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Markov invariant dynamics

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ABSTRACT

We consider, $\mathcal{M}(I)$, a certain class of Markov interval maps with domain contained in the interval I, whose associated transition matrix A_f, necessarily primitive, plays a crucial role in the study of the underlying dynamics.
We study the problem of deciding when a restriction $g := f|_J$ of a map $f \in \mathcal{M}(I)$ to a subset $J \subset I$ is in the class $\mathcal{M}([J])$, where $[J]$ is the minimal closed interval containing J. We establish natural conditions on $J \subset I$ so that $g = f|_J$ is a Markov map. Then we tackle the central problem of deciding when the matrix A_g is primitive in this framework. We are able to enumerate the sets J, satisfying the referred conditions, through a systematic process of elimination of the rows/columns of the state splitting of A_f associated to the so called removable states, which ensures the primitivity of the matrices A_g.

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1. Introduction

In this paper, we consider Markov discrete dynamical systems arising from the iteration of interval maps, belonging to a particular class denoted by $\mathcal{M}(I)$ (Definition 2.1). This class of maps formalize and establish certain conditions which are common in discrete dynamical systems in order to provide several basic results and to settle a clear context in the interplay between iterated maps of the interval, symbolic dynamics and C*-algebras. Weakening some of those conditions will lead to non-trivial generalizations of our previous work on these subjects, see [3] or [4] and the references therein. If $f \in \mathcal{M}(I)$ then $\mathrm{dom}(f) = \bigcup_{j=1}^{n} I_j \subset I$, being $I_1, ..., I_n$ a Markov partition for f, for some natural n and $\mathrm{im}(f) = I$. The maximal invariant set, contained in I, is typically a Cantor set denoted by Ω_f. Thus, the discrete dynamical system can be properly defined by the pair (Ω_f, f). The states of this system are identified with the intervals which constitute the Markov partition for f and there is a $0-1$ matrix, A_f, canonically determined by the action of f on its Markov partition. Therefore, it is natural to consider the subshift of finite type $\left(\Sigma_{A_f}, \sigma\right)$ to codify the orbit structure of (Ω_f, f), in fact the systems are topologically conjugated. Consider now a collection of closed subintervals J_i, $i = 1, ..., m$ so that $J = \cup_{i=1}^{m} J_i \subset I$. Let $[J]$ be the minimal interval satisfying $J \subset [J]$. One of the main objective on our paper is to determine the conditions for which the map $g := f_{|J}$ belongs to $\mathcal{M}([J])$. This problem is related with the study of the invariant Cantor sets for $f \in \mathcal{M}(I)$ and the classification of the Markov subsystems (Ω_g, g) of (Ω_f, f), with $\Omega_g \subset \Omega_f$, $g := f_{|J}$. In fact, if $J \subset I$ satisfies certain conditions so that $g := f_{|J}$ belongs to $\mathcal{M}([J])$ then the maximal invariant Cantor set for g, Ω_g, is an invariant Cantor set for f, that is, $\Omega_g \subset \Omega_f$ and $f(\Omega_g) = \Omega_f$. Moreover, in this case, $\Sigma_{A_g} \subset \Sigma_{A_g}$ and $\sigma\left(\Sigma_{A_g}\right) = \Sigma_{A_g}$.

The method we use, to solve the problem, is to transform a refined Markov partition for the given map f, removing appropriate intervals, and use it to support a newly defined interval map g, on J, in the spirit of our previous works and [4] [5]. The main problem to solve is to guarantee the primitivity of the matrix A_g, associated with g.

The change of Markov partitions and its impact on the underlying dynamics is an important issue in dynamical systems and its applications, see for example [13] or [1]. In [11], Raith study the influence of small changes, imposed in the partition of expanding piecewise monotone interval maps, on the Hausdorff dimension of invariant sets. In particular, he shows that, given $t \in [0, 1]$, there is an invariant set for the interval map with Hausdorff dimension equal to t. Following the same perspective, in [12], Raith studies restrictions of an interval map f with given positive topological entropy, less than the topological entropy of f. A similar result was presented earlier, by Krieger for subshifts of finite type. He proves, in [7], that given a subshift of finite type, Σ_A, with A primitive and a minimal expansive homeomorphism on a Cantor set with lesser topological entropy than the topological entropy of Σ_A then the referred homeomorphism is topologically conjugated to a subsystem of Σ_A. The minimality condition can be replaced by a certain condition on low period orbits of the homeomorphism ([7]). As a consequence, if there is a subshift of finite type $\Sigma_{\tilde{A}}$ so that the topological entropy of $\Sigma_{\tilde{A}}$ is less than the topological entropy of Σ_A then we expect to find subsystems of Σ_A topologically conjugated to $\Sigma_{\tilde{A}}$.

Here, we go further with respect to Markov systems since we are able to enumerate certain Markov subsystems of a given one. This is accomplished identifying the rows/columns which may be removed from a given primitive matrix still getting a primitive matrix. The removal of such rows/columns of a transition matrix, A_f, corresponds to the removal of intervals from the Markov partition of f, getting a restriction of f which belongs to $\mathcal{M}$. Note that each row/column of A_f is associated to a Markov state and to an interval in the Markov partition of f.

The singular points of $f \in \mathcal{M}(I)$ are the boundary points of its Markov partition. The refinements of the Markov partition of f using points in the orbits of the singular points of f does not affect the dynamical system (Ω_f, f), since the refinement of the Markov partition for these points corresponds to state splitting operations on the corresponding transition matrix, see [4]. If the refinement is made removing other type of orbits, or intervals, then the obtained dynamical system is usually not topologically equivalent to the original, see [13] and [5].

In [5] we have considered the dynamics of interval maps whose maximal invariant set is a non trivial Cantor set, that is, a *cookie-cutter* set with non-integer Hausdorff dimension, [6]. In this case, the map f must possess a nontrivial escape set. Removing appropriate intervals from refined Markov partitions of a map f, as referred above, give us restrictions of the map f with nontrivial escape sets - at least the removed intervals are contained in the escape set. When this removal does not affect the Markov property, neither the primitivity of the new transition matrix, the obtained restrictions belong to the class $\mathcal{M}$, fulfilling our objective. Therefore, we aim to determine which conditions guarantee the Markov and primitivity properties of the referred restrictions.

The paper is organized as follows. In the section 2.1 we introduce all the preliminary notions, definitions and auxiliary results. In particular, we explain in detail the refinements of the Markov partitions and the use of the state splitting transformation, in the context of the subshift of finite type and in the context of interval maps.

We establish some conditions for g to be Markov. In particular, if the domain of g is a refinement of the Markov partition of f, using pre-images of the singular points of f, then the Markov partition of g is obtained through finite unions of f-cylinders. This is accomplished in the section 3, in the Lemma 3.2. Next, the main problem to deal with is to determine which conditions guarantee that the transition matrix A_g associated with the partition of the domain of g will be primitive, which ensures, together with the conditions of the Lemma 3.2, that $g \in \mathcal{M}([J])$. This problem is considered in the section 4, where the main result, Theorem 4.10, is stated and proven.

Finally, we indicate a method which allows, given a particular map $f \in \mathcal{M}(I)$, to find domains $J \subset I$ for which $g := f_{|J} \in \mathcal{M}([J])$ and this can be performed in a systematic way, using the concept of removable state (Definition 4.6). In the Theorem 4.10, a necessary and sufficient condition is given for a state to be removable. The candidates for J

are built removing intervals which are Markov for the refined partitions and which are associated with the removable states.

Using the concept of degree of a partition - basically it measures the degree of the refinement of a Markov partition (Definition 3.1) - we describe a method which allows to enumerate the Markov subsystems of a given Markov system as consequence of the Theorem 4.10. This method can be performed in the abstract context of subshifts of finite type or in the context of Markov interval maps. Moreover, these subsystems, with fixed degree, can be arranged in a partially ordered set. In the section 5, we apply our results for the family of β-shifts, considering the cases $\beta = 2$, $\beta = \frac{1+\sqrt{5}}{2}$ for the degrees, $r = 1$, $r = 2$ and $r = 3$. This is a topic which we plan to address our attention in a future work, for a complete classification.

2. Preliminaries

2.1. Markov maps and escape sets

We consider the class of interval maps whose domain is based on partitions of an interval I, as in [5].

Definition 2.1. Let $I \subset \mathbb{R}$ be an interval. A map f is in the class $\mathcal{M}(I)$ if it satisfies the properties $(P1)$, $(P2)$, $(P3)$, $(P4)$, presented below:

- (P1) [Existence of a finite partition in the domain of f] There is a partition $C = \{I_1, ..., I_n\}$ of closed intervals with $\#(I_i \cap I_j) \leq 1$ for $i \neq j$, $\mathrm{dom}(f) = \bigcup_{j=1}^{n} I_j \subset I$ and $\mathrm{im}(f) = I$.
- (P2) [Markov property] For every $i = 1, ..., n$ the set $f(I_i) \cap \left(\bigcup_{j=1}^{n} I_j\right)$ is a non-empty union of intervals from C.
- (P3) [Expansive map] $f_{|I_j} \in \mathcal{C}^1(I_j)$, monotone and $|f'_{|I_j}(x)| > b > 1$, for every $x \in I_j, j = 1, ..., n$, and some b.
- (P4) [Aperiodicity] For every interval $J \in C$ there is a natural number q_J such that $\mathrm{dom}(f) \subset f^{q_J}(J)$.

The minimal partition C satisfying the Definition 2.1 is denoted by C_f. We remark that the Markov property $(P2)$ allows us to encode the transitions between the intervals in the so-called (Markov) transition $n \times n$ matrix $A_f = (a_{ij})$, defined as follows:

$$a_{ij} = \begin{cases} 1 \text{ if } f(\mathring{I}_i) \supset \mathring{I}_j, \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

where $\mathring{J}$ denotes the interior of a set J.

Let Γ be an ordered set

$$\Gamma = \{c_0^-, c_0^+, c_1^-, c_1^+, ..., c_{n-1}^-, c_{n-1}^+, c_n^-, c_n^+\}, \tag{2}$$

of real numbers, with $n \in \mathbb{N}$, such that

$$c_0^- \leq c_0^+ < c_1^- \leq c_1^+ < c_2^- \leq ... < c_{n-1}^- \leq c_{n-1}^+ < c_n^- \leq c_n^+. \tag{3}$$

The boundary points of a partition C arising from a map $f \in \mathcal{M}(I)$, as in $(P1)$, constitute naturally a set Γ as in (2) satisfying (3). In fact, considering Γ as in (2), the related partition $C = \{I_1, ..., I_n\}$, is given by

$$I_1 = \left[c_0^+, c_1^-\right], \ ..., \ I_j = \left[c_{j-1}^+, c_j^-\right], ..., I_n = \left[c_{n-1}^+, c_n^-\right]. \tag{4}$$

We consider also the collection of open intervals $\{E_1, ..., E_{n-1}\}$, with

$$E_0 = \left]c_0^-, c_0^+\right[, \ E_1 = \left]c_1^-, c_1^+\right[, \ ..., \ E_{n-1} = \left]c_{n-1}^-, c_{n-1}^+\right[, \ E_n = \left]c_n^-, c_n^+\right[, \tag{5}$$

so that

$$I := [c_0^-, c_n^+] = \left(\cup_{j=1}^{n} I_j\right) \bigcup \overline{\left(\cup_{j=1}^{n-1} E_j\right)}. \tag{6}$$

In this case, the set Γ is called set of the *singular points of* f and, from the Markov property, Γ satisfies $f(\Gamma) \subset \Gamma$.

A map $f \in \mathcal{M}(I)$ uniquely determines (together with the minimal partition $C_f = \{I_1, ..., I_n\}$):

(i) The f-invariant set

$$\Omega_f := \{x \in I : f^k(x) \in \text{dom}(f) \text{ for all } k = 0, 1, ...\}.$$

(ii) The collection of open intervals $\{E_0, E_1, ..., E_{n-1}, E_n\}$, such that

$$I \setminus \bigcup_{j=1}^{n} I_j = \bigcup_{j=1}^{n-1} E_j.$$

(iii) The transition matrix $A_f = (a_{ij})_{i,j=1,..,n}$.

The intervals $I_1, ..., I_n$ are called the *Markov intervals* of f whereas $E_1, \ldots, E_{n-1}$, are called the *escape intervals* (possibly empty) of f.

Note that Ω_f, a Cantor set, is the set of points that remain in dom(f) under iteration of f and the open set

$$E_f := I \setminus \Omega_f = \bigcup_{k=0}^{\infty} f^{-k} \left(\bigcup_{j=1}^{n-1} E_j\right) \tag{7}$$

is usually called the *escape set*. Every point in E_f will eventually fall, under iteration of f, into the interior of some interval E_j (where f is not defined) and the iteration process ends. We may say that x is in the escape set E_f of f if and only if there is $k \in \mathbb{N}$ such that $f^k(x) \notin \mathrm{dom}(f)$. If $c_j^- = c_j^+$, for some j, then $E_j = \emptyset$ and c_j is a singular point, either a critical point or a discontinuity point of f. Note that if $c_j^- = c_j^+$, the notation c_j^-, c_j^+ represents the side limits of the point c_j, that is,

$$c_j^{\pm} = \lim_{\varepsilon \to 0^{\pm}} c_j + \varepsilon.$$

However, if $c_j^- < c_j^+$, that is, $E_j \neq \emptyset$, we use the notation $c_j^{-,-}$, $c_j^{-,+}$, $c_j^{+,-}$, $c_j^{+,+}$ to denote the corresponding side limits, that is,

$$c_j^{\pm,\pm} = \lim_{\varepsilon \to 0^{\pm}} c_j^{\pm} + \varepsilon.$$

The *orbit* of a point $x \in \Omega_f$ is the set $\mathrm{orb}_f(x) := \left\{ f^k(x) : k \in \mathbb{N} \right\}$. We will consider the equivalence relation R_f, on Ω_f, defined by

$$R_f := \{(x,y) : \ f^n(x) = f^m(y) \text{ for some } n, m \in \mathbb{N}_0\}. \tag{8}$$

The relation R_f is a countable equivalence relation in the sense that the equivalence class $R_f(x)$ of $x \in I$, is a countable set. We denote $x \sim y$ whenever $(x,y) \in R_f$, and $R_f(x)$ is called the *generalized orbit of* $x \in I$.

2.2. Symbolic dynamics for Markov maps and escape sets

In the following we encode the orbit (under f) of a point via symbolic dynamics.

Definition 2.2. The *address map* $ad : \bigcup_{j=1}^{n} \mathring{I}_j \to \{1, 2, ..., n\}$ is defined as follows: $ad(x)=i$ if $x \in \mathring{I}_i$, where $\mathring{I}_j$ denotes the interior of I_j. The *itinerary map* $it_f : \bigcup_{j=1}^{n} \mathring{I}_j \to \{1, 2, ..., n\}^{\mathbb{N}_0}$ is defined as $it_f(x)=(ad(f^n(x))_{n=0,1,...}$.

Let Σ_{A_f} be the subspace of $\{1, 2, \ldots, n\}^{\mathbb{N}}$ given by $it_f(\cup_{j=1}^{n} I_j)$, which is invariant under the *shift map* defined as

$$\begin{gathered} \sigma : \{1, 2, \ldots, n\}^{\mathbb{N}} \to \{1, 2, \ldots, n\}^{\mathbb{N}} \\ \sigma(i_1 i_2 \ldots) = (i_2 i_3 \ldots). \end{gathered} \tag{9}$$

We will use just σ instead of $\sigma_{|\Sigma_{A_f}}$. Naturally,

$$\Sigma_{A_f} = \left\{ i_1 i_2 \ldots : a_{i_k i_{k+1}} = 1, \, k \in \mathbb{N} \right\}$$

and $it_f \circ f = \sigma \circ it_f$. The pair $\left(\Sigma_{A_f}, \sigma\right)$ is a subshift of finite type, characterized by the matrix A_f. Since f is expansive the dynamical system (Ω_f, f) is topologically conjugated to $\left(\Sigma_{A_f}, \sigma\right)$ via the itinerary map it_f.

A sequence in $\{1, 2, \ldots, n\}^{\mathbb{N}}$ is called *admissible*, with respect to f, or f-*admissible*, if it occurs as an itinerary of some point x in Ω_f, *i.e.*, if it belongs to Σ_{A_f}. An *admissible word* is a finite subsequence of some admissible sequence. The set of admissible words of size k is denoted by $\mathcal{W}_k$, or by $\mathcal{W}_{A_f,k}$, depending on the context. The words of size 1 coincide with the symbols in the alphabet $\{1, 2, ..., n\}$. Each symbol correspond to a *state* of the system $\left(\Sigma_{A_f}, \sigma\right)$ and also of the system (Ω_f, f), at some time instant.

A periodic sequence, or a *cycle*, in Σ_{A_f} is an admissible sequence which is the repetition *ad infinitum* of a certain admissible word

$$\gamma = \gamma_1 ... \gamma_l \gamma_1 ... \gamma_l ... = (\gamma_1 ... \gamma_l)^{\infty} .$$

The set of cycles is denoted by $\Sigma_{A_f}^{per}$. The size of the cycle γ is the minimal natural number l so that $\gamma = (\gamma_1 ... \gamma_l)^{\infty}$ and the word $\gamma_1 ... \gamma_l$ is called *representative* of the cycle. Note that any of the words $\gamma_2 ... \gamma_l \gamma_1, \ldots, \gamma_l \gamma_1 ... \gamma_{l-1}$ are representative of the same cycle. A word is called *simple* if every symbol in the word is distinct, with the same meaning we use the notion *simple cycle*. A cycle γ is simple if its representative words cannot be decomposed as concatenation or juxtaposition of representative words of smaller cycles.

To each admissible word $\xi \in \mathcal{W}_k$, $k > 0$, corresponds a special type of interval denoted by I_ξ and called an f-*cylinder*. If $\xi = \xi_1 ... \xi_k$ then the interval I_ξ is defined

$$I_\xi = I_{\xi_1 ... \xi_k} := \left\{ x \in I_{\xi_1} : f(x) \in I_{\xi_2}, f^2(x) \in I_{\xi_3}, ..., f^{k-1}(x) \in I_{\xi_k} \right\} .$$

Note that if $k > 1$ then $f\left(I_{\xi_1 ... \xi_k}\right) = I_{\xi_2 ... \xi_k}$. Moreover, the f-cylinders are obtained as pre-images of the intervals partition

$$I_{\xi_1 ... \xi_k} = I_{\xi_1} \cap f^{-1}\left(I_{\xi_2 ... \xi_k}\right) ,$$

see the Proposition 2 in [3].

Associated to each map $f \in \mathcal{M}(I)$ with transition matrix $A_f = (a_{ij})$ we define a digraph G_{A_f} with vertex set C_f and edge set

$$\left\{ (I_i, I_j) \in C_f^2 : a_{ij} = 1 \right\} .$$

From this definition, the adjacency matrix of G_{A_f} is precisely the transition matrix A_f. Therefore, we have a natural correspondence between the orbits of f, the sequences in Σ_{A_f} and the paths on G_{A_f}. The cycles of Σ_{A_f} and therefore the periodic orbits of f, correspond to closed paths on G_{A_f}.

The *follower* of a state i is any state j so that $a_{ij} = 1$. The *predecessor* of a state j is any state i so that $a_{ij} = 1$. We then define the set of followers by

$$fol(i) := \{ j : a_{ij} = 1 \}$$

and the set of predecessors

$$pre(j) := \{i : a_{ij} = 1\}.$$

The *state splitting* applied to a certain state $s \in \{1, 2, ..., s, ..., n\}$ is an operation which gives us a new alphabet

$$\{1, 2, ..., s-1, s+1, ..., n\} \cup \{(sj) : a_{sj} = 1\},$$

in which the state s is divided (splitted) in the states represented by (sj), for every j so that $a_{sj} = 1$, that is, the state s with its followers j. The transitions in the new alphabet are codified in a matrix denoted by $\Phi_s(A) = (b_{ij})$, indexed by

$$\{1, 2, ..., s-1, s+1, ..., n\} \cup \{(sj) : a_{sj} = 1\}$$

and defined by

$$\begin{cases} b_{ij} = a_{ij} \\ b_{(si)j} = \delta_{ij} \\ b_{i(sj)} = a_{is} \\ b_{(si)(sj)} = \delta_{is}. \end{cases} \tag{10}$$

This is an adaptation of the state splitting definition found in [8] or [9]. The state splitting of A, on s, $\Phi_s(A)$ has a corresponding transformation, with respect to f, in writing a f-cylinder associated with s as the union of smaller cylinders, in this case

$$I_s = \bigcup_{j \in fol(s)} I_{sj}.$$

Moreover, the subshift $\Sigma_{\Phi_s(A)}$ is topologically conjugated to Σ_A, see [8].

Matrices A for which there exists a positive integer m such that all the entries of A^m are non-zero are called primitive. We note that the matrix A_f is primitive (thus irreducible) whenever $f \in \mathcal{M}(I)$ (Definition 2.1, property $P4$).

Remark 1. The inverse operation of the state splitting is the amalgamation of a set of states. The states which can be amalgamated are those which have disjoint follower sets and identical predecessor sets. A matrix is in total amalgamation form if there are no states satisfying these conditions, see [8].

Example 2.3. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

with respect to the alphabet $\{1,2,3\}$. The state splitting applied to the state 1 gives us the matrix

$$\Phi_1(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ with alphabet } \{(12), 2, 3\}.$$

On the other hand, the state splitting applied to the state 3 gives us the matrix

$$\Phi_3(A) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \text{ with alphabet } \{1, 2, (31), (32), (33)\}.$$

2.3. Non minimal Markov partitions for f

Let $f \in \mathcal{M}(I)$, with domain $\cup_{j=1}^{n} I_j \subset I$, transition matrix A_f and invariant set Ω_f. Consider the set $\mathcal{W}_k$ of f-admissible words of size $k \geq 1$. Let $k \geq 1$ and consider f^k obtained iterating k times the map f. The map f^k belongs also to $\mathcal{M}(I)$. In fact, the domain of f^k is contained in the domain of f, which itself is contained in the interval I. The invariant Cantor set is $\Omega_{f^k} = \Omega_f$ and the minimal Markov partition for f^k is given by

$$C_{f^k} = \{I_\xi : \xi \in \mathcal{W}_k\},$$

which is a refinement of $C_f = \{I_1, ..., I_n\}$, since $I_i = \cup_{i\eta \in \mathcal{W}_k} I_{i\eta}$. The Markov property for f^k arise from

$$f(I_{\xi_1 \ldots \xi_r}) = I_{\xi_2 \ldots \xi_r} = \cup_{j \in fol(\xi_r)} I_{\xi_2 \ldots \xi_r j}. \tag{11}$$

The collection C_{f^k}, with natural $k \geq 1$, is also a Markov partition for f, nevertheless is not minimal with respect to f. When used with this perspective, with the intervals in C_{f^k} enumerated by the f-admissible words of size k, we denote it by $C_f^{(k)} := C_{f^k}$. In particular, $C_f^{(1)} = C_f$.

Example 2.4. Consider the α, β-family of maps studied in [10], $f(x) = \beta x + \alpha \bmod 1$. If β is the maximal solution of $\beta^3 - \beta^2 - \beta - 1 = 0$ and $\alpha = \beta^{-3}$, $\beta = 1.83929\ldots$, $\alpha = 0.160713\ldots$, then $f \in \mathcal{M}([0,1])$ and $C_f = \{I_1, I_2, I_3\}$, with $I_1 = [0, \alpha], I_2 = [\alpha, \alpha\beta + \alpha], I_3 = [\alpha\beta + \alpha, 1]$. The previous equation, on β, arises from the eigenvalue equation $\det(A_f - I\beta) = 0$, with the transition matrix given by

$$A_f = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The value of α can be determined through the Perron right eigenvector of A_f, see [10], or more directly from the periodicity condition of the orbit of 0^+, $orb_f(0^+) = \{0^+, \alpha, \alpha\beta + \alpha\}$, that is,

$$f(\alpha\beta + \alpha) = 0^+ \Leftrightarrow \alpha\left(\beta^2 + \beta + 1\right) - 1 = 0 \Leftrightarrow \alpha = \frac{1}{\beta^2 + \beta + 1} = \beta^{-3}.$$

The map f^2 belongs also to $\mathcal{M}([0,1])$ and its minimal Markov partition can be written in terms of the f-cylinders associated with f-admissible words of size 2, that is, $C_{f^2} = \{I_{12}, I_{23}, I_{31}, I_{32}, I_{33}\}$. Each $I_{\xi_1\xi_2}$ is the interval of points x which belong to I_{ξ_1} so that $f(x) \in I_{\xi_2}$. For this particular case, $I_{12} = [0, \alpha]$, $I_{23} = [\alpha, \alpha\beta + \alpha]$, $I_{31} = \left[\alpha\beta + \alpha, \beta^{-1}\right]$, $I_{32} = \left[\beta^{-1}, \beta^{-1}(1 + \alpha\beta)\right]$, $I_{33} = \left[\beta^{-1}(1 + \alpha\beta), 1\right]$. Moreover, note that $I_1 = I_{12}$, $I_2 = I_{23}$ (there are only one word of size 2 beginning with symbol 1 or 2) and $I_3 = I_{31} \cup I_{32} \cup I_{33}$. The map f^2 is given explicitly by

$$f^2(x) = \begin{cases} \beta^2 & \text{if } x \in I_{12} \\ \beta^2 x + \alpha & \text{if } x \in I_{23} \\ \beta^2 x + \alpha\beta & \text{if } x \in I_{31} \\ \beta^2 x + \alpha\beta & \text{if } x \in I_{32} \\ \beta^2 x + \alpha\beta + \alpha & \text{if } x \in I_{33}. \end{cases}$$

The transition matrix associated to f, with respect to the partition $C_f^{(k)}$, is defined by

$$A_f^{(k)} := (a_{\xi,\eta})_{\xi,\eta \in \mathcal{W}_k}, \text{ where } a_{\xi,\eta} := \begin{cases} 1 & \text{if } f(\mathring{I}_\xi) \supset \mathring{I}_\eta, \\ 0 & \text{otherwise.} \end{cases}$$

If $k = 1$, then $A_f^{(k)}$ is the Markov transition matrix defined in eq. (1). Thanks to (11) we have

$$a_{\xi,\eta} = \begin{cases} 1 & \text{if } \xi_2 \ldots \xi_k = \eta_1 \ldots \eta_{k-1} \\ 0 & \text{if } \xi_2 \ldots \xi_k \neq \eta_1 \ldots \eta_{k-1}, \end{cases} \quad \text{where } \xi = \xi_1 \ldots \xi_k, \eta = \eta_1 \ldots \eta_k \in \mathcal{W}_k.$$

The matrix $A_f^{(k)}$ may also be obtained through systematic state splitting applied to every state $i \in \mathcal{W}_k$, see [4].

Example 2.5. Consider the α, β-family of maps, $f(x) = \beta x + \alpha \bmod 1$, referred in the Example 2.4 above, with β being the maximal solution of $\beta^3 - \beta^2 - \beta - 1 = 0$ and

$\alpha = \beta^{-3}$, $\beta = 1.83929\ldots$, $\alpha = 0.160713\ldots$ with $C_f = \{I_1, I_2, I_3\}$, with $I_1 = [0, \alpha], I_2 = [\alpha, \alpha\beta + \alpha], I_3 = [\alpha\beta + \alpha, 1]$. The map f, considered on the refined partitions

$$C_f^{(2)} = \{I_{12}, I_{23}, I_{31}, I_{32}, I_{33}\},$$
$$C_f^{(3)} = \{I_{123}, I_{231}, I_{232}, I_{233}, I_{312}, I_{323}, I_{331}, I_{332}, I_{333}\},$$

gives the following transition matrices

$$A_f^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad A_f^{(3)} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

which are obtained by the state splitting on $1, 2, 3$, in the case $A_f \to A_f^{(2)}$ and by the state splitting on the states $12, 23, 31, 32, 33$, in the case $A_f^{(2)} \to A_f^{(3)}$. In other words,

$$A_f^{(2)} = \Phi_3 \circ \Phi_2 \circ \Phi_1 (A_f),$$
$$A_f^{(3)} = \Phi_{33} \circ \Phi_{32} \circ \Phi_{31} \circ \Phi_{23} \circ \Phi_{12} \left(A_f^{(2)}\right).$$

Note that, since the follower set of 1 and 2 are singular we have, in fact, $A_f^{(2)} = \Phi_3 (A_f)$, since Φ_1 and Φ_2 does not affect the form of the splitted matrix. The same reasoning lead us to

$$A_f^{(3)} = \Phi_{33} \circ \Phi_{23} \left(A_f^{(2)}\right).$$

3. Conditions for Markov invariant dynamics

Given a closed set $J \subset I \subset \mathbb{R}$, we denote $[J] = [\min J, \max J]$. Here, we search for conditions, given $J \subset I$, to have $g := f_{|_J} \in \mathcal{M}([J])$.

Let

$$\widetilde{\Gamma} = \{d_0^-, d_0^+, d_1^-, d_1^+, ..., d_{m-1}^-, d_{m-1}^+, d_m^-, d_m^+\}$$

be an ordered set of real numbers so that $[d_0, d_m] \subset I$ and

$$d_0^- \leq d_0^+ < d_1^- \leq d_1^+ < d_2^- \leq ... < d_{m-1}^- \leq d_{m-1}^+ < d_m \leq d_m^+. \tag{12}$$

Given $\widetilde{\Gamma}$ as above, we define the collection of closed intervals $C_{\widetilde{\Gamma}} = \{J_1, ..., J_m\}$, with

$$J_1 = \left[d_0^+, d_1^-\right], \; ..., \; J_i = \left[d_{i-1}^+, d_i^-\right], ..., J_m = \left[d_{m-1}^+, d_m^-\right]. \tag{13}$$

Next, let $J = \cup_{i=1}^m J_i$ and we consider the collection of open intervals

$$\{E_0, E_1, ..., E_m\},$$

corresponding to eventual escape intervals for $g := f_{|J}$, each one defined by

$$E_0 = \left]d_0^-, d_0^+\right[, \; E_1 = \left]d_1^-, d_1^+\right[, \; ..., \; E_m = \left]d_m^-, d_m^+\right[, \tag{14}$$

in such a way that $[J] = [d_0, d_m] = (\cup_{i=1}^m J_i) \bigcup \overline{\left(\cup_{i=1}^{m-1} E_i\right)}$. In this case $\mathrm{dom}\,(g) = J$.

Let us analyze the conditions, on $\widetilde{\Gamma}$, for $g \in \mathcal{M}\,([J])$. First, assume that the orbits of the points in $\widetilde{\Gamma}$ are contained in the set J, that is,

$$f^k\left(\widetilde{\Gamma}\right) \subset J, \text{ for every } k > 0, \tag{15}$$

and for every $i \in \{0, 1, ..., m-1\}$, there must exist $j_i \in \{0, 1, ..., n-1\}$ so that

$$c_{j_i}^+ < d_i^+ < d_{i+1}^- < c_{j_i+1}^-, \tag{16}$$

assuming $d_0 = d_0^+$ and $d_m = d_m^-$, for enumeration purposes (to facilitate the notation we further assume $d_0 = d_0^+ = d_0^-$, $d_m = d_m^- = d_m^+$).

From the conditions (12), (15) and (16) we have that $f_{|J_i}$, $i = 1, ..., m$ is continuous, monotone and its image is contained in $[J]$. In fact, using the notation from (16) we have $f_{|J_i} = f_{j_i+1}$, the branch of the function f on $I_{j_i+1} \supset J_i$.

We further consider an additional condition that each point $d_i^{\pm}$, $i = 0, 1, ..., m$, belongs to the generalized orbits of the singular points of f, that is,

$$d_i^{\pm} \in R_f\,(\Gamma)\,, \; i = 0, 1, ..., m, \tag{17}$$

where Γ is constituted by the boundary points of the partition C_f, satisfying (3).

The condition (17) implies that each interval J_i is a finite union of f-cylinders, contained in a unique larger f-cylinder, which is a consequence of (16).

Since $f \in \mathcal{M}\,(I)$, satisfies the Markov property (P2), then every point in Γ is periodic (or pre-periodic) and this fact can be expressed by the following statement: there is a natural number $s \geq 0$ so that $f^{s+1}\,(\Gamma) = f^s\,(\Gamma)$. Recall that the boundary points of the intervals in $C_f^{(r)}$ are the points in $f^{-r}\,(\Gamma)$, for some natural $r \geq 1$.

Definition 3.1. Let $r \in \mathbb{N}$. An element in $f^{-r}\,(\Gamma)$ is called a pre-image of degree r. The degree of $\widetilde{\Gamma}$, with respect to Γ, is the maximal degree of the elements in $\widetilde{\Gamma}$, which are pre-images of Γ, under f.

Note that this definition makes sense since we impose (17).

The following result give necessary conditions for restriction $g := f_{|J}$ belong to $\mathcal{M}([J])$.

Lemma 3.2. *Consider the set $J = \cup_{i=1}^{m} J_i$, arising from $\widetilde{\Gamma}$ through (12) and (13). Let $g := f_{|J}$. If $\widetilde{\Gamma}$ satisfies (15), (16) and (17) then g satisfy the properties (P1), (P2) and (P3), from the definition (2.1). Moreover, there is a Markov partition for g constituted by intervals in $C_f^{(r)}$ which are contained in $\cup_{i=1}^{m} J_i$, where r is the degree of $\widetilde{\Gamma}$.*

Proof. From conditions (12), (15), and (16), it is possible to define the map $g := f_{|\cup_{i=1}^{m} J_i}$, f restricted to $\cup_{i=1}^{m} J_i$, so that each branch $f_{|J_i}$, $i = 1, ..., m$ is continuous, monotone. Consider i with $1 \leq i \leq m$. If (17) is verified then J_i is an interval which is a union of f-cylinder sets. Its boundary points $\partial J_i = \{\min(J_i), \max(J_i)\}, i = 1, ..., m$, which are pre-images of the singular points of f, Γ, constitute $\widetilde{\Gamma}$. Let r be the maximal degree of these pre-images, that is, the maximal exponent r so that $\partial J_i \subset f^{-r}(\Gamma)$, for every $i = 1, ..., m$. Since f is Markov, there is some natural number $s \geq 0$ so that $f^{s+1}(\Gamma) = f^{s}(\Gamma)$. Therefore, the boundary points in $\widetilde{\Gamma} = \partial \cup_{i=1}^{m} J_i$ satisfy $f^{r+s+1}(\partial \cup_{i=1}^{m} J_i) = f^{r+s}(\partial \cup_{i=1}^{m} J_i)$ and are contained in $\cup_{i=1}^{m} J_i = \mathrm{dom}(g)$, by (15). Therefore g is Markov and the partition of $J = \cup_{i=1}^{m} J_i$ into a union of intervals from $C_f^{(r)}$ constitute a Markov partition for g. □

Note that the partition obtained from Lemma 3.2 is not necessarily minimal. Nevertheless, it has the property that is constituted by f-cylinder sets of the same degree, that is, those which are indexed by f-admissible words of the same size r. This means that this r is a number depending exclusively on $\widetilde{\Gamma}$.

The transition matrix, associated with g through this process, is denoted by B_g. It may be not in totally amalgamated form (which in that case the partition would be minimal) and may not be primitive. We consider this problem in the next section.

Example 3.3. Recall $f = \beta x + \alpha \bmod 1$, with $\beta^3 - \beta^2 - \beta - 1 = 0$ and $\alpha = \beta^{-3}$, from the Example 2.5:

$$\Gamma = \{c_0^-, c_0^+, c_1^-, c_1^+, c_2^-, c_2^+, c_3^-, c_3^+\},$$

with $c_0^{\pm} = 0$, $c_1^{\pm} = \alpha$, $c_2^{\pm} = \alpha\beta + \alpha$, $c_3^{\pm} = 1$. Now, consider the refined partition

$$C_f^{(3)} = \{I_{123}, I_{231}, I_{232}, I_{233}, I_{312}, I_{323}, I_{331}, I_{332}, I_{333}\}.$$

Let us consider the set $J = \cup_{i=1}^{6} J_i$ with $J_1 = I_{123}$, $J_2 = I_{231}$, $E_2 = (I_{232} \overset{\circ}{\cup} I_{233})$, $J_3 = I_{312}$, $J_4 = I_{323}$, $E_4 = \overset{\circ}{I}_{331}$, $J_5 = I_{332}$, $J_6 = I_{333}$. Note that $E_0 = E_1 = E_3 = E_5 = E_6 = \emptyset$, following the definition (14). In this case $\widetilde{\Gamma} = \partial\left(\cup_{i=1}^{6} J_i\right)$ and its degree is 3. The set $\widetilde{\Gamma}$ can be written explicitly through

$$\widetilde{\Gamma}=\{d_0^-,d_0^+,d_1^-,d_1^+,d_2^-,d_2^+,d_3^-,d_3^+,d_4^-,d_4^+,d_5^-,d_5^+,d_6^-,d_6^+\},$$

with $d_0^-=d_0^+$, $d_1^-=d_1^+$, $d_3^-=d_3^+$, $d_5^-=d_5^+$, $d_6^-=d_6^+$, so that

$$\begin{aligned}&J_1=\left[d_0^+,d_1^-\right],\ J_2=\left[d_1^+,d_2^-\right],\ E_2=\left]d_2^-,d_2^+\right[,J_3=\left[d_2^+,d_3^-\right],\\&J_4=\left[d_3^+,d_4^-\right],\ E_4=\left]d_4^-,d_4^+\right[,\ J_5=\left[d_4^+,d_5^-\right],\ J_6=\left[d_5^+,d_6^-\right].\end{aligned}$$

Now, from the construction of the partition $C_f^{(3)}$, we have

$$\begin{array}{ll}f\left(J_1\right)=f\left(I_{123}\right)=I_{23}, & f\left(J_1\right)\cap J=J_2,\\ f\left(J_2\right)=f\left(I_{231}\right)=I_{31}, & f\left(J_2\right)\cap J=J_3,\\ f\left(J_3\right)=f\left(I_{312}\right)=I_{12}, & f\left(J_3\right)\cap J=J_1,\end{array}$$

$$\begin{array}{ll}f\left(J_4\right)=f\left(I_{323}\right)=I_{23}, & f\left(J_1\right)\cap J=J_2,\\ f\left(J_5\right)=f\left(I_{332}\right)=I_{32}, & f\left(J_1\right)\cap J=J_4,\\ f\left(J_6\right)=f\left(I_{333}\right)=I_{33}, & f\left(J_1\right)\cap J=J_5\cup J_6.\end{array}$$

The above equalities, written in terms of the boundary points, are

$$\begin{array}{lll}f\left(d_0^+\right)=d_1^+\in J_2, & f\left(d_1^+\right)=d_2^+\in J_3, & f\left(d_2^+\right)=d_0^+\in J_1,\\ f\left(d_1^-\right)=d_2^{+,-}=d_2^+\in J_3, & f\left(d_2^-\right)=d_3^-\in J_3, & f\left(d_3^-\right)=d_1^-\in J_1,\end{array}$$

$$\begin{array}{lll}f\left(d_3^+\right)=d_1^+\in J_2, & f\left(d_4^+\right)=d_3^+\in J_4, & f\left(d_5^+\right)=d_4^{-,+}=d_4^-\in J_4,\\ f\left(d_4^-\right)=d_2^{+,-}\in J_3, & f\left(d_5^-\right)=d_4^{-,-}=d_4^-\in J_4, & f\left(d_6^-\right)=d_6^-\in J_6.\end{array}$$

Note that $f\left(d_1^-\right)=d_2^{+,-}$ means that $f\left(d_1^- - \varepsilon\right)\in E_2$ (escape) for arbitrary small $\varepsilon>0$. In this case

$$\begin{aligned}f\left(\widetilde{\Gamma}\right)&=\left\{d_0^+,d_1,d_2^+,d_3,d_4^-,d_6^-\right\},\\ f^2\left(\widetilde{\Gamma}\right)&=\left\{d_0^+,d_1,d_2^+,d_3,d_4^-,d_6^-\right\},\end{aligned}$$

that is, $f^2\left(\widetilde{\Gamma}\right)=f\left(\widetilde{\Gamma}\right)$. The set $\widetilde{\Gamma}$ satisfies the conditions (15), (16) and (17), and $g=f_{|J}$ is Markov. The matrix associated with $g=f_{|J}$ is

$$B_g=\begin{pmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&1\end{pmatrix},$$

which is reducible. Therefore, although g is Markov, the matrix B_g is not primitive, (P4) fails, and $g\notin\mathcal{M}\left([J]\right)$. See the graph of f in the Fig. 1, with the partition $C_f^{(3)}$ indicated, and the removed intervals.

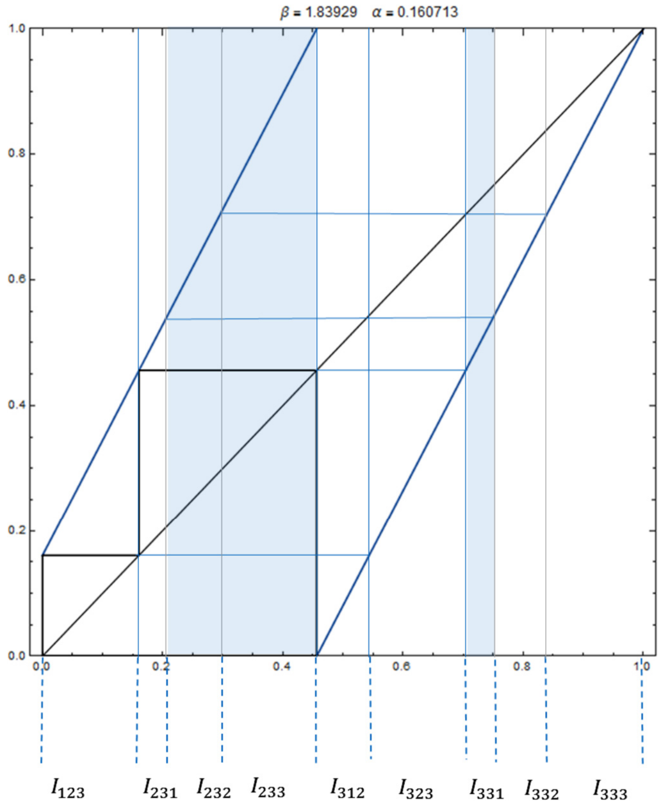

Fig. 1. Graph of f, from the Example 3.3, with the partition $C_f^{(3)}$ indicated. Note that $J = \cup_{i=1}^{6} J_i$, with $J_1 = I_{123}$, $J_2 = I_{231}$, $E_2 = (I_{232} \overset{\circ}{\cup} I_{233})$, $J_3 = I_{312}$, $J_4 = I_{323}$, $E_4 = \overset{\circ}{I_{331}}$, $J_5 = I_{332}$, $J_6 = I_{333}$, and $g = f_{|J}$.

Example 3.4. Again, from the Example 2.5, let $f = \beta x + \alpha \bmod 1$, with $\beta^3 - \beta^2 - \beta - 1 = 0$ and $\alpha = \beta^{-3}$. Consider the partition

$$C_f^{(3)} = \{I_{123}, I_{231}, I_{232}, I_{233}, I_{312}, I_{323}, I_{331}, I_{332}, I_{333}\}.$$

Let us consider the set $J = \cup_{i=1}^{5} J_i$ with $J_1 = I_{232}$, $J_2 = I_{233}$, $E_2 = \overset{\circ}{I_{312}}$, $J_3 = I_{323}$, $E_3 = \overset{\circ}{I_{331}}$, $J_4 = I_{332}$, $J_5 = I_{333}$. Let $\widetilde{\Gamma} = \partial\left(\cup_{i=1}^{5} J_i\right)$, its degree is 3. The set $\widetilde{\Gamma}$ can be written explicitly through

$$\widetilde{\Gamma} = \{d_0^-, d_0^+, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_5^+\},$$

with $d_0^- = d_0^+$, $d_1^- = d_1^+$, $d_3^- = d_3^+$, $d_4^- = d_4^+$, $d_5^- = d_5^+$, so that

$$J_1 = \left[d_0^+, d_1^-\right], \; J_2 = \left[d_1^+, d_2^{-,-}\right], \; E_2 = \left]d_2^{-,+}, d_2^+\right[, J_3 = \left[d_2^+, d_3^{-,-}\right],$$
$$E_3 = \left]d_3^{-,+}, d_3^+\right[, J_4 = \left[d_3^+, d_4^-\right], J_5 = \left[d_4^+, d_5^-\right].$$

The above equalities, written in terms of the boundary points, are

$$\begin{array}{c} f\left(d_0^+\right) = d_2^+ \in J_3 \\ f\left(d_1^-\right) = d_3^{-,-} = d_3^- \in J_3 \end{array}$$

$$\begin{array}{l} f\left(d_1^+\right) = d_3^+ \in J_4 \\ f\left(d_2^{-,-}\right) = d_5^- \in J_5 \end{array}, \begin{array}{l} f\left(d_2^+\right) = d_0^+ \in J_1 \\ f\left(d_3^-\right) = d_2^{-,-} = d_2^- \in J_2 \end{array},$$

$$\begin{array}{l} f\left(d_3^+\right) = d_2^+ \in J_3 \\ f\left(d_4^-\right) = d_3^{-,-} = d_3^- \in J_3 \end{array}, \begin{array}{l} f\left(d_4^+\right) = d_3^+ \in J_4 \\ f\left(d_5^-\right) = d_5^- \in J_5 \end{array}$$

In this case

$$\begin{array}{l} f\left(\widetilde{\Gamma}\right) = \left\{d_0^+, d_2^-, d_2^+, d_3^-, d_3^+, d_5^-\right\}, f^2\left(\widetilde{\Gamma}\right) = \left\{d_0^+, d_2^-, d_2^+, d_5^-\right\}, \\ f^3\left(\widetilde{\Gamma}\right) = \left\{d_0^+, d_2^+, d_5^-\right\}, f^4\left(\widetilde{\Gamma}\right) = \left\{d_0^+, d_2^+, d_5^-\right\}, \end{array}$$

that is, $f^4\left(\widetilde{\Gamma}\right) = f^3\left(\widetilde{\Gamma}\right)$. The set $\widetilde{\Gamma}$ satisfies the conditions (15), (16) and (17), and $g = f_{|J}$ is Markov. The matrix associated with $g = f_{|J}$ is

$$B_g = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

which is primitive. Therefore, in this case $g \in \mathcal{M}([J])$. The partition for B_g is not minimal. The minimal Markov partition is

$$J_1 = I_{123},\ J_2 \cup J_3 = I_{23},\ J_4 = I_{312},\ J_5 = I_{331}$$

with transition matrix

$$A_g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

See the graph of f in the Fig. 2, with the partition $C_f^{(3)}$ indicated, and the removed intervals.

4. Sufficient conditions for primitivity

The main problem to overcome, at this point, is the primitivity of the transition matrix obtained from the restriction $g = f_{|J}$. To deal with this problem let us introduce

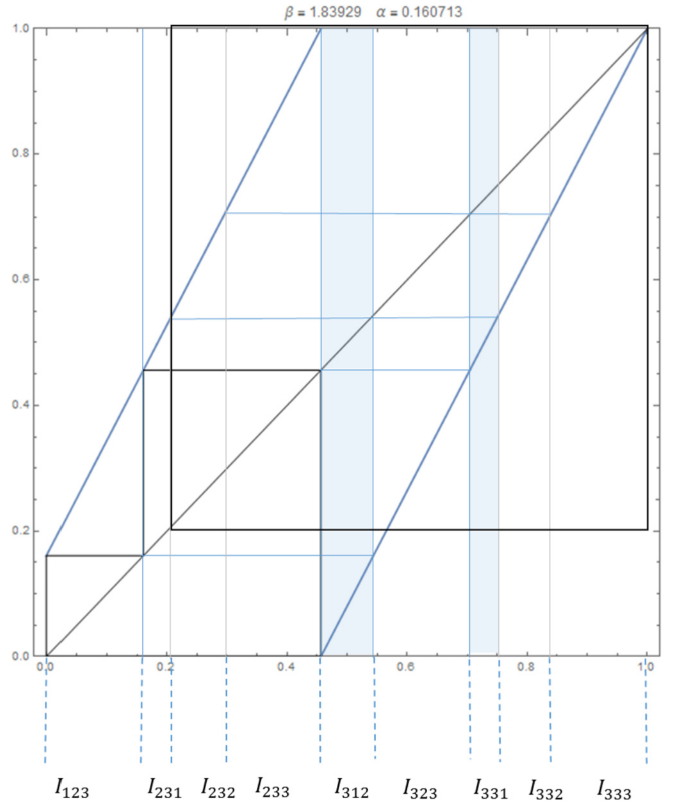

Fig. 2. Graph of f, from the Example 3.4, with the partition $C_f^{(3)}$ indicated. Note that $J = \cup_{i=1}^{5} J_i$ with $J_1 = I_{232}$, $J_2 = I_{233}$, $E_2 = \overset{\circ}{I}_{312}$, $J_3 = I_{323}$, $E_3 = \overset{\circ}{I}_{331}$, $J_4 = I_{332}$, $J_5 = I_{333}$, and $g = f_{|J}$.

additional notation regarding cycles in subshifts of finite type and use simultaneously the notions related to the digraph G_{A_f} arising from A_f. In what follows we use the term *state* and *vertex* indistinctly and any path in the digraph G_{A_f} is associated with an admissible word in Σ_{A_f}. Moreover, a cycle or periodic orbit for f is identified with a cycle or periodic admissible word for Σ_{A_f} and identified with a cycle or a closed path for G_{A_f}.

The size of a path $\xi_1 ... \xi_r$ is the number of edges which compose it and it is also equal to the number of transitions between states or symbols in the associated word, in this case, $r-1$. Note that the size of the corresponding word is r and for a cycle the two notions, the size of a closed path and the size of corresponding word, coincide.

Definition 4.1. (i) The distance, d_{ij}, between two states i, j, is the minimum number of edges (state transitions) necessary to pass from i to j through a path in the digraph G_{A_f}. In other words, is the minimal number, $d \in \mathbb{N}$, for which there is a word $\xi_1 ... \xi_{d-1} \in \mathcal{W}_{d-1}$ (eventually the empty word with $d = 1$) so that $i\xi_1 ... \xi_{d-1} j \in \mathcal{W}_{d+1}$.

(ii) The diameter d_A of A (of the digraph G_A) is the largest distance between pairs of states with respect to A.

Note that with this definition if $a_{ij} = 1$ then $d_{ij} = 1$ and we set $d_{ii} = 0$ by default. If a matrix A is reducible then there are pairs of states which are not connected therefore their distance can be seen as infinite. If A is irreducible and not primitive there are pairs of states which are connected by paths with certain type of sizes. If A is primitive then there is a certain natural number q so that for every pair i, j and for every $k > q$ there is a path of size k between i and j. Or, what is equivalent, for every $k > q$ there is a word of size $k - 1$, $\xi_1 ... \xi_{k-1} \in \mathcal{W}_{k-1}$, with $i\xi_1 ... \xi_{k-1} j \in \mathcal{W}_{k+1}$. That is, for a primitive matrix there are paths of any size greater than q, connecting any pair of vertices of G_{A_f}.

A cycle for A_f containing every symbol in $\{1, 2, 3, ..., n\}$ is called a *complete cycle*. A cycle for A_f containing every symbol in $\{1, 2, 3, ..., n\}$ except one is called an *almost complete cycle*.

Lemma 4.2. *Let A be a $0 - 1$ matrix. The matrix A is irreducible if and only if there is a complete cycle.*

Proof. Straightforward. □

Remark 2. The above means that if f is a Markov map with irreducible transition matrix A_f then there is a periodic orbit visiting each Markov interval of the partition C_f.

We say that two cycles are *relatively prime* if its sizes are relatively prime numbers.

Lemma 4.3. *Let A be a $0 - 1$ matrix. The matrix A is primitive if and only if there is a complete cycle γ, not necessarily simple, and there is at least another cycle η relatively prime to γ.*

Proof. If A is primitive then there is a certain natural q so that every pair of states are connected through a path of arbitrary size greater than q. Now, consider that there is a cycle γ of size a containing every symbol and there is another cycle η of size b so that $\gcd(a, b) = 1$. Consider a pair of states $i \neq j$. Some of the paths between i and j can be written as follows: assume the cycle γ is starting in i (the cycle contains every symbol, in particular it contains i and j at least once), that is, $\gamma_1 = i$, $\gamma_{t+1} = j$, for some $t \geq d_{ij}$. Recall, from the Definition 4.1, that d_{ij} is the distance between the two states i and j and there can be another shorter path between i, j. If not we have $t = d_{ij}$. We can then write $\gamma = (i\gamma_2\gamma_3 ... \gamma_t j \gamma_{t+2} ... \gamma_a)^\infty$. Let $\eta = (\eta_1 ... \eta_b)^\infty$. Knowing that every symbol in η is contained in γ there is a state s so that $\gamma_s = \eta_1$. Therefore, a path from i to j can be written as

$$\gamma^{m_1} i\gamma_2 ... \gamma_{s-1} \eta^{m_2} \gamma_s \gamma_{s+1} .. \gamma_a \ i \ \gamma_2 \gamma_3 ... \gamma_{t-1} j,$$

for some natural numbers $m_1, m_2 \in \mathbb{N}$, since $\eta_b\eta_1$ is admissible and $\eta_1 = \gamma_s$, allowing the transition between the cycle γ to η and from the cycle η to γ. The size of the word is

$$m_1 a + s - 1 + m_2 b + (a - s) + 1 + t = (m_1 + 1)\, a + m_2 b + t + 1,$$

and therefore the size of the path, from i to j, is

$$(m_1 + 1)\, a + m_2 b + t.$$

Since $\gcd(a, b) = 1$ the set $\{m_1 a + m_2 b : m_1, m_2 \in \mathbb{N}\}$ is a numerical semigroup with Frobenius number given by $F(a, b) = ab - a - b$, see [2]. The Frobenius number of a numerical semigroup is the greatest natural number not belonging to the semigroup. This means that every natural number greater than $F(a, b)$ belongs to the numerical semigroup and is written as $m_1 a + m_2 b$ for certain $m_1, m_2 \in \mathbb{N}$. Since

$$m_1 a + m_2 b + t \geq m_1 a + m_2 b + d_{ij},$$

we have paths connecting i,j of any size larger than $F(a, b) + d_{ij}$. Considering the sizes of the paths linking an arbitrary pair i,j we have surely paths of size $F(a, b) + d_A$, where d_A is the diameter of A, see the Definition 4.1. Therefore, A is primitive.

Now, assume that A is irreducible and suppose that the existing cycles (which are more than one from irreducibility) all have a common divisor $q > 1$. Let again γ be the cycle containing every symbol, with size a. Since any path from i to j can be written as

$$\gamma^{m_1} i\gamma_2 ... \gamma_{s-1}\eta^{m_2}\gamma_s\gamma_{s+1} .. \gamma_a \ i \ \gamma_2\gamma_3 ... \gamma_{t-1} j,$$

for any other cycle η and for some natural numbers $m_1, m_2 \in \mathbb{N}$, the size of this path, from i to j, is given by

$$(m_1 + 1)\, a + m_2 b + t = m_3 q + d_{ij}.$$

The sizes of the paths between i,j have fixed remainder mod q, therefore A is irreducible periodic of period q and not primitive. $\square$

Definition 4.4. Given a transition matrix A, a state i for which $\#fol\,(i) = 1$ is called *contractive singular state*. A state i for which $\#pre(i) = 1$ is called *expansive singular states*. Those states which satisfy both $\#fol\,(i) = 1$, $\#pre\,(i) = 1$ are called neutral *singular* states. In short *cs*-state, *ns*-state, *es*-state. Two distinct states are called *related* if there is a transition between them, either in one direction or in the other.

The follower of a contractive singular state and the predecessor of a expansive singular state constitute also important types of states to consider. The set of related states of i

is the union of the set of followers and predecessors of i, excluding the self followers and the self predecessors, that is,

$$rel(i) = (fol(i) \cup pre(i)) \setminus (fol(i) \cap pre(i)).$$

Let A be a transition matrix. We denote $(A)_s$ the submatrix of A obtained removing the row and the column corresponding to the state s.

Proposition 4.5. *Let A be a $0-1$ primitive matrix. If s is a follower of a contractive singular state or a predecessor of an expansive singular state then $(A)_s$ is not irreducible, in particular, is not primitive.*

Proof. If we remove s, a follower of a *cs*-state, this state with respect to $(A)_s$ is a sink (state without exit). If we remove s, a predecessor of a *es*-state, this state with respect to $(A)_s$ is a source (state for which there is no return). In either cases $(A)_s$ is not irreducible and thus not primitive. $\square$

Definition 4.6. We call s a *removable state* of a primitive matrix A if $(A)_s$ is primitive. If, given A, there is a *removable* state then A is called *strongly primitive*.

Proposition 4.7. *Let A be a $0-1$ primitive matrix. If there are only two cycles in the conditions of the Lemma 4.3 then A is not strongly primitive.*

Proof. If there are only two cycles if we remove a state s then one of the cycles is broken. Therefore, $(A)_s$ is not primitive. $\square$

Proposition 4.8. *Let A be a $0-1$ irreducible matrix. There is a cycle almost complete for A, that is, it contains every state except one, s, if and only if $(A)_s$ is irreducible.*

Proof. If there is a cycle containing every states except s, then the same cycle exists in $(A)_s$ and contains every state for $(A)_s$, therefore from the Lemma 4.2 $(A)_s$ is irreducible. On the other hand, if $(A)_s$ is irreducible for a certain s then the complete cycle for $(A)_s$ is an almost complete cycle for A. $\square$

Lemma 4.9. *Let A be a $0-1$ primitive matrix. If there is a state s and two relatively prime cycles not containing s, one of the cycles is almost complete for A, then $(A)_s$ is primitive. In other words, s is removable and A is strongly primitive. The reverse is also true.*

Proof. If we remove the state s the almost complete cycle turns a complete cycle of size a for $(A)_s$, the cycle of size b not contains s so is not affected in $(A)_s$, therefore $(A)_s$ is in the conditions of Lemma 4.3 and thus $(A)_s$ is primitive. On the other hand, let A be strongly primitive with a removable state s. Therefore $(A)_s$ is primitive and there must be at least two relatively prime cycles, if not $(A)_s$ would be irreducible and non primitive. $\square$

Theorem 4.10. *Let $r > 1$ and consider $A_f^{(r)}$ the transition matrix associated with f with respect to the partition $C_f^{(r)}$. Let $J_\gamma = I \backslash I_\gamma$ with $\gamma \in \mathcal{W}_r(A_f)$, then:*
(a) $g = f_{|J_\gamma} \in \mathcal{M}([J_\gamma])$ if and only if γ is removable regarding $A_f^{(r)}$.
(b) γ is removable if and only if there are two cycles relatively prime not containing γ, one of each is almost complete for $A_f^{(r)}$.

Proof. First observe that the matrix A_g is obtained from $A_f^{(r)}$ deleting the row and the column corresponding to the word γ, followed by an eventual amalgamation if possible. The matrix $A_f^{(r)}$ is indexed by the words $\xi \in \mathcal{W}_r(A_f)$ and by hypothesis is primitive. Let B be the matrix $B = \left(A_f^{(r)}\right)_\gamma$ obtained from $A_f^{(r)}$ deleting the row and the column corresponding to the word γ. The matrix B is primitive if and only if γ is removable. Since J_γ satisfies the conditions for Lemma 3.2, (a) follows, and A_g is the total amalgamated matrix topologically equivalent to B. On the other hand the existence of two cycles not containing γ, being one cycle almost complete for $A_f^{(r)}$, of size a, and the other cycle of size b so that $\gcd(a, b) = 1$, is equivalent, from the Lemma 4.9, to the fact that the state γ is removable. $\square$

Example 4.11. Consider $f(x) = \beta x + \alpha \mod 1$, with $\beta^5 - \beta - 1 = 0$ and $\alpha = \left(1 + \beta + \beta^2 + \beta^3 + \beta^4\right)^{-1}$. The alphabet associated with the partition is $\{1, 2, 3, 4, 5\}$. There are exactly two simple cycles: $(12345)^\infty$, which is complete, and $(2345)^\infty$ which is almost complete. It is not possible to remove any state. See the Fig. 3. Therefore A_f is not strongly primitive.

$$A_f = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, there is no $r > 1$ so that $A^{(r)}$ is strongly primitive.

Example 4.12. Recall from the Example 2.5, $f(x) = \beta x + \alpha \mod 1$, with $\beta^3 - \beta^2 - \beta - 1 = 0$ and $\alpha = \beta^{-3}$ (Fig. 4). Let $r = 3$. The corresponding partition

$$C_f^{(3)} = \{I_{123}, I_{231}, I_{232}, I_{233}, I_{312}, I_{323}, I_{331}, I_{332}, I_{333}\}.$$

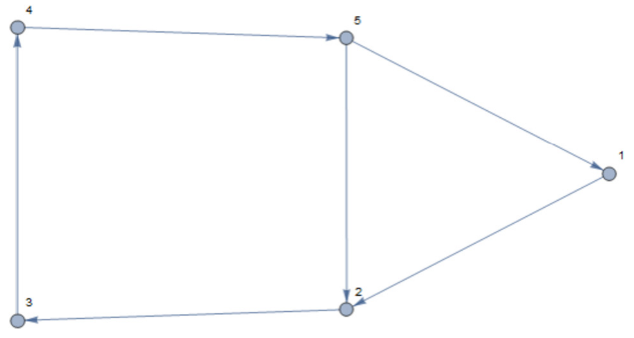

Fig. 3. Digraph of the Example 4.11, associated with A_f, with two simple cycles $(12345)^\infty$ and $(1234)^\infty$.

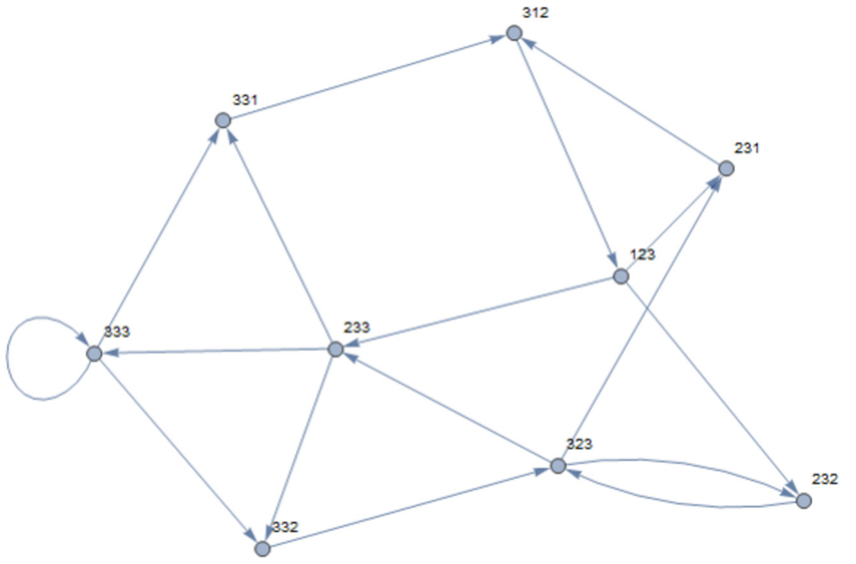

Fig. 4. Digraph of the Example 4.12, associated with $A_f^{(3)}$.

Let us consider $\gamma = 232$. There is an almost complete cycle, not containing γ, for example

$$[(123)\,(233)\,(332)\,(323)\,(233)\,(333)\,(331)\,(312)\,(123)\,(231)\,(312)]^\infty$$

of size 11. Consider also the cycle $[(123)\,(231)\,(312)]^\infty$, not containing γ, of size 3. Therefore, γ is removable, we are in the conditions of the Theorem 4.10 and consequently $g = f_{|J_{232}} \in \mathcal{M}\left([0,1]\right)$, since $[J_{232}] = [0,1]$. In fact, let

$$B_g = \left(A_f^{(3)}\right)_{232} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The transition matrix of g, on the Markov minimal partition, A_g, obtained through amalgamation from B_g, is given by

$$A_g = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

and is primitive.

Remark 3. In the paper [5] we have introduced the escape matrix $\widehat{A}$ which codifies the transitions for the escape set. These are essentially matrices with 0 rows associated to the intervals which are escape intervals, $E_1, ..., E_m$ as in (5). When we eliminate a removable state, as above, associated with a word γ, we may see this process as a transformation of the interval I_γ into an escape interval.

Using the previous results, we propose a method to enumerate the Markov subsystems of a given one (Ω_f, f), whose Markov intervals are obtained as finite unions of f-cylinders. The method follows:

First, fix a natural r and identify the removable states, γ, of $A_f^{(r)}$. Obtain $B_\gamma = \left(A_f^{(r)}\right)_\gamma$, removing from $A_f^{(r)}$ the removable state γ. Next, identify the removable states for B_γ and repeat the process, using $r+1$ as the degree.

For each fixed natural number $r \in \mathbb{N}$ the process ends at some point since the set of the removable states is finite.

Naturally, we obtain a poset of invariant subshifts of the given subshift, for each fixed degree r. In this way, we obtain also the interval maps $g \in \mathcal{M}([J])$, which are restrictions of a given map $f \in \mathcal{M}(I)$, with $J \subset I$. We illustrate the procedure in the next section, with several examples using the family $f(x) = \beta x \bmod 1$.

5. Family $f(x) = \beta x \bmod 1$

5.1. β-shift for $\beta = 2$

Let $f(x) = 2x \bmod 1$ in $\mathcal{M}([0,1])$. Let us analyze the possible subdynamics, up to $r = 3$. The transition matrix for f, with respect to the partition $C_f = \{I_1, I_2\}$, is

$$A_f = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $r = 2$. Then

$$A_f^{(2)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

and $C_f^{(2)} = \{I_{11}, I_{12}, I_{21}, I_{22}\}$. The singular states are 11 and 22. The followers and the predecessors of 11 and 22 cannot be removed, these are the states $12, 21$. On the other hand the simple cycles are

$$[(12)(21)]^{\infty}, [(11)(12)(21)]^{\infty},$$
$$[(12)(22)(21)]^{\infty}, [(11)(12)(22)(21)]^{\infty}.$$

Therefore, the removable states are precisely 11 and 22. Let us observe which primitive matrices we obtain:

$$B_{11} = \left(A_f^{(2)}\right)_{11} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad B_{22} = \left(A_f^{(2)}\right)_{22} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

In this case, $\Sigma_{B_{11}}$ is topologically conjugated to $\Sigma_{B_{22}}$. When removing one removable state the remaining state is no longer removable. If we remove both states we obtain a permutation matrix

$$\left(A_f^{(3)}\right)_{11,22} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now, let $r = 3$ and consider the transition matrix

$$A_f^{(3)} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

with $C_f^{(3)} = \{I_{111}, I_{112}, I_{121}, I_{122}, I_{211}, I_{212}, I_{221}, I_{222}\}$. The singular states are 111 and 222. Therefore, the followers and the predecessors of 111 and 222 cannot be removed, which are the states $122, 221, 211, 112$. Moreover, the smallest simple cycles are

$$[(121)(212)]^\infty, [(111)(121)(212)]^\infty, [(222)(212)(121)]^\infty,$$
$$[(111)(112)(121)(211)]^\infty, [(222)(221)(212)(122)]^\infty, \ldots$$

and the removable states are 111, 121, 212, 222. The matrices which we obtain, eliminating each removable state, are:

$$\left(A_f^{(3)}\right)_{111} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \left(A_f^{(3)}\right)_{121} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

$$\left(A_f^{(3)}\right)_{212} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \left(A_f^{(3)}\right)_{222} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The subshifts obtained are invariant subshifts of finite type of the original one, that is, $\Sigma_{B_\gamma} \subset \Sigma_{A_f^{(3)}}$. Moreover, if we proceed, fixing $r = 3$, we obtain the partial ordered set, with inclusion, of the invariant subshifts of the original subshift $\Sigma_{A_f^{(3)}}$, which is topologically conjugated to Σ_{A_f}, see the Fig. 5.

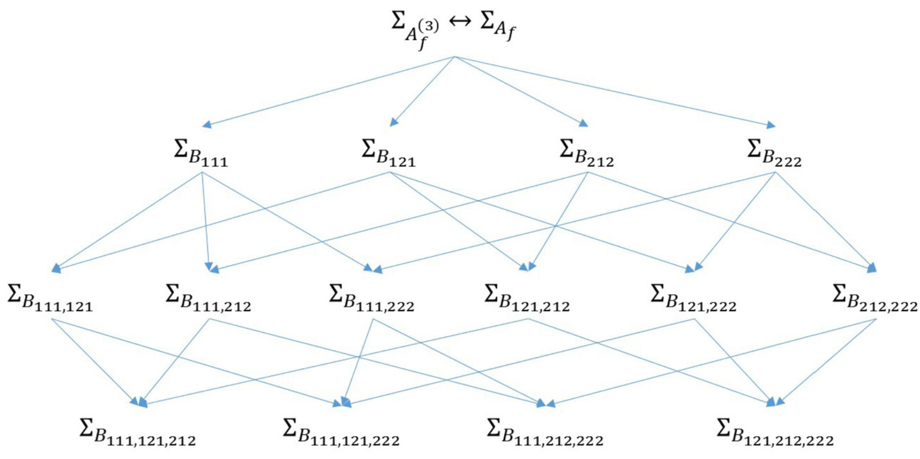

Fig. 5. The poset organized by invariant subspaces based on $A_f^{(3)}$, removing appropriate states $\gamma \in \mathcal{W}_3$. From the β-shift with $\beta = 2$.

Note, for example, if we remove 112 the obtained matrix is not primitive

$$B_{112} = \left(A_f^{(3)}\right)_{112} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

If we remove three removable states, the remaining state is no longer removable. If we remove it, we obtain a permutation matrix

$$B_{111,121,212,222} = \left(A_f^{(3)}\right)_{111,121,212,222} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which codifies the transitions between the remaining states, according to

$$(112) \to (122) \to (221) \to (211) \to (112).$$

In the Fig. 6, as an example, we present the graph of f and of $g = f_{|J_{121}}$.

5.2. β-shift for $\beta = \frac{1+\sqrt{5}}{2}$

Let $f(x) = \beta x \bmod 1$ in $\mathcal{M}([0,1])$, with $\beta = \frac{1+\sqrt{5}}{2}$, the golden number. Its transition matrix, with respect to the $C_f = \{I_1, I_2\}$, is

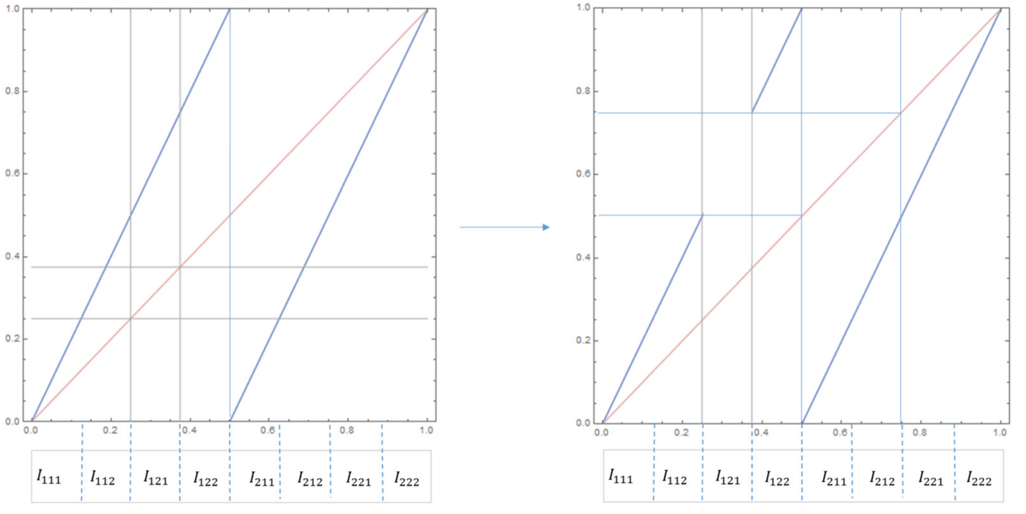

Fig. 6. Graphs of f and $g = f_{|J_{121}}$ in the case $J_{121} = I \backslash I_{121}$.

$$A_f = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $r = 2$. Then

$$A_f^{(2)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

and $C_f^{(2)} = \{I_{11}, I_{12}, I_{21}\}$. The state 11 is singular neutral. The followers and the predecessors of 11 are the states 12 and 21 which cannot be removed. Since the cycles are

$$[(11)(12)(21)]^{\infty},\ [(12)(21)]^{\infty},$$

the state 11 is not removable. In fact

$$B_{11} = \left(A_f^{(2)}\right)_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is not primitive.

Let $r = 3$. Then

$$A_f^{(3)} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

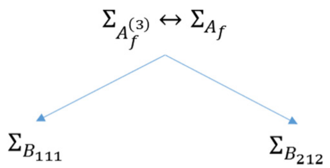

Fig. 7. The poset organized by invariant subspaces based on $A_f^{(3)}$, removing appropriate states $\gamma \in \mathcal{W}_3$. From the β-shift with $\beta = \frac{1+\sqrt{5}}{2}$.

and $C_f^{(3)} = \{I_{111}, I_{112}, I_{121}, I_{211}, I_{212}\}$. The states 111, 212 are singular neutral states, the state 211 is singular expansive and 122 is singular contractive. Therefore, the followers and the predecessors of 111, the followers of 112 and the predecessors of 211 cannot be removed, which are the states $211, 112, 121$. Moreover, the simple cycles are

$$[(121)(212)]^{\infty}\,, \;\; [(112)(121)(211)]^{\infty}\,, \;\; [(111)(112)(121)(211)]^{\infty}\,.$$

The removable states are then 111, 212. The obtained matrices (all primitive) are the following (Fig. 7)

$$B_{111} = \left(A_f^{(3)}\right)_{111} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_{212} = \left(A_f^{(3)}\right)_{212} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

When removing one state the remaining state is no longer removable, since there are no extra cycles left. If we remove both states we obtain a permutation matrix

$$\left(A_f^{(3)}\right)_{111,212} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Declaration of competing interest

There is no competing interest.

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