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Stochastic differential equations harvesting policies: Allee effects, logistic-like growth and profit optimization

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Abstract

In this article, stochastic differential equations are used to model the dynamics of a harvested population in the presence of weak Allee effects. Two optimal harvesting policies are presented, one with variable effort based on optimal control theory, which is for practical reasons inapplicable in a random environment, and the other with constant effort and easily applicable. For a logistic-like model with weak Allee effects, we show that the optimal policy based on constant effort implies, in a suitable range of effort values, the existence of a steady-state stochastic equilibrium with a stationary density, obtained explicitly here, for the population size. With this new result, we compare the performance of both policies in terms of the profit obtained over a finite time horizon. Using realistic data from a harvested population and a logistic-type growth model, we quantify the profit reduction when choosing the optimal policy based on constant effort instead of the optimal policy based on variable effort. We also study the influence of the Allee effects strength.

KEYWORDS

Allee effects, harvesting policies, logistic-like model, profit optimization, stochastic differential equations

1 | INTRODUCTION

In a random environment, the logistic growth model for a harvested population can be described by the stochastic differential equation (SDE)

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)dt - qE(t)X(t)dt + \sigma X(t)dW(t),\ X(0) = x, \tag{1}$$

where $X(t)$ is the population size at time t, measured as biomass or as number of individuals, $r > 0$ is the intrinsic growth rate of the population, $K > 0$ is the carrying capacity of the environment, $q > 0$ is the catchability coefficient, $E(t) \geq 0$ is the fishing effort, $\sigma > 0$ measures the strength of environmental fluctuations, $W(t)$ is a standard Wiener process and $X(0) = x > 0$ represents the population size at time 0. The environmental fluctuations affect the *per capita* natural growth rate. For model (1), the *per capita* average natural growth rate is $r(1 - X(t)/K)$ and the *per capita* average net growth rate is $r(1 - X(t)/K) - qE(t)$. The harvesting term, named yield per unit time, is defined as $H(t) := qE(t)X(t)$. In the absence of Allee effects, a generalized logistic growth model with harvesting can be found in Shah.[1] Here, however, we will study instead a logistic-like growth model where the population is under the influence of Allee effects. This means that, for low

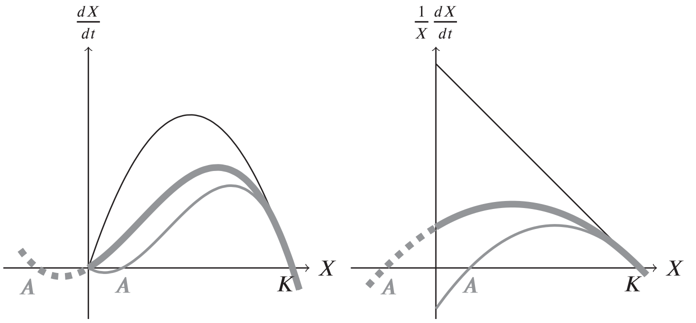

FIGURE 1 Total population growth (left side) and *per capita* growth (right side) in the absence of fishing for the deterministic logistic model without Allee effects (black thin lines) and for the deterministic logistic-like growth model under weak (gray thicker line) and strong Allee effects (gray thinner lines)

values of the population size, we observe *per capita* growth rates lower than the high rates one would expect considering the higher availability of resources per individual. The presence of Allee effects when the population size is low may be due to several causes, such as the difficulty in finding mating partners or in setting up an effective pack-hunting size or, in the case of prey species, in constructing a strong enough group defense against predators (see Allee[2]). The study of population general SDE growth models without harvesting and under Allee effects (weak and strong) can be seen in Carlos and Braumann[3] and references therein. Considering strong Allee effects would lead the population to extinction, even in the absence of harvesting (ie, $E(t) = 0$), since the average natural growth rate would be negative for low population sizes (see Carlos and Braumann[3] and also, for specific closed and open population models, Dennis et al[4]). Therefore, we will consider only weak Allee effects.

The existence of Allee effects requires the modification of Equation (1) to

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t) - A}{K - A}\right)dt - qE(t)X(t)dt + \sigma X(t)dW(t), \tag{2}$$

with $X(0) = x > 0$ and where $A \in (-K, 0)$ represents the Allee parameter measuring the strength of the weak Alee effects. The closer A is to 0, the more intense is the Allee effect. On the contrary, the closer A is to $-K$, the less intense is the Allee effect. Taking $A \to -\infty$ leads to the logistic model. Strong Allee effects occur when $A \in (0, K)$ and they will not be considered here. For this new model, the *per capita* average natural growth rate is $r\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t)-A}{K-A}\right)$, the total average natural growth rate is $rX(t)\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t)-A}{K-A}\right)$ and the *per capita* average net growth rate is $r\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t)-A}{K-A}\right) - qE(t)$.

Equation (2) assumes that the natural growth rate follows a logistic-like model inspired by a similar deterministic model (see, for instance, Dennis[5]). However, without changing the logistic-like model for the average natural growth rate dynamics, we use a different parametrization of that model in order to allow easier comparisons with the logistic model without Allee effects (see Carlos and Braumann[3]). In particular, the logistic model and the logistic-like model here considered have in common the same carrying capacity K and the same slope of the natural growth rate at $X = K$.

Figure 1 shows, for the deterministic case ($\sigma = 0$) and in the absence of fishing, two examples of the logistic-like model with strong and weak Allee effects. The total population growth curve and the *per capita* growth curve are depicted, respectively on the left and on the right side of the figure. We also show, for comparison purposes, similar curves for the logistic model without Allee effects. For the model with Allee effects it is easily seen that, at low population sizes, the *per capita* growth rates are not at their maximum value as they would ordinarily be based on the *per capita* resource abundance. For values of $A \leq -K$ (not depicted), the *per capita* growth rates at low population sizes are at their maximum value and so, technically, we do not speak of having Allee effects. However, those rates are still depressed when compared to the logistic model, which is only reached when $A \to -\infty$.

In previous work (see Brites and Braumann[6-10]), we discussed the use of a variable effort optimal policy vs a constant effort optimal sustainable policy, considering the Gompertz and the logistic models, with the purpose of deriving harvesting policies based on profit optimization. We have shown that the optimal policy with variable effort, obtained using optimal control methods, has several shortcomings, namely: (i) the effort depends on the randomly varying population size, implying the estimation of the population size in each time instant, which is a costly, time consuming, and inaccurate task; and (ii) these policies are inapplicable from the practical and social point of view. In fact, the effort is highly variable and may even have frequent periods of no harvesting or harvesting at the maximum possible rate.

On the contrary, the optimal sustainable policy based on constant effort has strong advantages: (i) leads to sustainable and very easily applicable fishing policies; (ii) population is driven to a stationary regimen when $t \to +\infty$; and (iii) does not require knowledge of population size. The only disadvantage of this policy is the reduction in profit, which we show to be slight for the models and data considered (see Brites and Braumann[6-10]). The incorporation of Allee effects on the dynamics of harvested populations living in randomly varying environments was, however, not included in our previous articles and is the innovative purpose of the current work.

This article is organized in the following way: In Section 2, we describe the formulation of the stochastic optimal control problem (SOCP) and determination of the optimal variable effort policy. In Section 3, we obtain the optimal sustainable policy based on SDEs theory. Section 4 refers to the comparisons of the above policies, and the effect of the Allee parameter A on the profits. Finally, some concluding remarks are given in Section 5.

2 | OPTIMAL POLICY WITH VARIABLE EFFORT

The process to obtain an optimal policy with variable effort based on profit optimization is a SOCP and was already formulated in Brites and Braumann[7,10] considering a logistic growth model. The profit per unit time, $\Pi(t)$, is now defined as the difference between sales revenues per unit time, $R(t) = (p_1 - p_2H(t))H(t)(p_1 > 0, p_2 \geq 0)$, and fishing costs per unite time, $C(t) = (c_1 + c_2E(t))E(t)(c_1 > 0, c_2 > 0)$, that is, $\Pi(t) := R(t) - C(t)$, while in Brites and Braumann[7] we had assumed $p_2 = 0$.

To derive the SOCP with a logistic-like model in the presence of Allee effects, we follow Brites and Braumann,[7] assuming now $p_2 \geq 0$ and replacing the *per capita* average natural growth rate, which previously followed the logistic model $r\left(1 - \frac{X(t)}{K}\right)$, by the logistic-like rate with Allee effects $r\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t)-A}{K-A}\right)$. The result is a SOCP consisting in maximizing the expected accumulated discounted profit per unit time over a finite time interval $[0, T]$:

$$V^* := J^*(x, 0) = \max_{\substack{E(\tau) \\ 0\leq\tau\leq T}} J(x, 0) = \max_{\substack{E(\tau) \\ 0\leq\tau\leq T}} \mathbb{E}_{0,x}\left[\int_0^T e^{-\delta\tau}\Pi(\tau)d\tau\right], \tag{3}$$

subject to the population dynamics given by Equation (2), to the control restrictions $0 \leq E_{\min} \leq E(t) \leq E_{\max} < \infty$ and to a terminal condition $J(X(T), T) = 0$. Note that we use the short notation $\mathbb{E}[\ldots|X(t) = y] = \mathbb{E}_{t,y}[\ldots]$ and

$$J(y, t) := \mathbb{E}_{t,y}\left[\int_t^T e^{-\delta(\tau-t)}\Pi(\tau)d\tau\right] \tag{4}$$

is, at time t, the expected discounted future profits when the population size at that time is y. The parameter $\delta > 0$ refers to a discount rate accounting for interest rate and cost of opportunity losses and for other social rates. In addition, we assume that optimization starts at time $t = 0$ and harvesting continues up to the time horizon T.

The above SOCP can be solved by applying stochastic dynamic programming theory through Bellman's principle of optimality (see Bellman[11]). In terms of optimization theory the problem resorts to finding the effort (ie, the control) that maximizes the present value $V := J(x, 0)$, subject to the growth dynamics given by Equation (2) and to the constrains on effort and the terminal condition given above. The maximizer that leads to the maximum V^* will be called the optimal variable effort and will be denoted by $E^*(t)$.

The Hamilton-Jacobi-Bellman (HJB) equation (see Hanson[12]) associated to the SOCP is

$$\begin{aligned} -\frac{\partial J^*(X(t),t)}{\partial t} &= \left(p_1qX(\tau) - c_1 - (p_2q^2X^2(\tau) + c_2)E^*(\tau)\right)E^*(\tau) - \delta J^*(X(t),t) \\ &+ \frac{\partial J^*(X(t),t)}{\partial X(t)}\left(rX(t)\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t)-A}{K-A}\right) - qE^*(t)X(t)\right) + \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}\sigma^2X^2(t), \end{aligned} \tag{5}$$

and the optimal variable effort is

$$E^*(t) = \begin{cases} E_{\min}, & \text{if } E^*_{\text{free}}(t) < E_{\min} \\ E^*_{\text{free}}(t), & \text{if } E_{\min} \leq E^*_{\text{free}}(t) \leq E_{\max} \\ E_{\max}, & \text{if } E^*_{\text{free}}(t) > E_{\max}, \end{cases}$$

where

$$E^*_{\text{free}}(t) = \frac{\left(p_1 - \frac{\partial J^*(X(t),t)}{\partial X(t)}\right) qX(t) - c_1}{2\left(p_2 q^2 X(t)^2 + c_2\right)}$$

is the unconstrained effort (see Hanson and Ryan[13]).

Equation (5) does not have an explicit solution and needs to be solved numerically. A Crank-Nicolson discretization scheme was used as in Brites and Braumann.[6-10]

3 | OPTIMAL SUSTAINABLE POLICY WITH CONSTANT EFFORT

For the logistic-like model with weak Allee effects and a constant effort fishing policy $E(t) \equiv E \geq 0$, the dynamics of a population is described by the autonomous SDE

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t) - A}{K - A}\right) dt - qEX(t)dt + \sigma X(t)dW(t),\ X(0) = x. \tag{6}$$

To avoid extinction, we continue to assume the presence of weak Allee effects, that is, $-K < A < 0$. Our state space is $(0, +\infty)$, with boundaries $X = 0$ and $X = +\infty$.

The drift and diffusion coefficients (see, for instance, Braumann[14] or Øksendal[15]) associated to Equation (6) are, respectively, $a(X) = rX\left(1 - \frac{X}{K}\right)\left(\frac{X-A}{K-A}\right) - qEX$ and $b^2(X) = \sigma^2 X^2$. Both coefficients are continuous functions with respect to X, so the unique solution of Equation (6) is, up to a possible explosion time, a homogeneous diffusion process (see, for instance, Arnold[16] and Braumann[14]). The scale and speed densities are, respectively, with $c > 0$ an arbitrary constant,

$$s(X) = \exp\left(-\int_c^X \frac{2a(\theta)}{b^2(\theta)} d\theta\right) = CX^{-\alpha+\beta E-1} \exp\left\{\gamma(X - (K + A))^2\right\}$$

and

$$m(X) = \frac{1}{b^2(X)s(X)} = DX^{\alpha-\beta E-1} \exp\left\{-\gamma(X - (K + A))^2\right\},$$

where $\alpha = \frac{2rA}{\sigma^2(A-K)} - 1$, $\beta = \frac{2q}{\sigma^2}$, $\gamma = \frac{r}{\sigma^2 K(K-A)}$ and $C > 0$ and $D > 0$ are constants. From the scale and speed densities, one can define the scale function $S(X) = \int_d^X s(z)dz$ and the speed function $M(X) = \int_d^X m(z)dz$, where $d > 0$ is an arbitrary constant. Such functions are similar to distribution functions in the sense that the speed and scale measures of intervals of the form $(u, v]$ $(u, v \in [0, +\infty))$ are determined by $S(u, v] = S(v) - S(u)$ and $M(u, v] = M(v) - M(u)$.

When $X = 0$ is nonattractive, there is a zero probability of having $X(t) = 0$ for some finite t or $X(t) \to 0$ as $t \to +\infty$, that is, there is nonextinction from the mathematical point of view. On the other hand, the nonattractiveness of $X = +\infty$ implies that there is a zero probability of having $X(t) = +\infty$ for some finite t or $X(t) \to +\infty$ as $t \to +\infty$ and so explosions have zero probability of occurring. When both boundaries are nonattractive, the trajectories of $X(t)$ tend to be pushed towards the interior of the state space whenever they approach the boundaries, avoiding extinction and explosions and opening the possibility of a stochastic equilibrium in the sense of $X(t)$ converging in distribution, as $t \to +\infty$, to a random variable (r.v.) X_∞.

We now study the behavior of the boundaries for model (6).

Proposition 1. *Let $-K < A < 0$, $\alpha = \frac{2rA}{\sigma^2(A-K)} - 1$, $\beta = \frac{2q}{\sigma^2}$ and $\gamma = \frac{r}{\sigma^2 K(K-A)}$. Then, for the SDE (6):*

1. *the boundary $X = +\infty$ is nonattractive and the solution of (6) exists and is unique.*
2. *if $\alpha - \beta E < 0$, the boundary $X = 0$ is attractive and mathematical extinction will occur with probability one.*
3. *if $\alpha - \beta E \geq 0$, the boundary $X = 0$ is nonattractive and mathematical extinction has zero probability of occurring.*

Proof.

1. We first show that the boundary $+\infty$ is nonattractive. A necessary and sufficient condition for that is $S(z,+\infty)=+\infty$ for some $z>0$ (see, for instance, Karlin and Taylor[17]).

   Considering $z>0$, the scale measure of a neighborhood $(z,+\infty)$ of $+\infty$ is

$$\begin{aligned} S(z,+\infty) &= \int_z^{+\infty} CX^{-\alpha+\beta E-1}\exp\left\{\gamma(X-(K+A))^2\right\}dX \\ &= C\int_z^{+\infty}\exp\left\{-(\alpha-\beta E+1)\ln X+\gamma(X-(K+A))^2\right\}dX=+\infty, \end{aligned}$$

   because $\gamma>0$ and so $-(\alpha-\beta E+1)\ln X+\gamma(X-(K+A))^2\to+\infty$ as $X\to+\infty$.

   Since the coefficients of (6) are C^1 functions, one knows that the solution exists and is unique up to a possible explosion time. In this case, explosion has a zero probability of occurring due to the nonattractiveness of $+\infty$, and so the solution exists and is unique for all $t\geq 0$.
2. Let $\alpha-\beta E<0$. We show that the boundary $X=0$ is attractive. A necessary and sufficient condition is that $S[0,z)<+\infty$ for some $z>0$ (see, for instance, Karlin and Taylor[17]).

   The scale measure $S(0,z]$ of a small neighborhood of the zero boundary is, assuming that $0<z<K+A$ (note that $K+A>0$),

$$S(0,z]=\int_0^z CX^{-\alpha+\beta E-1}\exp\left\{\gamma(X-(K+A))^2\right\}dX\leq C\exp\left\{\gamma(K+A)^2\right\}\int_0^z X^{-\alpha+\beta E-1}dX<+\infty.$$

3. Assume $\alpha-\beta E\geq 0$. We show that the boundary $X=0$ is nonattractive. A necessary and sufficient condition is that $S[0,z)=+\infty$ for some $z>0$ (see, for instance, Karlin and Taylor[17]).

   In fact,

$$S(0,z]=\int_0^z CX^{-\alpha+\beta E-1}\exp\left\{\gamma(X-(K+A))^2\right\}dX\geq C\exp\left\{\gamma(z-(K+A))^2\right\}\int_0^z X^{-\alpha+\beta E-1}dX=+\infty.$$

■

We will now show the existence of a stochastic equilibrium when $\alpha-\beta E>0$.

Proposition 2. *Let $\alpha-\beta E>0$ with $\alpha=\frac{2rA}{\sigma^2(A-K)}-1$, $\beta=\frac{2q}{\sigma^2}$, $\gamma=\frac{r}{\sigma^2K(K-A)}$ and $-K<A<0$. Then $M(0,+\infty)<+\infty$, the process is ergodic and converges in distribution to a r.v. X_∞ with probability density function, called the stationary density, given by*

$$p(X)=\frac{m(X)}{\int_0^{+\infty}m(z)dz}=\frac{X^{\alpha-\beta E-1}\exp\{-\gamma(X-(K+A))^2\}}{\int_0^{+\infty}z^{\alpha-\beta E-1}\exp\{-\gamma(z-(K+A))^2\}dz},\quad 0<X<+\infty. \tag{7}$$

Proof. Since both boundaries are nonattractive, we just have to show that the scale measure is finite, that is, $M(0,\infty)=\int_0^{+\infty}m(z)dz<+\infty$ (see Gihmann and Skorohod[18]). One can write

$$M(0,\infty)=M_1+M_2+M_3=\int_0^{L_1}m(z)dz+\int_{L_1}^{L_2}m(z)dz+\int_{L_2}^{+\infty}m(z)dz, \tag{8}$$

with $0<L_1<L_2<L_3<+\infty$ and $L_3>\max\{1,K+A\}$. We will show that each of the three integrals in (8) is finite. Since $\alpha-\beta E>0$, we have

$$M_1=\int_0^{L_1}DX^{\alpha-\beta E-1}\exp\left\{-\gamma(X-(K+A))^2\right\}dX\leq D\int_0^{L_1}X^{\alpha-\beta E-1}dX=+\infty.$$

Let η be positive and larger than $\alpha - \beta E - 1$. Then, for $X > L_3$, we have $(\alpha - \beta E - 1)\ln X < \eta \ln X < \eta X$. Putting $\mu = K + A + \frac{\eta}{2\gamma}$ and $\nu = \frac{1}{\sqrt{2\gamma}}$, we have

$$\begin{aligned} M_3 &= \int_{L_3}^{+\infty} DX^{\alpha-\beta E-1} \exp\left\{-\gamma(X-(K+A))^2\right\} dX \\ &= D\int_{L_3}^{+\infty} \exp\left\{(\alpha-\beta E-1)\ln X - \gamma(X-(K+A))^2\right\} dX \\ &\leq D\int_{L_3}^{+\infty} \exp\left\{\eta X - \gamma(X-(K+A))^2\right\} dX \\ &= D\exp\left\{\eta\left(K+A+\frac{\eta}{4\gamma}\right)\right\}\sqrt{\frac{\pi}{\gamma}}\int_{L_3}^{+\infty}\frac{1}{\nu\sqrt{2\pi}}\exp\left\{-\frac{(X-\mu)^2}{2\nu^2}\right\} dX \\ &\leq D\exp\left\{\eta\left(K+A+\frac{\eta}{4\gamma}\right)\right\}\sqrt{\frac{\pi}{\gamma}} < +\infty, \end{aligned}$$

because the integrand of the last integral is the p.d.f. of a Gaussian r.v. with mean μ and variance ν^2.

As for

$$M_2 = \int_{L_1}^{L_2} m(X)dX,$$

it is finite since m is a continuous function in the close interval $[L_1, L_2]$. ∎

From Propositions 1 and 2, and since we wish to avoid extinction and to insure a stochastic equilibrium with a stationary density, we will assume from now on that $\alpha - \beta E > 0$ which, since $E \geq 0$, is equivalent to $0 \leq E < \frac{r}{q}\left(\frac{A}{A-K} - \frac{\sigma^2}{2r}\right)$. The stationary density is given by (7).

The first and second moments of X_∞ are obtained, respectively, as

$$\mathbb{E}[X_\infty] = \int_0^{+\infty} xp(x)dx = \frac{I_1(E)}{I_0(E)}$$

and

$$\mathbb{E}[X_\infty^2] = \int_0^{+\infty} x^2p(x)dx = \frac{I_2(E)}{I_0(E)},$$

where

$$I_j(E) = \int_0^{+\infty} z^{\alpha-\beta E+j-1} \exp\left\{-\gamma(z-(K+A))^2\right\} dz.$$

The steady-state optimization problem is similar to the logistic case without Allee effects (see, for instance, Brites and Braumann[7,10]) and consists in maximizing the expected sustainable profit per unit time, that is, to determine

$$\max_{E\geq 0} \mathbb{E}[\Pi_\infty] = \max_{E\geq 0}\left\{\left(p_1q\frac{I_1(E)}{I_0(E)} - c_1\right)E - \left(p_2q^2\frac{I_2(E)}{I_0(E)} + c_2\right)E^2\right\},$$

in case there is a maximum in the admissible range $0 \leq E < \frac{r}{q}\left(\frac{A}{A-K} - \frac{\sigma^2}{2r}\right)$.

The optimal sustainable effort, denoted by E^{**}, can be obtained by solving the equation $d\mathbb{E}[\Pi_\infty]/dE = 0$ such that the solution satisfies $d^2\mathbb{E}[\Pi_\infty]/dE^2 < 0$, which requires numerical methods. Finally, the optimal expected sustainable profit per unit time is given by

$$\mathbb{E}[\Pi_\infty^{**}] = \left(p_1q\frac{I_1(E^{**})}{I_0(E^{**})} - c_1\right)E^{**} - \left(p_2q^2\frac{I_2(E^{**})}{I_0(E^{**})} + c_2\right)E^{**2},$$

and the corresponding sustainable expected population is given by

$$\mathbb{E}[X_\infty^{**}] = \frac{I_1(E^{**})}{I_0(E^{**})}.$$

4 | COMPARISON OF POLICIES

Profit comparisons between the optimal variable effort policy and the optimal sustainable policy with constant effort, both considering a logistic-like growth model with weak Allee effects, are very similar to the comparisons made in the case of the logistic model without Allee effects. Following Brites and Braumann,[8] let

$$\Pi^*(t) = (p_1 q X(t) - c_1)E^*(t) - (p_2 q^2 X^2(t) + c_2)E^{*^2}(t)$$

and

$$\Pi^{**}(t) = (p_1 q X(t) - c_1)E^{**} - (p_2 q^2 X^2(t) + c_2)E^{**^2}$$

be, respectively, the profit per unit time under the optimal variable effort $E^*(t)$ and the profit per unit time under the optimal sustainable effort E^{**}. To perform comparisons we define the expected accumulated discounted profit, for both policies, as

$$V^* := \mathbb{E}_{0,x}\left[\int_0^T e^{-\delta\tau}\Pi^*(\tau)d\tau\right] \quad \text{and} \quad V^{**} := \mathbb{E}_{0,x}\left[\int_0^T e^{-\delta\tau}\Pi^{**}(\tau)d\tau\right]. \tag{9}$$

To compute V^* and V^{**}, we resort to Monte Carlo simulations of the population, based on an Euler scheme and a 1000 sample paths, and obtaining the corresponding efforts and profits. We have used realistic biological and economic parameters from the Pacific halibut (*Hippoglossus hippoglossus*) based on Hanson and Ryan.[13] Some parameters, for which no information was available, were borrowed from similar studies. The full list of parameters, also used to run the simulations for the logistic model without Allee effects, is shown in Table 1, which also presents (last six lines) the values used for the application of the Crank-Nicolson discretization scheme applied to solve the HJB equation given by (5). The time and space grid was designed with $n = 150$ intervals for time (with a time step of $\Delta t = 4$ months) and with $m = 75$ intervals for the state space (with a space step $\Delta x = 2.15 \times 10^6$ kg).

The resulting profit values (9) using the parameters presented in Table 1 are shown in Table 2. We have considered five scenarios S_{A1} to S_{A5} corresponding to different values of A. The logistic model without Allee effects corresponds to $A \to -\infty$ and is labeled as scenario S_0.

From Table 2 one can see that scenario S_{A1}, corresponding to a quite extreme $A = -0.10K$, has a catastrophic behavior in terms of profit when we apply the optimal sustainable policy. In fact, the profit reduction is 61.9% in comparison with the optimal policy. Scenarios S_{A2} to S_{A5} correspond to consider weaker Allee effects and we might see that the profit values under weak Allee effects are lower than in the model without Allee effects, corresponding to the base scenario S_0, but are approaching the S_0 values as A decreases. Of course, in the limit, one would actually reach S_0, corresponding to $A = -\infty$. The same happens with the profit differences between the optimal variable policy and the optimal sustainable policy (Δ values). We notice that, with a few exceptions, the standard deviations have very small variations across the various scenarios.

Figures 2 and 3 show, respectively, for scenarios S_0 and S_{A3}, what could happen when applying the optimal variable effort harvesting policy (left side) and the optimal constant effort sustainable policy (right side), in terms of the evolution, from time $t = 0$ to time $t = T = 50$ years, of the expected population size (top), optimal effort (middle), and profit per unit time (bottom). The black thin lines show, for both policies, one path for the population, effort and profit per unit time, randomly chosen among the 1000 simulated sample paths. The thicker gray lines refer to the mean of the 1000 sample paths (estimating the expected values) and the black dashed lines presents the constant values at steady-state from the sustainable policy given by the expressions at the end of Section 3. For other scenarios, sample trajectories and means have qualitatively similar behaviors and, therefore, are not shown.

TABLE 1 Parameter values used in the simulations

Item	Description	Value	Unit
r	Intrinsic growth rate (a)	0.71	year^{-1}
K	Carrying capacity (a)	80.5×10^6	kg
q	Catchability coefficient (a)	3.30×10^{-6}	SFU^{-1}year^{-1}
$E_{\min}$	Minimum fishing effort (b)	0	SFU
$E_{\max}$	Maximum fishing effort (b)	$0.7r/q$	SFU
A	Allee parameter	(c)	kg
σ	Strength of environmental fluctuations (b)	0.2	year$^{-1/2}$
x	Initial population size (b)	$0.5K$	kg
δ	Discount factor (b)	0.05	year^{-1}
p_1	Linear price parameter (a)	1.59	\$kg^{-1}
p_2	Quadratic price parameter (b)	5×10^{-9}	\$year · kg^{-2}
c_1	Linear cost parameter (a)	96×10^{-6}	\$SFU^{-1}year^{-1}
c_2	Quadratic cost parameter (a)	0.10×10^{-6}	\$SFU^{-2}year^{-1}
T	Time horizon (b)	50	year
n	Number of time subintervals (b)	150	
Δt	Amplitude of time subintervals (b)	4	month
$X_{\max}$	Maximum population level (b)	$2K$	kg
m	Number of subintervals for the space state (b)	75	
Δx	Amplitude of space state subintervals (b)	2.15×10^6	kg

Note: Parameters with (a) and the definition of SFU (Standardized Fishing Unit) can be found in Hanson and Ryan.[13] Parameters with (b) were borrowed from Brites,[6] from where this table was adapted. (c) Indicates that we use several values for A corresponding to different simulated scenarios (see Table 2).

TABLE 2 Profit values for the scenarios $S_{Ai}, i = 1, \ldots, 5$

Scenario	A	V^*	SD	V^{**}	SD	Δ (%)
S_{A1}	$-0.10K$	218.790	44.700	83.405	7.397	−61.9
S_{A2}	$-0.25K$	248.307	45.209	186.924	25.281	−24.7
S_{A3}	$-0.50K$	277.986	46.088	237.896	35.734	−14.4
S_{A4}	$-0.75K$	296.141	44.655	261.851	36.193	−11.6
S_{A5}	$-0.95K$	307.457	43.234	276.037	36.206	−10.2
S_0	$-\infty$	413.586	38.322	396.424	34.948	−4.1

Note: Besides the expected values, we also present the standard deviations (SD). The percent relative difference between both policies is denoted by Δ. Currency values are in million dollars. For comparison purposes, we show the information for the basic scenario S_0 of the logistic model without Allee effects.

The optimal policy with variable effort presents less variability in terms of the population size when compared with the alternative policy. The opposite occurs if we look to the effort and profit sample paths, which show huge and frequent variations, ranging from harvesting with maximum effort to no harvesting at all. The latter case is, indeed, a major problem since it foresees (sometimes long) periods where the fishery is closed, implying social costs as unemployment and subsidies. In addition, abrupt and frequent changes in effort are not compatible with the logistic of fisheries, as said in Section 1. These are some of the shortcomings of the optimal variable effort policy, which occur with and without Allee effects. On the contrary, the optimal sustainable policy with constant effort does not suffer from such shortcomings, has

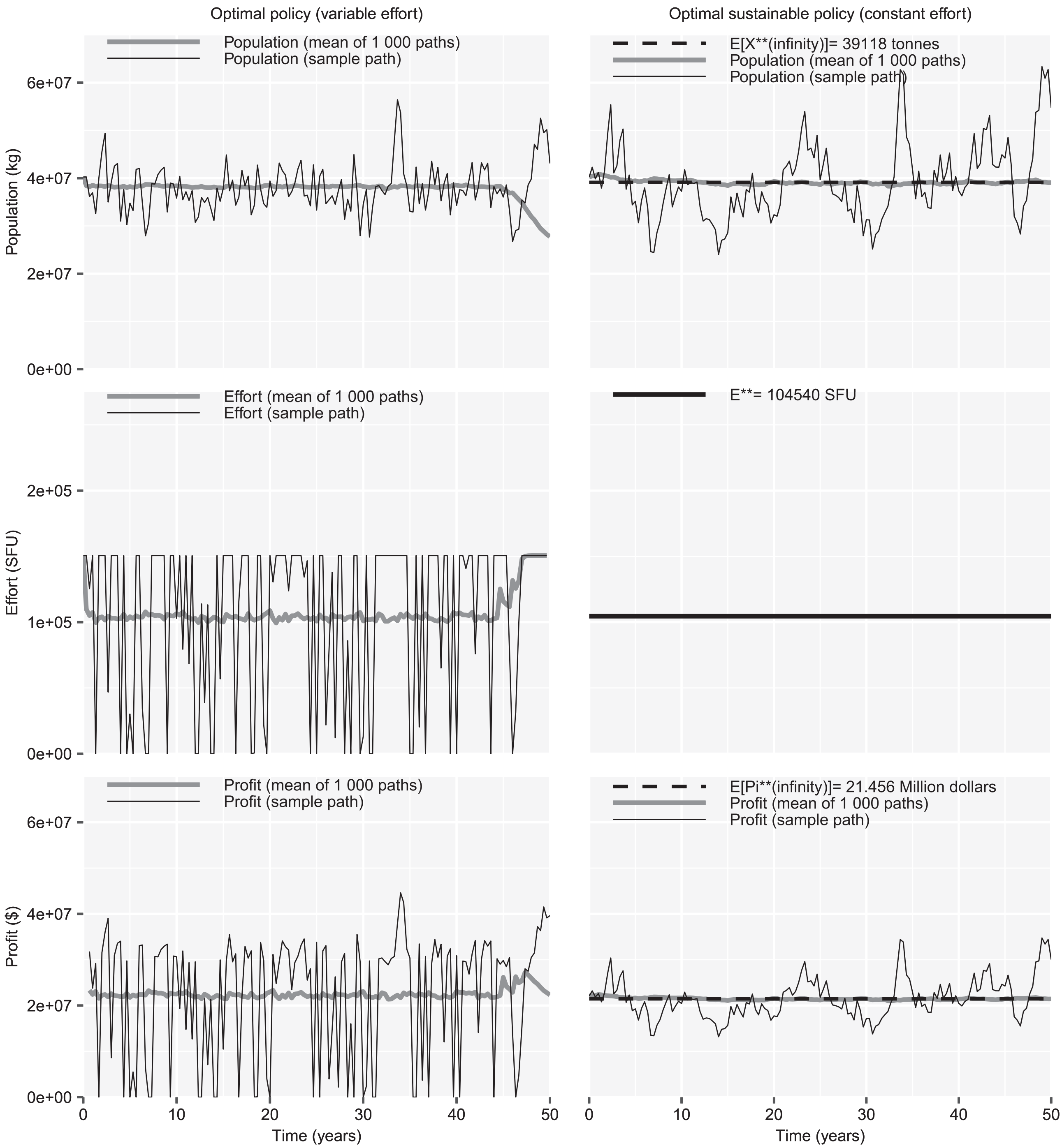

FIGURE 2 Mean and randomly chosen sample path for the population, the effort and the profit per unit time for the application of the logistic model without Allee effects (scenario S_0). The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side. Image adapted from Brites and Braumann[7]

the advantage of being easily applicable, drives the population to a stationary regime when $t \to +\infty$ and there is no need to estimate population sizes at each time instant (which is mandatory for the optimal policy with variable effort).

Furthermore, in typical situations, such as the absence of Allee effects, there is only a small reduction in profit compared with the optimal variable effort inapplicable policy. Considering weak mild Allee effects causes higher, but still small, profit differences. Allee effects reduce population growth and decreases the optimal efforts and profits for both policies. These effects become more noticeable when the Allee effects become more intense.

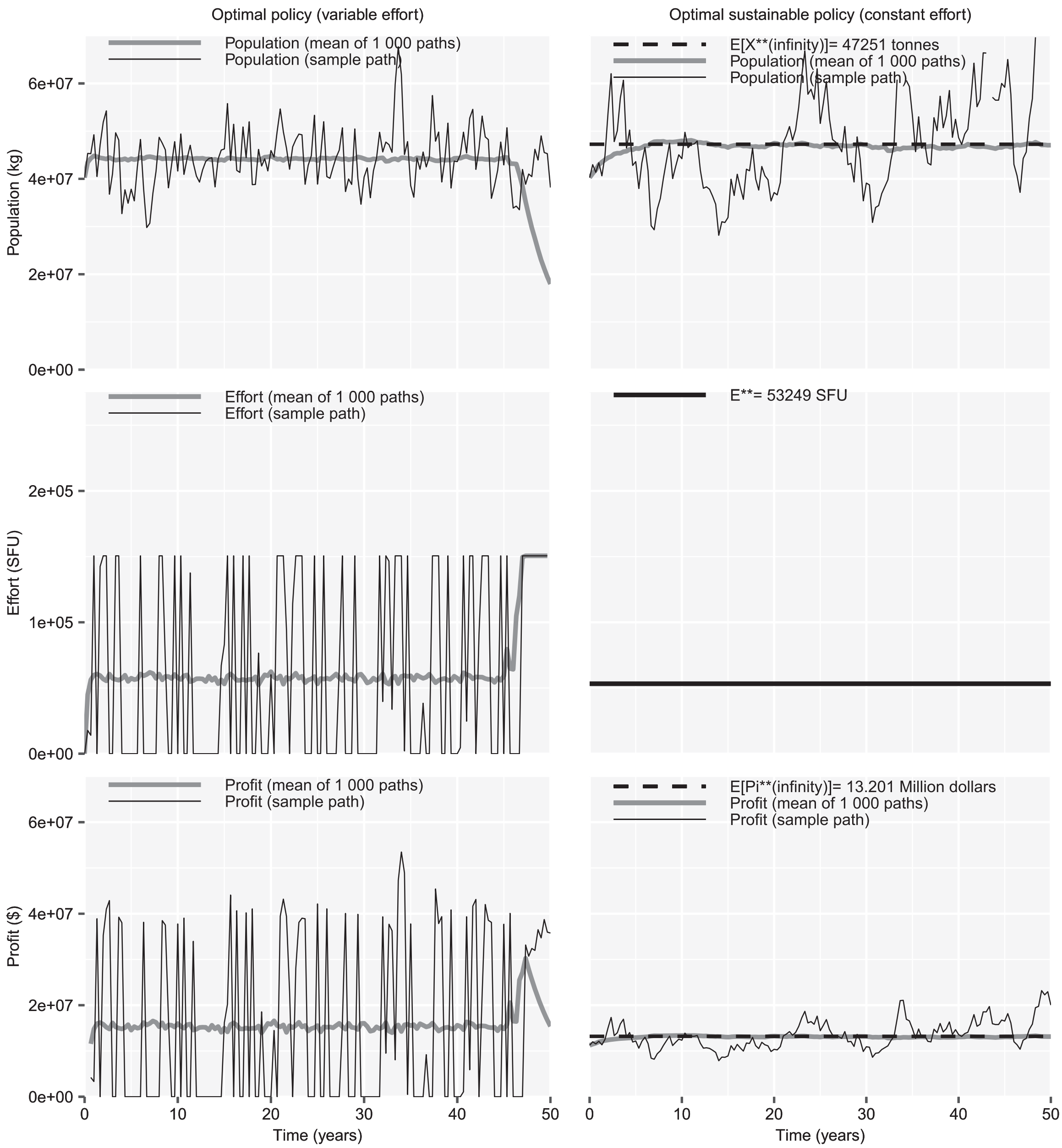

FIGURE 3 Mean and randomly chosen sample path for the population, the effort and the profit per unit time for the application of the logistic-like model with Allee effects (scenario S_{A3}). The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side

5 | CONCLUSIONS

In this article we have worked with a SDE logistic-type population growth model under the influence of weak Allee effects. For this growth model, we have formulated the problems of the optimal variable effort policy and of the optimal constant effort sustainable policy. For the constant effort model we showed that, if the effort is not too high, namely if $0 \leq E < \frac{r}{q}\left(\frac{A}{A-K} - \frac{\sigma^2}{2r}\right)$, the state space boundaries are nonattractive and there is a stationary density for the population size, for which we have found an expression.

Both optimal policies were applied for the realistic parameters values used in the basic scenario S_0 (the basic scenario of the logistic model without Allee effects). To see the influence of the weak Allee effects when comparing both policies, we have simulated five scenarios with variations on the Allee parameter A. We have seen that, as A becomes smaller, the Allee effects have less influence on both policies and, therefore, the policies tend to behave as in the scenario without Allee effects. When A increases (approaching zero), the Allee effects become more pronounced and imply huge differences in terms of profit values when comparing both harvesting policies; the profit becomes, for both types of policies, substantially lower than in the model without Allee effects.

So, although the logistic model (without Allee effects) is the common paradigm in fishery applications, the possible presence of Allee effects should be checked since they may, depending on their strength, have a considerable impact in profit and in designing appropriate fishing policies.

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