Universidade de Évora - Instituto de Investigação e Formação Avançada

Programa de Doutoramento em Matemática

Área de especialização | Matemática e Aplicações

Tese de Doutoramento

Problemas impulsivos de ordem superior não-lineares e funcionais

Rui Manuel Silva Carapinha

Orientador | Feliz Manuel Minhós

Évora 2021

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Évora 2021

Nonlinear and Functional Higher Order Impulsive Problems

Rui Manuel Silva Carapinha

Contents

To my dear mother, Joaquina.

To Ivone and Francisco, my father.

Acknowledgements

To Professor Feliz Manuel Minhós for his scientific support, professionalism and honorable friendship that made everything possible.

And to the University of Évora for the welcoming and monitoring throughout the journey.

Problemas impulsivos de ordem superior não lineares e funcionais

RESUMO

A abordagem teórica dos problemas do tipo impulsivo tem assumido importância crescente no mundo científico e industrial de hoje, pelas respostas que oferece aos problemas de pesquisa e produção industrial, nos quais as ocorrências de descontinuidades abruptas e saltos funcionais são decisivos nos respectivos contextos. Em suma, problemas impulsivos representam fenómenos em que ocorrem mudanças repentinas nas suas propriedades dinâmicas .

Esses problemas são frequentes no estudo da dinâmica populacional, na otimização de processos de produção, na biologia, nos problemas de estudo do comportamento dos fatores ambientais, na medicina e na farmacologia, entre outros ramos da Ciência.

Nesta teses são abordados problemas de ordem superior com valores na fronteira, com mudanças instantâneas na função incógnita e nas suas derivadas.

Nos últimos anos, os operadores Laplaciano e as suas variantes como, por exemplo, p-Laplaciano e phi-Laplaciano, têm sido aplicados em várias situações mas poucas vezes no caso descontínuo. Os três primeiros capítulos da tese procuram contribuir para colmatar esta falha.

O caso periódico e a situação em que o domínio de definição não é limitado, são mais delicados e exigem um tipo de abordagem diferente. Nos Capítulos 4, 5 e 6 apresentam-se técnicas e métodos topológicos que permitem abordar estes tipos de problemas.

Nonlinear and Functional Higher Order Impulsive Problems

ABSTRACT

The theoretical approach of impulsive problems has assumed an increasing importance in the scientific and industrial world today, due to its answers to the problems of research and industrial production, in which the occurrences of abrupt discontinuities and functional leaps are decisive in the respective contexts. In short, impulsive problems represent phenomena in which sudden changes in their dynamic properties occur.

These problems are frequent in the study of population dynamics, in the optimization of production processes, in biology, in problems of studying the behavior of environmental factors, in medicine and pharmacology, among other branches of science.

This thesis addresses problems of a higher order with boundary values problems with instantaneous changes in the unknown function and its derivatives.

In recent years, Laplacian operators and their variants, such as p-Laplacian and Phi-Laplacian, have been applied in several situations, but rarely in the discontinuous case. The first three chapters of the thesis seek to contribute to fill this gap.

The periodic case and the situation in which the domain of definition is not bounded, are more delicate and require a different type of approach. In Chapters 4, 5 and 6, topological techniques and methods are presented that allow to approach these types of problems.

Introduction

The theoretical approach of the impulsive type problems has assumed increasing importance in the scientific and industrial world today, for the answers it offers to the research problems, industrial production,..., in which the occurrences of abrupt discontinuities, and functional leaps, are decisive in the respective contexts. In short, impulsive problems adequately represent phenomena in which sudden changes in the dynamic properties of behavior occur.

These problems are frequent in the study of population dynamics, the optimization of production processes, in biology, in the problems of studying the behavior of environmental factors, in medicine and pharmacology.

For the first time, to our knowledge, problems of first-order differential equations with instantaneous changes depending on an unknown function and its first and second derivatives are addressed, considering the cases in which the respective jumps at each moment depend not only on the value of the function, in a given instant, but also of its speed.

In recent years, the p-Laplacian and phi-Laplacian operators have been applied in semi-linear and quasi-linear differential equations, and especially in differential equations of the non-linear type. However, problems of the impulsive type with general nonlinear equations and general impulsive effects, have been very rare. With this thesis we intend to fulfill this gap, and to present different methods and approaches, applying lower and upper solutions method, which proved to be especially suitable for impulsive problems with boundary values.

The present thesis is structured in six chapters, as follows:

1 - Higher Order Nonlinear Impulsive Boundary Value Problems

This first chapter contains some two-point impulsive boundary value problems composed by a fully differential equation, which higher-order contains an increasing homeomorphism, two-point boundary conditions, and impulsive effects.

We point out that the impulsive functions are given via multivariate generalized functions, including impulses on the referred homeomorphism.

The method used applies lower and upper solutions technique together with fixed point theory. Therefore we have not only the existence of solutions but also the localization and qualitative data on their behavior. Moreover, a Nagumo condition will play a key role in the arguments to estimate the second derivative.

2 - Half-linear impulsive problems with more general jump conditions

Separated impulsive problems with a fully third order differential equation, including an increasing homeomorphism, and impulsive conditions given by more general functions: the impulsive effect in the first derivative can now depend on the value of the second derivative on the impulsive moment. Moreover, it is considered a general interval $[a, b]$ for the variation of the time variable, which introduce some modifications on the solution of the linear problem.

The key arguments are similar to the previous chapter.

3 - Semi-Linear Impulsive Higher Order Boundary Value Problems

In this chapter we generalize the method used in two previous chapters, considering a two-point $n^{th}-$order impulsive boundary value problems, with a strongly nonlinear fully differential equation with an increasing homeomorphism.

It is stressed that the impulsive effects are defined by very general functions, that can depend on the unknown function and its derivatives, till order $n-1$.

The arguments are based on the lower and upper solutions method, together with Leray-Schauder fixed point theorem.

An application, for $n = 4$, to estimate the bending of a one-sided clamped beam under some impulsive forces, is given in the last section of the chapter.

4 - Periodic third order boundary value problems with generalized impulsive conditions

Periodic boundary value problems require a different approach, mainly because the linear problem associated is not invertible. In particular, third-order periodic problems are more delicate, if we want some localization data on the unknown function or its derivatives.

Several authors apply fixed point theory, topological and coincidence degree, lower and upper solutions, cone theory, etc. In this chapter, we use an iterative technique together with mathematical induction. This problem covers cases where the jumps in each moment depend not only on the value of the function on the impulsive instant but also on the velocity and the convexity of the solution in the referred moment.

The main tools rely on a perturbed and truncated auxiliary problem, on an iterative technique, not necessarily monotone, as in [118], and lower and upper solutions method.

We point out that, the nonlinear part must verify only a local monotone condition (see 4.11) and no assumption on its periodicity or asymptotic growth is needed.

5 - Third-order generalized discontinuous impulsive problems on the half-line

In this chapter, we improve the existing results in the literature by presenting weaker sufficient conditions for the solvability of a third-order impulsive problem on the half-line, with generalized impulse effects. More precisely, our nonlinearities do not need to be positive, nor sublinear, and the monotone assumptions are local ones.

Our method makes use of some truncation and perturbed techniques and on the equiconvergence at the infinity and the impulsive points.

The last section contains an application to a boundary layer flow problem over a stretching sheet with and without heat transfer.

6 - Functional coupled systems with generalized impulsive conditions and application to a SIRS-type model

In this chapter, we consider a first-order coupled impulsive system of equations with functional boundary conditions, subject to the generalized impulsive effects.

It is pointed out that this problem generalizes the classical boundary assumptions, allowing two-point or multipoint conditions, nonlocal and integro-differential ones or global arguments, as maxima or minima, among others.

Our method is based on lower and upper solutions technique together with the fixed point theory.

The main theorem is applied to a SIRS model were, to the best of our knowledge, for the first time it includes impulsive effects combined with global data and the asymptotic behavior of the unknown functions.

Chapter 1

Higher Order Nonlinear Impulsive Boundary Value Problems

1.1 Introduction

This first chapter contains some two-point impulsive boundary value problems composed by a fully differential equation, which higher-order contains an increasing homeomorphism, two-point boundary conditions, and generalized impulsive conditions, given via multivariate generalized functions, including impulses on the homeomorphism.

Consider the following two point boundary value problem with one-dimensional ϕ-Laplacian:

$$\begin{cases} (\phi(u''(t)))' + q(t)f(t, u(t), u'(t), u''(t)) = 0, \ t \in J', \\ u(0) = A, \\ u'(0) = B, \\ u''(1) = C, \ A, B, C \in \mathbb{R}, \end{cases} \tag{1.1}$$

where

(A_1) ϕ is an increasing homeomorfism such that $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$,

(A_2) $q \in C([0, 1])$ with $q > 0$, $J := [0, 1]$, $J' = J\backslash\{t_1, ..., t_n\}$ and $\int_0^1 q(s)ds < \infty$, $f \in C([0, 1]\times \mathbb{R}^3, \mathbb{R})$,

with the impulsive conditions

$$\begin{cases} \Delta u(t_k) = I_{1k}(u(t_k), u'(t_k)), \ k = 1, 2, ...n, \ \Delta u^{(i)}|_{t=t_k} = u^{(i)}(t_k^+) - u^{(i)}(t_k^-), \\ \Delta u'(t_k) = I_{2k}(u(t_k), u'(t_k)) \\ \Delta\phi(u''(t_k)) = I_{3k}(u(t_k), u'(t_k), u''(t_k)), \end{cases} \tag{1.2}$$

being $I_{ik} \in C(\mathbb{R}^2, \mathbb{R})$, $i = 1, 2$, and $I_{3k} \in C(\mathbb{R}^3, \mathbb{R})$, with t_k fixed points such that $0 < t_1 < t_2 < ... < t_n < 1$.

Our method applies lower and upper solutions technique together with fixed point theory. Therefore we have not only the existence of solutions but also the localization and qualitative data on their behavior. Moreover, a Nagumo condition will play a key role in the arguments to estimate the second derivative.

For the classical approach to impulsive differential equations we can refer, as example, [9, 12, 53, 54, 75, 79, 91] and the references therein. Most of the arguments apply critical point theory and variational techniques ([58, 112]), fixed point results in cones ([96, 104]), bifurcation theory ([81, 108]), and lower and upper solutions method ([42, 97]).

In this chapter we consider, as far as we know, by the first time the impulsive effects with dependence on the unknown variable and its derivative, and even on its second derivative for the impulses on the homeomorphism ϕ, which includes the Laplacian or p-Laplacian cases.

Section 1.2 contains an uniqueness result for an associated problem to (2.1)-(2.3) and the definition of lower and upper solutions, with strict inequalities in some boundary and impulsive conditions. In Section 1.3 the main existence and localization result is obtained via an truncation and perturbation methods (suggested in [55, 71]) lower and upper solution technique and fixed point theory. Last chapter section provides an example where the impulses depend on the function and on its variation.

1.2 Auxiliary results

Let

$$PC[0,1] = \left\{ \begin{array}{c} u : u \in C([0,1], \mathbb{R}) \text{ continuous for } t \neq t_k, u(t_k) = u(t_k^-), u(t_k^+) \\ \text{exists for } k = 1, 2, ..., n \end{array} \right\}$$

and $PC^2[0,1] = \{u : u''(t) \in PC[0,1]\}$.

Then $PC^2[0,1]$ is a Banach Space with norm

$$||u(t)|| = \max\{||u||_\infty, ||u'||_\infty, ||u''||_\infty\},$$

where

$$||w||_\infty = \sup_{0 \le t \le 1} |w(t)|.$$

As a solution u of problem (1.1)-(1.2) one should consider $u(t) \in E$, where $E := PC[0,1] \cap C^2(J')$.

Lower and upper solutions will be given by next definition:

Definition 1.1 *A function $\alpha(t) \in E$ with $\phi(\alpha''(t)) \in PC^2[0,1]$ is a lower solution of problem (1.1)-(1.2) if*

$$\begin{cases} (\phi(\alpha''(t)))' + q(t)f(t,\alpha(t),\alpha'(t),\alpha''(t)) \geqslant 0 \\ \Delta\alpha(t_k) \leqslant I_{1k}(\alpha(t_k),\alpha'(t_k)) \\ \Delta\alpha'(t_k) > I_{2k}(\alpha(t_k),\alpha'(t_k)) \\ \Delta\phi(\alpha''(t_k)) > I_{3k}(\alpha(t_k),\alpha'(t_k),\alpha''(t_k)) \\ \alpha(0) \leqslant A, \\ \alpha'(0) \leqslant B, \\ \alpha''(1) < C. \end{cases} \tag{1.3}$$

A function $\beta(t) \in E$ such that $\phi(\beta''(t)) \in PC^2[0,1]$ and satisfies the opposite inequalities above, is an upper solution of (1.1)-(1.2).

For the linear problem associated to (1.1) it is possible to evaluate explicitly its solution, as it is shown in next lemma:

Lemma 1.2 *The problem composed by the differential equation*

$$(\phi(u''(t)))' + v(t) = 0 \tag{1.4}$$

and conditions

$$\begin{cases} \Delta u(t_k) = I_{1k}(u(t_k),u'(t_k)) \\ \Delta u'(t_k) = I_{2k}(u(t_k),u'(t_k)) \\ \Delta\phi(u''(t_k)) = I_{3k}(u(t_k),u'(t_k),u''(t_k)) \\ u(0) = A, \\ u'(0) = B, \\ u''(1) = C \end{cases} \tag{1.5}$$

has a unique solution given by

$$\begin{aligned} u(t) \quad = \quad & A + Bt + \sum_{t>t_k} I_{1k}(u(t_k),u'(t_k)) + \sum_{t>t_k} I_{2k}(u(t_k),u'(t_k))\ t \\ & + \int_0^t \int_0^\mu \phi^{-1}\left[\phi(C) + \int_\zeta^1 v(s)ds - \sum_{\zeta<t_k} I_{3k}(u(t_k),u'(t_k),u''(t_k))\right] d\zeta d\mu. \end{aligned}$$

Proof. For $t \in (t_n, 1]$, integrating (1.4) from t to 1, we have

$$u''(t) = \phi^{-1}\left(\phi(C) + \int_t^1 v(s)ds\right). \tag{1.6}$$

For $t \in (t_{n-1}, t_n]$, with $t_0 := 0$, by integration of (1.4), it is obtained by (1.6),

$$\begin{aligned}
u''(t) &= \phi^{-1}\left[\int_t^{t_n} v(s)ds + \phi\left(u''\left(t_n^-\right)\right)\right] \\
&= \phi^{-1}\left[\int_t^{t_n} v(s)ds + \phi\left(u''\left(t_n^+\right)\right) - I_{3n}(u(t_n), u'(t_n), u''(t_n))\right] \\
&= \phi^{-1}\left[\phi(C) + \int_{t_n}^1 v(s)ds + \int_t^{t_n} v(s)ds - I_{3n}(u(t_n), u'(t_n), u''(t_n))\right] \\
&= \phi^{-1}\left[\phi(C) + \int_t^1 v(s)ds - I_{3n}(u(t_n), u'(t_n), u''(t_n))\right].
\end{aligned}$$

Therefore by induction, for $t \in (0,1)$, we get

$$u''(t) = \phi^{-1}\left[\phi(C) + \int_t^1 v(s)ds - \sum_{t<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k))\right]. \tag{1.7}$$

By integration of (1.7) in $[0, t_1]$,

$$u'(t_1^-) = B + \int_0^{t_1} \phi^{-1}\left[\phi(C) + \int_\mu^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k))\right] d\mu. \tag{1.8}$$

Integrating (1.7) for $t \in (t_1, t_2]$, and applying (1.2) and (1.8),

$$\begin{aligned}
u'(t) &= u'(t_1^+) + \int_{t_1}^t \phi^{-1}\left[\phi(C) + \int_\mu^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k))\right] d\mu \\
&= u'(t_1^-) + I_{21}(u(t_1), u'(t_1)) \\
&\quad + \int_{t_1}^t \phi^{-1}\left[\phi(C) + \int_\mu^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k))\right] d\mu \\
&= I_{21}(u(t_1), u'(t_1)) + B \\
&\quad + \int_0^t \phi^{-1}\left[\phi(C) + \int_\mu^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k))\right] d\mu.
\end{aligned}$$

So, for $t \in [0,1]$,

$$\begin{aligned}
u'(t) &= \sum_{t>t_k} I_{2k}(u(t_k), u'(t_k)) + B \qquad (1.9) \\
&\quad + \int_0^t \phi^{-1}\left[\phi(C) + \int_\mu^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k))\right] d\mu
\end{aligned}$$

Integrating (1.9) for $t \in [0, t_1]$,

$$u(t_1^-) = A + \left(\sum_{t > t_k} I_{2k}(u(t_k), u'(t_k)) + B \right) t_1 + \int_0^{t_1} \int_0^r \phi^{-1} \left[\phi(c) + \int_\mu^1 v(s) ds - \sum_{\mu < t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k)) \right] d\mu dr \tag{1.10}$$

Integrating (1.9) for $t \in (t_1, t_2]$, by (1.10),

$$\begin{aligned} u(t) &= u(t_1^-) + \left(\sum_{t > t_k} I_{2k}(u(t_k), u'(t_k)) + B \right)(t - t_1) \\ &\quad + \int_{t_1}^{t} \int_0^\mu \phi^{-1} \left[\phi(C) + \int_\tau^1 v(s) ds - \sum_{\mu < t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k)) \right] d\tau d\mu \\ &= A + \left(\sum_{t > t_k} I_{2k}(u(t_k), u'(t_k)) + B \right) t \\ &\quad + \int_0^t \int_0^\mu \phi^{-1} \left[\phi(C) + \int_\tau^1 v(s) ds - \sum_{\mu < t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k)) \right] d\tau d\mu \end{aligned}$$

Finally, for $t \in [0, 1]$,

$$\begin{aligned} u(t) &= A + \sum_{t > t_k} I_{1k}(u(t_k), u'(t_k)) + \left(\sum_{t > t_k} I_{2k}(u(t_k), u'(t_k)) + B \right) t \\ &\quad + \int_0^t \int_0^\mu \phi^{-1} \left[\phi(C) + \int_\tau^1 v(s) ds - \sum_{\mu < t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k)) \right] d\tau d\mu. \end{aligned} \tag{1.11}$$

■

Define the continuous functions $\delta_i(t, u^{(i)}(t))$, for $i = 0, 1$, such that

$$\delta_i(t, u^{(i)}) = \begin{cases} \beta^{(i)}(t), & u^{(i)}(t) \geqslant \beta^{(i)}(t) \\ u^{(i)}(t), & \alpha^{(i)}(t) \leqslant u^{(i)}(t) \leqslant \beta^{(i)}(t) \\ \alpha^{(i)}(t), & u^{(i)}(t) \leqslant \alpha^{(i)}(t) \end{cases}$$

and consider the following modified and perturbed problem

$$\begin{cases} (\phi(u''(t)))' + q(t) f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \frac{d}{dt}\delta_1(t, u'(t))) \\ + \frac{\delta_1(t, u'(t)) - u'(t)}{1 + |u'(t) - \delta_1(t, u'(t))|} = 0 \\ \Delta u(t_k) = I_{1k}(\delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k))) \\ \Delta u'(t_k) = I_{2k}(\delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k))) \\ \Delta \phi(u''(t_k)) = I_{3k}(\delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)), \frac{d}{dt}\delta_1(t_k, u'(t_k))) \\ u(0) = A, \\ u'(0) = B, \\ u''(1) = C. \end{cases} \tag{1.12}$$

For this modified problem we have the following existence and localization result:

Lemma 1.3 *Assume that $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of (1.1), respectively, with $\alpha'(t) \leqslant \beta'(t)$, the continuous function f satisfies*

$$f(t,\alpha(t),y,z) \leq f(t,u(t),y,z) \leq f(t,\beta(t),y,z), \tag{1.13}$$

for $\alpha \leq u \leq \beta$, and fixed $(y,z) \in \mathbb{R}^2$. If

$$I_{1k}(\alpha(t_k),\alpha'(t_k)) \leqslant I_{1k}(u(t_k),u'(t_k)) \leqslant I_{1k}(\beta(t_k),\beta'(t_k)), \tag{1.14}$$

and

$$I_{2k}(\alpha(t_k),y) \geq I_{2k}(u(t_k),y) \geq I_{2k}(\beta(t_k),y), \tag{1.15}$$

for $k = 1,...,n$, $\alpha(t_k) \leqslant u(t_k) \leqslant \beta(t_k)$, $\alpha'(t_k) \leqslant u'(t_k) \leqslant \beta'(t_k)$ and fixed $y \in \mathbb{R}$, then every $u(t)$ solution of problem(1.12) verifies

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \text{ and } \alpha'(t) \leqslant u'(t) \leqslant \beta'(t), \text{ for } t \in [0,1].$$

Proof. To prove the second inequalities suppose, by contradiction, that there is $t \in [0,1]$ such that $u'(t) > \beta'(t)$. Therefore

$$\sup_{t\in[0,1]} (u'(t) - \beta'(t)) := u'(\bar{t}_0) - \beta'(\bar{t}_0) > 0. \tag{1.16}$$

As by boundary conditions, $u'(0) - \beta'(0) \leqslant 0$, then $\bar{t}_0 \neq 0$. In the same way $u''(1^-) - \beta''(1^-) < 0$ and then $\bar{t}_0 \neq 1$.

Let $t_0 = 0$ and $t_{n+1} = 1$. As the $\max\limits_{t\in[0,1]} (u' - \beta')(t)$ can not be achieved for $t = 1$ because of boundary conditions, only two cases must be considered:

Case 1: Assume that there is $p \in \{1,2,...,n\}$ such that $\bar{t}_0 \in (t_p, t_{p+1})$.

Define

$$\bar{t}_1 = \max_{t\in(t_p,\bar{t}_0)} \{t : (u' - \beta')(t) \leqslant 0\}$$

and

$$\bar{t}_2 = \min_{t\in(\bar{t}_0,t_p)} \{t : (u' - \beta')(t) \leqslant 0\}$$

If $(u' - \beta')(t) > 0$, $\forall t \in (t_p, \bar{t}_0)$ then consider $\bar{t}_1 = t_p$. Analogously for $(u' - \beta')(t) > 0$, $\forall t \in (\bar{t}_0, t_{p+1})$ then define $\bar{t}_2 = t_{p+1}$.

Therefore, by (1.13), for all $t \in (\bar{t}_1, \bar{t}_2)$,

$$\begin{aligned}
(\phi(u''(t)))' - (\phi(\beta''(t))' &\geqslant -q(t)\ f\left(t, \delta_0(t,u), \delta_1(t,u'), \frac{d}{dt}\delta_1(t,u')\right) \\
&\quad -\frac{\delta_1(t,u') - u'(t)}{1 + |u'(t) - \delta_1(t,u')|} + q(t) f\left(t, \beta(t), \beta'(t), \beta''(t)\right) \\
&= -q(t) f(t, \delta_0(t,u), \beta'(t), \beta''(t)) \\
&\quad -\frac{\beta'(t) - u'(t)}{1 + |u'(t) - \beta'(t)|} + q(t) f(t, \beta(t), \beta'(t), \beta''(t)) \\
&\geqslant -q(t) f(t, \beta(t), \beta'(t), \beta''(t)) - \frac{\beta'(t) - u'(t)}{1 + |u'(t) - \beta'(t)|} \\
&\quad + q(t) f(t, \beta(t), \beta'(t), \beta''(t)) \\
&= \frac{u'(t) - \beta'(t)}{1 + |u'(t) - \beta'(t)|} > 0.
\end{aligned}$$

So $\phi(u''(t)) - \phi(\beta''(t))$ is increasing for all $t \in (\bar{t}_1, \bar{t}_2)$.

For $t \in]\bar{t}_0, \bar{t}_2]$,

$$0 = \phi(u''(\bar{t}_0)) - \phi(\beta''(\bar{t}_0)) < \phi(u''(t)) - \phi(\beta''(t))$$

and $u''(t) > \beta''(t)$. Therefore $(u' - \beta')(t)$ is increasing in $]\bar{t}_0, t_2]$, which contradicts (1.16).

Case 2: Suppose that there is $p \in \{0, 1, 2, ..., n-1\}$ such that

$$\sup_{t \in [0,1]} (u'(t) - \beta'(t)) := u'(t_p) - \beta'(t_p) > 0 \tag{1.17}$$

or

$$\max_{t \in [0,1]} (u'(t) - \beta'(t)) := u'(t_{p+1}) - \beta'(t_{p+1}) > 0. \tag{1.18}$$

If (1.17) happens then

$$u''(t_p^+) - \beta''(t_p^+) \leq 0$$

and, for $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned}
u''(t) - \beta''(t) &\leq 0, \\
u'(t) - \beta'(t) &> 0, \ \forall t \in (t_p, t_p + \varepsilon).
\end{aligned} \tag{1.19}$$

So, for $t \in (t_p, t_p + \varepsilon) \subset [t_p, t_{p+1}]$,

$$\begin{aligned}
(\phi(u''(t)))' - (\phi(\beta''(t)))' &\geqslant -q(t) f\left(t, \delta_0(t,u), \delta_1(t,u'), \frac{d}{dt}\delta_1(t,u')\right) \\
&\quad -\frac{\beta'(t) - u'(t)}{1 + |u'(t)|} + q(t) f(t, \beta(t), \beta'(t), \beta''(t)) \\
&\geqslant \frac{u'(t) - \beta'(t)}{1 + |u'(t)|} > 0.
\end{aligned}$$

There is $\varepsilon > 0$ such that by integration on $t \in (t_p, t_p + \varepsilon)$ we get that $u''(t) > \beta''(t)$, which contradicts (1.19).

Assuming (1.18), we have

$$\max_{t\in[0,1]}(u'(t) - \beta'(t)) = u'(t_{p+1}) - \beta'(t_{p+1}) = u'(t_{p+1}^-) - \beta''(t_{p+1}^-) > 0$$

and, by (1.2) and (1.15), we achieve to the contradiction.

$$\begin{aligned} 0 &\geq u'(t_{p+1}^+) - \beta'(t_{p+1}^+) - \left[u'(t_{p+1}^-) - \beta''(t_{p+1}^-)\right] \\ &> I_{2p}(\delta_0(t_{p+1}, u), \delta_1(t_{p+1}, u')) - I_{2p}(\beta(t_{p+1}), \beta'(t_{p+1})) \\ &= I_{2p}(\delta_0(t_{p+1}, u), \beta'(t_{p+1})) - I_{2p}(\beta(t_{p+1}), \beta'(t_{p+1})) \geq 0. \end{aligned}$$

Therefore $u'(t) \leqslant \beta'(t)$, for $t \in [0,1]$. By similar arguments it can be proved the remaining inequality and therefore

$$\alpha'(t) \leqslant u'(t) \leqslant \beta'(t), \text{ for } t \in [0,1]. \tag{1.20}$$

By integration of (1.20) for $t \in [0, t_1]$,

$$\alpha(t) \leqslant u(t) - u(0) + \alpha(0) \leq u(t). \tag{1.21}$$

Integrating (1.20) for $t \in]t_1, t_2]$, we have, by (1.14) and (1.21),

$$\begin{aligned} \alpha(t) &\leqslant u(t) - u(t_1^+) + \alpha(t_1^+) \\ &\leq u(t) - I_{11}\left(\delta_0(t_1, u), \delta_1(t_1, u')\right) - u(t_1^-) + I_{11}\left(\alpha(t_1), \alpha'(t_1)\right) + \alpha(t_1^-) \\ &\leq u(t). \end{aligned}$$

By recurrence, we can prove analogously, that

$$\alpha(t) \leqslant u(t), \ \forall t \in [t_k, t_{k+1}], \text{ for } k = 1, 2, ..., n.$$

So $\alpha(t) \leqslant u(t)$, $\forall t \in [0,1]$. Applying similar arguments it can be proved the remaining inequality and, therefore,

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \text{ for } t \in [0,1].$$

■

To control the growth of the second derivative we apply a Nagumo-type condition:

Definition 1.4 *A function f satisfies a Nagumo condition related to a pair of functions $\gamma, \Gamma \in PC[0,1] \cap C^2(J')$, with $\gamma' \leqslant \Gamma'$, if exists a function $\psi : C([0,+\infty),]0,+\infty))$ such that:*

$$|f(t,x,y,z)| \leqslant \psi(|z|), \ \forall (t,x,y,z) \in F \tag{1.22}$$

with

$$F = \{(t,x,y,z) \in [0,1] \times \mathbb{R}^3 : \gamma(t) \leqslant x \leqslant \Gamma(t), \gamma'(t) \leqslant y \leqslant \Gamma'(t)\}$$

and such that

$$\int_{\phi(\mu)}^{+\infty} \frac{ds}{\psi(\phi^{-1}(s))} > \int_0^1 q(s)ds,$$

where

$$\mu := \max_{k=0,1,2,\dots,n} \left\{ \left| \frac{\Gamma'(t_{k+1}) - \gamma'(t_k)}{t_{k+1} - t_k} \right|, \left| \frac{\gamma'(t_{k+1}) - \Gamma'(t_k)}{t_{k+1} - t_k} \right| \right\}.$$

From Nagumo condition it is possible to have an *a priori* estimation on the second derivative:

Lemma 1.5 *Let α and β be lower and upper solutions of problem (1.1)-(1.2) such that $\alpha \le \beta$ and $\alpha' \le \beta'$ in $[0,1]$. If the continuous function $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies a Nagumo condition in the set F, then there is $N \ge \mu > 0$ such that every solution u of the differential equation*

$$(\phi(u''(t)))' + q(t)f(t, u(t), u'(t), u''(t)) = 0, \text{ in } J', \tag{1.23}$$

verifies $||u''||_\infty \le N$.

Proof. Let $u(t)$ be a solution of (1.23) such that

$$\alpha(t) \leqslant u(t) \leqslant \beta(t) \text{ and } \alpha'(t) \leqslant u'(t) \leqslant \beta'(t), \text{ for } t \in [0,1].$$

By the Mean Value Theorem, there exists $\eta_0 \in (t_k, t_{k+1})$ with

$$u''(\eta_0) = \frac{u'(t_{k+1}) - u'(t_k)}{t_{k+1} - t_k}, \text{ with } k = 0, 1, 2, ..., n.$$

Moreover,

$$-N \le -\mu \le \frac{\alpha'(t_{k+1}) - \beta'(t_k)}{t_{k+1} - t_k} \le u''(\eta_0) \le \frac{\beta'(t_{k+1}) - \alpha'(t_k)}{t_{k+1} - t_k} \le \mu \le N.$$

If $|u''(t)| \le N$ in $[0,1]$, the proof is complete.

Assume that there is $\tau \in [0,1]$ such that $|u''(\tau)| > N$.

Consider the case where $u''(\tau) > N$. Therefore there is η_1 such that $u''(\eta_1) = N$.

If $\eta_0 < \eta_1$, suppose, without loss of generality, that

$$u''(t) > 0 \text{ and } u''(\eta_0) \le u''(t) \le N, \text{ for } t \in [\eta_0, \eta_1].$$

So

$$|\phi(u''(t))| = |q(t)f(t, u(t), u'(t), u''(t))| \leqslant q(t)|\psi(u''(t))|, \text{ for } t \in [\eta_0, \eta_1],$$

and, by (1.22),

$$\begin{aligned}
\int_{\phi(u''(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s)} &\le \int_{\eta_0}^{\eta_1} \frac{|(\phi(u''(t))'|}{\psi(u''(t))} dt = \int_{\eta_0}^{\eta_1} \frac{|q(t)f(t, u(t), u'(t), u''(t))|}{\psi(u''(t))} dt \\
&\leqslant \int_{\eta_0}^{\eta_1} q(t)dt < \int_0^1 q(t)dt.
\end{aligned}$$

As $u''(\eta_0) \leq \mu < N$, by the monotony of ϕ,

$$\phi(u''(\eta_0)) \leqslant \phi(\mu)$$

and

$$\int_{\phi(u''(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi_p^{-1}(s)} \geqslant \int_{\phi(\mu)}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s)} > \int_0^1 q(t)dt$$

which leads to a contradiction.

The other cases, that is,

$$u''(\tau) > N \text{ with } \eta_1 < \eta_0,$$

and

$$u''(\tau) < -N \text{ with } \eta_0 < \eta_1 \text{ or } \eta_1 < \eta_0,$$

follow the same arguments to obtain a contradiction.

Therefore

$$|u''(t)| \leq N, \text{ for } t \in [0,1].$$

■

We recall the classical Shauder's fixed point theorem:

Theorem 1.6 *Let M be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $T : M \to M$ is a compact operator. Then T as at least one fixed point in M.*

1.3 Main result

The main theorem gives the existence and the localization of solution, and its first and second derivatives, for the initial problem:

Theorem 1.7 *Suppose that assumptions (A_1)-(A_3) hold and there are α and β be lower and upper solutions, respectively, of problem (1.1)-(1.2) such that $\alpha \leq \beta$ and $\alpha' \leq \beta'$ in $[0,1]$.*
If the continuous function $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies a Nagumo condition and verifies (1.13) and the impulsive functions I_{ik} satisfy (1.14) and (1.15), then problem (1.1)-(1.2) has at least one solution $u \in E$, such that

$$\alpha(t) \leqslant u(t) \leqslant \beta(t),\ \alpha'(t) \leqslant u'(t) \leqslant \beta'(t) \text{ and } -N \leqslant u''(t) \leqslant N, \text{ for } t \in [0,1]$$

Proof. Consider the modified and perturbed problem (1.12). To obtain a solution for problem (1.12) is equivalent to find a function $u \in E$ such that

$$\begin{aligned} u(t) \quad &= \quad A + Bt + \sum_{t>t_k} I_{1k}^*(u(t_k), u'(t_k)) + \sum_{t>t_k} I_{2k}^*(u(t_k), u'(t_k))\ t \\ &\quad + \int_0^t \int_0^\mu \phi^{-1}\left[\phi(c) + \int_\zeta^1 F_u(s)ds - \sum_{\zeta<t_k} I_{3k}^*(u(t_k), u'(t_k), u''(t_k))\right] d\zeta d\mu. \end{aligned}$$

where

$$\begin{aligned} F_u(s) \quad &: \quad = q(s) f(s, \delta_0(s, u(s)), \delta_1(s, u'(s)), \frac{d}{ds}\delta_1(s, u'(s)) \\ & \quad + \frac{\delta_1(s, u'(s)) - u'(s)}{1 + |u'(s) - \delta_1(s, u'(s))|}, \\ I_{ik}^*(u(t_k), u'(t_k)) \quad &: \quad = I_{ik}(\delta_0(t_k, u(t_k), \delta_1(t_k, u'(t_k)), \ i = 1, 2, \\ I_{3k}^*(u(t_k), u'(t_k), u''(t_k)) \quad &: \quad = I_{3k}(\delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)), \frac{d}{dt}\delta_1(t_k, u'(t_k))). \end{aligned}$$

Such function will be obtained as a fixed point of the operator $T : E \to E$, given by

$$\begin{aligned} T(u)(t) \quad : \quad &= A + Bt + \sum_{t > t_k} I_{1k}^*(u(t_k), u'(t_k)) + \sum_{t > t_k} I_{2k}^*(u(t_k), u'(t_k)) \ t \\ & + \int_0^t \int_0^\mu \phi^{-1} \left[\phi(c) + \int_\zeta^1 F_u(s) ds - \sum_{\zeta < t_k} I_{3k}^*(u(t_k), u'(t_k), u''(t_k)) \right] d\zeta d\mu. \end{aligned}$$

As T is completely continuous, by Theorem 1.6, T has a fixed point $u \in E$ which is a solution of (1.12).

By Lemma 1.2 and Lemma 1.3, this function $u \in E$ is also a solution of the problem (1.1). ■

1.4 Example

Consider the problem composed by the differential equation

$$\frac{u'''(t)}{1 + (u''(t))^2} + \arctan(u) - 6\,(u'(t))^3 - 2\sqrt[3]{u''(t) + 1} = 0, \quad \text{in } [0,1] \setminus \left\{ \frac{1}{2} \right\}, \tag{1.24}$$

the impulsive impulses given, for $t_1 = \frac{1}{2}$, by

$$\begin{cases} \Delta u(\frac{1}{2}) = u(\frac{1}{2}) + u'(\frac{1}{2}), \\ \Delta u'(\frac{1}{2}) = -u(\frac{1}{2}) + u'(\frac{1}{2}) \\ \Delta \phi(u''(\frac{1}{2})) = u(\frac{1}{2}), \end{cases} \tag{1.25}$$

and the boundary conditions

$$\begin{cases} u(0) = A, \\ u'(0) = B, \\ u''(1) = C, \ . \end{cases} \tag{1.26}$$

Problem (1.24), (1.25), (1.26) is a particular case of problem (1.1)-(1.2) with

$$\begin{aligned} \phi(w) &= \arctan(w) \\ q(t) &\equiv 1 \\ f(t,x,y,z) &= \arctan(x) - 6y^3 - 2\sqrt[3]{z+1} \\ I_{11}(x,y) &= x+y \\ I_{21}(x,y) &= -x+y \\ I_{31}(x,y,z) &= x. \end{aligned}$$

For $A \in [-1,0]$, $B \in [-2,1]$ and $C \in]0,6[$, the functions

$$\alpha(t) = \begin{cases} -2t-1 & , \quad 0 \leq t \leq \frac{1}{2} \\ -t-6 & , \quad \frac{1}{2} < t \leq 1 \end{cases}$$

and

$$\beta(t) = \begin{cases} t^3+t & , \quad 0 \leq t \leq \frac{1}{2} \\ t^3+3t+2 & , \quad \frac{1}{2} < t \leq 1 \end{cases}$$

are, respectively, lower and upper solutions of problem (1.24), (1.25), (1.26), considering

$$\alpha'(t) = \begin{cases} -2 & , \quad 0 \leq t \leq \frac{1}{2} \\ -1 & , \quad \frac{1}{2} < t \leq 1, \end{cases}$$

$$\beta'(t) = \begin{cases} 3t^2+1 & , \quad 0 \leq t \leq \frac{1}{2} \\ 3t^2+3 & , \quad \frac{1}{2} < t \leq 1, \end{cases}$$

$\alpha''(t) \equiv 0$ and $\beta''(t) = 6t$, in $[0,1]$.

As the assumptions of Theorem 1.7 are fulfilled, therefore there is a solution of problem (1.24), (1.25), (1.26), for $A \in [-1,0]$, $B \in [-2,1]$ and $C \in]0,6[$, such that

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \alpha'(t) \leqslant u'(t) \leqslant \beta'(t), \text{ in } [0,1].$$

Chapter 2

Half-linear impulsive problems with more general jump conditions

2.1 Introduction

Consider the separated boundary value problem which includes the third-order fully differential equation

$$(\phi(u''(t)))' + q(t)f(t, u(t), u'(t), u''(t)) = 0, t \in [a,b]\backslash\{t_1, ..., t_n\}, \tag{2.1}$$

where ϕ is an increasing homeomorphism with $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$, $q \in C([a,b])$ with $q > 0$ and $\int_a^b q(s)ds < \infty$, $f \in C([a,b] \times \mathbb{R}^3, \mathbb{R})$, and the boundary conditions

$$u(a) = A, \; u'(a) = B, \; u''(b) = C, \quad A, B, C \in \mathbb{R}. \tag{2.2}$$

The impulsive conditions are given by

$$\begin{array}{rcl} \Delta u(t_k) & = & I_{1k}(t_k, u(t_k), u'(t_k)), \\ \Delta u'(t_k) & = & I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)), \\ \Delta \phi(u''(t_k)) & = & I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)), \end{array} \tag{2.3}$$

where $k = 1, 2, ...n$, $\Delta\phi(u''(t_k)) = \phi(u''(t_k^+)) - \phi(u''(t_k^-))$, $I_{ik} \in C([a,b] \times \mathbb{R}^2, \mathbb{R})$, $i = 1, 2$, and $I_{3k} \in C([a,b] \times \mathbb{R}^3, \mathbb{R})$, with t_k fixed points such that $a < t_1 < t_2 < ... < t_n < b$.

Boundary value problems with impulsive jumps had been object of attention in recent literature, not only due to applications of real phenomena, but also in theoretical studies where some sudden discontinuities happen. See, for example, [9, 54, 79, 91] and the references therein.

The most common methods apply a variational approach and critical point techniques ([58, 112]), cones theory ([96, 104]), bifurcation theorems ([81, 108]), and upper and lower solutions geometry ([42, 97]).

In the recent work [12], the authors obtains the existence of solution for the differential system

$$\begin{aligned} (\phi_p(\rho(t)u'(t)))' &= f(t,u(t),v(t)), \quad a.e. t \in \mathbb{R}, \\ (\phi_q(\varsigma(t)v'(t)))' &= g(t,u(t),v(t)), \quad a.e. t \in \mathbb{R}, \end{aligned}$$

with the boundary conditions

$$\lim_{t\to\pm\infty} u(t) = 0, \qquad \lim_{t\to\pm\infty} v(t) = 0,$$

and the impulsive effects

$$\begin{aligned} \Delta u(t_k) &= I_k(t_k, u(t_k), v(t_k)), \ k \in \mathbb{Z}, \\ \Delta v(t_k) &= J_k(t_k, u(t_k), v(t_k)), \ k \in \mathbb{Z}, \end{aligned}$$

via integral operators and Schauder fixed point theorem.

Motivated by this result and the above works, we consider a nonlinear third order problem with generalized impulsive effects.

To the best of our knowledge, it is the first result where the impulsive functions can depend on several variables, such as the unknown variable and its first and second derivatives. Moreover, we point out that our results can be applied not only to classical ϕ-Laplacian operators, but also to singular ones, in the sense of [3, 16].

The chapter is organized in the following way: in Section 2.2 it is presented an uniqueness result for an adequate problem associated to (2.1)-(2.3) and some definitions to be used forward. Section 2.3 contains the main result: an existence and location theorem. We point out that this result is obtained without sign or asymptotic type assumptions, due to some truncation and perturbation techniques, suggested for instance in [23, 34, 55, 71], applying upper and lower solutions technique and fixed point theory. Last section presents an example where the impulsive conditions can depend on the unknown function and on its variation given by first and second derivatives.

2.2 Preliminary results

Consider the sets

$$PC^m[a,b] = \left\{ \begin{array}{c} u : u \in C^m([a,b],\mathbb{R}) \text{ for } t \neq t_k, u^{(i)}(t_k) = u^{(i)}(t_k^-), u^{(i)}(t_k^+) \\ \text{exists for } k = 1,2,...,n, \text{ and } i = 0,1,...,m \end{array} \right\},$$

which is a Banach Space with the norm

$$||u|| = \max\{||u^{(i)}||_\infty, i = 0,1,...,m\},$$

where

$$||w||_\infty = \sup_{a\le t\le b} |w(t)|.$$

By a solution u of problem (2.1)-(2.3) we mean $u(t) \in E_2$, with $E_2 := PC^2[a,b]$, verifying (2.1), the boundary conditions (2.2), and the impulse effects (2.3).

Next lemma provides an uniqueness result for an adequate problem related to (2.1), the boundary conditions (2.2) and the impulsive effects (2.3).

Lemma 2.1 *The problem defined by the differential equation*

$$(\phi(u''(t)))' + v(t) = 0 \tag{2.4}$$

and conditions (2.2), (2.3), has a unique solution given by

$$\begin{aligned}
u(t) \quad = \quad & A + B\,(t-a) + \sum_{t_k<t} I_{1k}(t_k, u(t_k), u'(t_k)) \\
& + \sum_{t_k<t} I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)\,(t-t_k) \\
& + \int_a^t \int_a^r \phi^{-1}\left[\phi(C) + \int_\mu^b v(s)ds - \sum_{\mu<t_k} I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k))\right] d\mu dr.
\end{aligned}$$

Proof. Consider $t \in (t_n, b]$. By integration of (2.4) from t to b,

$$u''(t) = \phi^{-1}\left(\phi(C) + \int_t^b v(s)ds\right). \tag{2.5}$$

For $t \in (t_{n-1}, t_n]$, integrating (2.4) and by (2.5),

$$\begin{aligned}
u''(t) \quad = \quad & \phi^{-1}\left[\int_t^{t_n} v(s)ds + \phi\left(u''\left(t_n^-\right)\right)\right] \\
= \quad & \phi^{-1}\left[\int_t^{t_n} v(s)ds + \phi\left(u''\left(t_n^+\right)\right) - I_{3n}(t_n, u(t_n), u'(t_n), u''(t_n))\right] \\
= \quad & \phi^{-1}\left[\phi(C) + \int_{t_n}^b v(s)ds + \int_t^{t_n} v(s)ds - I_{3n}(t_n, u(t_n), u'(t_n), u''(t_n))\right] \\
= \quad & \phi^{-1}\left[\phi(C) + \int_t^b v(s)ds - I_{3n}(t_n, u(t_n), u'(t_n), u''(t_n))\right].
\end{aligned}$$

So, by mathematical induction, with $t_0 = a$, for $t \in (a,b)$,

$$u''(t) = \phi^{-1}\left[\phi(C) + \int_t^b v(s)ds - \sum_{t<t_k} I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k))\right]. \tag{2.6}$$

Integrating last equality of (2.6) in $[a, t_1]$,

$$u'(t_1^-) = B + \int_a^{t_1} \phi^{-1}\left[\phi(C) + \int_\mu^b v(s)ds - \sum_{\mu<t_k} I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k))\right] d\mu. \tag{2.7}$$

From the integration of (2.6) with $t \in (t_1, t_2]$, by (2.3) and (2.7),

$$\begin{aligned} u'(t) &= u'(t_1^+) + \int_{t_1}^{t} \phi^{-1} \left[\phi(C) + \int_{\mu}^{b} v(s)ds - \sum_{\mu < t_k} I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)) \right] d\mu \\ &= u'(t_1^-) + I_{21}(t_1, u(t_1), u'(t_1), u''(t_1)) \\ &\quad + \int_{t_1}^{t} \phi^{-1} \left[\phi(C) + \int_{\mu}^{b} v(s)ds - \sum_{\mu < t_k} I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)) \right] d\mu \\ &= I_{21}(t_1, u(t_1), u'(t_1), u''(t_1)) + B \\ &\quad + \int_{a}^{t} \phi^{-1} \left[\phi(C) + \int_{\mu}^{b} v(s)ds - \sum_{\mu < t_k} I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)) \right] d\mu. \end{aligned}$$

Therefore, for $t \in [a, b]$,

$$\begin{aligned} u'(t) &= \sum_{t_k < t} I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)) + B \\ &\quad + \int_{a}^{t} \phi^{-1} \left[\phi(C) + \int_{\mu}^{b} v(s)ds - \sum_{\mu < t_k} I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)) \right] d\mu \end{aligned} \tag{2.8}$$

Integrating (2.8) for $t \in [a, t_1]$,

$$\begin{aligned} u(t_1^-) &= A + B\ \ (t_1 - a) \\ &\quad + \int_{a}^{t_1} \int_{a}^{r} \phi^{-1} \left[\phi(c) + \int_{\mu}^{b} v(s)ds - \sum_{\mu < t_k} I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)) \right] d\mu dr \end{aligned} \tag{2.9}$$

Integrating (2.8) for $t \in (t_1, t_2]$, by (2.9),

$$\begin{aligned} u(t) &= u(t_1^+) + \left(\sum_{t_k < t} I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)) + B \right) (t - t_1) \\ &\quad + \int_{t_1}^{t} \int_{a}^{r} \phi^{-1} \left[\phi(C) + \int_{\mu}^{b} v(s)ds - \sum_{\mu < t_k} I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)) \right] d\mu dr \\ &= A + I_{11}(t_1, u(t_1), u'(t_1)) + I_{21}(t_1, u(t_1), u'(t_1), u''(t_1))\,(t - t_1) + B\,(t - a) \\ &\quad + \int_{a}^{t} \int_{a}^{r} \phi^{-1} \left[\phi(C) + \int_{\mu}^{b} v(s)ds - \sum_{\mu < t_k} I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)) \right] d\mu dr \end{aligned}$$

So, for $t \in [a, b]$,

$$u(t) = A + \sum_{t_k < t} I_{1k}(t_k, u(t_k), u'(t_k)) + \sum_{t_k < t} I_{2k}(t_k, u(t_k), u'(t_k) u''(t_k))\,(t - t_k)$$

$$+ B\,(t - a) + \int_{a}^{t} \int_{a}^{r} \phi^{-1} \left[\phi(C) + \int_{\mu}^{b} v(s)ds - \sum_{\mu < t_k} I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)) \right] d\mu dr.$$

■

Next definition states the admissible lower and upper functions:

Definition 2.2 *A function $\alpha(t) \in E_2$ with $\phi(\alpha''(t)) \in PC^1[a,b]$ is a lower solution of problem (2.1), (2.2), (2.3) if*

$$\begin{cases} (\phi(\alpha''(t)))' + q(t)f(t,\alpha(t),\alpha'(t),\alpha''(t)) \geq 0 \\ \Delta\alpha(t_k) \leq I_{1k}(t_k,\alpha(t_k),\alpha'(t_k)) \\ \Delta\alpha'(t_k) \geq I_{2k}(t_k,\alpha(t_k),\alpha'(t_k),z), \;\; z \in \mathbb{R}, \\ \Delta\phi(\alpha''(t_k^+)) \geq I_{3k}(t_k,\alpha(t_k),\alpha'(t_k),\alpha''(t_k)) \\ \alpha(a) \leq A, \\ \alpha'(a) \leq B, \\ \alpha''(b) < C. \end{cases} \tag{2.10}$$

A function $\beta(t) \in E_2$ with $\phi(\beta''(t)) \in PC^1[a,b]$ and satisfying the reversed inequalities, is an upper solution of (2.1)-(2.3).

A Nagumo-type growth condition will be the key to bound the second derivatives of the unknown function:

Definition 2.3 *A function f verifies a Nagumo-type growth condition related to some functions $\gamma, \Gamma \in PC^1[a,b]$, with $\gamma' \leq \Gamma'$, if there is $\psi : C([0,+\infty),]0,+\infty))$ such that:*

$$|f(t,x,y,z)| \leq \psi(|z|), \;\; \textit{for} \;\; \textit{all} \; (t,x,y,z) \in F \tag{2.11}$$

where

$$F = \{(t,x,y,z) \in [a,b] \times \mathbb{R}^3 : \gamma(t) \leq x \leq \Gamma(t), \gamma'(t) \leq y \leq \Gamma'(t)\},$$

and

$$\int_{\phi(\mu)}^{+\infty} \frac{ds}{\psi(\phi^{-1}(s))} > \int_a^b q(s)ds,$$

with

$$\mu := \max_{k=0,1,2,\ldots,n} \left\{ \left| \frac{\Gamma'(t_{k+1}) - \gamma'(t_k)}{t_{k+1} - t_k} \right|, \left| \frac{\gamma'(t_{k+1}) - \Gamma'(t_k)}{t_{k+1} - t_k} \right| \right\}.$$

The arguments of the proof require the following lemma, given in [73]:

Lemma 2.4 *For $v, w \in C(I)$ such that $v(x) \leq w(x)$, for every $x \in I$, define*

$$q(x,u) = max\{v, min\{u,w\}\}.$$

Then, for each $u \in C^1(I)$ the next two properties hold:

(a) $\frac{d}{dx}q(x,u(x))$ *exists for a.e. $x \in I$.*

(b) *If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$ then*

$$\frac{d}{dx}q(x,u_m(x)) \to \frac{d}{dx}q(x,u(x)) \; \textit{for a.e.} \; x \in I.$$

2.3 Main result

The main theorem is an existence and localization result, as it guarantees not only the existence of solutions but provides also some qualitative properties on this solution.

Theorem 2.5 *Let $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying a Nagumo-type growth condition. Assume that there are α and β lower and upper solutions, respectively, of problem (2.1)-(2.3) such that $\alpha'(t) \leq \beta'(t)$ in $[a,b]$.*
If f verifies

$$f(t,\alpha(t),y,z) \leq f(t,x,y,z) \leq f(t,\beta(t),y,z), \tag{2.12}$$

for $\alpha \leq x \leq \beta$, and fixed $(y,z) \in \mathbb{R}^2$, and the impulsive functions verify

$$I_{1k}(t_k,\alpha(t_k),\alpha'(t_k)) \leq I_{1k}(t_k,x,y) \leq I_{1k}(t_k,\beta(t_k),\beta'(t_k)), \tag{2.13}$$

for $k = 1,...,n$, $\alpha(t_k) \leq x \leq \beta(t_k)$, $\alpha'(t_k) \leq y \leq \beta'(t_k)$,

$$I_{2k}(t_k,\alpha(t_k),\alpha'(t_k),z) \geq I_{2k}(t_k,x,y,z) \geq I_{2k}(t_k,\beta(t_k),\beta'(t_k),z), \tag{2.14}$$

for $k = 1,...,n$, $\alpha(t_k) \leq x \leq \beta(t_k)$, $\alpha'(t_k) \leq y \leq \beta'(t_k)$, and fixed $z \in \mathbb{R}$, and

$$I_{3k}(t_k,\alpha(t_k),y,z) \geq I_{3k}(t_k,x,y,z) \geq I_{3k}(t_k,\beta(t_k),y,z), \tag{2.15}$$

for $k = 1,...,n$, $\alpha(t_k) \leq x \leq \beta(t_k)$, and fixed $(y,z) \in \mathbb{R}^2$, then problem (2.1)-(2.3) has at least one solution $u \in E$, such that

$$\alpha(t) \leq u(t) \leq \beta(t),\ \ \alpha'(t) \leq u'(t) \leq \beta'(t) \ \ \text{and} \ \ -N \leq u''(t) \leq N,\ \text{for } t \in [a,b].$$

To prove this theorem we need some auxiliary results:
Define the continuous functions $\delta_i(t,u^{(i)}(t))$, for $i = 0,1$, such that

$$\delta_i(t,u^{(i)}) = \begin{cases} \beta^{(i)}(t), & u^{(i)}(t) \geq \beta^{(i)}(t) \\ u^{(i)}(t), & \alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t) \\ \alpha^{(i)}(t), & u^{(i)}(t) \leq \alpha^{(i)}(t) \end{cases}$$

and consider the following modified and perturbed equation

$$\begin{gathered}(\phi(u''(t)))' + q(t)f(t,\delta_0(t,u(t)),\delta_1(t,u'(t)),\frac{d}{dt}\delta_1(t,u'(t)) \\ +\frac{\delta_1(t,u'(t)) - u'(t)}{1+|u'(t)-\delta_1(t,u'(t))|} = 0,\end{gathered} \tag{2.16}$$

coupled with the truncated impulsive conditions

$$\begin{aligned} \Delta u(t_k) &= I_{1k}(t_k,\delta_0(t_k,u(t_k),\delta_1(t_k,u'(t_k)), \\ \Delta u'(t_k) &= I_{2k}(t_k,\delta_0(t_k,u(t_k)),\delta_1(t_k,u'(t_k),\frac{d}{dt}\delta_1(t_k,u'(t_k)), \\ \Delta\phi(u''(t_k)) &= I_{3k}(t_k,\delta_0(t_k,u(t_k)),\delta_1(t_k,u'(t_k)),\frac{d}{dt}\delta_1(t_k,u'(t_k))). \end{aligned} \tag{2.17}$$

and boundary conditions (2.2).

Remark that by Lemma 2.4 the auxiliary problem (2.16), (2.17), (2.2) is well defined.

Next lemma will prove that every solution of problem (2.16), (2.17), (2.2) will be a solution of problem (2.1)-(2.3), too :

Lemma 2.6 *Suppose that $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of problem (2.1)-(2.3), respectively, with $\alpha'(t) \leq \beta'(t)$, the continuous function f satisfies (2.12) and the impulsive functions I_{ik}, $i = 1, 2$, satisfy (2.13) and (2.14), then every $u(t)$ solution of problem (2.16), (2.17), (2.2) verifies*

$$\alpha(t) \leq u(t) \leq \beta(t), \text{ and } \alpha'(t) \leq u'(t) \leq \beta'(t), \text{ for } t \in [a, b].$$

Proof. To prove the second part of last inequality assume, by contradiction, that there is $t \in [a, b]$ such that $u'(t) > \beta'(t)$. Therefore

$$\sup_{t\in[a,b]} (u'(t) - \beta'(t)) := u'(\bar{t}_0) - \beta'(\bar{t}_0) > 0. \tag{2.18}$$

By boundary conditions, $u'(a) - \beta'(a) \leq 0$, and therefore $\bar{t}_0 \neq a$. In the same way $u''(b^-) - \beta''(b^-) < 0$ and then $\bar{t}_0 \neq b$. Let $t_0 = a$ and $t_{n+1} = b$. Moreover, by (2.2) and (2.10), the $\max_{t\in[a,b]} (u' - \beta')(t)$ can not be achieved for $t = b$. So two cases must be studied:

Case 1: Assume that there is $p \in \{1, 2, ..., n\}$ such that $\bar{t}_0 \in (t_p, t_{p+1})$.

Therefore,

$$\max_{t\in(t_p,t_{p+1})} (u'(t) - \beta'(t)) := u'(\bar{t}_0) - \beta'(\bar{t}_0) > 0,$$

and

$$u''(\bar{t}_0) - \beta''(\bar{t}_0) = 0.$$

Choose $\varepsilon > 0$, sufficiently small, such that

$$u'(t) - \beta'(t) > 0 \text{ and } u''(t) - \beta''(t) \leq 0, \ \forall t \in (\bar{t}_0, \bar{t}_0 + \varepsilon). \tag{2.19}$$

By (2.12), for all $t \in (\bar{t}_0, \bar{t}_0 + \varepsilon)$,

$$\begin{aligned}
(\phi(u''(t)))' - (\phi(\beta''(t))' \quad &\geq \quad -q(t)\ f\left(t, \delta_0(t, u), \delta_1(t, u'), \frac{d}{dt}\delta_1(t, u')\right) \\
&\quad -\frac{\delta_1(t, u') - u'(t)}{1 + |u'(t) - \delta_1(t, u')|} + q(t)f\left(t, \beta(t), \beta'(t), \beta''(t)\right) \\
&= \quad -q(t)f(t, \delta_0(t, u), \beta'(t), \beta''(t)) \qquad (2.20) \\
&\quad -\frac{\beta'(t) - u'(t)}{1 + |u'(t) - \beta'(t)|} + q(t)f((t, \beta(t), \beta'(t), \beta''(t)) \\
&\geq \quad -q(t)f(t, \beta(t), \beta'(t), \beta''(t)) - \frac{\beta'(t) - u'(t)}{1 + |u'(t) - \beta'(t)|} \\
&\quad +q(t)f((t, \beta(t), \beta'(t), \beta''(t)) \\
&= \quad \frac{u'(t) - \beta'(t)}{1 + |u'(t) - \beta'(t)|} > 0.
\end{aligned}$$

So $\phi(u''(t)) - \phi(\beta''(t)$ is increasing for all $t \in (\bar{t}_0, \bar{t}_0 + \varepsilon)$, and, by (2.19), we obtain the contradiction in $(\bar{t}_0, \bar{t}_0 + \varepsilon)$:

$$0 = \phi(u''(\bar{t}_0)) - \phi(\beta''(\bar{t}_0)) < \phi(u''(t)) - \phi(\beta''(t)) \leq 0.$$

Case 2: Suppose that there is $p \in \{1, 2, ..., n\}$ such that

$$\sup_{t\in[a,b]} (u'(t) - \beta'(t)) := u'(t_p) - \beta'(t_p) > 0.$$

As, by Definition 2.2 and (2.14),

$$\begin{aligned} \Delta u'(t_p) &= I_{2,p}(t_p, \delta_0(t_p, u), \delta_1(t_p, u'), \frac{d}{dt}\delta_1(t_p, u'')) \\ &\geq I_{2,p}(t_p, \beta(t_p), \beta'(t_p), \frac{d}{dt}\delta_1(t_p, u'')) \geq \Delta\beta'(t_p), \end{aligned}$$

then

$$\sup_{t\in[a,b]} (u'(t) - \beta'(t)) := u'(t_p^+) - \beta'(t_p^+) > 0. \tag{2.21}$$

Consider $\eta > 0$, sufficiently small, such that

$$u'(t) - \beta'(t) > 0,\ u''(t) - \beta''(t) \leq 0,\ \forall t \in (t_p, t_p + \eta). \tag{2.22}$$

For $t \in (t_p, t_p + \eta)$, and, by the monotonicity of $\phi(u''(t)) - \phi(\beta''(t))$, as in (2.20), by (2.22), Definition 2.2 and (2.15), we obtain the contradiction

$$\begin{aligned} 0 &< \int_{t_p^+}^{t} (\phi(u''(s)))' - (\phi(\beta''(s))'ds \\ &= \phi(u''(t)) - \phi(u''(t_p^+)) - \phi(\beta''(t)) + \phi(\beta''(t_p^+)) \\ &\leq -\phi(u''(t_p^+)) + \phi(\beta''(t_p^+)) \\ &\leq -I_{3p}(t_p, \delta_0(t_p, u(t_p)), \beta'(t_p), \beta''(t_p)) \\ &\quad +I_{3p}(t_p, \beta(t_p), \beta'(t_p), \beta''(t_p)) \leq 0. \end{aligned}$$

Therefore $u'(t) \leq \beta'(t)$, for $t \in [a, b]$. By similar arguments it can be proved the remaining inequality and therefore

$$\alpha'(t) \leq u'(t) \leq \beta'(t),\ \text{for } t \in [a, b]. \tag{2.23}$$

By integration of (2.23) for $t \in [a, t_1]$,

$$\alpha(t) \leq u(t) - u(a) + \alpha(a) \leq u(t). \tag{2.24}$$

Integrating (2.23) for $t \in]t_1, t_2]$, we have, by (2.13) and (2.24),

$$\begin{aligned} \alpha(t) &\leq u(t) - u(t_1^+) + \alpha(t_1^+) \\ &\leq u(t) - I_{11}\left(t_1, \delta_0(t_1, u), \delta_1(t_1, u')\right) - u(t_1^-) + I_{11}\left(t_1, \alpha(t_1), \alpha'(t_1)\right) + \alpha(t_1^-) \\ &\leq u(t). \end{aligned}$$

By recurrence, it can be shown, analogously, that

$$\alpha(t) \le u(t), \ \forall t \in [t_k, t_{k+1}], \text{ for } k = 1, 2, ..., n.$$

So $\alpha(t) \le u(t)$, $\forall t \in [a, b]$. Applying similar arguments it can be proved the remaining inequality and, therefore,

$$\alpha(t) \le u(t) \le \beta(t), \text{ for } t \in [a, b].$$

■

Lemma 2.7 *Let α and β be lower and upper solutions of problem (2.1)-(2.3) such that $\alpha'(t) \le \beta'(t)$ in $[a, b]$. If the continuous function $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies a Nagumo condition in the set F, referred to α and β, then there is $N \ge \mu > 0$ such that every solution u of the differential equation (2.1) verifies $||u''||_\infty \le N$.*

Proof. Let $u(t)$ be a solution of (2.1) such that

$$\alpha(t) \le u(t) \le \beta(t) \text{ and } \alpha'(t) \le u'(t) \le \beta'(t), \text{ for } t \in [a, b].$$

By the Mean Value Theorem, there exists $\eta_0 \in (t_k, t_{k+1})$ with

$$u''(\eta_0) = \frac{u'(t_{k+1}) - u'(t_k)}{t_{k+1} - t_k}, \text{ with } k = 0, 1, 2, ..., n.$$

Moreover,

$$-N \le -\mu \le \frac{\alpha'(t_{k+1}) - \beta'(t_k)}{t_{k+1} - t_k} \le u''(\eta_0) \le \frac{\beta'(t_{k+1}) - \alpha'(t_k)}{t_{k+1} - t_k} \le \mu \le N.$$

If $|u''(t)| \le N$ in $[a, b]$, the proof is complete.

Assume that there is $\tau \in [a, b]$ such that $|u''(\tau)| > N$.

Consider the case where $u''(\tau) > N$. Therefore there is η_1 such that $u''(\eta_1) = N$.

If $\eta_0 < \eta_1$, suppose, without loss of generality, that

$$u''(t) > 0 \text{ and } u''(\eta_0) \le u''(t) \le N, \text{ for } t \in [\eta_0, \eta_1].$$

So

$$|\phi(u''(t))| = |q(t) f(t, u(t), u'(t), u''(t))| \le q(t)|\psi(u''(t))|, \text{ for } t \in [\eta_0, \eta_1],$$

and, by (2.11),

$$\begin{aligned} \int_{\phi(u''(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s)} &\le \int_{\eta_0}^{\eta_1} \frac{|(\phi(u''(t))'|}{\psi(u''(t))} dt = \int_{\eta_0}^{\eta_1} \frac{|q(t) f(t, u(t), u'(t), u''(t))|}{\psi(u''(t))} dt \\ &\le \int_{\eta_0}^{\eta_1} q(t) dt < \int_a^b q(t) dt. \end{aligned}$$

As $u''(\eta_0) \leq \mu < N$, by the monotony of ϕ,

$$\phi(u''(\eta_0)) \leq \phi(\mu)$$

and

$$\int_{\phi(u''(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi_p^{-1}(s)} \geq \int_{\phi(\mu)}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s)} > \int_a^b q(t)dt$$

which leads to a contradiction.

The other cases, that is, $u''(\tau) > N$ with $\eta_1 < \eta_0$, and $u''(\tau) < -N$ with $\eta_0 < \eta_1$ or $\eta_1 < \eta_0$, follow the same arguments to obtain a contradiction.

Therefore $|u''(t)| \leq N$, for $t \in [a,b]$. ■

Proof of Theorem 2.5:

Consider the modified and perturbed problem (2.16), (2.17), (2.2).

Obtain a solution for problem (2.16), (2.17), (2.2) is equivalent to find a function $u \in E$ such that

$$\begin{aligned} u(t) \quad = \quad & A + B\,(t-a) + \sum_{t_k<t} I_{1k}^*(t_k, u(t_k), u'(t_k)) + \sum_{t_k<t} I_{2k}^*(t_k, u(t_k), u'(t_k), u''(t_k)\,(t-t_k) \\ & + \int_a^t \int_a^r \phi^{-1}\left[\phi(C) + \int_\mu^b F_u(s)ds - \sum_{\mu<t_k} I_{3k}^*(t_k, u(t_k), u'(t_k), u''(t_k))\right] d\mu dr. \end{aligned}$$

where

$$\begin{aligned} F_u(s) \quad : \quad & = q(s) f(s, \delta_0(s, u(s)), \delta_1(s, u'(s)), \frac{d}{ds}\delta_1(s, u'(s)) \\ & + \frac{\delta_1(s, u'(s)) - u'(s)}{1 + |u'(s) - \delta_1(s, u'(s))|}, \\ I_{1k}^*(t_k, u(t_k), u'(t_k)) \quad : \quad & = I_{1k}(t_k, \delta_0(t_k, u(t_k), \delta_1(t_k, u'(t_k)), \\ I_{2k}^*(t_k, u(t_k), u'(t_k), u''(t_k) \quad : \quad & = I_{2k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)), \frac{d}{dt}\delta_1(t_k, u'(t_k))), \\ I_{3k}^*(t_k, u(t_k), u'(t_k), u''(t_k)) \quad : \quad & = I_{3k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)), \frac{d}{dt}\delta_1(t_k, u'(t_k))). \end{aligned}$$

Define the operator $T : E \to E$ by

$$\begin{aligned} T(u)(t) \quad : \quad & = A + B\,(t-a) + \sum_{t_k<t} I_{1k}^*(t_k, u(t_k), u'(t_k)) \\ & + \sum_{t_k<t} I_{2k}^*(t_k, u(t_k), u'(t_k), u''(t_k)) \;\; (t-t_k) \\ & + \int_a^t \int_a^r \phi^{-1}\left[\phi(c) + \int_\mu^b F_u(s)ds - \sum_{\mu<t_k} I_{3k}^*(t_k, u(t_k), u'(t_k), u''(t_k))\right] d\mu dr. \end{aligned}$$

As T is completely continuous, by Schauder's fixed point theorem, T has a fixed point $u \in E_2$ which is a solution of (2.16), (2.17), (2.2).

By Lemma 2.1, this function $u \in E$ is also a solution of the problem (2.1)-(2.3).■

Remark 2.8 *Notice that Theorem 2.5 still holds if ϕ be a singular $\phi-$Laplacian operator, in the sense applied in [16] and [3], that is, if there is $0 < a < +\infty$ such that*

$$\phi :]-a,a[\to \mathbb{R}.$$

2.4 Example

Consider the problem composed by the differential equation

$$\frac{(u''(t))^2\, u'''(t)}{216} + \arctan(u) - (u'(t)+1)^3 - 4\sqrt[3]{u''(t)+1} = 0, \quad \text{in } [0,1]\setminus\left\{\frac{1}{2}\right\}, \tag{2.25}$$

the boundary conditions

$$u(0) = A,\ u'(0) = B,\ u''(1) = C, \tag{2.26}$$

for $A \in [-\frac{1}{2}, 4]$, $B \in [0,1]$ and $C \in]-2, 6[$, and the impulse effects given, for $t_1 = \frac{1}{2}$, by

$$\begin{cases} \Delta u(\frac{1}{2}) = u(\frac{1}{2}) + u'(\frac{1}{2}), \\ \Delta u'(\frac{1}{2}) = -u(\frac{1}{2}) + u''(\frac{1}{2}) \\ \Delta\phi(u''(\frac{1}{2})) = 2u'(\frac{1}{2}). \end{cases} \tag{2.27}$$

Remark that constant functions are not solutions of this problem (2.25), (2.27), (2.26), which is a particular case of problem (2.1)-(2.3) with $a = 0$, $b = 1$, and

$$\begin{aligned} \phi(w) &= \frac{w^3}{648},\ q(t) \equiv 1, \\ f(t,x,y,z) &= \arctan(x) - (y+1)^3 - 4\sqrt[3]{z+1} \\ I_{11}(t,x,y) &= x + y, \\ I_{21}(t,x,y,z) &= -x + z, \\ I_{31}(t,x,y,z) &= 2y. \end{aligned}$$

For $A \in [-\frac{1}{2}, 4]$, $B \in [0,1]$ and $C \in]-2, 6[$, the functions

$$\alpha(t) = \begin{cases} -2t^2 - \frac{1}{2} &, \ t \le \frac{1}{2} \\ -t^2 - 3 &, \ \frac{1}{2} < t \end{cases} \quad \text{and} \quad \beta(t) = \begin{cases} t^3 + t + 4 &, \ t \le \frac{1}{2} \\ t^3 + 3t + 10 &, \ \frac{1}{2} < t, \end{cases}$$

are, respectively, lower and upper solutions of problem (2.25), (2.27), (2.26), considering

$$\alpha'(t) = \begin{cases} -4t &, \ t \le \frac{1}{2} \\ -2t &, \ \frac{1}{2} < t \end{cases}, \quad \beta'(t) = \begin{cases} 3t^2 + 1 &, \ t \le \frac{1}{2} \\ 3t^2 + 3 &, \ \frac{1}{2} < t \end{cases},$$

$$\alpha''(t) = \begin{cases} -4 &, \ t \le \frac{1}{2} \\ -2 &, \ \frac{1}{2} < t \end{cases} \quad \text{and} \quad \beta''(t) = 6t, \text{ in } [0,1].$$

As the assumptions of Theorem 2.5 are fulfilled, therefore there is a solution of problem (2.25), (2.27), (2.26), such that

$$\alpha(t) \le u(t) \le \beta(t), \quad \alpha'(t) \le u'(t) \le \beta'(t), \text{ in } [0,1].$$

Chapter 3

Semi-Linear Impulsive Higher Order Boundary Value Problems

3.1 Introduction

This chapter studies the two point boundary value problem composed by the one-dimensional ϕ-Laplacian equation

$$(\phi(u^{(n-1)}(t)))' + q(t)f(t, u(t), ..., u^{(n-1)}(t)) = 0, \ t \in [a,b]\backslash\{t_1, ..., t_m\}, \quad (3.1)$$

where ϕ is an increasing homeomorphism such that $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$, $q \in L^\infty[a,b]$ is a positive function and $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function, together with the boundary conditions

$$u^{(j)}(a) = A_j, \ u^{(n-1)}(b) = B, \ j = 0, 1, ..., n-2, \quad A_j, B \in \mathbb{R}, \quad (3.2)$$

and the impulsive conditions

$$\begin{aligned} \Delta u^{(i)}(t_k) &= I_{i,k}(t_k, u(t_k), u'(t_k), ..., u^{(n-1)}(t_k)), \ i = 0, 1, ..., n-2, \\ \Delta\phi(u^{(n-1)}(t_k)) &= I_{n-1,k}(t_k, u(t_k), u'(t_k), , ..., u^{(n-1)}(t_k)), \quad (3.3) \end{aligned}$$

being $\Delta u^{(i)}(t_k) = u^{(i)}(t_k^+) - u^{(i)}(t_k^-)$, $i = 0, 1, ..., n-1$, $k = 1, 2, ..., m$, $I_{i,k} \in C([a,b] \times \mathbb{R}^n, \mathbb{R})$, and t_k fixed points such that $a = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = b$.

Impulsive boundary value problems have been studied by many authors where it is highlighted the huge possibilities of applications to phenomena where a sudden variation happens. Indeed, these types of jumps occur in different areas such as population dynamics, engineering, control, and optimization theory, medicine, ecology, biology and biotechnology, economics, pharmacokinetics, and many other fields.

From a large number of items existent in the literature on classical impulsive differential problems, we mention, for instance, [24, 53, 54, 63, 66, 75] and the references therein. The most applied arguments are based on critical point theory and variational methods ([51, 58, 112]), fixed point theory on cones ([96, 105]), bifurcation results ([81, 108]), and upper and lower solutions techniques suggested on ([23, 56, 72, 97]).

In the last years, p-Laplacian and ϕ-Laplacian operators have been applied to semi-linear, quasi-linear, and strongly nonlinear differential equations, in singular and regular cases, increasing the range of theoretical and practical applications, as it can be seen, for example, in [12, 35, 55, 60, 103, 115] and in their references. However, impulsive problems with this type of nonlinear differential equations are scarce.

In [30], it is studied the third order differential equation

$$(\phi(u''(t)))' + q(t)f(t, u(t), u'(t), u''(t)) = 0, t \in [a, b]\backslash\{t_1, ..., t_n\},$$

with ϕ is an increasing homeomorphism, $q \in C([a, b])$ with $q > 0$, $f \in C([a, b] \times \mathbb{R}^3, \mathbb{R})$, the two-point boundary conditions

$$u(a) = A,\ u'(a) = B,\ u''(b) = C, \quad A, B, C \in \mathbb{R},$$

and the impulsive effects are given by

$$\begin{aligned} \Delta u(t_k) &= I_{1k}(t_k, u(t_k), u'(t_k)), \\ \Delta u'(t_k) &= I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)), \\ \Delta\phi(u''(t_k)) &= I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)), \end{aligned}$$

where $k = 1, 2, ...n$, $I_{1k} \in C([a, b] \times \mathbb{R}^2, \mathbb{R})$, and $I_{ik} \in C([a, b] \times \mathbb{R}^3, \mathbb{R})$, $i = 2, 3$.

In this work, we found a method that allows generalizing the above results to higher-order boundary value problems with impulsive functions, depending not only on the unknown function but also on its derivatives till order $n - 1$. To best of our knowledge, it is the first time, where such nonlinear higher-order problems are considered with this type of generalized impulsive functions.

This work is organized in the following way: Section 3.2 contains the functional framework, some definitions and an explicit form for the solution of the associated homogeneous problem. Section 3.3 presents the main existence and localization theorem obtained via lower and upper solutions technique and a fixed point theorem. The last section gives a technique to estimate the bending of a one-sided clamped beam under some impulsive forces and how it can be obtained some qualitative data about its variation.

3.2 Definitions and preliminary results

Let

$$PC^{n-1}[a, b] = \left\{ \begin{array}{c} u : u \in C^{n-1}([a, b]; \mathbb{R}) \text{ for } t \neq t_k, u^{(i)}(t_k) = u^{(i)}(t_k^-), u^{(i)}(t_k^+) \\ \text{exists for } k = 1, 2, ..., m, \text{ and } i = 0, 1, ..., n - 1 \end{array} \right\}.$$

Denote $X := PC^{n-1}[a,b]$. Then X is a Banach Space with norm

$$||u||_X = \max\{||u^{(i)}||_\infty, i = 0,1,...,n-1\},$$

where

$$||w||_\infty = \sup_{a\leq t\leq b} |w(t)|.$$

Defining $J := [a,b]$ and $J' = J\backslash\{t_1,...,t_m\}$, for a solution u of problem (3.1)-(2.3) one should consider $u(t) \in E$, where

$$E_3 := PC^{n-1}(J) \cap C^n(J').$$

Next lemma provides an uniqueness result an adequate problem related to (3.1)-(2.3).

Lemma 3.1 *For $v \in PC[a,b]$, the problem composed by the differential equation*

$$(\phi(u^{(n-1)}(t)))' + v(t) = 0 \tag{3.4}$$

together with conditions (3.2), (2.3), has a unique solution given by

$$u(t) = \sum_{i=0}^{n-2}\left(\left[A_i + \sum_{k\,:\,t_k<t} I_{i,k}\left(t_k, u(t_k),...,u^{(n-1)}(t_k)\right)\right]\frac{(t-a)^{n-2-i}}{(n-2-i)!}\right)$$
$$+\int_a^t \frac{(t-s)^{n-2}}{(n-2)!}\phi^{-1}\left(\phi(B) + \int_s^b v(r)dr - \sum_{k\,:\,t_k>s} I_{n-1,k}\left(t_k, u(t_k),...,u^{(n-1)}(t_k)\right)\right)ds.$$

Proof. Integrating the differential equation (3.4) for $t \in (t_m, b]$ we get, by (3.2),

$$\phi\left(u^{(n-1)}(t)\right) = \phi(B) + \int_{t_m}^b v(s)ds. \tag{3.5}$$

By integration of (3.4) for $t \in (t_{m-1}, t_m]$ one has by (3.5)

$$\begin{aligned}\phi\left(u^{(n-1)}(t)\right) &= \int_t^{t_m} v(s)ds - I_{n-1,m}\left(t_m, u(t_m),...,u^{(n-1)}(t_m)\right) + \phi\left(u^{(n-1)}\left(t_m^+\right)\right)\\ &= \phi(B) - I_{n-1,m}\left(t_m, u(t_m),...,u^{(n-1)}(t_m)\right) + \int_t^b v(s)ds\end{aligned}$$

and so,

$$u^{(n-1)}(t) = \phi^{-1}\left(\phi(B) - I_{n-1,m}\left(t_m, u(t_m),...,u^{(n-1)}(t_m)\right) + \int_t^b v(s)ds\right)$$

Therefore, for $t \in [a,b]$, we have

$$u^{(n-1)}(t) = \phi^{-1}\left(\phi(B) + \int_t^b v(s)ds - \sum_{k\,:\,t_k>t} I_{n-1,k}\left(t_k, u(t_k),...,u^{(n-1)}(t_k)\right)\right). \tag{3.6}$$

Integrating (3.6), for $t \in [a, t_1]$,

$$u^{(n-2)}(t) = A_{n-2} + \int_a^t \phi^{-1}(\phi(B) + \int_s^b v(r)dr -$$

$$\sum_{k\,:\,t_k>s} I_{n-1,k}\left(t_k, u(t_k), ..., u^{(n-1)}(t_k)\right)\Bigg) ds.$$

By integration of (3.6) on $(t_1, t_2]$ and (2.3)

$$u^{(n-2)}(t) = I_{n-2,1}\left(t_1, u(t_1), ..., u^{(n-1)}(t_1)\right)$$

$$+ \int_{t_1}^t \left(\phi^{-1}\left(\phi(B) + \int_s^b v(r)dr - \sum_{k\,:\,t_k>s} I_{n-1,k}\left(t_k, u(t_k), ..., u^{(n-1)}(t_k)\right)\right)\right) ds$$

Therefore, for $t \in [a, b]$,

$$u^{(n-2)}(t) = \sum_{k\,:\,t_k<t} \left(I_{n-2,k}\left(t_k, u(t_k), ..., u^{(n-1)}(t_k)\right)\right) + A_{n-2}$$

$$+ \int_a^t \left(\phi^{-1}\left(\phi(B) + \int_s^b v(r)dr - \sum_{k\,:\,t_k>s} I_{n-1,k}\left(t_k, u(t_k), ..., u^{(n-1)}(t_k)\right)\right)\right) ds.$$

Following the same method, by iterate integrations and (2.3), we obtain for $t \in [a, b]$

$$u(t) = \sum_{i=0}^{n-2}\left(\left[A_i + \sum_{k\,:\,t_k<t} I_{i,k}\left(t_k, u(t_k), ..., u^{(n-1)}(t_k)\right)\right]\frac{(t-a)^{n-2-i}}{(n-2-i)!}\right) +$$

$$\int_a^t \frac{(t-s)^{n-2}}{(n-2)!}\phi^{-1}\left(\phi(B) + \int_s^b v(s)ds - \sum_{k\,:\,t_k>s} I_{n-1,k}\left(t_k, u(t_k), ..., u^{(n-1)}(t_k)\right)\right) ds.$$

■

Lower and upper functions will play a key role in our method, and they are defined as it follows:

Definition 3.2 *A function $\alpha(t) \in E_3$ with $\phi(\alpha^{(n-1)}(t)) \in PC^1[a, b]$ is a lower solution of problem (3.1), (3.2), (2.3) if*

$$\begin{cases} \left(\phi\left(\alpha^{(n-1)}(t)\right)\right)' + q(t)f\left(t, \alpha(t), \alpha'(t), ..., \alpha^{(n-1)}(t)\right) \geq 0 \\ \alpha^{(j)}(a) \leq A_j, \; j = 0, 1, ..., n-2, \\ \alpha^{(n-1)}(b) \leq B \\ \Delta\alpha^{(i)}(t_k) \leq I_{i,k}(t_k, \alpha(t_k), ..., \alpha^{(n-1)}(t_k)), \; i = 0, 1, ..., n-3, \\ \Delta\alpha^{(n-2)}(t_k) > I_{n-2,k}(t_k, \alpha(t_k), ..., \alpha^{(n-1)}(t_k)) \\ \Delta\phi(\alpha^{(n-1)}(t_k)) > I_{n-1,k}(t_k, \alpha(t_k), ..., \alpha^{(n-1)}(t_k)), \end{cases} \tag{3.7}$$

for $k = 1, 2, ..., m$.
A function $\beta(t) \in E_3$ such that $\phi(\beta^{(n-1)}(t)) \in PC^1[a, b]$ is an upper solution of (3.1)-(2.3) if it satisfies the opposite inequalities.

To control the derivative $u^{(n-1)}(t)$ it will be applied the Nagumo condition:

Definition 3.3 *A L^1-Carathéodory function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Nagumo condition related to a pair of functions $\gamma, \Gamma \in E_3$, with $\gamma^{(i)}(t) \leq \Gamma^{(i)}(t)$, for $i = 0,1,...,n-2$, and $t \in [a,b]$, if there exists a function $\psi : C([0,+\infty),]0,+\infty))$ such that*

$$|f(t,x_0,x_1,...,x_{n-1})| \leq \psi(|x_{n-1}|), \text{ for all } (t,x_0,x_1,...,x_{n-1}) \in S \tag{3.8}$$

with

$$S := \{(t,x_0,x_1,...,x_{n-1}) \in [a,b] \times \mathbb{R}^n : \gamma^{(i)}(t) \leq x_i \leq \Gamma^{(i)}(t), i = 0,1,...,n-2\},$$

and

$$\int_{\phi(\mu)}^{+\infty} \frac{ds}{\psi(\phi^{-1}(s))} > \int_a^b q(s)ds, \tag{3.9}$$

where

$$\mu := \max_{k=0,1,2,...,m} \left\{ \left| \frac{\Gamma^{(n-2)}(t_{k+1}) - \gamma^{(n-2)}(t_k)}{t_{k+1}-t_k} \right|, \left| \frac{\gamma^{(n-2)}(t_{k+1}) - \Gamma^{(n-2)}(t_k)}{t_{k+1}-t_k} \right| \right\}.$$

From Nagumo condition we deduce an *a priori* estimation for $u^{(n-1)}(t)$:

Lemma 3.4 *If the L^1-Carathéodory function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Nagumo condition in the set S, referred to the functions γ and Γ, then there is $N \geq \mu > 0$ such that every solution u of the differential equation (3.1) verifies $||u^{(n-1)}||_\infty \leq N$.*

Proof. Let $u(t)$ be a solution of (3.1) such that

$$\gamma^{(i)}(t) \leq u^{(i)}(t) \leq \Gamma^{(i)}(t), \text{ for } i = 0,1,...,n-2 \text{ and } t \in [a,b].$$

By the Mean Value Theorem, there exists $\eta_0 \in (t_k, t_{k+1})$ with

$$u^{(n-1)}(\eta_0) = \frac{u^{(n-2)}(t_{k+1}) - u^{(n-2)}(t_k)}{t_{k+1}-t_k}, \text{ with } k = 0,1,2,...,m.$$

Moreover,

$$\begin{aligned} -N &\leq -\mu \leq \frac{\gamma^{(n-2)}(t_{k+1}) - \Gamma^{(n-2)}(t_k)}{t_{k+1}-t_k} \leq u^{(n-1)}(\eta_0) \\ &\leq \frac{\Gamma^{(n-2)}(t_{k+1}) - \gamma^{(n-2)}(t_k)}{t_{k+1}-t_k} \leq \mu \leq N. \end{aligned} \tag{3.10}$$

If $\left|u^{(n-1)}(t)\right| \leq N$ for every $t \in [a,b]$, the proof is complete.

On the contrary, assume that there is $\tau \in [a,b]$ such that $\left|u^{(n-1)}(\tau)\right| > N$. Consider the case where $u^{(n-1)}(\tau) > N$. Therefore there is η_1 such that $u^{(n-1)}(\eta_1) = N$. Suppose, without loss of generality, that $\eta_0 < \eta_1$. So,

$$u^{(n-1)}(t) > 0 \text{ and } u^{(n-1)}(\eta_0) \leq u^{(n-1)}(t) \leq N, \text{ for } t \in [\eta_0, \eta_1].$$

So

$$|\phi(u^{(n-1)}(t))| = |q(t)f(t,u(t),...,u^{(n-1)}(t))| \leq q(t)|\psi(u^{(n-1)}(t))|, \text{ for } t \in [\eta_0,\eta_1],$$

and

$$\int_{\phi(u^{(n-1)}(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s))} \leq \int_{\eta_0}^{\eta_1} \frac{|(\phi(u^{(n-1)}(t))'|}{\psi(u^{(n-1)}(t))}dt$$
$$= \int_{\eta_0}^{\eta_1} \frac{|q(t)f(t,u(t),...,u^{(n-1)}(t))|}{\psi(u^{(n-1)}(t))}dt \leq \int_{\eta_0}^{\eta_1} q(t)dt < \int_a^b q(t)dt.$$

As $u^{(n-1)}(\eta_0) \leq \mu < N$, by the monotony of ϕ,

$$\phi(u^{(n-1)}(\eta_0)) \leq \phi(\mu)$$

and, by (3.9),

$$\int_{\phi(u^{(n-1)}(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s))} \geq \int_{\phi(\mu)}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s))} > \int_a^b q(t)dt$$

which leads to a contradiction.

The other cases, that is, $u^{(n-1)}(\tau) > N$ with $\eta_1 < \eta_0$, and $u^{(n-1)}(\tau) < -N$ with $\eta_0 < \eta_1$ or $\eta_1 < \eta_0$, follow the same arguments to obtain a contradiction.

Therefore $\left|u^{(n-1)}(t)\right| \leq N,$ for $t \in [a,b]$. ■

Forward, in our method, we will use the following lemma, given in [73]:

3.3 Existence and localization result

The main result is an existence and localization theorem, as it provides not only the existence of solutions but also some of its qualitative properties.

Theorem 3.5 *Suppose that there are α and β lower and upper solutions, respectively, of problem (3.1)-(2.3) such that*

$$\alpha^{(n-2)}(t) \leq \beta^{(n-2)}(t), \text{ for } t \in [a,b].$$

Assume that the L^1-Carathéodory function $f : [a,b]\times\mathbb{R}^n \to \mathbb{R}$ satisfies a Nagumo condition, related to α and β, and verifies

$$f(t,\alpha(t),...,\alpha^{(n-3)}(t)),y,z) \leq f(t,x_0,...,x_{n-1}) \leq f(t,\beta(t),...,\beta^{(n-3)}(t),y,z), \tag{3.11}$$

for $\alpha^{(i)}(t) \leq x_i \leq \beta^{(i)}(t)$, for $i = 0,...,n-3$, and fixed $(y,z) \in \mathbb{R}^2$.
Moreover, if the impulsive functions satisfy

$$I_{j,k}(t_k,\alpha(t_k),...,\alpha^{(n-1)}(t_k)) \leq I_{j,k}(t_k,x_0,...,x_{n-1}) \leq I_{j,k}(t_k,\beta(t_k),...,\beta^{(n-1)}(t_k)), \tag{3.12}$$

for $j = 0, ..., n-3$, $\alpha^{(i)}(t_k) \leq x_i \leq \beta^{(i)}(t_k)$, *for* $i = 0, 1, ..., n-2$, $k = 1, 2, ..., m$, *and*

$$\begin{aligned} I_{n-2,k}(t_k, \alpha(t_k), ..., \alpha^{(n-3)}(t_k), y, z) &\geq I_{n-2,k}(t_k, x_0, ..., x_{n-3}, y, z) \qquad (3.13) \\ &\geq I_{n-2,k}(t_k, \beta(t_k), ..., \beta^{(n-3)}(t_k), y, z) \end{aligned}$$

for $\alpha^{(i)}(t) \leq x_i \leq \beta^{(i)}(t)$, *for* $i = 0, ..., n-3$, *and fixed* $(y, z) \in \mathbb{R}^2$, *then problem (3.1)-(2.3) has at least one solution* $u \in E$, *such that*

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t) \quad \text{for } i = 0, 1, ..., n-2 \text{ and } \quad -N \leq u^{(n-1)}(t) \leq N,$$

for $t \in [a, b]$ *and* N *given by (3.10).*

Proof. Define the continuous functions δ_i, for $i = 0, 1, ..., n-2$,

$$\delta_i(t, u^{(i)}(t)) = \begin{cases} \beta^{(i)}(t), & u^{(i)}(t) \geq \beta^{(i)}(t) \\ u^{(i)}(t), & \alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t) \\ \alpha^{(i)}(t), & u^{(i)}(t) \leq \alpha^{(i)}(t) \end{cases}$$

and consider the following modified and perturbed equation

$$\begin{aligned} &(\phi(u^{(n-1)}(t)))' + q(t) f\left(t, \delta_0(t, u(t)), ..., \delta_{n-2}(t, u^{(n-2)}(t)), \frac{d}{dt}\left(\delta_{n-2}(t, u^{(n-2)}(t))\right)\right) \qquad (3.14) \\ &+ \frac{\delta_{n-2}(t, u^{(n-2)}(t)) - u^{(n-2)}(t)}{1 + |u^{(n-2)}(t) - \delta_{n-2}(t, u^{(n-2)}(t))|} = 0, \end{aligned}$$

coupled with boundary conditions (3.2) and the truncated impulsive conditions , for $i = 0, 1, ..., n-2$,

$$\begin{aligned} \Delta u^{(i)}(t_k) &= I_{i,k}\left(\begin{array}{c} t_k, \delta_0(t_k, u(t_k)), ..., \delta_{n-2}(t_k, u^{(n-2)}(t_k), \\ \frac{d}{dt}(\delta_{n-2}(t_k, u^{(n-2)}(t_k))) \end{array}\right) := I^*_{i,k}(t_k), \\ \Delta\phi(u^{(n-1)}(t)) &= I_{n-1,k}\left(\begin{array}{c} t_k, (t_k, u(t_k)), ..., \delta_{n-2}(t_k, u^{(n-2)}(t_k)), \\ \frac{d}{dt}(\delta_{n-2}(t_k, u^{(n-2)}(t_k))) \end{array}\right) := I^*_{n-1,k}(t_k). \end{aligned} \qquad (3.15)$$

Define the operator $T : E_3 \to E_3$ by

$$\begin{aligned} T(u)(t) \quad : \quad &= \sum_{i=0}^{n-2}\left(\left[A_i + \sum_{k\,:\,t_k<t} I^*_{i,k}\right] \frac{(t-a)^{n-2-i}}{(n-2-i)!}\right) \\ &+ \int_a^t \frac{(t-s)^{n-2}}{(n-2)!} \phi^{-1}\left(\phi(B) + \int_s^b v(s)ds - \sum_{k\,:\,t_k>s} I^*_{n-1,k}\right) ds. \end{aligned}$$

By Lemma 3.1, it is clear that the fixed points of T, u_*, are solutions of the initial problem (3.1)-(3.15), if they verify

$$\alpha^{(i)}(t) \leq u_*^{(i)}(t) \leq \beta^{(i)}(t), \text{ for } t \in [a, b] \text{and } i = 0, 1, ..., n-2.$$

As T is completely continuous, by Schauder's fixed point theorem, T has a fixed point $u \in E$, which is a solution of (3.14), (3.2), (3.15). To prove that this solution verifies

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \text{ for } t \in [a,b], \text{ and } i = 0,1,...,n-2,$$

suppose, by contradiction, that, for $i = n-2$, there is $t \in [a,b]$ such that

$$u^{(n-2)}(t) > \beta^{(n-2)}(t).$$

Define $\zeta \in [a,b]$ as

$$\sup_{t\in[a,b]} (u^{(n-2)}(t) - \beta^{(n-2)}(t)) := u^{(n-2)}(\zeta) - \beta^{(n-2)}(\zeta)) > 0. \tag{3.16}$$

By (3.2) and Definition 3.2, $u^{(n-2)}(a) - \beta^{(n-2)}(a) \leq 0$, then $\zeta \neq a$. On the other hand $u^{(n-1)}(b) - \beta^{(n-1)}(b) < 0$ and then $\zeta \neq b$, by (3.16).

Therefore $\zeta \in]a,b[$.

Case 1: Assume that there is $p \in \{1,2,...,m\}$ such that $\zeta \in (t_p, t_{p+1})$.

Consider $\epsilon > 0$ small enough such that

$$u^{(n-2)}(t) - \beta^{(n-2)}(t) > 0 \text{ and } u^{(n-1)}(t) - \beta^{(n-1)}(t) \leq 0, \text{ for } t \in (\zeta, \zeta+\epsilon). \tag{3.17}$$

Therefore, by (3.11) and (3.17), for all $t \in (\zeta, \zeta+\epsilon)$, we have the following contradiction

$$\begin{aligned}
0 &\geq \phi\left(u^{(n-1)}(t)\right)' - \phi\left(\beta^{(n-1)}(t)\right)' \\
&\geq -q(t)f\left(t, \delta_0(t,u(t)), ..., \delta_{n-2}(t,u^{(n-2)}(t)), \frac{d}{dt}\left(\delta_{n-2}(t,u^{(n-2)}(t))\right)\right) \\
&- \frac{\delta_{n-2}\left(t,u^{(n-2)}(t)\right) - u^{(n-2)}(t)}{1+\left|u^{(n-2)}(t) - \delta_{n-2}\left(t,u^{(n-2)}(t)\right)\right|} + q(t)f\left(t,\beta(t),...,\beta^{(n-1)}(t)\right) \\
&= -q(t)f\left(t, \delta_0(t,u(t)), ..., \delta_{n-3}\left(t,u(t)\right), \beta^{(n-2)}(t), \beta^{(n-1)}(t)\right) \\
&- \frac{\beta^{(n-2)}(t) - u^{(n-2)}(t)}{1+\left|u^{(n-2)}(t) - \beta^{(n-2)}(t)\right|} + q(t)f\left(t,\beta(t),...,\beta^{(n-1)}(t)\right) \\
&\geq -q(t)f\left(t,\beta(t),...,\beta^{(n-1)}(t)\right) - \frac{\beta^{(n-2)}(t) - u^{(n-2)}(t)}{1+\left|u^{(n-2)}(t) - \beta^{(n-2)}(t)\right|} \\
&+ q(t)f\left(t,\beta(t),...,\beta^{(n-1)}(t)\right) = \frac{u^{(n-2)}(t) - \beta^{(n-2)}(t)}{1+\left|u^{(n-2)}(t) - \beta^{(n-2)}(t)\right|} > 0.
\end{aligned}$$

Case 2: Consider that there exists $k \in \{1, 2, ..., m\}$ such that, or

$$\max_{t\in[a,b]} \left(u^{(n-2)}(t) - \beta^{(n-2)}(t)\right) := u^{(n-2)}(t_k^-) - \beta^{(n-2)}(t_k^-) > 0 \tag{3.18}$$

or

$$\sup_{t\in[a,b]} \left(u^{(n-2)}(t) - \beta^{(n-2)}(t)\right) := u^{(n-2)}(t_k^+) - \beta^{(n-2)}(t_k^+) > 0. \tag{3.19}$$

If (3.18) holds, then

$$\Delta \left(u^{(n-2)}(t) - \beta^{(n-2)}(t)\right) \leq 0$$

and, by (3.13) and Definition 3.2, we have the contradiction

$$\begin{aligned}
0 &\geq \Delta u^{(n-2)}(t_k) - \Delta\beta^{(n-2)}(t_k) = I^*_{n-2,k} - \Delta\beta^{(n-2)}(t_k) \\
&= I_{n-2,k} \left(\delta_0(t, u(t)), ..., \delta_{n-3}\left(t, u(t)\right), \beta^{(n-2)}(t), \beta^{(n-1)}(t)\right) - \Delta\beta^{(n-2)}(t_k) \\
&\geq I_{n-2,k} \left(t_k, \beta(t_k), ..., \beta^{(n-1)}(t_k)\right) - \Delta\beta^{(n-2)}(t_k) > 0.
\end{aligned}$$

Consider now (3.19). So, there is $\epsilon > 0$ such that , for $t \in (t_k, t_k + \epsilon)$,

$$u^{(n-1)}(t) - \beta^{(n-1)}(t) \leq 0,$$

and the arguments follow by the same technique as in Case 1, to have

$$u^{(n-2)}(t) \leq \beta^{(n-2)}(t), \forall t \in [a, b].$$

To prove that $u^{(n-2)}(t) \geq \alpha^{(n-2)}(t), \forall t \in [a, b]$, the method is similar. Therefore

$$\alpha^{(n-2)}(t) \leq u^{(n-2)}(t) \leq \beta^{(n-2)}, \text{for } t \in [a, b].$$

Integrating the first inequality in $[a, t_1]$, we have

$$\begin{aligned}
\alpha^{(n-3)}(t) &\leq u^{(n-3)}(t) - u^{(n-3)}(a) + \alpha^{(n-3)}(a) \\
&= u^{(n-3)}(t) - A_{n-3} + \alpha^{(n-3)}(a) \leq u^{(n-3)}(t).
\end{aligned} \tag{3.20}$$

For $t \in (t_1, t_2]$, by (3.12) and (3.20),

$$\begin{aligned}
\alpha^{(n-3)}(t) &\leq u^{(n-3)}(t) - u^{(n-3)}(t_1^+) + \alpha^{(n-3)}(t_1^+) \\
&\leq u^{(n-3)}(t) - I^*_{n-3,1}(t_1) - u^{(n-3)}(t_1) \\
&+ I_{n-3,1}\left(t_1, \alpha(t_1), ..., \alpha^{n-1}(t_1)\right) + \alpha^{(n-3)}(t_1) \\
&\leq u^{(n-3)}(t) - I^*_{n-3,1}(t_1) + I_{n-3,1}\left(t_1, \alpha(t_1), ..., \alpha^{(n-1)}(t_1)\right) \\
&\leq u^{(n-3)}(t).
\end{aligned}$$

Applying this method for each interval $(t_k, t_{k+1}]$, $k = 2, ..., m$, we obtain

$$\alpha^{(n-3)}(t) \leq u^{(n-3)}(t), \forall t \in [a, b],$$

and, by the same technique,

$$\beta^{(n-3)}(t) \geq u^{(n-3)}(t), \forall t \in [a, b].$$

By iteration of these arguments, we conclude

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \text{ for } i = 0, 1, ..., n-2, \text{ and } t \in [a, b].$$

The estimation $\left|u^{(n-1)}(t)\right| \leq N$ is a trivial consequence of Lemma 3.4. ■

3.4 Estimation for the bending of one-sided clamped beam under impulsive effects

Problems related to beam structures and especially beams that support some forces as impulses, are part of a vast field of investigation in boundary value problems theory, see, for example, [89, 76, 110, 106, 113].

In this application we consider a model to describe the bending of a beam with length $L > 1$, given by the fourth-order equation

$$\frac{EI}{A} u^{(4)}(x) + \frac{3}{2} \sqrt[3]{u'(x)} |u''(x)| - ku(x) - \gamma u'''(x) = 0, \text{ for } x \in]0, L[\,, \tag{3.21}$$

where $E > 0$ is the Young's modulus, $I > 0$ the mass moment of inertia, $A > 0$ the cross section area, $k > 0$ the tension of a spring force vertically applied on the beam, and $\gamma > 0$ the shear force coefficient.

At the end points the behavior of the beam is given by the following boundary conditions

$$u(0) = 0,\ u'(0) = 1,\ u''(0) = 0,\ u'''(L) = 0, \tag{3.22}$$

meaning that the beam is clamped on the left end side.

For clearance, we consider only one moment of impulse which occurs at $t_1 = 1$. The impulsive effects are given by generalized functions with dependence on the unknown function itself, and on several derivatives till order three,

$$\begin{array}{rcl} \Delta u(1) & = & u(1) + u'(1) - 2u''(1) - u'''(1) \\ \Delta u'(1) & = & u(1) + u'(1) - 2u''(1) - u'''(1) \\ \Delta u''(1) & = & -u(1) - u'(1) + u''(1) + 5u'''(1) - 1 \\ \Delta u'''(1) & = & u(1) - u'(1) + u''(1) + u'''(1) - 1. \end{array} \tag{3.23}$$

This problem (3.21)-(3.23) is a particular case of (3.1)-(2.3) with $[a, b] = [0, L]$, $n = 4$,

$$f(x, y_0, y_1, y_2, y_3) = \frac{A}{EI} \left(\frac{3}{2} \sqrt[3]{y_1} |y_2| - ky_0 - \gamma y_3 \right), \tag{3.24}$$

$\phi(w) = w, q(t) \equiv 1, m = 1,\ t_1 = 1$, and the impulsive functions given by

$$\begin{aligned}
I_{0,1}(t_1, w_0, w_1, w_2, w_3) &= w_0 + w_1 - 2w_2 - w_3 \\
I_{1,1}(t_1, w_0, w_1, w_2, w_3) &= w_0 + w_1 - 2w_2 - w_3 \\
I_{2,1}(t_1, w_0, w_1, w_2, w_3) &= -w_0 - w_1 + w_2 + 5w_3 - 1 \\
I_{3,1}(t_1, w_0, w_1, w_2, w_3) &= w_0 - w_1 + w_2 + w_3 - 1.
\end{aligned}$$

As a numeric example we can consider $A = 1$, $EI = 1$, $k = 1$, $\gamma = 6$, $L = 2$. In this case, the continuous functions

$$\alpha(x) = 0, \beta(x) = \frac{x^3}{6} + x^2 + x,\ \text{for } x \in [0, 2],$$

are, respectively, lower and upper solutions of problem (3.24)-(3.23), according to Definition 3.16. In fact, for $\alpha(x) \equiv 0$ the inequalities are trivially satisfied and for β, we have,

$$\begin{aligned}
\beta(0) = 0,\ \beta'(0) = 1,\ \beta''(0) = 2 > 0,\ \beta'''(2) = 1 > 0, \\
\Delta\beta(1) = 0 \geq \beta(0) + \beta'(0) - 2\beta''(0) - \beta'''(0) = -\frac{4}{3}, \\
\Delta\beta'(1) = 0 \geq \beta(0) + \beta'(0) - 2\beta''(0) - \beta'''(0) = -\frac{4}{3}, \\
\Delta\beta''(1) = 0 < -\beta(0) - \beta'(0) + \beta''(0) + 5\beta'''(0) - 1 = \frac{4}{3} \\
\Delta\beta'''(1) = 0 < \beta(0) - \beta'(0) + \beta''(0) + \beta'''(0) - 1 = \frac{5}{3}.
\end{aligned}$$

The nonlinear part $f(x, y_0, y_1, y_2, y_3)$, given by (3.24), verifies a Nagumo condition on the set

$$S_* = \left\{ \begin{array}{c} (t, y_0, y_1, y_2, y_3) \in [0, 2] \times \mathbb{R}^n : 0 \leq y_0 \leq \frac{x^3}{6} + x^2 + x, \\ 0 \leq y_1 \leq \frac{x^2}{2} + 2x + 1, 0 \leq y_2 \leq x + 2 \end{array} \right\}$$

with

$$\mu = \max\left\{ \left|\beta''(2)\right|, \left|\beta''(0)\right|, \left|\beta''(1)\right| \right\} = 4,,$$

$$\psi(|y_3|) := |y_3| + \frac{22}{3},$$

and

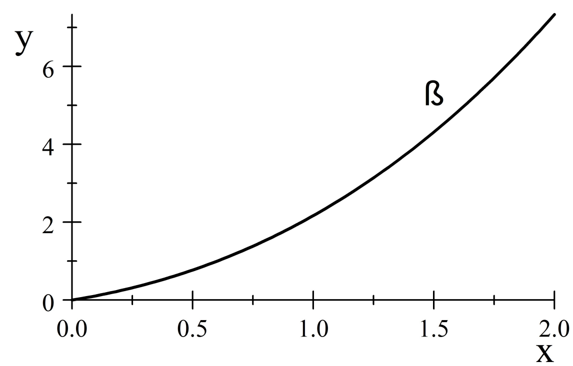
$$\int_{\mu}^{+\infty} \frac{ds}{s + \frac{22}{3}} = +\infty > \int_0^L 1\ ds = L.$$

Moreover f is nondecreasing on y_0 and, by Theorem 3.5, there exists a solution $u(x)$ of problem (3.21)-(3.23) such that

$$\alpha^{(i)}(x) \leqslant u^{(i)}(x) \leqslant \beta^{(i)}(x), i = 0, 1, 2,\ \text{for } x \in [0, 2],$$

that is

$$
\begin{aligned}
0 &\leq u(x) \leq \frac{x^3}{6} + x^2 + x, \\
0 &\leq u'(x) \leq \frac{x^2}{2} + 2x + 1, \\
0 &\leq u''(x) \leq x + 2, \text{ for } x \in [0, 2].
\end{aligned}
$$



Chapter 4

Periodic third order boundary value problems with generalized impulsive conditions

4.1 Introduction

This section presents a nonlinear periodic third order impulsive problem composed by the fully differential equation

$$u'''(t) = f(t, u(t), u'(t), u''(t)) \tag{4.1}$$

for a.e. $t \in J \backslash \{t_1, ..., t_m\}$ with $J := [0,1]$, where $f : J \times \mathbb{R}^3 \to \mathbb{R}$ is a L^1-Carathéodory function, the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(1), \ i = 0, 1, 2, \tag{4.2}$$

and the impulsive effects given by some generalized functions with dependence on the nonlinear function and its first and second derivatives, in the form

$$u^{(i)}\left(t_j^+\right) = I_{ij}\left(u(t_j), u'(t_j), u''(t_j)\right), \tag{4.3}$$

where $u^{(i)}\left(t_j^+\right) = \lim_{t \to t_j^+} u^{(i)}(t)$, $i = 0, 1, 2$, for $j = 1, ..., m$, $t_j \in (0,1)$ such that $0 = t_0 < t_1 < ... < t_m < t_{m+1} = 1$ and $I_{ij} : \mathbb{R}^3 \to \mathbb{R}$ are continuous and nondecreasing functions in all variables.

Different types of third order boundary value problems (separated, periodic, multipoint, with delays, integro-differential, functional,...) have been studied by many authors and several methods, such as fixed point theory, topological and coincidence degree, lower and upper solutions, cone theory,..., and can describe real phenomena in medicine, physics, agriculture, biology, economics,...(see, for example, [2, 6, 10, 11, 28, 45, 45, 52, 57] and the references therein).

Impulsive problems are particularly well adapted to models where there have sudden changes at some moments, and they have been the subject of growing interest (see, for instance, [5, 40, 42, 94, 109]). These jump situations may happen in many fields, such as, population dynamics, control theory, chemistry, ... (see, for example, [9, 80, 91]).

To our best knowledge, it is the first time where third order periodic impulsive problems are considered with the instantaneous changes, depending on the unknown function and its first and second derivatives, given by generalized functions. In this way, problem (4.1)-(4.3) covers cases where the jumps in each moment depend not only on the value of the function on this instant, but also on the velocity and the convexity of the solution in the referred moment.

A particular case of the above problem is applied to a mathematical model concerning the thyroid-pituitary system. In short, the anterior lobe of the pituitary gland produces the hormone thyrotropin under the influence of the Thyrotropin Releasing Factor secreted by the hypothalamus. The thyrotropin, in turn, causes the thyroid gland to produce a thyroid enzyme which when activated produces the hormone thyroxine. This hormone has a negative feedback effect on the secretion of thyrotropin from pituitary. For the first time in the literature an impulsive problem, with generalized impulsive conditions, is used in this type of reaction-diffusion phenomena. This mathematical theory is very useful in the study of causes and clinical treatment of periodic catatonic schizophrenia.

The main tools rely on a perturbed and truncated auxiliary problem, on an iterative technique, not necessarily monotone, as in [118], and lower and upper solutions method. We point out that, the nonlinear part must verify only a local monotone condition (see (4.11)) and no assumption on its periodicity or asymptotic growth is needed.

The work is organized as it follows: in Section 4.2 we describe the class of functions to be considered and an explicit expression for the solution of the associated linear problem. Section 4.3 contains the main result: an existence and localization theorem, that is, with some qualitative information on the solution. In Section 4.4 we present an example to illustrate the potentialities of the main theorem. Last section contains an application to the thyroid-pituitary homeostatic mechanism with impulses, which can be seen as the moments of administration of adequate drugs. As far as we know, it is the first time where such model is considered with impulsive moments.

4.2 Preliminary results

This section contains some notations, definitions and auxiliary results, to be used forward.

For $m \in \mathbb{N}$, let $0 = t_0 < t_1 < ... < t_m < t_{m+1} = 1$, $D = \{t_1, ..., t_m\}$ and

$$u(t_j^{\pm}) := \lim_{t \to t_j^{\pm}} u(t).$$

Definition 4.1 *Denote by $PC(J)$ the set of functions $u : J \to \mathbb{R}$ continuous on $J \setminus D$ where $u\left(t_j^+\right)$ and $u\left(t_j^-\right)$ exist with $u\left(t_j^-\right) = u\left(t_j\right)$, for $j = 1, 2, ..., m$.*

For $u \in PC(J)$, we define the norm by

$$\|u\| = \sup_{t \in J} |u(t)|.$$

Consider $PC^l(J)$, $l = 1, 2$, as the space of the real-valued functions u, such that $u^{(l)} \in PC(J)$, $u^{(l)}\left(t_j^+\right)$ and $u^{(l)}\left(t_j^-\right)$ exist with $u^{(l)}\left(t_j^-\right) = u^{(l)}\left(t_j\right)$, for $l = 0, 1, 2$ and $j = 1, 2, ..., m$.

Therefore $u \in PC^2(J)$ can be written as

$$u(t) = \begin{cases} u_0(t) & \text{if } t \in [0, t_1], \\ u_1(t) & \text{if } t \in (t_1, t_2], \\ \vdots & \\ u_m(t) & \text{if } t \in (t_m, 1], \end{cases} \tag{4.4}$$

where $u_i := u_i|_{(t_i, t_{i+1}]}$ with $u_i \in AC^2(t_i, t_{i+1}]$ for $i = 0, 1, ..., m$.

Denote, for $n \in \mathbb{N}$,

$$PC_D^n(J) = \left\{ \begin{array}{c} u \in PC^n(J) : u^{(j)} \in AC(t_i, t_{i+1}], j = 0, 1, ..., n, \\ i = 0, 1, ..., m \end{array} \right\}$$

and for each $u \in PC_D^n(J)$ we set the norm

$$\|u\|_D = \|u\| + \|u'\| + \cdots + \left\|u^{(n)}\right\|$$

Moreover for $p \in L^1(J)$ we consider the usual norm

$$\|p\|_1 := \int_J |p(t)| \, dt.$$

Throughout this work the following hypothesis will be assumed :

(A1) $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is a L^1-Carathéodory function, that is, $f(t, \cdot, \cdot, \cdot)$ is a continuous function for a. e. $t \in J$;

$f(\cdot, y_0, y_1, y_2)$ is measurable for $(y_0, y_1, y_2) \in \mathbb{R}^3$; and for every $M > 0$ there is a real-valued function $\psi_M \in L^1([0, 1])$ such that

$$|f(t, y_0, y_1, y_2)| \leq \psi_M(t), \text{ for a. e. } t \in [0, 1]$$

and for every $(y_0, y_1, y_2) \in \mathbb{R}^3$ with $|y_i| \leq M$, for $i = 0, 1, 2$;

(A2) the real valued functions I_{ij}, for $i = 0, 1, 2$ and $j = 1, ..., m$ are nondecreasing in all variables.

Definition 4.2 *A function $u \in PC_D^2(J)$ is a solution of (4.1)-(4.3) if it satisfies (4.1) almost everywhere in $J \setminus D$, the periodic conditions (4.2) and the impulse conditions (4.3).*

Next Lemma will give the unique solution for a linear Cauchy problem:

Lemma 4.3 *Let $p : [0,1] \to \mathbb{R}$ such that $p \in L^1([0,1])$. Then for each interval $(t_k, t_{k+1}]$, $j = 0, 1, ..., m$, and $a_j,\ b_j,\ c_j \in \mathbb{R}$, the initial value problem composed by the equation*

$$u'''(t) = p(t), \ \text{for } t \in J \setminus \{t_1, ..., t_m\} \tag{4.5}$$

and the conditions

$$u(t_j^+) = a_j, \quad u'(t_j^+) = b_j, \quad u''(t_j^+) = c_j, \tag{4.6}$$

has a unique solution $u_j \in C^2(t_j, t_{j+1}]$, given by

$$u_j(t) = a_j + b_j(t - t_j) + c_j \frac{(t - t_j)^2}{2} + \int_{t_j}^{t} \frac{(t - r)^2}{2} p(r)\, dr, \tag{4.7}$$

for $t \in (t_j, t_{j+1}]$.
Therefore, $u \in PC_D^2(J)$, given by (4.4), is the unique solution of

$$u'''(t) = p(t), \ \text{for a.e. } t \in [0,1], \tag{4.8}$$

verifying (4.6), for each $j = 0, 1, ..., m$.

Proof. The solution $u(t)$ given by (4.7) can be obtained by iterate integrations of (4.5).
Remark that $t_0 = 0 = t_0^+$ and $t_{m+1} = 1$. ■
Strict lower and upper solutions are defined by the following inequalities:

Definition 4.4 *A function $\alpha \in PC_D^3(J)$ is said to be a strict lower solution of the problem (4.1)-(4.3) if:*

(i) $\alpha'''(t) > f(t, \alpha(t), \alpha'(t), \alpha''(t))$, *for a.e.* $t \in (0,1)$.

(ii) $\alpha(0) \le \alpha(1), \quad \alpha'(0) \le \alpha'(1), \quad \alpha''(0) \le \alpha''(1)$,

(iii) $\alpha(t_j^+) \le I_{0j}(\alpha(t_j), \alpha'(t_j), \alpha''(t_j))$,
$\alpha'(t_j^+) \le I_{1j}(\alpha(t_j), \alpha'(t_j), \alpha''(t_j))$,
$\alpha''(t_j^+) \le I_{2j}(\alpha(t_j), \alpha'(t_j), \alpha''(t_j))$, $j = 1, ..., m$.

A function $\beta \in PC_D^3(J)$ is a strict upper solution of problem (4.1)-(4.3) if the reversed inequalities hold.

4.3 Existence and localization result

The main theorem provides not only the existence of a solution, but also gives some qualitative data about its behavior:

Theorem 4.5 *Let $\alpha, \beta \in PC_D^3(J)$ be, respectively, strict lower and upper solutions of (4.1)-(4.3) such that*

$$\alpha''(t) \le \beta''(t) \ \text{ on } J \setminus D \tag{4.9}$$

and

$$\alpha^{(i)}(0) \le \beta^{(i)}(0), \ i = 0, 1. \tag{4.10}$$

Assume that

$$f(t, \alpha(t), \alpha'(t), y_2) \ge f(t, y_0, y_1, y_2) \ge f\left(t, \beta(t), \beta'(t), y_2\right), \tag{4.11}$$

for fixed $(t, y_2) \in J \times \mathbb{R}$, $\alpha^{(i)}(t) \le y_i \le \beta^{(i)}(t)$, for $i = 0, 1$.

If conditions (A1) and (A2) hold, then the problem (4.1)-(4.3) has a solution $u(t) \in PC_D^3(J)$, such that

$$\alpha^{(i)}(t) \le u^{(i)}(t) \le \beta^{(i)}(t), \ \text{ on } J, \ \text{ for } i = 0, 1, 2.$$

Remark 4.6 *From (4.10), we have $\alpha^{(i)}(t) \le \beta^{(i)}(t)$, for $i = 0, 1$, and every $t \in J$.*

Proof. Consider the following modified problem composed by the equation

$$u'''(t) = f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \delta_2(t, u''(t))), \tag{4.12}$$

for $t \in (0, 1)$ and $t \neq t_j$, $j = 1, ..., m$, where the continuous functions $\delta_i : \mathbb{R}^2 \to \mathbb{R}$, for $i = 0, 1, 2$ are given by

$$\delta_i(t, y_i) = \begin{cases} \beta^{(i)}(t) & , & y_i > \beta^{(i)}(t) \\ y_i & , & \alpha^{(i)}(t) \le y_i \le \beta^{(i)}(t) \\ \alpha^{(i)}(t) & , & y_i < \alpha^{(i)}(t) \end{cases} \tag{4.13}$$

with the boundary conditions (4.2) and the impulse conditions (4.3).

To prove the solvability of problem (4.12),(4.2),(4.3) it is applied an iterative technique, not necessarily monotone.

Let $(u_n)_{n \in \mathbb{N}}$ be the sequence in $PC_D^3(J)$ defined as it follows

$$u_0(t) = \alpha(t), \tag{4.14}$$

and for $n = 1, 2, ...$, $u_n(t)$ is the solution of the problem composed by the equation

$$u_n'''(t) = f\left(t, \delta_0(t, u_{n-1}(t)), \delta_1\left(t, u_{n-1}'(t)\right), \delta_2(t, u_n''(t))\right), \tag{4.15}$$

for a.e. $t \in [0,1]$, with the boundary conditions

$$u_n(0) = u_{n-1}(1), \quad u_n'(0) = u_{n-1}'(1), \quad u_n''(0) = u_{n-1}''(1), \tag{4.16}$$

and the impulsive conditions, for $j = 1, ..., m$,

$$\begin{aligned} u_n\left(t_j^+\right) &= I_{0j}\left(u_{n-1}(t_j), u_n'(t_j), u_n''(t_j)\right), \\ u_n'\left(t_j^+\right) &= I_{1j}\left(\delta_0(t_j, u_{n-1}(t_j)), u_{n-1}'(t_j), u_n''(t_j)\right), \\ u_n''\left(t_j^+\right) &= I_{2j}\left(\delta_0(t_j, u_{n-1}(t_j)), \delta_1\left(t_j, u_{n-1}'(t_j)\right), u_{n-1}''(t_j)\right). \end{aligned} \tag{4.17}$$

By Lemma 4.3, the sequence $(u_n)_{n\in\mathbb{N}}$, is well defined, as for each n the nonlinear part of (4.15) is L^1-Carathéodory and bounded.

Remark that the initial value problem (4.15)-(4.17) will become the periodic impulsive problem (4.1)-(4.3), if the two following claims hold:

- *Every solution $u_n(t)$ of the problem (4.15)-(4.17) verifies*

  $$\alpha^{(i)}(t) \le u_n^{(i)}(t) \le \beta^{(i)}(t), \text{ for } i = 0, 1, 2,$$

  for all $n \in \mathbb{N}$ and every $t \in J$, which implies that

  $$\delta_i\left(t, u_n^{(i)}(t)\right) = u_n^{(i)}(t), \text{ for } i = 0, 1, 2, \ n \in \mathbb{N} \textit{ and every } t \in J,$$

  and, consequently, (4.15) become

  $$u_n'''(t) = f\left(t, u_{n-1}(t), u_{n-1}'(t), u_n''(t)\right), \text{ for a.e. } t \in [0,1].$$

- *There is a subsequence of $(u_n)_{n\in\mathbb{N}}$, denoted by simplicity as (u_n), uniformly convergent to $u \in PC_D^2$, solution of problem (4.1)-(4.3).*

These claims are proven in the following steps:

<u>*Step 1*</u> - *Every solution of problem (4.15)-(4.17) verifies*

$$\alpha^{(i)}(t) \le u_n^{(i)}(t) \le \beta^{(i)}(t), \text{ for } i = 0, 1, 2, \tag{4.18}$$

for all $n \in \mathbb{N}$ and every $t \in J$.

Let u_n be a solution of the problem (4.15)-(4.17). The proof of inequalities (4.18) will be done by mathematical induction. Let us begin to prove that $\alpha''(t) \le u_n''(t) \le \beta''(t)$, $\forall n \in N$, $\forall t \in J$.

For $n = 0$, by (4.14), and $i = 2$,

$$\alpha''(t) = u_0''(t) \le \beta''(t), \text{ for } t \in J,$$

and by, Remark 4.6,

$$\alpha^{(i)}(t) = u_0^{(i)}(t) \le \beta^{(i)}(t), \text{ for } i = 0, 1, \text{ and } t \in J. \tag{4.19}$$

Suppose that, for $k = 1, ..., n-1$ and every $t \in J$, we have

$$\alpha''(t) \leq u_k''(t) \leq \beta''(t). \tag{4.20}$$

For $t = 0$, by (4.16), (4.20) and Definition 4.4, we get

$$u_n''(0) = u_{n-1}''(1) \geq \alpha''(1) \geq \alpha''(0).$$

If $t = t_j^+$, $j = 1, ..., m$, from (4.17), (A_2), (4.20) and Definition 4.4, then

$$\begin{aligned} u_n''\left(t_j^+\right) &= I_{2j}\left(\delta_0(t_j, u_{n-1}(t_j)), \delta_1\left(t_j, u_{n-1}'(t_j)\right), u_{n-1}''(t_j)\right) \\ &\geq I_{2j}\left(\alpha(t_j), \alpha'(t_j), \alpha''(t_j)\right) \geq \alpha''\left(t_j^+\right). \end{aligned}$$

For $t \in]t_j, t_{j+1}]$, $j = 1, 2, ..., m$, suppose, by contradiction, that there exists $t^* \in]t_j, t_{j+1}]$ such that $\alpha''(t^*) > u_n''(t^*)$ and define

$$\min_{t \in]t_j,\, t_{j+1}]} u_n''(t) - \alpha''(t) := u_n''(t^*) - \alpha''(t^*) < 0.$$

From (4.11), (4.15) and Definition 4.4, we get the following contradiction

$$\begin{aligned} 0 &= u_n'''(t^*) - \alpha'''(t^*) \\ &= f\left(t^*, \delta_0(t^*, u_{n-1}(t^*)), \delta_1\left(t^*, u_{n-1}'(t^*)\right), \alpha''(t^*)\right) - \alpha'''(t^*) \\ &\leq f\left(t^*, \alpha(t^*), \alpha'(t^*), \alpha''(t^*)\right) - \alpha'''(t^*) < 0. \end{aligned}$$

Then $u_n''(t) \geq \alpha''(t)$, for all $n \in \mathbb{N}$ and every $t \in J$. In the same way it can be shown that $u_n''(t) \leq \beta''(t)$, $\forall t \in J, \forall n \in \mathbb{N}$, and so (4.18) is proved for $i = 2$.

Let us prove now, again by induction, that

$$\alpha'(t) \leq u_n'(t) \leq \beta'(t), \ \forall t \in J, \forall n \in \mathbb{N}. \tag{4.21}$$

Assume that for $k = 1, ..., n-1$ and every $t \in J$,

$$\alpha'(t) \leq u_k'(t) \leq \beta'(t). \tag{4.22}$$

Then for $t \in [0, t_1]$, by integration of the inequality $u_n''(t) \geq \alpha''(t)$ in $[0, t]$ we have

$$u_n'(t) - u_n'(0) \geq \alpha'(t) - \alpha'(0).$$

By (4.16) and (4.22),

$$\begin{aligned} u_n'(t) &\geq \alpha'(t) - \alpha'(0) + u_{n-1}'(1) \\ &\geq \alpha'(t) - \alpha'(0) + \alpha'(1) \geq \alpha'(t), \end{aligned}$$

and so, $u_n'(t) \geq \alpha'(t)$, for all $t \in [0, t_1]$.

If $t = t_j^+$, $j = 1, ..., m$, by (4.17), (A_2) and Definition 4.4,

$$\begin{aligned} u_n'\left(t_j^+\right) &= I_{1j}\left(\delta_0(t_j, u_{n-1}(t_j)), u_{n-1}'(t_j), u_n''(t_j)\right) \\ &\geq I_{1j}\left(\alpha(t_j), \alpha'(t_j), \alpha''(t_j)\right) \geq \alpha'\left(t_j^+\right). \end{aligned}$$

For $t \in \left]t_j, t_{j+1}\right]$, $j = 1, 2, ..., m$, from an integration of the same inequality in $\left]t_j, t\right]$, and, by (4.17) and Definition 4.4,

$$\begin{aligned}
\alpha'(t) &\leq u_n'(t) + \alpha'\left(t_j^+\right) - u_n'\left(t_j^+\right) \\
&= u_n'(t) + \alpha'\left(t_j^+\right) - I_{1j}\left(\delta_0(t_j, u_{n-1}(t_j)), u_{n-1}'(t_j), u_n''(t_j)\right) \\
&\leq u_n'(t) + I_{1j}\left(\alpha(t_j), \alpha'(t_j), \alpha''(t_j)\right) - I_{1j}\left(\alpha(t_j), \alpha'(t_j), \alpha''(t_j)\right) \\
&= u_n'(t).
\end{aligned}$$

Therefore $u_n'(t) \geq \alpha'(t)$, for all $n \in \mathbb{N}$ and every $t \in J$.

Using similar arguments it can be proved that $u_n'(t) \leq \beta'(t)$ and so (4.21) is proved.

The remaining inequalities in (4.18) can be proved as above, by integration of (4.21) in $[0, t]$, for all cases of t, applying the correspondent hypothesis of induction, conditions (4.16), (4.17), (A_2), and Definition 4.4.

Step 2 - There is a subsequence of $(u_n)_{n\in\mathbb{N}}$, denoted by simplicity as (u_n), uniformly convergent to $u \in PC_D^2$, solution of problem (4.1)-(4.3).

For $i = 0, 1, 2$, let $C_i = \max\left\{\left\|\alpha^{(i)}\right\|, \left\|\beta^{(i)}\right\|\right\}$. So there exists $M > 0$, with $M := C_0 + C_1 + C_2$ and for all $n \in \mathbb{N}$

$$\|u_n\|_D \leq M. \tag{4.23}$$

Let Ω be a compact subset of $\mathbb{R}^3$ given by

$$\Omega = \left\{(w_0, w_1, w_2) \in \mathbb{R}^3 : |w_i| \leq C_i,\ i = 0, 1, 2\right\}.$$

As f is a L^1-Carathéodory function in Ω, then there exists a real-valued function $h_M(t) \in L^1(J)$, such that

$$|f(t, w_0, w_1, w_2)| \leq h_M(t), \text{ for every } (w_0, w_1, w_2) \in \Omega. \tag{4.24}$$

By Step 1 and (4.23), $(u_n, u_n', u_n'') \in \Omega$, for all $n \in \mathbb{N}$. From (4.15) and (4.24) we obtain

$$|u_n'''(t)| \leq h_M(t), \text{ for a.e. } t \in J,$$

hence $u_n'''(t) \in L^1(J)$.

By integration in J we have that

$$\begin{aligned}
u_n''(t) &= u_n''(0) + \int_0^t u_n'''(s)\, ds \\
&\quad + \sum_{0<t_j\leq t}\left(I_{2j}\left(u_{n-1}(t_j), u_{n-1}'(t_j), u_{n-1}''(t_j)\right) - u_n''(t_j^-)\right),
\end{aligned}$$

therefore $u_n'' \in AC(t_j, t_{j+1})$ and $u_n \in PC_D^2(J)$.

Moreover $u_n''(s)$ is equicontinuous in $PC_D(J)$. Indeed, assuming without loss of generality, that $t_* \le t^*$, then

$$|u_n''(t^*) - u_n''(t_*)| \le \left| \int_{t_*}^{t^*} u_n'''(s)\,ds \right|$$
$$+ \left| \sum_{t_* < t_j \le t^*} \left(I_{2j}\left(u_{n-1}(t_j), u_{n-1}'(t_j), u_{n-1}''(t_j)\right) - u_n''(t_j^-)\right) \right| \longrightarrow 0,$$

as $t_* \to t^*$. Therefore, by Ascoli-Arzèla Theorem, there exists a subsequence of u_n, denoted by simplicity as u_n, which converges to $u \in PC_D^2(J)$. Then $(u, u', u'') \in \Omega$.

Using the Lebesgue dominated convergence theorem and Step 1, for $t \in (t_j, t_{j+1})$,

$$\int_{t_j}^{t} f\left(s, u_{n-1}(s), u_{n-1}'(s), u_n''(s)\right) ds$$

is convergent to

$$\int_{t_j}^{t} f\left(s, u(s), u'(s), u''(s)\right) ds$$

as $n \to \infty$.

So, for $j = 0, 1, ..., m$, as $n \to \infty$,

$$u_{jn}''(t) = u_n''(t_j) + \int_{t_j}^{t} f\left(s, u_{n-1}(s), u_{n-1}'(s), u_n''(s)\right) ds$$

is uniformly convergent to

$$u_j''(t) = u''(t_j) + \int_{t_j}^{t} f\left(s, u(s), u'(s), u''(s)\right) ds.$$

As the function f is L^1-Carathéodory function in (t_j, t_{j+1}), then $u_j''(t) \in AC(t_j, t_{j+1})$. Therefore, for u defined as in (4.4), we have $u \in PC_D^2(J)$ and u is a solution of equation (4.1).

To prove that u is a solution of the boundary value problem (4.1)-(4.3) we note that (u_n) is uniformly convergent to $u \in PC_D^2(J)$, and so in each branch $u_{jn} \to u_j$ uniformly in $C^2(t_j, t_{j+1}]$, for $j = 0, 1, ..., m$. Therefore, for $i = 0, 1, 2$, by (4.16),

$$u^{(i)}(0) = \lim_{n\to+\infty} u_n^{(i)}(0) = \lim_{n\to+\infty} u_{n-1}^{(i)}(1) = u^{(i)}(1).$$

Moreover, by the continuity of the impulsive functions I_{ij}, for $i = 0, 1, 2$ and $j = 1, ..., m$, Step 1, and (4.17),

$$\begin{aligned} u^{(i)}\left(t_j^+\right) &= \lim_{n\to+\infty} I_{ij}\left(u_n(t_j), u_n'(t_j), u_n''(t_j)\right)) \\ &= \lim_{n\to+\infty} I_{ij}\left(u_{n-1}(t_j), u_{n-1}'(t_j), u_{n-1}''(t_j)\right)) \\ &= I_{ij}\left(u(t_j), u'(t_j), u''(t_j)\right)). \end{aligned}$$

So, u verifies (4.3), and problem (4.1)-(4.3) has a solution $u(t) \in PC_D^2(J)$, such that

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \text{ for } i = 0, 1, 2,$$

for $t \in J$. ■

4.4 Example

Consider the third order fully differential equation

$$u'''(t) = -e^{u(t)} - (u'(t))^3 + 280\sqrt[5]{u''(t)}, \tag{4.25}$$

with the periodic boundary conditions (4.2) and the impulsive functions

$$\begin{aligned} u\left(\frac{1}{2}^+\right) &= \frac{1}{10}u\left(\frac{1}{2}\right) + \frac{1}{100}\left[u'\left(\frac{1}{2}\right) + u''\left(\frac{1}{2}\right)\right], \\ u'\left(\frac{1}{2}^+\right) &= \frac{1}{10}\left[u\left(\frac{1}{2}\right) + u'\left(\frac{1}{2}\right) + u''\left(\frac{1}{2}\right)\right], \\ u''\left(\frac{1}{2}^+\right) &= \frac{1}{5}\left[u\left(\frac{1}{2}\right) + u'\left(\frac{1}{2}\right)\right] + \frac{1}{10}u''\left(\frac{1}{2}\right). \end{aligned} \tag{4.26}$$

This problem is a particular case of (4.1)-(4.3) with

$$f(t, y_0, y_1, y_2) = -e^{y_0} - (y_1)^3 + 280\sqrt[5]{y_2},$$

for all $t \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, $m = 1$, $t_1 = \frac{1}{2}$ and the nondecreasing functions I_{i1}, $i = 0, 1, 2$, are given by

$$\begin{aligned} I_{01}(a,b,c) &= \frac{1}{10}a + \frac{1}{100}(b+c), \\ I_{11}(a,b,c) &= \frac{1}{10}(a+b+c), \\ I_{21}(a,b,c) &= \frac{1}{5}(a+b) + \frac{1}{10}c. \end{aligned} \tag{4.27}$$

The piecewise continuous $\alpha, \beta \in PC_D^2(J)$, with $D = \left\{\frac{1}{2}\right\}$, defined as

$$\alpha(t) = \begin{cases} 10^{-10}\left(-t^2 - 2t - 2\right) & , t \in \left[0, \frac{1}{2}\right] \\ 10^{-10}\left(-t^2 - \frac{1}{4}\right) & , t \in \left(\frac{1}{2}, 1\right] \end{cases}$$

and

$$\beta(t) = \begin{cases} t^2 + 4t + 3 & , t \in \left[0, \frac{1}{2}\right] \\ t^2 + \frac{t}{2} + \frac{1}{4} & , t \in \left(\frac{1}{2}, 1\right] \end{cases},$$

are lower and upper solutions, respectively, for problem (4.25), (4.2), (4.26), assuming

$$\alpha'(t) = \begin{cases} 10^{-10}(-2t - 2) & , t \in \left[0, \frac{1}{2}\right] \\ -2 \times 10^{-10}t & , t \in \left(\frac{1}{2}, 1\right] \end{cases}, \quad \beta'(t) = \begin{cases} 2t + 4 & , t \in \left[0, \frac{1}{2}\right] \\ 2t + \frac{1}{2} & , t \in \left(\frac{1}{2}, 1\right] \end{cases},$$

and $\alpha''(t) = -2 \times 10^{-10}$, $\beta''(t) = 2$, for $t \in [0,1]$.

As f satisfies assumption (A_1) and (4.11), the jump functions (4.27) verify (A_2), then, by Theorem 4.5 there is a periodic solution $u(t) \in PC_D^3(J)$ of problem (4.25), (4.2), (4.26), such that

$$\begin{aligned} 10^{-10}\left(-t^2-2t-2\right) &\leq u(t) \leq t^2+4t+3,\ t \in \left[0,\frac{1}{2}\right], \\ 10^{-10}\left(-t^2-\frac{1}{4}\right) &\leq u(t) \leq t^2+\frac{t}{2}+\frac{1}{4},\ t \in \left(\frac{1}{2},1\right], \end{aligned}$$

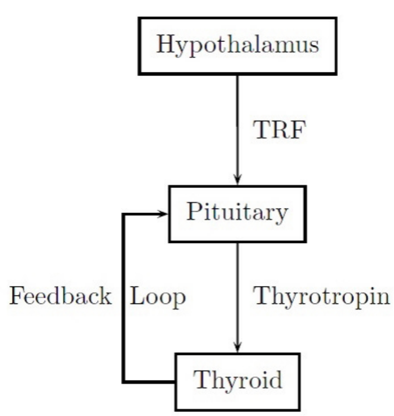
that is $u(t)$ lies in this branchand, for $u'(t)$,

$$\begin{aligned} 10^{-10}(-2t-2) &\leq u'(t) \leq 2t+4,\ t \in \left[0,\frac{1}{2}\right], \\ -2\times 10^{-10}t &\leq u'(t) \leq 2t+\frac{1}{2},\ t \in \left(\frac{1}{2},1\right], \end{aligned}$$

Remark that this solution can not be a trivial periodic one, as constants do not verify (4.25).

4.5 Application to the thyroid-pituitary homeostatic mechanism

The anterior lobe of pituitary gland produces the hormone thyrotropin under the influence of the Thyrotropin Releasing Factor (TRF) secreted by the hypothalamus in the brain. Thyrotropin, when it reaches the thyroid gland, activates a thyroid enzyme which, in turn, catalyzes the shedding of thyroxine from the colloidal follicles of the thyroid gland into the blood stream. This hormone has a negative feedback effect on the secretion of thyrotropin from pituitary. This mechanism can be depicted as in the following block diagram.



Abnormal steady-state thyroxine levels in the bloodstream can cause system malfunction leading to various types of physical and mental disorders. Physical disorders include different forms of hypo- and hyperthyroidism. A system malfunction leading to a severe mental disorder is known as periodic catatonic schizophrenia. In this disease, the symptoms vary with remarkably regular periodicity. This has been studied at length in [82].

The mathematical model to describe this negative feedback mechanism was introduced in [62], and, more recently, studied in [13], is the following

$$
\begin{aligned}
\frac{dP}{dt} &= \begin{cases} c - h\theta - gP & , \text{ if } \theta \leq \frac{c}{h} \\ -gP & , \text{ if } \theta > \frac{c}{h}, \end{cases} \\
\frac{dE}{dt} &= mP - kE, \\
\frac{d\theta}{dt} &= aE - b\theta,
\end{aligned}
$$

where P, E and θ represent the concentrations of thyrotropin, activated enzyme and thyroxine, respectively, b, g and k represent the loss constants of thyroxine, thyrotropin and activated enzyme, respectively, a, h, m are constants expressing the sensitivities of the glands to stimulation or inhibition; c is the rate of production of thyrotropin in the absence of thyroid inhibition. All constants are assumed to be positive.

This model is very useful in the study of causes and clinical treatment of periodic catatonic schizophrenia.

As it is suggested in [62], from the above equations it can be obtained the third order differential equation describing the variation of thyroid hormone,thyroxine, with time

$$
\eta_3 \frac{d^3\theta(t)}{dt^3} - \eta_2 \frac{d^2\theta(t)}{dt^2} + \eta_1 \frac{d\theta(t)}{dt} + \xi_1\theta(t) + \xi_2 \frac{\theta(t)}{1 + n\ \theta(t)} + \xi_3 = 0, \tag{4.28}
$$

with η_i, ξ_i are positive constants, for $i = 1, 2, 3$.

As far as we know, this work studies for the first time, an impulsive periodic problem for this pituitary-thyroid mechanism model. Indeed together with equation (4.28) and periodic conditions (4.2) we consider the impulsive conditions

$$
\begin{aligned}
\theta\left(t_j^+\right) &= \kappa_1\theta(t_j) + \kappa_2\theta'(t_j) + \kappa_3\theta''(t_j), \\
\theta'\left(t_j^+\right) &= \kappa_4\theta(t_j) + \kappa_5\theta'(t_j) + \kappa_6\theta''(t_j), \\
\theta''\left(t_j^+\right) &= \kappa_7\theta(t_j) + \kappa_8\theta'(t_j) + \kappa_9\theta''(t_j),
\end{aligned} \tag{4.29}
$$

for $j = 1, ..., m$, and $\kappa_i > 0$, $i = 1, 2, ..., 9$, meaning, in the model context, that new quantity of thyroxine and the increment of thyroxine introduced in the blood stream at moments t_j, are both directly proportional to the level of thyroxine already existent at the instant t_j, $\theta(t_j)$, and to the previous increments, $\theta'(t_j)$.

It is clear that the equation (4.28) is a particular case of (4.1), with

$$f\left(t, y_0, y_1, y_2\right)=\frac{\eta_2}{\eta_3} y_2-\frac{\eta_1}{\eta_3} y_1-\frac{1}{\eta_3}\left(\xi_1 \theta(t)+\xi_2 \frac{\theta(t)}{1+n\ \theta(t)}+\xi_3\right),$$

verifying (4.11), and the impulsive conditions (4.29) are an example of the generalized impulsive conditions (4.3),

$$\begin{aligned}
I_{0j}\left(\theta(t_j), \theta'(t_j), \theta''(t_j)\right) &:= \kappa_1 \theta(t_j)+\kappa_2 \theta'(t_j)+\kappa_3 \theta''(t_j), \\
I_{1j}\left(\theta(t_j), \theta'(t_j), \theta''(t_j)\right) &:= \kappa_4 \theta(t_j)+\kappa_5 \theta'(t_j)+\kappa_6 \theta''(t_j), \\
I_{2j}\left(\theta(t_j), \theta'(t_j), \theta''(t_j)\right) &:= \kappa_7 \theta(t_j)+\kappa_8 \theta'(t_j)+\kappa_9 \theta''(t_j),
\end{aligned}$$

with $\kappa_i \geq 0$, $i=1,2,...,9$.

The functions $\alpha, \beta \in PC_D^3(J)$ given by

$$\alpha(t):=\begin{cases} \frac{1}{4} & , \quad x \leq \frac{1}{2} \\ -0.025 & , \quad x>\frac{1}{2}\end{cases}$$

and

$$\beta(t):=\begin{cases} -x^3+2x^2+4x+6 & , \quad x \leq \frac{1}{2} \\ -\frac{1}{2}x^3+2x^2+\frac{3}{2}x-1 & , \quad x>\frac{1}{2},\end{cases}$$

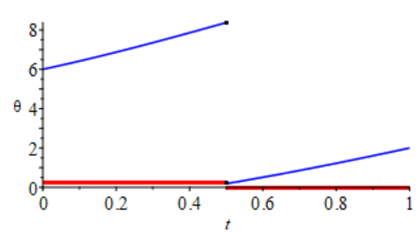
are, respectively, lower and upper solutions of the problem (4.28), (4.2), (4.29), according Definition 4.4, verifying (4.9) and (4.10), for

$$\begin{aligned}
\frac{\eta_2}{\eta_3} &= \frac{3}{2},\ \frac{\eta_1}{\eta_3}=\frac{1}{10},\ \frac{\xi_1}{\eta_3}=\frac{1}{10},\ \frac{\xi_2}{\eta_3}=1,\ \xi_3=0, \\
\kappa_1 &= \frac{1}{100},\ \kappa_2=\frac{1}{200}=\kappa_3,\ \kappa_4=\frac{1}{10}=\kappa_5=\kappa_6, \\
\kappa_7 &= \frac{1}{5},\ \kappa_8=\frac{1}{100},\ \kappa_9=\frac{1}{10}.
\end{aligned}$$

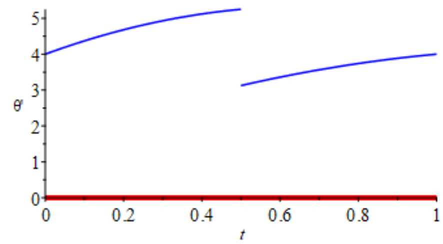
Then, by Theorem 4.5, there is a solution $\theta(t)$ of problem (4.28), (4.2), (4.29) such that

$$\alpha^{(i)}(t) \leq \theta^{(i)}(t) \leq \beta^{(i)}(t), \text{ on } [0,1], \text{ for } i=0,1,2,$$

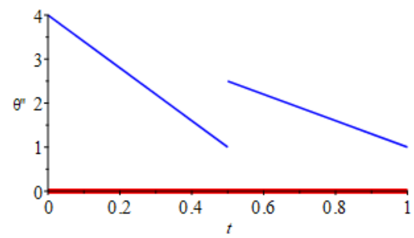
that is lying in the following strips:



Localization of function $\theta(t)$



Localization of function $\theta'(t)$



Localization of $\theta''(t)$.

From this localization result we may say that the nontrivial solution $\theta(t)$ of (4.28), (4.2), (4.29) is nondecreasing and a nonconcave function.

Chapter 5

Third-order generalized discontinuous impulsive problems on the half-line

This chapters deals with the boundary value problem composed by the third-order differential equation on the half real line

$$u'''(t) = f\left(t, u(t), u'(t), u''(t)\right), \ t \in [0, +\infty) \tag{5.1}$$

where $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is a L^1- Carathéodory function, together with the boundary conditions

$$u(0) = A, \ u'(0) = B, \ u''(+\infty) = C, \tag{5.2}$$

with $A, B, C \in \mathbb{R}$, $u''(+\infty) := \lim\limits_{t \to +\infty} u''(t)$, and the impulsive effects given by the generalized functions

$$\begin{aligned} \Delta u(t_k) &= I_{0k}\left(t_k, u(t_k), u'(t_k), u''(t_k)\right) \\ \Delta u'(t_k) &= I_{1k}\left(t_k, u(t_k), u'(t_k), u''(t_k)\right) \\ \Delta u''(t_k) &= I_{2k}\left(t_k, u(t_k), u'(t_k), u''(t_k)\right) \end{aligned} \tag{5.3}$$

with $0 = t_0 < t_1 < t_2 < ... < t_k < ...$, $k \in \mathbb{N}$, such that $\lim\limits_{k \to +\infty} t_k = +\infty$, and $I_{ik} : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ Catathéodory sequences for $i = 0, 1, 2$ and $k \in \mathbb{N}$.

We point out that the technique presented in this work can be easily adapted to n^{th} order problems, with obvious changes, of the type

$$u^{(n)}(t) = f\left(t, u(t), ..., u^{(n-1)}(t)\right), \ t \in [0, +\infty), \tag{5.4}$$

$$u^{(i)}(0) = A_i, \ \ u^{(n-1)}(+\infty) = B, \ A_i, B \in \mathbb{R}, \ i = 0, 1, ..., n-2, \tag{5.5}$$

$$\Delta u^{(j)}(t_k) = I_{ik}\left(t_k, u(t_k), ..., u^{(n-1)}(t_k)\right), j = 0, 1, ..., n-1, k \in \mathbb{N}. \tag{5.6}$$

The option for order three here, is due to clearance reasons, to highlight the method and not make the reading more difficult with a heavy notation.

These higher-order boundary value problems with asymptotic conditions can model some real phenomena as gas pressure in a semi-infinite porous medium, the draining or coating fluid-flow problems and other evolution of physical processes. Likewise, they are useful in more theoretical studies such as on nonlinear elliptic equations, to prove the existence of radially symmetric solutions, or heteroclinic and homoclinic solutions of differential equations or coupled systems of differential and integral equations. As related works we mention, for instance, [21, 22, 24, 25, 29, 33, 84, 93].

As the infinite interval is noncompact, the discussion about sufficient conditions for the solvability of boundary value problems on the half-line is more delicate. In the literature the main methods to obtain existence results are the extension of continuous solutions on finite intervals via a diagonalization process, lower and upper solutions and fixed point theorems in Banach weighted spaces (see [14, 15, 38, 85] and their references).

Impulsive problems, that is, situations where a sudden variations happens, had have an important development in last decades, mostly due to their applicability to real life phenomena. See, for example, [9, 12, 29, 54, 58, 79, 81, 108, 112] and the references therein.

In [116], the authors consider a problem similar to (5.4)-(5.6) where the nonlinearity and all the impulsive functions must be sublinear and nondecreasing in all space variables. It is proved the existence of positive solutions by cone theory and Mönch fixed point theorem, together with a monotone iterative technique.

Motivated by this work we study problem (5.1)-(5.3) under weaker conditions, not only on the nonlinearity but also on the impulsive functions. Indeed, being more specific:

- the nonlinearity f is a L^1- Carathéodory function, meaning that could be discontinuous on time and, eventually, superlinear near the origin or at $+\infty$. Moreover, there is not a monotone assumption on f for higher order derivative, while in the other variables there is only the restriction of a local monotony in some strip.

- the impulsive functions I_{ik}, with $i = 0, 1, 2$, are local monotone, that is, the monotony is required only on a strip. Moreover, the sequence I_{2k} has a different monotony of [116] and no monotone assumption at all on the highest order variable .

- the solutions may have negative values.

Our method relies on lower and upper solutions technique, which reveals to be adequate to these impulsive boundary value problems, adding to the existence of solution its localization and some qualitative data on its behavior as well. We apply some truncation and perturbation techniques suggested, for example, in

[2, 47, 48], together with equiconvergence on the $+\infty$ and on the impulsive points, as it appears in [18].

The chapter is organized in the following way: Section 5.2 contains the definition of the Banach spaces, the corresponding weighted norms, and other auxiliary results as well. In Section 5.3 it is presented the main theorem: an existence and localization result, where it is proved the existence of at least a solution, and some bounds on the first and second derivatives. Last section, has an application to a boundary layer flow problem over a stretching sheet with and without heat transfer.

5.1 Definitions and auxiliary results

A key argument of our method is based on a weighted space with some asymptotic assumptions.

Consider the spaces

$$PC^2[0,+\infty] = \left\{ \begin{array}{c} u : u \in C^2([0,+\infty];\mathbb{R}) \text{ for } t \neq t_k, u^{(i)}(t_k) = u^{(i)}(t_k^-), u^{(i)}(t_k^+) \\ \text{exists for } k = 1,2,...,m, \text{ and } i = 0,1,2 \end{array} \right\}.$$

and

$$X = \left\{ x \in PC^2[0,+\infty) : \lim_{t\to+\infty} \frac{x^{(i)}(t)}{w_i(t)} \text{ exists, } i = 0,1,2 \right\}$$

with $w_i(t) = 1 + t^{2-i}$, and the norm $\|y\| = \max\{\|y\|_0, \|y\|_1, \|y\|_2\}$, where

$$\|y\|_0 = \sup_{0\le t<+\infty} \left\{ \frac{|y(t)|}{1+t^2} \right\}, \|y\|_1 = \sup_{0\le t<+\infty} \left\{ \frac{|y'(t)|}{1+t} \right\}, \|y\|_2 = \sup_{0\le t<+\infty} \left\{ \frac{|y''(t)|}{2} \right\}.$$

Therefore $(X, \|.\|)$ is a Banach space.

The nonlinearities will have the regularity of L^1- Carathéodory function defined as it follows:

Definition 5.1 *A function $f : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is L^1- Carathéodory if it verifies*

i) for each $(x,y,z) \in \mathbb{R}^3$, $t \mapsto f(t,x,y,z)$ is measurable on $[0,+\infty)$;

ii) for almost every $t \in [0,+\infty), (x,y,z) \mapsto f(t,x,y,z)$ is continuous in $\mathbb{R}^3$;

iii) for each $\rho > 0$, there exists a positive function $\psi_\rho \in L^1[0,+\infty)$ such that, for $\max\{\|x\|_0, \|y\|_1, \|z\|_2\} < \rho$,

$$|f(t,x,y,z)| \le \psi_\rho(t), \text{ a.e. } t \in [0,+\infty).$$

The impulsive effects are given in terms of sequences of functions as in next definition:

Definition 5.2 *A sequence $(w_n)_{n\in\mathbb{N}}$ is a Carathéodory sequence if* *(i) for each* $(x,y,z) \in \mathbb{R}^3$, $(x,y,z) \to w_n(x,y,z)$ *is continuous for all* $n \in \mathbb{N}$*;*

(ii) for each $\rho > 0$, *there are nonnegative constants* $\lambda_{n\rho} \geq 0$ *with* $\sum_{n=1}^{+\infty} \lambda_{n\rho} < +\infty$ *such that for* $|x| < \rho(1+t^2), |y| < \rho(1+t), |z| < 2\rho$, *for* $t \in [0,+\infty[$, *we have*

$$|w_n(x,y,z)| \leq \lambda_{n\rho}, \text{ for every } n \in \mathbb{N}.$$

Next lemma gives the exact solution for the associated linear and homogeneous problem:

Lemma 5.3 *If* $e \in L^1[0,+\infty)$, *then the boundary value problem*

$$\begin{cases} u'''(t) = e(t), \quad t \in (0,+\infty), \\ u(0) = A, \; u'(0) = B, \; u''(+\infty) = C. \end{cases} \tag{5.7}$$

has a unique solution in X. *Moreover, this solution can be expressed as*

$$u(t) = A + Bt + \frac{Ct^2}{2} \tag{5.8}$$

$$+ \sum_{k\,:\,t>t_k} \left[\begin{array}{c} I_{0,k}(t_k, u(t_k), u'(t_k), u''(t_k)) + I_{1,k}(t_k, u(t_k), u'(t_k), u''(t_k))(t-t_k) \\ +I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)) \frac{(t-t_k)^2}{2} \end{array} \right]$$

$$-\frac{t^2}{2} \sum_{k=1}^{+\infty} I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)) - \int_0^{+\infty} G(t,s)\, e(s) ds,$$

where $G(t,s)$ *is the Green function of the homogeneous problem associated to (5.7), given by*

$$G(t,s) = \begin{cases} \frac{1}{2}s^2 - st, & 0 \leq s \leq t \\ -\frac{1}{2}t^2 & t \leq s \leq +\infty. \end{cases} \tag{5.9}$$

The proof follows from standard integrations and usual arguments and is omitted.

The following theorem, to be used forward, gives a general criterion for relative compactness:

Theorem 5.4 *([18]) Let* $M \subset C_\infty = \left\{ x \in C[0,+\infty) : \lim_{t\to+\infty} x(t) \text{ exists} \right\}$. *Then* M *is relatively compact if the following conditions hold:*

1. *all functions from* M *are uniformly bounded;*
2. *all functions from* M *are equicontinuous on any compact interval of* $[0,+\infty)$*;*
3. *all functions from* M *are equiconvergent at infinity, that is, for any given* $\epsilon > 0$, *there exists a* t_ϵ *such that* $|x(t) - x(+\infty)| < \epsilon$, *for all* $t > t_\epsilon$ *and* $x \in M$.

5.2 Main Result

In this section we prove the existence of at least one solution for the problem (5.1)-(5.3), applying lower and upper solutions method and, moreover, some data on its behavior and variation are given.

First we define lower and upper functions for the impulsive problems:

Definition 5.5 *Given $A, B, C \in \mathbb{R}$, a function $\alpha \in X$ is a lower solution of problem (5.1)-(5.3) if*

$$\begin{cases} \alpha'''(t) \geq f\left(t, \alpha(t), \alpha'(t), \alpha''(t)\right), t \in [0, +\infty), \\ \alpha(0) \leq A, \\ \alpha'(0) \geq B, \\ \alpha''(+\infty) \leq C, \\ \Delta\alpha(t_k) \leq I_{0k}\left(t_k, \alpha(t_k), \alpha'(t_k), \alpha''(t_k)\right) \\ \Delta\alpha'(t_k) > I_{1k}\left(t_k, \alpha(t_k), \alpha'(t_k), \alpha''(t_k)\right) \\ \Delta\alpha''(t_k) > I_{2k}\left(t_k, \alpha(t_k), \alpha'(t_k), \alpha''(t_k)\right), \end{cases}$$

with $k \in \mathbb{N}$.
A function $\beta \in X$ is an upper solution if it verifies the reversed inequalities.

Forward, the following assumption will play a key role:

(A) There is $\xi > 0$ such that

$$\xi \geq \max \left\{ \begin{array}{c} \|\alpha\|_0, \|\beta\|_0, \|\alpha'\|_1, \left\|\beta'\right\|_1, \|\alpha''\|_2, \left\|\beta''\right\|_2, \\ |A| + \frac{|B|+|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{0k\xi} + \sum_{k=1}^{+\infty} \lambda_{1k\xi} + \sum_{k=1}^{+\infty} \lambda_{2k\xi} \\ +M_0\left(\frac{\pi}{2} + \int_0^{+\infty} \psi_\xi(s)ds\right), \\ |B| + |C| + \sum_{k=1}^{+\infty} \lambda_{1k\xi} + 2\sum_{k=1}^{+\infty} \lambda_{2k\xi} \\ +M_1\left(\frac{\pi}{2} + \int_0^{+\infty} \psi_\xi(s)ds\right), \\ \frac{|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{2k\xi} + \frac{1}{2}\int_0^{+\infty} \psi_\xi(s)ds + \frac{\pi}{4} \end{array} \right\},$$

where, for $I_{ik} : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ Catathéodory sequences, $i = 0, 1, 2$, $k \in \mathbb{N}$,

$$\left|I_{ik}\left(t_k, y_0, y_1, y_2\right)\right| \leq \sum_{k=1}^{+\infty} \lambda_{ik\xi} < +\infty,$$

when

$$|y_0| < \xi(1+t^2), |y_1| < \xi(1+t), |y_2| < 2\xi, \text{ for } t \in [0, +\infty[,$$

$$M_0 := \sup_{t\in[0,+\infty[} \frac{|G(t,s)|}{1+t^2}, M_1 := \sup_{t\in[0,+\infty[} \frac{\left|\frac{\partial G}{\partial t}(t,s)\right|}{1+t},$$

for $s \in [0, +\infty[$, and

$$|f(t,x,y,z)| \leq \psi_\xi(t), \ a.e. \ t \in [0, +\infty),$$

when $\max\left\{\|x\|_0, \|y\|_1, \|z\|_2\right\} < \xi$.

The existence and localization result is given by next theorem:

Theorem 5.6 *Consider $A, B, C \in \mathbb{R}$. Assume that there are α and β lower and upper solutions of problem (5.1)-(5.3) such that*

$$\alpha''(t) \leq \beta''(t), \ \forall t \in [0, +\infty). \tag{5.10}$$

Let $f : [0, +\infty] \times \mathbb{R}^4 \to \mathbb{R}$ be a $L^1 -$ Carathéodory function with

$$f\left(t, \alpha(t), \alpha'(t), y_2\right) \geq f\left(t, y_0, y_1, y_2\right) \geq f\left(t, \beta(t), \beta'(t), y_2\right) \tag{5.11}$$

for $t \in [0, +\infty]$, $\alpha(t) \leq y_0 \leq \beta(t)$, $\alpha'(t) \leq y_1 \leq \beta'(t)$, and $y_2 \in \mathbb{R}$.
Assume that $I_{ik} : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ are Catathéodory sequences, for $i = 0, 1, 2$, $k \in \mathbb{N}$, such that

$$I_{0k}\left(t_k, \alpha(t_k), \alpha'(t_k), \alpha''(t_k)\right) \leq I_{0k}\left(t_k, y_0, y_1, y_2\right) \leq I_{0k}\left(t_k, \beta(t_k), \beta'(t_k), \beta''(t_k)\right), \tag{5.12}$$

for $\alpha^{(i)}(t_k) \leq y_i \leq \beta^{(i)}(t_k)$, $i = 0, 1, 2$,

$$I_{1k}\left(t_k, \alpha(t_k), \alpha'(t_k), \alpha''(t_k)\right) \leq I_{1k}\left(t_k, y_0, y_1, y_2)\right) \leq I_{1k}\left(t_k, \beta(t_k), \beta'(t_k), \beta''(t_k)\right), \tag{5.13}$$

for $\alpha^{(i)}(t_k) \leq y_i \leq \beta^{(i)}(t_k)$, $i = 0, 1, 2$,

$$I_{2k}\left(t_k, \beta(t_k), \beta'(t_k), y_2\right) \leq I_{2k}\left(t_k, y_0, y_1, y_2)\right) \leq I_{2k}\left(t_k, \alpha(t_k), \alpha'(t_k), y_2\right), \tag{5.14}$$

for $\alpha^{(i)}(t_k) \leq y_i \leq \beta^{(i)}(t_k)$, $i = 0, 1$, $y_2 \in \mathbb{R}$.
If there is $\xi > 0$ such that assumption (A) holds, then there is at least $u(t) \in X$ solution of (5.1)-(5.3) such that

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \ \forall t \in [0, +\infty], \ i = 0, 1, 2.$$

Proof. Let $\alpha, \beta \in X$ be, respectively, lower and upper solutions of (5.1)-(5.3) verifying (5.10).

Notice that the relations $\alpha(t) \leq \beta(t)$ and $\alpha'(t) \leq \beta'(t)$, $\forall t \in [0, +\infty[$, are obtained by integration from (5.10) and the boundary conditions (5.2).

Consider the modified and perturbed problem composed by the differential equation

$$\begin{aligned} u'''(t) &= f\left(t, \delta_0\left(t, u(t)\right), \delta_1\left(t, u'(t)\right), \delta_2\left(t, u''(t)\right)\right) \\ &\quad + \frac{1}{1+t^2} \frac{u''(t) - \delta_2\left(t, u''(t)\right)}{1 + \left|u''(t) - \delta_2\left(t, u''(t)\right)\right|}, \end{aligned} \tag{5.15}$$

for $t \in [0, +\infty)$, where the functions $\delta_j : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $j = 0, 1, 2$ are given by

$$\delta_j(t, u^{(j)}(t)) = \begin{cases} \beta^{(j)}(t) & , u^{(j)}(t) > \beta^{(j)}(t) \\ u^{(j)}(t) & , \alpha^{(j)}(t) \leq u^{(j)}(t) \leq \beta^{(j)}(t) \\ \alpha^{(j)}(t) & , u^{(j)}(t) < \alpha^{(j)}(t), \end{cases}$$

the boundary conditions (5.2), and the truncated impulsive effects

$$\Delta u^{(j)}(t_k) = I_{j,k}\left(t_k, \delta_0\left(t_k, u(t_k)\right), \delta_1(t_k, u'(t_k)), \delta_2\left(t_k, u''(t_k)\right)\right), \; j = 0, 1, 2. \tag{5.16}$$

For clearance we divide the proof into claims.

CLAIM 1: *Problem (5.15),(5.2), (5.16) has at least one solution.*

Define the operator $\mathcal{T} : X \to X$ given by

$$\mathcal{T}u(t) = A + Bt + \frac{Ct^2}{2}$$
$$+ \sum_{k\,:\,t>t_k} \left[\begin{array}{c} I_{0,k}\left(t_k, u(t_k), u'(t_k), u''(t_k)\right) + I_{1,k}\left(t_k, u(t_k), u'(t_k), u''(t_k)\right)(t - t_k) \\ +I_{2k}\left(t_k, u(t_k), u'(t_k), u''(t_k)\right) \frac{(t-t_k)^2}{2} \end{array} \right]$$
$$-\frac{t^2}{2} \sum_{k=1}^{+\infty} I_{2k}\left(t_k, u(t_k), u'(t_k), u''(t_k)\right) - \int_0^{+\infty} G(t,s)\; F\left(u(s)\right) ds,$$

with $G(t,s)$ given by (5.9), and

$$F\left(u(s)\right) := f\left(s, \delta_0\left(s, u(s)\right), \delta_1\left(s, u'(s)\right), \delta_2(s, u''(s))\right) + \frac{1}{1+t^2} \frac{u''(s) - \delta_2\left(t, u''(s)\right)}{1 + \left|u''(s) - \delta_2\left(s, u''(s)\right)\right|}.$$

By Lemma 5.3, the fixed points of $\mathcal{T}$ are solutions of problem (5.15), (5.2) and (5.16). So it is enough to prove that $\mathcal{T}$ has a fixed point.

For an easy reading, forward we denote

$$I_{i,k} := I_{i,k}\left(t_k, u(t_k), u'(t_k), u''(t_k)\right), \text{ for } i = 0, 1, 2.$$

(1) *$\mathcal{T} : X \to X$ is well defined.*

Take

$$\rho > \max\left\{\|\alpha\|_0, \|\beta\|_0, \|\alpha'\|_1, \left\|\beta'\right\|_1, \|\alpha''\|_2, \left\|\beta''\right\|_2\right\}. \tag{5.17}$$

As f is a L^1-Carathéodory function, by Definition 5.1, for $u \in X$ with $\|u\| < \rho$,

$$\int_0^{+\infty} \left|F\left(u(s)\right)\right| ds \le \int_0^{+\infty} \left(\psi_\rho(s) + \frac{1}{1+t^2}\right) ds \le \int_0^{+\infty} \psi_\rho(s) ds + \frac{\pi}{2} < +\infty,$$

and then $F\left(u(s)\right) \in L^1([0, +\infty[)$.

By Lebesgue Dominated Theorem and Definition 5.2,

$$\lim_{t\to+\infty} \frac{|\mathcal{T}u(t)|}{1+t^2} \le \frac{|C|}{2} + \sum_{k=1}^{+\infty} |I_{2k}| \le \frac{|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{2k\rho} < +\infty.$$

Analogously,

$$\lim_{t\to+\infty} \frac{|\mathcal{T}u'(t)|}{1+t} \le |C| + 2\sum_{k=1}^{+\infty} |I_{2k}| \le |C| + 2\sum_{k=1}^{+\infty} \lambda_{2k\rho} < +\infty,$$

and

$$\lim_{t\to+\infty}\frac{|(\mathcal{T}u)''(t)|}{2}\leq\frac{1}{2}\left(|C|+2\sum_{k=1}^{+\infty}\lambda_{2k\rho}+\int_0^{+\infty}\psi_\rho(s)ds\right)<+\infty.$$

Therefore $\mathcal{T}u\in X$.

(2) $\mathcal{T}$ *is continuous.*

For any convergent sequence $u_n\to u$ in X, there exists $r_1>0$ such that $\|u_n\|<r_1$, we have

$$\begin{aligned}\|\mathcal{T}u_n-\mathcal{T}u\| &= \max\left\{\|\mathcal{T}u_n-\mathcal{T}u\|_0,\|(\mathcal{T}u_n)'-(\mathcal{T}u)'\|_1,\|(\mathcal{T}u_n)''-(\mathcal{T}u)''\|_2,\right\}\\ &\leq \int_0^{+\infty}\max\{M_0,M_1\}\,|F(u_n(s))-F(u(s))|\,ds\\ &\leq \int_0^{+\infty}|F(u_n(s))-F(u(s))|\,ds\longrightarrow 0\ ,\quad n\to+\infty.\end{aligned}$$

(3) $\mathcal{T}$ is compact.

Let $D\subset X$ be any bounded subset. Therefore there is $R>0$ such that $\|u\|<R,\forall u\in D$.

Then

$$\begin{aligned}\|\mathcal{T}u\|_0 &= \sup_{t\in[0,+\infty[}\frac{|\mathcal{T}u(t)|}{1+t^2}\leq|A|+\frac{|B|+|C|}{2}+\sum_{k=1}^{+\infty}|I_{0k}|\\ &\quad+\sum_{k=1}^{+\infty}|I_{1k}|+\sum_{k=1}^{+\infty}|I_{2k}|+\int_0^{+\infty}\sup_{t\in[0,+\infty[}\frac{|G(t,s)|}{1+t^2}\psi_R(s)ds\\ &\leq|A|+\frac{|B|+|C|}{2}+\sum_{k=1}^{+\infty}\lambda_{0kR}+\sum_{k=1}^{+\infty}\lambda_{1kR}+\sum_{k=1}^{+\infty}\lambda_{2kR}+M_0\left(\frac{\pi}{2}+\int_0^{+\infty}\psi_R(s)ds\right)\\ &<+\infty,\end{aligned}$$

$$\begin{aligned}\|\mathcal{T}u\|_1 &= \sup_{t\in[0,+\infty[}\frac{|\,(\mathcal{T}u(t))'\,|}{1+t}\\ &\leq|B|+|C|+\sup_{t\in[0,+\infty[}\frac{1}{1+t}\sum_{k\,:\,t>t_k}|I_{1k}|+\sup_{t\in[0,+\infty[}\sum_{k\,:\,t>t_k}|I_{2k}|\frac{t-t_k}{1+t}\\ &\quad+\sup_{t\in[0,+\infty[}\frac{t}{1+t}\sum_{k=1}^{+\infty}|I_{2k}|+\int_0^{+\infty}M_1\,|F\left(u(s)\right)|\,ds\\ &\leq|B|+|C|+\sum_{k=1}^{+\infty}\lambda_{1kR}+2\sum_{k=1}^{+\infty}\lambda_{2kR}+M_1\left(\frac{\pi}{2}+\int_0^{+\infty}\psi_p(s)ds\right)\\ &<+\infty\end{aligned}$$

and

$$\|\mathcal{T}u\|_2 = \sup_{t\in[0,+\infty[} \frac{|\left(\mathcal{T}u(t)\right)''|}{2} \le \frac{|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{2kR} + \frac{1}{2}\int_0^{+\infty} \psi_R(s)ds + \frac{\pi}{4}$$
$$< +\infty$$

So $\mathcal{T}$ is uniformly bounded. Moreover, $\mathcal{T}$ is equicontinuous on each interval $]t_k, t_{k+1}]$, as for $t_1, t_2 \in]t_k, t_{k+1}]$, with $t_1 < t_2$, without loss of generality, we have when $t_1 \to t_2$,

$$\begin{aligned}
\left|\frac{\mathcal{T}u(t_1)}{1+t_1^2} - \frac{\mathcal{T}u(t_2)}{1+t_2^2}\right| \quad &\le \quad \left|\frac{Bt_1 + \frac{C}{2}t_1^2}{1+t_1^2} - \frac{Bt_2 + \frac{C}{2}t_2^2}{1+t_2^2}\right| \\
&+ \left| \begin{array}{l} \displaystyle\sum_{k\,:\,t_1>t_k} \left[I_{0k} + I_{1k}(t_1 - t_k) + I_{2k}\frac{(t_1-t_k)^2}{2}\right] \\ \displaystyle -\sum_{k\,:\,t_2>t_k} \left[I_{0k} + I_{1k}(t_2 - t_k) + I_{2k}\frac{(t_2-t_k)^2}{2}\right] \end{array} \right| \\
&+ \frac{1}{2}\left|t_1^2 - t_2^2\right| \sum_{k=1}^{+\infty} |I_{2k}| + \int_0^{+\infty} \left|\frac{G(t_1,s)}{1+t_1^2} - \frac{G(t_2,s)}{1+t_2^2}\right| (\psi_R(s)+1)\,ds \\
\longrightarrow \quad & 0,
\end{aligned}$$

$$\begin{aligned}
\left|\frac{(\mathcal{T}u)'(t_1)}{1+t_1} - \frac{(\mathcal{T}u)'(t_2)}{1+t_2}\right| \quad &\le \quad \left|\frac{Ct_1}{1+t_1} - \frac{Ct_2}{1+t_2}\right| + \\
&+ \left|\sum_{k\,:\,t_1>t_k} \left[I_{1k} + I_{2k}(t_1 - t_k)\right] - \sum_{k\,:\,t_2>t_k} \left[I_{1k} + I_{2k}(t_2 - t_k)\right]\right| \\
&+ |t_1 - t_2| \sum_{k=1}^{+\infty} |I_{2k}| \\
&+ \int_0^{+\infty} \left|\frac{\frac{\partial G}{\partial t}(t_1,s)}{1+t_1^2} - \frac{\frac{\partial G}{\partial t}(t_2,s)}{1+t_2^2}\right| (\psi_R(s)+1)\,ds \\
\longrightarrow \quad & 0,
\end{aligned}$$

The function $\frac{\partial^2 G}{\partial t^2}(t,s)$ is not continuous for $s = t$ but the jump is controlled by 1. Then

$$\begin{aligned}
\left|\frac{(\mathcal{T}u)''(t_1)}{2} - \frac{(\mathcal{T}u)''(t_2)}{2}\right| \quad &\le \quad \frac{1}{2}\sum_{k\,:\,t_1<t_k<t_2} |I_{2k}| + \left|\int_{t_1}^{+\infty} F(u(s))\,ds - \int_{t_2}^{+\infty} F(u(s))\,ds\right| \\
&\le \quad \frac{1}{2}\sum_{k\,:\,t_1<t_k<t_2} |I_{2k}| + \int_{t_1}^{t_2} (\psi_R(s)+1)\,ds \\
&\longrightarrow \quad 0 \text{ as } t_1 \to t_2.
\end{aligned}$$

To prove that $\mathcal{T}D$ is equiconvergent at infinity we apply, as $t \to +\infty$,

$$\begin{aligned}
\left|\frac{\mathcal{T}u(t)}{1+t^2} - \lim_{t\to+\infty}\frac{\mathcal{T}u(t)}{1+t^2}\right| \quad &\leq \quad \left|\frac{A+Bt}{1+t^2} + \frac{Ct^2}{2+2t^2} - \frac{C}{2}\right| + \\
&\quad + \frac{1}{1+t^2}\left|\begin{array}{c} \sum\limits_{k\,:\,t>t_k}\left[I_{0k} + I_{1k}(t_1-t_k) + I_{2k}\frac{(t_1-t_k)^2}{2}\right] \\ -\frac{1}{2}\sum_{k=1}^{+\infty} I_{2k} \end{array}\right| \\
&\quad + \left|\frac{1}{2}\frac{t^2}{1+t^2}\sum_{k=1}^{+\infty}|I_{2k}| - \frac{1}{2}\sum_{k=1}^{+\infty}|I_{2k}|\right| \\
&\quad + \int_0^{+\infty}\left|\frac{G(t,s)}{1+t^2} + \frac{1}{2}\right|(\psi_R(s)+1)\,ds \\
&\longrightarrow \quad 0,
\end{aligned}$$

$$\begin{aligned}
\left|\frac{(\mathcal{T}u)'(t)}{1+t} - \lim_{t\to+\infty}\frac{(\mathcal{T}u)'(t)}{1+t}\right| \quad &\leq \quad \left|\frac{B+Ct}{1+t} - C\right| \\
&\quad + \frac{1}{1+t}\left|\sum_{k\,:\,t>t_k}\left[I_{1k} + I_{2k}(t_1-t_k)\right] - \sum_{k=1}^{+\infty} I_{2k}\right| \\
&\quad + \left|\frac{t}{1+t}\sum_{k=1}^{+\infty}|I_{2k}| - \sum_{k=1}^{+\infty}|I_{2k}|\right| \\
&\quad + \int_0^{+\infty}\left|\frac{\frac{\partial G}{\partial t}(t,s)}{1+t} + 1\right|(\psi_R(s)+1)\,ds \\
&\longrightarrow \quad 0,
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{(\mathcal{T}u)''(t)}{2}\lim_{t\to+\infty}\frac{(\mathcal{T}u)''(t)}{2}\right| \quad &\leq \quad \frac{1}{2}\left|\sum_{k\,:\,t>t_k} I_{2k} - \sum_{k=1}^{+\infty} I_{2k}\right| \\
&\quad + \int_0^{+\infty}\left|\frac{\partial^2 G}{\partial t^2}(t,s) + 1\right|(\psi_R(s)+1)\,ds \\
&\longrightarrow \quad 0, \text{ as } t \to +\infty\ .
\end{aligned}$$

Finally, to prove that $\mathcal{T}D$ is equiconvergent at the impulsive moments we apply, as $t \to t_i^+$, for $i \in \mathbb{N}$,

$$
\begin{aligned}
\left|\frac{\mathcal{T}u(t)}{1+t^2} - \lim_{t\to t_i^+}\frac{\mathcal{T}u(t)}{1+t^2}\right| \leq & \left|\frac{Bt+\frac{C}{2}t^2}{1+t^2} - \frac{Bt_i+\frac{C}{2}t_i^2}{1+t_i^2}\right| + \\
& + \left| \begin{array}{l} \frac{1}{1+t^2}\sum_{k\,:\,t>t_k}\left(I_{0k}+I_{1k}(t-t_k)+I_{2k}\frac{(t-t_k)^2}{2}\right) \\ -\frac{1}{1+t_i^2}\sum\limits_{k\,:\,t_i^+>t_k}\left(I_{0k}+I_{1k}(t_i^+-t_k)+I_{2k}\frac{(t_i^+-t_k)^2}{2}\right) \end{array} \right| \\
& + \left|\left(-\frac{t^2}{1+t^2}+\frac{t_i^2}{1+t_i^2}\right)\sum_{k=1}^{+\infty} I_{2k}\right| \\
& + \int_0^{+\infty}\left|\frac{G(t,s)}{1+t^2} - \frac{G(t_i^+,s)}{1+t_i^2}\right|(\psi_R(s)+1))\,ds \\
& \longrightarrow 0,
\end{aligned}
$$

uniformly on $u \in D$, as $t \longrightarrow t_i^+$,

$$
\begin{aligned}
\left|\frac{(\mathcal{T}u)'(t)}{1+t} - \lim_{t\to t_i^+}\frac{(\mathcal{T}u)'(t)}{1+t}\right| \leq & \left|\frac{Ct}{1+t} - \frac{Ct_i}{1+t_i}\right| \\
& + \left|\frac{1}{1+t}\sum_{k\,:\,t>t_k}\left[I_{1k}+I_{2k}(t-t_k)\right] - \frac{1}{1+t_i}\sum_{t_i^+>t_k}\left[I_{1k}+I_{2k}(t_i-t_k)\right]\right| \\
& + \left|\left(-\frac{t}{1+t}+\frac{t_i}{1+t_i}\right)\sum_{k=1}^{+\infty} I_{2k}\right| \\
& + \int_0^{+\infty}\left|\frac{\frac{\partial}{\partial t}G(t,s)}{1+t^2} - \frac{\frac{\partial}{\partial t}G(t_i^+,s)}{1+t_i^2}\right|(\psi_R(s)+1))\,ds \\
& \longrightarrow 0, \text{ as } t \longrightarrow t_i^+,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{(\mathcal{T}u)''(t)}{2} - \lim_{t\to t_i^+}\frac{(\mathcal{T}u)''(t)}{2}\right| \leq & \left|\frac{1}{2}\sum_{k\,:\,t>t_k} I_{2k} - \frac{1}{2}\sum_{k\,:\,t_i^+>t_k} I_{2k}\right| \\
& + \frac{1}{2}\left|\int_t^{+\infty} F(u(s))\,ds - \int_{t_i}^{+\infty} F(u(s))\,ds\right| \\
& \leq \frac{1}{2}\left|\sum_{k\,:\,t>t_k} I_{2k} - \sum_{k\,:\,t_i^+>t_k} I_{2k}\right| + \frac{1}{2}\left|\int_{t_i}^{t}(\psi_R(s)+1)ds\right| \\
& \longrightarrow 0,
\end{aligned}
$$

uniformly on $u \in D$, as $t \longrightarrow t_i^+$.

So, by Theorem 5.4, $\mathcal{T}D$ is relatively compact.

To apply Schauder's Fixed Point Theorem, it must be verified that $\mathcal{T}: D \to D$.

CLAIM 2: *For some nonempty, closed, bounded and convex subset $D \subset X$, $\mathcal{T}D \subseteq D$.*

In Step (3) of the previous Claim 1, take $R > 0$ such that

$$R \geq \max \left\{ \begin{array}{c} \rho, |A| + \frac{|B|+|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{0kR} + \sum_{k=1}^{+\infty} \lambda_{1kR} + \sum_{k=1}^{+\infty} \lambda_{2kR} \\ +M_0 \left(\frac{\pi}{2} + \int_0^{+\infty} \psi_R(s)ds \right), \\ |B| + |C| + \sum_{k=1}^{+\infty} \lambda_{1kR} + 2\sum_{k=1}^{+\infty} \lambda_{2kR} \\ +M_1 \left(\frac{\pi}{2} + \int_0^{+\infty} \psi_R(s)ds \right), \\ \frac{|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{2kR} + \frac{1}{2} \int_0^{+\infty} \psi_R(s)ds + \frac{\pi}{4} \end{array} \right\},$$

with ρ given by (5.17).

From the calculus in Claim 1, for every $u \in D$ such that $\|u\| < R$, we have $\mathcal{T}D \subseteq D$.

As $\mathcal{T}$ is completely continuous then by Schauder's Fixed Point Theorem, $\mathcal{T}$ has at least one fixed point $u \in X$.

From Lemma 5.3, (5.15) and (5.16), this fixed point will be a solution of the initial problem (5.1)-(5.3) if

$$\alpha^{(i}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \ i = 0, 1, 2, \ \forall t \in [0, +\infty[.$$

CLAIM 3: *Every solution of problem (5.15), (5.16), (5.2), verifies*

$$\alpha^{(i}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \ i = 0, 1, 2, \ \forall t \in [0, +\infty[.$$

Let u be a solution of problem (5.15), (2), (5.16).

Suppose by contradiction that there is

$$u''(t) < \alpha''(t),$$

and define

$$\inf_{t \in [0,+\infty[} u''(t) - \alpha''(t) := u''(t_*) - \alpha''(t_*) < 0$$

Remark that $t_* \neq +\infty$, as by *(5.2)* and Definition 5.5, $u''(+\infty) - \alpha''(+\infty) \geq 0$.

If $t_* = 0$, the following contradiction holds, by (5.11) and Definition 5.5:

$$\begin{aligned} 0 &\leq u'''(0) - \alpha'''(0) = \\ &= f\left(0, \delta_0(0, u(0)), \delta_1(0, u'(0)), \delta_2(0, u''(0))\right) + \frac{u''(0) - \alpha''(0)}{1 + |u''(0) - \alpha''(0)|} - \alpha'''(0) \\ &< f\left(0, \delta_0(0, u(0)), \delta_1(0, u'(0)), \delta_2(0, u''(0))\right) - \alpha'''(0) \\ &\leq f\left(0, \alpha(0), \alpha'(0), \alpha''(0)\right) - \alpha'''(0) \leq 0. \end{aligned}$$

Therefore $t_* \neq 0$.

Consider now that t_* is between two consecutive impulses. That is, there is a $p \in \mathbb{N}$ such that $t_* \in]t_p, t_{p+1}[$.

Then

$$u'''(t_*) = \alpha'''(t_*), \ u''(t_*) - \alpha''(t_*) < 0$$

and we have, by (5.11) and Definition 5.5, the contradiction

$$\begin{aligned}
0 &= u'''(t_*) - \alpha'''(t_*) \\
&= f\left(t_*, \delta_0\left(t_*, u(t_*)\right), \delta_1(t_*, u'(t_*)), u''(t_*)\right) + \frac{1}{1+t_*^2} \frac{u''(t_*) - \delta_2(t_*, u(t_*))}{|u''(t_*) - \delta_2(t_*, u(t_*))| + 1} \\
&- \alpha'''(t_*) \\
&\leq f(t_*, \alpha(t_*), \alpha'(t_*), \alpha''(t_*)) + \frac{1}{1+t_*^2} \frac{u''(t_*) - \alpha''(t_*)}{|u''(t_*) - \alpha''(t_*)| + 1} - \alpha'''(t_*) \\
&< f(t_*, \alpha(t_*), \alpha'(t_*), \alpha''(t_*)) - \alpha''(t_*) \leq 0.
\end{aligned}$$

Assume now that the infimum is attained in the impulsive moment. So, we have two cases: $t_* = t_q^- = t_q^-$ or $t_* = t_q^+$. Firstly, consider that there is $q \in \mathbb{N}$ where

$$\min_{t \in [0,+\infty[} (u''(t) - \alpha''(t)) := u''(t_q) - \alpha''(t_q) < 0. \tag{5.18}$$

Then this contradiction holds

$$\begin{aligned}
0 &\leq \Delta(u'' - \alpha'')(t_q) \\
&= I_{2,q}\left(t_q, \delta_0(t_q, u(t_q)), \delta_1(t_q, u'(t_q)), \delta_2(t_q, u''(t_q))\right) - \Delta\alpha''(t_q) \\
&= I_{2,q}\left(t_q, \delta_0(t_q, u(t_q)), \delta_1(t_q, u'(t_q)), \alpha''(t_q)\right) - \Delta\alpha''(t_q) \\
&\leq I_{2,q}\left(t_q, \alpha(t_q), \alpha'((t_q), \alpha''(t_q)\right) - \Delta\alpha''(t_q) < 0.
\end{aligned}$$

In the second case, assume that

$$\inf_{t \in [0,+\infty[} u''(t) - \alpha''(t) := u''(t_*^+) - \alpha''(t_*^+) < 0.$$

Consider $\varepsilon > 0$ small enough such that

$$(u'' - \alpha'')(t) < 0, \ u'''(t^+) - \alpha'''(t^+) \geq 0, \quad \text{for } t \in]t_q, t_q + \varepsilon[.$$

So, for $t \in]t_q, t_q + \varepsilon[$, it can be obtained a contradiction following the same arguments as for $t_* \in]t_p, t_{p+1}[$.

Therefore

$$\alpha''(t) \leq u''(t), \quad \text{for } t \in [0, +\infty[.$$

By a similar technique it can be proved that $u''(t) \leq \beta''(t)$, for $t \in (0, +\infty[$, and then

$$\alpha''(t) \leq u''(t) \leq \beta''(t), \quad \text{for } t \in [0, +\infty[. \tag{5.19}$$

Integrating the first inequality of (5.19) for $t \in [0, t_1]$, by (5.2) and Definition (5.5),

$$\alpha'(t) \leq u'(t) + \alpha'(0) - u'(0) \leq u'(t). \tag{5.20}$$

By integration in $]t_1,+\infty[$, (5.14), (5.20) and Definition (5.5), we have for $t \in]t_1,+\infty[$

$$\begin{aligned}\alpha'(t) &\le u'(t) + \alpha'(t_1^+) - u'(t_1^+)\\ &= u'(t) + \alpha'(t_1^+) - I_{11}\left(t_1, \delta_0(t_1,u(t_1)), \delta_1(t_1,u'(t_1)), \delta_2(t_1,u''(t_1))\right) - u'(t_1)\\ &\le u'(t) + I_{11}\left(t_1,\alpha(t_1),\alpha'(t_1),\alpha''(t_1)\right) + \alpha'(t_1)\\ &- I_{11}\left(t_1,\delta_0(t_1,u(t_1)),u'(t_1),u''(t_1)\right) - u'(t_1)\\ &\le u'(t) + I_{11}\left(t_1,\alpha(t_1),\alpha'(t_1),\alpha''(t_1)\right) - I_{11}\left(t_1,\delta_0(t_1,u(t_1)),u'(t_1),u''(t_1)\right)\\ &\le u'(t).\end{aligned}$$

Analogously, one can show that

$$u'(t) \le \beta'(t), \;\forall t \in [0,+\infty[$$

and, then,

$$\alpha'(t) \le u'(t) \le \beta'(t), \text{ for } t \in [0,+\infty[. \tag{5.21}$$

Integrating the first inequality of (5.21) on $[0,t_1]$, we have

$$\alpha(t) \le u(t) - u(0) + \alpha(0) \le u(t)$$

and on $]t_1,+\infty[$, by (5.12) and Definition (5.5),

$$\begin{aligned}\alpha(t) &\le u(t) + \alpha(t_1^+) - u(t_1^+)\\ &\le u(t) + I_{01}\left(t_1,\alpha(t_1),\alpha'(t_1),\alpha''(t_1)\right) + \alpha(t_1)\\ &- I_{01}\left(t_1,\delta_0(t_1,u(t_1)),\delta_1(t_1,u'(t_1)),\delta_2(t_1,u''(t_1))\right) - u(t_1)\\ &\le u(t) + I_{01}\left(t_1,\alpha(t_1),\alpha'(t_1),\alpha''(t_1)\right) - I_{01}\left(t_1,u(t_1),u'(t_1),u''(t_1)\right)\\ &\le u(t)\end{aligned}$$

So, $\alpha(t) \le u'(t), \forall t \in [0,+\infty[$, and the remaining inequality $u(t) \le \beta(t), \forall t \in [0,+\infty[$, can be proved using the same technique. ■

5.3 Example

Problems of boundary layer flow over a stretching sheet, with and without heat transfer, are a topic that arouses growing interest in the literature (see, for example [70, 83, 88, 90]). These works deal with a boundary value problem of normal stagnation point flow impinging on a stretching sheet, governed by the parameter b which represents the ratio of the strain rate of the stagnation flow to that of the stretching sheet. The existent numerical studies on the basic flow shows that a solution exists for all values of $b > 0$.

In [59], it is studied the third order differential equation

$$f''' + f\ f'' - (f')^2 + b^2 = 0, \tag{5.22}$$

together with the boundary conditions

$$f(0)=0,\ f'(0)=1,\ f'(\infty)=b. \tag{5.23}$$

Motivated by this work, in this application, we prove the solvability of the impulsive third order problem composed by a differential equation similar to (5.22),

$$u'''(t)=\begin{cases} -\frac{0.01}{1+t^2}\left(\sqrt[3]{u(t)}\sqrt{|u''(t)|}+sgn(u'(t))\sqrt{|u'(t)|}-sgn(u''(t))b^2\right) \\ \text{if } 0\le t\le 1, \\ -\frac{0.01}{1+t^2}\left(\sqrt[3]{u(t)}\sqrt{|u''(t)|}+sgn(u'(t))\sqrt{|u'(t)|}+10sgn(u''(t))b^2\right) \\ \text{if } 0\le t\le 1, \end{cases}, \tag{5.24}$$

with $b\in\mathbb{R}\setminus\{0\}$, where $u(t)$ represents the flow speed across a time t, together with the asymptotic boundary conditions

$$u(0)=A,\ u'(0)=B,\ u''(+\infty)=0, \tag{5.25}$$

for $A,B\in\mathbb{R}$, and the impulsive effects with the form

$$\begin{aligned} \Delta u(t_k) &= \frac{1}{(t_k)^3}\left(\lambda_{01}\sqrt[3]{u(t_k)}+\lambda_{02}\left(u'(t_k)\right)+\lambda_{03}\sqrt[3]{u''(t_k)}\right) \\ \Delta u'(t_k) &= \frac{1}{(t_k)^3}\left(\lambda_{11}\left(u(t_k)\right)+\lambda_{12}\sqrt[3]{u'(t_k)}\right) \\ \Delta u''(t_k) &= \frac{1}{(t_k)^3}\left(\lambda_{21}\sqrt[3]{u(t_k)}+\lambda_{22}\left(u'(t_k)\right)+\lambda_{23}\left(u''(t_k)\right)+sgn(u''(t_k))*5t_k\right), \end{aligned} \tag{5.26}$$

where $\lambda_{ij}\in\mathbb{R}$, for $i=0,1,2$ and $j=1,2,3$, and $k\in\mathbb{N}$.

Remark that:

1. The null function is not a solution of (5.24).
2. In (5.24), from a theoretical point of view, the parameter b could be non-positive.
3. For functions $u\in X$, the condition $u''(+\infty)=0$ implies that $u'(+\infty)$ is finite.
4. Neither (5.24) nor (5.26) are covered by Theorem 3.1 of [116], as they are not sublinear and have different monotonies.
5. The problem (5.24)-(5.26) is a particular case of the initial problem (5.1)-(5.3), with $C=0$,

$$f(t,y_0,y_1,y_2)=\begin{cases} -\frac{0.01}{1+t^2}\left(\sqrt[3]{y_0}\sqrt{|y_2|}+sgn(y_1)\sqrt{|y_1|}-sgn(y_2)b^2\right) \\ \text{if } 0\le t\le 1, \\ -\frac{0.01}{1+t^2}\left(\sqrt[3]{y_0}\sqrt{|y_2|}+sgn(y_1)\sqrt{|y_1|}+10\ sgn(y_2)b^2\right) \\ \text{if } t>1, \end{cases}, \tag{5.27}$$

$$\begin{aligned}
I_{0,1}(t_k, w_0, w_1, w_2) &= \frac{1}{(t_k)^3}\left(\lambda_{01}\sqrt[3]{w_0} + \lambda_{02}(w_1) + \lambda_{03}\sqrt[3]{w_2}\right),\\
I_{1,1}(t_k, w_0, w_1, w_2) &= \frac{1}{(t_k)^3}\left(\lambda_{11}(w_0) + \lambda_{12}\sqrt[3]{w_1}\right),\\
I_{2,1}(t_k, w_0, w_1, w_2) &= \frac{1}{(t_k)^3}\left(\lambda_{21}\sqrt[3]{w_0} + \lambda_{22}(w_1) + \lambda_{23}(w_2) + sgn(w_2) * 5t_k\right).
\end{aligned}$$

As a numeric example, let us consider $b = -1$, $A = B = 0$, $t_k = k$, $k \in \mathbb{N}$, and adequate values for the parameters.

In this case, the impulsive conditions are given by

$$\begin{aligned}
\Delta u(k) &= \frac{1}{k^3}\left(0.001^k\sqrt[3]{u(k)} + 0.001^k(u'(k)) + 0.001^k\sqrt[3]{u''(k)}\right)\\
\Delta u'(k) &= \frac{1}{k^3}\left(0.1^k(u(k)) + 0.1^k\sqrt[3]{u'(k)}\right)\\
\Delta u''(k) &= \frac{1}{k^3}\left(-0.1^k\sqrt[3]{u(k)} - 0.1^k(u'(k)) + 0.1^k(u''(k)) + sgn(u''(k)) * 5k\right),
\end{aligned} \tag{5.28}$$

and the piecewise functions $\alpha, \beta \in X$ defined as

$$\begin{aligned}
\alpha(t) &= \begin{cases} 1.5t^3 - 5t^2 & \text{if } 0 \le t \le 1\\ -0.1\left(\frac{1}{2t} + 16t\right) - 3k & \text{if } t \in]k, k+1], k \ge 1,\end{cases}\\
\beta(t) &= \begin{cases} -1.5t^3 + 5t^2 & \text{if } t \le 1\\ 0.1\left(\frac{1}{2t} + 16t\right) + 3k & \text{if } t \in]k, k+1], k \ge 1,\end{cases}
\end{aligned}$$

are, respectively, lower and upper solutions of problem (5.24), (5.25), (5.28), according to Definition 5.5, verifying (5.10).

Moreover, the nonlinear part given by (5.27) verifies (5.11), the impulsive functions $I_{i,1} : \mathbb{R}^4 \mapsto \mathbb{R}$, $i = 0, 1, 2$,

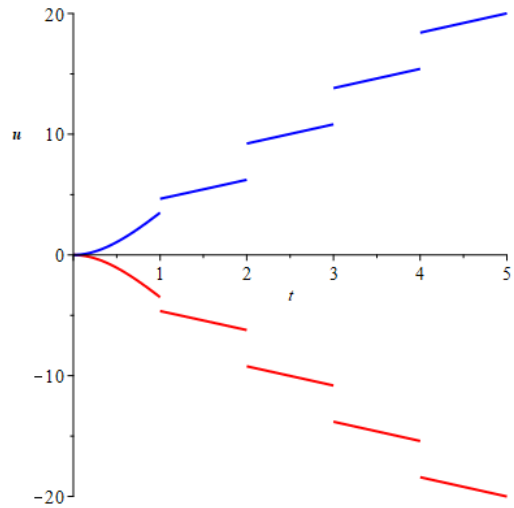
$$\begin{aligned}
I_{0,1}(k, w_0, w_1, w_2) &= \frac{1}{k^3}\left(0.001^k\sqrt[3]{w_0} + 0.001^k w_1 + 0.001^k\sqrt[3]{w_2}\right)\\
I_{1,1}(k, w_0, w_1, w_2) &= \frac{1}{k^3}\left(0.1^k w_0 + 0.1^k\sqrt[3]{w_1}\right)\\
I_{2,1}(k, w_0, w_1, w_2) &= \frac{1}{k^3}\left(-0.1^k\sqrt[3]{w_0} - 0.1^k w_1 + 0.1^k w_2 + sgn(w_2) * 5k\right)
\end{aligned}$$

satisfy (5.12), (5.13) and (5.14), and assumption (A) holds for $\xi \ge 24.245$.

Then, by Theorem 5.6, there exists a solution $u \in X$ of problem (5.24), (5.25), (5.28), in the strip

$$\alpha(t) \le u(t) \le \beta(t), \text{ for } t \in]0, +\infty),$$

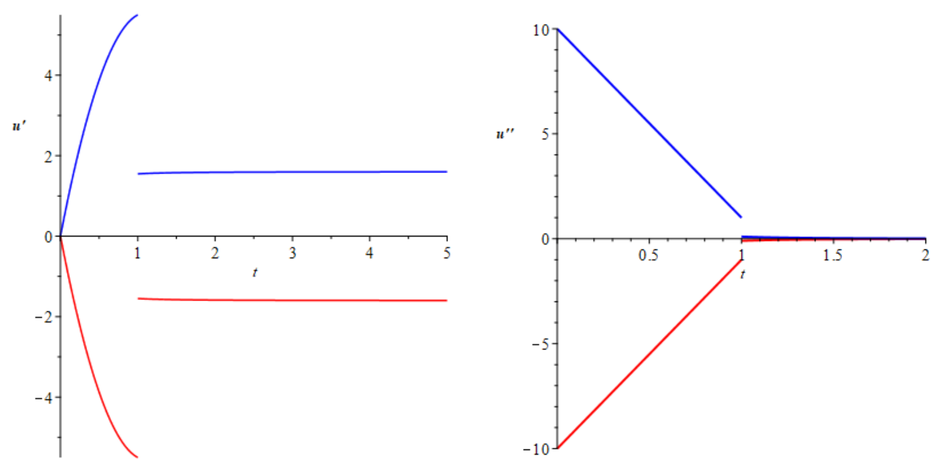
that is



From the localization part of Theorem 5.6 we can also have some data on the first and second derivatives:

$$\alpha'(t) \leq u'(t) \leq \beta'(t), \;\; \alpha''(t) \leq u''(t) \leq \beta''(t), \;\; \text{for } t \in]0, +\infty),$$

that is



Chapter 6

Functional coupled systems with generalized impulsive conditions and application to a SIRS-type model

6.1 Introduction

The study of impulsive boundary value problems is richer than the related differential equations theory without impulses, and has strategic importance in multiple current scientific fields, from sociology and medical sciences to generalized industry production, or in any other real-world phenomena where sudden variations occur.

The classic impulsive theory can be seen in [19, 92]. In the last two decades, a vast literature on impulsive differential problems has been produced, such as, [1, 7, 12, 26, 29, 36, 51, 53, 54, 63, 66, 67, 75, 99, 100], only to mention a few.

Functional problems composed by differential equations and conditions with global dependence on the unknown variable, generalize the usual boundary value problems, and can include equations and/or conditions with deviating arguments, delays or advances, nonlinear or nonlocal, increasing in this way the range of applications. The readers interested in results in this direction, on bounded or unbounded domains, may look for in [4, 24, 37, 39, 43, 44, 46, 49, 68, 95] and the references therein.

Recently, coupled systems have been studied by many authors, not only from a theoretical point of view but also due to the huge applications in many sciences and fields, with several methods and approaches. We recommend to the interested readers, for instance, [8, 17, 20, 31, 32, 41, 50, 61, 64, 101].

Motivated by the results contained in some of the above references, in this work, we consider the first-order coupled impulsive system of equations

$$\begin{cases} y_1'(t) = g_1(t, y_1(t), y_2(t), y_3(t)) \\ y_2'(t) = g_2(t, y_1(t), y_2(t), y_3(t)) \\ y_3'(t) = g_3(t, y_1(t), y_2(t), y_3(t)) \end{cases} \tag{6.1}$$

a.e. $t \in [a,b] \setminus \{t_j\}$, where t_j are fixed points, $j = 1, 2, ..., n$, $g_i : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ are L^1-Carathéodory functions, for $i = 1, 2, 3$, with the functional boundary conditions

$$\begin{array}{l} B_1(y_1, y_2, y_3) = 0, \\ B_2(y_1, y_2, y_3) = 0, \\ B_3(y_1, y_2, y_3) = 0, \end{array} \tag{6.2}$$

where $B_i : C[a,b] \to \mathbb{R}$, $i = 1, 2, 3$, are continuous functions, and verifying the generalized impulsive conditions

$$\begin{cases} \Delta y_1(t_j) = H_{1j}(t_j, y_1(t_j), y_2(t_j), y_3(t_j)) \\ \Delta y_2(t_j) = H_{2j}(t_j, y_1(t_j), y_2(t_j), y_3(t_j)) \\ \Delta y_3(t_j) = H_{3j}(t_j, y_1(t_j), y_2(t_j), y_3(t_j)), \end{cases} \tag{6.3}$$

where $H_{ij} : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions for $i = 1, 2, 3$, $j = 1, 2, ..., n$, with $\Delta y_i(t_j) = y_i(t_j^+) - y_i(t_j^-)$, and t_j fixed points such that $a := t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} := b$.

As far as we know, it is the first time where those three features are taken together to have a coupled impulsive system with functional boundary conditions and generalized impulsive effects, which one including, eventually, impulses on the three unknown functions. We underline two novelties of this chapter:

1. Condition (6.2) generalizes the classical boundary assumptions, allowing two-point or multipoint conditions, nonlocal and/or integro-differential ones or global arguments, as maxima or minima, among others. In this way new types of problems and applications could be considered, enabling greater and wider information on the problems studied.

2. The main theorem is applied to a SIRS model where, to the best of our knowledge, for the first time it includes impulsive effects combined with global, local, and asymptotic behavior of the unknown functions.

Our method is based on lower and upper solutions technique together with the fixed point theory. In short, the main result is obtained studying a perturbed and truncated system, with modified boundary and impulsive conditions, and applying the Schauder's fixed point theorem to a completely continuous vectorial operator. Moreover, the work contains a method to overcome the nonlinearities monotony through a combination with adequate changes in the definition of lower and upper solutions.

The work is structured in the following way: Section 6.2 contains the functional framework, definitions, and other known properties. The main result is in Section 6.3, where the proof is divided into steps, for the reader's convenience.

In Section 6.4 it is shown a method where the definition of coupled lower and upper functions can be used to obtain different versions of the main theorem, with different monotone characteristics on the nonlinearities. The last section contains an application to a vital dynamic SIRS-type model, representing the dynamic epidemiological evolution of Susceptible (S), Infected (I), Recovered (R), and newly infected individuals in a population on a normalized period, subject to impulsive effects and global restrictions.

6.2 Definitions and auxiliary results

Define $y_i(t_k^{\pm}) := \lim_{t \to t_k^{\pm}} y_i(t)$, for $i = 1, 2, 3$, consider the set

$$PC\left([a,b]\right) = \left\{ \begin{array}{c} y : y \in C([a,b], \mathbb{R}^3) \text{ continuous for } t \neq t_j, \\ y(t_j) = y(t_j^+) - y(t_j^-), \\ y(t_j^+) \text{ exists for } j = 1, 2, ..., n \end{array} \right\},$$

and the space $X^3 := \left(PC\left([a,b]\right)\right)^3$ equipped with the norm

$$\left\|(y_1, y_2, y_3)\right\|_{X^3} = \max\left\{\|y_1\|, \|y_2\|, \|y_3\|\right\},$$

where

$$\|y\| := \sup_{t \in [a,b]} |y(t)|.$$

It is clear that $(X^3, \|.\|_{X^3})$ is a Banach space.

The triple (y_1, y_2, y_3) is a solution of problem (6.1)-(6.3) if $(y_1, y_2, y_3) \in X^3$ and verifies conditions (6.1), (6.2) and (6.3).

Definition 6.1 *A function $w : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$, for $i = 1, 2, 3$ is L^1- Carathéodory if*

i) *for each $(x, y, z) \in \mathbb{R}^3$, $t \mapsto w(t, x, y, z)$ is measurable on $[a, b]$;*

ii) *for a.e. $t \in [a, b]$, $(x, y, z) \mapsto w(t, x, y, z)$ is continuous on $\mathbb{R}^3$;*

iii) *for each $\rho > 0$, there exists a positive function $\psi_\rho \in L^1([a,b])$ and for $(x, y, z) \in \mathbb{R}^3$ such that*

$$\max\left\{|x|, |y|, |z|\right\} < \rho, \tag{6.4}$$

one has

$$|w(t, x, y, z)| \leq \psi_\rho(t), \ a.e. x \in [a, b]. \tag{6.5}$$

In this work the definition of lower and upper solutions plays a key role in our method.

Next definition will be used in the main theorem:

Definition 6.2 *Consider the PC^1-functions $\alpha_i, \beta_i : [a,b] \to \mathbb{R}$, $i = 1,2,3$. The triple $(\alpha_1, \alpha_2, \alpha_3) \in X^3$ is a lower solution of the problem (6.1)-(6.3) if*

$$\alpha_i'(t) \leq g_i\left(t, \alpha_1(t), \alpha_2(t), \alpha_3(t)\right), \text{ for } i = 1,2,3, \tag{6.6}$$

$$\begin{aligned} B_1\left(\alpha_1, \alpha_2, \alpha_3\right) &\geq 0 \\ B_2\left(\alpha_1, \alpha_2, \alpha_3\right) &\geq 0 \\ B_3\left(\alpha_1, \alpha_2, \alpha_3\right) &\geq 0, \end{aligned}$$

and, for $j = 1,2,...,n$,

$$\begin{aligned} \Delta\alpha_1(t_j) &\leq H_{1j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)) \\ \Delta\alpha_2(t_j) &\leq H_{2j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)) \\ \Delta\alpha_3(t_j) &\leq H_{3j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)). \end{aligned}$$

The triple $(\beta_1, \beta_2, \beta_3) \in X^3$ is an upper solution of the problem (6.1)-(6.3) if the reversed inequalities hold.

6.3 Main result

The main result will provide the existence of, at least, a solution for the problem (6.1)-(6.3).

Theorem 6.3 *Assume that there are α and β lower and upper solutions of problem (6.1)-(6.3), according Definition 6.2, such that*

$$\alpha_i(t) \leq \beta_i(t), \ \forall t \in [a,b], \text{ for } i = 1,2,3. \tag{6.7}$$

Let $g_i : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$, $i = 1,2,3$ be L^1-Carathéodory functions, not identically null, on the set

$$\left\{(t, y_i) \in [a,b] \times \mathbb{R} : \alpha_i(t) \leq y_i \leq \beta_i(t), i = 1,2,3\right\},$$

and

$$g_1\left(t, y_1, \alpha_2(t), \alpha_3(t)\right) \leq g_1\left(t, y_1, y_2, y_3\right) \leq g_1\left(t, y_1, \beta_2(t), \beta_3(t)\right) \tag{6.8}$$

for $t \in [a,b] \setminus \{t_j\}$, $j \in \{1,2,\ldots,n\}$, $y_1 \in \mathbb{R}$, $\alpha_i(t) \leq y_i \leq \beta_i(t)$, $i = 2,3$,

$$g_2\left(t, \alpha_1(t), y_2, \alpha_3(t)\right) \leq g_2\left(t, y_1, y_2, y_3\right) \leq g_2\left(t, \beta_1(t), y_2, \beta_3(t)\right) \tag{6.9}$$

for $t \in [a,b] \setminus \{t_j\}$, $j \in \{1,2,\ldots,n\}$, $y_2 \in \mathbb{R}$, $\alpha_i(t) \leq y_i \leq \beta_i(t)$, $i = 1,3$,

$$g_3\left(t, \alpha_1(t), \alpha_2(t), y_3\right) \leq g_3\left(t, y_1, y_2, y_3\right) \leq g_3\left(t, \beta_1(t), \beta_2(t), y_3\right) \tag{6.10}$$

for $t \in [a,b] \setminus \{t_j\}$, $j \in \{1,2,\ldots,n\}$, $y_3 \in \mathbb{R}$, $\alpha_i(t) \leq y_i \leq \beta_i(t)$, $i = 1,2$, with

$$B_i\left(\alpha_1, \alpha_2, \alpha_3\right) \leq B_i\left(y_1, y_2, y_3\right)) \leq B_i\left(\beta_1, \beta_2, \beta_3\right), \tag{6.11}$$

for $\alpha_i \leq y_i \leq \beta_i$, $i = 1, 2, 3$.
Assume that $H_{ij} : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ *satisfy*

$$\begin{aligned} H_{ij}\left(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)\right) &\leq H_{ij}\left(t_j, w_1, w_2, w_3)\right) \\ &\leq H_{ij}\left(t_j, \beta_1(t_j), \beta_2(t_j), \beta_3(t_j)\right), \end{aligned} \tag{6.12}$$

for $\alpha_i(t_j) \leq w_i \leq \beta_i(t_j)$, $i = 1, 2, 3$, $j \in \{1, 2, \ldots, n\}$.
If there is $\rho > 0$ *such that*

$$\max_{i=1,2,3} \left[\max\left\{\|\alpha_i\|, \|\beta_i\|\right\} + \sum_{j=1}^{n} \left|H_{ij}\left(t_j, w_1, w_2, w_3)\right)\right| + \int_a^b \left(\psi_\rho^i(s) + 1\right) ds \right] < \rho,$$

then there exists at least a triple $(y_1, y_2, y_3) \in X^3$, *solution of (6.1)-(6.3), such that*

$$\alpha_i(t) \leq y_i(t) \leq \beta_i(t), \ \forall t \in [a, b], \ \textit{for } i = 1, 2, 3.$$

Proof. Let $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \in X^3$ be, respectively, lower and upper solutions of (6.1)-(6.3), as in Definition 6.2, verifying (6.7).

Consider the continuous truncatures $\delta_i : [a, b] \times \mathbb{R} \to \mathbb{R}, i = 1, 2, 3$, denoted, for short, as $\delta_i(t)$, defined by

$$\delta_i(t) := \delta_i(t, y_i(t)) = \begin{cases} \beta_i(t) & , y_i(t) > \beta_i(t) \\ y_i(t) & , \alpha_i(t) \leq y_i(t) \leq \beta_i(t) \\ \alpha_i(t) & , y_i(t) < \alpha_i(t), \end{cases} \tag{6.13}$$

and consider the modified and perturbed problem composed by the differential system

$$\begin{aligned} y_1'(t) &= g_1\left(t, \delta_1(t), \delta_2(t), \delta_3(t)\right) + \frac{y_1(t) - \delta_1(t)}{1 + |y_1(t) - \delta_1(t)|}, \\ y_2'(t) &= g_2\left(t, \delta_1(t), \delta_2(t), \delta_3(t)\right) + \frac{y_2(t) - \delta_2(t)}{1 + |y_2(t) - \delta_2(t)|}, \\ y_3'(t) &= g_3\left(t, \delta_1(t), \delta_2(t), \delta_3(t)\right) + \frac{y_3(t) - \delta_3(t)}{1 + |y_3(t) - \delta_3(t)|}, \end{aligned} \tag{6.14}$$

for $t \in [a, b] \setminus \{t_j\}$, $j \in \{1, 2, \ldots, n\}$, together with the truncated boundary conditions

$$y_i(a) = \delta_i(a, y_i(a) + B_i\left(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3)\right)), \tag{6.15}$$

for $i = 1, 2, 3$, with

$$\delta_i^*(w) = \begin{cases} \beta_i & , w > \beta_i \\ w & , \alpha_i \leq w \leq \beta_i \\ \alpha_i & , w < \alpha_i, \end{cases}$$

and the truncated impulsive conditions

$$\Delta y_i(t_j) = H_{ij}\left(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j)\right), \tag{6.16}$$

for $j \in \{1, 2, \ldots, n\}$.

Claim 1: *The problem (6.14), (6.15), (6.16) has at least a solution.*

This claim will be proved by the fixed point theory, applied to the vectorial operator

$$\mathcal{T} : X^3 \longrightarrow X^3$$

given by

$$\mathcal{T}(y_1, y_2, y_3) = (\mathcal{T}_1(y_1, y_2, y_3), \mathcal{T}_2(y_1, y_2, y_3), \mathcal{T}_3(y_1, y_2, y_3)),$$

where, for $i = 1, 2, 3$,

$$\mathcal{T}_i : X^3 \longrightarrow X$$

defined as

$$\begin{aligned}
\mathcal{T}_i(y_1, y_2, y_3)(t) \quad : \quad &= \delta_i(a, y_i(a) + B_i(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3))) \\
&+ \sum_{j\,:\,t_j<t} H_{ij}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j)) \\
&+ \int_a^t g_i(s, \delta_1(s), \delta_2(s), \delta_3(s)) + \frac{y_i(s) - \delta_i(s)}{1 + |y_i(s) - \delta_i(s)|} ds.
\end{aligned}$$

It is clear that the fixed points of $\mathcal{T}$, that is, the set of the fixed points of $\mathcal{T}_i$, for $i = 1, 2, 3$, are solutions of the problem (6.14), (6.15), (6.16).

As g_i are L^1-Carathéodory functions, H_{ij} and the truncatures δ_i, δ_i^* are continuous, therefore $\mathcal{T}_i$ are well defined and continuous. Therefore, $\mathcal{T}$ is well defined and continuous.

Consider a bounded set $D \subset X^3$. So there is $k > 0$ such that $\|(x, y, z)\|_{X^3} < k$, for $(x, y, z) \in D$.

$\mathcal{T}_i D$ is uniformly bounded, as, for $i = 1, 2, 3$,

$$\begin{aligned}
\|\mathcal{T}_i(y_1, y_2, y_3)\| \quad &= \quad \sup_{t\in[a,b]} |\mathcal{T}_i(y_1, y_2, y_3)(t)| \\
&\leq \quad \sup_{t\in[a,b]} |\delta_i(a, y_i(a) + B_i(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3)))| \\
&\quad + \sum_{j\,:\,t_j<t} |H_{ij}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j))| \\
&\quad + \int_a^t \left| g_i(s, \delta_1(s), \delta_2(s), \delta_3(s)) + \frac{y_i(s) - \delta_i(s)}{1 + |y_i(s) - \delta_i(s)|} \right| ds \\
&\leq \quad \max_{i=1,2,3} \left[\max\{\|\alpha_i\|, \|\beta_i\|\} + \sum_{j=1}^n |H_{ij}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j))| \right] \\
&\quad + \max_{i=1,2,3} \int_a^b \left(\psi_k^i(s) + 1\right) ds < \infty,
\end{aligned}$$

where ψ_k^i are the positive functions given by Definition 6.1.

$\mathcal{T}_i D$ is equicontinuous because, for $i = 1, 2, 3$, and $t_1, t_2 \in [a, b]$ with $t_1 < t_2$ (without loss of generality),

$$\begin{aligned} & \left|\mathcal{T}_i\left(y_1, y_2, y_3\right)\left(t_1\right) - \mathcal{T}_i\left(y_1, y_2, y_3\right)\left(t_2\right)\right| \\ \leq & \sum_{j\,:\,t_1<t_j<t_2}\left|H_{ij}\left(t_j, \delta_1\left(t_j\right), \delta_2\left(t_j\right), \delta_3\left(t_j\right)\right)\right| + \int_{t_1}^{t_2}\left(\psi_k^i(s)+1\right) ds \rightarrow 0, \end{aligned}$$

as $t_1 \rightarrow t_2$.

$\mathcal{T}_i D$ is equiconvergent on the impulsive moments, as

$$\begin{aligned} & \left|\mathcal{T}_i\left(y_1, y_2, y_3\right)(t) - \lim_{t \rightarrow t_j^+} \mathcal{T}_i\left(y_1, y_2, y_3\right)(t)\right| \\ \leq & \left|\sum_{j\,:\,t_j<t}\left|H_{ij}\left(t_j, \delta_1\left(t_j\right), \delta_2\left(t_j\right), \delta_3(t)\right)\right| - \sum_{j\,:\,t_j<t_j^+}\left|H_{ij}\left(t_j, \delta_1\left(t_j\right), \delta_2\left(t_j\right), \delta_3(t)\right)\right|\right| \\ & + \left|\int_a^t\left(\psi_k^i(s)+1\right) ds - \int_a^{t_j^+}\left(\psi_k^i(s)+1\right) ds\right| \rightarrow 0, \end{aligned}$$

when $t \rightarrow t_j^+$. Therefore $\mathcal{T}_i$ and $\mathcal{T}$ are compact operators.

Consider now the closed, bounded and convex set $\Omega \subset X^3$, defined by

$$\Omega = \left\{w \in X^3 : \|\mathcal{T} w\|_{X^3} \leq R\right\}$$

with $R > 0$ such that

$$\begin{aligned} R \quad > & \max_{i=1,2,3}\left[\max\left\{\left\|\alpha_i\right\|, \left\|\beta_i\right\|\right\} + \sum_{j=1}^n\left|H_{ij}\left(t_j, \delta_1\left(t_j\right), \delta_2\left(t_j\right), \delta_3\left(t_j\right)\right)\right|\right] \\ & + \max_{i=1,2,3} \int_a^b\left(\psi_R^i(s)+1\right) ds. \end{aligned}$$

By the above calculus, $\mathcal{T}\,\Omega \subset \Omega$, and from Schauder´s Fixed Point Theorem, $\mathcal{T}$ has a fixed point $y^* = \left(y_1^*, y_2^*, y_3^*\right)$, which is solution of the problem (6.14), (6.15), (6.16).

Claim 2: *This function* $y^* = \left(y_1^*, y_2^*, y_3^*\right)$ is a solution of *problem (6.1)-(6.3), too.*

To prove this claim it is enough to show that, for every solution $\left(y_1, y_2, y_3\right) \in X^3$ of problem (6.14), (6.15), (6.16), the following inequalities hold:

$$\alpha_i(t) \leq y_i(t) \leq \beta_i(t), \text{ for } i = 1, 2, 3, \text{ and } t \in [a, b], \tag{6.17}$$

$$\alpha_i(a) \leq y_i(a) + B_i\left(\delta_1(a, y_1^*(a)), \delta_2(a, y_2^*(a)), \delta_3(a, y_3^*(a))\right) \leq \beta_i(a).$$

Let $y := (y_1, y_2, y_3) \in X^3$ be a solution of the problem (6.14), (6.15), (6.16).

To prove the first inequality of (6.17), for $i = 1$, assume that there is $t \in [a, b]$ such that $\alpha_1(t) - y_1(t) > 0$, and define

$$\sup_{t \in [a,b]} (\alpha_1(t) - y_1(t)) := \alpha_1(t_*) - y_1(t_*) > 0. \tag{6.18}$$

Remark that $t_* \neq a$, as, by (6.15) and (6.13),

$$\alpha_1(a) - y_1(a) = \alpha_1(a) - \delta_1(a, y_i(a) + B_1\left(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3)\right)) \leq 0.$$

If t_* is between two consecutive impulses, that is, $t \in]t_p, t_{p+1}]$, for fixed $p = 0, 1, ..., n$, then $\alpha_1'(t_*) - y_1'(t_*) = 0$, by (6.15), (6.8) and Definition 6.2, this contradiction is achieved

$$\begin{aligned}
0 &\leq \alpha_1'(t_*) - y_1'(t_*) \\
&= \alpha_1'(t_*) - g_1\left(t_*, \alpha_1(t_*), \delta_2(t_*), \delta_3(t_*)\right) - \frac{y_1(t_*) - \alpha_1(t_*)}{1 + |y_1(t_*) - \alpha_1(t_*)|} \\
&< \alpha_1'(t_*) - g_1\left(t_*, \alpha_1(t_*), \delta_2(t_*), \delta_3(t_*)\right) \\
&\leq \alpha_1'(t_*) - g_1\left(t_*, \alpha_1(t_*), \alpha_2(t_*), \alpha_3(t_*)\right) \leq 0.
\end{aligned}$$

If t_* is an impulsive moment, that is, there is $j \in \{1, 2, ..., n\}$ such that $t_* = t_j^+$, then, by (6.16), (6.12) and Definition 6.2 we have

$$\begin{aligned}
0 &\leq \Delta\alpha_1(t_j) - \Delta y_1(t_j) \\
&= \Delta\alpha_1(t_j) - H_{1j}\left(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j)\right) \\
&\leq \Delta\alpha_1(t_j) - H_{1j}\left(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)\right) \leq 0.
\end{aligned}$$

Therefore

$$\Delta y_1(t_j) - \Delta\alpha_1(t_j) = 0,$$

that is, there are no jumps at any point t_j. Then, by (6.18),

$$0 \leq \alpha_1'(t_j^-) - y_1'(t_j^-),$$

and the contradiction is obtained as in the previous case.

Therefore $\alpha_1(t) \leq y_1(t)$, for $t \in [a, b]$. With the same arguments it can be proved that $y_1(t) \leq \beta_1(t)$, for $t \in [a, b]$.

A similar technique can be applied for functions g_2 and g_3, applying conditions (6.9) and/or (6.10), respectively.

Suppose now, by contradiction, that

$$\alpha_i(a) > y_i(a) + B_i\left(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3)\right). \tag{6.19}$$

Then, by (6.15),

$$y_i(a) = \delta_i(a, y_i(a) + B_i\left(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3)\right)) = \alpha_i(a),$$

which is in contradiction with (6.19), by (6.11) and Definition 6.2,

$$\begin{aligned} 0 &= y_i(a) - \alpha_i(a) \\ &> B_i\left(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3)\right) - B_i\left(\alpha_1, \alpha_2, \alpha_3\right) \geq 0, \end{aligned}$$

for $i = 1, 2, 3$.

The remaining inequalities can be proved with similar arguments. ■

6.4 Relation between monotonies and lower and upper definitions

The monotone assumptions required on the nonlinearities and on the impulsive functions, by conditions (6.8)-(6.10) and (6.12), although local, can seem too restrictive. Indeed, these monotonies can be modified since they are combined with different definitions of coupled lower and upper solutions, following the method described in this section.

Definition 6.4 *Consider the PC^1-functions $\alpha_i, \beta_i : [a, b] \to \mathbb{R}$, $i = 1, 2, 3$. The triples $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \in X^3$ are coupled lower and upper solutions of the problem (6.1)-(6.3) if*

$$\begin{aligned} \alpha_1'(t) &\leq g_1\left(t, \alpha_1(t), \beta_2(t), \alpha_3(t)\right), \\ \alpha_i'(t) &\leq g_i\left(t, \alpha_1(t), \alpha_2(t), \alpha_3(t)\right), \text{ for } i = 2, 3, \end{aligned}$$

$$\begin{aligned} \beta_1'(t) &\geq g_1\left(t, \beta_1(t), \alpha_2(t), \beta_3(t)\right), \\ \beta_i'(t) &\geq g_1\left(t, \beta_1(t), \beta_2(t), \beta_3(t)\right), \text{ for } i = 2, 3, \end{aligned}$$

and, for $j = 1, 2, ..., n$,

$$\begin{aligned} \Delta\alpha_1(t_j) &\leq H_{1j}(t_j, \beta_1(t_j), \alpha_2(t_j), \alpha_3(t_j)), \\ \Delta\alpha_2(t_j) &\leq H_{2j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)), \\ \Delta\alpha_3(t_j) &\leq H_{3j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \beta_3(t_j)), \end{aligned} \tag{6.20}$$

$$\begin{aligned} \Delta\beta_1(t_j) &\geq H_{1j}(t_j, \alpha_1(t_j), \beta_2(t_j), \beta_3(t_j)), \\ \Delta\beta_2(t_j) &\geq H_{2j}(t_j, \beta_1(t_j), \beta_2(t_j), \beta_3(t_j)), \\ \Delta\beta_3(t_j) &\geq H_{3j}(t_j, \beta_1(t_j), \beta_2(t_j), \alpha_3(t_j)). \end{aligned}$$

The inequalities for boundary conditions are similar to Definition 6.2.

With this definition the assumption on the local monotony of function g_1 and on the impulsive functions H_{1j} and H_{3j}, can be replaced, as in the following version of Theorem 6.3:

Theorem 6.5 *Assuming that all the assumptions of Theorem 6.3, with coupled lower and upper solutions defined as in Definition 6.4, and (6.8) replaced by*

$$g_1\left(t, \beta_1(t), \beta_2(t), \alpha_3(t)\right) \leq g_1\left(t, y_1, y_2, y_3\right) \leq g_1\left(t, \alpha_1, \alpha_2(t), \beta_3(t)\right)$$

for $t \in [a,b] \setminus \{t_j\}$, $j \in \{1,2,\ldots,n\}$, $y_1 \in \mathbb{R}$, $\alpha_i(t) \leq y_i \leq \beta_i(t)$, $i = 2,3,$ *and (6.12) by*

$$H_{1j}\left(t_j, \beta_1(t_j), \alpha_2(t_j), \alpha_3(t_j)\right) \leq H_{1j}\left(t_j, w_1, w_2, w_3)\right) \leq H_{1j}\left(t_j, \alpha_1(t_j), \beta_2(t_j), \beta_3(t_j)\right),$$

$$H_{2j}\left(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)\right) \leq H_{2j}\left(t_j, w_1, w_2, w_3)\right) \leq H_{2j}\left(t_j, \beta_1(t_j), \beta_2(t_j), \beta_3(t_j)\right),$$

$$H_{3j}\left(t_j, \alpha_1(t_j), \alpha_2(t_j), \beta_3(t_j)\right) \leq H_{3j}\left(t_j, w_1, w_2, w_3)\right) \leq H_{3j}\left(t_j, \beta_1(t_j), \beta_2(t_j), \alpha_3(t_j)\right),$$

for $\alpha_i(t_j) \leq w_i \leq \beta_i(t_j)$, $i = 1,2,3$, $j \in \{1,2,\ldots,n\}$. *Then there is at least* $(y_1, y_2, y_3) \in X^3$ *solution of (6.1)-(6.3) such that*

$$\alpha_i(t) \leq y_i(t) \leq \beta_i(t), \ \forall t \in [a,b], \ \textit{for } i = 1,2,3.$$

Proof. The proof of Theorem 6.3 holds and it remains to prove the relation that every solution $(y_1, y_2, y_3) \in X^3$ of *problem (6.14), (6.15), (6.16) verifies*

$$\alpha_1(t) \leq y_1(t) \leq \beta_1(t), \text{ for } t \in [a,b].$$

Assume that there is $t \in [a,b]$ such that $\alpha_i(t) - y_i^*(t) > 0$ and define

$$\sup_{t \in [a,b]} \left(\alpha_i(t) - y_i^*(t)\right) := \alpha_i(t_*) - y_i^*(t_*) > 0.$$

Consider t_* between two consecutive impulses, that is, $t \in]t_p, t_{p+1}]$. Then, by (6.15), (6.8) and Definition 6.2, this contradiction is achieved

$$\begin{aligned}
0 &\leq \alpha_1'(t) - y_1'^*(t) \\
&= \alpha_1'(t) - g_1\left(t, \alpha_1(t), \delta_2(t), \delta_3(t)\right) - \frac{y_1^*(t) - \alpha_1(t)}{1 + |y_1^*(t) - \alpha_1(t)|} \\
&< \alpha_1'(t) - g_1\left(t, \alpha_1(t), \delta_2(t), \delta_3(t)\right) \\
&\leq \alpha_1'(t) - g_1\left(t, \alpha_1(t), \beta_2(t), \alpha_3(t)\right) \leq 0.
\end{aligned}$$

In the impulsive points, case where $t_* = t_j^+$, by (6.20),

$$\begin{aligned}
0 &\leq \Delta\alpha_1\left(t_j\right) - \Delta y_1\left(t_j\right) \\
&= \Delta\alpha_1\left(t_j\right) - H_{1j}\left(t_j, \delta_1\left(t_j\right), \delta_2\left(t_j\right), \delta_3\left(t_j\right)\right) \\
&\leq \Delta\alpha_1\left(t_j\right) - H_{1j}\left(t_j, \beta_1\left(t_j\right), \alpha_2\left(t_j\right), \alpha_3\left(t_j\right)\right) \leq 0.
\end{aligned}$$

Analogously, for $\Delta\left(\alpha_3 - y_3\right)\left(t_j\right)$. ■

Following these arguments, we can obtain different versions of Theorem 6.3, combining adequate definitions of coupled lower and upper solutions, as in Definition 6.4, and alternative monotone assumptions on g_2, g_3, and on the impulsive functions H_{ij}.

6.5 Application to a vital dynamic SIRS model

The study of epidemiological phenomena via compartmental models is currently a special concern as it simplifies the mathematical modeling of infectious diseases. These types of models try to predict, for instance, how a disease spreads, the duration of an epidemic, the variation of the number of infected people, and other epidemiological parameters. So, they are important tools to help the definition of rules for public health interventions and how they may affect the outcome of the epidemic.

The classic SIR model is a basic compartmental model where the population is divided into three groups: susceptible (S), infected (I), and recovered (R). People may change groups, but the SIR model assumes that the population gains lifelong immunity to some disease upon recovery. This is true for some infectious diseases, such as measles, mumps, or rubella, but it is not the case for some airborne diseases, such as seasonal influenza, where the individual's immunity may wane over time. In this situation, the SIRS model is more adequate as it allows that the recovered individuals can return to a susceptible state and be infected again.

These compartmental models were introduced in the early 20th century, by Kermack and McKendrick in 1927, ([98]), but since then many authors study these topics, under different and varied features, objectives and techniques. As examples, we mention only some recent works on the field: [65, 69, 86, 87, 102, 105, 111, 117].

Motivated by the works above, we apply our technique to a vital dynamic SIRS system composed by the differential equations

$$\left\{\begin{array}{l} S'(t) = -\beta S(t)I(t) + \lambda R(t) \\ I'(t) = \beta S(t)I(t) - (\mu + d)\, I(t) \\ R'(t) = \mu I(t) - \lambda R(t), \end{array}\right. \tag{6.21}$$

for t in a normalized interval $[0,1]$, β, μ representing the infection and recover rates; λ the rate of recovered individuals becoming susceptible again, and d the death number by infection.

Our method allows to consider global and asymptotic data as a particular case of functional boundary conditions:

$$\begin{array}{rcl} \inf\limits_{t\in\left]\frac{1}{4},1\right]} S(t) & = & \lim\limits_{t\to\left(\frac{1}{4}\right)^+} S(t), \\ \max\limits_{t\in[0,1]} I(t) & = & I(1), \\ \sup\limits_{t\in\left]\frac{1}{4},1\right]} R(t) & = & \lim\limits_{t\to\left(\frac{1}{4}\right)^+} R(t), \end{array} \tag{6.22}$$

and generalized impulsive functions, with only one impulsive moment, for the sake of clarity,

$$\begin{aligned}
\Delta S\left(\frac{1}{4}\right) &= -5\beta S\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right) + \lambda R\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right) \\
\Delta I\left(\frac{1}{4}\right) &= 5\beta S\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right) - kI(\frac{1}{4}) \qquad (6.23)\\
\Delta R\left(\frac{1}{4}\right) &= -\lambda R\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right) + kI(\frac{1}{4}).
\end{aligned}$$

It is clear that problem (6.21)-(6.23) is a particular case of problem (6.1)-(6.3), with $y_1 = S, y_2 = I, y_3 = R, a = 0, b = 1, n = j = 1, t_1 = \frac{1}{4}$,

$$\begin{aligned}
&g_1(t, S(t), I(t), R(t)) = -\beta S(t)I(t) + \lambda R(t) \\
&g_2(t, S(t), I(t), R(t)) = \beta S(t)I(t) - (\mu + d)\, I(t) \\
&g_3(t, S(t), I(t), R(t)) = \mu I(t) - \lambda R(t)
\end{aligned}$$

the functional boundary conditions

$$\begin{aligned}
B_1(S, I, R) &= \inf_{t\in\left]\frac{1}{4},1\right]} S(t) - \lim_{t\to\left(\frac{1}{4}\right)^+} S(t) = 0 \\
B_2(S, I, R) &= \max_{t\in[0,1]} I(t) - I(1) = 0 \\
B_3(S, I, R) &= \sup_{t\in\left]\frac{1}{4},1\right]} R(t) - \lim_{t\to\left(\frac{1}{4}\right)^+} R(t) = 0
\end{aligned}$$

and the impulsive effects

$$\begin{aligned}
&\Delta S(\tfrac{1}{4}) = H_{11}(S(\tfrac{1}{4}), I(\tfrac{1}{4}), R(\tfrac{1}{4})) \;= -5\beta S\left(\tfrac{1}{4}\right) I(\tfrac{1}{4}) + \lambda R\left(\tfrac{1}{4}\right) I(\tfrac{1}{4}) \\
&\Delta I(\tfrac{1}{4}) = H_{21}(S(\tfrac{1}{4}), I(\tfrac{1}{4}), R(\tfrac{1}{4})) = 5\beta S(\tfrac{1}{4}) I(\tfrac{1}{4}) - kI(\tfrac{1}{4}) \\
&\Delta R(\tfrac{1}{4}) = H_{31}(S(\tfrac{1}{4}), I(\tfrac{1}{4}), R(\tfrac{1}{4})) = -\lambda R\left(\tfrac{1}{4}\right) I(\tfrac{1}{4}) + kI(\tfrac{1}{4}).
\end{aligned}$$

As a numeric example we consider the rates $\beta_1 = 0.1$, $\lambda_1 = 0.44$, $\mu_1 = 0.93$, $d_1 = 0.2$ before the impulsive moment, that is for $0 \leq t \leq \frac{1}{4}$, and $\beta_2 = 0.0001$, $\lambda_2 = 0.1$, $\mu_2 = 0.162$, $d_2 = 0$, after the impulsive effect, i.e., for $\frac{1}{4} < t \leq 1$.

For these values, the triple null functions $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$ and the piecewise one $(\beta_1, \beta_2, \beta_3)$ given by

$$\beta_1(t) = \begin{cases} 112000t^6 + 17120t^5 + 4600t^4 - 400t^3 - 20t^2 + 20t + 110, 0 \leq t \leq \frac{1}{4}, \\ -14500t^6 + 20000t^5 + 10000t^4 + 14600t^3 - 20t^2 + 6t - 260, \frac{1}{4} < t \leq 1, \end{cases}$$

$$\beta_2(t) = \begin{cases} 112000t^6 + 17120t^5 + 4600t^4 - 400t^3 - 20t^2 + 20t, 0 \leq t \leq \frac{1}{4}, \\ -14500t^6 + 20000t^5 + 10000t^4 + 14600t^3 - 20t^2 + 6t - 200, \frac{1}{4} < t \leq 1, \end{cases}$$

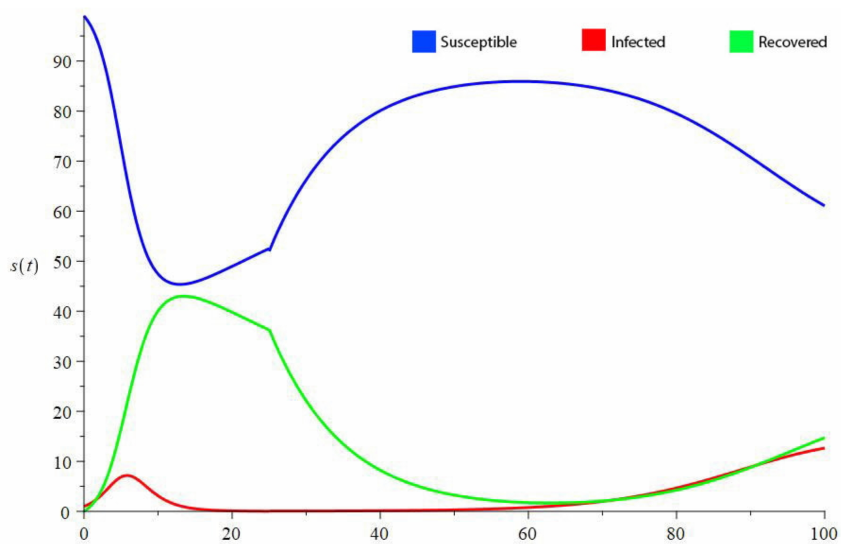
$$\beta_3(t) = \begin{cases} 112000t^6 + 17120t^5 + 4600t^4 - 400t^3 - 20t^2 + 20t + 1, 0 \leq t \leq \frac{1}{4}, \\ -14500t^6 + 20000t^5 + 10000t^4 + 14600t^3 - 20t^2 + 6t - 250, \frac{1}{4} < t \leq 1, \end{cases}$$

are, respectively, lower and upper solutions of problem (6.21)-(6.23), according to Definition 6.4.

All the assumptions of Theorem 6.3 are fulfilled for $62.35 < \rho < 79.19$, and, therefore, there is a solution of problem (6.21)-(6.23), such that

$$\begin{aligned} 0 &\leq S(t) \leq \beta_1(t), \\ 0 &\leq I(t) \leq \beta_2(t), \\ 0 &\leq R(t) \leq \beta_3(t), \text{ for } t \in [0,1]. \end{aligned}$$

Applying an adequate mathematical software, these inequalities can be illustrated by the graph of the correspondent solution, given in the next figure.



Representation of epidemic behavior with a impulsive moment at 0,25 of a time segment.

Bibliography

[1] A. Abdel-Rady, A. El-Sayed, S. Rida, I. Ameen, *On some impulsive differential equations*, Math. Sci. Lett. 1 No. 2, (2012) 105-113.

[2] A. Cabada, F. Minhós, A. Santos, *Solvability for a third order discontinuous fully equation with nonlinear functional boundary conditions,* J. Math. Anal. Appl. 322 (2006) 735–748

[3] A. Cabada, J. Ángel Cid, *Heteroclinic solutions for non-autonomous boundary value problems with singular* Φ*-Laplacian operators*, Discrete and Continuous Dynamical Systems, Supplement 2009, 118–122.

[4] A. Cabada, J. Fialho, F. Minhós, *Extremal solutions to fourth order discontinuous functional boundary value problems.* Math. Nachr. 2013, 286, 1744–1751.

[5] A. Cabada, J. Tomeček, *Extremal solutions for nonlinear functional* ϕ*-Laplacian impulsive equations.* Nonlinear Anal. 67 (2007) 827–841.

[6] A. Cabada, *The method of lower and upper solutions for third-order periodic boundary value problems*, J. Math. Anal. Appl., 195 (1995) 568–589.

[7] A. Catlla, D. Schaeffer, T. Witelski, E. Monson, A. Lin, *On Spiking Models for Synaptic Activity and Impulsive Differential Equations*, Society for Industrial and Applied Mathematics, Vol. 50, No. 3, (2008) 553-569.

[8] A. El-Sayed, R. Ahmed, *Solvability of a coupled system of functional integro-differential equations with infinite point and Riemann–Stieltjes integral conditions,* Applied Mathematics and Computation, Volume 3701, (2020) Article 124918.

[9] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[10] A. R. Aftabizadeh, J.-M. Xu, C. P. Gupta, *Periodic boundary value problems for third order ordinary differential equations*, Nonlinear Anal. Theory, Methods and Applications A, vol. 14, no. 1, pp. 1–10, 1990.

[11] A.U. Afuwape, *Nonlinear perturbations of differential operators with non-trivial kernel and applications to third-order periodic boundary value problems*, J. Math. Anal. Appl. 143 (1989) 35–56.

[12] B. Liu, J. Yu, *Existence of solution for m-point boundary value problems of second-order differential systems with impulses,* Appl. Math. Comput., 125, (2002), 155-175.

[13] B. Mukhopadhyay, R. Bhattacharyya, *A mathematical model describing the thyroid-pituitary axis with time delays in hormone transportation*, Applications of Mathematics, Vol. 51 (2006), No. 6, 549–564

[14] B. Yan, D. O'Regan, R.P. Agarwal, *Unbounded solutions for singular boundary value problems on the semi-infinite interval: Upper and lower solutions and multiplicity*, J.Comput.Appl.Math. **197** (2006) 365-386.

[15] C. Bai, C. Li, *Unbounded upper and lower solution method for third-order boundary-value problems on the half-line*, Electronic Journal of Differential Equations, vol. 2009, no. **119**, pp. 1–12, 2009.

[16] C. Bereanu, J. Mawhin, *Boundary-value problems with non-surjective* Φ-*Laplacian and one-sided bounded nonlinearity*, Adv. Differential Equations 11 (2006) 35–60.

[17] C. Çelik, H. Merdan, *Hopf bifurcation analysis of a system of coupled delayed-differential equations,* Applied Mathematics and Computation, Volume 219, Issue 1215, (2013) 6605-6617 .

[18] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, NewYork, 1973.

[19] D.D. Bainov, P.S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical, Harlow, 1993.

[20] D. Lu, A. Seadawy, M. Iqbal, *Mathematical methods via construction of traveling and solitary wave solutions of three coupled system of nonlinear partial differential equations and their applications,* Results in Physics, Volume 11 (2018) 1161-1171 .

[21] E.O. Tuck, L.W. Schwartz, *A boundary value problem from draining and coating flows involving a third-order differential equation relevant to draining and coating flows*, SIAM Rev., **32** (1990) 453-469.

[22] F. Bernis, L.A. Peletier, *Two problems from draining flows involving third-order ordinary differential equations*, SIAM J. Math. Anal., **27** (1996), 515-527.

[23] F. Minhós, *Location results: an under used tool in higher order boundary value problems*, International Conference on Boundary Value Problems: Mathematical Models in Engineering, Biology and Medicine, American Institute of Physics Conference Proceedings, Vol. 1124, 244-253, 2009

[24] F. Minhós, H. Carrasco, *High Order Boundary Value Problems on Unbounded Domains: Types of Solutions, Functional Problems and Applications*, Trends in Abstract and Applied Analysis, Vol. 5, World Scientific, 2017, ISBN 978-981-3209-90-9.

[25] F. Minhós, H. Carrasco, *Lidstone-type problems on the whole real line and homoclinic solutions applied to infinite beams*, Neural Computing & Application, 2020, DOI: 10.1007/s00521-020-04732-x.

[26] F. Minhós, *Impulsive problems on the half-line with infinite impulse moments*, Lithuanian Mathematical Journal, Vol. 57, No. 1, January, 2017, pp. 69–79, DOI 10.1007/s10986-017-9344-5.

[27] F. Minhós, J. Fialho, *On the solvability of some fourth-order equations with functional boundary conditions*, Discrete Contin. Dyn. Syst., 2009, suppl., 564–573.

[28] F. Minhós, *Periodic Solutions for a Third Order Differential Equation under Conditions on the Potential*, Portugaliae Mathematica, Vol. 55, Fasc. 4 (1998) 475-484.

[29] F. Minhós, R. Carapinha, *Half-linear impulsive problems for classical and singular φ-Laplacian with generalized impulsive conditions*, Journal of Fixed Point Theory and Applications, 20 (3), 117, 2018, DOI: 10.1007/s11784-018-0598-2

[30] F. Minhós, R. Carapinha, *On higher order nonlinear impulsive boundary value problems*, Discrete Contin. Dyn. Syst. 2015, Dynamical Systems, Differential Equations and Applications 10th AIMS Conference. Suppl., 851–860.

[31] F. Minhós, R. de Sousa, Impulsive coupled systems with generalized jump conditions, Nonlinear Analysis: Modelling and Control, Vol. 23, No. 1, (2018) 103–119.

[32] F. Minhós, R. de Sousa, *Nonlinear higher order differential and integral coupled systems: Impulsive and Integral Equations on Bounded and Unbounded Domains*, Trends in Abstract and Applied Analysis, Vol 10, World Scientific, 2020, DOI: 10.1142/11961, ISBN 978-981-122-512-3 .

[33] F. Minhós, R. de Sousa, *Solvability of coupled systems of generalized Hammerstein-type integral equations in the real line*, Mathematics, 2020, 8, 111; doi:10.3390/math8010111.

[34] F. Minhós, T. Gyulov, A. I. Santos, *Lower and upper solutions for a fully nonlinear beam equations*, Nonlinear Anal., 71 (2009) 281–292.

[35] G.C. Yang, H.B. Feng, *New results of positive solutions for the Sturm-Liouville problem*, Boundary Value Problems (2016) 2016:64, DOI 10.1186/s13661-016-0571-1 .

[36] G. Infante, P. Pietramala, *Nonnegative solutions for a system of impulsive BVPs with nonlinear nonlocal BCs*, Nonlinear Analysis: Modelling and Control 19(3) (2014).

[37] H. Carrasco, F. Minhós, *Sufficient conditions for the solvability of some third order functional boundary value problems on the half-line*, Comment. Math. Univ. Carolin. 58, 4 (2017) 443–459, DOI 10.14712/1213-7243.2015.220 .

[38] H. Lian, P. Wang, W. Ge, *Unbounded upper and lower solutions method for Sturm-Liouville boundary value problem on infinite intervals*, Nonlinear Anal., **70** (2009) 2627–2633.

[39] I. Karaca, *On positive solutions for second-order boundary value problems of functional differential equations,* Applied Mathematics and Computation, Volume 219, Issue 1015 (2013) 5433-5439 .

[40] I. Rachůnková, M. Tvrdý, *Existence results for impulsive second-order periodic problems*. Nonlinear Anal. 59 (2004) 133–146

[41] J. Fialho, F. Minhós, *First order coupled systems with functional and periodic boundary conditions: existence results and application to an SIRS model*, Axioms 2019, 8(1), 23, https://doi.org/10.3390/axioms8010023.

[42] J. Fialho, F. Minhós, *High order Boundary Value Problems: Existence, localization and multiplicity results*, Mathematics Research Developments Series, Nova Science Publishers, Inc., New York, 2014.

[43] J. Fialho, F. Minhós, *Higher order periodic impulsive problems*, Dynamical Systems, Differential Equations and Applications, AIMS Proceedings, (2015) 446-454.

[44] J. Fialho, F. Minhós, *Multiplicity and location results for second order functional boundary value problems*, Dyn. Syst. Appl. 2014, 23, 453–464.

[45] J. Fialho, F. Minhós, *On higher order fully periodic boundary value problems*, J. Math. Anal. Appl., 395 (2012) 616-625.

[46] J. Graef, L. Kong, F. Minhós, *Higher order boundary value problems with φ -Laplacian and functional boundary conditions*. Comput. Math. Appl. 2011, 61, 236–249.

[47] J. Graef, L. Kong, F. Minhós, *Higher order boundary value problems with Φ-Laplacian and functional boundary conditions*, Computers and Mathematics with Applications, 61 (2011) 236–249.

[48] J. Graef, L. Kong, F. Minhós, J. Fialho, *On the lower and upper solutions method for higher order functional boundary value problems*, Applicable Analysis and Discrete Mathematics, vol. 5, n^o 1 (2011) 133-146

[49] J. Han, Y. Liu, J. Zhao, *Integral boundary value problems for first order nonlinear impulsive functional integro-differential differential equations,* Applied Mathematics and Computation, Volume 218, Issue 91 (2012) 5002-5009.

[50] J. Henderson, R. Luca, *Systems of Riemann–Liouville fractional differential equations with coupled integral boundary conditions,* Boundary Value Problems for Systems of Differential, Difference and Fractional Equations, 2016, Pages 229-298.

[51] J. J. Nieto, J. Uzal, *Nonlinear second-order impulsive differential problems with dependence on the derivative via variational structure*, Journal of Fixed Point Theory and Applications, 2020.

[52] J. Mawhin, *Degré topologique et solutions périodiques des systèmes différentiels non linéaires*, Bull. Soc. Roy. Sci. Liège 38 (1969) 308–398.

[53] J. Nieto, D. O'Regan, *Variational approach to impulsive differential equations,* Nonlinear Anal. RWA, (2009), 680-690

[54] J. Nieto, R. López, *Boundary value problems for a class of impulsive functional equations*, Comput. Math. Appl. 55 (2008) 2715-2731

[55] J. R. Graef, L. Kong, F. Minhós, *Higher order boundary value problems with phi-Laplacian and functional boundary conditions,* Comp. Math. Appl., 61 (2011) 236-249.

[56] J. Graef, L. Kong, F. Minhós, J. Fialho, *On the lower and upper solutions method for higher order functional boundary value problems*, Applicable Analysis and Discrete Mathematics, vol. 5, n^o 1 (2011) 133-146.

[57] J. Ren, S. Siegmund, Y. Chen, *Positive periodic solutions for third-order nonlinear differential equations*, Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 66, pp. 1–19.

[58] J. Xiao, J. Nieto, Z. Luo, *Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods,* Communications in Nonlinear Science and Numerical Simulation, 17, (2012), 426-432.

[59] J.Paullet, P.Weidman, *Analysis of Stagnation Point Flow Toward a Stretching Sheet*, School of Science, Pennsylvania State University at Erie, Erie, PA 16563-0203, USA.

[60] J. Zhao, B. Sun, Yu Wang, *Existence and iterative solutions of a new kind of Sturm-Liouville-type boundary value problem with one-dimensional p-Laplacian*, Boundary Value Problems (2016) 2016:86, DOI 10.1186/s13661-016-0590-y

[61] L. Carichino, G. Guidoboni, M. Szopos, *Energy-based operator splitting approach for the time discretization of coupled systems of partial and ordinary differential equations for fluid flows: The Stokes case,* Journal of Computational Physics, Volume 3641 (2018) 235-256

[62] L. Danziger, G. L. Elmergreen, *The thyroid-pituitary homeostatic mechanism.* Bull. Math. Biophys. 18 (1956), 1–13.

[63] L. Skóra, *Boundary value problem for the second orderimpulsive delay differential equations*, Ann. Univ. Paedagog. Crac. Stud. Math. 14 (2015), 27-35DOI: 10.1515/aupcsm-2015-0002.

[64] L. Sunny, R. Roy, V. Vijesh, *An accelerated technique for solving a coupled system of differential equations for a catalytic converter in interphase heat transfer,* Journal of Mathematical Analysis and ApplicationsVolume 445, Issue 11 (2017) 318-336 .

[65] M. Barro, A. Guiro, D. Ouedraogo, *Optimal control of a SIR epidemic model with general incidence function and a time delays,* CUBO A Mathematical Journal, Vol.20, N^o 02, (2018) 53–66.

[66] M. Benchohra, J. Henderson, S. Ntouyas, *Impulsive Differential Equations and Inclusions,* Contemporary Mathematics and Its Applications,Hindawi Publishing Corporation, 2006

[67] M. Benchohra, B. Slimani, *Existence and uniqueness of solutions to impulsive fractional differential equations*, Electron. J. Differ. Equ., (10), (2009) 1-11.

[68] M. Domínguez-Pérez, R. Rodríguez-López, *Multipoint boundary value problems of Neumann type for functional differential equations,* Nonlinear Analysis: Real World ApplicationsVolume 13, Issue 4, (2012) 1662-1675 .

[69] M. Meehan, D. Cocks, J. Trauer, E.McBryde, *Coupled, multi-strain epidemic models of mutating pathogens,* arXiv:1611.04204v2 [q-bio.PE] 8 May 2017.

[70] M. Reza, A.S. Gupta, *Steady two-dimensional oblique stagnation-point flow towards a stretching surface*, Fluid Dyn. Res. 37 (2005) 334–340.

[71] M. R.Grossinho, F. Minhós, A.I. Santos, *A note on a class of problems for a higher-order fully nonlinear equation under one-sided Nagumo-type condition,* Nonlinear Anal., 70 (2009) 4027-4038.

[72] M. R. Grossinho, F. Minhós, A. I. Santos, *A note on a class of problems for a higher order fully nonlinear equation under one sided Nagumo type condition*, Nonlinear Anal., 70 (2009) 4027–4038

[73] M. X. Wang, A. Cabada, J. J. Nieto, *Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions,* Ann. Polon. Math. **58** (1993), 221–235.

[74] P. Amster, P. De Napoli, M. C. Mariani, *Periodic solutions of a resonant third-order equation*, Nonlinear Anal. Theory, Methods and Applications A, vol. 60, no. 3, pp. 399–410, 2005.

[75] P. Chen, X. Tang, *Existence and multiplicity of solutions for second-order impulsive differential equations with Dirichlet problems,* Appl. Math. Comput. 218 (2012) 1775-11789.

[76] P. Kang, Z. Wei, *Existence of positive solutions for systems of bending elastic beam equations*, Electron. J. Differential Equations, **19** (2012).

[77] P. Omari, M. Trombetta, *Remarks on the lower and upper solutions method for second- and third-order periodic boundary value problems*, Appl. Math. Comp.,vol. 50, no. 1, pp. 1–21, 1992.

[78] R. Agarwal, D. O'Reagan, *A coupled system of boundary value problems.* Appl. Anal. 1998, 69, 381–385.

[79] R. Agarwal, D. O'Regan, *Multiple nonnegative solutions for second order impulsive differential equations*, Appl. Math. Comput. 155 (2000) 51-59.

[80] R. Bellman, *Mathematical Methods in Medicine*, World Scientific, Singapore, 1983.

[81] R. Ma, B. Yang, Z. Wang, Positive periodic solutions of first-order delay differential equations with impulses, Appl. Math. Comput. 219 (2013) 6074–6083.

[82] R. Gjessing, A. Bernhardsen, H. Froshaug, *Investigation of amino acids in periodic catatonic patients.* J. Ment. Sci. 104 (1958), 188–200.

[83] R. Nazar, N. Amin, D. Filip, I. Pop, *Stagnation point flow of a micropolar fluid towards a stretching sheet*, Int. J. Non-Linear Mech. 39 (2004) 1227–1235.

[84] R.P. Agarwal, D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publisher, Glasgow 2001.

[85] R. P. Agarwal, D. O'Regan, *Non-linear boundary value problems on the semi-infinite interval: an upper and lower solution approach*, Mathematika, vol. **49**, no. 1-2, (2002) 129–140.

[86] S. O'Regan, T. Kelly, A. Korobeinikov, M. O'Callaghan. A. Pokrovskii, *Lyapunov functions for SIR and SIRS epidemic models*, Applied Mathematics Letters, 23 (2010) 446–448.

[87] S. Rajasekar, M. Pitchaimani, *Ergodic stationary distribution and extinction of a stochastic SIRS epidemic model with logistic growth and nonlinear incidence,* Applied Mathematics and Computation, Volume 377, 15 (2020), 125143, https://doi.org/10.1016/j.amc.2020.125143.

[88] T. Chiam, *Stagnation-point flow towards a stretching plate*, J. Phys. Soc. Jpn. 63 (1994) 2443–2444.

[89] T.S. Jang, *A new semi-analytical approach to large deflections of Bernoulli–Euler-v.Karman beams on a linear elastic foundation: Nonlinear analysis of infinite beams*, International Journal of Mechanical Sciences **66** (2013) 22–32.

[90] T.R. Mahapatra, A.S. Gupta, *Heat transfer in stagnation-point flow towards a stretching sheet*, Heat Mass Transfer 38 (2002) 517–521.

[91] V. Lakshmikantham, D. Baĭnov, P. Simeonov, *Theory of impulsive differential equations*. Series in Modern Applied Mathematics, 6. World Scientific Publishing Co., Inc., 1989.

[92] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[93] W. C. Troy, *Solutions of third-order differential equations relevant to draining and coating flows*, SIAM J. Math. Anal., **24** (1993), 155-171.

[94] W. Ding, J. Mi, M. Han, *Periodic boundary value problems for the first order impulsive functional differential equations*, Appl. Math. Comput. 165 (2005) 433–446.

[95] W. Jiang, J. Qiu, B. Sun, *Conjugate boundary value problems with functional boundary conditions at resonance,* Bound Value Probl 2019, 69 (2019). https://doi.org/10.1186/s13661-019-1182-4 .

[96] X. Hao, L. Liu, Y. WU, Positive solutions for nth-order singular nonlocal boundary value problems, Bound. Value Probl. (2007) 10, Article ID 74517.

[97] X. Liu, D. Guo, *Method of upper and lower solutions for second-order impulsive integro-differential equations in a Banach space,* Comput. Math. Appl., 38 (1999), 213–223.

[98] W. O. Kermack, A. G. McKendrick, *Contributions to the Mathematical Theory of epidemics–I. 1927,* Bull. Math. Biol., 53(1-2) (1991), 33-55. doi: 10.1007/BF02464423.

[99] X. Li, M. Bohner, C-K. Wang *Impulsive differential equations: Periodic solutions and applications*, Automatica 52 (2015) 173–178014.

[100] X. Li, S. Song, *Impulsive control for existence, uniqueness and global stability of periodic solutions of recurrent neural networks with discrete and continuously distributed delays*, IEEE Transactions on Neural Networks and Learning Systems, 24 (6) (2013) 868–877, doi: 10.1109/TNNLS.2012.2236352.

[101] X. Liu, X. Zhang, *The weak maximum principle for a class of strongly coupled elliptic differential systems,* Journal of Functional Analysis, Volume 263, Issue 71 (2012) 1862-1886 .

[102] X. Wang, *An SIRS Epidemic Model with Vital Dynamics and a Ratio-Dependent Saturation Incidence Rate,* Discrete Dynamics in Nature and Society, Volume 2015, Article ID 720682, 9 pages, http://dx.doi.org/10.1155/2015/720682 .

[103] X. Wang, Q. Liu, D. Qian, *Existence and multiplicity results for some nonlinear problems with singular φ-Laplacian via a geometric approach,* Boundary Value Problems (2016) 2016:47, DOI 10.1186/s13661-016-0554-2.

[104] X. Zhang, M. Feng, W. Ge, *Existence of solutions of boundary value problems with integral boundary conditions for second-order impulsive integro-differential equations in Banach spaces,* J. Comput. Appl. Math., 233, (2010), 1915-1926

[105] X. Zhang, X. Wang, H. Huo, *Extinction and stationary distribution of a stochastic SIRS epidemic model with standard incidence rate and partial immunity,* Physica A: Statistical Mechanics and its Applications, Volume 531, 1 (2019), 121548, https://doi.org/10.1016/j.physa.2019.121548 .

[106] Y. Li, *Existence of positive solutions for the cantilever beam equations with fully nonlinear terms*, Nonlinear Analysis: Real World Applications **27** (2016) 221–237.

[107] Y. Li, Q. Li, *Positive Periodic Solutions of Third-Order Ordinary Differential Equations with Delays*, Abstract and Applied Analysis, 2014, 8 pages

[108] Y. Liu, D. O'Regan, Multiplicity results using bifurcation techniques for a class of boundary value problems of impulsive differential equations, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 1769–1775.

[109] Y. Liu, W. Ge, *Solutions of a generalized multi-point conjugate BVPs for higher order impulsive differential equations.* Dynam. Systems Appl. 14 (2005) 265–279.

[110] Y. Kuo, S. Lee, *Deflection of non-uniform bearns resting on a nonlinear elastic foundation.* Comput Struct , 51 (1994) 513-519.

[111] Y. Muroya, H. Li, T. Kuniya, *Complete global analysis of an SIRS epidemic model with graded cure and incomplete recovery rates*, J. Math.Anal.Appl. 410 (2014) 719–732.

[112] Y. Tian, W. Ge, *Variational methods to Sturm–Liouville boundary value problem for impulsive differential equations,* Nonlinear Analysis: Theory, Methods & Applications, 72, (2010), 277-287.

[113] Y. Wang, *Some Fourth Order Equations Modeling Suspension Bridges*, Ph.D. dissertation, Politecnico di Milano, 2015

[114] Y. Wu, *Existence nonexistence and multiplicity of periodic solutions for a kind of functional differential equation with parameter*, Nonlinear Anal. Theory, Methods and Applications A, vol. 70, no. 1, pp. 433–443, 2009.

[115] Y. Xin, Z. Cheng, *Positive periodic solution for φ-Laplacian Rayleigh equation with strong singularity*, Boundary Value Problems (2016) 2016:139, DOI 10.1186/s13661-016-0648-x .

[116] Y. Xu, H. Zhang, *Positive solutions of an infinite boundary value problem for nth-order nonlinear impulsive singular integro-differential equations in Banach spaces*, Applied Mathematics and Computation 218 (2012) 5806–5818.

[117] Y. Yang, L. Zou, T. Zhang, Y. Xu, *Dynamical Analysis of a Diffusive SIRS Model with General Incidence Rate*, Discrete and Continuous Dynamical Systems, Series B, Volume 25, Number 7, July 2020 pp. 2433-2451, doi:10.3934/dcdsb.2020017.

[118] Z. Benbouziane, A. Boucherif, S. Bouguima, *Existence result for impulsive third order periodic boundary value problems*, Appl. Math. Comput. 206 (2008) 728–737.

[119] Z. Zhang, Z. Wang, *Periodic solutions of the third order functional differential equations*, J. Math. Anal. Appl. 292 (2004) 115–134.