Journal of Statistical Computation and Simulation

ISSN: 0094-9655 (Print) 1563-5163 (Online) Journal homepage: https://www.tandfonline.com/loi/gscs20

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To cite this article: M. Ivette Gomes, Frederico Caeiro, Fernanda Figueiredo, Lígia Henriques-Rodrigues & Dinis Pestana (2020) Reduced-bias and partially reduced-bias mean-of-order-*p* value-at-risk estimation: a Monte-Carlo comparison and an application, Journal of Statistical Computation and Simulation, 90:10, 1735-1752, DOI: 10.1080/00949655.2020.1746787

To link to this article: https://doi.org/10.1080/00949655.2020.1746787

Published online: 31 Mar 2020.

JOURNAL OF STATISTICAL COMPUTATION AND SIMULATION
2020, VOL. 90, NO. 10, 1735–1752
https://doi.org/10.1080/00949655.2020.1746787

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Reduced-bias and partially reduced-bias mean-of-order-*p* value-at-risk estimation: a Monte-Carlo comparison and an application

M. Ivette Gomes[a], Frederico Caeiro[b], Fernanda Figueiredo[c,d], Lígia Henriques-Rodrigues[d,e] and Dinis Pestana[a]

[a]CEAUL, FCUL, Universidade de Lisboa, Lisboa, Portugal; [b]CMA, FCT, Universidade Nova de Lisboa, Caparica, Portugal; [c]Faculdade de Economia, Universidade do Porto, Porto, Portugal; [d]CEAUL, Lisboa, Portugal; [e]Universidade de Évora, Évora, Portugal

ABSTRACT

On the basis of a sample of either independent, identically distributed or possibly weakly dependent and stationary random variables from an unknown model F with a heavy right-tail function, and for any small level q, the value-at-risk (VaR) at the level q, i.e. the size of the loss that occurs with a probability q, is estimated by new semi-parametric reduced-bias procedures based on the mean-of-order-p of a set of k quotients of upper order statistics, with p an adequate real number. After a brief reference to the asymptotic properties of these new VaR-estimators, we proceed to an overall comparison of alternative VaR-estimators, for finite samples, through large-scale Monte-Carlo simulation techniques. Possible algorithms for an adaptive VaR-estimation, an application to financial data and concluding remarks are also provided.

ARTICLE HISTORY
Received 15 October 2019
Accepted 19 March 2020

KEYWORDS
Bias reduction; heavy right-tails; heuristic methods; Monte-Carlo simulation; semi-parametric estimation; statistics of extremes; value-at-risk estimation

AMS 2000 SUBJECT CLASSIFICATIONS
Primary 62G32; Secondary 65C05

1. Introduction

The risk of a high loss that occurs rarely is a primordial parameter of extreme events. A possible and common indicator of such a risk is the *value-at-risk* (VaR), which can be identified with the size of the loss that occurs with a small probability, q, $0 < q < 1$. For any unknown *cumulative distribution function* (CDF) F underlying the available sample, and denoting by $F^{\leftarrow}(z) := \inf\{x : F(x) \geq z\}$ the generalized inverse function of F, we are thus dealing with a (very high) quantile,

$$\chi_{1-q} \equiv \mathrm{VaR}_q := F^{\leftarrow}(1-q) =: U(1/q), \quad \text{where} \quad U(t) = F^{\leftarrow}(1-1/t), \quad t \geq 1, \quad (1)$$

is the so-called reciprocal *right-tail quantile function* (RTQF). With n denoting the size of the available sample, $(X_1, \ldots, X_n)$, we often have $nq \leq 1$, and this justifies theoretically the assumption that $q = q_n \to 0$, as $n \to \infty$. We thus want to extrapolate beyond the sample, being then in the area of statistical *extreme value theory* (EVT). Since in real applications in the areas of biostatistics, environment, finance, insurance and statistical quality

CONTACT M. Ivette Gomes ivette.gomes@fc.ul.pt CEAUL, FCUL, Universidade de Lisboa, Lisboa, Portugal

control, among others, one often encounters heavy right-tails, we shall assume that, for some $\xi > 0$, the *right-tail function* (RTF), $\overline{F}(x) := 1 - F(x)$, satisfies the condition $\overline{F}(x) = 1 - F(x) \sim c\, x^{-1/\xi}$, as $x \to \infty$, for some positive constant c, where $a(z) \sim b(z)$ means that $a(z)/b(z) \to 1$, as $z \to \infty$. Equivalently, and for some $C > 0$, $U(t) \sim C\, t^{\xi}$, as $t \to \infty$. Slightly more restrictively, we shall here assume to be working in Hall-Welsh class of models [1]. Such an assumption means that, as $t \to \infty$, and with $C,\ \xi > 0$, $\rho < 0$ and β non-zero,

$$U(t) = Ct^{\xi}\left(1 + A(t)/\rho + o\left(t^{\rho}\right)\right), \quad \text{with} \quad A(t) = \xi\beta t^{\rho}. \tag{2}$$

This is a wide class of models, which contains most of the heavy-tailed parents useful in applications. Note that (2) implies a Paretian-type RTF, being ξ a positive version of the general EV *index* (EVI) for maxima, the primary parameter of large extreme events.

1.1. The main limiting EV distributions for maxima

Given a random sample $(X_1, \ldots, X_n)$ from F (Gnedenko [2]), or even more generally a weakly dependent stationary sample from F (Leadbetter et al. [3]), if there exist attraction coefficients (a_n, b_n), $a_n > 0$ and $b_n \in \mathbb{R}$, such that the sequence of linearly normalized maxima, $\{(X_{n:n} - b_n)/a_n\}_{n\geq 1}$, converges to a non-degenerate *random variable* (RV), such an RV is of the type of a general EV RV, with CDF

$$\mathrm{EV}_{\xi}(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}),\ 1+\xi x > 0, & \text{if } \xi \neq 0, \\ \exp(-\exp(-x)),\ x \in \mathbb{R}, & \text{if } \xi = 0. \end{cases} \tag{3}$$

It is then said that F is in the max-domain of attraction of EV_{ξ}, and the notation $F \in \mathcal{D}_M(\mathrm{EV}_{\xi})$ is used. If $\xi > 0$, as happens, among others, for the Student-t_{ν}, with ν degrees of freedom ($\xi = 1/\nu$), quite common in the field of finance, F has an associated heavy RTF, of a negative polynomial type, i.e. of a Pareto-type. We shall then use the notation $\mathcal{D}_M(\mathrm{EV}_{\xi>0}) =: \mathcal{D}_M^{+}$.

1.2. Classic semi-parametric VaR and EVI–estimators

For these heavy right-tailed or Paretian-type models, and with Q standing for quantile, the most common semi-parametric VaR-estimator was proposed by Weissman [4]. It is defined as

$$\mathrm{Q}_{\hat{\xi}}^{(q)}(k) := X_{n-k:n}\left(\frac{k}{nq}\right)^{\hat{\xi}} =: X_{n-k:n}\, r_n^{\hat{\xi}}, \quad r_n \equiv r_n(k;q) = \frac{k}{nq}, \tag{4}$$

with $X_{n-k:n}$ the $(k+1)$th upper *order statistic* (OS), $1 \leq k < n$, being strongly dependent on $\hat{\xi}$, any consistent estimator for ξ. Other semi-parametric procedures for the estimation of high quantiles can be found in the books by Beirlant et al. [5] and de Haan and Ferreira [6], among others.

For heavy RTFs, i.e. when we work in $\mathcal{D}_M^{+}$, the most common EVI-estimator, often used in (4) for a semi-parametric quantile estimation, is the Hill estimator $\hat{\xi} = \hat{\xi}(k) =: \mathrm{H}(k)$

[7], given by,

$$\mathrm{H}(k) := \frac{1}{k}\sum_{i=1}^{k} V_{ik}, \quad V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}, \quad 1 \le i \le k < n. \tag{5}$$

Indeed, in $\mathcal{D}_M^+$, the log-excesses V_{ik}, $1 \le i \le k$, in (5), are approximately the k descending OSs of a sample of size k from an exponential CDF with mean value ξ. And this is one of the reasons for the average in (5).

The estimators in (4) and in (5) are *asymptotic* estimators, i.e. they provide useful estimates, respectively of VaR_q in (1) and $\xi > 0$ in (3), only when the sample size n is high and for an adequate value of k. Indeed, consistency is achieved when we work with an *intermediate* sequence of integers,

$$k = k_n \to \infty, \quad k \in [1, n), \quad \text{with } k/n \to 0, \text{ as } n \to \infty. \tag{6}$$

The plugging in (4) of the Hill estimator, $\mathrm{H}(k)$, leads to the so-called Weissman-Hill quantile or VaR_q-estimator, denoted by $\mathrm{Q}_{\mathrm{H}}^{(q)}(k)$. To study the asymptotic behaviour of $\mathrm{Q}_{\mathrm{H}}^{(q)}(k)$, as well as of alternative VaR_q-estimators, it is useful to impose a second-order expansion on the RTF, $\overline{F} = 1 - F$, or on the RTQF, U, like the one in (2).

Since the Hill estimator is the logarithm of the *geometric mean* (or *mean-of-order-0*) of $U_{ik} := X_{n-i+1:n}/X_{n-k:n}$, $1 \le i \le k < n$, Brilhante et al. [8], and almost simultaneously and independently [9,10], considered as basic statistics the power *mean-of-order-p* (MO_p) of U_{ik}, $1 \le i \le k$, $p \in \mathbb{R}_0^+$. More generally, Gomes and Caeiro [11] and Caeiro et al. [12] worked with $p \in \mathbb{R}$ and the class of EVI-estimators,

$$\mathrm{H}_p(k) \equiv \mathrm{MO}_p(k) := \begin{cases} \left(1 - \left(\dfrac{1}{k}\displaystyle\sum_{i=1}^{k} U_{ik}^p\right)^{-1}\right) \Big/ p, & \text{if } p < 1/\xi,\ p \ne 0, \\ \dfrac{1}{k}\displaystyle\sum_{i=1}^{k} \ln U_{ik} = \mathrm{H}(k), & \text{if } p = 0. \end{cases} \tag{7}$$

The restriction $p < 1/\xi$ in (7) ensures the consistency of the MO_p EVI-estimators. This class of MO_p EVI-estimators depends now on this *tuning* parameter $p \in \mathbb{R}$, and was shown to be highly flexible. If we plug in (4) the MO_p EVI-estimator, $\mathrm{H}_p(k)$, we get the so-called MO_p quantile or VaR_q-estimator, with the obvious notation, $\mathrm{Q}_{\mathrm{H}_p}^{(q)}(k)$, studied for finite samples in Gomes et al. [13].

1.3. Bias-corrected EVI/VaR-estimators and a new VaR estimation procedure

The MO_p EVI-estimators in (7) can often have a high asymptotic bias, and bias reduction has recently been a vivid topic of research in the area of statistical EVT. For recent overviews of this topic, see Beirlant et al. [14] and Gomes and Guillou [15]. Working just for technical simplicity in the particular class of Hall-Welsh models in (2), the asymptotic distributional representation of $\mathrm{H}_p(k)$ for $p = 0$ led Caeiro et al. [16] to directly remove the dominant component of the bias of the Hill EVI-estimator, given by $\xi\beta(n/k)^\rho/(1-\rho)$ whenever (2) holds, considering the *corrected-Hill* (CH) EVI-estimators,

$$\mathrm{CH}(k) \equiv \mathrm{CH}(k;\hat\beta,\hat\rho) := \mathrm{H}(k)\left(1-\frac{\hat\beta}{1-\hat\rho}\left(\frac{n}{k}\right)^{\hat\rho}\right). \tag{8}$$

The EVI-estimators in (8) are *minimum-variance reduced-bias* (MVRB) provided that we consider adequate estimators, $(\hat\beta,\hat\rho)$, of the vector of second-order parameters (β,ρ) in (2), fully described in Gomes and Pestana [17], among others, where a reliable algorithm has been provided, based on the use of a simple class of ρ-estimators in Fraga Alves et al. [18], and the associated β-estimators introduced in Gomes and Martins [19]. Gomes and Pestana [17] have used the EVI-estimators in (8) to build RB-CH VaR_q-estimators, which we denote by $\mathrm{Q}^{(q)}_{\mathrm{CH}}(k)$. Since the main topic under consideration in this article is related to RB VaR-estimation for Paretian-type RTFs, we further mention the articles by Matthys and Beirlant [20], Matthys et al. [21], Gomes and Figueiredo [22] and Caeiro and Gomes [23,24], where RB VaR-estimation is discussed.

Gomes et al. [12] have further recently suggested for a positive p the class of EVI-estimators

$$\mathrm{CH}_p(k) \equiv \mathrm{CH}_p(k;\hat\beta,\hat\rho) \equiv \mathrm{RBMO}_p(k) := \mathrm{H}_p(k)\left(1-\frac{\hat\beta(1-p\mathrm{H}_p(k))}{1-\hat\rho-p\mathrm{H}_p(k)}\left(\frac{n}{k}\right)^{\hat\rho}\right), \tag{9}$$

which can be considered as a generalization to a real p of the $\mathrm{CH} \equiv \mathrm{CH}_0$ class of EVI-estimators in (8). It is thus sensible to work with the new class of VaR_q-estimators,

$$\mathrm{Q}^{(q)}_{\mathrm{CH}_p}(k) := X_{n-k:n}\, r_n^{\mathrm{CH}_p(k;\hat\beta,\hat\rho)} \left[\mathrm{Q}^{(q)}_{\mathrm{CH}_0}(k) \equiv \mathrm{Q}^{(q)}_{\mathrm{CH}}(k)\right], \tag{10}$$

with $r_n \equiv r_n(q)$ and $\mathrm{CH}_p(k;\hat\beta,\hat\rho)$ respectively given in (4) and (9), being p any adequate real number.

Further note that, working with values of p such that the asymptotic normality of the estimators in (7) was known to held at the time, i.e. with $0 \le p < 1/(2\xi)$, Brilhante et al. [25] proved that there is an optimal value of p, denoted by p_M, an explicit function of ξ and ρ, which maximizes the asymptotic efficiency of the class of EVI-estimators in (7). And the same result holds if we more generally consider any real p. The asymptotic behaviour of $\mathrm{H}^*(k) := \mathrm{H}_{p_M}(k)$, with $\mathrm{H}_p(k)$ given in (7), has led Gomes et al. [26] to introduce a *partially* RB (PRB) class of MO_p EVI-estimators based on $\mathrm{H}_p(k)$, in (7), with the functional expression

$$\mathrm{PRB}_p(k) \equiv \mathrm{PRB}_p(k;\hat\beta,\hat\rho) := \mathrm{H}_p(k)\left(1-\frac{\hat\beta(1-\varphi_{\hat\rho})}{1-\hat\rho-\varphi_{\hat\rho}}\left(\frac{n}{k}\right)^{\hat\rho}\right),$$
$$\varphi_\rho := 1-\rho/2-\sqrt{(1-\rho/2)^2-1/2}. \tag{11}$$

On the basis of a large-scale simulation study, it was shown in the aforementioned paper that the PRB EVI-estimators, in (11), are also able to outperform the CH EVI-estimators, in (8), for a large variety of models. For the study and asymptotic comparison under a third-order framework of different classes of RB-MO_p EVI-estimators, including the ones in (9) and (11), see Caeiro et al. [27], where updated references on second-order parameters' estimation can be found. Despite of only partially RB, and with r_n and $\mathrm{PRB}_p(k;\hat\beta,\hat\rho)$ respectively given in (4) and (11), we thus think sensible to work not only with the new class of VaR-estimators in (10), but also with

$$\mathrm{Q}^{(q)}_{\mathrm{PRB}_p}(k) := X_{n-k:n}\, r_n^{\mathrm{PRB}_p(k;\hat\beta,\hat\rho)}, \tag{12}$$

already considered in Gomes et al. [28].

Since $p_{\mathrm{M}} = \varphi_\rho/\xi$, with φ_ρ given in (11), just as done in Gomes et al. [12] for the EVI-estimation, and in Gomes et al. [28] for the PRB VaR-estimation, we shall also consider the EVI-estimators

$$\begin{aligned} \mathrm{CH}^*(k) &\equiv \mathrm{CH}^*(k;\hat\beta,\hat\rho) := \mathrm{CH}_{p^*_{\mathrm{M}}}(k), \quad \mathrm{PRB}^*(k) \equiv \mathrm{PRB}^*(k;\hat\beta,\hat\rho) \\ &:= \mathrm{PRB}_{p^*_{\mathrm{M}}}(k), \quad p^*_{\mathrm{M}} = \varphi_{\hat\rho}/\xi^*, \\ \xi^* &= \mathrm{CH}(\hat k_{0|\mathrm{H}}), \quad \hat k_{0|\mathrm{H}} := \min\left(n-1, \left\lfloor \left((1-\hat\rho)^2 n^{-2\hat\rho}/(-2\hat\rho\hat\beta^2)\right)^{1/(1-2\hat\rho)} \right\rfloor + 1\right), \end{aligned} \tag{13}$$

with $\hat k_{0|\mathrm{H}}$ a k-estimate associated with minimum *mean square error* (MSE), i.e. an estimate of $k_{0|\mathrm{H}} := \arg\min_k \mathrm{MSE}(\mathrm{H}(k))$, as suggested in Hall [29], and where $\lfloor x \rfloor$ denotes the integer part of x. Just as done by Gomes et al. [28], now on the basis of the two adaptive EVI-estimators in (13), we further work with the associated VaR-estimators, obviously denoted by

$$\mathrm{Q}^{(q)}_{\mathrm{CH}^*}(k) := X_{n-k:n}\, r_n^{\mathrm{CH}^*(k;\hat\beta,\hat\rho)} \quad \text{and} \quad \mathrm{Q}^{(q)}_{\mathrm{PRB}^*}(k) := X_{n-k:n}\, r_n^{\mathrm{PRB}^*(k;\hat\beta,\hat\rho)}. \tag{14}$$

1.4. Scope of the article

The scope of this article is to overall compare the aforementioned VaR_q-estimators, replacing, in (4), $\mathrm{Q}^{(q)}_{\hat\xi}(k)$ by the VaR_q-estimators $\mathrm{Q}^{(q)}_{\mathrm{CH}_p}(k)$, $\mathrm{Q}^{(q)}_{\mathrm{PRB}_p}(k)$ and $(\mathrm{Q}^{(q)}_{\mathrm{CH}^*}(k), \mathrm{Q}^{(q)}_{\mathrm{PRB}^*}(k))$, respectively given in (10), (12) and (14), based on the corresponding EVI-estimators, respectively given in (9), (11) and (13). Provided that we choose the value of p that provides the highest asymptotic efficiency for $\mathrm{H}_p(k)$ (see [25]), the new VaR-estimators in (10) have an asymptotic MSE (AMSE) smaller than the AMSE of Weissman-Hill VaR-estimators for all k, being also obviously able to overpass the CH$\equiv$ CH_0 VaR_q-estimator in Gomes and Pestana [17]. Consequently, the new RB-MO_p VaR-estimators in (10) are reliable alternatives to the previous VaR-estimators not only around optimal levels but for all k. Anyway, given the outstanding behaviour of the PRB VaR-estimators, which are not generally RB, just as shown in Gomes et al. [28], we shall also take into account the PRB-MO_p VaR-estimators in (12). The outline of the paper is the following. In Section 2, after a brief reference to general first and second-order conditions under a heavy-tailed framework, the asymptotic behaviour of the classes of EVI and VaR-estimators under study is discussed. Section 3 is devoted to a large-scale Monte-Carlo simulation that enables the derivation of the finite-sample distributional properties of the new classes of RB VaR-estimators, compared to the Weissman-Hill, CH and PRB VaR-estimators. Algorithms for an adaptive

VaR_q-estimation and an application to financial data are provided in Section 4. Finally, in Section 5, a few overall conclusions are drawn.

2. A brief discussion on the asymptotic behaviour of EVI and VaR-estimators

After a brief reference, in Section 2.1, to general first and second-order conditions for a heavy RTF, we deal, in Section 2.2, with known results on the asymptotic behaviour of the EVI-estimators under consideration. A parallel exposition is performed in Section 2.3 for the VaR-estimators. For a normal RV, with mean value μ and variance σ^2, the notation $\mathcal{N}(\mu, \sigma^2) = \mu + \sigma\mathcal{N}(0, 1)$ is used.

2.1. *First and second-order conditions for heavy right-tails*

In the area of EVT and whenever working with large values, a CDF F is often said to be *heavy-tailed* whenever the associated RTF $\overline{F}$ is a regularly varying function with a negative index of regular variation equal to $-1/\xi, \xi > 0$ (Gnedenko [2]), with ξ the EVI, given in (3). We then use the notation $\overline{F} \in \mathcal{R}_{-1/\xi}$ (see, among others, Bingham et al. [30], for details on regular variation). Equivalently, the RTQF U, defined in (1), is of regular variation with index ξ (de Haan [31]), i.e. $U \in \mathcal{R}_\xi$, the first-order condition under which we work in this article. Then, $\ln U(tx) - \ln U(t) - \xi \ln x \to 0$, for all $x > 0$, and as $t \to \infty$, and the second-order parameter ρ (≤ 0) measures the aforementioned rate of convergence to zero. Such a rate can be measured through a function A such that $|A| \in \mathcal{R}_\rho$ (Geluk and de Haan [32]). Such a second-order condition has been widely accepted as the appropriate one to specify the RTF of a Pareto-type distribution in a semi-parametric way, and easily enables the derivation of the non-degenerate bias of EVI and VaR-estimators, under such a framework. Further developments of the topic can be found in the books by Beirlant et al. [5] and de Haan and Ferreira [6], as well as in Fraga Alves et al. [33], among others.

If we assume that $\rho < 0$, we are then in the class of models in (2), and the slowly varying function $\ell(\cdot)$, in $\overline{F}(x) = x^{-1/\xi}\,\ell(x)$, behaves asymptotically as a constant. Note thus that when we assume (2) we are excluding models with $\rho = 0$. We are also excluding the Pareto model itself, associated with $\rho = -\infty$, but for such a model no bias-reduction is needed. Indeed, the Hill is already an RB EVI-estimator for underlying Pareto models.

2.2. *The EVI-estimators*

Just as proved in Brilhante et al. [8] and Gomes and Caeiro [11], the result obtained in de Haan and Peng [34] for the Hill EVI-estimator in (5), or equivalently, $p = 0$ in (7), can be generalized for any adequate real p, as indicated below. Under the validity of the first-order condition, $U \in \mathcal{R}_\xi$, and for intermediate k, i.e. whenever (6) holds, $\mathrm{H}_p(k)$, given in (7), is consistent for the estimation of ξ whenever $p < 1/\xi$. If we further assume the existence of a function $A(\cdot)$ that measures the rate of convergence in the first-order condition,

$$\sqrt{k}\left(\mathrm{H}_p(k) - \xi\right) \stackrel{d}{=} \mathcal{N}\left(0, \frac{\xi^2(1-p\xi)^2}{1-2p\xi}\right) + \frac{\sqrt{k}\,A(n/k)(1-p\xi)}{1-\rho-p\xi}(1+o_{\mathbb{P}}(1)), \qquad (15)$$

for $p < 1/(2\xi)$, where the bias can be small, moderate or large, in the sense that it can go respectively to zero, a constant or infinity, as $n \to \infty$. Straightforwardly from (15), if we further assume that $\sqrt{k}A(n/k) \to \lambda_A$, finite,

$$\sqrt{k}\left(\mathrm{H}_p(k) - \xi\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(\frac{\lambda_A(1-p\xi)}{1-\rho-p\xi}, \frac{\xi^2(1-p\xi)^2}{1-2p\xi}\right).$$

Theorem 3.1 in Gomes et al. [12] enables us to state that under the same conditions as above, and the adequate conditions on the (β,ρ)-estimation,

$$\sqrt{k}\left(\mathrm{CH}_p(k) - \xi\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \frac{\xi^2(1-p\xi)^2}{1-2p\xi}\right),$$

i.e. $\mathrm{CH}_p(k)$, in (9) outperforms $\mathrm{H}_p(k)$, in (7) for all k. For the EVI-estimators in (11), Theorem 2 in Gomes et al. [26] enable us to guarantee that

$$\sqrt{k}\left(\mathrm{PRB}_p(k) - \xi\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(\frac{\lambda_A(p\xi-\varphi_\rho)}{(1-\rho-p\xi)(1-\rho-\varphi_\rho)}, \frac{\xi^2(1-p\xi)^2}{1-2p\xi}\right),$$

with a null mean value only if $p\xi = \varphi_\rho$, φ_ρ given in (11). Further note that if we consider $\mathrm{CH}^*(k)$ or $\mathrm{PRB}^*(k)$, in (13), Theorem 3.1 in Gomes et al. [12] (for $\mathrm{CH}^*(k)$) and Proposition 2.1 in Gomes et al. [28] (for $\mathrm{PRB}^*(k)$) enable us to say that

$$\sqrt{k}\left(\mathrm{CH}^*(k) - \xi\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \frac{\xi^2(1-\varphi_\rho)^2}{1-2\varphi_\rho}\right) \quad \text{and}$$

$$\sqrt{k}\left(\mathrm{PRB}^*(k) - \xi\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \frac{\xi^2(1-\varphi_\rho)^2}{1-2\varphi_\rho}\right),$$

with φ_ρ given in (11), i.e. both $\mathrm{CH}^*(k)$ and $\mathrm{PRB}^*(k)$ have the same limiting behaviour.

2.3. The VaR-estimators

For models in (2), and if we further assume that $\lim_{n\to\infty}\sqrt{k}A(n/k) = \lambda_A \in \mathbb{R}$, finite, the asymptotic behaviour of $\mathrm{Q}_{\mathrm{H}}^{(q)}(k)$ is well-known (Weissman [4]). We have

$$\frac{\sqrt{k}}{\ln r_n}\left(\frac{\mathrm{Q}_{\mathrm{H}}^{(q)}(k) - \mathrm{VaR}_q}{\mathrm{VaR}_q}\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(\frac{\lambda_A}{1-\rho}, \xi^2\right),$$

with $r_n \equiv r_n(k;q)$ defined in (4), $A(\cdot)$ the function in (2), and

$$q = q_n \to 0, \quad \text{with} \quad \ln\left(n\, q_n\right) = o\left(\sqrt{k}\right) \quad \text{and} \quad nq_n = o\left(\sqrt{k}\right). \tag{16}$$

Under these same conditions, for $p = 0$ in (9), if (β,ρ) are consistently estimated so that $\hat{\rho} - \rho = o_{\mathbb{P}}(1/\ln n)$, then [17, Theorem 5.1]

$$\frac{\sqrt{k}}{\ln r_n}\left(\frac{\mathrm{Q}_{\mathrm{CH}}^{(q)}(k) - \mathrm{VaR}_q}{\mathrm{VaR}_q}\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \xi^2\right).$$

Again under the same aforementioned conditions, and for any real $p < 1/(2\xi)$ [28, Theorem 2.1],

$$\frac{\sqrt{k}}{\ln r_n}\left(\frac{\mathrm{Q}^{(q)}_{\mathrm{PRB}_p}(k) - \mathrm{VaR}_q}{\mathrm{VaR}_q}\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(\frac{\lambda_A(p\xi - \varphi_\rho)}{(1-\rho-p\xi)(1-\rho-\varphi_\rho)}, \frac{\xi^2(1-p\xi)^2}{1-2p\xi}\right).$$

We further get

$$\frac{\sqrt{k}}{\ln r_n}\left(\frac{\mathrm{Q}^{(q)}_{\mathrm{PRB}^*}(k) - \mathrm{VaR}_q}{\mathrm{VaR}_q}\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(0, \frac{\xi^2(1-\varphi_\rho)^2}{1-2\varphi_\rho}\right).$$

If we consider the classes of VaR_q-estimators $\mathrm{Q}^{(q)}_{\mathrm{CH}_p}(k)$, in (10), and $\mathrm{Q}^{(q)}_{\mathrm{CH}^*}(k)$, in (14), trivial adaptations of the results in Gomes and Figueiredo [22], Gomes and Pestana [17], and essentially Gomes et al. [28] enable us to state, without the need of a proof, the following theorem.

Theorem 2.1: *In* Hall-Welsh *class of models in (2), for intermediate k, i.e. k-values such that (6) holds, if* $\sqrt{k}\,A(n/k) \to \lambda_A$, *finite, possibly non-null, if we further consistently estimate the vector of second-order parameters* (β,ρ) *so that* $\hat\rho - \rho = o_{\mathbb{P}}(1/\ln n)$ *and whenever (16) holds, we can guarantee that for any real* $p < 1/(2\xi)$,

$$\frac{\sqrt{k}}{\ln r_n}\left(\frac{\mathrm{Q}^{(q)}_{\mathrm{CH}_p}(k) - \mathrm{VaR}_q}{\mathrm{VaR}_q}\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(0, \frac{\xi^2(1-p\xi)^2}{1-2p\xi}\right), \quad \textit{and}$$

$$\frac{\sqrt{k}}{\ln r_n}\left(\frac{\mathrm{Q}^{(q)}_{\mathrm{CH}^*}(k) - \mathrm{VaR}_q}{\mathrm{VaR}_q}\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(0, \frac{\xi^2(1-\varphi_\rho)^2}{1-2\varphi_\rho}\right),$$

with VaR_q, r_n, $\mathrm{Q}^{(q)}_{\mathrm{CH}_p}(k)$, φ_ρ *and* $\mathrm{Q}^{(q)}_{\mathrm{CH}^*}(k)$, *given in (1), (4), (10), (11) and (14), respectively.*

3. Multi-sample Monte-Carlo simulations

For the new classes of VaR-estimators, $\mathrm{Q}^{(q)}_{\mathrm{CH}_p}(k)$, in (10), and $\mathrm{Q}^{(q)}_{\mathrm{CH}^*}(k)$, in (14), but including also the classes $\mathrm{Q}^{(q)}_{\mathrm{PRB}_p}(k)$ and $\mathrm{Q}^{(q)}_{\mathrm{PRB}^*}(k)$, respectively given in (12) and (14), large-scale multi-sample Monte-Carlo simulation experiments of size 5000×20 have now been implemented. Following closely the simulation study in Gomes et al. [28], we have considered sample sizes $n = 100(100)500, 1000(1000)5000$, and $\xi = 0.1, 0.25, 0.5$ and 1, from a large variety of heavy tailed models, among which we mention the *generalized Pareto* (GP) models, with CDF, $F(x) = \mathrm{GP}_\xi(x) = 1 + \ln \mathrm{EV}_\xi(x) = 1 - (1+\xi x)^{-1/\xi}$, $x \geq 0\ (\rho = -\xi)$, and the $\mathrm{Burr}_{\xi,\rho}$ model, with CDF, $F(x) = 1 - (1 + x^{-\rho/\xi})^{1/\rho}$, $x \geq 0$, the two models considered here for illustration.

The mean value (E) and *root* MSE (RMSE) of the normalized VaR-estimators under consideration, as a function of the sample fraction, k/n, with k related to the number of upper OSs involved in the estimation, $1 \leq k < n$, have been simulated for each of the aforementioned models and for each value of n. Just as an illustration, we present Figures 1 and 2,

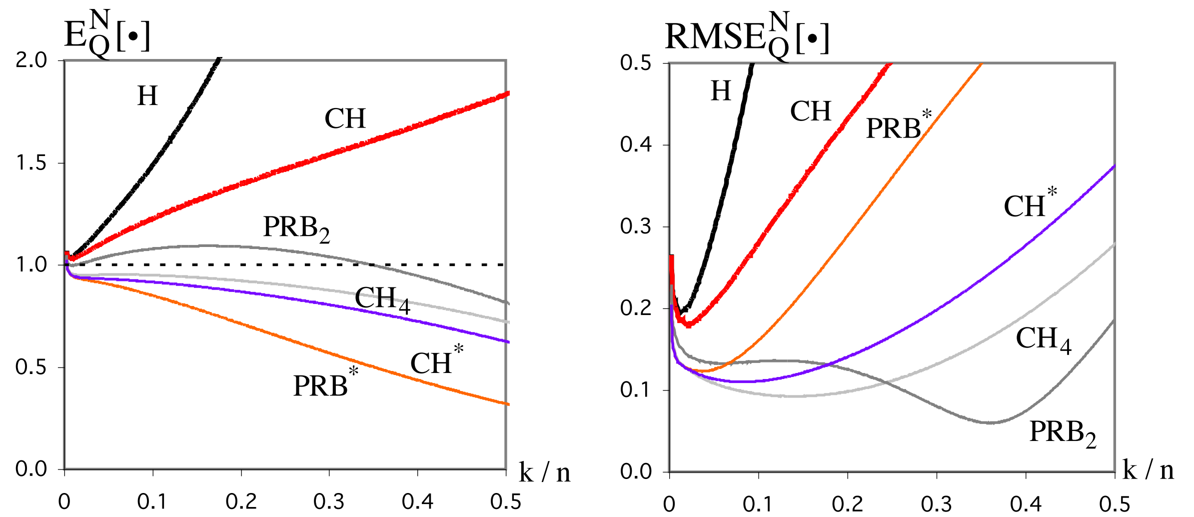

Figure 1. Mean values (*left*) and RMSEs (*right*) of the normalized VaR-estimators under consideration for an underlying $GP_{0.1}$ parent ($\rho = -0.1$).

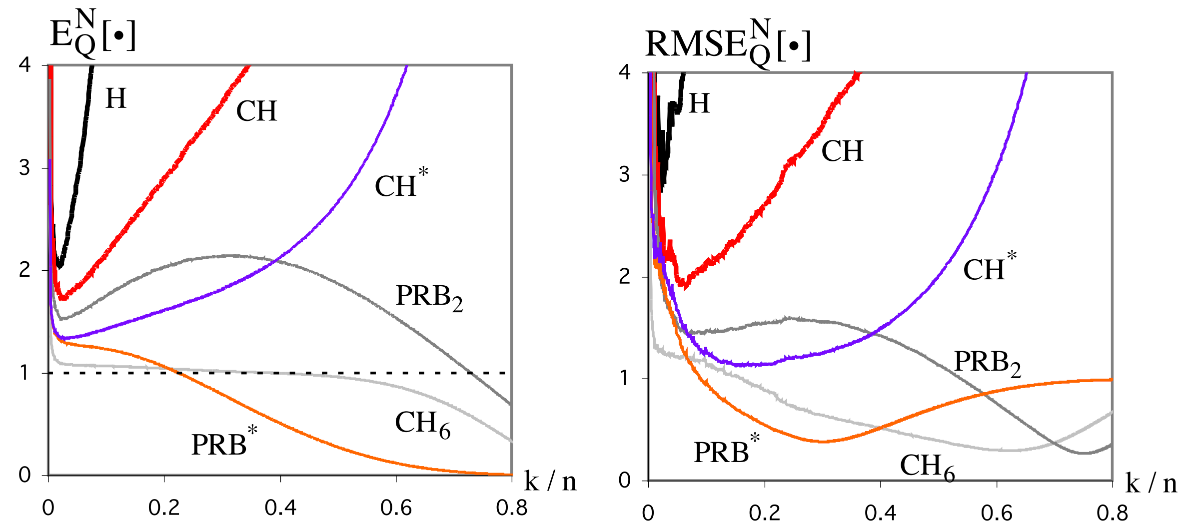

Figure 2. Mean values (*left*) and RMSEs (*right*) of the normalized VaR-estimators under consideration for an underlying $\mathrm{Burr}_{1,-0.25}$ parent.

associated with $GP_{0.1}$ and $\mathrm{Burr}_{1,-0.25}$ parents. In these figures, we show, for $n = 1000$, $q = 1/n$, and on the basis of the first 5000 runs, the simulated patterns of normalized mean value, $E_Q^N := E_Q[\cdot]/\mathrm{VaR}_q$, and $\mathrm{RMSE}_Q^N := \mathrm{RMSE}_Q[\cdot]/\mathrm{VaR}_q$, of $Q_{\hat{\xi}}^{(q)}(k)/\mathrm{VaR}_q$, based on $Q_{\hat{\xi}}^{(q)}(k)$ in (4), with $\hat{\xi}$ replaced by both CH_p, in (9), for $p = p_\ell = \ell/(16\xi)$, $\ell = 1(1)8$, representing only the best value of $\ell \neq 0$, associated with minimum RMSE, but including also $\ell = 0$, corresponding to CH, in (8). We further replace $\hat{\xi}$ by $PRB_p(k)$, in (9), for $p = p_\ell = \ell/(16\xi)$, $\ell = 1(1)8$, the values where optimal behaviour was detected for $PRB_p(k)$ in Gomes et al. [28], again representing only the best ℓ-value. We have also plotted the PRB*, CH* and H VaR-estimators.

3.1. Behaviour at simulated optimal levels

The Weissman-Hill VaR-estimator $Q_H^{(q)}(k) \equiv Q_{H_0}^{(q)}(k)$ has been computed at the simulated value of $k_{0|H_0}^{(q)} := \arg\min_k \text{RMSE}\big(Q_{H_0}^{(q)}(k)\big)$. Such a value is denoted by Q_{00}. We have also computed Q_{p0}, generally denoting the VaR-estimators both in (10) and in (12) at optimal levels, for the aforementioned values of p, and the simulated indicators,

$$\text{REFF}_{p|0} := \text{RMSE}\,(Q_{00})/\text{RMSE}\,\big(Q_{p0}\big),$$

with REFF standing for *relative efficiency*. Similar REFF-indicators have also been computed for the VaR-estimators based on $\text{CH} = \text{CH}_0$ EVI-estimators, in (8), and for the VaR-estimators in (14). The higher these indicators are, the better the associated VaR-estimators perform, compared to Q_{00}.

As an illustration of the results obtained for the different VaR-estimators under consideration, we first present Tables 1–2. In the last row, we provide the RMSE of Q_{00}, denoted by RMSE_{00}, so that we can easily recover the RMSE of all other estimators. The subsequent rows provide the REFF-indicators of the VaR-estimators under consideration. The highest REFF-indicator is *underlined* and **bolded**. Information on 95% confidence intervals (CIs), computed on the basis of the 20 replicates with 5000 runs each, is also provided in all tables.

Table 1. Simulated REFF-indicators of $Q_{\text{CH}|0}^{(q)}$, $Q_{\text{CH}^*|0}^{(q)}$, $Q_{\text{PRB}^*|0}^{(q)}$, $Q_{\text{CH}_{p_\ell}|0}^{(q)}$ and $Q_{\text{PRB}_{p_\ell}|0}^{(q)}$, for $p_\ell = \ell/(16\xi)$, $\ell = 2(2)8$, and simulated RMSE of Q_{00}, $q = 1/n$ (last row), for $\text{GP}_{0.1}$ parents ($\rho = -0.1$), together with 95% CIs.

	$\text{GP}_{0.1}$ parent					
n	100	200	500	1000	2000	5000
CH	1.442 ± 0.0131	1.220 ± 0.0109	1.117 ± 0.0050	1.079 ± 0.0032	1.059 ± 0.0032	1.038 ± 0.0022
CH*	1.594 ± 0.0100	1.553 ± 0.0119	1.570 ± 0.0157	1.813 ± 0.0121	2.680 ± 0.0163	5.819 ± 0.0436
PRB*	1.581 ± 0.0107	1.542 ± 0.0108	1.537 ± 0.0150	1.621 ± 0.0105	1.938 ± 0.0121	3.058 ± 0.0240
CH_2	$\underline{\mathbf{1.677}} \pm 0.0127$	$\underline{\mathbf{2.475}} \pm 0.0210$	$\underline{\mathbf{2.468}} \pm 0.0248$	1.868 ± 0.0137	1.644 ± 0.0127	1.529 ± 0.0122
CH_4	1.606 ± 0.0105	1.570 ± 0.0133	1.662 ± 0.0169	2.158 ± 0.0141	3.745 ± 0.0198	$\underline{\mathbf{7.695}} \pm 0.0501$
CH_6	1.515 ± 0.0113	1.491 ± 0.0105	1.459 ± 0.0147	1.439 ± 0.0091	1.442 ± 0.0100	1.638 ± 0.0139
CH_8	1.375 ± 0.0112	1.377 ± 0.0113	1.377 ± 0.0138	1.362 ± 0.0088	1.343 ± 0.0092	1.304 ± 0.0108
PRB_2	1.653 ± 0.0114	1.750 ± 0.0150	2.387 ± 0.0251	$\underline{\mathbf{3.323}} \pm 0.0243$	$\underline{\mathbf{4.600}} \pm 0.0266$	7.010 ± 0.0468
PRB_4	1.593 ± 0.0100	1.556 ± 0.0119	1.577 ± 0.0152	1.751 ± 0.0114	2.251 ± 0.0130	3.751 ± 0.0281
PRB_6	1.481 ± 0.0113	1.477 ± 0.0107	1.452 ± 0.0145	1.433 ± 0.0089	1.430 ± 0.0102	1.518 ± 0.0131
PRB_8	1.336 ± 0.0111	1.354 ± 0.0113	1.368 ± 0.0137	1.358 ± 0.0088	1.341 ± 0.0090	1.303 ± 0.0108
RMSE_{00}	0.320 ± 0.0024	0.269 ± 0.0023	0.224 ± 0.0020	0.199 ± 0.0013	0.179 ± 0.0010	0.157 ± 0.0008

For an illustration of the bias of the new VaR-estimators at optimal levels, see Tables 3–4. The simulated mean values at optimal levels of the VaR-estimators under study are there presented, for the same values of n. Again, 95% CIs are provided, and among the estimators considered, the one providing the smallest squared bias is *underlined* and written in **bold**.

For a better visualization of the tables above, we represent Figures 3–4.

4. Algorithms for an adaptive VaR_q-estimation and an application

4.1. Sample path stability algorithms

In this section we introduce algorithms for an adaptive selection of the tuning parameters k (Algorithm 4.1) and (p, k) (Algorithm 4.2) for a VaR-estimation. The motivation for the

Table 2. Simulated REFF-indicators of $\mathrm{Q}^{(q)}_{\mathrm{CH}|0}$, $\mathrm{Q}^{(q)}_{\mathrm{CH}^*|0}$, $\mathrm{Q}^{(q)}_{\mathrm{PRB}^*|0}$, $\mathrm{Q}^{(q)}_{\mathrm{CH}_{p_\ell}|0}$ and $\mathrm{Q}^{(q)}_{\mathrm{PRB}_{p_\ell}|0}$, for $p_\ell = \ell/(16\xi)$, $\ell = 2(2)8$, and simulated RMSE of Q_{00}, $q = 1/n$ (last row), for $\mathrm{Burr}_{1,-0.25}$ parents, together with 95% CIs.

n	100	200	500	1000	2000	5000
CH	11.159 ± 0.7776	4.871 ± 0.3331	2.020 ± 0.0712	1.618 ± 0.0254	1.391 ± 0.0254	1.230 ± 0.0108
CH*	9.089 ± 1.1817	16.58 ± 1.0118	8.827 ± 0.2865	5.731 ± 0.1417	4.205 ± 0.1016	3.217 ± 0.0530
PRB*	18.696 ± 1.1668	11.479 ± 0.6641	8.864 ± 0.2858	8.469 ± 0.2477	9.23 ± 0.1992	11.748 ± 0.1632
CH_2	15.918 ± 1.0683	7.575 ± 0.4482	2.985 ± 0.0985	2.087 ± 0.0523	1.672 ± 0.0353	1.410 ± 0.0134
CH_4	19.089 ± 1.1817	**16.58** ± 1.0118	8.827 ± 0.2865	5.731 ± 0.1417	4.205 ± 0.1016	3.217 ± 0.0530
CH_6	18.829 ± 1.1384	11.25 ± 0.7124	8.412 ± 0.2912	11.22 ± 0.3156	**16.376** ± 0.3734	**16.036** ± 0.2548
CH_8	18.408 ± 1.1806	11.52 ± 0.7376	7.260 ± 0.2452	6.155 ± 0.1618	5.716 ± 0.1393	7.225 ± 0.1554
PRB_2	19.105 ± 1.2155	12.432 ± 0.7457	**10.355** ± 0.3632	**11.981** ± 0.3291	14.943 ± 0.3433	12.309 ± 0.1969
PRB_4	18.696 ± 1.1668	11.479 ± 0.6641	8.464 ± 0.2858	8.469 ± 0.2477	9.237 ± 0.1992	11.748 ± 0.1632
PRB_6	19.035 ± 1.1872	11.235 ± 0.7159	7.262 ± 0.2494	6.311 ± 0.1951	5.876 ± 0.2001	6.406 ± 0.3358
PRB_8	**19.218** ± 1.1694	11.464 ± 0.7069	7.205 ± 0.2416	5.834 ± 0.1533	4.848 ± 0.1183	4.263 ± 0.0822
RMSE_{00}	13.5412 ± 0.7960	7.430 ± 0.4543	4.257 ± 0.1466	3.242 ± 0.0799	2.491 ± 0.0586	1.890 ± 0.0291

Table 3. Simulated mean values (at optimal levels) of $\mathrm{Q}^{(q)}_{00}$, $\mathrm{Q}^{(q)}_{\mathrm{CH}|0}$, $\mathrm{Q}^{(q)}_{\mathrm{CH}^*|0}$, $\mathrm{Q}^{(q)}_{\mathrm{PRB}^*|0}$, $\mathrm{Q}^{(q)}_{\mathrm{CH}_{p_\ell}|0}$ and $\mathrm{Q}^{(q)}_{\mathrm{PRB}_{p_\ell}|0}$, for $p_\ell = \ell/(16\xi)$, $\ell = 2(2)8$, $q = 1/n$ and $\mathrm{GP}_{0.1}$ underlying parents, together with 95% CIs.

n	100	200	500	1000	2000	5000
H	1.085 ± 0.0030	1.070 ± 0.0041	1.064 ± 0.0036	1.058 ± 0.0035	1.054 ± 0.0033	1.053 ± 0.0030
CH	**0.989** ± 0.0060	**1.057** ± 0.0053	1.060 ± 0.0038	1.058 ± 0.0028	1.058 ± 0.0030	1.053 ± 0.0029
CH*	0.894 ± 0.0047	0.905 ± 0.0031	0.905 ± 0.0012	0.924 ± 0.0007	0.965 ± 0.0004	0.997 ± 0.0002
PRB*	0.892 ± 0.0009	0.907 ± 0.0027	0.913 ± 0.0015	0.919 ± 0.0008	0.944 ± 0.0006	0.983 ± 0.0003
CH_2	0.897 ± 0.0024	0.937 ± 0.0011	**1.007** ± 0.0015	**1.000** ± 0.0024	**1.001** ± 0.0008	0.995 ± 0.0007
CH_4	0.894 ± 0.0033	0.904 ± 0.0035	0.910 ± 0.0012	0.943 ± 0.0004	0.986 ± 0.0003	**0.999** ± 0.0001
CH_6	0.859 ± 0.0008	0.894 ± 0.0009	0.910 ± 0.0029	0.915 ± 0.0028	0.916 ± 0.0014	0.926 ± 0.0006
CH_8	0.818 ± 0.0008	0.857 ± 0.0008	0.893 ± 0.0009	0.911 ± 0.0008	0.922 ± 0.0021	0.927 ± 0.0019
PRB_2	0.900 ± 0.0028	0.921 ± 0.0016	0.968 ± 0.0012	0.990 ± 0.0008	0.997 ± 0.0003	0.999 ± 0.0002
PRB_4	0.897 ± 0.0048	0.905 ± 0.0024	0.911 ± 0.0012	0.928 ± 0.0009	0.959 ± 0.0006	0.991 ± 0.0003
PRB_6	0.849 ± 0.0009	0.889 ± 0.0009	0.913 ± 0.0027	0.917 ± 0.0018	0.919 ± 0.0017	0.923 ± 0.0008
PRB_8	0.808 ± 0.0009	0.851 ± 0.0008	0.890 ± 0.0009	0.910 ± 0.0008	0.923 ± 0.0017	0.927 ± 0.0020

Table 4. Simulated mean values (at optimal levels) of $\mathrm{Q}^{(q)}_{00}$, $\mathrm{Q}^{(q)}_{\mathrm{CH}|0}$, $\mathrm{Q}^{(q)}_{\mathrm{CH}^*|0}$, $\mathrm{Q}^{(q)}_{\mathrm{PRB}^*|0}$, $\mathrm{Q}^{(q)}_{\mathrm{CH}_{p_\ell}|0}$ and $\mathrm{Q}^{(q)}_{\mathrm{PRB}_{p_\ell}|0}$, for $p_\ell = \ell/(16\xi)$, $\ell = 2(2)8$, $q = 1/n$ and $\mathrm{Burr}_{1,-0.25}$ underlying parents, together with 95% CIs.

n	100	200	500	1000	2000	5000
H	4.340 ± 0.1588	3.323 ± 0.1815	2.589 ± 0.0828	2.348 ± 0.0653	2.050 ± 0.0500	1.889 ± 0.0354
CH	**0.829** ± 0.0149	1.399 ± 0.0127	1.981 ± 0.0463	1.933 ± 0.0358	1.848 ± 0.0319	1.756 ± 0.0219
CH*	0.388 ± 0.0189	0.716 ± 0.0042	0.857 ± 0.0149	0.744 ± 0.0215	0.721 ± 0.0230	0.709 ± 0.0193
PRB*	0.414 ± 0.0181	0.507 ± 0.0136	0.665 ± 0.0057	0.793 ± 0.0107	0.898 ± 0.0035	0.966 ± 0.0014
CH_2	0.703 ± 0.0046	**1.097** ± 0.0092	1.675 ± 0.0191	1.732 ± 0.0288	1.709 ± 0.0390	1.657 ± 0.0304
CH_4	0.388 ± 0.0189	0.716 ± 0.0042	**0.857** ± 0.0149	0.744 ± 0.0215	0.721 ± 0.0230	0.709 ± 0.0193
CH_6	0.423 ± 0.0124	0.472 ± 0.0104	0.601 ± 0.0060	0.841 ± 0.0031	**0.974** ± 0.0020	**0.990** ± 0.0009
CH_8	0.449 ± 0.0088	0.516 ± 0.0079	0.557 ± 0.0059	0.599 ± 0.0042	0.683 ± 0.0046	0.857 ± 0.0052
PRB_2	0.443 ± 0.0179	0.591 ± 0.0112	0.786 ± 0.0075	**0.904** ± 0.0044	0.968 ± 0.0022	0.985 ± 0.0024
PRB_4	0.414 ± 0.0181	0.507 ± 0.0136	0.665 ± 0.0057	0.793 ± 0.0107	0.898 ± 0.0035	0.966 ± 0.0014
PRB_6	0.451 ± 0.0102	0.486 ± 0.0090	0.562 ± 0.0067	0.633 ± 0.0123	0.727 ± 0.0181	0.851 ± 0.0210
PRB_8	0.445 ± 0.0120	0.506 ± 0.0104	0.559 ± 0.0073	0.587 ± 0.0040	0.626 ± 0.0047	0.697 ± 0.0078

algorithms comes from the higher stability on k of the estimates of several RB and PRB EVI-estimators (see [26,35,36]). Let $(x_1, \ldots, x_n)$ be an observed sample, and let $\mathrm{S}(k) \equiv \mathrm{S}^{(p)}(k)$ be a consistent estimator of any parameter of extreme events, where p is a tuning parameter, and k is another tuning parameter related to the threshold. Some estimators are only valid

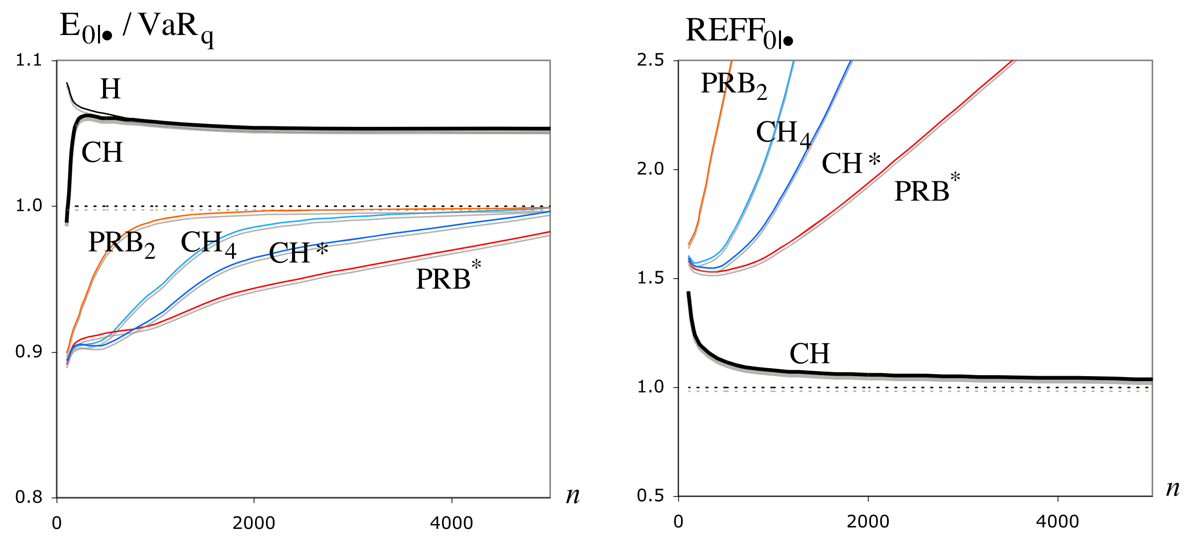

Figure 3. Normalized mean values (*left*) and REFF-indicators (*right*) of the VaR_q-estimators under study, at optimal levels, for $q = 1/n$ and $GP_{0.1}$ parents.

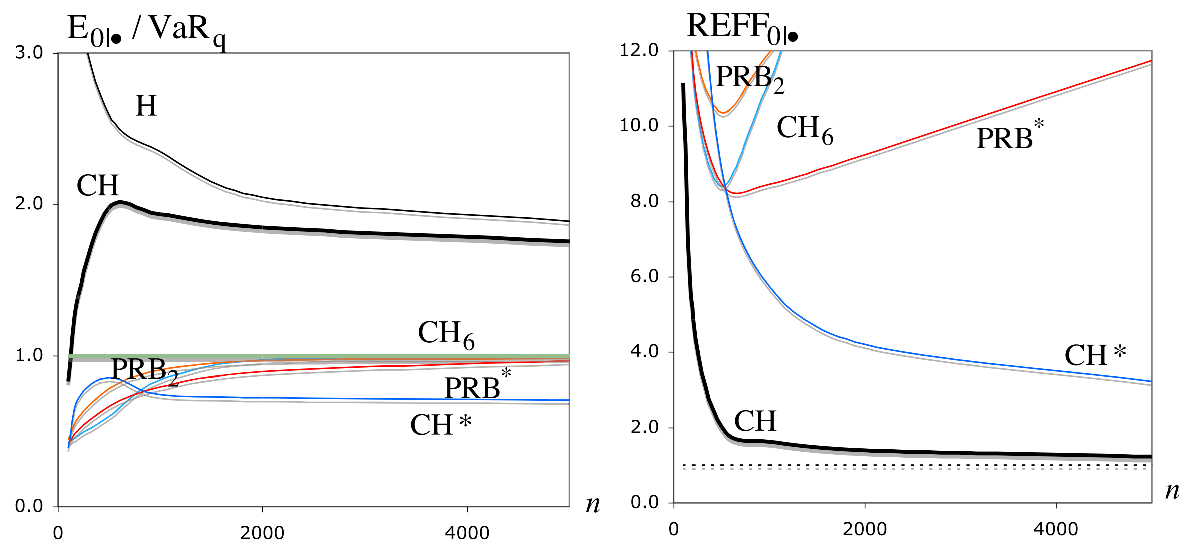

Figure 4. Normalized mean values (*left*) and REFF-indicators (*right*) of the VaR_q-estimators under study, at optimal levels, for $q = 1/n$ and $BURR_{1,-0.25}$ parents.

if we use the sub-sample containing all positive values. In the following, n denotes either the size of the sample or, if necessary, the size of the sub-sample of positive values. The heuristic algorithms are the following:

Algorithm 4.1 (Heuristic choice of k):

(1) Compute $\mathrm{S}(k)$ for $k = 1, 2 \ldots, n-1$.
(2) Next, obtain j, the minimum non-negative integer value, such that the rounded values of $S(k)$ to j decimal places are distinct. Define $a_k^{(\mathrm{S})}(j) = \mathrm{round}(\mathrm{S}(k), j)$, $k = 1, 2, \ldots, n-1$, the rounded values of $\mathrm{S}(k)$ to j decimal places.

(3) Consider the different sets of k-values associated with equal consecutive values of $a_k^{(\mathrm{S})}(j)$, obtained in Step (2). Set $k_{\min}^{(\mathrm{S})}$ and $k_{\max}^{(\mathrm{S})}$ the minimum and maximum values, respectively, of the set with the largest range. The largest run size is denoted by $l_{\mathrm{S}} := k_{\max}^{(\mathrm{S})} - k_{\min}^{(\mathrm{S})}$.
(4) Consider next all estimates, $\mathrm{S}(k)$, $k_{\min}^{(\mathrm{S})} \le k \le k_{\max}^{(\mathrm{S})}$, now with $d = 1$ extra decimal place, i.e. compute $a_k^{(\mathrm{S})}(j+1)$. Obtain the mode of $a_k^{(\mathrm{S})}(j+1)$ and denote by $\mathcal{K}_{\mathrm{S}}$ the set of associated k-values.
(5) Let n_{S} denote the number of elements of $\mathcal{K}_{\mathrm{S}}$. Take k^* as the $\lfloor (n_{\mathrm{S}}+1)/2 \rfloor$th ascending order statistic, i.e. the sample median of $\mathcal{K}_{\mathrm{S}}$. The adaptive estimate is the value $\mathrm{S}^* = \mathrm{S}(k^*)$.

Let us now assume that $\mathrm{S}(k) \equiv \mathrm{S}^{(p)}(k)$ also depends on the tuning parameter $0 \le p < 1/\xi$.

Algorithm 4.2 (Heuristic choice of k and p):

(1) Compute ξ^* given in (13) and consider $p_\ell = \ell/(16\xi^*)$, $\ell = 0(1)15$.
(2) For each $\ell = 0(1)15$, apply Steps (1) up to (3) from Algorithm 4.1 and obtain the associated largest run size, $l_{\mathrm{S}}^{(\ell)}$, $\ell = 0(1)15$.
(3) Choose $p_0 := \arg\max_\ell (l_{\mathrm{S}}^{(\ell)})/(16\xi^*)$.
(4) Apply Steps (4) and (5) from Algorithm 4.1 with $\mathrm{S}(k) \equiv \mathrm{S}^{(p_0)}(k)$, $k = 1, \dots, n-1$.

Remark 4.1: The previous algorithms are valid for $\mathrm{S}(k)$, either an EVI or a VaR semi-parametric estimator. For the VaR estimation, preliminary simulation studies suggested that the algorithms have better estimation performance if $\mathrm{S}(k)$ is the logarithm of the VaR estimator.

4.2. A small-scale Monte Carlo simulation study

To study the sensitivity of the algorithms in Section 4.1, a small-scale simulation procedure, with 5000 samples of size $n = 1000$, has been implemented. We have considered the models used in the simulation study in Section 3. For each model, we have applied the algorithms to the logarithm of the VaR-estimators (see Remark 4.1) based upon the Weissman estimator in (4) and the EVI-estimators H, CH, CH_p, PRB_p and $(\mathrm{PRB}^*, \mathrm{CH}^*)$, respectively given in (5), (8), (9), (11) and (13), and we have considered $q = 0.001$. For the $\mathrm{Q}_{\hat\xi}^{(q)}(k)$ estimators with $\hat\xi$ replaced by both CH_p or PRB_p, in (10) and (12) respectively, the choice of p was done accordingly with Algorithm 4.2. All results here presented are related to the normalized estimates $\ln(\mathrm{Q}_{\hat\xi}^{(q)}(k)) - \ln(\mathrm{VaR}_q)$, which should be close to 0. In Figure 5, we show box-and-whiskers plots of the simulated adaptive estimates of the normalized ln-VaR. The small square, usually inside the box, represents the corresponding simulated mean value. As expected, and mainly due to the asymptotic behaviour of the estimators, the Weissman-Hill VaR-estimator has usually a strong positive bias. The results clearly demonstrate the

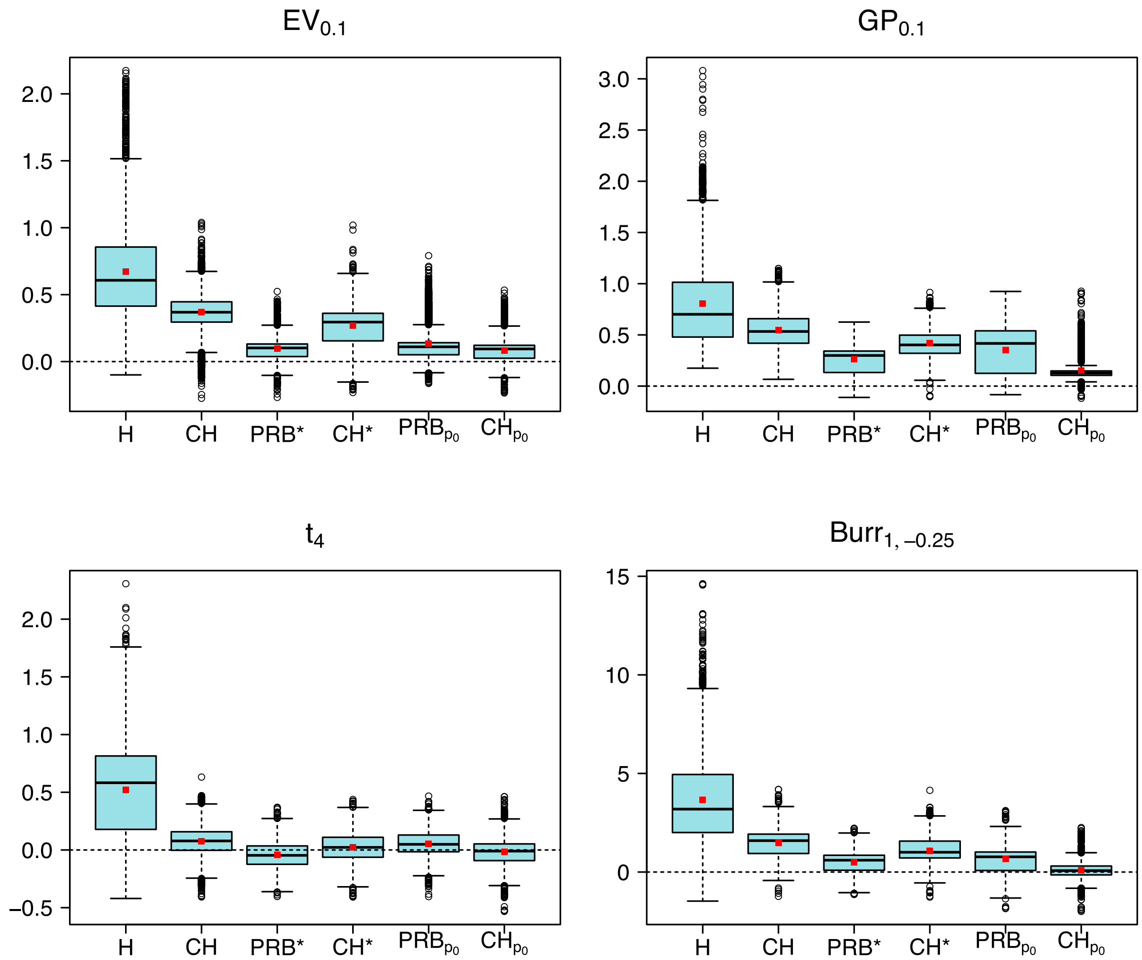

Figure 5. Box-and-whiskers plot of the normalized adaptive $\mathrm{VaR}_{0.001}$ estimates obtained through the use of the Algorithms, for the $\mathrm{EV}_{0.1}$ (*top-left*), the $\mathrm{GP}_{0.1}$ (*top-right*), the Student-t_4 (*bottom-left*) and the $\mathrm{Burr}_{1,-0.25}$ (*bottom-right*) models.

improved performance of the new VaR-estimators, when compared to the Weissman-Hill and even to the Weissman-CH estimators. The best performance is often achieved by the new VaR-estimator $\mathrm{Q}^{(q)}_{\mathrm{CH}_p}(k)$ in (10).

Table 5 presents the simulated RMSE of the adaptive normalized logarithm of the VaR-estimators under consideration in this work. For each model here considered, the smallest RMSE is written in **bold**. Among the considered estimators, $\mathrm{Q}^{(q)}_{\mathrm{CH}_p}(k)$ is the one providing, for all considered models, the smallest RMSE.

Table 5. Simulated RMSE of adaptive estimates obtained through Algorithms 4.1 and 4.2 for the different normalized VaR estimators under study.

	H	CH	PRB*	CH*	PRB_{p_0}	CH_{p_0}
$\mathrm{EV}_{0.1}$	0.7516	0.3967	0.1346	0.3075	0.1854	**0.1142**
$\mathrm{GP}_{0.1}$	0.9169	0.5736	0.2894	0.4369	0.4191	**0.1832**
Student t_4	0.6709	0.1516	**0.1202**	0.1287	0.1237	**0.1202**
$\mathrm{Burr}_{1,-0.25}$	4.2695	1.6269	0.7053	1.2468	0.9543	**0.5629**

4.3. An application to financial data: the BOVESPA stock index

We shall now apply the VaR-estimators under analysis to the daily BOVESPA Brazilian Stock Exchange Index (IBOVESPA). The data was collected from January 2, 2004 through June, 23, 2016, from www.ipeadata.gov.br, with a size $n = 3082$.

We have performed Engle's ARCH test for the presence of ARCH effects in the log-returns' series (see [37,38]), and the ARCH/GARCH model was not rejected. The estimated GARCH(1,1) model was,

$$y_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = 0.0622088 + 0.910216\sigma_{t-1}^2 + 0.0695197 y_{t-1}^2,$$

where y_t are the negative log-returns, ϵ_t the white noise disturbance, and σ_t^2 the variance forecast. We have considered the standardized log-returns, $y_t^* = y_t/\sigma_t$. There was next no significant evidence in support of GARCH effects for the standardized return series (p−value $> 25\%$). A stationary setup for the standardized log-returns has been assumed, and an analysis of those standardized log-returns, y^*, has been performed. Working with $n_0 = 1490$ positive values of the negative log-returns of the IBOVESPA data set, the associated second-order estimates are $\hat{\rho} = -0.723$ and $\hat{\beta} = 1.027$. In Figure 6 we present the sample path of the estimates, as a function of k, provided by the log $\mathrm{VaR}_{0.001}$ estimators under consideration.

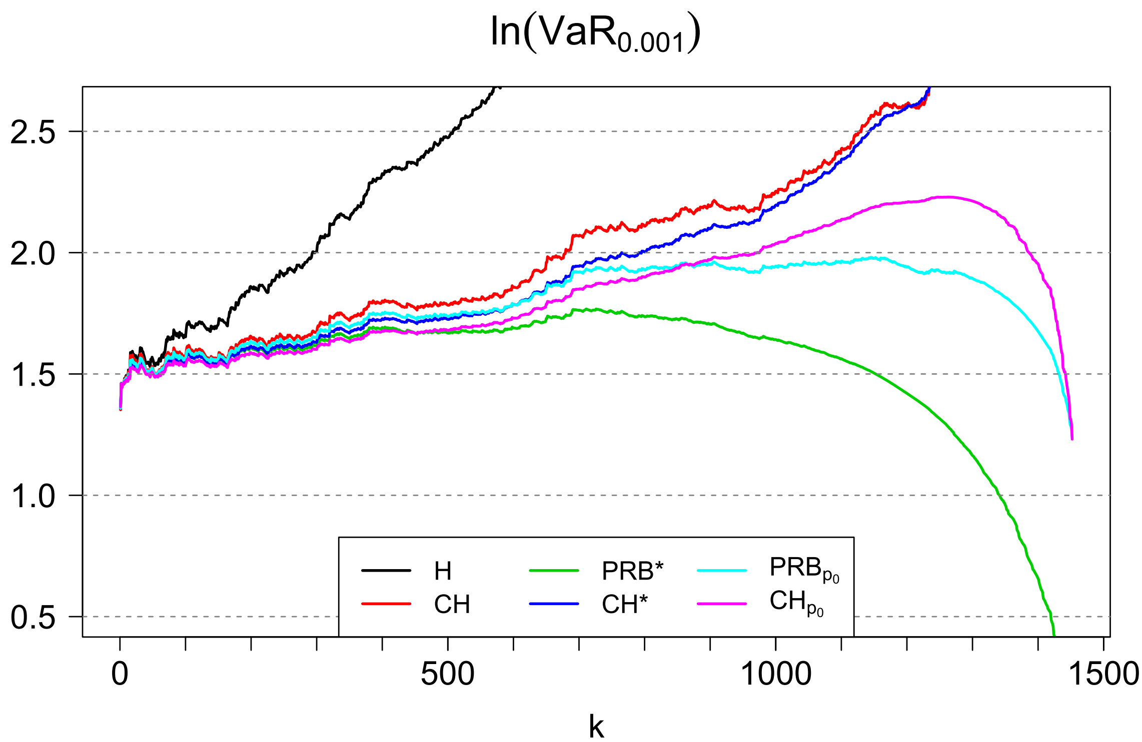

Figure 6. Logarithm of the $\mathrm{VaR}_{0.001}$-estimates, for IBOVESPA data.

As expected, the Weissman-Hill VaR estimator exhibits a strong positive bias. The remaining VaR estimators exhibit more stable sample paths as functions of k, due to their smaller bias. In Table 6, the adaptive estimates of ln-$\mathrm{VaR}_{0.001}$ provided by the algorithms in Section 4.1 are presented, together with the chosen values of p_0, whenever applicable,

Table 6. Adaptive estimates of the logarithm of the $\mathrm{VaR}_{0.001}$ with the related parameters p and k for the different estimators here considered.

	H	CH	PRB*	CH*	PRB_{p_0}	CH_{p_0}
n_T	86	231	573	252	480	261
p_0	–	–	–	–	0.555	1.387
k^*	124	482	610	430	930	502
$\ln(\widehat{\mathrm{VaR}}_{0.001})$	1.707	1.789	1.695	1.714	1.932	1.680

and the threshold k^*. Notice that, due to the existence of a small stability region for small k, Algorithm 4.1 provided an adaptive Weissman-Hill VaR estimate very close to the estimates provided by most of the remaining VaR estimators. Also, the existence of a second large stability region of the estimates provided by Weissman-PRB_{p_0} leads to what we think to be an over-estimation of the $\mathrm{VaR}_{0.001}$. Although the algorithm does not work well with the PRB_p estimator when applied to the data set under study, it works quite well for all other estimators, in particular, for the CH_p estimator. Therefore, for a best choice of the estimate, we must look to the estimates provided by competitor estimators, based on a similar and large k^*, and discard the unlikely ones, in this case the PRB_{p_0} and PRB*. The consideration of other case-studies and simulated samples, out of the scope of this article, would possibly help to clarify the problem, and to lead us to the suggestion of an adaptive choice based on the minimization of estimated RMSE rather than sample-path stability.

5. Overall remarks

- It has been clear for a long time that Weissman-Hill VaR-estimators lead to a high over-estimation of VaR. The MVRB CH VaR-estimators are in most cases a nice alternative, more regarding RMSE rather than bias, but the use of the extra tuning parameter $p \in \mathbb{R}$ and the $\mathrm{CH}_p \equiv \mathrm{MO}_p$ methodology can provide a much more adequate VaR-estimation. But we cannot forget the PRB VaR-estimation, which despite of not generally RB, can work even better than the CH_p VaR-estimation.
- The obtained results lead us to strongly advise the use of the quantile estimators $\mathrm{Q}_{\mathrm{CH}_p}$ and $\mathrm{Q}_{\mathrm{PRB}_p}$ for an adequate choice of p, provided by Algorithm 4.2 in Section 4.1, a reliable heuristic procedure related to sample path stability.
- For small values of $|\rho|$ the use of $\mathrm{Q}_{\mathrm{CH}_p}$, with an adequate value of p, always enables a reduction in RMSE regarding the Weissman-Hill estimator and even the CH VaR_q-estimator. Moreover, the bias is also reduced comparatively with the bias of the Weissman-Hill VaR-estimator with the obtention of estimates closer to the target value VaR_q.
- The patterns of the estimators' sample paths are always of the same type, in the sense that for all k the VaR_q-estimators, $\mathrm{Q}^{(q)}_{\mathrm{CH}_p}$ and $\mathrm{Q}^{(q)}_{\mathrm{PRB}_p}$ decrease as p increases.
- The simulation results obtained for a reasonable large class of heavy-tailed models, and partially presented in Section 3, enable us to advance that the high stability of sample paths achieved with the RB-MO_p EVI-estimates for large values of p, out of the scope of asymptotic normality (see [8]), is no longer achieved for these RB-MO_p VaR-estimates. Anyway, the proposed estimators perform better than the classical one, i.e. the one where the H EVI-estimator is used, as well as the one where the RB-CH EVI-estimator is considered as the basis of the VaR_q-estimation. Moreover, for most of the

simulated models the RB-MO$_p$ EVI-estimators exhibit a much more stable mean value pattern, as a function of k, the number of upper OSs used, and a smaller RMSE (higher REFF, comparatively with the CH VaR-estimation).

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

Research partially supported by FLAD, Fundação Luso-Americana para o Desenvolvimento, COST Action IC1408, and National Funds through FCT—Fundação para a Ciência e a Tecnologia, projects UID/MAT/0006/2019, UIDB/0006/2020 (CEAUL) and UID/MAT/0297/2019, UIDB/0297/2020 (CMA/UNL); European Cooperation in Science and Technology.

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