numerical semigroups modulo 2.

A numerical semigroup S is prefect if $\{x-1, x+1\} \subseteq S$ then $x \in S$ (see for instance [MR19], [MR20b]). Observe that every equidistant modularly numerical semigroup is a perfect numerical semigroup. In fact, if S is equidistant numerical semigroup modulo a and $\{x-1, x+1\} \subseteq S$ then $x \in S$, because otherwise $\mathrm{next}_S(x-1) = x+1$ and then $\mathrm{next}_S(x-1)-(x-1)-1 = x+1-x+1-1 = 1$ that is not a multiple of a.

We briefly outline the structure of this chapter. In Section 2, we will order the elements of $\mathsf{E}(a)$ in a tree rooted in $\mathbb{N}$. We will characterize the children of a vertex and this will allow us to build recursively the elements of $\mathsf{E}(a)$.

As a consequence of the results of Section 2, we will be presented in Section 3 an algorithmic procedure to compute all elements in $\mathsf{E}(a)$ with a given multiplicity and genus. In Section 4, we get an algorithm for computing the set $\mathsf{E}(a)$ with a given Frobenius number and we also characterize the maximal elements of this set.

In Section 5, we characterize the elements in $\mathsf{E}(a)$ with maximal embedding dimension.

2. The tree associated to $\mathsf{E}(a)$

In this section, we characterize the set $\mathsf{E}(a)$, for a given a integer greater than or equal to 2. We see how this set can be arranged in a tree.

Lemma 71. *If $S \in \mathsf{E}(a)$ then $\mathrm{m}(S) \equiv 1 (mod\ \ a)$.*

Proof. Clearly, $\mathrm{next}_S(0) = \mathrm{m}(S)$ and thus $\mathrm{m}(S)-0-1$ is a multiple of a. Therefore, we have that $\mathrm{m}(S) \equiv 1 (\mathrm{mod}\ \ a)$. □

It is clear that $\mathbb{N}$ is a numerical semigroup that belongs to $\mathsf{E}(a)$. If $S \in \mathsf{E}(a)$ and $S \neq \mathbb{N}$ then $\mathrm{m}(S) \geq 2$, and so there exists $k \in \mathbb{N}\backslash\{0\}$ such that $\mathrm{m}(S) = 1+ka$. Therefore, $a < \mathrm{m}(S)$.

Lemma 72. *If $S \in \mathsf{E}(a)\backslash\{\mathbb{N}\}$, then $\{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} \subseteq \mathbb{N}\backslash S$.*

Proof. Since $\mathrm{F}(S) \geq \mathrm{m}(S) - 1 \geq a$ we have that $\{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} \subseteq \mathbb{N}\backslash\{0\}$. Assume that there exists $i \in \{1, \ldots, a-1\}$ such that $\mathrm{F}(S) - i \in S$ and let $t = \min\{i \in \{1, \ldots, a-1\} \mid \mathrm{F}(S) - i \in S\}$. Then $\mathrm{next}_S(\mathrm{F}(S) - t) = \mathrm{F}(S) + 1$. As $S \in \mathsf{E}(a)$ then $\mathrm{F}(S) + 1 - (\mathrm{F}(S) - t) - 1$ is a multiple of a. Hence $t = la$ for some $l \in \mathbb{N}\backslash\{0\}$ and consequently $t \geq a$, which is impossible. □

Lemma 73. *If $S \in \mathsf{E}(a)\backslash\{\mathbb{N}\}$, then $S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} \in \mathsf{E}(a)$.*

Proof. Since $a < \mathrm{m}(S)$, by applying Lemma 72, we get that $T = S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\}$ is a numerical semigroup. Let $s \in S$ such that $\mathrm{next}_S(s) = \mathrm{F}(S) + 1$. In order to conclude the proof, it suffices to show that $\mathrm{next}_T(s) - s - 1$ is a multiple of a. As $\mathrm{next}_T(s) = \mathrm{F}(S) - (a-1)$ then $\mathrm{next}_T(s) - s - 1$ is multiple of a if and only if $\mathrm{F}(S) - (a-1) - s - 1 = \mathrm{F}(S) - s - a$ is multiple of a. But this is true, because if $S \in \mathsf{E}(a)$ then $\mathrm{F}(S) + 1 - s - 1 = \mathrm{next}_S(s) - s - 1$ is multiple of a. □

The previous result can be viewed as a procedure to construct a sequence $\{S_n \mid n \in \mathbb{N}\}$ of elements in $\mathsf{E}(a)$. Given a numerical semigroup S, we define recursively

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n), \mathrm{F}(S_n)-1, \ldots, \mathrm{F}(S_n)-(a-1)\} & \text{if } S_n \neq \mathbb{N} \\ \mathbb{N} & \text{otherwise.} \end{cases}$

The next result can be easily proved.

Proposition 74. *If $S \in \mathsf{E}(a)$ and $\{S_n \mid n \in \mathbb{N}\}$ is the previous sequence, then there exists $k \in \mathbb{N}$ such that $S_k = \mathbb{N}$. Furthermore, for every $i \in \{0, 1, \ldots, k-1\}$ then $\mathrm{card}(S_{i+1}\backslash S_i) = a$ and $k = \frac{\mathrm{g}(S)}{a}$.*

We illustrate the previous result with an example.

Example 75. In the following sequence of elements of $\mathsf{E}(2)$ we have $k = 3$.

$$S_0 =< 5, 6, 7 >\subsetneq S_1 =< 5, 6, 7, 8, 9 >\subsetneq S_2 =< 3, 4, 5 >\subsetneq S_3 = \mathbb{N}.$$

We define the **graph** $G(\mathsf{E}(a))$ as graph whose vertices are the elements of $\mathsf{E}(a)$ and $(S,T) \in \mathsf{E}(a) \times \mathsf{E}(a)$ is an edge if $T = S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\}$. From Proposition 74, we deduce the following.

Theorem 76. *The graph $G(\mathsf{E}(a))$ is a **tree** with root equal to $\mathbb{N}$.*

Note that the tree $G(\mathsf{E}(a))$ can be constructed recursively, from the root $\mathbb{N}$ in each step we are joining each of the vertices with its children. Our next goal is to characterize the children of an arbitrary vertex in the tree $G(\mathsf{E}(a))$. We distinguish two cases depending on whether or not the vertex is $\mathbb{N}$.

Lemma 77. *The vertex $\mathbb{N}$ has a unique child in the tree $G(\mathsf{E}(a))$ that is $\{0, a+1, \rightarrow\}$.*

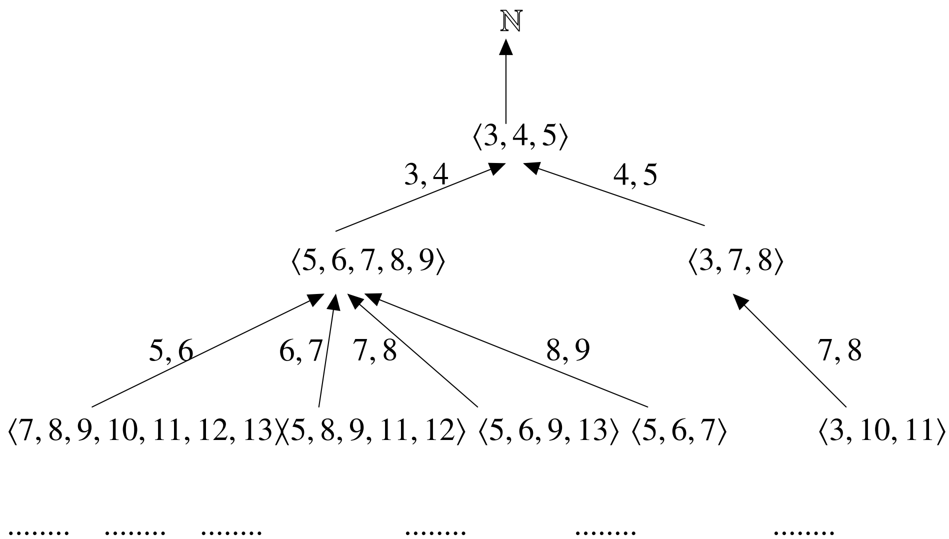
Proof. If S is a child of $\mathbb{N}$, then $S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} = \mathbb{N}$ and thus $S = \{0, a+1, \rightarrow\}$. □

Proposition 78. *Let $T \in \mathsf{E}(a)\backslash\{\mathbb{N}\}$. Then the set of children of T, in the tree $G(\mathsf{E}(a))$, is equal to $\{T\backslash\{x, x+1, \ldots, x+(a-1)\} \mid \{x, x+1, \ldots, x+(a-1)\} \subseteq \mathrm{msg}(T)$ and $x > \mathrm{F}(T)\}$.*

Proof. If $T \in \mathsf{E}(a)\backslash\{\mathbb{N}\}$, then by Lemma 71, we have that $\mathrm{m}(T) > a$. Since S is a child of T then $T = S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\}$. Hence $\{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} \subseteq \mathrm{msg}(T)$ and $\mathrm{F}(S)-(a-1) > \mathrm{F}(T)$.

Conversely, if $\{x, x+1, \ldots, x+(a-1)\} \subseteq \mathrm{msg}(T)$ and $x > \mathrm{F}(T)$ then, using repeatedly Lemma 31, we obtain that $S = T\backslash\{x, x+1, \ldots, x+(a-1)\}$ is a numerical semigroup with $\mathrm{F}(S) = x + a - 1$. Therefore, $T = S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\}$ and thus S is a child of T. □

Example 79. Let us construct recursively the tree $G(\mathsf{E}(2))$.



The numbers that appear on either side of the edges is the elements that we remove from the semigroup to obtain its child. For example $\langle 3,4,5\rangle\backslash\{3,4\} = \langle 5,6,7,8,9\rangle$.

Observe that $\mathsf{E}(a)$ has infinite cardinality, because $\{0, ka+1, \rightarrow)\} \in \mathsf{E}(a)$ for all $k \in \mathbb{N}$.

3. The set $\mathsf{E}(a)$ with a given multiplicity and genus

We will denote by $\mathsf{E}(a,m) = \{S \in \mathsf{E}(a) \mid \mathrm{m}(S) = m\}$. By Lemma 71, we obtain that $\mathsf{E}(a,m) \neq \emptyset$ if and only if $m = ka+1$ for some $k \in \mathbb{N}$. It is clear that $\mathsf{E}(a,1) = \{\mathbb{N}\}$.

From now on, assume that $m = ka+1$ with $k \in \mathbb{N}\backslash\{0\}$. It is clear that maximum element in $\mathsf{E}(a)$ is $\langle m, m+1, \ldots, 2m-1\rangle = \{0, m, \rightarrow\}$. From Lemma 73, we deduce the next result.

Lemma 80. *If $S \in \mathsf{E}(a,m)$ and $S \neq \{0, m, \rightarrow\}$, then $S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} \in \mathsf{E}(a,m)$.*

The previous lemma allows us to define recurrently the sequence $\{S_n \mid n \in \mathbb{N}\}$ of elements of $S \in \mathsf{E}(a, m)$. If $S \in \mathsf{E}(a, m)$, then

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n), \mathrm{F}(S_n) - 1, \ldots, \mathrm{F}(S_n) - (a-1)\} & \text{if } S_n \neq \{0, m, \rightarrow\} \\ \{0, m, \rightarrow\} & \text{otherwise.} \end{cases}$

The next result has immediate proof.

Proposition 81. *If $S \in \mathsf{E}(a, m)$ and $\{S_n \mid n \in \mathbb{N}\}$ is the previous sequence, then there exists $k \in \mathbb{N}$ such that $S_k = \{0, m, \rightarrow\}$. Moreover, for every $i \in \{0, 1, \ldots, k-1\}$ then $\mathrm{card}(S_{i+1} \backslash S_i) = a$ and $k = \frac{\mathrm{g}(S) - (m-1)}{a}$.*

As a consequence of Lemma 71 and Proposition 81, we have the following result.

Corollary 82. *If $S \in \mathsf{E}(a, m)$ then $\mathrm{g}(S)$ is a multiple of a.*

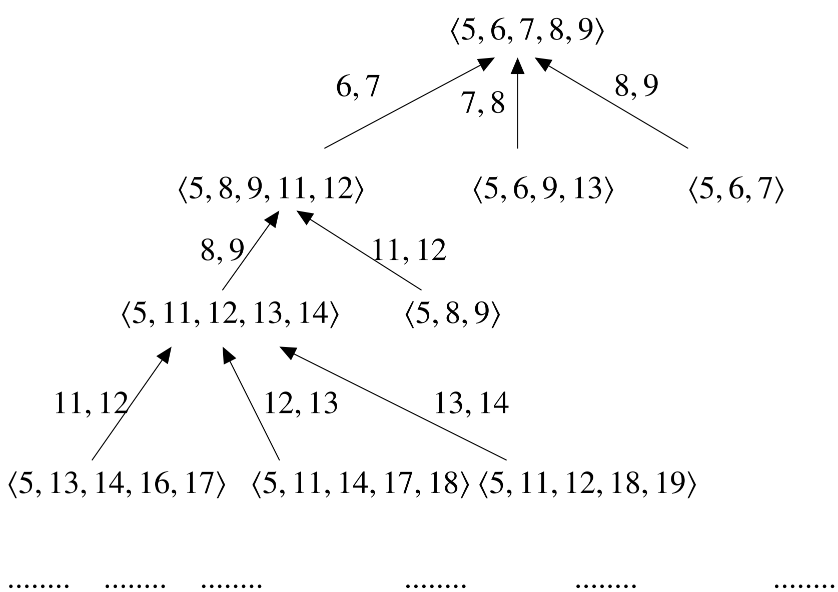
We define the graph $G(\mathsf{E}(a, m))$ as follows: $\mathsf{E}(a, m)$ is its set of vertices and $(S, T) \in \mathsf{E}(a) \times \mathsf{E}(a)$ is an edge if $T = S \cup \{\mathrm{F}(S), \mathrm{F}(S) - 1, \ldots, \mathrm{F}(S) - (a-1)\}$.

The following result is a consequence of Propositions 78 and 81.

Theorem 83. *The graph $G(\mathsf{E}(a, m))$ is a tree rooted in $\{0, m, \rightarrow\}$. Moreover, the set of children of T is equal to*

$$\{T \backslash \{x, x+1, \ldots, x+(a-1)\} \mid \{x, x+1, \ldots, x+(a-1)\} \subseteq msg(T), \\ x > \mathrm{F}(T)) \text{ and } x \neq m\}.$$

Example 84. We are going to build the tree $G(\mathsf{E}(2, 5))$.



Observe that for $t \in \mathbb{N}\backslash\{0\}$ we have that $\langle m\rangle \cup \{tm, \rightarrow\} \in \mathsf{E}(a,m)$ and so $\mathsf{E}(a,m)$ has infinite cardinality. Our next aim in this section will be to show an algorithm that allows us to compute the set $\mathsf{E}(a,m)$ with a given genus.

Let G be a tree with root and v one of its vertices. We define the **depth** of the vertex v as the length of the path that connects v to the root of G, denoted by $d_G(v)$. Given $k \in \mathbb{N}$, denote by

$$N(G,k) = \{v \mid d_G(v) = k\}.$$

We define the **height** of the tree G by $h(G) = \max\{k \in \mathbb{N} \mid N(G,k) \neq \emptyset\}$.

As a consequence of Proposition 81 we have the following result.

Proposition 85. *Let $S \in \mathsf{E}(a,m)$. Then $S \in N(G(\mathsf{E}(a,m)),k)$ if and only if $\mathrm{g}(S) = m - 1 + ak$.*

The next lemma follows immediately from the definitions.

Lemma 86. *If $k \in \mathbb{N}$, then $N(G(\mathsf{E}(a,m)),k+1) = \{S \mid S$ is a child of an element in $N(G(\mathsf{E}(a,m)),k)\}$.*

We are already able to give an algorithm that allows us to compute the set $\mathsf{E}(a,m)$ with a given genus. Note that $S \in \mathsf{E}(a,m)$ then $\{1,\ldots,m-1\} \subseteq \mathbb{N}\backslash S$ and thus $\mathrm{g}(S) \geq m-1$. Moreover, by Corollary 82, we have that $\mathrm{g}(S)$ is a multiple of a.

Algorithm 1

INPUT: a, m and g nonnegative integers such that $2 \leq a \leq m-1 \leq g$, $\mathrm{m} \equiv 1 (\mathrm{mod}\ \ a)$ and $g \equiv 0 (\mathrm{mod}\ \ a)$.
OUTPUT: The set $\{S \in \mathsf{E}(a,m) \mid \mathrm{g}(S) = g\}$.

1: Set $i = m-1$ and $A = \{\langle m, m+1, \ldots, 2m-1\rangle\}$
2: **while** True **do**
3: **if** $i = g$ **then**
4: **return** A
5: **for** $S \in A$ **do**
6: $B_S = \{\{x, x+1, \ldots, x+(a-1)\} \subseteq \mathrm{msg}(S), x > F(S) \text{ and } x \neq m\}$
7: $A := \bigcup_{S\in A} \{S\backslash\{x, x+1, \ldots, x+(a-1)\} \mid \{x, x+1, \ldots, x+(a-1)\} \in B_S\}$, $i = i + a$ and go to step 2.

Example 87. Let us compute the set $\{S \in \mathsf{E}(2,5) \mid \mathrm{g}(S) = 8\}$ using Algorithm 1.

(1) Set $i = 4$ and $A = \{\langle 5,6,7,8,9\rangle\}$.
(2) The first loop constructs $B_{\langle 5,6,7,8,9\rangle} = \{\{6,7\},\{7,8\},\{8,9\}\}$ then $A = \{\langle 5,6,7,8,9\rangle\backslash\{6,7\}, \langle 5,6,7,8,9\rangle\backslash\{7,8\}, \langle 5,6,7,8,9\rangle\backslash\{8,9\}\}$, $i = 6$,
(3) the second loop constructs $B_{\langle 5,8,9,11,12\rangle} = \{\{8,9\},\{11,12\}\}$, $B_{\langle 5,6,9,13\rangle} = \emptyset$ and $B_{\langle 5,6,7\rangle} = \emptyset$ then
$A = \{\langle 5,8,9,11,12\rangle\backslash\{8,9\}, \langle 5,8,9,11,12\rangle\backslash\{11,12\}\}$, $i = 8$.

Hence $\{S \in \mathsf{E}(2,5) \mid \mathrm{g}(S) = 8\} = \{\langle 5,11,12,13,14\rangle, \langle 5,8,9\rangle\}$.

4. The set $\mathsf{E}(a)$ with a given Frobenius number

Our aim in this section will be to show an algorithm that allows us to compute the set $\mathsf{E}(a)$ with a given Frobenius number. If S is a numerical semigroup such that $S \neq \mathbb{N}$, then we have that $\mathrm{F}(S) \geq \mathrm{m}(S) - 1$. Besides, by Lemma 71, if $S \in \mathsf{E}(a)\backslash\{\mathbb{N}\}$ then $\mathrm{m}(S) = 1 + ka$ for some $k \in \mathbb{N}\backslash\{0\}$. Since $\mathrm{m}(S) \leq \mathrm{F}(S) + 1$ and $\mathrm{m}(S) = 1 + ka$ then $k \leq \frac{\mathrm{F}(S)}{a}$.

Given a rational number q we denote by $\lfloor q \rfloor$ its integer part, that is, $\lfloor q \rfloor = \max\{z \in \mathbb{Z} \mid z \le q\}$. We can announce the following result.

Lemma 88. *Let* $\mathrm{F}(S)$ *be an integer greater than or equal to two. Then* $\{S \in \mathsf{E}(a) \mid \mathrm{F}(S) = F\} = \bigcup_{k=1}^{\lfloor \frac{F}{a} \rfloor} \{S \in \mathsf{E}(a, ka+1) \mid \mathrm{F}(S) = F\}$.

Clearly, $\{S \in \mathsf{E}(a, ka+1) \mid \mathrm{F}(S) = ka\} = \{0, ka+1 \rightarrow\}$.

Lemma 89. *Let* $k \in \mathbb{N}\backslash\{0\}$ *such that* $ka+1 < F$. *Then* $\{S \in \mathsf{E}(a, ka+1) \mid \mathrm{F}(S) = F\} \neq \emptyset$ *if and only if* F *mod* $(ka+1) \notin \{0, 1, \ldots, a-1\}$.

Proof. *Necessity.* If $S \in \mathsf{E}(a, ka+1)$ and $\mathrm{F}(S) = F$ then, applying Lemma 72, we have that $\{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} \subseteq \mathbb{N}\backslash S$. Therefore, $\{\mathrm{F}(S), \mathrm{F}(S)-1, \ldots, \mathrm{F}(S)-(a-1)\} \cap \langle ka+1 \rangle = \emptyset$ and so $F \bmod (ka+1) \notin \{0, 1, \ldots, a-1\}$.

Sufficiency. Assume that $r = F \bmod (ka+1)$. Then $F = q(ka+1)+r$ for some $q \in \mathbb{N}\backslash\{0\}$ and $r \in \{a, \ldots, ka\}$. Hence we obtain that

$$S = \{0, ka+1, 2(ka+1), \ldots, q(ka+1), q(ka+1)+1, \ldots, \\ q(ka+1)+r-a, F+1, \rightarrow\} \in \mathsf{E}(a)$$

and $\mathrm{F}(S) = F$. □

Now by using Lemmas 88 and 89, in order to compute the set $\mathsf{E}(a)$ with a given Frobenius number F it is enough to give an algorithm that compute this set with m a positive integer and verify that $2 \le a \le m-1$, $m < F$, $m \equiv 1 (\bmod\ a)$ and $(F \bmod m) \notin \{0, 1, \ldots, a-1\}$.

Algorithm 2

INPUT: a, m and F nonnegative integers such that $2 \leq a \leq m-1$,
$m < F$, $m \equiv 1 (\mathrm{mod}\ a)$ and $(F \bmod m) \notin \{0, 1, \dots, a-1\}$.

OUTPUT: The set $\{S \in \mathsf{E}(a, m) \mid \mathrm{F}(S) = F\}$.

$B = \emptyset$ and $A = \{\langle m, m+1, \dots, 2m-1\rangle\}$
while True **do**
  for $S \in A$ **do**
    Compute $B_S = \{\{x, x+1, \dots, x+(a-1)\} \subseteq \mathrm{msg}(S) \mid x \neq m,$
      $x > F(S)$ and $x + (a-1) \leq F\}$
    $B := B \cup \{S \setminus \{x, x+1, \dots, x+(a-1)\} \mid S \in A,$
      $\{x, x+1, \dots, x+(a-1)\} \in B_S$ and $x + (a-1) = F\}$
    $A := \bigcup_{S \in A} \{S \setminus \{x, x+1, \dots, x+(a-1)\} \mid$
      $\{x, x+1, \dots, x+(a-1)\} \in B_S$ and $x + (a-1) < F\}$
  if $A = \emptyset$ **then**
    return B

Next, we illustrate this method with an example.

Example 90. Let us compute the set $\{S \in \mathsf{E}(2) \mid \mathrm{F}(S) = 12\}$.

First, by using Lemma 88, we have that $\{S \in \mathsf{E}(2) \mid \mathrm{F}(S) = 12\} = \bigcup_{m \in \{3,5,7,9,11,13\}} \{S \in \mathsf{E}(2, m) \mid \mathrm{F}(S) = 12\}$. From Lemma 89, we obtain that $\{S \in \mathsf{E}(2, 3) \mid \mathrm{F}(S) = 12\} = \emptyset$ and $\{S \in \mathsf{E}(2, 11) \mid \mathrm{F}(S) = 12\} = \emptyset$. Moreover, by the observation made after the Lemma 88, we get that $\{S \in \mathsf{E}(2, 13) \mid \mathrm{F}(S) = 12\} = \{0, 13, \rightarrow\}$. Therefore, by using Algorithm 2, we have to compute the set $\{S \in \mathsf{E}(2, m) \mid \mathrm{F}(S) = 12\}$ with $m \in \{5, 7, 9\}$.

For example we will calculate the set $\{S \in \mathsf{E}(2, 5) \mid \mathrm{F}(S) = 12\}$.

(1) Start $B = \emptyset$ and $A = \{\langle 5, 6, 7, 8, 9\rangle\}$.
(2) The first loop constructs $B_{\langle 5,6,7,8,9\rangle} = \{\{6, 7\}, \{7, 8\}, \{8, 9\}\}$
 then $B = \emptyset$,
(3) next constructs $A = \{\langle 5, 6, 7, 8, 9\rangle \setminus \{6, 7\}, \langle 5, 6, 7, 8, 9\rangle \setminus \{7, 8\}, \langle 5, 6, 7, 8, 9\rangle \setminus \{8, 9\}\}$,
(4) the second loop constructs $B_{\langle 5,8,9,11,12\rangle} = \{\{8, 9\}, \{11, 12\}\}$, $B_{\langle 5,6,9,13\rangle} = \emptyset$ and $B_{\langle 5,6,7\rangle} = \emptyset$
 then $B = \{\langle 5, 8, 9, 11, 12\rangle \setminus \{11, 12\}\}$,
(5) next constructs $A = \{\langle 5, 8, 9, 11, 12\rangle \setminus \{8, 9\}\}$,

(6) the third loop constructs $B_{\langle 5,11,12,13,14\rangle} = \{\{11, 12\}\}$
then $B = \{\langle 5, 8, 9\rangle, \langle 5, 11, 12, 13, 14\rangle\backslash\{11, 12\}\}$,
(7) next constructs $A = \emptyset$,
(8) $\{S \in \mathsf{E}(2, 5) \mid \mathrm{F}(S) = 12\} = \{\langle 5, 8, 9\rangle, \langle 5, 13, 14, 16, 17\rangle\}$.

Next, we are interested in characterizing the maximal elements in the set $\{S \in \mathsf{E}(a) \mid \mathrm{F}(S) = F\}$. The next result is well known.

Lemma 91. [RGSGGJM03, Lemma 10] *Let S and T be two numerical semigroups such that $S \subsetneq T$ and $x = \max(T\backslash S)$. Then $S \cup \{x\}$ is a numerical semigroup.*

Proposition 92. *Let $\{S, T\} \subseteq \mathsf{E}(a)$ such that $S \subsetneq T$, $x = \max(T\backslash S)$ and let $s \in S$ such that $s < x < next_S(s)$. If $\{x_1 < x_2 < \cdots < x_r\} = \{t \in T \mid s < t < next_S(s)\}$ then $S \cup \{x_1, x_2, \ldots, x_r\} \in \mathsf{E}(a)$, r is a multiple of a and $S \cup \{x_r, x_{r-1}, \ldots, x_{r-(a-1)}\} \in \mathsf{E}(a)$.*

Proof. By repeatedly applying Lemma 91, we get that $S \cup \{x_1, x_2, \ldots, x_r\}$ is a numerical semigroup. Moreover, as $\{S, T\} \subseteq \mathsf{E}(a)$ such that $S \subseteq T$, we deduce that $S \cup \{x_1, x_2, \ldots, x_r\} \in \mathsf{E}(a)$.

Since $s < x_1 < \cdots < x_r < \mathrm{next}_S(s)$ are consecutive elements of T and $T \in \mathsf{E}(a)$, then there exist $\{k_1, \ldots k_{r+1}\} \subseteq \mathbb{N}$ such that $x_1 = s + k_1a + 1, x_2 = s + k_1a + 1 + k_2a + 1, \ldots, x_r = s+k_1a+1+\cdots+k_ra+1$ and thus $\mathrm{next}_S(s) = s+k_1a+1+\cdots+k_ra+1+k_{r+1}a+1$. Therefore, $\mathrm{next}_S(s) - s = k_1a + 1 + \cdots + k_ra + 1 + k_{r+1}a + 1$. As $S \in \mathsf{E}(a)$ then $\mathrm{next}_S(s)-s-1 = ta$ for some $t \in \mathbb{N}$. Consequently $k_1a+1+\cdots+k_ra+1+k_{r+1}a+1 = ta+1$. Then $(k_1 + \cdots + k_{r+1})a + r + 1 = ta + 1$ and thus r is a multiple of a. Assume that $r = la$ for some $l \in \mathbb{N}\backslash\{0\}$

To conclude the proof, we check that $S \cup \{x_{(l-1)a+1}, x_{(l-1)a+2}, \ldots, x_{(l-1)a+a}\} \in \mathsf{E}(a)$. In order to see this, it is enough to see that $x_{(l-1)a+1} - s - 1$ is a multiple of a. This is true because $x_{(l-1)a+1} - s - 1 = s + k_1a + 1 + k_2a + 1 + \cdots + k_{(l-1)a+1}a + 1 - s - 1 = (k_1 + \cdots + k_{(l-1)+1})a + (l-1)a + 1 - 1$ is a multiple of a. □

Given a sequence of nonnegative integers $n_1 < n_2 < \cdots < n_p$, we say that it is equidistant modulo a if $n_{i+1} - n_i - 1$ is a multiple of a for all $i \in \{1, \ldots, p-1\}$.

Let S be a numerical semigroup. An element of $s \in S$ is called *a-refinable* if there exists $\{x_1 < x_2 < \cdots < x_a\} \subseteq \{x \in \mathbb{N} \mid s < x < \mathrm{next}_S(s) \text{ and } x_a < \mathrm{F}(S)\}$ such that $S \cup \{x_1, x_2, \ldots, x_a\}$ is a numerical semigroup and the sequence $s, x_1, x_2, \ldots, x_a, \mathrm{next}_S(s)$ is equidistant modulo a. We denote by $\mathcal{R}(S) = \{s \in S \mid S \textit{ is a-refinable}\}$.

Theorem 93. *Let $S \in \mathsf{E}(a)$ with $\mathrm{F}(S) = F$. Then S is a maximal element in the set $\{T \in \mathsf{E}(a) \mid \mathrm{F}(T) = F\}$ if and only if $\mathcal{R}(S) = \emptyset$.*

Proof. *Necessity*. If $\mathcal{R}(S) \neq \emptyset$, then there exists $s \in \mathcal{R}(S)$. Hence there exist $\{x_1 < x_2 < \cdots < x_a\} \subseteq \mathbb{N}\backslash\{0\}$ such that the sequence $s < x_1 < x_2 < \cdots < x_a < \mathrm{next}_S(s)$ is equidistant modulo a with $x_a < F$ and $S \cup \{x_1, x_2, \ldots, x_a\}$ is a numerical semigroup. We deduce that $S \cup \{x_1, x_2, \ldots, x_a\} \in \mathsf{E}(a)$ with $\mathrm{F}(S \cup \{x_1, x_2, \ldots, x_a\}) = F$ contradicting the maximality of S.

Sufficiency. If we suppose that S is not maximal, then there exists $T \in \mathsf{E}(a)$ with $\mathrm{F}(T) = F$ and $S \subsetneq T$. Let $x = \max(T\backslash S)$ and $s \in S$ such that $s < x < \mathrm{next}_S(s)$. By applying Proposition 92, we obtain that $s \in \mathcal{R}(S)$ and thus $\mathcal{R}(S) \neq \emptyset$. □

5. The elements of $\mathsf{E}(a)$ with maximal embedding dimension

The following result can be deduced from [**BDF97**, Proposition I.2.9].

Lemma 94. *Let S be a numerical semigroup. Then S is a MED-semigroup if and only if $\{s - \mathrm{m}(S) \mid s \in S\backslash\{0\}\}$ is a numerical semigroup.*

Proposition 95. *Let S be a MED-semigroup. Then $S \in \mathsf{E}(a)$ if and only if $T = \{s - \mathrm{m}(S) \mid s \in S\backslash\{0\}\}$ is an element of $\mathsf{E}(a)$ and $\mathrm{m}(S) \equiv 1 (mod\ \ a)$.*

Proof. *Necessity*. By Lemmas 94 and 71, we have that T is a numerical semigroup and $\mathrm{m}(S) \equiv 1 (\mathrm{mod}\ \ a)$, respectively. To conclude the proof it suffices to see that $\mathrm{next}_T(t) - t - 1$ is a multiple of a for every $t \in T$. If $t \in T$, then there exists $s \in S\backslash\{0\}$ such that $t = s - \mathrm{m}(S)$ and so $\mathrm{next}_T(t) = \mathrm{next}_S(t) - \mathrm{m}(S)$. Therefore, $\mathrm{next}_T(t) - t - 1 = \mathrm{next}_S(s) - \mathrm{m}(S) - (s - \mathrm{m}(S)) - 1 = \mathrm{next}_S(s) - s - 1$ is a multiple of a, because $S \in \mathsf{E}(a)$.

Sufficiency. Let us see that $S \in \mathsf{E}(a)$, that is, if $s \in S$ then $\mathrm{next}_S(s) - s - 1$ is a multiple of a. If $s = 0$, then $\mathrm{next}_S(s) - 0 - 1 = \mathrm{m}(S) - 1$ is a multiple of a. If $s \neq 0$, then $s - m(s) = t \in T$ and $\mathrm{next}_T(t) = \mathrm{next}_S(s) - \mathrm{m}(S)$. Hence $\mathrm{next}_S(s) - s - 1 = \mathrm{next}_T(t) - t - 1$ is multiple of a, because $T \in \mathsf{E}(a)$. □

From Lemma 94, it is easy to deduce the following result.

Lemma 96. *Let S be a numerical semigroup and $x \in S\backslash\{0\}$. Then $S(x) = (\{x\} + S) \cup \{0\}$ is a MED-semigroup with multiplicity x. Moreover, every MED-semigroup is of this form.*

Proposition 97. *Let $S \in \mathsf{E}(a)$ and $x \in S\backslash\{0\}$ such that $x \equiv 1(mod \ \ a)$. Then $S(x) = (\{x\} + S) \cup \{0\}$ is an equidistant MED-semigroup modulo a. Moreover, every equidistant MED-semigroup modulo a is of this form.*

Proof. By Lemma 96, we obtain that $S(x)$ is a MED-semigroup with multiplicity x. Clearly $S = \{s - x \mid s \in S(x)\backslash\{0\}\}$ and thus, applying Proposition 95, we obtain that $S(x)$ is equidistant modulo a.

Let T be an equidistant MED-semigroup modulo a. Then by Proposition 95, we get that $S = \{T - \mathrm{m}(T) \mid t \in T\backslash\{0\}\} \in \mathsf{E}(a)$ and $\mathrm{m}(T) \equiv 1(\mathrm{mod} \ \ a)$. Therefore, $T = (\{\mathrm{m}(T)\} + S) \cup \{0\}$ with $S \in \mathsf{E}(a)$ and $\mathrm{m}(T) \in S$ such that $\mathrm{m}(T) \equiv 1(\mathrm{mod} \ \ a)$. □

From [**Ros03**], we can deduce the next result.

Proposition 98. *Le S be a numerical semigroup, $n \in S\backslash\{0\}$ and $T = (\{n\} + S) \cup \{0\}$. Then the following conditions hold:*

(1) *T is MED-semigroup*
(2) $\mathrm{m}(T) = n$.
(3) $\mathrm{F}(T) = \mathrm{F}(S) + n$.
(4) $\mathrm{g}(T) = \mathrm{g}(S) + n - 1$.
(5) $\mathrm{msg}(T) = \mathrm{Ap}(S, n) + \{n\}$.

Example 99. It is clear that $S = \langle 5, 8, 9\rangle$ is an equidistant numerical semigroup modulo 2. We have that 9 is an element in S such that $9 \equiv 1 \pmod{2}$. Then from Proposition 97 we obtain that $T = (\{9\} + S) \cup \{0\}$ is is an equidistant MED-semigroup modulo 2. Since $\mathrm{F}(S) = 12$, $g(S) = 8$ and $\mathrm{Ap}(S, 9) = \{0, 5, 8, 10, 13, 15, 16, 20, 21\}$, by Proposition 98 we have that $\mathrm{F}(T) = 12 + 9 = 21$, $\mathrm{g}(T) = 8 + 9 - 1 = 16$ and $\mathrm{msg}(T) = \{9, 14, 17, 19, 22, 24, 25, 29, 30\}$.

CHAPTER 3

Numerical semigroups with concentration

1. Numerical semigroups with concentration two

In this section, we study the class of numerical semigroups with concentration 2. We give algorithms to calculate the whole set of this class of semigroups with a given multiplicity, genus or Frobenius number. Separately, we prove that this class of semigroups verifies Wilf's conjecture. This section has been already published in [**RBT22b**].

1.1. Definitions and preliminaries. If S is a numerical semigroup and s an element in S, we denote by $\mathrm{next}_S(s)$ the integer $\min\{x \in S \mid s < x\}$. We define the **concentration of a numerical semigroup** S as $\mathsf{C}(S) = \max\{\mathrm{next}_S(s) - s \mid s \in S\backslash\{0\}\}$. The least nonnegative integer belonging to S is called the multiplicity, denoted by $\mathrm{m}(S)$. Clearly, we have that if S is a numerical semigroup with concentration 1 then $S = \{0, \mathrm{m}(S), \rightarrow\}$. Remember that S is called the **ordinary numerical semigroup** and is denoted by $\mathcal{O}_m$.

Our aim in this section is to study the **numerical semigroups with concentration 2**.

This section is organized as follows. In Subsection 1.2 we give a characterization of numerical semigroups with concentration 2 in terms of its minimal system of generators. For $m \in \mathbb{N}\backslash\{0, 1\}$ we denote by $\mathsf{C}_2[m]$ the set of all numerical semigroups with concentration 2 and multiplicity m, that is,

$$\mathsf{C}_2[m] = \{S \mid S \text{ is a numerical semigroup}, \mathsf{C}(S) = 2 \text{ and } \mathrm{m}(S) = m\}.$$

In this subsection, we will order the elements of $\mathsf{C}_2[m]$ to construct a tree with root. This ordering will provide us with an algorithmic procedure that allows us to recursively build the elements $\mathsf{C}_2[m]$.

We started subsection 1.3 by seeing that $\mathsf{C}_2[m]$ is a finite set if and only if m is odd. Besides we give an algorithm that allows us to compute all elements of $\mathsf{C}_2[m]$ with a given genus.

Given S a numerical semigroup, we define $\mathrm{N}(S)$ to be the set $\{s \in S \mid s < \mathrm{F}(S)\}$ and $\mathrm{n}(S)$ its cardinality.

In 1978, Wilf conjectured (see [**Wil78**]) every numerical semigroup S satisfies $\mathrm{g}(S) \leq (\mathrm{e}(S) - 1)\mathrm{n}(S)$. This question is still widely open and it is one of the most important problems in numerical semigroups theory. A very good source of the state of the art of this problem is [**Del18**]. Our aim in subsection 1.4 will be to prove that numerical semigroups with concentration 2 verify Wilf's conjecture.

By using the terminology of [**RB03**], a numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it. A numerical semigroup is a symmetric numerical semigroup (pseudo-symmetric, resp.) if it is irreducible and its Frobenius number is odd (even, resp). This class of numerical semigroups are probably the numerical semigroups that have been more studied in the literature (see [**Kun70**] and [**BDF97**]).

Given a positive integer F, denote by

$$\mathsf{C}_2(F) = \{S \mid S \text{ is a numerical semigroup}, \mathsf{C}(S) = 2 \text{ and } \mathrm{F}(S) = F\}$$

$$\text{and } I(\mathsf{C}_2(F)) = \{S \in \mathsf{C}_2(F) \mid S \text{ is a irreducible numerical semigroup}\}.$$

In subsection 1.5 we define an equivalence relation $\sim$ over $\mathsf{C}_2(F)$ such that $\mathsf{C}_2(F)/\!\sim\; = \{[S] \mid S \in I(\mathsf{C}_2(F))\}$ where [S] denotes the equivalence class of S with respect to $\sim$. Hence, in order to compute all the elements in $\mathsf{C}_2(F)$ it is enough to determine all elements in $I(\mathsf{C}_2(F))$ and, for each $S \in I(\mathsf{C}_2(F))$, to compute the class

$[S]$. As a consequence of this study, we give an algorithm that allows us to calculate the whole set of $\mathsf{C}_2(F)$.

1.2. The tree associated to $\mathsf{C}_2[m]$. We started this subsection by presenting several characterizations for the numerical semigroups with concentration 2.

Proposition 100. *Let S be a numerical semigroup such that S is not the ordinary. The following conditions are equivalent:*

(1) $\mathsf{C}(S) = 2$.

(2) $h + 1 \in S$ *for all* $h \in \mathbb{N}\backslash S$ *such that* $h > \mathrm{m}(S)$.

(3) $\{s + 1, s + 2\} \cap S \neq \emptyset$ *for all* $s \in S\backslash\{0\}$.

(4) $\{x + 1, x + 2\} \cap S \neq \emptyset$ *for all* $x \in \mathrm{msg}(S)$.

Proof. (1) *implies* (2). Let $s \in S\backslash\{0\}$ such that $s < h < \mathrm{next}_S(s)$ with $h > \mathrm{m}(S)$ and thus $\mathrm{next}_S(s) - s \leq 2$. Hence $h + 1 = \mathrm{next}_S(s) \in S$.

(2) *implies* (3). For $s \in S\backslash\{0\}$ we have $s + 1 > \mathrm{m}(S)$. If $s + 1 \in \mathbb{N}\backslash S$ then by (2), we conclude that $s + 2 \in S$.

(3) *implies* (4). Trivial.

(4) *implies* (1). Suppose that $\mathrm{msg}(S) = \{n_1, n_2, \ldots, n_e\}$. If $s \in S\backslash\{0\}$, then there exists $(\lambda_1, \ldots, \lambda_e) \in \mathbb{N}^e\backslash\{(0, \ldots, 0)\}$ such that $s = \lambda_1 n_1 + \cdots + \lambda_e n_e$. Let $\lambda_i \neq 0$ with $i \in \{1, \ldots, e\}$. As by hypothesis $\{n_i + 1, n_i + 2\} \cap S \neq \emptyset$, if $n_i + 1 \in S$ then $s + 1 = \lambda_1 n_1 + \cdots + (\lambda_i - 1)n_i + \cdots + \lambda_e n_e + n_i + 1$ and thus $s + 1 \in S$. In the same way, if $n_i + 2 \in S$ we obtain that $s + 2 \in S$. Hence $\mathrm{next}_S(s) - s \leq 2$, that is, $\mathsf{C}(S) = 2$.

□

Example 101. Using the previous proposition we deduce that $S = \langle 5, 7, 9\rangle$ is a numerical semigroup with $\mathsf{C}(S) = 2$, because $\{5 + 2, 7 + 2, 9 + 1\} \subseteq S$.

Given m belonging to $\mathbb{N}\backslash\{0, 1\}$, we denote by

$\mathsf{C}_2[m] = \{S \mid S$ is a numerical semigroup, $\mathsf{C}(S) = 2$ and $\mathrm{m}(S) = m\}$ and

$\overline{\mathsf{C}_2[m]} = \{S \mid S$ is a numerical semigroup, $\mathsf{C}(S) \leq 2$ and $\mathrm{m}(S) = m\}$.

The next result characterizes the set $\overline{\mathsf{C}_2[m]}$ and it has an immediate proof.

Proposition 102. *If* $m \in \mathbb{N}\backslash\{0,1\}$*, then* $\overline{\mathsf{C}_2[m]} = \mathsf{C}_2[m] \cup \{\mathcal{O}_m\}$.

Lemma 103. *If* $m \in \mathbb{N}\backslash\{0,1\}$ *and* $S \in \mathsf{C}_2[m]$*, then* $S \cup \{F(S)\} \in \overline{\mathsf{C}_2[m]}$.

The previous result enable us, given an element $S \in \overline{\mathsf{C}_2[m]}$, to define recursively the following sequence of elements in $\overline{\mathsf{C}_2[m]}$:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n)\} & \text{if } S_n \neq \mathcal{O}_m \\ \mathcal{O}_m & \text{otherwise.} \end{cases}$

The next is a direct consequence of $\mathrm{g}(S_{n+1}) < g(S_n)$ whenever $S_n \neq \mathcal{O}_m$.

Proposition 104. *If* $m \in \mathbb{N}\backslash\{0,1\}$*,* $S \in \overline{\mathsf{C}_2[m]}$ *and* $\{S_n \mid n \in \mathbb{N}\}$ *is the previous sequence of numerical semigroups, then there exists* $k \in \mathbb{N}$ *such that* $S_k = \mathcal{O}_m$.

We define the **graph** $G(\overline{\mathsf{C}_2[m]})$ as the graph whose vertices are elements of $\overline{\mathsf{C}_2[m]}$ and $(S,T) \in \overline{\mathsf{C}_2[m]} \times \overline{\mathsf{C}_2[m]}$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$. As a consequence of Proposition 104, we deduce the following.

Theorem 105. *If* $\mathrm{m} \in \mathbb{N}\backslash\{0,1\}$*, then the graph* $G(\overline{\mathsf{C}_2[m]})$ *is a* ***tree*** *with root equal to* $\mathcal{O}_m$.

Previous results allow us to construct recursively the elements of the set $\overline{\mathsf{C}_2[m]}$, starting in $\mathcal{O}_m$, we connect each vertex with its children. We will characterize the children of an arbitrary vertex of this tree.

Proposition 106. *Let* $m \in \mathbb{N}\backslash\{0,1\}$ *and* $S \in \overline{\mathsf{C}_2[m]}$*. Then the set of children of* S *in the tree* $G(\overline{\mathsf{C}_2[m]})$ *is equal to* $\{S\backslash\{x\} \mid x \in msg(S),\ x \geq \mathrm{F}(S)+2\}$.

Proof. If $x \in \mathrm{msg}(S)$ and $x \geq \mathrm{F}(S)+2$, then by applying (3) of Proposition 100 and Lemma 31 we have that $S\backslash\{x\} \in \mathsf{C}_2[m]$. Hence $S\backslash\{x\}$ is a child of S with $F(S\backslash\{x\}) = x$.

Conversely, if T is a child of S, then $T \in \overline{\mathsf{C}_2[m]}$ and $S = T \cup \{F(T)\}$. Hence we deduce that $T = S\backslash\{F(T)\}$. By Lemma 31, we have that $F(T) \in \mathrm{msg}(S)$ and since

$S = T \cup \{F(T)\}$ then $F(S) < F(T)$. Since $T \in \overline{\mathsf{C}_2[m]}$ then, by (3) Proposition 100, we obtain that $F(T) - 1 \in T$. Therefore, $F(T) - 1 \in S$ and consequently $F(T) \geq F(S) + 2$. $\square$

Example 107. Let us construct the tree $G(\overline{\mathsf{C}_2[3]})$.

$$\begin{tikzcd} & \langle 3,4,5\rangle & \\ \langle 3,5,7\rangle \arrow[ur, "4"] & & \langle 3,4\rangle \arrow[ul, "5"'] \\ \langle 3,5\rangle \arrow[u, "7"] & & \end{tikzcd}$$

1.3. The genus of the elements in $\mathsf{C}_2[m]$. It is clear that, in the tree $G(\overline{\mathsf{C}_2[m]})$, the elements of $\mathsf{C}_2[m]$ with minimum **genus** are the children of O_m. Consequently, we obtain the following result.

Proposition 108. *If $m \in \mathbb{N}\backslash\{0,1\}$ and $S \in \mathsf{C}_2[m]$, then $\mathrm{g}(S) \geq m$. Furthermore, we have the following equality $\{S \in \mathsf{C}_2[m] \mid \mathrm{g}(S) = m\} = \{O_m\backslash\{m+i\} \mid i \in \{1,\ldots,m-1\}\}$.*

From the previous characterization, it is natural to ask which are the elements of $\mathsf{C}_2[m]$ with maximum genus. As a consequence of the next proposition, we will see that if m is even then $\mathsf{C}_2[m]$ contains elements of any genus greater than or equal to m.

If S is a numerical semigroup, then $\mathbb{N}\backslash S$ is a finite set and thus we conclude our next result.

Lemma 109. *If S is a numerical semigroup, then $\{T \mid T$ is a numerical semigroup and $S \subseteq T\}$ has a finite number of elements.*

Proposition 110. *Let $m \in \mathbb{N}\backslash\{0,1\}$. Then $\mathsf{C}_2[m]$ is finite if and only if m is odd.*

PROOF. *Necessity*. Assume that m is even and $n \in \mathbb{N}$ and denote by $S(n) = \langle \{m\} + \{2.k \mid k \in \mathbb{N}\} \rangle \cup \{n, \rightarrow\}$. Clearly, we have that $S(n)$ is an element of $\mathsf{C}_2[m]$ for all $n \geq m+2$ and so $\mathsf{C}_2[m]$ is an infinite set.

Sufficiency. If $S \in \mathsf{C}_2[m]$, then by Proposition 100, we deduce that $\{m+1, m+2\} \cap S \neq \emptyset$. Hence, either $\langle m, m+1 \rangle \subseteq S$ or $\langle m, m+2 \rangle \subseteq S$. Since m is odd we have that $\langle m, m+1 \rangle$ and $\langle m, m+2 \rangle$ are numerical semigroups. Therefore, we can conclude that $\mathsf{C}_2[m] \subseteq \{T \mid T \text{ is a numerical semigroup and } \langle m, m+1 \rangle \subseteq T\} \cup \{T \mid T \text{ is a numerical semigroup and } \langle m, m+2 \rangle \subseteq T\}$. By applying now Lemma 109 we get that $\mathsf{C}_2[m]$ is a finite set. □

As a consequence of the previous proposition, we obtain the following result.

Corollary 111. *If $m \in \mathbb{N} \backslash \{0, 1\}$ such that m is even, then the set of the genus of the elements in $\mathsf{C}_2[m]$ is equal to $\{m, \rightarrow\}$.*

Now our aim is to give an algorithm to compute all elements in the set $\mathsf{C}_2[m]$ with fixed genus. To this end, we need to introduce some concepts and results.

Remember that $N(G, k) = \{v \mid d_G(v) = k\}$ denotes the set of vertices v with **depth** k.

The next result is easy to prove.

Proposition 112. *Let $m \in \mathbb{N} \backslash \{0, 1\}$ and $k \in \mathbb{N}$. Then the following conditions hold.*

(1) $N(G(\overline{\mathsf{C}_2[m]}), k) = \left\{S \in \overline{\mathsf{C}_2[m]} \mid \mathrm{g}(S) = m - 1 + k\right\}$.

(2) $N(G(\overline{\mathsf{C}_2[m]}), k+1) = \left\{S \mid S \textit{ is a child of an element in } N(G(\overline{\mathsf{C}_2[m]}), k))\right\}$.

(3) If m is odd, then

$$\{\mathrm{g}(S) \mid S \in \mathsf{C}_2[m]\} = \left\{m, m+1, \ldots, m + h(G(\overline{\mathsf{C}_2[m]})) - 1\right\}.$$

We are already in a condition to present the announced algorithm.

Algorithm 113.

INPUT: Integers m, g such that $1 \leq m - 1 \leq g$.

OUTPUT: The set $\left\{S \in \overline{\mathsf{C}_2[m]} \mid \mathrm{g}(S) = g\right\}$

(1) $A = \{\langle m, m+1, \ldots, 2m-1\rangle\}$, $i = m-1$.
(2) If $i = g$ then return A.
(3) For each $S \in A$ compute $B_S = \left\{T \mid T \text{ is a child of } S \in G(\overline{\mathsf{C}_2[m]})\right\}$.
(4) If $\bigcup_{S\in A} B_S = \emptyset$, then return $\emptyset$.
(5) $A := \bigcup_{S\in A} B_S$, $i = i+1$ and go to step 2.

Example 114. Let us compute the set $\left\{S \in \overline{\mathsf{C}_2[4]} \mid \mathrm{g}(S) = 5\right\}$.

(1) Start with $A = \langle 4, 5, 6, 7\rangle$, $i = 3$.
(2) The first loop constructs $B_{\langle 4,5,6,7\rangle} = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle, \langle 4, 5, 6\rangle\}$ and then $A = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle, \langle 4, 5, 6\rangle\}$, $i = 4$.
(3) The second loop constructs $B_{\langle 4,6,7,9\rangle} = \{\langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle\}$, $B_{\langle 4,5,7\rangle} = \emptyset$ and $B_{\langle 4,5,6\rangle} = \emptyset$ and then $A = \{\langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle\}$, $i = 5$.

Hence $\left\{S \in \overline{\mathsf{C}_2[4]} \mid \mathrm{g}(S) = 5\right\} = \{\langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle\}$.

We finished this subsection by putting two problems:

(1) What is the cardinality of $\mathsf{C}_2[m]$ if m is odd and belongs to $\mathbb{N}\backslash\{0, 1\}$?
(2) What is the height of the tree $G(\overline{\mathsf{C}_2[m]})$ if m is odd and belongs to $\mathbb{N}\backslash\{0, 1\}$?

1.4. Wilf's conjecture. Our first aim in this subsection is to prove that every numerical semigroup with concentration 2 satisfies **Wilf's conjecture**.

Using the terminology introduced in [**RB22**] a numerical semigroup S is **elementary** if $F(S) < 2\mathrm{m}(S)$. Let us start by recalling the following result of Kaplan in [**Kap12**, Proposition 26].

Lemma 115. *Every elementary numerical semigroup satisfies Wilf's conjecture.*

As a consequence of [**Syl84**] and [**FGH86**] we have the following result.

Lemma 116. *If S is a numerical semigroup with $\mathrm{e}(S) \in \{2, 3\}$, then S satisfies Wilf's conjecture.*

Lemma 117. *If $S \in \mathsf{C}_2[m]$ and $\mathrm{F}(S) > 2m$, then $\mathrm{n}(S) \geq \frac{m}{2} + 2$.*

Proof. Let $A = \{m = a_1 < a_2 < \cdots < 2m = a_k\} = \{s \mid s \in S \text{ and } m \leq s \leq 2m\}$. Since $A \subseteq \mathrm{N}(S)\backslash\{0\}$ we get that $\mathrm{n}(S) \geq \#A + 1$. On the other hand, as $S \in \mathsf{C}_2[m]$ then $a_{i+1} - a_i \leq 2$ for all $i \in \{1, \ldots, k-1\}$. Then we have that $m = (a_k - a_{k-1}) + (a_{k-1} - a_{k-2}) + \cdots + (a_2 - a_1) \leq 2(k-1)$. Therefore $\#A = k \geq \frac{m}{2} + 1$ and thus $\mathrm{n}(S) \geq \frac{m}{2} + 2$. □

Theorem 118. *Every numerical semigroup with concentration* 2 *satisfies Wilf's conjecture.*

Proof. Taking into account Lemmas 115 and 116, we assume that $\mathrm{F}(S) > 2m$ and $\mathrm{e}(S) \geq 4$. We need to show that if $S \in \mathsf{C}_2[m]$ then $\mathrm{g}(S) \leq (\mathrm{e}(S) - 1)\mathrm{n}(S)$. By Proposition 100 we have that, if $h \in \mathbb{N}\backslash S$ and $h \geq m$ then $h + 1 \in S$. Therefore, the correspondence

$$f : \{h \in \mathbb{N}\backslash S \mid h \geq m\} \to \mathrm{N}(S)\backslash\{0\},$$

defined by $f(h) = h + 1$ if $h \neq \mathrm{F}(S)$ and $f(\mathrm{F}(S)) = m$ is an injective map. Hence $\mathrm{g}(S) \leq m + \mathrm{n}(S) - 2$. As by Lemma 117 $\mathrm{n}(S) \geq \frac{m}{2} + 2$ this forces $2\mathrm{n}(S) \geq m + 4 \geq m - 2$. Then we obtain that $\mathrm{g}(S) \leq m + \mathrm{n}(S) - 2 \leq 3\mathrm{n}(S) \leq (\mathrm{e}(S) - 1)\mathrm{n}(S)$, because $\mathrm{e}(S) \geq 4$. This proves that S verifies Wilf's Conjecture. □

Taking advantage of the introduction of elementary numerical semigroups, we give an algorithm to compute the set of all elementary numerical semigroups with concentration 2 and multiplicity m, that is,

$$\mathsf{EC}_2[m] = \{S \mid S \in \mathsf{C}_2[m] \text{ and } S \text{ is an elementary numerical semigroup}\}.$$

The next result is easy to prove and it can be deducted from [**Zha10**, Proposition 2.1].

Lemma 119. *Let* $m \in \mathbb{N}\backslash\{0, 1\}$ *and let* $A \subseteq \{m + 1, \ldots, 2m - 1\}$. *Then* $\{0, m\} \cup A \cup \{2m, \to\}$ *is an elementary numerical semigroup with multiplicity m. Furthermore, every elementary numerical semigroup with multiplicity m is of this form.*

Given $m \in \mathbb{N}\backslash\{0, 1\}$, we denote by

$$\overline{\mathsf{EC}_2[m]} = \{S \mid S \text{ is elementary semigroup, } \mathsf{C}(S) \leq 2 \text{ and } \mathrm{m}(S) = m\}.$$

It is easy to prove our next result.

Lemma 120. *Let $m \in \mathbb{N} \backslash \{0, 1\}$. Then the following conditions hold:*

(1) $\overline{\mathsf{EC}_2[m]} = \mathsf{EC}_2[m] \cup \{\mathcal{O}_m\}$.

(2) If $S \in \mathsf{EC}_2[m]$, *then* $S \cup \{F(S)\} \in \overline{\mathsf{EC}_2[m]}$.

Given $S \in \overline{\mathsf{EC}_2[m]}$, by using Lemma 120, we can define recursively the following sequence of elements in $\overline{\mathsf{EC}_2[m]}$.

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n)\} & \text{if } S_n \neq \mathcal{O}_m \\ \mathcal{O}_m & \text{otherwise.} \end{cases}$

The next result has immediate proof.

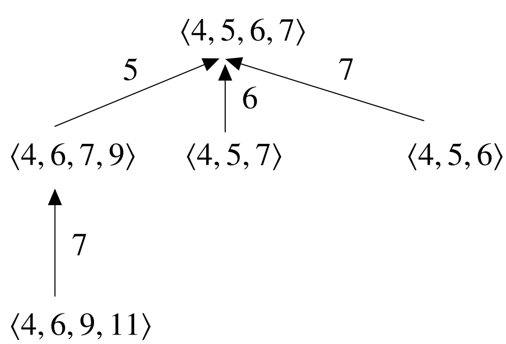
Lemma 121. *If* $m \in \mathbb{N} \backslash \{0, 1\}$, $S \in \overline{\mathsf{EC}_2[m]}$ *and* $\{S_n \mid n \in \mathbb{N}\}$ *is the previous sequence of numerical semigroups, then there exists* $k \in \mathbb{N}$ *such that* $S_k = \mathcal{O}_m$.

We can define a new graph $G(\overline{\mathsf{EC}_2[m]})$ as graph whose vertices are the elements of $\overline{\mathsf{EC}_2[m]}$ and $(S, T) \in \overline{\mathsf{EC}_2[m]} \times \overline{\mathsf{EC}_2[m]}$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$.

As a consequence of Lemma 121 and Proposition 106 we have the following result.

Proposition 122. *If* $m \in \mathbb{N} \backslash \{0, 1\}$, *then the graph* $G(\overline{\mathsf{EC}_2[m]})$ *is a tree rooted in* $\mathcal{O}_m$. *Moreover, the set of children of the vertice* S *in the tree is the set* $\{S \backslash \{x\} \mid x \in msg(S),\ \mathrm{F}(T) + 2 \leq x \leq 2m - 1\}$.

Example 123. Let us construct the tree $G(\overline{\mathsf{EC}_2[4]})$.



On the same line as the previous subsection, we finished this subsection by putting two problems :

(1) What is the cardinality of $\mathsf{EC}_2[m]$ if m belongs to $\mathbb{N}\backslash\{0, 1\}$?
(2) What is the height of the tree $G(\overline{\mathsf{EC}_2[m]})$ if m belongs to $\mathbb{N}\backslash\{0, 1\}$?

1.5. The Frobenius number. Our aim in this subsection is to give an algorithm to compute the whole set of numerical semigroups with concentration 2 and with fixed **Frobenius number**.

Proposition 124. [**BR12b**, Lemma 4] *Let S be a numerical semigroup with the Frobenius number F. Then:*

(1) S is irreducible if and only if S is maximal in the set of all the numerical semigroups with the Frobenius number F.
(2) If $h = \max\left\{x \in \mathbb{N}\backslash S \mid F - x \notin S \text{ and } x \neq \frac{F}{2}\right\}$, then $S \cup \{h\}$ is a numerical semigroups with Frobenius number F.
(3) S is irreducible if and only if $\left\{x \in \mathbb{N}\backslash S \mid F - x \notin S \text{ and } x \neq \frac{F}{2}\right\} = \emptyset$.

The following result has immediate proof.

Lemma 125. *Let S be a numerical semigroup with concentration 2, $x \in \mathbb{N}\backslash S$, $x \neq \mathrm{F}(S)$ and let $S \cup \{x\}$ be a numerical semigroup, then $S \cup \{x\}$ is a numerical semigroup with concentration 2 and Frobenius number $\mathrm{F}(S)$.*

Given $F \in \mathbb{N}\backslash\{0, 1\}$, we denote by

$$\mathsf{C}_2(F) = \{S \mid S \text{ is a numerical semigroup}, \mathsf{C}(S) = 2 \text{ and } \mathrm{F}(S) = F\}.$$

Let S be a non-irreducible numerical semigroup. Denote by $\alpha(S)$ the integer $\max\left\{x \in \mathbb{N}\backslash S \mid \mathrm{F}(S) - x \notin S \text{ and } x \neq \frac{\mathrm{F}(S)}{2}\right\}$.

As a consequence of Lemma 125 and (2) of Proposition 124, we can define recursively the following sequence of elements of $\mathsf{C}_2(F)$:

- $S_0 = S$,

- $S_{n+1} = \begin{cases} S_n \cup \{\alpha(S_n)\} & \text{if } S_n \text{ is non-irreducible} \\ S_n & \text{otherwise.} \end{cases}$

Taking into account the previous results the next result is easy to prove.

Proposition 126. *Let $F \in \mathbb{N}\backslash\{0,1\}$, $S \in \mathsf{C}_2(F)$ and let $\{S_n \mid n \in \mathbb{N}\}$ be the previous sequence. Then there exists a positive integer k such that S_k is an irreducible numerical semigroup.*

For $k \in \backslash\{0\}$ we will call S_k the irreducible numerical semigroup associated with S and it will be denoted by $\mathscr{V}(S)$.

We define the following equivalence relation over $\mathsf{C}_2(F)$:

$$S \sim T \text{ if and only if } \mathscr{V}(S) = \mathscr{V}(T).$$

We denote the equivalence class of $S \in \mathsf{C}_2(F)$ modulo $\sim$ by $[S] = \{T \in \mathsf{C}_2(F) \mid S \sim T\}$ and the quotient set $\mathsf{C}_2(F)/\!\sim = \{[S] \mid S \in \mathsf{C}_2(F)\}$.

Denote by $\mathcal{I}(\mathsf{C}_2(F)) = \{S \in \mathsf{C}_2(F) \mid S \text{ is irreducible}\}$.

As a consequence of Proposition 126 we have the following result.

Theorem 127. *If $F \in \mathbb{N}\backslash\{0,1\}$, then the quotient set $\mathsf{C}_2(F)/\!\sim = \{[S] \mid S \in \mathcal{I}(\mathsf{C}_2(F))\}$. Moreover, if $\{S,T\} \subseteq \mathcal{I}(\mathsf{C}_2(F))$ and $S \neq T$ then $[S] \cap [T] = \emptyset$.*

In view of Theorem 127, in order to determine explicitly the elements in the set $\mathsf{C}_2(F)$ we need:

1) an algorithm to compute the set $\mathcal{I}(\mathsf{C}_2(F))$;
2) an algorithm to compute the class $[S]$, for each $S \in \mathcal{I}(\mathsf{C}_2(F))$.

An efficient procedure to compute the set of irreducible numerical semigroups with the Frobenius number F is given in [**BR13**]. Using Proposition 100, we can decide if a numerical semigroup is or is not of concentration 2. Therefore we have solved 1).

Now we will focus on solving 2). Let $\Delta \in \mathcal{I}(\mathsf{C}_2(F))$. We define the graph $G([\Delta])$ whose vertices are the elements of $[\Delta]$ and $(S,T) \in [\Delta] \times [\Delta]$ is an edge if and only if $T = S \cup \{\alpha(S)\}$.

By convention, we write $\alpha(S) = -\infty$ whenever S is irreducible because in this case, $\alpha(S)$ does not exist.

Proposition 128. *If $F \in \mathbb{N}\backslash\{0,1\}$ and $\Delta \in \mathcal{I}(\mathsf{C}_2(F))$, then $G([\Delta])$ is a tree rooted in Δ. Moreover, the set of children of a vertex T is equal to*

$$\Big\{T\backslash\{x\} \mid x \in \mathrm{msg}(T),\ \tfrac{F}{2} < x < F,\ \alpha(T) < x \text{ and } \{x-1, x+1\} \subseteq T \text{ or } x = \mathrm{m}(T)\Big\}.$$

Proof. If S is a child of T, then $T = S \cup \{\alpha(S)\}$ and thus $S = T\backslash\{\alpha(S)\}$. By Lemma 31, we have that $\alpha(S) \in \mathrm{msg}(T)$. It is clear that $\frac{F}{2} < \alpha(S) < F$ and $\alpha(S) = \mathrm{m}(T)$ or $\{\alpha(S)-1, \alpha(S)+1\} \subseteq T$. Also we have that $\alpha(T) < \alpha(S)$.

Conversely, if $x \in \mathrm{msg}(T)$, $\frac{F}{2} < x < F$ and $\{x-1, x+1\} \subseteq T$ or $x = \mathrm{m}(T)$ then $T\backslash\{x\} \in \mathsf{C}_2(F)$. If $\alpha(T) < x$ then $\mathrm{g}(T\backslash\{x\}) = x$. Hence $T = T\backslash\{x\} \cup \{\alpha(T\backslash\{x\})\}$ and so $T\backslash\{x\}$ is a child of T. $\square$

Example 129. Applying Proposition 124, we have that $\Delta = \langle 5,6,7,8\rangle \in \mathcal{I}(\mathsf{C}_2(9))$. Now by applying Proposition 128, we can construct $G([\Delta])$.

$$\begin{tikzcd}
 & & \langle 5,6,7,8\rangle & \\
 & \langle 6,7,8,10,11\rangle \arrow[ur, "5"] & \langle 5,7,8,11\rangle \arrow[u, "6"] & \langle 5,6,8\rangle \arrow[ul, "7"'] \\
\langle 7,8,10,11,12,13\rangle \arrow[ur, "6"] & & \langle 6,8,10,11,13,15\rangle \arrow[ul, "7"'] & \\
\langle 8,10,11,12,13,14,15,17\rangle \arrow[u, "7"] & & &
\end{tikzcd}$$

An edge $S \to T$ is labelled x whenever S is obtain from T by removing x, ie, $S = T\backslash\{x\}$.

2. Numerical semigroups with fixed multiplicity and concentration

In this section, we study the class of **numerical semigroups with multiplicity m and concentration less or equal to** k, denoted by $\mathsf{C}_k[m]$. We give algorithms to calculate the whole set $\mathsf{C}_k[m]$ with a given genus or Frobenius number. In addition, we prove that if $S \in \mathsf{C}_k[m]$ with $k \leq \sqrt{\frac{m}{2}}$, then S verifies the Wilf's conjecture. The results of this section are submitted for publication.

2.1. Definitions and preliminaries. If S is a numerical semigroup and $s \in S \setminus \{0\}$, we have that $\mathrm{next}_S(s) = \min\{x \in S \mid s < x\}$ and $\mathrm{prev}_S(s) = \max\{x \in S \mid x < s\}$.

Given m and k positive integers, we denote by

$$\mathscr{L}(m) = \{S \mid S \text{ is a numerical semigroup with } \mathrm{m}(S) = m\}$$

$$\text{and by,} \quad \mathsf{C}_k[m] = \{S \mid S \in \mathscr{L}(m) \text{ with } \mathsf{C}(S) \leq k\}.$$

Observe that if $k \geq m$, then $\mathsf{C}_k[m] = \mathscr{L}(m)$. The purpose of the present section is to study the set of numerical semigroups $\mathsf{C}_k[m]$ when $m \geq 3$ and $k \in \{2, \dots, m-1\}$.

In [**RGS09**, Corollary 2.8] it is shown that every submonoid of $(\mathbb{N}, +)$ has a unique minimal system of generators, which is finite. We denote by $\mathrm{msg}(M)$ the minimal system of generators of M, its cardinality is called the embedding dimension of M denoted by $\mathrm{e}(M)$.

This section is organized as follows. In Subsection 2.2, we will show that if S is a numerical semigroup, then $\mathsf{C}(S) = \max\{\mathrm{next}_S(x) - x \mid x \in \mathrm{msg}(S)\}$. Also, we will see that if $S \in \mathsf{C}_k[m]$ and $S \neq \mathcal{O}_m$ then $S \cup \{\mathrm{F}(S)\} \in \mathsf{C}_k[m]$. This will allow us to order the elements of $\mathsf{C}_k[m]$ making a tree with root $\mathcal{O}_m$. We will characterize the children of an arbitrary vertex of this tree and this will give us an algorithmic procedure to compute the elements $\mathsf{C}_k[m]$ with a given genus. Besides, we will prove that $\mathsf{C}_k[m]$ is an infinite set if and only if there exists $d \in \{2, \dots, k\}$ wherein d divides m.

Given $S \in \mathscr{L}(m)$, denote by $\theta(S) = S \cap \{m+1, \ldots, 2m-1\}$. An (k,m)-set is a set A fulfilling that $A = \theta(S)$, for some $S \in \mathsf{C}_k[m]$. We started the Subsection 2.3, by proving that the set $\mathsf{C}_k[m]$ is equal to $\{S \in \mathscr{L}(m) \mid A \subseteq S \text{ for some } (k,m) - \text{set } A\}$. From this fact, we will show that $\mathsf{C}_k[m]$ is the union of finitely many Frobenius pseudo-varieties (see [**RR15**]).

Remember that a numerical semigroup is elementary if $\mathrm{F}(S) < 2m$. Denote by $\mathcal{E}(m) = \{S \in \mathscr{L}(m) \mid S \text{ is elementary}\}$ and by $\mathcal{E}(\mathsf{C}_k[m]) = \{S \in \mathsf{C}_k[m] \mid S \text{ is elementary}\}$. In Subsection 2.3, we will prove that the set $\mathcal{E}(\mathsf{C}_k[m])$ is equal to $\{\{0\} \cup \{m, m+x_1, m+x_1+x_2, \ldots, m+x_1+x_2+\cdots+x_p\} \cup \{2m, \rightarrow\} \mid (x_1, x_2, \ldots, x_p, x_{p+1}) \in \{1, \ldots, k\}^{p+1} \text{ and } x_1+x_2+\cdots+x_{p+1} = m\}$. As a consequence of this, we will give an algorithm to compute the whole set $\mathcal{E}(\mathsf{C}_k[m])$ with a given genus and Frobenius number.

Denote by $\mathsf{C}_k[m,F] = \{S \in \mathsf{C}_k[m] \mid \mathrm{F}(S) = F\}$. In Subsection 2.4, we will give an algorithm to compute whole set $\mathsf{C}_k[m,F]$ (note that the case $\mathrm{F}(S) < 2m$ has been studied in the Subsection 2.3). Using the terminology introduced in [**RB03**] a numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it. Denote by $\mathscr{I}(\mathsf{C}_k[m,F]) = \{S \in \mathsf{C}_k[m,F] \mid S \text{ is irreducible}\}$. In this subsection, we define an equivalence relation $\sim$ over $\mathsf{C}_k[m,F]$ such that $\mathsf{C}_k[m,F]/\!\sim\, = \{[S] \mid S \in \mathscr{I}(\mathsf{C}_k[m,F])\}$ where $[S]$ denotes the equivalence class of S with respect to $\sim$.

As a consequence of this result, in order to determine explicitly the elements in the set $\mathsf{C}_k[m,F]$ we need:

1) an algorithm to compute the set $\mathscr{I}(\mathsf{C}_k[m,F])$;
2) an algorithm to compute the class $[S]$, for each $S \in \mathscr{I}(\mathsf{C}_k[m,F])$.

Since (1) is solved in the same way as in [**BOR21**], we only need to solve (2).

An element s in S is a small element if $s < \mathrm{F}(S)$. Denote by $\mathrm{N}(S)$ the set of all small elements in S and by $\mathrm{n}(S)$ its cardinality. In Subsection 2.5, we will show that if $S \in \mathsf{C}_k[m]$ with $k \leq \sqrt{\frac{m}{2}}$, then S satisfies Wilf's conjecture.

2.2. The tree associated to $\mathsf{C}_k[m]$. Throughout this section, m and k are positive integers such that $2 \leq k \leq m-1$. From [**RGS09**, Lemma 2.3] we can deduce the following result.

Lemma 130. *Let M be a submonoid of $(\mathbb{N},+)$ such that $M \neq \{0\}$ and $M^* = M\backslash\{0\}$. Then $\mathrm{msg}(M) = M^*\backslash(M^* + M^*)$.*

The next result gives us characterizations for numerical semigroups with multiplicity m and concentration less or equal to k.

Proposition 131. *Let S be a numerical semigroup with $\mathrm{m}(S) = m$. The following conditions are equivalent:*

(1) S belongs to $\mathsf{C}_k[m]$.
(2) if $h \in \mathbb{N}\backslash S$ such that $h > \mathrm{m}(S)$, then $\{h+1,\dots,h+k-1\} \cap S \neq \emptyset$.
(3) $\{s+1,\dots,s+k\} \cap S \neq \emptyset$ for all $s \in S\backslash\{0\}$.
(4) $\{x+1,\dots,x+k\} \cap S \neq \emptyset$ for all $x \in \mathrm{msg}(S)$.

Proof. (1) *implies* (2). Let $s \in S\backslash\{0\}$ such that $s < h < \mathrm{next}_S(s)$. As $h > \mathrm{m}(S)$, then $s \geq m$ and thus $\mathrm{next}_S(s) - s \leq k$. Hence, $h + (\mathrm{next}_S(s) - h) \in S$ and $\mathrm{next}_S(s) - h \in \{1,\dots,k-1\}$.

(2) *implies* (3). If $s+1 \notin S$ then by (2), we deduce that $\{s+2,\dots,s+k\} \cap S \neq \emptyset$.

(3) *implies* (4). Trivial.

(4) *implies* (1). If $s \in S\backslash\{0\}$, then there exists $x \in \mathrm{msg}(S)$ and $t \in S$ such that $s = x+t$. Let $i \in \{1,\dots,k\}$ such that $x+i \in S$. Then $s+i = x+i+t \in S$ and thus $\mathrm{next}_S(s) - s \leq s+i-s \leq k$. □

From the proof of the previous proposition, we obtain the following.

Corollary 132. *If S is a numerical semigroup, then* $\mathsf{C}(S) = \max\{next_S(x) - x \mid x \in \mathrm{msg}(S)\}$.

Example 133. Using the previous results, we deduce that $S = \langle 5, 7, 9\rangle$ is a numerical semigroup with $\mathsf{C}(S) = 2$, because $\max\{\mathrm{next}_S(5) - 5, \mathrm{next}_S(7) - 7, \mathrm{next}_S(9) - 9\} = \max\{7 - 5, 9 - 7, 10 - 9\} = 2$

The following result can be deduced from (2) Proposition 131.

Corollary 134. *If $\{S, T\} \subseteq \mathscr{L}(m)$ such that $S \subset T$ and $S \in \mathsf{C}_k[m]$, then $T \in \mathsf{C}_k[m]$*

It is well known that if S is a numerical semigroup such that $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is a numerical semigroup. As a consequence of Corollary 134, we have the next result.

Corollary 135. *If $S \in \mathsf{C}_k[m]$ such that $S \neq \mathcal{O}_m$, then $S \cup \{\mathrm{F}(S)\} \in \mathsf{C}_k[m]$*

The previous result enable us, given an element $S \in \mathsf{C}_k[m]$, to define recursively the following sequence of elements in $\mathsf{C}_k[m]$:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n)\} & \text{if } S_n \neq \mathcal{O}_m \\ \mathcal{O}_m & \text{otherwise.} \end{cases}$

The next result is easy to prove.

Proposition 136. *If $S \in \mathsf{C}_k[m]$ and $\{S_n \mid n \in \mathbb{N}\}$ is the previous sequence of numerical semigroups, then $S_{\mathrm{g}(S)-m+1} = \mathcal{O}_m$.*

We define the **graph** $G(\mathsf{C}_k[m])$ as the graph whose vertices are elements of $\mathsf{C}_k[m]$ and $(S, T) \in \mathsf{C}_k[m] \times \mathsf{C}_k[m]$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$.

As a consequence of Proposition 136, we have the following result.

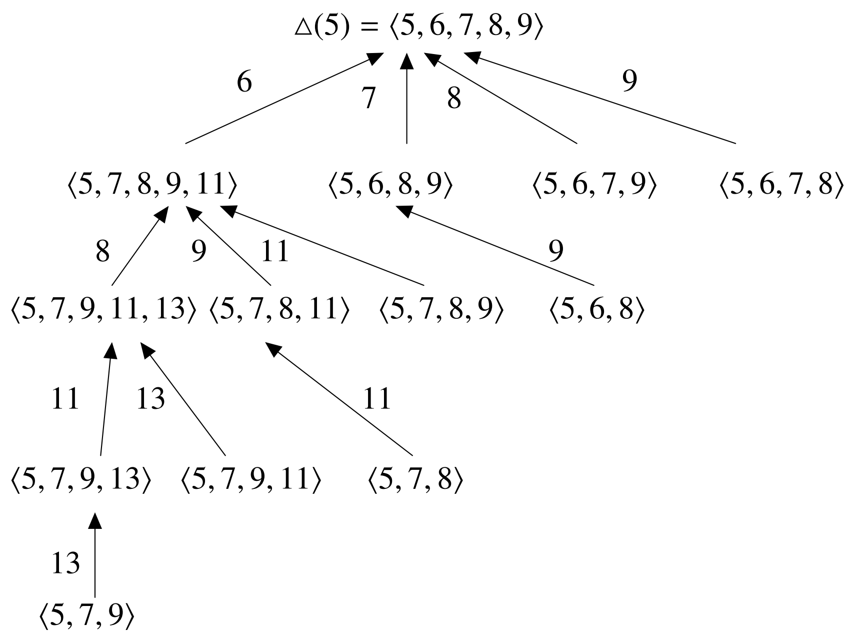
Theorem 137. *The graph $G(\mathsf{C}_k[m])$ is a* ***tree*** *with root equal to $\mathcal{O}_m$.*

From this, it is possible to construct recursively the elements of the set $G(\mathsf{C}_k[m])$, starting in $\mathcal{O}_m$, we connect each vertex with its children. Hence, we need to characterize the children of an arbitrary vertex of this tree.

The following result is easy to prove.

Proposition 138. *If $S \in \mathsf{C}_k[m]$, then the set of children of S in the tree $G(\mathsf{C}_k[m])$ is equal to* $\{S\setminus\{x\} \mid x \in msg(S),\ x \geq \mathrm{F}(S)+2\} \cup \{S\setminus\{\mathrm{F}(S)+1\} \mid \mathrm{F}(S)+1 \in msg(S),\ \mathrm{F}(S)+1 \neq m$ *and* $\mathrm{F}(S)+2-prev_S(\mathrm{F}(S)+1) \leq k\}$.

Example 139. By using the previous proposition, let us construct the tree $G(\mathsf{C}_2[5])$.



We have that $G(\mathsf{C}_2[5])$ is finite, in fact by [**RBT22b**, Proposition 12] we already knew that $G(\mathsf{C}_2[5])$ is finite but, for example, $G(\mathsf{C}_2[4])$ is infinite. Our next goal is to characterize the pair of positive integers (k, m) such that $\mathsf{C}_k[m]$ is finite.

Hence we can announce the next result.

Proposition 140. *The set $\mathsf{C}_k[m]$ is infinite if and only if there exists d divisor of m such that $2 \leq d \leq k$.*

Proof. *Necessity.* If $S \in \mathsf{C}_k[m]$, then $\mathrm{next}_S(m) \in \{m+1, \ldots, m+k\}$ and so $\mathsf{C}_k[m] \subseteq \{S \mid S$ is a numerical semigroup and $\langle m, m+i\rangle \subseteq S,$ for some $i \in \{1, \ldots, k\}\}$.

By using Lemma 109, we can deduce that there exists $i \in \{1, \ldots, k\}$ such that $\gcd(m, m+i) = d \neq 1$. Hence, d is a divisor of m such that $2 \leq d \leq k$.

Sufficiency. For each $t \in \mathbb{N}$ define $S(t) = \{0\} \cup (\{m\} + \langle d \rangle) \cup \{m + t.d, \rightarrow\}$. Is clear that $S(t)$ is a numerical semigroup in $\mathsf{C}_k[m]$ with $\mathrm{F}(S(t)) = m + t.d - 1$. Therefore, $\mathsf{C}_k[m]$ is an infinite set. $\square$

Example 141. Let $d = 2$, $m = 14$ and $k = 3$. As 2 divides 14 and $2 \leq 3$, by Proposition 140, we have that $\mathsf{C}_3[14]$ has infinite carnality.

On the other hand, none of the elements of $\{2, 3, 4\}$ divides 25, then we have that $\mathsf{C}_4[25]$ has finite cardinality.

Let us finish this subsection by giving an algorithm that allows us to compute the whole set $\mathsf{C}_k[m]$ with a given genus.

The following result is easy to prove.

Lemma 142. *With the above notation, we have that:*

(1) $N(G(\mathsf{C}_k[m]), n) = \{S \in \mathsf{C}_k[m] \mid \mathrm{g}(S) = m - 1 + n\}$,

(2) $N(G(\mathsf{C}_k[m]), n + 1) = \{S \mid S$ *is a child of an element in* $N(G(\mathsf{C}_k[m], n))\}$.

Algorithm 143.

Input: Integers m, g such that $g \geq m - 1$.

Output: The set $\{S \in \mathsf{C}_k[m] \mid \mathrm{g}(S) = g\}$

(1) $A = \{\langle m, m+1, \ldots, 2m-1 \rangle\}$, $i = m - 1$.

(2) If $i = g$ then return A.

(3) For each $S \in A$ compute $B_S = \{T \mid T$ is a child of $S \in G(\mathsf{C}_k[m])\}$.

(4) If $\bigcup_{S \in A} B_S = \emptyset$, then return $\emptyset$.

(5) $A := \bigcup_{S \in A} B_S$, $i = i + 1$ and go to step 2.

Example 144. Let us compute the set $\{S \in \mathsf{C}_3[4] \mid \mathrm{g}(S) = 6\}$.

(1) Start with $A = \{\langle 4, 5, 6, 7 \rangle\}$, $i = 3$.

(2) The first loop constructs $B_{\langle 4,5,6,7\rangle} = \{\langle 4,6,7,9\rangle, \langle 4,5,7\rangle, \langle 4,5,6\rangle\}$ and then $A = \{\langle 4,6,7,9\rangle, \langle 4,5,7\rangle, \langle 4,5,6\rangle\}$, $i = 4$.

(3) The second loop constructs $B_{\langle 4,6,7,9\rangle} = \{\langle 4,7,9,10\rangle, \langle 4,6,9,11\rangle, \langle 4,6,7\rangle\}$, $B_{\langle 4,5,7\rangle} = \{\langle 4,5,11\rangle\}$ and $B_{\langle 4,5,6\rangle} = \emptyset$ and then $A = \{\langle 4,7,9,10\rangle, \langle 4,6,9,11\rangle, \langle 4,6,7\rangle, \langle 4,5,11\rangle\}$, $i = 5$.

(4) The third loop constructs $B_{\langle 4,7,9,10\rangle} = \{\langle 4,7,10,13\rangle, \langle 4,7,9\rangle\}$, $B_{\langle 4,6,9,11\rangle} = \{\langle 4,6,11,13\rangle, \langle 4,6,9\rangle\}$, $B_{\langle 4,6,7\rangle} = \emptyset$ and $B_{\langle 4,5,11\rangle} = \{\langle 4,5\rangle\}$ and then $A = \{\langle 4,7,10,13\rangle, \langle 4,7,9\rangle, \langle 4,6,11,13\rangle, \langle 4,6,9\rangle, \langle 4,5\rangle\}$, $i = 6$.

(5) Return
$\{\langle 4,7,10,13\rangle, \langle 4,7,9\rangle, \langle 4,6,11,13\rangle, \langle 4,6,9\rangle, \langle 4,5\rangle\}$.

2.3. (k,m)-sets. Recall that, a (k,m)-set is the set $\theta(S) = S \cap \{m+1, \ldots, 2m-1\}$ with $S \in \mathsf{C}_k[m]$. The next result characterizes the set $\mathsf{C}_k[m]$.

Theorem 145. *With the above notation,* $\mathsf{C}_k[m] = \{S \in \mathscr{L}(m) \mid A \subseteq S$ *for some* $(k,m)-$*set* $A\}$.

Proof. If $S \in \mathsf{C}_k[m]$, then S is a numerical semigroup with multiplicity m and $\theta(S) \subseteq S$.

Conversely, if A is a (k,m)-set, $S \in \mathscr{L}(m)$ such that $A \subseteq S$, we distinguish two cases:

(1) If $\gcd(\{m\} \cup A) = 1$, then $T = \langle \{m\} \cup A\rangle \in \mathscr{L}(m)$. From (4) Proposition 131, we deduce that $T \in \mathsf{C}_k[m]$. Since $S \in \mathscr{L}(m)$, $T \subseteq S$ and $T \in \mathsf{C}_k[m]$, then by Corollary 134, we have that $S \in \mathsf{C}_k[m]$.

(2) If $\gcd(\{m\} \cup A) = d \neq 1$, then $T = \langle \{m\} \cup A\rangle \cup \{\mathrm{F}(S)+1, \rightarrow\} \in \mathsf{C}_k[m]$ and $T \subseteq S$. By applying again Corollary 134, we get that $S \in \mathsf{C}_k[m]$.

□

The next result is easy to prove.

Lemma 146. *If* $A \subseteq \{m, \rightarrow\}$, *then* $\mathcal{P}(A) = \{S \in \mathscr{L}(m) \mid A \subseteq S\}$ *is a Frobenius pseudo-variety with* $\max(\mathcal{P}(A)) = \mathcal{O}_m$.

Note that, if the set $\text{Minimals}_{\subseteq}\{A \mid A \text{ is a } (k,m)-\text{set}\} = \left\{A_1, A_2, \ldots, A_p\right\}$, then as a consequence of Theorem 145, we have that $\mathsf{C}_k[m] = \bigcup_{i=1}^{p} \mathcal{P}(A_i)$. From this fact follows the next result.

Proposition 147. *With the above notation, $\mathsf{C}_k[m]$ is the union of finitely many Frobenius pseudo-varieties.*

Our next goal is to give an algorithm to compute all (k,m)-sets, given positive integers k and m. To this end, we need to introduce some concepts and results.

Given a (k,m)-set A such that $A \neq \{m+1, m+2, \ldots, 2m-1\}$, denote by

$$\mathcal{B}(A) = \max(\{m+1, m+2, \ldots, 2m-1\} \setminus A).$$

The following result is easy to prove.

Lemma 148. *If A is a (k,m)-set and $A \subseteq B \subseteq \{m+1, m+2, \ldots, 2m-1\}$, then B is also a (k,m)-set.*

The previous result enable us, given a (k,m)-set A , to define recursively the following sequence of (k,m)-sets:

- $A_0 = A$,
- $A_{n+1} = \begin{cases} A_n \cup \{\mathcal{B}(A_n)\} & \text{if } A_n \neq \{m+1, m+2, \ldots, 2m-1\} \\ \{m+1, m+2, \ldots, 2m-1\} & \text{otherwise.} \end{cases}$

Let $\mathscr{C}(k,m) = \{A \mid A \text{ is a } (k,m)-\text{set}\}$. We define the graph $G(\mathscr{C}(k,m))$ as the graph whose vertices are elements of $\mathscr{C}(k,m)$ and $(X,Y) \in \mathscr{C}(k,m) \times \mathscr{C}(k,m)$ is an edge if $Y = X \cup \{\mathcal{B}(X)\}$. From the previous results, it is easy to prove the next one.

Proposition 149. *The graph $G(\mathscr{C}(k,m))$ is a tree with root equal to $\{m+1, m+2, \ldots, 2m-1\}$. Moreover, if A is a (k,m)-set, then the set of children of A in the tree $G(\mathscr{C}(k,m))$ is equal to $\{A \setminus \{a\} \mid \{a, a+1, \ldots, 2m-1\} \subseteq A$ and either $a \leq m+k-1$ or $\{a-1, a-2, \ldots, a-(k-1)\} \cap A \neq \emptyset\}$.*

Example 150. By using Proposition 149, let us construct the tree $G(\mathscr{C}(2,4))$.

$$\begin{tikzcd}
& \{5,6,7\} & \\
\{6,7\} \arrow[ur, "5"] & \{5,7\} \arrow[u, "6"'] & \{5,6\} \arrow[ul, "7"'] \\
\{6\} \arrow[u, "7"'] & &
\end{tikzcd}$$

Observe that $\mathrm{Minimals}_{\subseteq}(\mathscr{C}(2,4)) = \{\{6\},\{5,7\}\}$. Hence, by Theorem 145, we obtain that $\mathsf{C}_2[4] = \{S \mid S \in \mathscr{L}(4) \text{ with } \{6\} \subset S \text{ or } \{5,7\} \subset S\}$. Moreover, by Proposition 147, $\mathsf{C}_2[4]$ is the union of the Frobenius pseudo-varieties $\{S \mid S \in \mathscr{L}(4) \text{ with } \{6\} \subset S\}$ and $\{S \mid S \in \mathscr{L}(4) \text{ with } \{5,7\} \subset S\}$.

Now, our goal in this subsection is to compute all (k,m)-sets with a given cardinality.

Proposition 151. *Let p be a positive integer. The following conditions are equivalents.*

(1) A is an (k,m)-set with cardinality p,

(2) $A = \{m + x_1, m + x_1 + x_2, \dots, m + x_1 + x_2 + \cdots + x_p \mid (x_1, x_2, \dots, x_p, x_{p+1}) \in \{1,\dots,k\}^{p+1}$ and $x_1 + x_2 + \cdots + x_p + x_{p+1} = m\}$.

Proof. 1) *implies* 2). Assume that $A = \left\{a_1 < a_2 < \cdots < a_p\right\}$, $x_1 = a_1 - m$, $x_{i+1} = a_{i+1} - a_i$ for all $i \in \{1,\dots,p-1\}$ and $x_{p+1} = 2m - a_p$. Then, we can conclude that $(x_1, x_2, \dots, x_p, x_{p+1}) \in \{1,\dots,k\}^{p+1}$, $x_1 + x_2 + \cdots + x_p + x_{p+1} = m$ and $A = \left\{m + x_1, m + x_1 + x_2, \dots, m + x_1 + x_2 + \cdots + x_p\right\}$.

2) *implies* 1). It is clear that, under desired conditions, every $A = \{m + x_1, m + x_1 + x_2, \dots, m + x_1 + x_2 + \cdots + x_p\} \subseteq \{m+1, m+2, \dots, 2m-1\}$. Then, we have that $S = \{0, m\} \cup A \cup \{2m, \rightarrow\} \in \mathsf{C}_k[m]$ and $A = \theta(S)$. Hence, A is an (k,m)-set with cardinality p. $\square$

Given $q \in \mathbb{Q}$ and $p \in \mathbb{N}\backslash\{0\}$, we denote by $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$ and by $n(k,m,p) = \#\{A \mid A \text{ is an } (k,m) - \text{set with cardinality } p\}$.

As a consequence of Proposition 151, we obtain the following result.

Corollary 152. *With the above notation, we have that* $n(k, m, p) \neq 0$ *if and only if* $\lceil \frac{m}{k} \rceil - 1 \leq p \leq m - 1$. *Furthermore,* $n(k, m, p) = \#\{(x_1, x_2, \ldots, x_{p+1}) \in \{1, \ldots, k\}^{p+1} \mid x_1 + x_2 + \cdots + x_p + x_{p+1} = m\}$.

Example 153. By using Proposition 151 let us calculate:

(1) the set $\{A \mid A \text{ is a } (2, 4) - \text{set with cardinality } 2\}$, which is equal to $\{\{4 + x_1, 4 + x_1 + x_2\} \mid (x_1, x_2, x_3) \in \{1, 2\}^3 \text{ and } x_1 + x_2 + x_3 = 4\}$. Since $\{(x_1, x_2, x_3) \in \{1, 2\}^3 \mid x_1 + x_2 + x_3 = 4\} = \{(1, 1, 2), (1, 2, 1), (2, 1, 1)\}$, then the solution is $\{\{5, 6\}, \{5, 7\}, \{6, 7\}\}$;

(2) the set $\{A \mid A \text{ is a } (2, 4) - \text{set with cardinality } 1\}$, which is equal to $\{\{4 + x_1\} \mid (x_1, x_2) \in \{1, 2\}^2 \text{ and } x_1 + x_2 = 4\}$. As $\{(x_1, x_2) \in \{1, 2\}^2 \mid x_1 + x_2 = 4\} = \{(2, 2)\}$, then the solution is equal to $\{\{6\}\}$.

Recall that S is an elementary numerical semigroup if $\mathrm{F}(S) < 2m(S)$. We denote by $\mathcal{E}(m)$ the set of elementary numerical semigroups with multiplicity m.

Proposition 154. [**RB22**, Lemma 1] *Let* A *be a subset of* $\{m + 1, \ldots, 2m - 1\}$. *Then* $\{0, m\} \cup A \cup \{2m, \rightarrow\}$ *is an elementary numerical semigroup with multiplicity* m. *Moreover, every elementary numerical semigroup with multiplicity* m *is of this form.*

Denote by

$$\mathcal{E}(\mathsf{C}_k[m]) = \{S \in \mathsf{C}_k[m] \mid S \text{ is elementary}\}.$$

Thus, by using Theorem 145 and Propositions 151 and 154, we deduce the next result.

Proposition 155. *With the above notation,* $\mathcal{E}(\mathsf{C}_k[m]) = \{\{0\} \cup \{m, m + x_1, m + x_1 + x_2, \ldots, m + x_1 + x_2 + \cdots + x_p\} \cup \{2m, \rightarrow\} \mid (x_1, x_2, \ldots, x_p, x_{p+1}) \in \{1, \ldots, k\}^{p+1} \text{ and } x_1 + x_2 + \cdots + x_{p+1} = m\}$.

As a consequence of the previous proposition, we have the next result.

Corollary 156. *Let g be a positive integer. Then, the set of $\{S \in \mathcal{E}(\mathsf{C}_k[m])$ with $\mathrm{g}(S) = g\}$ is equal to $\{\{0\} \cup \{m, m + x_1, m + x_1 + x_2, \ldots, m + x_1 + x_2 + \cdots + x_{2m-g-2}\} \cup \{2m, \rightarrow\ \} \mid (x_1, x_2, \ldots, x_p, x_{2m-g-1}) \in \{1, \ldots, k\}^{2m-g-1}$ and $x_1 + x_2 + \cdots + x_{2m-g-1} = m\}$.*

Example 157. Let us calculate all numerical semigroups S in $\mathcal{E}(\mathsf{C}_3[5])$ with $\mathrm{g}(S) = 6$. By Corollary 156, we have that $\{S \in \mathcal{E}(\mathsf{C}_3[5])$ with $\mathrm{g}(S) = 6\} = \{\{0\} \cup \{5, 5 + x_1, 5 + x_1 + x_2\} \cup \{10, \rightarrow\ \} \mid (x_1, x_2, x_3) \in \{1, 2, 3\}^3$ and $x_1 + x_2 + x_3 = 5\}$. As $\{(x_1, x_2, x_3) \in \{1, 2, 3\}^3$ and $x_1 + x_2 + x_3 = 5\} = \{(1, 1, 3), (1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1), (3, 1, 1)\}$, then $\{S \in \mathcal{E}(\mathsf{C}_3[5]) \mid \mathrm{g}(S) = 6\} = \{0\} \cup A \cup \{10, \rightarrow\}$ such that A belongs to $\{\{5, 6, 7\}, \{5, 6, 8\}, \{5, 6, 9\}, \{5, 7, 8\}, \{5, 7, 9\}, \{5, 8, 9\}\}$.

Given a positive integer F, denote by

$$\mathsf{C}_k[m, F] = \{S \in \mathsf{C}_k[m] \mid \mathrm{F}(S) = F\}.$$

We will finish this subsection by studying the elementary elements in $\mathsf{C}_k[m, F]$. The next result is easy to prove.

Proposition 158. *With the above notation, we have the following:*

(1) If $F = m - 1$ then $\mathsf{C}_k[m, F] = \{\mathcal{O}_m\}$;

(2) If $F = m + r$ with $1 \leq r \leq k - 1$, then $\mathsf{C}_k[m, F] = \{\{0, m\} \cup A \cup \{m + r + 1, \rightarrow\ \} \mid A \subseteq \{m + 1, \ldots, m + r - 1\}\}$;

(3) If $F = m + r$ with $k \leq r < m$, then $\mathsf{C}_k[m, F] = \{\{0, m\} \cup A \cup \{m + r + 1, \rightarrow\ \} \mid A = \{m + x_1, m + x_1 + x_2, \ldots, m + x_1 + x_2 + \cdots + x_p\}$, $(x_1, x_2, \ldots, x_p, x_{p+1}) \in \{1, \ldots, k\}^{p+1}$ with $x_1 + x_2 + \cdots + x_p + x_{p+1} = r + 1$ and $x_{p+1} \geq 2\}$.

Example 159. Let us calculate the set of numerical semigroups $\mathsf{C}_2[5, 8]$. By Proposition 158, we get that $\mathsf{C}_2[5, 8] = \{\{0, 5\} \cup A \cup \{9, \rightarrow\} \mid A = \{5 + x_1\}$, $(x_1, x_2) \in \{1, 2\}^2$ with $x_1 + x_2 = 4$ and $x_2 \geq 2\} \cup \{\{5 + x_1, 5 + x_1 + x_2\}$, $(x_1, x_2, x_3) \in \{1, 2\}^3$ with $x_1 + x_2 + x_3 = 4$ and $x_3 \geq 2\}$. Then, $\mathsf{C}_2[5, 8] = \{\{0, 5\} \cup A \cup \{9, \rightarrow\} \mid A \in \{\{7\}, \{6, 7\}\}\}$.

2.4. Non elementary elements of $\mathsf{C}_k[m,F]$. The aim of this subsection is to give an algorithm to compute all elements in the set $\mathsf{C}_k[m,F]$ with $F > 2m$.

If S is a numerical semigroup, then $\mathrm{g}(S) \geq \frac{\mathrm{F}(S)+1}{2}$ (see, [**RGS09**, Lemma 2.14]. As a consequence of Corollary 22, we obtain that the irreducible numerical semigroups are those with the least possible genus in terms of their Frobenius number.

Given S a nonirreducible numerical semigroup, here we will denote by $\alpha(S) = \max\left\{x \in \mathbb{N}\backslash S \mid \mathrm{F}(S) - x \notin S \text{ and } x \neq \frac{\mathrm{F}(S)}{2}\right\}$. If S is an irreducible numerical semigroup, then by definition $\alpha(S) = 0$. Observe that, if $\alpha(S) \neq 0$ then $\frac{\mathrm{F}(S)}{2} < \alpha(S) < \mathrm{F}(S)$.

As a consequence of Corollary 134 and Proposition 124, if $S \in \mathsf{C}_k[m,F]$, we can define the following sequence of the elements in $\mathsf{C}_k[m,F]$:

- $S_0 = S$,
- $S_{n+1} = S_n \cup \{\alpha(S_n)\}$

It is easy to prove the next result.

Proposition 160. *Let $S \in \mathsf{C}_k[m,F]$ and let $(S_n)_{n\in\mathbb{N}}$ be the previous sequence. Then there exists a nonnegative integer p such that S_p is an irreducible numerical semigroup in $\mathsf{C}_k[m,F]$.*

We denote by $\mathbf{I}(S)$ the irreducible S_p obtained from a numerical semigroup S.

We define the following equivalence relation over $\mathsf{C}_k[m,F]$:

$$S \sim T \text{ if and only if } \mathbf{I}(S) = \mathbf{I}(T).$$

Denote the equivalence class modulo $\sim$ by $[S] = \{T \in \mathsf{C}_k[m,F] \mid S \sim T\}$ and by $\mathscr{I}(\mathsf{C}_k[m,F]) = \{S \in \mathsf{C}_k[m,F] \mid S \text{ is irreducible}\}$. As a consequence of Proposition 160, we deduce the next result.

Theorem 161. *The quotient set $\mathsf{C}_k[m,F]/\!\sim\, = \{[S] \mid S \in \mathscr{I}(\mathsf{C}_k[m,F])\}$. Moreover, if $\{S,T\} \subseteq \mathscr{I}(\mathsf{C}_k[m,F])$ and $S \neq T$ then $[S] \cap [T] = \emptyset$*

In view of Theorem 161, in order to determine explicitly the elements in the set $\mathsf{C}_k[m, F]$ we need:

1) an algorithm to compute the set $\mathscr{I}(\mathsf{C}_k[m, F])$.
2) an algorithm to compute the class $[S]$, for each $S \in \mathscr{I}(\mathsf{C}_k[m, F])$.

In [**BOR21**] it was given an efficient algorithm to compute all irreducible numerical semigroups with fixed multiplicity m and Frobenius number F. By using Proposition 131, we choose those with concentration less or equal to k and thus we solve 1). Our goal is to provide an algorithm to solve 2).

Let ∇ be an element in $\mathscr{I}(\mathsf{C}_k[m, F])$. Let $G([\nabla])$ be the graph with vertex set $[\nabla]$ and $(S, T) \in [\nabla] \times [\nabla]$ is an edge if and only if $T = S \cup \{\alpha(S)\}$ and $\alpha(S) \neq 0$.

Proposition 162. *If $\nabla \in \mathscr{I}(\mathsf{C}_k[m, F])$, then the graph $G([\nabla])$ is a tree with root equal to ∇. Moreover, the children of a vertex T is $\{T \backslash \{x\} \mid x \in \operatorname{msg}(T),\ \frac{F}{2} < x < F,\ \alpha(T) < x,\ \mathrm{next}_T(x) - \mathrm{prev}_T(x) \leq k$ and $x > m\}$.*

Proof. If S is a child of T, then $T = S \cup \{\alpha(S)\}$ with $\alpha(S) \neq 0$ and thus $T \backslash \{\alpha(S)\} = S$. By Lemma 31, we have that $\alpha(S) \in \operatorname{msg}(T)$. Clearly, $\frac{F}{2} < \alpha(S) < F$, $\alpha(T) < \alpha(S)$ and $\mathrm{next}_T(\alpha(S)) - \mathrm{prev}_T(\alpha(S)) \leq k$.

Conversely, if $x \in \operatorname{msg}(T)$, $\frac{F}{2} < x < F$, $\mathrm{next}_T(x) - \mathrm{prev}_T(x) \leq k$, then we get that $T \backslash \{x\} \in \mathsf{C}_k[m, F]$. If $\alpha(T) < x$, then we have that $\alpha(T \backslash \{x\}) = x$. Therefore, $T = (T \backslash \{x\}) \cup \{\alpha(T \backslash \{x\}\}$ and thus $T \backslash \{x\}$ is a child of T. □

Algorithm 163.

Input: $\nabla \in \mathscr{I}(\mathsf{C}_k[m, F])$.

Output: The set $[\nabla]$.

1. $A = \{\nabla\}$ and $C = \{\nabla\}$.
2. For each $S \in C$ compute the set
   $B_S = \{T \mid T$ is child of S in the tree $G([\nabla])\}$.
3. $C = \bigcup_{S \in C} B_S$.

4. If $C = \emptyset$ then return A.
5. $A = A \cup C$ go to step 2.

Example 164. By using Propositions 131 and 22 we have that $\langle 5, 7, 9, 11\rangle \in \mathscr{I}(\mathsf{C}_3[5, 13])$. Let us compute $[\langle 5, 7, 9, 11\rangle]$.

- Start with $A = \{\langle 5, 7, 9, 11\rangle\}$ and $C = \{\langle 5, 7, 9, 11\rangle\}$.
- The first loop constructs $B_{\langle 5,7,9,11\rangle} = \{\langle 5, 7, 11\rangle, \langle 5, 7, 9\rangle\}$, then $C = \{\langle 5, 7, 11\rangle, \langle 5, 7, 9\rangle\}$ and thus $A = \{\langle 5, 7, 9, 11\rangle, \langle 5, 7, 11\rangle, \langle 5, 7, 9\rangle\}$
- The second loop constructs $B_{\langle 5,7,11\rangle} = \{\langle 5, 7, 16, 18\rangle, \}$ and $B_{\langle 5,7,9\rangle} = \emptyset$, then $C = \{\langle 5, 7, 16, 18\rangle, \}$ and thus $A = \{\langle 5, 7, 9, 11\rangle, \langle 5, 7, 11\rangle, \langle 5, 7, 9\rangle, \langle 5, 7, 16, 18\rangle\}$.
- The third loop constructs $B_{\langle 5,7,16,18\rangle} = \emptyset$, then $C = \emptyset$.
- Hence, $[\langle 5, 7, 9, 11\rangle] = \{\langle 5, 7, 9, 11\rangle, \langle 5, 7, 11\rangle, \langle 5, 7, 9\rangle, \langle 5, 7, 16, 18\rangle\}$.

2.5. Wilf's conjecture. The next result is in [**Eli18**, Corollary 6.5].

Lemma 165. *If S is a numerical semigroup with $\mathrm{F}(S) + 1 \leq 3\mathrm{m}(S)$, then S verifies Wilf's conjecture.*

It is clear that $\{0, 1, \ldots, \mathrm{F}(S)\} = \mathrm{N}(S) \cup (\mathbb{N}\backslash S)$ and so $\mathrm{F}(S) + 1 = \mathrm{g}(S) + \mathrm{n}(S)$. Hence, we have that $\mathrm{F}(S) + 1 \leq \mathrm{e}(S)\mathrm{n}(S)$ is another way to present Wilf's conjecture.

Theorem 166. *If $S \in \mathscr{L}(m)$, $p = \#\theta(S)$ and $2m \leq (p + 1)^2$, then S satisfies the Wilf's conjecture.*

Proof. Let $q \in \mathbb{N}$ and $r \in \{1, \ldots, m - 1\}$ such that $\mathrm{F}(S) = q.m + r$. If $q \in \{0, 1, 2\}$, then by Lemma 165, S satisfies the Wilf's conjecture.

Now, we suppose that $q \geq 3$. By Lemma 130, we know that $\theta(S) \cup \{m\} \subseteq \mathrm{msg}(S)$ and thus $p + 1 \leq \mathrm{e}(S)$. Clearly, we have that $\{0\}$, $\theta(S) \cup \{m\}$, $\{m\} + (\theta(S) \cup \{m\})$, $\{2m\} + (\theta(S) \cup \{m\})$, …, $\{(q - 2)m\} + (\theta(S) \cup \{m\})$ are disjoint subsets of the set $\mathrm{N}(S)$ and so we can conclude that $(q - 1)(p + 1) + 1 \leq \mathrm{n}(S)$.

As by hypothesis $2m \leq (p+1)^2$, then $2(q-1)m \leq (q-1)(p+1)^2$. If $q \geq 3$, then $q+1 \leq 2(q-1)$ and so we deduce that $(q+1)m \leq (q-1)(p+1)^2$. Since $r \leq m-1$, then we have that $\mathrm{F}(S)+1 = (q+1)m \leq (q-1)(p+1)^2$. As $p+1 \leq e(S)$ thus $F(S)+1 \leq (q-1)(p+1)e(S)$. By applying that $(q-1)(p+1)+1 \leq \mathrm{n}(S)$, we obtain that $\mathrm{F}(S)+1 \leq \mathrm{e}(S)(\mathrm{n}(S)-1)$. Hence, we get that $\mathrm{F}(S)+1 \leq \mathrm{e}(S)\mathrm{n}(S)$ and thus S verifies Wilf's conjecture. □

Corollary 167. *If $S \in \mathsf{C}_k[m]$ with $k \leq \sqrt{\frac{m}{2}}$, then S satisfies the Wilf's conjecture.*

Proof. If $S \in \mathsf{C}_k[m]$, by Proposition 151, then $\theta(S) = \{m+x_1, m+x_1+x_2, \ldots, m+x_1+x_2+\cdots+x_p \mid (x_1, x_2, \ldots, x_p, x_{p+1}) \in \{1, \ldots, k\}^{p+1}$ and $x_1+x_2+\cdots+x_p+x_{p+1} = m\}$. From Corollary 152, we have that $\#\theta(S) \geq \lceil \frac{m}{k} \rceil - 1$. If $k \leq \sqrt{\frac{m}{2}}$, then $2m \leq (\frac{m}{k})^2$ this implies that $2m \leq (p+1)^2$. By using Theorem 166, we can conclude that S satisfies Wilf's conjecture. □

Example 168. By applying the above corollary we get that if $S \in \mathsf{C}_5[100]$, then S satisfies Wilf's conjecture.

CHAPTER 4

Numerical semigroups without consecutive small elements

In this chapter, we study A**-semigroups**, that is, **numerical semigroups which have no consecutive elements less than the Frobenius number**. We give algorithms that allow the computation of the whole set of A-semigroups with a given genus, multiplicity and Frobenius number and from this we examine interesting families of A-semigroups which are Frobenius varieties, pseudo-varieties and R-varieties. Results of this chapter are published in [**RBT22c**].

1. Definitions and preliminaries

We say that S is an A**-semigroup** if it has no consecutive elements in $\mathrm{N}(S)$, that is, if $\{x, x+1\} \subseteq S$ then $\mathrm{F}(S) < x$.

Denote by $\mathscr{A}$ the set of all A-semigroups. In Section 2 we see that $\mathscr{A}$ **is a Frobenius variety**. This fact allows us to arrange the elements of $\mathscr{A}$ in a tree with root. We describe the children of any vertex of this tree and this will enable us to recursively construct the entire set of elements in $\mathscr{A}$.

An A-monoid is a submonoid of $(\mathbb{N}, +)$ which can be expressed as the intersection of A-semigroups. In Section 3 we show that a submonoid M of $(\mathbb{N}, +)$ is an A-monoid if and only if M is an A-semigroup or M is not a numerical semigroup. If $X \subseteq \mathbb{N}$, then we will see that there exists the least A-monoid that contains X, denoted by $A(X)$. If $M = A(X)$, then we will say that X is an A-system of generators of M. Moreover if $M \neq A(X)$ for every $Y \subsetneq X$, then we say that X is a **minimal** A**-system of generators** of M. In this section, we see that every A-monoid admits a unique finite minimal system of generators. We denote by $\mathrm{mAsg}(M)$ the minimal A-system of generators of M and we will see that $\mathrm{mAsg}(M) = \{x \in \mathrm{msg}(M) \mid M\backslash\{x\}$ is A-monoid$\}$. The A-embedding dimension of M is the cardinality of $\mathrm{mAsg}(M)$, denoted by $\mathrm{e}_A(M)$.

An A**MED-semigroup** is an A-semigroup which is a **MED-semigroup**. In Section 4, we see that there exists a one-to-one correspondence between the set of all numerical semigroups and the set of AMED-semigroups. Inspired by [**Arf48**], Lipman in [**Lip71**] introduces and motivates the study of **Arf** rings. The characterization of these rings via their value semigroups yields the Arf property for numerical semigroups. In this section, we also show that **every Arf numerical semigroup is** A**MED-semigroup**.

For a positive integer m, we denote by

$$\mathscr{A}(m) = \{S \mid S \text{ is an } A\text{-semigroup and } \mathrm{m}(S) = m\} \quad \text{and}$$

$$\mathscr{A}\mathscr{M}\mathscr{E}\mathscr{D}(m) = \{S \mid S \text{ is an AMED-semigroup and } \mathrm{m}(S) = m\}.$$

In section 5, we see that $\mathscr{A}(m)$ **and** $\mathscr{A}\mathscr{M}\mathscr{E}\mathscr{D}(m)$ **are Frobenius pseudo-variety**. This fact enables us to order the elements of $\mathscr{A}(m)$ and $\mathscr{A}\mathscr{M}\mathscr{E}\mathscr{D}(m)$ in a tree with root and, as a consequence, we give an algorithm to compute all elements of $\mathscr{A}(m)$ ($\mathscr{A}\mathscr{M}\mathscr{E}\mathscr{D}(m)$, respectively) with a given genus.

Given F and m positive integers, denote by

$$\mathscr{A}(m, F) = \{S \mid S \text{ is an } A\text{-semigroup with } \mathrm{m}(S) = m \text{ and } \mathrm{F}(S) = F\}.$$

In Section 6, we give an algorithmic method to compute $\mathscr{A}(m, F)$ with $F < 2m$. For the case $F > 2m$ we need to introduce the following notation:

- $\Delta(m, F) = \{\mathrm{F}(S) - i \mid i \in \{1, \ldots, \mathrm{m}(S) - 1\}\}$.
- If $S \in \mathscr{A}(m, F)$, then $B(S) = \{x \in \Delta(m, F) \mid x \notin S\}$.
- $\mathsf{B}(m, F) = \{B \subseteq \Delta(m, F)\backslash\langle m\rangle \mid \{x, x+1\} \cap B \neq \emptyset \text{ if } \{x, x+1\} \subseteq \Delta(m, F)\}$.
- If $B \in \mathsf{B}(m, F)$, then $\mathscr{A}(m, F, B) = \{S \mid S \text{ is an } A\text{-semigroup with } \mathrm{m}(S) = m, \mathrm{F}(S) = F \text{ and } B(S) = B\}$.

In this section, we will see that the set $\{\mathscr{A}(m,F,B) \mid B \in \mathsf{B}(m,F)\}$ is a partition of the set $\mathscr{A}(m,F)$. Therefore, to compute the whole $\mathscr{A}(m,F)$ it is sufficient to give an algorithm that computes $\mathscr{A}(m,F,B)$ for each $B \in \mathsf{B}(m,F)$.

In section 7 we will show that if $T \in \mathscr{A}(m,F,B)$, then $R(T) = \{S \in \mathscr{A}(m,F,B) \mid S \subseteq T\}$ **is a Frobenius R-variety**. As a consequence of this fact, it is possible to order, in a rooted tree, all the elements of $R(T)$ and also to give an algorithm that computes $R(T)$. Finally, we describe an algorithm to compute the set $\mathscr{A}(m,F,B)$. The idea is the following: first, we compute the Maximals($\mathscr{A}(m,F,B)$) and for each $T \in$ Maximals($\mathscr{A}(m,F,B)$) it is possible to compute the whole set $R(T)$.

2. Frobenius variety of A-semigroups

Our aim in this section is to see that the class of A-semigroups is a **Frobenius variety**. In order to see this, we need the next result which is easy to prove.

Lemma 169. *If S and T are numerical semigroups then $S \cap T$ is a numerical semigroup such that* $\mathrm{F}(S \cap T) = \max\{\mathrm{F}(S), \mathrm{F}(T)\}$.

As a consequence of previous lemma we have the following result.

Proposition 170. *The set $\mathscr{A} = \{S \mid S$ is an A-semigroup$\}$ is a Frobenius variety.*

We define the **graph** $G(\mathscr{A})$ as follows: $\mathscr{A}$ is the vertex set and $(S,T) \in \mathscr{A} \times \mathscr{A}$ is an edge if and only if $T = S \cup \{\mathrm{F}(S)\}$.

The next result is deduced from [**Ros08b**, Theorem 27]

Proposition 171. *The graph $G(\mathscr{A})$ is a **tree** with root $\mathbb{N}$. Furthermore, the set of children of vertex T is* $\{T \backslash \{x\} \mid x \in \mathrm{msg}(T),\ x > \mathrm{F}(T)$ *and* $T\backslash\{x\} \in \mathscr{A}\}$.

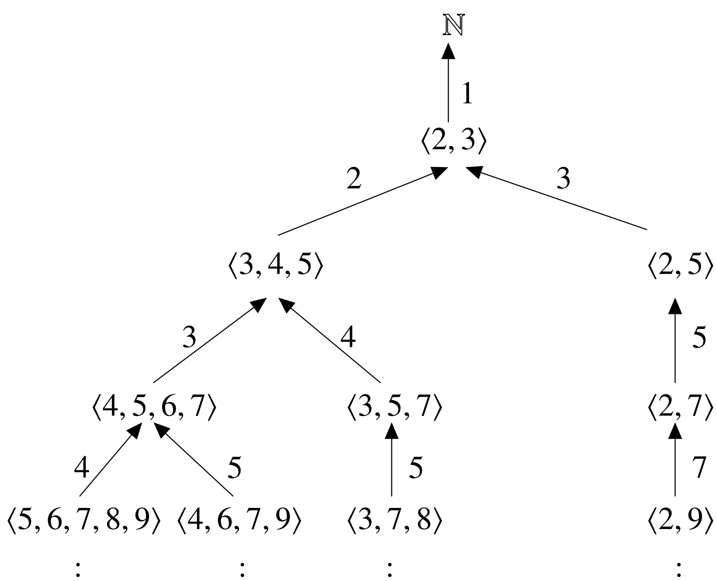
The following result is straightforward to prove.

Lemma 172. *Let M be a submonoid of* $(\mathbb{N},+)$ *and $x \in M$. Then $M\backslash\{x\}$ is a submonoid of* $(\mathbb{N},+)$ *if and only if* $x \in \mathrm{msg}(M)$.

As a consequence of the definition of an A-semigroup and Lemma 172, we obtain the following.

Proposition 173. *Let* $T \in \mathscr{A}$ *and* $x \in \mathrm{msg}(T)$ *such that* $x > \mathrm{F}(T)$. *Then* $T \backslash \{x\} \in \mathscr{A}$ *if and only if* $x \in \{\mathrm{F}(T)+1, \mathrm{F}(T)+2\}$.

In order to recursively build the tree $G(\mathscr{A})$, starting from $\mathbb{N}$, it suffices to compute the children of each vertex of $G(\mathscr{A})$. However, by applying Propositions 171 and 173, we can build $G(\mathscr{A})$. Since $G(\mathscr{A})$ has infinite cardinal it is not possible to build a whole tree, thus the hanging points indicate that the process continues.



Although all numerical semigroups we have built previously are MED-semigroups, it is not always so as we see in the next example.

Example 174. $S = \langle 5, 7, 11, 13 \rangle = \{0, 5, 7, 10, \rightarrow\}$ is an A-semigroup which is not MED-semigroup.

Moreover, in the previous tree, there are no leaves, but we will see that if we continue building the tree it has leaves.

Example 175. $S = \langle 7, 10, 22, 23, 25, 26\rangle = \{0, 7, 10, 14, 17, 20 \rightarrow\}$ is an A-semigroup such that $\mathrm{F}(S) = 19$. Since $\mathrm{msg}(S) \cap \{F(S) + 1, \mathrm{F}(S) + 2\} = \emptyset$, then applying Propositions 171 and 5 we get that S has no children, and so S is a leaf.

It is clear that if $k \in \mathbb{N}\backslash\{0\}$ then $\langle 2, 2k + 1\rangle$ is an A-semigroup. In the next proposition, we will show that these are the only A-semigroups with embedding dimension two.

Lemma 176. *If S is a symmetric numerical semigroup with $\mathrm{m}(S) \geq 3$, then S is not an A-semigroup.*

Proof. Since $\{1, 2\} \subseteq \mathbb{N}\backslash S$, then $\{\mathrm{F}(S) - 2, \mathrm{F}(S) - 1\} \subseteq \mathrm{N}(S)$, and thus S is not A-semigroup. □

Lemma 177. [**RGS09**, Corollary 4.7] *Every numerical semigroup with embedding dimension two is symmetric.*

As a consequence of Lemmas 176 and 177, we have the following.

Proposition 178. *The set of all A-semigroups of embedding dimension two is equal to* $\{\langle 2, 2k + 1\rangle \mid k \in \mathbb{N}\backslash\{0\}\}$.

We finish this section by posing the following problem: how to calculate all A-semigroups of embedding dimension three?

3. A-Monoids

Nonfinite intersections of A-semigroups are not necessarily an A-semigroup. To remember we rewrite example 40.

Example 179. Given $n \in \mathbb{N}$, denote by $S_n = \{0, n, \rightarrow\}$. It can be easily seen that S_n is an A-semigroup but $\bigcap_{n\in\mathbb{N}} S_n = \{0\}$ is not a numerical semigroup.

Clearly, the intersection of A-semigroups is always a submonoid of $(\mathbb{N}, +)$. This fact motivates the following definition: an A-monoid is a submonoid of $(\mathbb{N}, +)$ which can be expressed as an intersection of A-semigroups.

Proposition 180. *Let M be a submonoid of $(\mathbb{N}, +)$. Then M is an A-monoid if and only if M is an A-semigroup or M is not a numerical semigroup.*

Proof. *Necessity*. If M is a numerical semigroup, then $\mathbb{N} \backslash M$ is finite. Hence there exist finitely many A-semigroups that contain M. From Proposition 170 we obtain that M is an A-semigroup.

Sufficiency. If M is an A-semigroup, then M is an A-monoid. If M is not a numerical semigroup then $\gcd(M) \neq 1$. We deduce that $S_k = M \cup \{k, \rightarrow\}$ is A-semigroup for all $k \in \mathbb{N}$. Clearly $\bigcap_{k \in \mathbb{N}} S_k = M$ and thus M is an A-monoid. □

As a consequence of Lemma 41, Lemma 42 and Theorem 43 we have the following:

Proposition 181. *Every A-monoid M admits a unique minimal A-system of generators. Furthermore, this A-system of generators is contained in* $\mathrm{msg}(M)$.

We denote by $\mathrm{mAsg}(M)$ the minimal A-system of generators of A-monoid M. The A-embedding dimension of M is the cardinality of $\mathrm{mAsg}(M)$, denoted by $\mathrm{e}_A(M)$. As a consequence of Proposition 181 we have that $e_A(M) \leq \mathrm{e}(M)$.

The next result is deduced from Lemma 44.

Lemma 182. *Let M be an A-monoid and $x \in M$. The set $M \backslash \{x\}$ is an A-monoid if and only if $x \in$* $\mathrm{mAsg}(M)$.

By Proposition 180, we have that if M is an A-monoid then M is an A-semigroup or M is not a numerical semigroup and thus we have the following.

Proposition 183. *Let M be a A-monoid.*

(1) If M is not a numerical semigroup, then $\mathrm{mAsg}(M) = \mathrm{msg}(M)$

(2) If M is an A-semigroup, then $\mathrm{mAsg}(M) = \mathrm{msg}(M) \cap \{1, 2, \ldots, \mathrm{F}(M) + 2\}$.

Proof. (1) From Lemma 172 and Proposition 180 we deduce that if $x \in \mathrm{msg}(M)$ then $M\backslash\{x\}$ is an A-monoid. The proof follows by applying Proposition 181 and Lemma 182.

(2) Suppose that $x \in \mathrm{msg}(M)$. If $x < \mathrm{F}(M)$ then, from Lemma 172, we obtain that $M\backslash\{x\}$ is an A-semigroup. Now, if $x > \mathrm{F}(M)$ then, by Lemma 172 and Proposition 173, we have that $M\backslash\{x\}$ is an A-semigroup if and only if $x \in \{\mathrm{F}(M) + 1, \mathrm{F}(M) + 2\}$. The proof now follows by applying Proposition 181 and Lemma 182. □

Example 184. 1) Let M be a submonoid of $(\mathbb{N}, +)$ such that $\mathrm{msg}(M) = \{6, 10\}$. Since $\gcd\{6, 10\} \neq 1$ then M is not a numerical semigroup and so $\mathrm{mAsg}(M) = \{6, 10\}$.

2) Let M be a submonoid of $(\mathbb{N}, +)$ such that $\mathrm{msg}(M) = \{6, 10, 19, 21, 23\}$. We have that $M = \{0, 6, 10, 12, 16, 18 \rightarrow\}$ is an A-semigroup such that $\mathrm{F}(M) = 17$. By Proposition 183 we obtain that $\mathrm{mAsg}(M) = \{6, 10, 19\}$.

4. *A*MED-Semigroups

Recall that S is a **MED-semigroup** if $\mathrm{e}(S) = \mathrm{m}(S)$ and an AMED-semigroup is an A-semigroup which is a MED-semigroup.

The following result can be deduced from [**BDF97**, Proposition I.2.9]

Proposition 185. *Let S be a numerical semigroup. Then S is a MED-semigroup if and only* $\{s - \mathrm{m}(S) \mid s \in S\backslash\{0\}\}$ *is a numerical semigroup.*

As a consequence of the previous proposition, we have the following.

Corollary 186. *Let S be a numerical semigroup and $x \in S\backslash\{0\}$. Then* $S_x = (\{x\} + S) \cup \{0\}$ *is a MED-semigroup with* $\mathrm{m}(S_x) = x$, $\mathrm{F}(S_x) = \mathrm{F}(S) + x$ *and* $\mathrm{g}(S_x) = \mathrm{g}(S) + x - 1$. *Furthermore, every MED-semigroup is of this form.*

The next result shows whenever a numerical semigroup is replaced by an A-semigroup the two previous results remain valid.

Proposition 187. *Let S be a numerical semigroup. Then S is an AMED-semigroup if and only $T = \{s - \mathrm{m}(S) \mid s \in S\backslash\{0\}\}$ is an A-semigroup.*

Proof. *Necessity*. By applying Proposition 185, T is a numerical semigroup. If T is not A-semigroup, then exists $\{t, t+1\} \subseteq T$ and $t+1 < \mathrm{F}(T) = \mathrm{F}(S) - \mathrm{m}(S)$. Let $s \in S$ such that $t = s - \mathrm{m}(S)$. Then we have that $t + 1 = s + 1 - \mathrm{m}(S)$ and so $s + 1 \in S$. Therefore we obtain that $\{s, s+1\} \subseteq S$ and $s + 1 < \mathrm{F}(S)$. Consequently S is not A-semigroup.

$Sufficiency$. It is clear that $S = (\{\mathrm{m}(S)\} + T) \cup \{0\}$ and $\mathrm{m}(S) \in T\backslash\{0\}$. By Corollary 186, $\mathrm{F}(S) = \mathrm{F}(T) + \mathrm{m}(S)$. If S is not A-semigroup then there exists $\{s, s+1\} \subseteq S$ such that $s + 1 < F(S)$. If $t = s - \mathrm{m}(S)$, then $\{t, t+1\} \subseteq T$ and $t + 1 < \mathrm{F}(T)$, and so T is not A-semigroup. □

Corollary 188. *If S is an A-semigroup and $x \in S\backslash\{0\}$, then $T = (\{x\} + S) \cup \{0\}$ is an AMED-semigroup. Moreover, every AMED-semigroup is of this form.*

Proof. From Corollary 186, we have that T is an AMED-semigroup. If P is an AMED-semigroup, then by Proposition 187 we obtain that
$Q = \{p - \mathrm{m}(P) \mid p \in P\backslash\{0\}\}$ is A-semigroup. In order to conclude the proof, it suffices to note that $P = (\{\mathrm{m}(P)\} + Q) \cup \{0\}$ and $\mathrm{m}(P) \in Q\backslash\{0\}$. □

Lemma 189. [**Ros03**, Corollary 4] *Let S and T be two numerical semigroups, and let $s \in S\backslash\{0\}$ and $t \in T\backslash\{0\}$. If $(\{s\} + S) \cup \{0\} = (\{t\} + T) \cup \{0\}$, then $s = t$ and $S = T$.*

Let $\mathscr{S}_{em}$ be the set of all numerical semigroups and let $\mathscr{A}_{med}$ be the set of all AMED-semigroups. From this we deduce the following correspondence.

Proposition 190. *The map $\varphi : \mathscr{S}_{em} \to \mathscr{A}_{med}$, defined by $\varphi(S) = (\{2\mathrm{m}(S)\} + (2S \cup \{2\mathrm{F}(S) + 1, \to\})) \cup \{0\}$ is injective.*

Proof. For a given $S \in \mathscr{S}_{em}$, suppose that $P = 2S \cup \{2\mathrm{F}(S) + 1, \to\}$. Clearly P is an A-semigroup with $2\mathrm{m}(S) \in P\backslash\{0\}$. By applying Corollary 188, we obtain that

$(\{2\mathrm{m}(S)\} + P) \cup \{0\}$ is an AMED-semigroup and thus $\varphi(S) \in \mathscr{A}_{med}$. Furthermore, from Lemma 189, we get that φ is an injective map. □

The following result can be deduced from [**RGS09**, Proposition 3.1].

Lemma 191. *Let S be a numerical semigroup. Then S is a MED-semigroup if and only* $\mathrm{msg}(S) = (\mathrm{Ap}(S, \mathrm{m}(S))\backslash\{0\}) \cup \{\mathrm{m}(S)\}$.

Note that $\mathrm{m}(S) = \min(\mathrm{msg}(S))$ and we denote by $M(S) = \max(\mathrm{msg}(S))$.

Proposition 192. *If S is an AMED-semigroup, then*

$$\mathrm{mAsg}(S) = \{x \in \mathrm{msg}(S) \mid x \leq M(S) - \mathrm{m}(S) + 2\}.$$

Proof. From Lemmas 13 and 191 we obtain that $\mathrm{F}(S) = M(S) - \mathrm{m}(S)$. The proof now follows by using Proposition 183. □

Observe that, if S is a numerical semigroup with $\mathrm{m}(S) \geq 3$, then $M(S) - \mathrm{m}(S) + 2 < M(S)$ and thus by the previous result $M(S) \notin \mathrm{mAsg}(S)$. From this and since $\mathrm{e}(S) \leq \mathrm{m}(S)$ we have the next result.

Corollary 193. *If S is an AMED-semigroup with* $\mathrm{m}(S) \geq 3$, *then* $e_A(S) \leq \mathrm{m}(S) - 1$.

We propose the following problem: to characterize the whole class of AMED-semigroups such that $e_A(S) = \mathrm{m}(S) - 1$.

A numerical semigroup S is Arf if for all $x, y, z \in S$ with $z \leq y \leq x$, $x + y - z \in S$.

As a consequence of [**RGS09**, Proposition 3.12] we have the next result.

Lemma 194. *If S is an Arf numerical semigroup, then S is a MED-semigroup.*

A sequence $x_1, x_2, \ldots, x_n$ of positive integers is an Arf sequence if it fulfills the following:

(1) $2 \leq x_1 \leq x_2 \leq \cdots \leq x_n$.

(2) $x_{i+1} \in \{x_i, x_i + x_{i-1}, \ldots, x_i + \cdots + x_1, \rightarrow\}$ for all $i \in \{1, \ldots, n-1\}$.

Lemma 195. [GSHKR17, Proposition 1] *Let S be a proper subset of $\mathbb{N}$. Then S is an Arf numerical semigroup if and only if there exists an Arf sequence $x_1, x_2 \ldots, x_n$ such that*

$$S = \{0, x_n, x_n + x_{n-1}, \ldots, x_n + x_{n-1} + \cdots + x_1, \rightarrow\}.$$

As a consequence of Lemmas 194 and 195 we obtain the following result.

Proposition 196. *Every Arf numerical semigroup is an AMED-semigroup.*

Let us finish this section to talk about a special kind of Arf numerical semigroups which are those saturated numerical semigroups. Besides, in the literature, one finds many manuscripts devoted to the study of analytically irreducible one-dimensional local domains via their value semigroups. Following this line of research, the characterization of these rings in terms of their value semigroups yields the Arf and saturated numerical semigroups. Saturated rings were introduced in three different ways in [**Zar71a**, **Zar71b**, **Zar75**], [**PT69**], and [**Cam83**]. Though their definitions coincide for algebraically closed fields of zero characteristic.

For a numerical semigroup S and $s \in S \backslash \{0\}$, set

$$d_S(s) = \gcd\{x \in S \mid x \le s\}.$$

We say that a numerical semigroup S is saturated if $s + d_S(s) \in S$ for all $s \in S \backslash \{0\}$.

The next result can be deduced from [**RGS09**, Lemma 3.31]

Proposition 197. *If S is a saturated numerical semigroup, then S is an Arf numerical semigroup.*

The following result can be used to compute all numerical semigroups of this kind.

Proposition 198. [**RGSGGB04**, Theorem 6] *Let $n_1 < n_2 < \cdots < n_p$ be positive integers such that* $\gcd(n_1, n_2, \ldots, n_p) = 1$. *For every $i \in \{1, \ldots, p\}$, set* $d_i = \gcd(n_1, \ldots, n_i)$

and for all $j \in \{1, \cdots, p-1\}$ *denote*
$k_j = \max\left\{k \in \mathbb{N} \mid n_j + kd_j < n_{j+1}\right\}$. *Then*

$$\left\{0, n_1, n_1 + d_1, \ldots, n_1 + k_1 d_1, n_2, n_2 + d_2, \ldots, n_2 + k_2 d_2, \ldots, n_{p-1}, n_{p-1} + d_{p-1}, \ldots, n_{p-1} + k_{p-1} d_{p-1}, n_p, n_p + 1, \rightarrow\right\}$$

is a saturated numerical semigroup. Moreover, every saturated numerical semigroup is of this form.

5. A-semigroups with a given multiplicity

Recall that for m a positive integer, we denote by

$$\mathscr{A}(m) = \{S \mid S \text{ is an } A\text{-semigroup and } \mathrm{m}(S) = m\}.$$

The next result is easy to prove.

Proposition 199. *If m is a positive integer, then $\mathscr{A}(m)$ is a Frobenius pseudo-variety. Moreover,* $\max(\mathscr{A}(m)) = \{0, m, \rightarrow\}$.

We define the graph $G(\mathscr{A}(m))$ as follows: $\mathscr{A}(m)$ is the vertex set and $(S, T) \in \mathscr{A}(m) \times \mathscr{A}(m)$ is an edge if and only if $T = S \cup \{\mathrm{F}(S)\}$.

As a consequence of [**RR15**, Lemma 12 and Theorem 3] we have the following result.

Proposition 200. *The graph $G(\mathscr{A}(m))$ is a tree with root $\{0, m, \rightarrow\}$. Furthermore, the set of children of vertex S is equal to $\{S \backslash \{x\} \mid x \in \mathrm{msg}(S),\ x > \mathrm{F}(S)$ and $S \backslash \{x\} \in \mathscr{A}(m)\}$.*

As a consequence of Propositions 173 and 200 we have the next result.

Corollary 201. *If $S \in \mathscr{A}(m)$, then the set of the children of S is equal to* $\{S \backslash \{x\} \mid x \in \mathrm{msg}(S) \cap \{\mathrm{F}(S) + 1, \mathrm{F}(S) + 2\}$ *and* $x \neq m\}$.

By the way, by applying Propositions 200 and Corollary 201, we can recursively build the tree $G((\mathscr{A}(4)))$.

$$\begin{tikzcd}
 & & & \langle 4,5,6,7\rangle & \\
 & & & \langle 4,6,7,9\rangle \arrow[u, "5"] & \\
 & \langle 4,7,9,10\rangle \arrow[urr, "6"] & & & \langle 4,6,9,11\rangle \arrow[ul, "7"'] \\
 & \langle 4,9,10,11\rangle \arrow[u, "7"] & & & \langle 4,6,11,13\rangle \arrow[u, "9"] \\
 & \langle 4,10,11,13\rangle \arrow[u, "9"] & & & \langle 4,6,13,15\rangle \arrow[u, "11"] \\
\langle 4,11,13,14\rangle \arrow[ur, "10"] & & \langle 4,10,13,15\rangle \arrow[ul, "11"'] & & \langle 4,6,15,17\rangle \arrow[u, "13"] \\
\vdots & & \vdots & & \vdots
\end{tikzcd}$$

Note that the piece of tree we built all numerical semigroups are MED-semigroups. In fact, all elements in $\mathscr{A}(2)$, $\mathscr{A}(3)$, and $\mathscr{A}(4)$ are MED-semigroups as we see in the following result.

Proposition 202. *If $S \in \mathscr{A}(2) \cup \mathscr{A}(3) \cup \mathscr{A}(4)$, then S is a MED-semigroup.*

Proof. (1) If $S \in \mathscr{A}(2)$, by Proposition 178, then $S = \langle 2, 2k+1\rangle$ for some $k \in \mathbb{N}\backslash\{0\}$ and so S is a MED-semigroup.

(2) If $S \in \mathscr{A}(3)$ and $n = \min\{x \in S \mid x \neq 3k, \text{ and } k \in \mathbb{N}\backslash\{0\}\}$, then $S = \langle 3\rangle \cup \{n, \rightarrow\}$. We distinguish two cases.

(2.1) If $n = 3k+1$ with $k \in \mathbb{N}\backslash\{0\}$, then $S = \langle 3, 3k+1, 3k+2\rangle$ and thus S is a MED-semigroup.

(2.2) If $n = 3k+2$ with $k \in \mathbb{N}\backslash\{0\}$, then $S = \langle 3, 3k+2, 3k+4\rangle$ and so S is a MED-semigroup.

(3) Suppose that $S \in \mathscr{A}(4)$ and $\mathrm{Ap}(S,4) = \{0 = w(0), w(1), w(2), w(3)\}$. We distinguish three cases.

(3.1) If $w(1) = \min\{w(1), w(2), w(3)\}$, then clearly $S = \langle 4 \rangle \cup \{w(1), \rightarrow\}$. Therefore $S = \langle 4, w(1), w(1)+1, w(1)+2 \rangle$ and consequently S is a MED-semigroup.

(3.2) If $w(3) = \min\{w(1), w(2), w(3)\}$, then we have that $S = \langle 4 \rangle \cup \{w(3), \rightarrow\}$. Wherefore $S = \langle 4, w(3), w(3)+2, w(3)+3 \rangle$ and so S is a MED-semigroup.

(3.3) If $w(2) = \min\{w(1), w(2), w(3)\}$, then we get that $S = \langle 4, w(2) \rangle \cup \{\min\{w(1), w(3)\}, \rightarrow\}$. Hence, the minimal system of generators of S is either $\{4, w(2), w(1), w(1)+2\}$ or $\{4, w(2), w(3), w(3)+2\}$ depending on the $\min\{w(1), w(3)\}$ if it is equal to $w(1)$ or $w(3)$, respectively. In both cases, we obtain that S is a MED-semigroup.

□

Remark 203. Note that in the set $\mathscr{A}(5)$ there are numerical semigroups that are not MED-semigroups, as we can see in Example 174.

Observe that if $k \in \mathbb{N}\backslash\{0\}$, then the semigroup $S = \langle m \rangle \cup \{km, \rightarrow\}$ belongs to $\mathscr{A}(m)$. Therefore, for every $m \in \mathbb{N}\backslash\{0,1\}$ the set $\mathscr{A}(m)$ has infinite cardinality. An algorithmic procedure is given in [**RR15**, Algorithm 1] to compute the set of all elements of $\mathscr{A}(m)$ with a certain fixed genus.

Let m be a positive integer, we denote by

$$\mathscr{AMED}(m) = \{S \mid S \text{ is an } A\text{MED-semigroup and } \mathrm{m}(S) = m\}.$$

Our next task in this section will be to prove that $\mathscr{AMED}(m)$ is a **Frobenius pseudo-variety**. In order to do this we need to introduce some previous results.

Lemma 204. [**RGSGGB03**, Proposition 3] *Let S_1 and S_2 be two MED-semigroups with multiplicity m. Then $S_1 \cap S_2$ is a MED-semigroup of multiplicity m.*

The next result is easily deduced from [**RGSGGB03**, Lemma 10]

Lemma 205. *If S is a MED-semigroup with* $\mathrm{F}(S) > m(S)$*, then* $S \cup \mathrm{F}(S)$ *is also a MED-semigroup with the same multiplicity.*

As the semigroup $\{0, m, \rightarrow\}$ is the maximum element in $\mathscr{AMED}(m)$ (with respect to the inclusion order), then using Lemmas 204 and 205, we obtain the next result.

Proposition 206. *If m is a positive integer, then the set $\mathscr{AMED}(m)$ is a Frobenius pseudo-variety.*

Note that if $k \in \mathbb{N}\backslash\{0\}$, then the semigroup $S = \langle m\rangle \cup \{km, \rightarrow\}$ belongs to $\mathscr{AMED}(m)$. Hence, for every $m \in \mathbb{N}\backslash\{0, 1\}$ the set $\mathscr{AMED}(m)$ has infinite cardinality. Proposition 202 ensures that $\mathscr{AMED}(m) = \mathscr{A}(m)$ for $m \in \{2, 3, 4\}$. Note also that $\mathscr{AMED}(1) = \mathscr{A}(1) = \mathbb{N}$.

We define the graph $G(\mathscr{AMED}(m))$ as follows: $\mathscr{AMED}(m)$ is the vertex set and $(S, T) \in \mathscr{AMED}(m) \times \mathscr{AMED}(m)$ is an edge if and only if $T = S \cup \{\mathrm{F}(S)\}$.

As a consequence of [**RR15**, Lemma 12 and Theorem 3] we have:

Proposition 207. *The graph $G(\mathscr{AMED}(m))$ is a tree with root $\{0, m, \rightarrow\}$. Furthermore, the set of children of vertex S is* $\{S\backslash\{x\} \mid x \in \operatorname{msg}(S), x > \mathrm{F}(S)$ *and* $S\backslash\{x\} \in \mathscr{AMED}(m)\}$.

Combining Propositions 173 and 207, we get the following.

Corollary 208. *If $S \in \mathscr{AMED}(m)$, then the set of children of S is equal to* $\{S\backslash\{x\} \mid x \in \operatorname{msg}(S) \cap \{\mathrm{F}(S)+1, \mathrm{F}(S)+2\},\ x \neq m,$ *and* $x + m \in \operatorname{msg}(S\backslash\{x\})\}$.

By applying Propositions 207 and Corollary 208 we can recursively build the tree $G(\mathscr{AMED}(5))$.

$$\begin{tikzcd}
 & & & \langle 5,6,7,8,9\rangle & \\
 & & & \langle 5,7,8,9,11\rangle \arrow[u, "6"'] & \\
 & \langle 5,8,9,11,12\rangle \arrow[urr, "7"] & & & \langle 5,7,9,11,13\rangle \arrow[ul, "8"'] \\
\langle 5,9,11,12,13\rangle \arrow[ur, "8"] & & \langle 5,8,11,12,14\rangle \arrow[ul, "9"'] & & \\
\langle 5,11,12,13,14\rangle \arrow[u, "9"'] & & & & \\
\vdots & & & &
\end{tikzcd}$$

Applying again [**RR15**, Algorithm 1], we can compute the set of all elements of $\mathscr{AMED}(m)$ with a certain fixed genus.

6. A-semigroups with a given Frobenius number and multiplicity

Let F and m be positive integers. We denote by

$$\mathscr{A}(m, F) = \{S \mid S \text{ is an } A\text{-semigroup with } \mathrm{m}(S) = m \text{ and } \mathrm{F}(S) = F\}.$$

Proposition 209. *Let F and m be positive integers. Then $\mathscr{A}(m, F) \neq \emptyset$ if and only if $m - 1 \leq F$ and $m \nmid F$.*

Proof. The necessary condition is trivial, the sufficiency follows from the fact that $\langle m\rangle \cup \{F + 1, \rightarrow\}$ belongs to $\mathscr{A}(m, F)$. $\square$

In order to study the set $\mathscr{A}(m, F)$ we distinguish two cases depending on $F < 2m$ or $F > 2m$.

Lemma 210. [**RB22**, Lemma 2.1] *Let $m \in \mathbb{N}\backslash\{0,1\}$ and $A \subseteq \{m+1,\dots,2m-1\}$. Then $S = \{0,m\} \cup A \cup \{2m,\rightarrow\}$ is an **elementary numerical semigroup with multiplicity** m. Moreover, every elementary numerical semigroup with multiplicity m is of this form.*

The following characterization can be deduced from Lemma 210 and describe the whole set $\mathscr{A}(m,F)$ in the case $m-1 \leq F \leq 2m-1$ and $m \neq F$.

Proposition 211. *Let $m \in \mathbb{N}\backslash\{0,1\}$ and $F \in \{m-1, m+1, \dots, 2m-1\}$.*

(1) If $F = m-1$, then $\mathscr{A}(m,F) = \{\,\{0,m,\rightarrow\}\,\}$.
(2) If $F = m+k$ with $k \in \{1,2\}$, then $\mathscr{A}(m,F) = \{\,\{0,m,m+k+1 \rightarrow\}\,\}$.
(3) If $F = m+k$ with $3 \leq k \leq m-1$, then $\mathscr{A}(m,F) = \{\,\{0,m\} \cup A \cup \{m+k+1,\rightarrow\}$ such that $A \subseteq \{m+2,\dots,m+k-1\}$ and A does not contain consecutive elements$\}$.

Note that to compute the whole set $\mathscr{A}(m,F)$ is sufficient to give an algorithm which computes all subsets of a set $\{a, a+1, \dots, a+b\}$, given two positive integers a and b, which does not contain consecutive elements.

Observe that to give a subset of $\{a, a+1, \dots, a+b\}$ it is equivalent to give an element in $\{0,1\}^{b+1}$. In fact with this notation the element $(x_0, x_1, \dots, x_b)$ represents the subset $\{a+i \mid x_i = 1 \text{ and } i \in \{0,\dots,b\}\}$. Therefore we need an algorithm that computes all elements in $\{0,1\}^{b+1}$ which does not contain two 1s as consecutive numbers.

Algorithm 1

INPUT: n a positive integer.
OUTPUT: $\{(x_1,\dots,x_n) \in \{0,1\}^n \mid x_i \cdot x_{i+1} = 0, \text{ for all } i \in \{1,\dots,n-1\}\,\}$.

1: $A_1 = \{(0),(1)\}$.
2: $A_{k+1} = \{(a_1,\dots,a_k,1) \mid (a_1,\dots,a_k) \in A_k \text{ and } a_k = 0\} \cup$
$\{(a_1,\dots,a_k,0) \mid (a_1,\dots,a_k) \in A_k\}$.
3: **return** A_n.

Now let us compute $\mathscr{A}(m,F)$, considering the case where $F > 2m$.

Given a numerical semigroup S, denote by

$$\Delta(S) = \{\mathrm{F}(S) - i \mid i \in \{1, \ldots, \mathrm{m}(S) - 1\}\}.$$

Proposition 212. *Let S be a numerical semigroup. Then S is an A-semigroup if and only if $\{x, x+1\} \nsubseteq S$ if $\{x, x+1\} \subseteq \Delta(S)$.*

Proof. *Necessity*. Suppose that S is an A-semigroup. If $\{x, x+1\} \subseteq S$, then $\mathrm{F}(S) < x$ and thus $\{x, x+1\} \nsubseteq \Delta(S)$.

Sufficiency. If S is not an A-semigroup, then there exists $\{s, s+1\} \subseteq \mathrm{N}(S)$. Suppose that $k = \max\{n \in \mathbb{N} \mid s + 1 + n.\mathrm{m}(S) < \mathrm{F}(S)\}$. Therefore, it is clear that $\{s + k.\mathrm{m}(S), s + 1 + k.\mathrm{m}(S)\} \subseteq \Delta(S) \cap S$. □

Let $\Delta(m, F) = \{F - 1, \ldots, F - (m - 1)\}$. If $S \in \mathscr{A}(m, F)$, we denote by $B(S) = \{x \in \Delta(m, F) \mid x \notin S\}$. Note that as a consequence of Proposition 212, we have that $\{x, x+1\} \cap B(S) \neq \emptyset$ if $\{x, x+1\} \subseteq \Delta(m, F)$. Observe also that $\Delta(m, F)$ contains an element multiple of m which is in S.

Denote by

$$\mathsf{B}(m, F) = \{B \subseteq \Delta(m, F) \backslash \langle m \rangle \mid \{x, x+1\} \cap B \neq \emptyset \text{ if } \{x, x+1\} \subseteq \Delta(m, F)\}.$$

Given $B \in \mathsf{B}(m, F)$, denote by

$$\mathscr{A}(m, F, B) = \{S \mid S \text{ is an } A\text{-semigroup with } \mathrm{m}(S) = m, \mathrm{F}(S) = F \text{ and } B(S) = B\}.$$

Clearly $\mathscr{A}(m, F, B) \neq \emptyset$, because $\langle m \rangle \cup (\Delta(m, F) \backslash B) \cup \{F + 1, \rightarrow\} \in \mathscr{A}(m, F, B)$.

It is easy to prove the following result.

Proposition 213. *Let m and F be positive integers such that $F > 2m$ and $m \nmid F$.Then, the set $\{\mathscr{A}(m, F, B) \mid B \in \mathsf{B}(m, F)\}$ is a partition of the set $\mathscr{A}(m, F)$.*

7. Algorithm to compute $\mathscr{A}(m, F, B)$

Our aim in this section is to describe an algorithm to compute $\mathscr{A}(m, F, B)$, given m and F positive integers such that $m \geq 2$, $F > 2m$ and $B \in \mathsf{B}(m, F)$. The idea of this algorithm follows the following steps:

(1) First we will give an algorithm that computes all of the maximal elements in $\mathscr{A}(m, F, B)$;
(2) for $T \in \mathscr{A}(m, F, B)$, we will give an algorithm that computes all of the elements in $\mathscr{A}(m, F, B)$ which are contained in T.

Proposition 214. *Let C be a nonempty subset of $\mathbb{N} \backslash \{0\}$ and $m \in \mathbb{N} \backslash \{0\}$. Let $\mathsf{D}(C, m) = \{S \mid S$ is a numerical semigroup, $\mathrm{m}(S) = m$ and $S \cap C = \emptyset\}$ and $\mathsf{D}(C) = \{S \mid S$ is a numerical semigroup and $S \cap C = \emptyset\}$. If P is a maximal element in $\mathsf{D}(C, m)$, then there exists a maximal element T in $\mathsf{D}(C)$ such that $P = T \backslash \{1, 2, \ldots, m-1\}$.*

Proof. If P is not a maximal element in $\mathsf{D}(C)$, then there exists a maximal element T in $\mathsf{D}(C)$ such that $P \subsetneq T$. It is clear that $T \backslash \{1, 2, \ldots, m-1\}$ is in $\mathsf{D}(C, m)$ and $P \subseteq T \backslash \{1, 2, \ldots, m-1\}$. Since P is a maximal element in $\mathsf{D}(C, m)$, then $P = T \backslash \{1, 2, \ldots, m-1\}$. $\square$

If C is a nonempty subset of $\mathbb{N} \backslash \{0\}$, then [**RB19**, Algorithm 1] computes the set $\mathscr{M}(C) = \mathrm{Maximals}\{S \mid S$ is a numerical semigroup and $S \cap C = \emptyset\}$.

Algorithm 2

INPUT: m and F positive integers such that $m \geq 2$, $F > 2m$, $m \nmid F$ and $B \in \mathsf{B}(m, F)$.
OUTPUT: The set $\mathrm{Maximals}(\mathscr{A}(m, F, B))$.

1: using the [**RB19**, Algorithm 1] computes $\mathscr{M}(B \cup \{F\})$.
2: **return** $\mathrm{Maximals}\{T \backslash \{1, 2, \ldots, m-1\} \mid T \in \mathscr{M}(B \cup \{F\})\}$.

As the Algorithm 2 computes the set Maximals$(\mathscr{A}(m,F,B))$. Next, we describe an algorithm (Algorithm 3) which allows us to compute the set $R(T) = \{S \in \mathscr{A}(m,F,B) \mid S \subseteq T\}$ from $T \in \mathscr{A}(m,F,B)$.

If S and T are numerical semigroups and $S \subsetneq T$, then we denote by $\mathrm{F}_T(S) = \max(T \backslash S)$ called the *Frobenius number of S restricted to T*. By definition $\mathrm{F}_T(T) = -1$.

As a consequence of [**RGS09**, Lemma 4.35] we have the following result.

Lemma 215. *If S and T are numerical semigroups and $S \subsetneq T$, then $S \cup \{\mathrm{F}_T(S)\}$ is also a numerical semigroup.*

If T is an element in $\mathscr{A}(m,F,B)$, then $T = \max(R(T))$. On the other hand if $\{P,Q\} \subseteq \mathscr{A}(m,F,B)$, then $P \cap Q \in \mathscr{A}(m,F,B)$. As a consequence of these two statements and Lemma 215 we have the next result.

Proposition 216. *Let m and F be positive integers such that $m \geq 2$, $F > 2m$, $m \nmid F$ and $B \in \mathsf{B}(m,F)$. If $T \in \mathscr{A}(m,F,B)$, then $R(T)$ is a* Frobenius R-variety.

We define the graph $G(R(T))$ as the graph whose vertices are the elements of $R(T)$ and $(P,Q) \in R(T) \times R(T)$ is an edge if $Q = P \cup \{F_T(P)\}$.

The following result can be deduced from [**RR18**, Corollary 4.5]

Proposition 217. *Let m and F be positive integers such that $m \geq 2$, $F > 2m$, $m \nmid F$ and $B \in \mathsf{B}(m,F)$. If $T \in \mathscr{A}(m,F,B)$, then the graph $G(R(T))$ is a tree rooted in T. Moreover, the set of children of S is equal to $\{S \backslash \{x\} \mid x \in msg(S),\ x > F_T(S)$ and $S \backslash \{x\} \in R(T)\}$.*

Note that $S \backslash \{x\} \in R(T)$ if and only if $S \backslash \{x\} \in \mathscr{A}(m,F,B)$. As a consequence of Proposition 217 we have the next result.

Corollary 218. *Let m and F be positive integers such that $m \geq 2$, $F > 2m$ $m \nmid F$, $B \in \mathsf{B}(m,F)$and $T \in \mathscr{A}(m,F,B)$. Then, in the tree $G(R(T))$ the set of children of S is equal to $\{S \backslash \{x\} \mid x \in msg(S),\ F_T(S) < x < F,\ x \neq m$ and $x \notin \Delta(m,F)\}$.*

Example 219. Using Proposition 217 and Corollary 218, let us construct recursively the tree $G(R(T))$ with $T = \langle 6, 8, 10, 21, 23, 25\rangle \in \mathscr{A}(6, 19, \{15, 17\})$.

$$\begin{tikzcd}
& \langle 6, 8, 10, 21, 23, 25\rangle & \\
\langle 6, 10, 14, 21, 23, 25\rangle \arrow[ur, "8"] & & \langle 6, 8, 21, 23, 25\rangle \arrow[ul, "10"'] \\
\langle 6, 14, 16, 21, 23, 25\rangle \arrow[u, "10"'] & &
\end{tikzcd}$$

Algorithm 3

INPUT: m and F positive integers such that $m \geq 2$, $F > 2m$, $m \nmid F$ and $B \in \mathsf{B}(m, F)$.
OUTPUT: The set $\mathscr{A}(m, F, B)$.

1: using the Algorithm 2 computes $\mathscr{M} = \text{Maximals}(\mathscr{A}(m, F, B))$.
2: for each $T \in \mathscr{M}$ computes $R(T)$.
3: **return** $\bigcup_{T \in \mathscr{M}} R(T)$.

CHAPTER 5

Numerical semigroups coated with odd elements

A numerical semigroup S is **coated with odd elements (Coe-semigroup)**, if $\{x-1, x+1\} \subseteq S$ for all odd element x in S. In this chapter, we will study this kind of numerical semigroup. In particular, we are interested in the study of the Frobenius number, genus and embedding dimension of a numerical semigroup of this type. In addition, we solve the Frobenius problem for Coe-semigroups with embedding dimension three. The results of this chapter are submitted for publication.

1. Definitions and preliminaries

A numerical semigroup, S, is coated with odd elements, if $\{x-1, x+1\} \subseteq S$ for all odd element x in S. Henceforth known as **Coe-semigroup**, which will be the object of study in this work.

Denote by $\mathscr{C} = \{S \mid S \text{ is a Coe-semigroup}\}$. In Section 2, we will show that, if $S \in \mathscr{C}$ then $S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1\} \in \mathscr{C}$. This result will be used in Section 3, to order the elements of $\mathscr{C}$ in a rooted **tree**. We will characterize the children of a vertex in this tree and this will allow us to give an algorithm procedure to obtain recursively the elements of $\mathscr{C}$.

In Section 4, we will see that given k an odd positive integer, then $\mathscr{C}(k) = \{S \mid S \text{ is a Coe-semigroup and } k \in S\}$ is a finite set. Moreover, given p a positive integer, $\mathscr{C}(Frob \leq p) = \{S \mid S \text{ is a Coe-semigroup with } \mathrm{F}(S) \leq p\}$ and $\mathscr{C}(gen \leq p) = \{S \mid S \text{ is a Coe-semigroup with } \mathrm{g}(S) \leq p\}$ are also finite sets. Following the same line of the previous section, we are going to order these three families of semigroups in a **rooted tree**.

A Coe-monoid is a submonoid of $(\mathbb{N}, +)$ which can be expressed as an intersection of Coe-semigroups. It is clear that the intersection of Coe-monoids is a Coe-monoid. This allows us to introduce the smallest Coe-monoid containing a subset X of $\mathbb{N}$, denoted by $Coe(X)$. In Section 5, we will show that a submonoid M of $(\mathbb{N}, +)$ is an Coe-monoid if and only if either $M \subseteq \{2k \mid k \in \mathbb{N}\}$ or M is a Coe-semigroup. We will give an algorithm to compute $Coe(X)$ and will see that $Coe(X)$ is a Coe-semigroup if and only if X contains at least an odd element.

In Section 6, we will study the Coe-semigroups which are MED-semigroups.

Finally, in Section 7, we will study the Coe-semigroups with a unique odd minimal generator. We will show that this kind of numerical semigroups are equal to the set $\{T \mid T = 2S \cup (\{2s+1\} + 2S),\ S$ is a numerical semigroup and $\{s, s+1\} \subseteq S\}$. We will give formulas for $\mathrm{F}(T)$, $\mathrm{g}(T)$ and $\mathrm{e}(T)$ as a function of $\mathrm{F}(S)$, $\mathrm{g}(S)$ and $\mathrm{e}(S)$. From these results, we will prove that **if S verifies Wilf's conjecture, then T also** verifies the same conjecture. In addition, we will solve the Frobenius problem for Coe-semigroups with embedding dimension three.

2. First results

Note that $\mathbb{N}$ is a Coe-semigroup with $\mathrm{m}(\mathbb{N}) = 1$ and $\mathrm{F}(\mathbb{N}) = -1$. Hence, if S is a numerical semigroup and $S \neq \mathbb{N}$, then we let us consider that $\mathrm{m}(S) \in \mathbb{N}\backslash\{0, 1\}$ and $\mathrm{F}(S) \in \mathbb{N}\backslash\{0\}$.

Proposition 220. *If S is a Coe-semigroup and $S \neq \mathbb{N}$, then $\mathrm{m}(S)$ is an even integer and $\mathrm{F}(S)$ is an odd integer.*

Proof. As $S \neq \mathbb{N}$, then $\mathrm{m}(S) \geq 2$ and $\mathrm{F}(S) \geq 1$. If $\mathrm{m}(S)$ is odd, then we get that $\mathrm{m}(S) - 1 \in S$, a contradiction. If $\mathrm{F}(S)$ is even, then $\mathrm{F}(S) + 1$ is an odd element belonging to S. Hence, we obtain that $(\mathrm{F}(S)+1)-1 = \mathrm{F}(S) \in S$, a contradiction again. $\square$

Proposition 221. *Let S be a numerical semigroup. The following conditions are equivalent.*

(1) S is a Coe-semigroup.

(2) $\{x-1, x+1\} \subseteq S$ for all odd element x in $\mathrm{msg}(S)$.

Proof. 1) *implies* 2) Trivial.

2) *implies* 1) Let s be an odd element in S. Clearly, there exists an odd element x in $\mathrm{msg}(S)$, such that $s-x \in S$. Then, we obtain that $s-1 = x-1+s-x \in S$ and $s+1 = x+1+s-x \in S$ and so S is a Coe-semigroup. $\square$

Example 222. By applying Proposition 221, we have that $S = \langle 4, 6, 7\rangle$ is a Coe-semigroup, because $\{7-1, 7+1\} \subseteq S$.

The next result has immediate proof.

Lemma 223. *If S is a numerical semigroup such that $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ and $S \cup \{\mathrm{F}(S)-1, \mathrm{F}(S)\}$ are also numerical semigroups.*

Observe that, if S is a Coe-semigroup such that $S \neq \mathbb{N}$, then we have that $S \cup \{\mathrm{F}(S)\}$ is not necessarily a Coe-semigroup. In fact, $S = \{0, 6, 10, \rightarrow\}$ is a Coe-semigroup, but $S \cup \{\mathrm{F}(S)\} = \{0, 6, 9, 10, \rightarrow\}$ is not a Coe-semigroup. In addition, if S is a Coe-semigroup then $\mathrm{F}(S)-1$ may or may not belong to S. Indeed, we have that $S = \{0, 6, 10, \rightarrow\}$ is a Coe-semigroup such that $\mathrm{F}(S)-1 \notin S$ and $T = \{0, 6, 8, 10, \rightarrow\}$ is a Coe-semigroup such that $\mathrm{F}(T)-1 \in T$.

Lemma 224. *If S is a Coe-semigroup such that $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)-1, \mathrm{F}(S)\}$ is a Coe-semigroup.*

Proof. Applying Lemma 223, we obtain that $S \cup \{\mathrm{F}(S)-1, \mathrm{F}(S)\}$ is a numerical semigroup. We know that, by Proposition 220, $\mathrm{F}(S)$ is an odd integer. As $\{\mathrm{F}(S)-$

$1, \mathrm{F}(S)+1\} \subseteq S \cup \{\mathrm{F}(S)-1, \mathrm{F}(S)\}$, we deduce that $S \cup \{\mathrm{F}(S)-1, \mathrm{F}(S)\}$ is a Coe-semigroup. □

As a consequence of Lemma 224, we have all the ingredients needed to give a recursive way of calculating a sequence of Coe-semigroups. For a given a Coe-semigroup S:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n)-1, \mathrm{F}(S_n)\} & \text{if } S_n \neq \mathbb{N} \\ \mathbb{N} & \text{otherwise.} \end{cases}$

The next result is trivial.

Proposition 225. *If S is a Coe-semigroup, then there exists a sequence of Coe-semigroups, $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k = \mathbb{N}$. Furthermore, the cardinality of $S_{i+1} \backslash S_i$ is 1 or 2 for all $i \in \{0, 1, \ldots, k-1\}$.*

We will refer to the previous sequence as the chain of Coe-semigroups associated with S and k is the length of this chain, denoted by l_S.

The following result is easy to prove.

Proposition 226. *Let S be a Coe-semigroup. The cardinality of the set $\{x \in \mathbb{N} \backslash S \mid x \text{ is odd}\}$ is equal to l_S.*

Example 227. Clearly $S = \langle 6, 8, 13, 14, 15, 17\rangle = \{0, 6, 8, 12, \rightarrow\}$ is a Coe-semigroup. The chain of Coe-semigrops associated to S is the following: $S = S_0 \subsetneq S_1 = \{0, 6, 8, 10 \rightarrow\} \subsetneq S_2 = \{0, 6, 8, \rightarrow\} \subsetneq S_3 = \{0, 6, \rightarrow\} \subsetneq S_4 = \{0, 4 \rightarrow\} \subsetneq S_5 = \{0, 2 \rightarrow\} \subsetneq S_6 = \mathbb{N}$. The cardinality of $\{x \in \mathbb{N} \backslash S \mid x \text{ is odd}\} = \{1, 3, 5, 7, 9, 11\}$ is $6 = l_S$.

3. The tree of Coe-semigroups

Our main goal in this section will be to build the tree whose vertex set is $\mathscr{C} = \{S \mid S \text{ is a Coe-semigroup}\}$.

We define the **graph** $G(\mathscr{C})$ as the graph whose vertices are elements of $\mathscr{C}$ and $(S,T) \in \mathscr{C} \times \mathscr{C}$ is an edge if $T = S \cup \{\mathrm{F}(S) - 1, \mathrm{F}(S)\}$.

As a consequence of Proposition 225, we have the following.

Proposition 228. *The graph $G(\mathscr{C})$ is a **tree** with root equal to $\mathbb{N}$.*

It is clear that we can build recursively the tree $G(\mathscr{C})$, starting in $\mathbb{N}$ and we connect each vertex with its children. Hence, we need to characterize the children of an arbitrary vertex of this tree.

The next result is easy to prove.

Lemma 229. *Let S be a numerical semigroup such that $S \neq \mathbb{N}$ and $\{x, x+1\} \subseteq S$. Then $S\backslash\{x, x+1\}$ is a numerical semigroup if and only if $\{x, x+1\} \subseteq \mathrm{msg}(S)$.*

Lemma 230. *Let S be a Coe-semigroup and let x be an odd element in $\mathrm{msg}(S)$, such that $x > \mathrm{F}(S)$. Then $S\backslash\{x\}$ is a child of S in the tree $G(\mathscr{C})$.*

Proof. By Lemma 31, we have that $S\backslash\{x\}$ is a numerical with $\mathrm{F}(S\backslash\{x\}) = x$. Since x is odd then $S\backslash\{x\}$ is a Coe-semigroup and $x - 1 \in S$. Moreover, we can deduce that $(S\backslash\{x\} \cup \{\mathrm{F}(S\backslash\{x\}), \mathrm{F}(S\backslash\{x\}) - 1\}) = S\backslash\{x\} \cup \{x, x-1\} = S$ and thus $S\backslash\{x\}$ is a child of S in the tree $G(\mathscr{C})$. □

Lemma 231. *Let S be a Coe-semigroup and let T be a child of S in the tree $G(\mathscr{C})$ such that $\mathrm{g}(T) = \mathrm{g}(S) + 1$. Then, there exists an odd element x in $\mathrm{msg}(S)$ with $x > \mathrm{F}(S)$ such that $T = S\backslash\{x\}$.*

Proof. If T is a child of S such that $\mathrm{g}(T) = \mathrm{g}(S) + 1$, then we obtain that $T \cup \{\mathrm{F}(T)\} = S$ and so $T = S\backslash\{\mathrm{F}(T)\}$. By using Lemma 31, we have that $\mathrm{F}(T) \in \mathrm{msg}(S)$ and it is clear that $\mathrm{F}(S) < \mathrm{F}(T)$. Applying Proposition 220, we conclude that $\mathrm{F}(T)$ is odd. □

Lemma 232. *Let S be a Coe-semigroup such that $\{\mathrm{F}(S) + 1, \mathrm{F}(S) + 2\} \subseteq \mathrm{msg}(S)$. Then $S\backslash\{\mathrm{F}(S) + 1, \mathrm{F}(S) + 2\}$ is a child of S in the tree $G(\mathscr{C})$.*

Proof. By Lemma 229, we have that $T = S\backslash\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\}$ is a numerical semigroup such that $\mathrm{F}(T) = \mathrm{F}(S)+2$. By applying Proposition 220, $F(S)$ is odd and so $\mathrm{F}(T)$ is also odd. Hence, we deduce that T is a Coe-semigroup. As $S = T \cup \{\mathrm{F}(T), \mathrm{F}(T)-1\}$, then we get that T is a child of S. $\square$

Lemma 233. *Let S be a Coe-semigroup and let T be a child of S in the tree $G(\mathscr{C})$ such that $\mathrm{g}(T) = \mathrm{g}(S)+2$. Then, $\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\} \subseteq \mathrm{msg}(S)$ and $T = S\backslash\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\}$.*

Proof. If T is a child of S, then T is a Coe-semigroup and $S = T\cup\{\mathrm{F}(T), \mathrm{F}(T)-1\}$. Since $\mathrm{g}(T) = \mathrm{g}(S)+2$, this implies that $\mathrm{F}(T)-1 \notin T$. By applying Proposition 220, $F(T)$ is odd and so $\mathrm{F}(T)-2$ is also odd. As $\mathrm{F}(S)-1 \notin T$ and T is a Coe-semigroup, we obtain that $\mathrm{F}(T)-2 \notin T$ and thus $\mathrm{F}(S) = \mathrm{F}(T)-2$. Consequently, $T = S\backslash\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\}$ and by using Lemma 229, we conclude that $\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\} \subseteq \mathrm{msg}(S)$. $\square$

As a consequence of Lemmas 230, 231, 232 and 233 we obtain the following result.

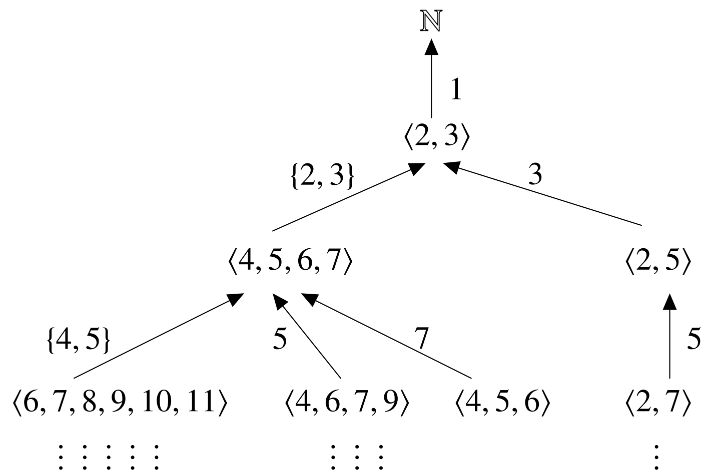
Theorem 234. *Let S be a Coe-semigroup, then set of children of S in the tree $G(\mathscr{C})$ is equal to:*

(1) $\{S\backslash\{x\} \mid x$ *is an odd element in* $\mathrm{msg}(S)$ *with* $x > \mathrm{F}(S)\}$ *if* $\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\} \nsubseteq \mathrm{msg}(S)$.

(2) $\{S\backslash\{x\} \mid x$ *is an odd element in* $\mathrm{msg}(S)$ *with* $x > \mathrm{F}(S)\} \cup \{S\backslash\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\}\}$ *if* $\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\} \subseteq \mathrm{msg}(S)$.

It is clear that for all $k \in \mathbb{N}\backslash\{0\}$ we have that $\{0, 2k, \rightarrow\}$ is a Coe-semigroup, this implies that the set $\mathscr{C}$ has infinite cardinality.

The last theorem can be used to recurrently construct the tree $G(\mathscr{C})$, starting in $\mathbb{N}$, containing the set of all Coe-semigroups.



Note that the numbers that appear on either side of the edges are the elements that we remove from the vertex to obtain its child.

4. Examples of finite trees

Given $k \in \mathbb{N}$ we denote by

$$\mathscr{C}(k) = \{S \mid S \text{ is a Coe-semigroup and } k \in S\}.$$

If k is an even positive integer and $n \in \{x \in \mathbb{N} \mid x > k\}$, then $(\{k\} + \langle 2\rangle) \cup \{0, n, \rightarrow\} \in \mathscr{C}(k)$ and so, in this case, $\mathscr{C}(k)$ has infinite cardinality. Our first aim in this section is to see that $\mathscr{C}(k)$ has finite cardinality when k is an odd positive integer.

If S is a numerical semigroup, then $\mathbb{N}\backslash S$ is a finite set and so we obtain the following result.

Lemma 235. *If S is a numerical semigroup, then $\{T \mid T$ is a numerical semigroup and $S \subseteq T\}$ is a finite set.*

Proposition 236. *If k is an odd positive integer, then $\mathscr{C}(k)$ is a nonempty finite set.*

Proof. Since $\mathbb{N} \in \mathscr{C}(k)$, then $\mathscr{C}(k) \neq \emptyset$. If $S \in \mathscr{C}(k)$, then we have that $\langle k-1, k, k+1\rangle \subseteq S$ and it is clear that $\langle k-1, k, k+1\rangle$ is a numerical semigroup. Besides,

$\mathscr{C}(k) \subseteq \{T \mid T$ is a numerical semigroup and $\langle k-1, k, k+1\rangle \subseteq T\}$. By Lemma 235, we can deduce that $\mathscr{C}(k)$ is a finite set. $\square$

The following result is easy to prove.

Lemma 237. *Let k be a positive integer and let $S \in \mathscr{C}(k)$ such that $S \neq \mathbb{N}$. Then $S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1\} \in \mathscr{C}(k)$.*

We define the graph $G(\mathscr{C}(k))$ as the graph whose vertices are elements of $\mathscr{C}(k)$ and $(S,T) \in \mathscr{C}(k) \times \mathscr{C}(k)$ is an edge if $T = S \cup \{\mathrm{F}(S)-1, \mathrm{F}(S)\}$.

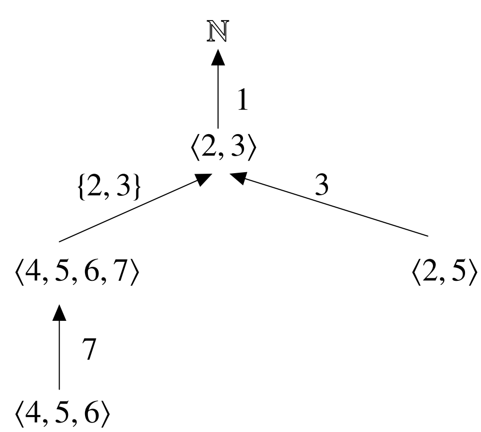
Using the same argument of Section 3, we have the following result.

Theorem 238. *If k is a positive integer, then the graph $G(\mathscr{C}(k))$ is a tree with root equal to $\mathbb{N}$. Furthermore, if $S \in \mathscr{C}(k)$ then the set of children of S in the tree $G(\mathscr{C}(k))$ is equal to:*

(1) $\{S\backslash\{x\} \mid x$ *is an odd element in* $\mathrm{msg}(S)$ *with* $x > \mathrm{F}(S)$ *and* $x \neq k\}$ *if* $\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\} \nsubseteq \mathrm{msg}(S)\backslash\{k\}$.

(2) $\{S\backslash\{x\} \mid x$ *is an odd element in* $\mathrm{msg}(S)$ *with* $x > \mathrm{F}(S)$ *and* $x \neq k\}$ $\cup \{S\backslash\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\}\}$ *if* $\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\} \subseteq \mathrm{msg}(S)\backslash\{k\}$.

Example 239. We are going to build the tree $G(\mathscr{C}(5))$.



Given a positive integer F, denote by

$$\mathscr{C}(Frob \leq F) = \{S \mid S \text{ is a Coe-semigroup with } \mathrm{F}(S) \leq F\}.$$

Note that if S is an element in $\mathscr{C}(Frob \leq F)$, then $\{F+1, \rightarrow\} \subseteq S$ and thus $\mathscr{C}(Frob \leq F)$ is a finite set.

The next result is easy to prove.

Lemma 240. *Let F be a positive integer and let $S \in \mathscr{C}(Frob \leq F)$ such that $S \neq \mathbb{N}$. Then $S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1\} \in \mathscr{C}(Frob \leq F)$.*

Now, we define the graph $G(\mathscr{C}(Frob \leq F))$ as follows: $\mathscr{C}(Frob \leq F)$ is its set of vertices and $(S,T) \in \mathscr{C}(Frob \leq F) \times \mathscr{C}(Frob \leq F)$ is an edge if $T = S \cup \{\mathrm{F}(S)-1, \mathrm{F}(S)\}$.

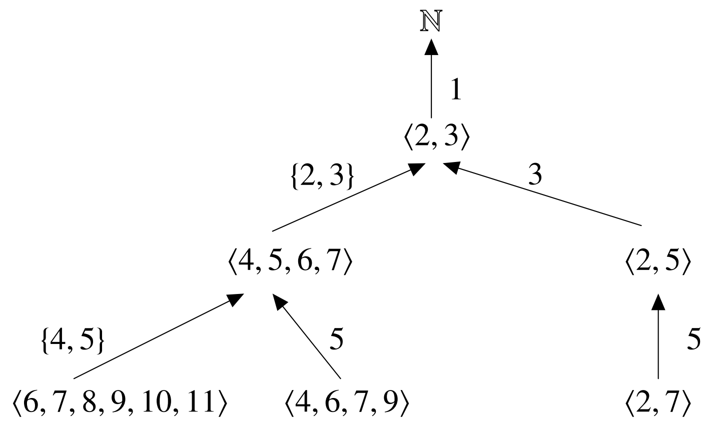
By using again the same argument of Section 3, we have the next result.

Theorem 241. *If F is a positive integer, then the graph $G(\mathscr{C}(Frob \leq F))$ is a tree with root equal to $\mathbb{N}$. Furthermore, if $S \in \mathscr{C}(Frob \leq F)$ then the set of children of S in the tree $G(\mathscr{C}(Frob \leq F))$ is equal to:*

(1) $\{S \setminus \{x\} \mid x$ *is an odd element in* $\mathrm{msg}(S)$ *with* $\mathrm{F}(S) < x \leq F\}$ *if* $\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\} \nsubseteq \{x \in \mathrm{msg}(S) \mid x \leq F\}$.

(2) $\{S \setminus \{x\} \mid x$ *is an odd element in* $\mathrm{msg}(S)$ *with* $\mathrm{F}(S) < x \leq F\}$ $\cup \{S \setminus \{\mathrm{F}(S)+1, \mathrm{F}(S)+2\}\}$ *if* $\{\mathrm{F}(S)+1, \mathrm{F}(S)+2\} \subseteq \{x \in \mathrm{msg}(S) \mid x \leq F\}$.

Example 242. We are going to build the tree $G(\mathscr{C}(Frob \leq 5))$.



Given a positive integer g, denote by

$$\mathscr{C}(gen \leq g) = \{S \mid S \text{ is a Coe-semigroup with } \mathrm{g}(S) \leq g\}.$$

In [**RGS09**, Lemma 2.14] it is shown that, if S is a numerical semigroup then $\mathrm{F}(S) \leq 2\mathrm{g}(S) - 1$. Therefore, we obtain that $\mathscr{C}(gen \leq g) \subseteq \mathscr{C}(Frob \leq 2g - 1)$ and so $\mathscr{C}(gen \leq g)$ is a finite set.

The next result is easy to prove.

Lemma 243. *Let g be a positive integer and let $S \in \mathscr{C}(gen \leq g)$ such that $S \neq \mathbb{N}$. Then $S \cup \{\mathrm{F}(S), \mathrm{F}(S) - 1\} \in \mathscr{C}(gen \leq g)$.*

We define the graph $G(\mathscr{C}(gen \leq g))$ as follows: $\mathscr{C}(gen \leq g)$ is its set of vertices and $(S, T) \in \mathscr{C}(gen \leq g) \times \mathscr{C}(gen \leq g)$ is an edge if $T = S \cup \{\mathrm{F}(S) - 1, \mathrm{F}(S)\}$.

By using again the same argument of Section 3, we have the next result.

Theorem 244. *If g is a positive integer, then the graph $G(\mathscr{C}(gen \leq g))$ is a tree with root equal to $\mathbb{N}$. Furthermore, if $S \in \mathscr{C}(gen \leq g)$ then the set of children of S in the tree $G(\mathscr{C}(gen \leq g))$ is equal to:*

(1) the set of children of S in the tree $G(\mathscr{C})$ if $\mathrm{g}(S) \leq g - 2$ (see Theorem 234).

(2) $\{S\backslash\{x\} \mid x$ *is an odd element in* $\mathrm{msg}(S)$ *with* $x > \mathrm{F}(S)\}$ *if* $\mathrm{g}(S) = g - 1$.

(3) the empty set if $\mathrm{g}(S) = g$.

Example 245. We are going to build the tree $G(\mathscr{C}(gen \leq 4))$.

$$\begin{tikzcd}
 & & \mathbb{N} & \\
 & & \langle 2,3\rangle\ (1) \arrow[u, "1"'] & \\
 & \langle 4,5,6,7\rangle\ (3) \arrow[ur, "\{2,3\}"] & & \langle 2,5\rangle\ (2) \arrow[ul, "3"'] \\
\langle 4,6,7,9\rangle\ (4) \arrow[ur, "5"] & \langle 4,5,6\rangle\ (4) \arrow[u, "7"'] & & \langle 2,7\rangle\ (3) \arrow[u, "5"'] \\
 & & & \langle 2,9\rangle\ (4) \arrow[u, "7"']
\end{tikzcd}$$

Observe that the numbers that appear on the right side of each of the semigroups are their genera.

5. Coe-monoids

It is well known that the intersection of finitely many numerical semigroups is a numerical semigroup. Hence, the next result is easy to prove.

Proposition 246. *The intersection of finitely many Coe-semigroups is a Coe-semigroup.*

Note that the previous result does not hold for infinite intersections. In fact, by using Proposition 221, we can deduce that $\langle 2, 2k+1\rangle$ is a Coe-semigroup for all $k \in \mathbb{N}$ and $\bigcap_{k\in\mathbb{N}}\langle 2, 2k+1\rangle = \langle 2\rangle$ is not a numerical semigroup.

Clearly, the intersection (finite or infinite) of numerical semigroups is a submonoid of $(\mathbb{N}, +)$. A Coe-monoid is a submonoid of $(\mathbb{N}, +)$ which can be expressed as an intersection of Coe-semigroups. Besides, the intersection of Coe-monoids is a Coe-monoid. In view of this, given X a subset of $\mathbb{N}$, we can define the Coe-monoid generated by X as the intersection of all Coe-monoids containing X, denoted by $Coe(X)$. Then, we have that $Coe(X)$ is the smallest Coe-monoid containing X.

The next result has immediate proof.

Proposition 247. *If X is a subset of $\mathbb{N}$, then $Coe(X)$ is the intersection of all Coe-semigroups that contain X.*

If M is a Coe-monoid and X is a subset of $\mathbb{N}$ such that $M = Coe(X)$, then we say that X is a coe-system of generators of M.

The following properties are a direct consequence of the definitions.

Lemma 248. *Let X and Y be subsets of $\mathbb{N}$ and let M be a Coe-monoid. Then the following conditions hold:*

(1) If $X \subseteq Y$ then $Coe(X) \subseteq Coe(Y)$,
(2) $Coe(X) = Coe(\langle X \rangle)$,
(3) $Coe(M) = M$,
(4) $Coe(X \backslash \{0\}) = Coe(X)$,
(5) $Coe(\emptyset) = \{0\}$.

Proposition 249. *Let M be a Coe-monoid such that $M \neq \{0\}$. Then there exists a nonempty finite subset of $\mathbb{N} \backslash \{0\}$ such that $M = Coe(X)$.*

Proof. We have that $\mathrm{msg}(M)$ is a nonempty finite subset $\mathbb{N} \backslash \{0\}$ such that $M = \langle \mathrm{msg}(M) \rangle$. By applying Lemma 248, we deduce that $M = Coe(\mathrm{msg}(M))$. □

As a consequence of the previous proposition, we have the following result.

Corollary 250. *The set formed by all Coe-monoids is equal to* $\{Coe(X) \mid X$ *is a nonempty finite subset of* $\mathbb{N}\backslash\{0\}\} \cup \{\{0\}\}$.

Given X a nonempty finite subset of $\mathbb{N}\backslash\{0\}$, our next aim in this section will be to show a procedure that allows us to compute $Coe(X)$.

Theorem 251. *Let M be a submonoid of* $(\mathbb{N}, +)$ *such that* $M \neq \{0\}$. *The following conditions are equivalent.*

(1) M is a Coe-monoid.
(2) $\{x-1, x+1\} \subseteq M$ *for all odd element x in M.*
(3) $\{x-1, x+1\} \subseteq M$ *for all odd element x in* $\mathrm{msg}(M)$.

Proof. The equivalence between conditions 2) and 3) is analogous to the proof of Proposition 221.

1) *implies* 2) If M is a Coe-monoid, then there exists a family $\{S_i\}_{i\in I}$ of Coe-semigroups such that $M = \bigcap_{i\in I} S_i$. If x is an odd element in M, then $x \in S_i$ for all $i \in I$. As $\{x-1, x+1\} \subseteq S_i$ for all $i \in I$, then we get that $\{x-1, x+1\} \subseteq \bigcap_{i\in I} S_i = M$.

2) *implies* 1) It is clear that $M_k = M \cup \{2k, \rightarrow\}$ is a Coe-semigroup for all $k \in \mathbb{N}$. Hence, $M = \bigcap_{k\in\mathbb{N}} M_k$ is a Coe-monoid. □

Corollary 252. *Let M be a submonoid of* $(\mathbb{N}, +)$. *Then M is a Coe-monoid if and only if either* $M \subseteq \langle 2 \rangle$ *or M is a Coe-semigroup.*

Proof. If M is a Coe-monoid that does not contain odd elements, then $M \subseteq \langle 2 \rangle$. On the other hand, if x is an odd element in M, then by Theorem 251, $\{x-1, x, x+1\} \subseteq M$. Therefore, we can conclude that M is a numerical semigroup which is a Coe-semigroup.

Conversely, if $M \subseteq \langle 2 \rangle$, then, by applying condition 2) of Theorem 251, we get that M is a Coe-monoid. Furthermore, if M is a Coe-semigroup, then it is a Coe-monoid. □

Corollary 253. *If X is a subset of $\mathbb{N}\backslash\{0\}$, then $Coe(X)$ is a submonoid of $(\mathbb{N}, +)$ generated by $X \cup \{x+1 \mid x$ is an odd element in $X\} \cup \{x-1 \mid x$ is an odd element in $X\}$.*

Proof. Let $A = X \cup \{x+1 \mid x \text{ is an odd element in } X\} \cup \{x-1 \mid x \text{ is an odd element in } X\}$. By condition 2) of Theorem 251, we get that $\langle A\rangle \subseteq Coe(X)$ and by condition 3) of Theorem 251, we obtain that $Coe(X) \subseteq A$. Whence, $\langle A\rangle = Coe(X)$. □

Example 254. By using Corollary 253, we obtain that $Coe(\{4,7\}) = \langle 4,6,7,8\rangle = \langle 4,6,7\rangle$.

The following result is easy to prove.

Corollary 255. *Let X be a subset of $\mathbb{N}\backslash\{0\}$. Then, $Coe(X)$ is a Coe-semigroup if and only if X contains at least one odd element.*

6. Coe-semigroups with maximal embedding dimension

From [**BDF97**, Proposition I.2.9] we can deduce the next result.

Lemma 256. *Let S be a numerical semigroup. Then S is an **MED-semigroup** if and only if $\{s - \mathrm{m}(S) \mid s \in S\backslash\{0\}\}$ is a numerical semigroup.*

Proposition 257. *Let S be a MED-semigroup such that $S \neq \mathbb{N}$. Then S is a Coe-semigroup if and only if $\mathrm{m}(S)$ is even and $T = \{-\mathrm{m}(S)\} + (S\backslash\{0\})$ is a Coe-semigroup.*

Proof. *Necessity*. From Lemma 256, we have that T is a numerical semigroup and by Proposition 220, $\mathrm{m}(S)$ is even. Let t be an odd element in T. Then there exists $s \in S$ such that $t = s - \mathrm{m}(S)$ and s is odd. Since S is a Coe-semigroup then $\{s-1, s+1\} \subseteq S$ and so $\{t-1, t+1\} = \{s-1-\mathrm{m}(S), s+1-\mathrm{m}(S)\} \subseteq T$. Hence, T is a Coe-semigroup.

Sufficiency. Let s be an odd element in S. Then $s - \mathrm{m}(S)$ is an odd element in T. As T is a Coe-semigroup then $\{s-1-\mathrm{m}(S), s+1-\mathrm{m}(S)\} \subseteq T$ and so $\{s-1, s+1\} \subseteq S$. Consequently, S is a Coe-semigroup. □

From Lemma 256, is easy to prove the following result.

Lemma 258. *If S is a numerical semigroup and $x \in S\backslash\{0\}$, then $S_x = (\{x\} + S) \cup \{0\}$ is a MED-semigroup with $\mathrm{m}(S_x) = x$. Moreover, every MED-semigroup is of this form.*

Proposition 259. *Let S be a Coe-semigroup and let x be an even element in $S\backslash\{0\}$. Then $S_x = (\{x\} + S) \cup \{0\}$ is a Coe-semigroup with maximal embedding dimension. Moreover, every Coe-semigroup with maximal embedding dimension, distinct of $\mathbb{N}$, is of this form.*

Proof. By using Lemma 258, we know that S_x is a MED-semigroup with $\mathrm{m}(S_x) = x$. Clearly, that $S = \{-x\} + (S_x\backslash\{0\})$ and by Proposition 257 we get that S_x is a Coe-semigroup.

Now, let T be a Coe-semigroup with maximal embedding dimension, distinct from $\mathbb{N}$. By Proposition 220, we have that $\mathrm{m}(T)$ is even and, by Proposition 257, we deduce that $Q = \{-\mathrm{m}(T)\} + (T\backslash\{0\})$ is a Coe-semigroup. Finally, $T = (\{\mathrm{m}(T)\} + Q) \cup \{0\}$ wherein Q is a Coe-semigroup and $\mathrm{m}(T)$ is an even element in Q. □

From [**Ros03**], we can deduce the next result.

Proposition 260. *Le S be a numerical semigroup, $x \in S\backslash\{0\}$ and $T = (\{x\} + S) \cup \{0\}$. Then the following conditions hold:*

(1) T *is a MED-semigroup*
(2) $\mathrm{m}(T) = x$.
(3) $\mathrm{F}(T) = \mathrm{F}(S) + x$.
(4) $\mathrm{g}(T) = \mathrm{g}(S) + x - 1$.
(5) $\mathrm{msg}(T) = \mathrm{Ap}(S, x) + \{x\}$.

Example 261. From Example 222 we have that $S = \langle 4, 6, 7\rangle = \{0, 4, 6, 7, 8, 10, \rightarrow\}$ is a Coe-semigroup. Since 6 is an even element in S, then by applying Propositions 259 and 260, we obtain that $T = (\{6\} + S) \cup \{0\}$ is a Coe-semigroup with maximal embedding dimension, $\mathrm{m}(T) = 6$, $\mathrm{F}(T) = 9 + 6 = 15$, $\mathrm{g}(T) = 5 + 6 - 1 = 10$ and $\mathrm{msg}(T) = \mathrm{Ap}(S, 6) + \{6\} = \{0, 4, 7, 8, 11, 15\} + \{6\} = \{6, 10, 13, 14, 17, 21\}$.

7. Coe-semigroups with an unique odd minimal generator

Our first aim in this section is to prove Theorem 264, which can be used to construct a whole set of Coe-semigroups with a unique odd minimal generator.

Lemma 262. *Let S be a numerical semigroup and $\{s, s+1\} \subseteq S$. Then $T = 2S \cup (\{2s+1\} + 2S)$ is a Coe-semigroup. Moreover, $2s+1$ is the unique odd minimal generator in T.*

Proof. The sum of two elements in $2S$ belongs to $2S$. The sum of two elements in $\{2s+1\} + 2S$ belongs to $2S$. The sum of an element in $2S$ and an element in $\{2s+1\} + 2S$ belongs to $\{2s+1\} + 2S$. As a result of the previous observations, we can conclude that T is a numerical semigroup. Besides, if $\mathrm{msg}(S) = \left\{n_1, \ldots, n_p\right\}$, then $\left\{2n_1, \ldots, 2n_p, 2s+1\right\}$ is a system of generators of T and $2s+1$ is the unique odd minimal generator in T. Since $\{s, s+1\} \subseteq S$, then $\{2s, 2s+2\} \subseteq T$ and so, by Proposition 221, we obtain that T is a Coe-semigroup. □

Lemma 263. *Let T be a Coe-semigroup, with $T \neq \mathbb{N}$, and x the unique odd minimal generator in T. Then there exists a numerical semigroup S such that $T = 2S \cup (\{x\} + 2S)$ and $\left\{\frac{x-1}{2}, \frac{x-1}{2} + 1\right\} \subseteq S$.*

Proof. If $A = \{a \in \mathrm{msg}(T) \mid a \text{ is even}\}$, then $\mathrm{msg}(T) = A \cup \{x\}$. Since T is a Coe-semigroup, then $\{x-1, x+1\} \subseteq T$ and so $\{x-1, x+1\} \subseteq \langle A \rangle$. As $\gcd\{x-1, x+1\} = 2$, then $\gcd\{A\} = 2$ and therefore $S = \langle\left\{\frac{a}{2} \mid a \in A\right\}\rangle$ is a numerical semigroup.

Since $\{x-1, x+1\} \subseteq \langle A \rangle$, then $2x = (x-1) + (x+1) \in \langle A \rangle$ and we obtain that $T = \langle A \rangle \cup (\{x\} + \langle A \rangle)$. Besides, we have that $\langle A \rangle = 2S$, then $T = 2S \cup (\{x\} + 2S)$. Finally, if $\{x-1, x+1\} \subseteq \langle A \rangle$, then we get that $\left\{\frac{x-1}{2}, \frac{x+1}{2}\right\} = \left\{\frac{x-1}{2}, \frac{x-1}{2} + 1\right\} \subseteq S$. □

As a consequence of Lemma's 262 and 263, we obtain the next result.

Theorem 264. *If S is a numerical semigroup and $\{s, s+1\} \subseteq S$, then $T = 2S \cup (\{2s+1\} + 2S)$ is a Coe-semigroup with an unique odd minimal generator. Moreover, every Coe-semigroup with a unique odd minimal generator is of this form.*

Proposition 265. *Let S be a numerical semigroup, and $\{s, s+1\} \subseteq S$, then $T = 2S \cup (\{2s+1\} + 2S)$. Then the following conditions hold:*

(1) $\mathrm{m}(T) = 2\mathrm{m}(S)$.
(2) $\mathrm{F}(T) = 2\mathrm{F}(S) + (2s+1)$.
(3) $\mathrm{g}(T) = 2\mathrm{g}(S) + s$.
(4) $\mathrm{msg}(T) = (2\mathrm{msg}(S)) \cup \{2s+1\}$.
(5) $\mathrm{e}(T) = \mathrm{e}(S) + 1$.

Proof. (1) Trivial.

(2) It is sufficient to see that all even elements greater than $2\mathrm{F}(S)$ and all odd elements greater than $2s+1+2\mathrm{F}(S)$ belong to T. Since $2s+1+2\mathrm{F}(S) \notin T$, then $\mathrm{F}(T) = 2s+1+2\mathrm{F}(S)$

(3) The cardinality of the set of the even elements not in T is equal to $\mathrm{g}(S)$. The set of the odd elements not in T is equal to $\{2k+1 \mid k \in \{0, 1, \ldots, s-1\}\} \cup \{2s+1+2x \mid x \in \mathbb{N}\backslash S\}$. Hence, we get that $\mathrm{g}(T) = 2\mathrm{g}(S) + s$.

(4) If $\mathrm{msg}(S) = \left\{n_1, \ldots, n_p\right\}$, then $T = \langle 2n_1, \ldots, 2n_p, 2s+1\rangle$. In fact, $2s+1 \notin \langle 2n_1, \ldots, 2n_p\rangle$. In order to conclude the proof, it suffices to show that $2n_1 \notin \langle 2n_2, \ldots, 2n_p, 2s+1\rangle$. Suppose that there exists $\left\{\lambda_1, \lambda_2, \ldots, \lambda_p\right\} \subseteq \mathbb{N}$ such that $2n_1 = \lambda_1(2s+1) + \lambda_2(2n_2) + \cdots + \lambda_p(2n_p)$. Hence, we get that λ_1 is even and $n_1 = \frac{\lambda_1}{2}(2s+1) + \lambda_2 n_2 + \cdots + \lambda_p n_p$, with $\left\{\frac{\lambda_1}{2}, \lambda_2, \ldots, \lambda_p\right\} \subseteq \mathbb{N}$. Since $\mathrm{msg}(S) = \left\{n_1, \ldots, n_p\right\}$ and $2s+1 \in S$, then $n_1 = 2s+1$. We have that $s = \frac{n_1-1}{2}$, and as $\{s, s+1\} \subseteq S$, then $\left\{\frac{n_1-1}{2}, \frac{n_1+1}{2}\right\} \subseteq S\backslash\{0\}$. Therefore, $\frac{n_1-1}{2} + \frac{n_1+1}{2} = n_1$ and so $n_1 \notin \mathrm{msg}(S)$, a contradiction.

(5) Follow directly from the (4).

□

Example 266. If $S = \langle 5, 7, 9\rangle$ then $\mathrm{m}(S) = 5$, $\mathrm{F}(S) = 13$, $\mathrm{g}(S) = 8$, $\mathrm{msg}(S) = \{5, 7, 9\}$ and $\mathrm{e}(S) = 3$. Since $\{14, 15\} \subseteq S$, then by Theorem 264, $T = 2S \cup (\{29\} + 2S)$ is a Coe-semigroup. By Proposition 265, we obtain that $m(T) = 10$, $\mathrm{F}(T) = 55$, $\mathrm{g}(T) = 30$, $\mathrm{msg}(T) = \{10, 14, 18, 29\}$ and $\mathrm{e}(T) = 4$.

Let S be a numerical semigroup. We say that s is a small element in S if $s < \mathrm{F}(S)$. Denote by $\mathrm{N}(S)$ the set of all small elements in S and by $\mathrm{n}(S)$ its cardinality. Note that $\mathrm{F}(S) + 1 = \mathrm{g}(S) + \mathrm{n}(S)$.

Proposition 267. *Let S be a numerical semigroup, $\{s, s+1\} \subseteq S$ and $T = 2S \cup (\{2s+1\} + 2S)$. If S verifies* ***Wilf's conjecture****, then T also verifies the same conjecture.*

Proof. As $\mathrm{n}(T) = \mathrm{F}(T) + 1 - \mathrm{g}(T)$, then by Proposition 265, we have that $\mathrm{n}(T) = 2\mathrm{F}(S) + (2s+1) + 1 - (2\mathrm{g}(S) + s) = 2(\mathrm{F}(S) + 1 - \mathrm{g}(S)) + s = 2\mathrm{n}(S) + s$. We have that T verifies Wilf's conjecture if

$$\mathrm{g}(T) \leq (\mathrm{e}(T) - 1)n(T) \Longleftrightarrow$$
$$2\mathrm{g}(S) + s \leq \mathrm{e}(S)(2\mathrm{n}(S) + s) \Longleftrightarrow$$
$$\mathrm{g}(S) \leq (\mathrm{e}(S) - 1)\mathrm{n}(S) + \mathrm{n}(S) + \frac{(\mathrm{e}(S) - 1)s}{2}.$$

Since S verifies Wilf's conjecture, then $\mathrm{g}(S) \leq (\mathrm{e}(S) - 1)\mathrm{n}(S)$, $\mathrm{n}(S) \geq 0$ and $\frac{(\mathrm{e}(S)-1)s}{2} \geq 0$ and therefore T also verifies the same conjecture. □

Example 268. If S is a numerical semigroup with $\mathrm{e}(S) = 3$, then by [**DM06**, Theorem 2.11], we have that S verifies Wilf's conjecture. If $\{s, s+1\} \subseteq S$, then by Proposition 265, we obtain that $T = 2S \cup (\{2s+1\} + 2S)$ is a Coe-semigroup which verifies Wilf's conjecture.

We finish this section by studying the class of Coe-semigroups with embedding dimension 1, 2, and 3. Clearly, the numerical semigroups, $\mathbb{N}$ and $\langle 2, 2k+1 \rangle$ with $k \in \mathbb{N} \backslash \{0\}$, are all Coe-semigroups with embedding dimension 1 and 2.

Lemma 269. *If S is a Coe-semigroup with $\mathrm{e}(S) = 3$, then S has a unique odd minimal generator.*

Proof. If $\mathrm{msg}(S) = \{n_1 < n_2 < n_3\}$, then by Proposition 220, we get that n_1 is an even integer. If n_2 and n_3 are two odd integers, as $\{n_2 - 1, n_2 + 1\} \subseteq T$, then

$\{n_2 - 1, n_2 + 1\} \subseteq \langle n_1 \rangle$ and so $n_1 = 2$. Therefore, $\mathrm{m}(S) = 2$ and $\mathrm{e}(S) \leq \mathrm{m}(S) = 2$ a contradiction. □

Proposition 270. *Let a and b be positive integers such that $2 \leq a < b$, $\gcd\{a, b\} = 1$ and $\{s, s+1\} \subseteq \langle a, b \rangle$. Then $T = \langle 2a, 2b, 2s+1 \rangle$ is a Coe-semigroup with embedding dimension 3. Moreover, every Coe-semigroup with embedding dimension 3 is of this form.*

Proof. As an immediate consequence of Theorem 264, Proposition 265 and Lemma 269. □

By applying Propositions 265, 270 and Example 14, we obtain the next result.

Proposition 271. *If S is a Coe-semigroup, $\mathrm{msg}(S) = \{n_1, n_2, n_3\}$ and n_3 an odd integer, then:*

(1) $\mathrm{F}(S) = n_3 + \frac{n_1 n_2}{2} - n_1 - n_2$.
(2) $\mathrm{g}(S) = \frac{n_3 - 1}{2} + (\frac{n_1}{2} - 1)(\frac{n_2}{2} - 1)$.

From [**RGS09**, Lemma 2.14] we know that if S is a numerical semigroup, then $\mathrm{g}(S) \geq \frac{\mathrm{F}(S)+1}{2}$. Following the terminology introduced in [**Kun70**] a numerical semigroup is symmetric if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$.

By Proposition 271, we have the following result.

Corollary 272. *If S is a Coe-semigroup with $\mathrm{e}(S) = 3$, then S is a symmetric numerical semigroup.*

CHAPTER 6

Numerical semigroups with distances no admissible between gaps greater than its multiplicity

In this chapter, we will study the sets of **numerical semigroups with distances no admissible between gaps greater than its multiplicity**, $\mathcal{P}(A)$, and the ones with fixed multiplicity, $\mathcal{P}(A,m)$. First, we order the elements of $\mathcal{P}(A)$ in a tree with root $\mathbb{N}$. Second, we found that $\gamma(X)$, a partition of the set $\mathcal{P}(A,m)$, is a Frobenius pseudo-variety and the set $\mathcal{P}(A,m)$ is a finite tree wherein its vertices are Frobenius pseudo-varieties. The results of this chapter are submitted for publication.

1. Definitions and preliminaries

We denote by $H(S) = \{x \in \mathbb{N} \backslash S \mid x > \mathrm{m}(S)\}$.

Let A be a nonempty set of $\mathbb{N}\backslash\{0\}$. An **$\mathsf{PD}(A)$-semigroup** is a numerical semigroup S such that $H(S) + A \subseteq S$. Our main purpose in this work is to study this class of numerical semigroups. In particular, we will study the sets $\mathcal{P}(A) = \{S \mid S \text{ is an } \mathsf{PD}(A)\text{-semigroup}\}$ and $\mathcal{P}(A,m) = \{S \in \mathcal{P}(A) \mid \mathrm{m}(S) = m\}$. Its study is clearly motivated by generalizing to other classes of semigroups studied before, such as:

(1) If we denote by $\mathcal{E}(m)$ the set of **elementary numerical semigroups with multiplicity** m, then we have that $\mathcal{E}(m) = \mathcal{P}(\{m\},m)$.
(2) In the first section of chapter 2, we studied **numerical semigroups with concentration two**. It is easy to see that this class coincides with the set $\mathcal{P}(\{1\},m)\backslash\mathcal{O}_m$.

This work is organized as follows. In Section 2, we will order the elements of $\mathcal{P}(A)$ to construct a tree with root $\mathbb{N}$. This ordering will provide us with an algorithmic procedure that allows us to recursively build the elements of $\mathcal{P}(A)$.

In Section 3, we will show that $\mathcal{P}(A,m)$ has infinite cardinality if and only if m is an even number and all elements in A are odd numbers.

If S is a numerical semigroup, we denote by $P(S) = \{x \in S \mid m < x < 2m\}$. A subset of $\{m+1, m+2, \ldots, 2m-1\}$, X, is an $\mathsf{PD}(A,m)$-set if $X = P(S)$ for some $\mathsf{PD}(A,m)$-semigroup. If X is a $\mathsf{PD}(A,m)$-set we denote by $\gamma(X) = \{S \in \mathsf{PD}(A,m) \mid P(S) = X\}$. In Section 4, we will see that the set $\{\gamma(X) \mid X \text{ is a } \mathsf{PD}(A,m)\text{-set}\}$ is a partition of the set $\mathcal{P}(A,m)$. Furthermore, following the notation introduced in [**RR15**], we will prove that $\gamma(X)$ is a Frobenius pseudo-variety. We will show that the elements of the set $\{\gamma(X) \mid X \text{ is an } \mathsf{PD}(A,m)\text{-set}\}$ can be ordered in a finite tree. From this, we will see the set $\mathcal{P}(A,m)$ is a finite tree wherein its vertices are Frobenius pseudo-varieties.

Finally, in Section 5 we will provide algorithms to produce all elements in $\mathcal{P}(A,m)$ with fixed genus or fixed Frobenius number.

2. The tree of $\mathsf{PD}(A)$-semigroups

Throughout this chapter A will be a nonempty set of $\mathbb{N}\backslash\{0\}$. The next result is easy to prove.

Lemma 273. *If S is a $\mathsf{PD}(A)$-semigroup and $S \neq \mathbb{N}$, then $S \cup \{F(S)\}$ is a $\mathsf{PD}(A)$-semigroup.*

The above result enable us, given a $\mathsf{PD}(A)$-semigroup S, to define recursively the following sequence of a $\mathsf{PD}(A)$-semigroups, as:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n)\} & \text{if } S_n \neq \mathbb{N} \\ \mathbb{N} & \text{otherwise.} \end{cases}$

The next result is trivial.

Proposition 274. *If S is a* PD(A)*-semigroup and* $\{S_n \mid n \in \mathbb{N}\}$ *is the previous sequence of numerical semigroups, then there exists* $k \in \mathbb{N}$ *such that* $S_k = \mathbb{N}$.

We define $G(\mathcal{P}(A))$ as the **graph** whose vertices are elements of $\mathcal{P}(A)$ and $(S, T) \in \mathcal{P}(A) \times \mathcal{P}(A)$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$. As a consequence of Lemma 274, we have the following.

Proposition 275. *The graph $G(\mathcal{P}(A))$ is a* ***tree*** *with root equal to* $\mathbb{N}$.

Clearly, we can construct the elements of the set $\mathcal{P}(A)$ recursively, starting in $\mathbb{N}$, we connect each vertex with its children. Therefore, we need to characterize the children of an arbitrary vertex of this tree.

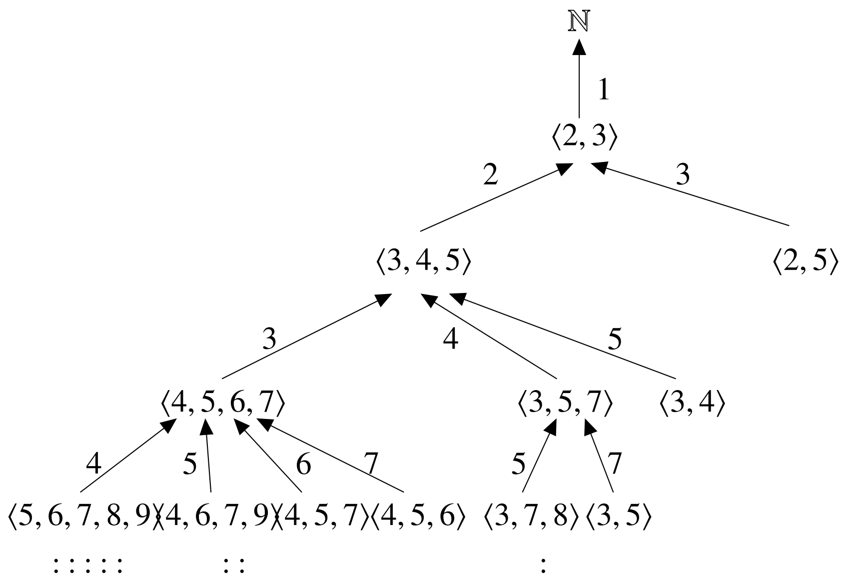
Proposition 276. *If $S \in \mathcal{P}(A)$, then the set of children of S in the tree $G(\mathcal{P}(A))$ is equal to* $\{S \setminus \{x\} \mid x \in \mathrm{msg}(S),\ x \geq \mathrm{F}(S)$ *and* $x - a \notin H(S \setminus \{x\}),\ \forall a \in A\}$.

Proof. If T is a child of S, then $T \in \mathcal{P}(A)$ and $S = T \cup \{F(T)\}$. Hence, we can deduce that $T = S \setminus \{F(T)\}$. Using Lemma 31, we have that $F(T) \in \mathrm{msg}(S)$ and as $S = T \cup \{F(T)\}$ then $F(S) < F(T)$. Moreover, since $T \in \mathcal{P}(A)$ and $\mathrm{F}(T) \notin T$, then $\mathrm{F}(T) - a \notin H(T)$ for all $a \in A$.

Conversely, suppose that $x \in \mathrm{msg}(S)$, $x > \mathrm{F}(S)$ and $x - a \notin H(S \setminus \{x\})$ for all $a \in A$. Then, by Lemma 31, $S \setminus \{x\}$ is a numerical semigroup with $\mathrm{F}(S \setminus \{x\}) = x$. By applying $x - a \notin H(S \setminus \{x\})$ for all $a \in A$ and $S \in \mathcal{P}(A)$, we deduce that $S \setminus \{x\} \in \mathcal{P}(A)$. Finally, as $S = S \setminus \{x\} \cup \mathrm{F}(S \setminus \{x\})$ then $S \setminus \{x\}$ is a child of S in the tree $G(\mathcal{P}(A))$. □

Observe that, since the **ordinary numerical semigroup** $O_m \in \mathcal{P}(A)$ for all $m \in \mathbb{N}$, then we get that the set $\mathcal{P}(A)$ has infinite cardinality.

Example 277. Let us construct the tree $G(\mathcal{P}(\{2\}))$.



3. $\mathsf{PD}(A)$-semigroups with a given multiplicity

From now on m denotes a positive integer greater than or equal to 2. Our first aim in this section is to see which conditions must m and A fulfill so that $\mathcal{P}(A, m)$ has infinite cardinality.

If S is a numerical semigroup, then $\mathbb{N} \backslash S$ is a finite set and so we get the following result.

Lemma 278. *Let S be a numerical semigroup, the set $\{T \mid T$ is a numerical semigroup and $S \subseteq T\}$ is a finite set.*

Lemma 279. *Let the hypothesis be as above. Then the following conditions hold:*

(1) $\{S \in \mathcal{P}(A, m) \mid m + 1 \in S\}$ *is a finite set.*

(2) $\{S \in \mathcal{P}(A, m) \mid 2m - 1 \in S\}$ *is a finite set.*

(3) If $x \in \{m + 1, m + 2, \ldots, 2m - 2\}$, *then*

$\{S \in \mathcal{P}(A, m) \mid \{x, x + 1\} \subseteq H(S)\}$ *is a finite set.*

(4) If $x \in \{m+1, m+2, \ldots, 2m-2\}$, then
$\{S \in \mathcal{P}(A,m) \mid \{x, x+1\} \subseteq S\}$ is a finite set.

Proof. (1) Since $\gcd\{m, m+1\} = 1$, then by Lemma 1, we get that $\langle m, m+1\rangle$ is a numerical semigroup with multiplicity m. It is clear that $\{S \in \mathcal{P}(A,m) \mid m+1 \in S\} \subseteq \{T \mid T \text{ is a numerical semigroup and } \langle m, m+1\rangle \subseteq T\}$ and, by Lemma 278, the last set is finite.

(2) The proof is similar to 1) using $2m-1$ in place of $m+1$.

(3) As $\{x, x+1\} \subseteq H(S)$, if $a \in A$, then $\{x+a, x+a+1\} \subseteq S$ and $\gcd\{x+a, x+a+1\} = 1$. To conclude the proof it is enough to note that $\{S \in \mathcal{P}(A,m) \mid \{x, x+1\} \subseteq H(S)\} \subseteq \{T \mid T \text{ is a numerical semigroup and } \langle x+a, x+a+1\rangle \subseteq T\}$ and, by Lemma 278, the last set is finite.

(4) Clearly $\{S \in \mathcal{P}(A,m) \mid \{x, x+1\} \subseteq S\} \subseteq \{T \mid T \text{ is a numerical semigroup and } \langle x, x+1\rangle \subseteq T\}$ and, again by Lemma 278, the last set is finite.

□

Lemma 280. *With the notation above, if $\mathcal{P}(A,m)$ has infinite cardinality, then m is an even number.*

Proof. By using (1) and (3) of Lemma 279, we deduce that if the set $\mathcal{P}(A,m)$ has infinite cardinality, then $\{S \in \mathcal{P}(A,m) \mid m+1 \notin S \text{ and } m+2 \in S\}$ is also an infinite set. Since $\{S \in \mathcal{P}(A,m) \mid m+1 \notin S \text{ and } m+2 \in S\} \subseteq \{T \mid T \text{ is a numerical semigroup and } \langle m, m+2\rangle \subseteq T\}$. By applying Lemmas 1 and 278, we obtain that $\gcd\{m, m+2\} \neq 1$. Therefore $\gcd\{m, m+2\} = 2$ and so m is an even number. □

Lemma 281. *With the notation above, if $\mathcal{P}(A,m)$ has infinite cardinality, then all elements in A are odd numbers.*

PROOF. For the same reasons as previously, we have that $\{S \in \mathcal{P}(A,m) \mid m+1 \notin S \text{ and } m+2 \in S\}$ is an infinite set. If $a \in A$, then we get that $\{m, m+2, m+1+a\} \subseteq S$ and $\{S \in \mathcal{P}(A,m) \mid m+1 \notin S \text{ and } m+2 \in S\} \subseteq \{T \mid T \text{ is a numerical semigroup and } \langle m, m+2, m+1+a\rangle \subseteq T\}$. By using Lemma 278, we can deduce that $\gcd\{m, m+2, m+1+a\} = \gcd\{m, 2, 1+a\} \neq 1$. Hence $\gcd\{m, 2, 1+a\} = 2$ and so a is an odd number. □

We are ready to show the above-announced result.

Theorem 282. *With the notation above, the set $\mathcal{P}(A,m)$ has infinite cardinality if and only if m is an even number and all elements in A are odd numbers.*

PROOF. *Necessity*. This is an immediate consequence of Lemmas 280 and 281.

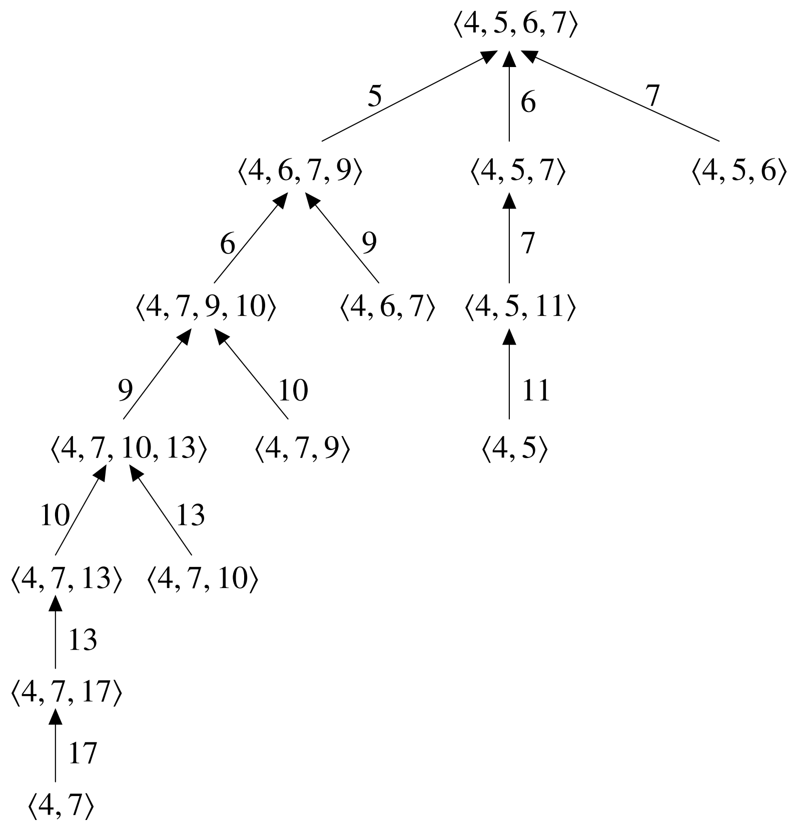
Sufficiency. Suppose that m is an even number. For each $n \in \{m, \rightarrow\}$ we denote by $S(n) = \{2k \mid k \in \left\{0, \frac{m}{2}, \rightarrow\right\}\} \cup \{n, \rightarrow\}$. It is easy to see that $S(n) \in \mathcal{P}(A,m)$ and thus $\mathcal{P}(A,m)$ has infinite cardinality. □

We define the graph $G(\mathcal{P}(A,m))$ as the graph whose vertices are elements of $\mathcal{P}(A,m)$ and $(S,T) \in \mathcal{P}(A,m) \times \mathcal{P}(A,m)$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$. In the same way, as in Section 2, we have the following result.

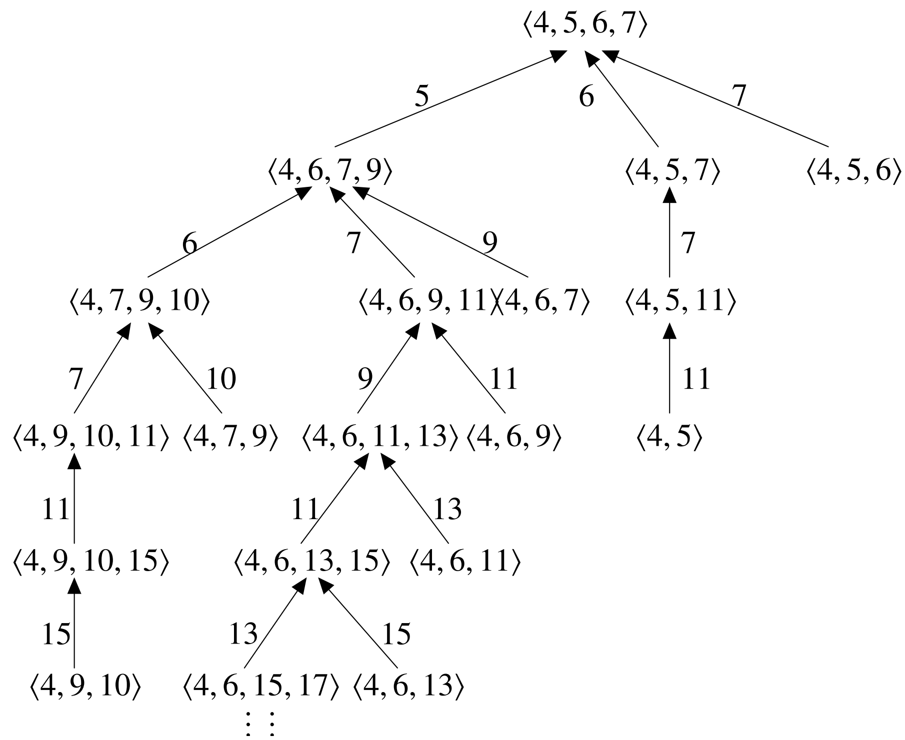
Proposition 283. *The graph $G(\mathcal{P}(A,m))$ is a tree with root equal to the* ***ordinary numerical semigroup*** *O_m. Furthermore, the set of children of S in the tree $G(\mathcal{P}(A,m))$ is equal to $\{S\backslash\{x\} \mid x \in \mathrm{msg}(S),\ x \neq m,\ x \geq \mathrm{F}(S) \text{ and } x-a \notin H(S\backslash\{x\}),\ \forall a \in A\}$.*

In the next examples, we are going to build the trees $G(\mathcal{P}(\{2\},4))$ and $G(\mathcal{P}(\{3\},4))$. Observe that by Theorem 282, we get that the first is finite and the second is infinite.

Example 284. We are going to build the finite tree $G(\mathcal{P}(\{2\},4))$.



Example 285. We are going to construct the infinite tree $G(\mathcal{P}(\{3\}, 4))$.



4. Partition of the set $\mathcal{P}(A, m)$

Given a numerical semigroup S, we denote by

$P(S) = \{x \in S \mid \mathrm{m}(S) + 1 \leq x \leq 2\mathrm{m}(S) - 1\}$ and by

$\overline{P(S)} = \{\mathrm{m}(S) + 1, \ldots, 2\mathrm{m}(S) - 1\} \setminus \{P(S)\}$.

Proposition 286. *Let S be a numerical semigroup. Then S is an $\mathsf{PD}(A)$-semigroup if and only if $\overline{P(S)} + A \subseteq S$.*

Proof. As $\overline{P(S)} \subseteq H(S)$, if $x \in \overline{P(S)}$ then $x \in H(S)$ and so $\{x\}+A \subseteq S$. Conversely, if $h \in H(S)$ then $i = h \bmod m \in \{1, \ldots, m-1\}$ and $m + i \in \overline{P(S)}$. Moreover, there exists $q \in \mathbb{N}$ such that $h = m + i + q.m$. Hence, $\{h\} + A = \{m + i\} + A + \{q.m\} \subseteq S$. $\square$

Let R be the equivalence relation defined on $\mathcal{P}(A, m)$ by

$$S \; R \; T \text{ if and only if } P(S) = P(T).$$

Let $[S]$ denote the class of $S \in \mathcal{P}(A, m)$ modulo R, i.e.,

$$[S] = \{T \in \mathcal{P}(A, m) \mid S\ R\ T\}.$$

Hence, the quotient set of $\mathcal{P}(A, m)$ induced by R is the set

$$\mathcal{P}(A, m)/R = \{[S] \mid S \in \mathcal{P}(A, m)\}.$$

The power set of a set X is the set of all subsets of X, denoted by $\mathbb{P}(X) = \{Y \mid Y \subseteq X\}$.

Proposition 287. *The correspondence*

$$\varphi : \mathcal{P}(A, m)/R \to \mathbb{P}(\{m + 1, \ldots, 2m - 1\})$$

such that $\varphi([S]) = \mathbb{P}(S)$ is an injective map.

Proof. Clearly, φ is a map, because if $[S] = [T]$ then $S\ R\ T$ and so $P(T) = P(S)$. Since $\varphi([S]) = \varphi([T])$ implies that $P(T) = P(S)$ and thus $[S] = [T]$, we get that φ is injective. $\square$

An $\mathsf{PD}(A, m)$-set is a subset, X, of $\{m + 1, \ldots, 2m - 1\}$ that verifies: if $a \in A$, $b \in \{m + 1, \ldots, 2m - 1\} \setminus X$ and $m + 1 \leq a + b \leq 2m - 1$, then $a + b \in X$.

Proposition 288. *If φ is the map defined in Proposition 287, then $Im(\varphi) = \{X \mid X$ is a $\mathsf{PD}(A, m)$-set$\}$.*

Proof. If $X \in Im(\varphi)$, then there exists $S \in \mathcal{P}(A, m)$ such that $P(S) = X$ and thus $X \subseteq \{m + 1, \ldots, 2m - 1\}$. Still, if $a \in A$, $b \in \{m + 1, \ldots, 2m - 1\} \setminus X$ then $b \in H(S)$ and so $a + b \in S$. Consequently, if $m + 1 \leq a + b \leq 2m - 1$, then $a + b \in X$. Hence, we obtain that X is an $\mathsf{PD}(A, m)$-set.

Conversely, if X is a $\mathsf{PD}(A, m)$-set, then we deduce that $S_X = \{0, m\} \cup X \cup \{2m, \rightarrow\} \in \mathcal{P}(A, m)$ and $P(S_X) = X$. Wherefore, $X \in Im(\varphi)$. $\square$

Given X a $\mathsf{PD}(A, m)$-set, we denote by $\gamma(X) = \{S \in \mathcal{P}(A, m) \mid P(S) = X\}$. As a consequence of Propositions 287 and 288, we establish the following result.

Theorem 289. *With notation above, the set* $\{\gamma(X) \mid X$ *is a* $\mathsf{PD}(A, m)$*-set*$\}$ *defines a (disjoint) partition of* $\mathcal{P}(A, m)$.

Our aim in this section is to prove that if X is a $\mathsf{PD}(A, m)$-set, then $\gamma(X)$ is a Frobenius pseudo-variety.

Proposition 290. *If* X *is a* $\mathsf{PD}(A, m)$*-set, then* $\gamma(X)$ *is a Frobenius pseudo-variety.*

Proof. Clearly, $S_X = \{0, m\} \cup X \cup \{2m, \rightarrow\}$ is the maximum element in the set $\mathcal{P}(A, m)$. If $\{S, T\} \subseteq \gamma(X)$, then $P(S) = X$ and $P(T) = X$. Hence, we can conclude that $P(S \cap T) = X$ and so $\overline{P(S \cap T)} = \overline{P(S)} = \overline{P(T)}$. By using Proposition 286, we have that $\overline{P(S)} + A = \overline{P(S \cap T)} + A \subseteq S \cap T$ and thus $S \cap T \in \mathcal{P}(A, m)$. Consequently, $S \cap T$ is an element of $\gamma(X)$.

If $S \in \gamma(X)$ and $S \neq S_X$, then we obtain that $\mathrm{F}(S) > 2m$ and thus $S \cup \{\mathrm{F}(S)\} \in \gamma(X)$. □

Following the notation introduced in [**BR12a**] a numerical semigroup S is elementary if $\mathrm{F}(S) < 2\mathrm{m}(S)$. In [**RB22**] a broad study of these semigroups is carried out which were also studied in [**KY13**] and [**Zha10**].

Proposition 291. *The following conditions are equivalent.*

(1) S *is an elementary numerical semigroup and* $S \in \mathcal{P}(A, m)$.
(2) $S = S_X$ *for some* $\mathsf{PD}(A, m)$*-set* X.

Proof. 1) *implies* 2). By Proposition 288, we obtain that $P(S)$ is a $\mathsf{PD}(A, m)$-set. If S is elementary, then $\{2m, \rightarrow\} \subseteq S$ and thus $S = \{0, m\} \cup P(S) \cup \{2m, \rightarrow\} = S_{P(S)}$.

2) *implies* 1). From the proof of Proposition 290, we have that $S_X \in \mathcal{P}(A, m)$ and so S_X is an elementary numerical semigroup. □

Let $\mathcal{E}(A, m) = \{S \in \mathcal{P}(A, m) \mid S$ is elementary$\} = \{S_X \mid X$ is an $\mathsf{PD}(A, m)$-set$\}$. The maximum element in the set $\mathcal{E}(A, m)$ is the ordinary numerical semigroup O_m. From Lemma 273, we can deduce the following result.

Lemma 292. *If $S \in \mathcal{E}(A, m)$ and $S \neq \mathcal{O}_m$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{E}(A, m)$.*

The previous result allows us, given $S \in \mathcal{E}(A, m)$ to define recursively the following sequence of elements in $\mathcal{E}(A, m)$ as:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n)\} & \text{if } S_n \neq \mathcal{O}_m \\ \mathcal{O}_m & \text{otherwise.} \end{cases}$

Now, we define the graph $G(\mathcal{E}(A, m))$ as the graph whose vertices are elements of $\mathcal{E}(A, m)$ and $(S, T) \in \mathcal{E}(A, m) \times \mathcal{E}(A, m)$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$. It is easy to prove the following result.

Proposition 293. *The graph $G(\mathcal{E}(A, m))$ is a finite tree with root equal to the ordinary numerical semigroup $\mathcal{O}_m$.*

It is clear that, if X is a $\mathsf{PD}(A, m)$-set, then $\gamma(X) = [S_X]$. As a consequence of Theorem 289 and Proposition 291, we obtain the next result.

Proposition 294. *With notation above, the set $\{[S] \mid S \in \mathcal{E}(A, m\}$ defines a (disjoint) partition of $\mathcal{P}(A, m)$.*

By using Propositions 290, 293, and 294 we can formulate the following result.

Corollary 295. *The set $\mathcal{P}(A, m)$ is a finite tree in which each vertex is a Frobenius pseudo-variety.*

5. Algorithms for computing all the elements in $\mathcal{P}(A, m)$

From Theorem 289, we deduce that $\{\gamma(X) \mid X \text{ is a } \mathsf{PD}(A, m)\text{-set}\}$ is a partition of the set $\mathcal{P}(A, m)$. Hence, in order to determine explicitly the elements in $\mathcal{P}(A, m)$ we will need:

(1) an algorithm to compute the set of all $\mathsf{PD}(A, m)$-set.

(2) an algorithm to compute the set $\gamma(X)$, given X an $\mathsf{PD}(A, m)$-set.

In the literature, one finds many algorithms devoted to computing the power set of a set $\{m+1,\dots,2m-1\}$, i.e., $\mathbb{P}(\{m+1,\dots,2m-1\})$. Moreover, it is easy to check whether an element in this set is or isn't an $\mathsf{PD}(A,m)$-set. As we have issue (1) solved, then we give the following example.

Example 296. Let us fully compute the set $\mathsf{PD}(\{2\},5)$-set. We need the set

$$\mathbb{P}(\{6,7,8,9\}) = \{\emptyset,\{6\},\{7\},\{8\},\{9\},\{6,7\},\{6,8\},\{6,9\},\{7,8\},\{7,9\},\\ \{8,9\},\{6,7,8\},\{6,7,9\},\{6,8,9\},\{7,8,9\},\{6,7,8,9\}\}.$$

Note that $X\subseteq\{6,7,8,9\}$ is a $\mathsf{PD}(\{2\},5)$-set if fulfills the following: if $6\notin X$ then $8\in X$ and if $7\notin X$ then $9\in X$. Hence, the $\mathsf{PD}(\{2\},5)$-set are

$$\{6,7\},\{6,9\},\{7,8\},\{8,9\},\{6,7,8\},\{6,7,9\},\{6,8,9\},\{7,8,9\},\{6,7,8,9\}.$$

Our main goal in this section is to solve the issue 2). By Proposition 290, we know that $S_X=\{0,m\}\cup X\cup\{2m,\rightarrow\}$ is the maximum element in $\gamma(X)$ and if $S\in\gamma(X)$ such that $S\neq S_X$, then $S\cup\{\mathrm{F}(S)\}\in\gamma(X)$. Moreover, S_X is the unique element in $\gamma(X)$ such that $\mathrm{F}(S_X)<2m$.

If X is a $\mathsf{PD}(A,m)$-set and $S\in\gamma(X)$, then we can define recursively the following sequence of elements in $\gamma(X)$:

- $S_0=S$,
- $S_{n+1}=\begin{cases} S_n\cup\{\mathrm{F}(S_n)\} & \text{if } \mathrm{F}(S_n)>2m\\ S_n & \text{otherwise.}\end{cases}$

The next result has immediate proof.

Lemma 297. *If X is a $\mathsf{PD}(A,m)$-set, $S\in\gamma(X)$ and $\{S_n\mid n\in\mathbb{N}\}$ is the previous sequence of numerical semigroups in $\gamma(X)$, then there exists $k\in\mathbb{N}$ such that $S_k=S_X$.*

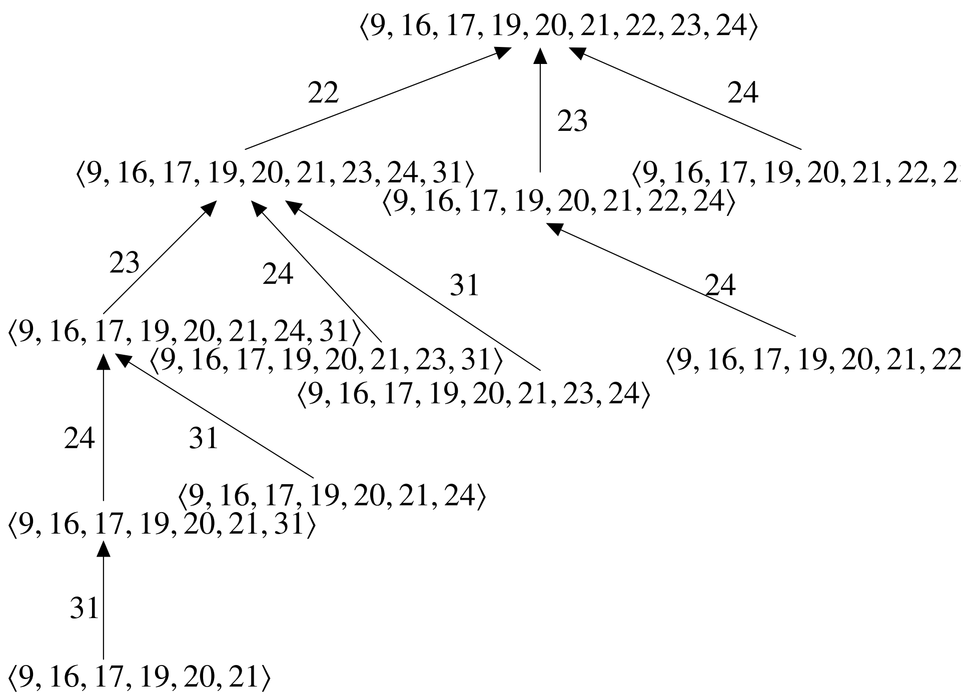
We define the graph $G(\gamma(X))$ as the graph whose vertices are elements of $\gamma(X)$ and $(S,T)\in\gamma(X)\times\gamma(X)$ is an edge if $T=S\cup\{\mathrm{F}(S)\}$. It is not hard to prove the following result.

Proposition 298. *If X is a $\mathsf{PD}(A, m)$-set, then the graph $G(\gamma(X))$ is a tree with root equal to S_X. Furthermore, the set of children of S in the tree $G(\gamma(X))$ is equal to* $\{S\setminus\{b\} \mid b \in \mathrm{msg}(S),\ b > \max\{\mathrm{F}(S), 2m\},$ *and* $b - a \notin H(S\setminus\{b\}),\ \forall a \in A\}$.

We illustrate the above results with the following example.

Example 299. Clearly the set $\{16, 17\}$ is a $\mathsf{PD}(\{6\}, 9)$-set. Let us compute the graph $G(\gamma(\{16, 17\}))$. By using Proposition 298, the graph $G(\gamma(\{16, 17\}))$ is a tree with root equal to $S_{\{16,17\}} = \{0, 9\} \cup \{16, 17\} \cup \{18, \rightarrow\} = \langle 9, 16, 17, 19, 20, 21, 22, 23, 24\rangle$.

Using again Proposition 298 we compute the children of each vertex.



Now our aim is to give an algorithm to compute all elements in $\mathcal{P}(A, m)$ with a given genus. For this purpose, we need to introduce some concepts and results.

Remember that $N(G, k) = \{v \mid d_G(v) = k\}$ denotes the set of vertices v with **depth** k.

The next result is easy to prove.

Proposition 300. *Let $m \in \mathbb{N}\backslash\{0, 1\}$ and $k \in \mathbb{N}$. Then the following conditions hold.*

(1) $N(G(\mathcal{P}(A, m)), k) = \{S \in \mathcal{P}(A, m) \mid \mathrm{g}(S) = m - 1 + k\}$.

(2) $N(G(\mathcal{P}(A, m)), k + 1) = \{S \mid S$ *is a child of an element in* $N(G(\mathcal{P}(A, m)), k)\}$.

(3) If $\mathcal{P}(A, m)$ *is an infinite set, then* $\{\mathrm{g}(S) \mid S \in \mathcal{P}(A, m)\} = \{m - 1, \rightarrow\}$.

(4) If $\mathcal{P}(A, m)$ *is a finite set, then* $\{\mathrm{g}(S) \mid S \in \mathcal{P}(A, m)\} = \{m - 1, m, \dots, m - 1 + h(G(\mathcal{P}(A, m)))\}$.

We are ready to present the advertised algorithm.

Algorithm 301. Input: g a positive integer greater than or equal to $m - 1$.

Output: The set $\{S \in \mathcal{P}(A, m) \mid \mathrm{g}(S) = g\}$.

1) Start with $i = m - 1$ and $X = \{\langle m, m + 1, \cdots, 2m - 1\rangle\}$.
2) If $i = g$, then return X.
3) For each $S \in X$ compute

   $B_S = \{T \mid T$ is a child of $S \in G(\mathcal{P}(A, m))\}$.
4) If the set $\bigcup_{S\in X} B_S = \emptyset$, return $\emptyset$.
5) $X = \bigcup_{S\in X} B_S$, $i = i + 1$ and go to step 2).

With Proposition 283 in mind, let us see in an example how our algorithm works.

Example 302. Let us compute the set $\{S \in \mathcal{P}(\{3\}, 4) \mid \mathrm{g}(S) = 6\}$.

(1) Start with $i = 3$ and $X = \langle 4, 5, 6, 7\rangle$.

(2) The first loop constructs $B_{\langle 4,5,6,7\rangle} = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle, \langle 4, 5, 6\rangle\}$ and then $X = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle, \langle 4, 5, 6\rangle\}$, $i = 4$.

(3) The second loop constructs

   $B_{\langle 4,6,7,9\rangle} = \{\langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle\}$,

   $B_{\langle 4,5,7\rangle} = \{\langle 4, 5, 11\rangle\}$ and $B_{\langle 4,5,6\rangle} = \emptyset$ and then

   $X = \{\langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle, \langle 4, 5, 11\rangle\}$, $i = 5$.

(4) The third loop constructs $B_{\langle 4,7,9,10\rangle} = \{\langle 4, 9, 10, 11\rangle, \langle 4, 7, 9\rangle\}$, $B_{\langle 4,6,9,11\rangle} = \{\langle 4, 6, 11, 13\rangle, \langle 4, 6, 9\rangle\}$ and $B_{\langle 4,6,7\rangle} = \emptyset$ and

$B_{\langle 4,5,11\rangle} = \{\langle 4, 5\rangle\}$ then

$X = \{\langle 4, 9, 10, 11\rangle, \langle 4, 7, 9\rangle, \langle 4, 6, 11, 13\rangle, \langle 4, 6, 9\rangle, \langle 4, 5\rangle\}$, $i = 6$.

(5) Return $\{S \in \mathcal{P}(\{3\}, 4) \mid \mathrm{g}(S) = 6\} =$

$= \{\langle 4, 9, 10, 11\rangle, \langle 4, 7, 9\rangle, \langle 4, 6, 11, 13\rangle, \langle 4, 6, 9\rangle, \langle 4, 5\rangle\}$.

We finish this section showing an algorithm that allows us to compute all elements in $\mathcal{P}(A, m)$, with a given Frobenius number. The operation of this algorithm is based on the fact that if S is a vertex of the tree $G(\mathcal{P}(A, m))$ then every descendant of S has a Frobenius number greater than $\mathrm{F}(S)$.

Algorithm 303. Input: F a positive integer greater than or equal to $m - 1$ and $m \nmid F$.

Output: The set $\{S \in \mathcal{P}(A, m) \mid \mathrm{F}(S) = F\}$

1) Start with $C = \emptyset$ and $X = \{\langle m, m + 1, \ldots, 2m - 1\rangle\}$
2) For each $S \in X$ compute
   $B_S = \{T \mid T$ is a child of $S \in G(\mathcal{P}(A, m))\}$,
   $C_S = \{T \in B_S \mid \mathrm{F}(T) = F\}$ and $D_S = \{T \in B_S \mid \mathrm{F}(T) < F\}$.
3) Do $C = C \cup \{\bigcup_{S\in X} C_S\}$.
4) If the set $\bigcup_{S\in X} D_S = \emptyset$, return C.
5) Do $X = \bigcup_{S\in X} D_S$ and go to step 2).

Example 304. Let us compute the set $\{S \in \mathcal{P}(\{3\}, 4) \mid \mathrm{F}(S) = 7\}$.

(1) Start with with $C = \emptyset$ and $X = \langle 4, 5, 6, 7\rangle$.

(2) The first loop constructs $B_{\langle 4,5,6,7\rangle} = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle, \langle 4, 5, 6\rangle\}$, $C_{\langle 4,5,6,7\rangle} = \{\langle 4, 5, 6\rangle\}$ and $D_{\langle 4,5,6,7\rangle} = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\}$ then
   $C = \{\langle 4, 5, 6\rangle\}$ and $X = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle\}$.

(3) The second loop constructs
   $B_{\langle 4,6,7,9\rangle} = \{\langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle\}$,
   $C_{\langle 4,6,7,9\rangle} = \{\langle 4, 6, 9, 11\rangle\}$ and $D_{\langle 4,6,7,9\rangle} = \{\langle 4, 7, 9, 10\rangle, \langle 4, 6, 7\rangle\}$,
   and it constructs $B_{\langle 4,5,7\rangle} = \{\langle 4, 5, 11\rangle\}$ and $C_{\langle 4,5,7\rangle} = \{\langle 4, 5, 11\rangle\}$ and $D_{\langle 4,5,7\rangle} = \emptyset$

then $C = \{\langle 4,5,6\rangle, \langle 4,6,9,11\rangle, \langle 4,5,11\rangle\}$ and
$X = \{\langle 4,7,9,10\rangle, \langle 4,6,7\rangle\}$.

(4) The third loop constructs $B_{\langle 4,7,9,10\rangle} = \{\langle 4,9,10,11\rangle, \langle 4,7,9\rangle\}$ and $C_{\langle 4,7,9,10\rangle} = \{\langle 4,9,10,11\rangle\}$ and $D_{\langle 4,7,9,10\rangle} = \{\langle 4,7,9\rangle\}$ and it constructs $B_{\langle 4,6,7\rangle} = \emptyset$ then
$C = \{\langle 4,5,6\rangle, \langle 4,6,9,11\rangle, \langle 4,5,11\rangle, \langle 4,9,10,11\rangle\}$, $X = \{\langle 4,7,9\rangle\}$.

(5) The fourth loop constructs $B_{\langle 4,7,9\rangle} = \emptyset$ then
$C = \{\langle 4,5,6\rangle, \langle 4,6,9,11\rangle, \langle 4,5,11\rangle, \langle 4,9,10,11\rangle\}$.

(6) Return $\{S \in \mathcal{P}(\{3\},4) \mid \mathrm{F}(S) = 7\} =$
$= \{\langle 4,5,6\rangle, \langle 4,6,9,11\rangle, \langle 4,5,11\rangle, \langle 4,9,10,11\rangle\}$.

CHAPTER 7

Frobenius R-variety of the numerical semigroups containing a given semigroup

In this chapter, we will study Frobenius R-variety of the **numerical semigroups containing a given one**. This study was published in [**RBT22a**].

1. Definitions and preliminaries

Throughout this chapter Δ will denote a numerical semigroup and

$$\mathrm{R}(\Delta) := \{S \mid S \text{ is a numerical semigroup and } S \subseteq \Delta\}.$$

If $S \in \mathrm{R}(\Delta)$ and $S \neq \Delta$, then we denote by $\mathrm{F}_\Delta(S) := \max(\Delta \backslash S)$ and $\mathrm{g}_\Delta(S) := \#(\Delta \backslash S)$ called the **Frobenius number and genus of S restricted to** Δ, respectively. By definition $\mathrm{F}_\Delta(\Delta) = -1$.

In section 2, we start by seeing that $\mathrm{R}(\Delta)$ is a Frobenius R-variety which can be arranged in a tree rooted. From this, given a nonnegative integer g, we exhibit an algorithmic process to determine the set $\{S \in \mathrm{R}(\Delta) \mid \mathrm{g}_\Delta(S) = g\}$.

We denote by

$$\alpha(\Delta, g) := \#\{S \in \mathrm{R}(\Delta) \mid \mathrm{g}_\Delta(S) = g\}.$$

To determine $\alpha(\mathbb{N}, g)$ is a classic problem that has been widely treated in the literature (see for example [**Eli10**], [**Kap12**], [**Zha10**] and [**Zha13**]). Some of these works are motivated by **Bras-Amorós's conjecture** [**BA08**] which says that $\alpha(\mathbb{N}, g) \leq \alpha(\mathbb{N}, g+1)$ for a fixed nonnegative integer g. This assumption is still open. In this section, we generalize Bras-Amorós's conjecture in as follows:

- Given Δ and $g \in \mathbb{N}$ then $\alpha(\Delta, g) \leq \alpha(\Delta, g+1)$?

In section 3 we generalize the Frobenius problem in the following way:

• Given Δ, to find formulas in terms of the elements in Δ and $\mathrm{msg}(S)$ to compute $\mathrm{F}_\Delta(S)$ and $\mathrm{g}_\Delta(S)$.

We will see how this new problem is open for embedding dimension two.

The set of **gaps of the first type** is defined as $N(S) = \{x \in \mathbb{N}\backslash S \mid F(S) - x \in S\}$ and the set of **gaps of the second type** as $\overline{N}(S) = \{x \in \mathbb{N}\backslash S \mid F(S) - x \notin S\}$. So, in a similar way, for $S \in \mathrm{R}(\Delta)$ we define $N_\Delta(S) = \{x \in S \mid \mathrm{F}_\Delta(S) - x \in \Delta\}$, $\overline{N}_\Delta(S) = \{x \in S \mid x < \mathrm{F}_\Delta(S) \text{ and } \mathrm{F}_\Delta(S) - x \notin \Delta\}$, $n_\Delta(S) = \#N_\Delta(S)$ and $\overline{n}_\Delta(S) = \#\overline{N}_\Delta(S)$.

Let $S \in \mathrm{R}(\Delta)$ such that $S \neq \Delta$. An integer $f \in \Delta\backslash S$ is called a **pseudo-Frobenius number** of S restricted to Δ if $f + S\backslash\{0\} \subseteq S$. We will denote by $\mathrm{PF}_\Delta(S)$ the set of pseudo-Frobenius numbers of S restricted to Δ and its carnality is the type of S restricted to Δ, denoted by $\mathrm{t}_\Delta(S)$.

In section 4 we show that $x \in \Delta\backslash S$ if and only if there exists $f \in \mathrm{PF}_\Delta(S)$ such that $f - x \in S$. As a consequence we obtain that the following inequality $\mathrm{g}_\Delta(S) \leq \mathrm{t}_\Delta(S)(\mathrm{n}_\Delta(S) + \overline{\mathrm{n}}_\Delta(S))$ holds.

Wilf conjecture [**Wil78**] can be reestated as $\mathrm{g}_\mathbb{N}(S) \leq (\mathrm{e}(S) - 1)(n_\mathbb{N}(S) + \overline{n}_\mathbb{N}(S))$. A deduction from Wilf's original question has already been presented on page 4. In section 4 we generalize the **Wilf's conjecture** in the following way:

• If $S \in \mathrm{R}(\Delta)$, then $\mathrm{g}_\Delta(S) \leq (\mathrm{e}(S) - 1)(n_\Delta(S) + \overline{n}_\Delta(S))$?

A numerical semigroup $S \in \mathrm{R}(\Delta)$ is $\mathrm{R}(\Delta)$-irreducible if it cannot be expressed as the intersection of two numerical semigroups of $\mathrm{R}(\Delta)$ properly containing it. In section 5 we see that if $S \in \mathrm{R}(\Delta)$ is an $\mathrm{R}(\Delta)$-irreducible numerical semigroup if and only if $\mathrm{PF}_\Delta(S) = \{F_\Delta(S)\}$ or $\mathrm{PF}_\Delta(S) = \left\{F_\Delta(S), \frac{F_\Delta(S)}{2}\right\}$. Furthermore, we show that $\mathrm{PF}_\Delta(S) = \{F_\Delta(S)\}$ (respectively $\mathrm{PF}_\Delta(S) = \left\{F_\Delta(S), \frac{F_\Delta(S)}{2}\right\}$) if and only if $\mathrm{g}_\Delta(S) = n_\Delta(S)$ (respectively $\mathrm{g}_\Delta(S) = n_\Delta(S) + 1$ and $\frac{F_\Delta(S)}{2} \in \Delta$).

2. Frobenius R-variety

Let Δ be a numerical semigroup and recall that $\mathrm{R}(\Delta)$ denotes the set of all numerical semigroups contained in Δ. Remember that the next result is deduced from [**RGS09**, Lemma 4.35]).

Lemma 305. *Let S and T be two numerical semigroups such that $S \subsetneq T$. Then $S \cup \{\max(T \backslash S)\}$ is also a numerical semigroup.*

As a consequence of the previous lemma, we obtain that if $S \in \mathrm{R}(\Delta)$ and $S \neq \Delta$ then $S \cup \{\mathrm{F}_\Delta(S)\} \in \mathrm{R}(\Delta)$. From here, we can deduce the following result (this is also a consequence of [**RR18**, Example 2.3]).

Proposition 306. *If Δ is a numerical semigroup then $\mathrm{R}(\Delta)$ is a Frobenius R-variety. Furthermore $\Delta = \max(\mathrm{R}(\Delta))$.*

We define the **graph** $G(\mathrm{R}(\Delta))$ as follows: $\mathrm{R}(\Delta)$ is its set of vertices and $(S, T) \in \mathrm{R}(\Delta) \times \mathrm{R}(\Delta)$ is a edge if $T = S \cup \{\mathrm{F}_\Delta(S)\}$.

The following result can be deduced from Theorem 70.

Theorem 307. *The graph $G(\mathrm{R}(\Delta))$ is a **tree** rooted in Δ. Moreover, the set of children of S is equal to $\{S \backslash \{x\} \mid x \in msg(S)$ and $x > \mathrm{F}_\Delta(S)\}$.*

In order to recurrently build the tree $G(\mathrm{R}(\Delta))$, starting from Δ, it is sufficient to compute the children of each vertex of $G(\mathrm{R}(\Delta))$. However, by applying Theorem 307, we can build the $G(\mathrm{R}(\langle 2, 5\rangle))$. Since $\mathrm{R}(\langle 2, 5\rangle)$ has infinite cardinal it is not possible to build a whole tree, thus the hanging points indicate that the process continues.

Example 308. We are going to build the $G(\mathrm{R}(\langle 2, 5\rangle))$. Thus, we obtain the next diagram, where each vertex is represented by its minimal system of generators as a numerical semigroup.

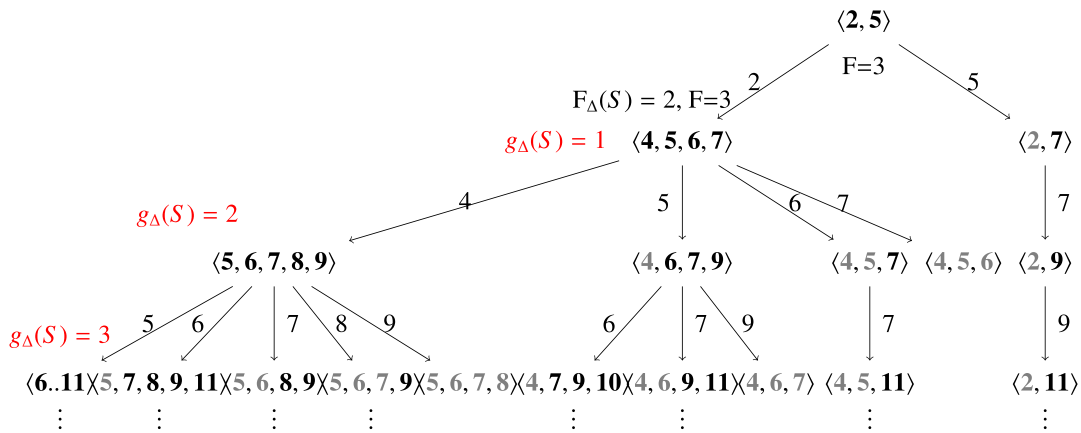


FIGURE 1. The first three layers of the tree $G(\mathrm{R}(\langle 2, 5\rangle))$.

The number that appears on either side of the edges is the element that we remove from the semigroup to obtain its child. Note that, this number becomes the Frobenius number of the new child restricted to $\langle 2, 5\rangle$.

The next result is easy to prove.

Proposition 309. *If $k \in \mathbb{N}$ then the following conditions hold.*

(1) $N(G(\mathrm{R}(\Delta)), k) = \{S \in \mathrm{R}(\Delta) \mid \mathrm{g}_\Delta(S) = k\}$.

(2) $N(G(\mathrm{R}(\Delta)), k+1) = \{S \in \mathrm{R}(\Delta) \mid S$ *is a child of an element in* $N(G(\mathrm{R}(\Delta)), k)\}$.

The following algorithm computes all the elements in $\mathrm{R}(\Delta)$ of a given genus restricted.

Algorithm 310.

INPUT: g nonnegative integer.

OUTPUT: The set $\{S \in \mathrm{R}(\Delta) \mid \mathrm{g}_\Delta(S) = g\}$

(1) $A = \{\Delta\}$, $i = 0$.

(2) If $i = g$ then return A.

(3) For each $S \in A$ compute the set $B_S = \{x \in \mathrm{msg}(S) \mid x > F_\Delta(S)\}$.

(4) $A := \bigcup_{S\in A} \{S\backslash\{x\} \mid x \in B_S\}$, $i = i + 1$ and go to step 2.

Example 311. Suppose that $\Delta = \langle 2, 5\rangle$. Let us compute the set $\{S \in \mathrm{R}(\Delta) \mid \mathrm{g}_\Delta(S) = 3\}$ using Algorithm 310.

(1) Start $\Delta = \langle 2, 5\rangle$, $i = 0$.
(2) the first loop constructs $B_{\langle 2,5\rangle} = \{2, 5\}$ then $A = \{\langle 4, 5, 6, 7\rangle, \langle 2, 7\rangle\}$, $i = 1$.
(3) the second loop constructs $B_{\langle 4,5,6,7\rangle} = \{4, 5, 6, 7\}$, $B_{\langle 2,7\rangle} = \{7\}$ then $A = \{\langle 5, 6, 7, 8, 9\rangle, \langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle, \langle 4, 5, 6\rangle, \langle 2, 9\rangle\}$, $i = 2$.
(4) the third loop constructs $B_{\langle 5,6,7,8,9\rangle} = \{5, 6, 7, 8, 9\}$, $B_{\langle 4,6,7,9\rangle} = \{6, 7, 9\}$, $B_{\langle 4,5,7\rangle} = \{7\}$, $B_{\langle 4,5,6\rangle} = \emptyset$, $B_{\langle 2,9\rangle} = \{9\}$ then $A = \{\langle 6, 8, 9, 10, 11\rangle, \langle 5, 7, 8, 9, 11\rangle, \langle 5, 6, 8, 9\rangle, \langle 5, 6, 7, 9\rangle, \langle 5, 6, 7, 8\rangle, \langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle, \langle 4, 5, 11\rangle, \langle 2, 11\rangle\}$.

Hence $\{S \in \mathrm{R}(\Delta) \mid \mathrm{g}_\Delta(S) = 3\} = \{\langle 6, 7, 8, 9, 10, 11\rangle, \langle 5, 7, 8, 9, 11\rangle, \langle 5, 6, 8, 9\rangle, \langle 5, 6, 7, 9\rangle, \langle 5, 6, 7, 8\rangle, \langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle, \langle 4, 5, 11\rangle, \langle 2, 11\rangle\}$.

Bras-Amorós has computed the number of semigroups of genus at most 50 and left us the next conjecture.

Conjecture 312. [**BA08**] *If $g \in \mathbb{N}$ then $\alpha(\mathbb{N}, g) \leq \alpha(\mathbb{N}, g + 1)$.*

Note that, by using Algorithm 310, we can compute the "first elements" of the sequence $\{\alpha(\Delta, g)\}_{g\in\mathbb{N}}$ and thus we generalize the **Bras-Amorós's conjecture** as follows.

Conjecture 313. *If $g \in \mathbb{N}$ then $\alpha(\Delta, g) \leq \alpha(\Delta, g + 1)$.*

3. Frobenius problem

Notice that the **Frobenius problem** consists in finding formulas in terms of the minimal generating set of a numerical semigroup S for $\mathrm{F}_\mathbb{N}(S)$ and $g_\mathbb{N}(S)$.

Now, we generalize the Frobenius problem in the following way: given a numerical semigroup Δ, to find formulas in terms of the elements in $\mathrm{msg}(S)$ to compute $\mathrm{F}_\Delta(S)$ and $\mathrm{g}_\Delta(S)$.

Observe, that this problem is open for embedding dimension two. Naturally the need arises to find formulas for $\mathrm{F}_\Delta(\langle a,b\rangle)$ and $\mathrm{g}_\Delta(\langle a,b\rangle)$ whether $\{a,b\} \subseteq \Delta\backslash\{0\}$ such that $\gcd(\langle a,b\rangle) = 1$.

Theorem 314. *Let $S \in \mathrm{R}(\Delta)$ such that $S \neq \Delta$, $n \in S\backslash\{0\}$, $w(i) \in \mathrm{Ap}(\Delta,n)$ and $w'(i) \in \mathrm{Ap}(S,n)$ for all $i \in \{1,\ldots,n-1\}$. Then there exists $(k_1,\ldots,k_{n-1}) \in \mathbb{N}^{n-1}$ such that $w'(i) = w(i) + k_i n$. Moreover, $\mathrm{F}_\Delta(S) = \max\{w(i) + k_i n \mid k_i \in \mathbb{N}\backslash\{0\}$ and $i \in \{1,\ldots,n-1\}\} - n$ and $\mathrm{g}_\Delta(S) = k_1 + \cdots + k_{n-1}$.*

Proof. Since $S \in \mathrm{R}(\Delta)$ we have that $w(i) \leq w'(i)$ for all $i \in \{1,\ldots,n-1\}$. As $w'(i) \equiv w(i) \bmod n$ then there exists $k_i \in \mathbb{N}$ such that $w'(i) = w(i) + k_i n$. Note that if $k_i = 0$ for all $i \in \{1,\ldots,n-1\}$ then $S = \Delta$. Hence, there exists some $k_i \neq 0$ with $i \in \{1,\ldots,n-1\}$.

By definition, we have $\mathrm{F}_\Delta(S) \in \Delta\backslash S$ and $\mathrm{F}_\Delta(S) + n \in S$, then $\mathrm{F}_\Delta(S) + n \in \mathrm{Ap}(S,n)\backslash\{0\}$. Hence, we obtain that $\mathrm{F}_\Delta(S) + n = w(i) + k_i n$ for some $i \in \{1,\ldots,n-1\}$ and $k_i \neq 0$. As a consequence, $\mathrm{F}_\Delta(S) \leq \max\{w(i) + k_i n \mid k_i \in \mathbb{N}\backslash\{0\}$ and $i \in \{1,\ldots,n-1\}\} - n$. On the other hand, if $k_i \neq 0$ with $i \in \{1,\ldots,n-1\}$, then $w(i) + k_i n - n \in \Delta\backslash S$ and thus $w(i) + k_i n - n \leq F_\Delta(S)$. We conclude that $\mathrm{F}_\Delta(S) = \max\{w(i) + k_i n \mid k_i \in \mathbb{N}\backslash\{0\}$ and $i \in \{1,\ldots,n-1\}\} - n$.

It is clear that $x \in \Delta\backslash S$ if and only if $w(x \bmod n) \leq x < w'(x \bmod n)$. Then $\Delta\backslash S = \{w(i) + t_i n \mid t_i \in \{0,\ldots,k_i - 1\},\ k_i \in \mathbb{N}\backslash\{0\},$ and $i \in \{1,\ldots,n-1\}\}$. Therefore, we get that $\mathrm{g}_\Delta(S) = k_1 + \cdots + k_{n-1}$. □

Let S be a numerical semigroup and $n \in S\backslash\{0\}$. Since $\mathrm{Ap}(\mathbb{N},n) = \{0 = w(0), 1 = w(1),\ldots,n-1 = w(n-1)\}$ and by Theorem 314 there exists $(k_1,\ldots,k_{n-1}) \in \mathbb{N}^{n-1}$ such that $\mathrm{Ap}(S,n)=\{0=w'(0),1+k_1n=w'(1),\dots,n-1+k_{n-1}n=w'(n-1)\}$. Whence,

$$\begin{aligned}\mathrm{g}_{\mathbb{N}}(S)&=k_1+\cdots+k_{n-1}=\\&=\frac{1}{n}(k_1n+1+\cdots+k_{n-1}n+n-1-(1+\cdots+n-1))=\\&=\frac{1}{n}\left(w'(1)+\cdots+w'(n-1)-\frac{n(n-1)}{2}\right).\end{aligned}$$

Consequently, $\mathrm{g}_{\mathbb{N}}(S)=\frac{1}{n}\left(\sum_{w'\in\mathrm{Ap}(S,n)}w'\right)-\frac{(n-1)}{2}$ which is the well-known formula given by Selmer in [**Sel77**].

Proposition 315. *Let* $S\in\mathrm{R}(\Delta)$. *Then* $\mathrm{n}_\Delta(S)\le\mathrm{g}_\Delta(S)\le\#\{x\in\Delta\backslash\{0\}\mid x\le\mathrm{F}_\Delta(S)\}$.

Proof. If $x\in S$, then $\mathrm{F}_\Delta(S)-x\notin S$. Therefore, the map

$$N_\Delta(S)\to\Delta\backslash S,\ x\mapsto\mathrm{F}_\Delta(S)-x$$

is injective, which proves that $\mathrm{n}_\Delta(S)\le\mathrm{g}_\Delta(S)$.

Clearly, if $x\in\Delta\backslash S$ then $x\in\Delta\backslash\{0\}$ and $x\le\mathrm{F}_\Delta(S)$. This implies that $\mathrm{g}_\Delta(S)\le\#\{x\in\Delta\backslash\{0\}\mid x\le\mathrm{F}_\Delta(S)\}$. □

Let S be a numerical semigroup. Then, we have that $N_{\mathbb{N}}(S)=\{x\in S\mid\mathrm{F}_{\mathbb{N}}(S)-x\in\mathbb{N}\}=\{x\in S\mid x<\mathrm{F}_{\mathbb{N}}(S)\}$ and the carnality of the set $\{x\in\mathbb{N}\backslash\{0\}\mid x\le\mathrm{F}_{\mathbb{N}}(S)\}$ is equal to $\mathrm{F}_{\mathbb{N}}(S)$. From Proposition 315, we can deduce the well-known inequalities $\#\{x\in S\mid x<\mathrm{F}_{\mathbb{N}}(S)\}\le\mathrm{g}_{\mathbb{N}}(S)\le\mathrm{F}_{\mathbb{N}}(S)$.

Proposition 316. *Let* $S\in\mathrm{R}(\Delta)$. *Then* $n_\Delta(S)+\mathrm{g}_\Delta(S)=\#(\Delta\backslash\{x\in S\mid\mathrm{F}_\Delta(S)-x\notin\Delta\})$.

Proof. Since $N_\Delta(S)$ and $\Delta\backslash S$ are disjoint sets, then $n_\Delta(S)+\mathrm{g}_\Delta(S)=\#(N_\Delta(S)\cup\Delta\backslash S)=\#(\{x\in S\mid\mathrm{F}_\Delta(S)-x\in\Delta\}\cup\Delta\backslash S)=\#(\Delta\backslash\{x\in S\mid\mathrm{F}_\Delta(S)-x\notin\Delta\})$. □

Let S be a numerical semigroup. Then, we have that $N_{\mathbb{N}}(S)=\{x\in S\mid x<\mathrm{F}_{\mathbb{N}}(S)\}$ and $\mathbb{N}\backslash\{x\in S\mid\mathrm{F}_{\mathbb{N}}(S)-x\notin\mathbb{N}\}=\mathbb{N}\backslash\{\mathrm{F}_{\mathbb{N}}(S)+1,\to\}=\{0,1,\dots,\mathrm{F}_{\mathbb{N}}(S)\}$. By applying the previous proposition, we can obtain the already known equality $\#\{x\in S\mid x<\mathrm{F}_{\mathbb{N}}(S)\}+\mathrm{g}_{\mathbb{N}}(S)=\mathrm{F}_{\mathbb{N}}(S)+1$.

As a consequence of Proposition 316, we obtain the following result.

Corollary 317. *Let* $S \in \mathrm{R}(\Delta)$*. Then* $\mathrm{g}_\Delta(S) + n_\Delta(S) + \overline{n}_\Delta(S) = \#\{x \in \Delta \mid x \leq \mathrm{F}_\Delta(S)\}$.

4. The pseudo-Frobenius numbers

Let $S \in \mathrm{R}(\Delta)$. Our next goal, in this section, is to give an algorithm method to compute the **pseudo-Frobenius numbers of** S **restricted to** Δ.

Proposition 318. *Let* $S \in \mathrm{R}(\Delta)$*. Then* $\mathrm{PF}_\Delta(S) = \mathrm{PF}_{\mathbb{N}}(S) \cap \Delta$.

Proof. If $f \in \mathrm{PF}_\Delta(S)$, then $f \in \Delta \backslash S$ and $f + S\backslash\{0\} \subseteq S$. Hence $f \in \mathbb{N}\backslash S$ and $f + S\backslash\{0\} \subseteq S$. This implies that $f \in \mathrm{PF}_{\mathbb{N}}(S) \cap \Delta$. For the other inclusion, if $f \in \mathrm{PF}_{\mathbb{N}}(S) \cap \Delta$ then $f \in \Delta\backslash S$ and $f + S\backslash\{0\} \subseteq S$. Whence $f \in \mathrm{PF}_\Delta(S)$. □

From the previous proposition, we have the following result, which gives an upper bound for the type of S restricted to Δ.

Corollary 319. *If* $S \in \mathrm{R}(\Delta)$*, then* $\mathrm{t}_\Delta(S) \leq \mathrm{t}_{\mathbb{N}}(S)$.

Observe that the formula for the pseudo-Frobenius numbers in terms of the Apéry sets, Proposition 16 from [**FGH86**, Proposition 7], can be written in the restricted form as $\mathrm{PF}_{\mathbb{N}}(S) = \{w - n \mid w \in \mathrm{Maximales}_{\leq_S} \mathrm{Ap}(S, n)\}$.

Remark 320. In numericalsgps GAP package [**DGSM20**] is given an algorithm to compute $\mathrm{PF}_{\mathbb{N}}(S)$. By applying Proposition 318, we obtain a method for computing $\mathrm{PF}_\Delta(S)$.

Indeed be $\Delta = \{0, 5, \rightarrow\}$ and $S = \Delta\backslash\{7, 8\}$.

```
gap> S := NumericalSemigroup(5, 6, 9, 13);
       <Numerical semigroup with 4 generators>
gap> PseudoFrobenius(S);
         [4, 7, 8]
```

We get that $\mathrm{PF}_\Delta(S) = \{4, 7, 8\} \cap \Delta = \{7, 8\}$.

As a consequence of Propositions 318 and 16 we obtain the following.

Corollary 321. *If* $S \in \mathrm{R}(\Delta)$*,* $\{f_1, f_2\} \subseteq \mathrm{PF}_\Delta(S)$ *and* $f_1 \neq f_2$*, then* $f_1 - f_2 \notin S$*.*

From [**RB02**, Proposition 12] we deduce the following result, which highlights the role that $\mathrm{PF}_\mathbb{N}(S)$ plays in a numerical semigroup.

Lemma 322. *Let* S *be a numerical semigroup and* $x \in \mathbb{N}$*. Then* $x \notin S$ *if and only if there exists* $f \in \mathrm{PF}_\mathbb{N}(S)$ *such that* $f - x \in S$*.*

The next result generalizes the previous lemma.

Theorem 323. *Let* $S \in \mathrm{R}(\Delta)$ *and* $x \in \Delta$*. Then* $x \notin S$ *if and only if there exists* $f \in \mathrm{PF}_\Delta(S)$ *such that* $f - x \in S$*.*

Proof. Suppose that $x \in \Delta \backslash S$. Then by Lemma 322, there exists $f \in \mathrm{PF}_\mathbb{N}(S)$ such that $f - x \in S$ and thus $f - x \in \Delta$. Hence, $f = x + (f - x) \in \Delta$ and so $f \in \Delta$. By applying Proposition 318, we conclude that $f \in \mathrm{PF}_\Delta(S)$.

Conversely, if $f \in \mathrm{PF}_\Delta(S)$ such that $f - x \in S$, as $f = x + (f - x) \notin S$, then we obtain that $x \notin S$. □

Let S be a numerical semigroup. By applying [**FGH86**, Theorem 20], we can deduce that $\mathrm{g}_\mathbb{N}(S) \leq \mathrm{t}_\mathbb{N}(S).\#\{x \in S \mid x < \mathrm{F}_\mathbb{N}(S)\}$. The following corollary generalizes the previous upper bound for the genus of S restricted to $\mathbb{N}$.

Corollary 324. *If* $S \in \mathrm{R}(\Delta)$*, then* $\mathrm{g}_\Delta(S) \leq \mathrm{t}_\Delta(S)\,(n_\Delta(S) + \overline{n}_\Delta(S))$*.*

Proof. If $x \in \Delta \backslash S$, by Theorem 323, there exists $f \in \mathrm{PF}_\Delta(S)$ such that $f - x \in S$. Denote by $f_x = \min\{f \in \mathrm{PF}_\Delta(S) \mid f - x \in S\}$. Then, the map

$$\Delta \backslash S \to \mathrm{PF}_\Delta(S) \times \{x \in S \mid x < F_\Delta(S)\},\ x \mapsto (f_x, f_x - x)\,.$$

is injective. Hence, we get that desired result

$\mathrm{g}_\Delta(S) \leq \#\mathrm{PF}_\Delta(S).\#\{x \in S \mid x < F_\Delta(S)\} = \mathrm{t}_\Delta(S)\,(n_\Delta(S) + \overline{n}_\Delta(S))$. □

We finished this section making the following generalization of **Wilf's conjecture**: if $S \in \mathrm{R}(\Delta)$, then it is true that $\mathrm{g}_\Delta(S) \leq (\mathrm{e}(S) - 1)(n_\Delta(S) + \overline{n}_\Delta(S))$?

5. Irreducibility

Recall that a numerical semigroup is **irreducible** if it can not be expressed as an intersection of two numerical semigroups containing it properly. Similarly we define $\mathrm{R}(\Delta)$-Irreducibility in section 1. In [**RB03**] it is proved that S is irreducible if and only if is maximal (with respect to set inclusion) in the set $\{T \in \mathrm{R}(\mathbb{N}) \mid \mathrm{F}_\mathbb{N}(T) = F_\mathbb{N}(S)\}$. Furthermore, remember that a numerical semigroup S is symmetric (respectively pseudo-symmetric) if it is irreducible and $\mathrm{PF}(S) = \{\mathrm{F}(S)\}$ (respectively $\mathrm{PF}(S) = \left\{\mathrm{F}(S), \frac{\mathrm{F}(S)}{2}\right\}$). Our aim in this section is to generalize these concepts and results.

Theorem 325. *Let $S \in \mathrm{R}(\Delta)$. The following conditions are equivalent.*

(1) S is $\mathrm{R}(\Delta)-$irreducible

(2) S is maximal in the set $\{T \in \mathrm{R}(\Delta) \mid \mathrm{F}_\Delta(T) = F_\Delta(S)\}$.

(3) S is maximal in the set $\{T \in \mathrm{R}(\Delta) \mid \mathrm{F}_\Delta(S) \notin T\}$

Proof. 1) *implies* 2) Let $T \in \mathrm{R}(\Delta)$ such that $S \subsetneq T$ and $\mathrm{F}_\Delta(T) = F_\Delta(S)$. By Proposition 306, we obtain that $S \cup \{F_\Delta(S)\} \in \mathrm{R}(\Delta)$. Since $(S \cup \{F_\Delta(S)\}) \cap T = S$ we deduce that $S = T$.

2) *implies* 3) Let $T \in \mathrm{R}(\Delta)$ such that $S \subseteq T$ and $\mathrm{F}_\Delta(S) \notin T$. Applying Proposition 306 repeatedly, we have that $\overline{T} = T \cup \{x \in \Delta \mid x > \mathrm{F}_\Delta(S)\} \in \mathrm{R}(\Delta)$ and $\mathrm{F}_\Delta(\overline{T}) = \mathrm{F}_\Delta(S)$. By 2) we conclude that $S = \overline{T}$ and so $S = T$.

3) *implies* 1) Let T_1 and T_2 be two numerical semigroups that contain S properly. Then, by 3) $\mathrm{F}_\Delta(S) \in T_1$ and $\mathrm{F}_\Delta(S) \in T_2$. Therefore $S \neq T_1 \cap T_2$ and so S is $\mathrm{R}(\Delta)$-irreducible. □

The set of **fundamental gaps** is defined as $FG(S) = \{x \in G(S) \mid \{2x, 3x\} \subset S\}$ and the set of **special gaps** as $SG(S) = \{x \in PF(S) \mid 2x \in S\}$. The latter can be seen as $SG(S) = \max_{\leq_S} FG(S)$.

Let $S \in \mathrm{R}(\Delta)$. An element $h \in \Delta\backslash S$ is said to be a **special gap of S restricted to Δ** if $S \cup \{h\}$ is a numerical semigroup. We denote by $SG_\Delta(S)$ the set of all special gaps of S restricted to Δ. Observe that if $S \neq \Delta$ then $\mathrm{F}_\Delta(S) \in SG_\Delta(S)$.

Theorem 326. *Let $S \in \mathrm{R}(\Delta)$ and $S \neq \Delta$. The following conditions are equivalent.*

(1) S is $\mathrm{R}(\Delta)-$irreducible

(2) $SG_\Delta(S) = \{\mathrm{F}_\Delta(S)\}$.

(3) $\#SG_\Delta(S) = 1$.

Proof. 1) *implies* 2) Suppose that $SG_\Delta(S) \neq \{\mathrm{F}_\Delta(S)\}$. This means that there exists $h \in SG_\Delta(S)$ such that $h \neq \mathrm{F}_\Delta(S)$. Hence $S \cup \{h\}$ and $S \cup \{\mathrm{F}_\Delta(S)\}$ are two elements in $\mathrm{R}(\Delta)$ that contain S properly. Then, we get that $(S \cup \{h\}) \cap (S \cup \mathrm{F}_\Delta(S)) = S$, contradicting that S is $\mathrm{R}(\Delta)$-irreducible.

2) *implies* 3) Trivial.

3) *implies* 1) Suppose that S is not $\mathrm{R}(\Delta)-$*irreducible*. From Theorem 325 (2), there exists $T \in \mathrm{R}(\Delta)$ such that $S \subsetneq T$ with $\mathrm{F}_\Delta(T) = F_\Delta(S)$. By Lemma 305, if h is the maximum element in $T\backslash S$ then $h \in SG_\Delta(S)$. However $\{h, \mathrm{F}_\Delta(S)\} \subseteq SG_\Delta(S)$ such that $h \neq \mathrm{F}_\Delta(S)$. This is a contradiction. □

We can see that the previous theorem is a generalization of [**RGS09**, Corollary 4.38]. The next result highlights the relation between the sets $\mathrm{PF}_\Delta(S)$ and $SG_\Delta(S)$.

Proposition 327. *If $S \in \mathrm{R}(\Delta)$, then the following conditions hold:*

(1) $SG_\Delta(S) = \{x \in \mathrm{PF}_\Delta(S) \mid 2x \in S\}$;

(2) $SG_\Delta(S) = SG_{\mathbb{N}}(S) \cap \Delta$.

Proof. 1) If $h \in SG_\Delta(S)$ then $h \in \Delta\backslash S$ and $S \cup \{h\}$ is a numerical semigroup. Whence, $2h \in S$ and for all $s \in S$ we have that $h + S \subseteq S\backslash\{0\}$ and thus $h \in \mathrm{PF}_\Delta(S)$. For the other inclusion, take $h \in \mathrm{PF}_\Delta(S)$, $2h \in S$. Then, $h \in \Delta\backslash S$ and $S \cup \{h\}$ is a numerical semigroup and thus $h \in SG_\Delta(S)$.

2) It is straightforward to use Proposition 318. □

Note that if $x \in \mathrm{PF}_\Delta(S)$ and $2x \notin S$, then $2x \in \mathrm{PF}_\Delta(S)$. This gives us the following result which is an alternative characterization of $SG_\Delta(S)$.

Proposition 328. *If $S \in \mathrm{R}(\Delta)$, then $SG_\Delta(S) = \{x \in \mathrm{PF}_\Delta(S) \mid 2x \notin \mathrm{PF}_\Delta\}$.*

Using Remark 320 and Proposition 328 we have an algorithm to compute the set $SG_\Delta(S)$.

Theorem 329. *Let $S \in \mathrm{R}(\Delta)$. Then S is $\mathrm{R}(\Delta)$-irreducible if and only if $\mathrm{PF}_\Delta(S) = \{\mathrm{F}_\Delta(S)\}$ or $\mathrm{PF}_\Delta(S) = \left\{\mathrm{F}_\Delta(S), \frac{\mathrm{F}_\Delta(S)}{2}\right\}$.*

Proof. *Necessity.* If S is $\mathrm{R}(\Delta)$-irreducible, then by Theorem 326 we have that $SG_\Delta(S) = \{\mathrm{F}_\Delta(S)\}$. By applying Propositions 327 and 328 we obtain that $SG_\Delta(S) = \{x \in \mathrm{PF}_\Delta(S) \mid 2x \in S\}=\{x \in \mathrm{PF}_\Delta(S) \mid 2x \notin \mathrm{PF}_\Delta(S)\} = \{\mathrm{F}_\Delta(S)\}$ and thus $\mathrm{PF}_\Delta(S) = \{\mathrm{F}_\Delta(S)\}$ or $\mathrm{PF}_\Delta(S) = \left\{\mathrm{F}_\Delta(S), \frac{\mathrm{F}_\Delta(S)}{2}\right\}$.

Sufficiency. If $\mathrm{PF}_\Delta(S) = \{\mathrm{F}_\Delta(S)\}$ or $\mathrm{PF}_\Delta(S) = \left\{\mathrm{F}_\Delta(S), \frac{\mathrm{F}_\Delta(S)}{2}\right\}$, then by Propositions 327 and 328 we have that $SG_\Delta(S) = \{\mathrm{F}_\Delta(S)\}$. Hence, applying Theorem 326 we obtain that S is $\mathrm{R}(\Delta)$-irreducible. □

If $S \in \mathrm{R}(\Delta)$ and $\mathrm{PF}_\Delta(S) = \{\mathrm{F}_\Delta(S)\}$ (respectively $\mathrm{PF}_\Delta(S) = \left\{\mathrm{F}_\Delta(S), \frac{\mathrm{F}_\Delta(S)}{2}\right\}$) then we say that S is a numerical semigroup $\mathrm{R}(\Delta)$-symmetric (respectively $\mathrm{R}(\Delta)$-pseudo-symmetric).

As a consequence of Theorem 329, we have that the $\mathrm{R}(\Delta)$-irreducible numerical semigroups gather both $\mathrm{R}(\Delta)$-symmetric and $\mathrm{R}(\Delta)$-pseudo-symmetric numerical semigroups.

Proposition 330. *Let $S \in \mathrm{R}(\Delta)$. Then S is $\mathrm{R}(\Delta)$-symmetric if and only if $\mathrm{g}_\Delta(S) = \mathrm{n}_\Delta(S)$.*

Proof. *Necessity.* By Proposition 315, we know that $\mathrm{g}_\Delta(S) \geq \mathrm{n}_\Delta(S)$. On the other hand, if S is $\mathrm{R}(\Delta)$-symmetric, then $\mathrm{PF}_\Delta(S) = \{\mathrm{F}_\Delta(S)\}$ and applying Theorem 323, the map

$$\Delta \backslash S \to N_\Delta(S),\ x \mapsto \mathrm{F}_\Delta(S) - x$$

is injective, which proves that $\mathrm{g}_\Delta(S) \leq \mathrm{n}_\Delta(S)$. Consequently $\mathrm{g}_\Delta(S) = \mathrm{n}_\Delta(S)$.

Sufficiency. If $\mathrm{g}_\Delta(S) = \mathrm{n}_\Delta(S)$, then the map

$$N_\Delta(S) \to \Delta\backslash S,\ s \mapsto \mathrm{F}_\Delta(S) - s$$

is bijective and thus $\Delta\backslash S = \{\mathrm{F}_\Delta(S) - s \mid s \in N_\Delta(S)\}$. By using Corollary 321, we deduce that $\mathrm{PF}_\Delta(S) = \{\mathrm{F}_\Delta(S)\}$. Hence S is $\mathrm{R}(\Delta)$-symmetric. $\square$

Note that $n_{\mathbb{N}}(S) + \mathrm{g}_{\mathbb{N}}(S) = \mathrm{F}_{\mathbb{N}}(S) + 1$. From Proposition 330, we deduce that a numerical semigroup S is $\mathrm{R}(\mathbb{N})$-symmetric if and only if $\mathrm{g}_{\mathbb{N}}(S) = \frac{\mathrm{F}_{\mathbb{N}}(S)+1}{2}$. This result already appears in [**RGS09**, Corollary 4.5 (1)].

Lemma 331. *Let* $S \in \mathrm{R}(\Delta)$ *with* S *being a* $\mathrm{R}(\Delta)$*-pseudo-symmetric and* $x \in \Delta\backslash S$ *with* $x \neq \frac{\mathrm{F}_\Delta(S)}{2}$. *Then* $\mathrm{F}_\Delta(S) - x \in S$.

Proof. If S is $\mathrm{R}(\Delta)$-pseudo-symmetric, then $\mathrm{PF}_\Delta(S) = \left\{\mathrm{F}_\Delta(S), \frac{\mathrm{F}_\Delta(S)}{2}\right\}$. By Theorem 323, for $x \in \Delta\backslash S$ we obtain $\mathrm{F}_\Delta(S) - x \in S$ or $\frac{\mathrm{F}_\Delta(S)}{2} - x \in S$. If $\frac{\mathrm{F}_\Delta(S)}{2} - x \in S$, then $\frac{\mathrm{F}_\Delta(S)}{2} - x \in S\backslash\{0\}$ and by applying that $\frac{\mathrm{F}_\Delta(S)}{2} \in \mathrm{PF}_\Delta(S)$ we obtain that $\frac{\mathrm{F}_\Delta(S)}{2} + \frac{\mathrm{F}_\Delta(S)}{2} - x \in S$. Hence $\mathrm{F}_\Delta(S) - x \in S$. $\square$

Proposition 332. *Let* $S \in \mathrm{R}(\Delta)$. *Then* S *is* $\mathrm{R}(\Delta)$*-pseudo-symmetric if and only if* $\mathrm{g}_\Delta(S) = \mathrm{n}_\Delta(S) + 1$ *and* $\frac{\mathrm{F}_\Delta(S)}{2} \in \Delta$.

Proof. If S is $\mathrm{R}(\Delta)$-pseudo-symmetric, by applying Propositions 315 and 330, then we obtain that $\mathrm{g}_\Delta(S) \geq \mathrm{n}_\Delta(S) + 1$. From Lemma 331, the map

$$(\Delta\backslash S)\backslash\{\frac{\mathrm{F}_\Delta(S)}{2}\} \to N_\Delta(S),\quad x \mapsto \mathrm{F}_\Delta(S) - x$$

is injective and thus $\mathrm{g}_\Delta(S) - 1 \leq \mathrm{n}_\Delta(S)$. Hence we get equality.

Conversely, If $\mathrm{g}_\Delta(S) = \mathrm{n}_\Delta(S) + 1$ and $\frac{\mathrm{F}_\Delta(S)}{2} \in \Delta$. The map

$$N_\Delta(S) \to (\Delta\backslash S)\backslash\{\frac{\mathrm{F}_\Delta(S)}{2}\},\quad x \mapsto \mathrm{F}_\Delta(S) - x$$

is bijective and thus $\Delta\backslash S = \{\mathrm{F}_\Delta(S) - x \mid x \in N_\Delta(S)\} \cup \left\{\frac{\mathrm{F}_\Delta(S)}{2}\right\}$. By Corollary 321, we deduce that $\mathrm{PF}_\Delta(S) = \left\{\mathrm{F}_\Delta(S), \frac{\mathrm{F}_\Delta(S)}{2}\right\}$. Consequently, S is $\mathrm{R}(\Delta)$-pseudo-symmetric. $\square$

Since $\mathrm{g}_{\mathbb{N}}(S) + \mathrm{n}_{\mathbb{N}}(S) = \mathrm{F}_{\mathbb{N}}(S) + 1$, then, by previous proposition, we have that S is R(Δ)-pseudo-symmetric if and only if $\mathrm{g}_{\mathbb{N}}(S) = \frac{\mathrm{F}_{\mathbb{N}}(S)+2}{2}$ and $\frac{\mathrm{F}_{\mathbb{N}}(S)}{2} \in \mathbb{N}$. Note that $\frac{\mathrm{F}_{\Delta}(S)}{2} \in \mathbb{N}$ if and only if $\mathrm{F}_{\mathbb{N}}(S)$ is even and so this statement is guaranteed because $\mathrm{g}_{\mathbb{N}}(S) = \frac{\mathrm{F}_{\mathbb{N}}(S)+2}{2}$. From this we get the result of [**RGS09**, Corollary 4.5 (2)], which states that S is R($\mathbb{N}$)-pseudo-symmetric if and only if $\mathrm{g}_{\mathbb{N}}(S) = \frac{\mathrm{F}_{\mathbb{N}}(S)+2}{2}$. Observe that in general the condition, given in Proposition 332, $\frac{\mathrm{F}_{\Delta}(S)}{2} \in \Delta$ is not superfluous as we will see in the next example.

Example 333. Let $\Delta = \{0, 5, \rightarrow\}$ and $S = \Delta\backslash\{7, 8\}$. Then we have that $\mathrm{g}_{\Delta}(S) = 2$, $\mathrm{n}_{\Delta}(S) = 1$ and $\mathrm{PF}_{\Delta}(S) = \{7, 8\}$. Therefore in this case $\mathrm{g}_{\Delta}(S) = \mathrm{n}_{\Delta}(S) + 1$ and S is not R(Δ)-pseudo-symmetric.

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