Around Hyperbolic Conservation Laws & Vanishing Capillarity-Viscosity Methods

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August 28, 2019, Roorkee, India

Acknowledgements

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Abstract

We will comment, review and try to motivate a few issues, perspectives and methods around hyperbolic conservation laws as the open 6th Hilbert problem and physical entropy solutions, smoothing perturbation or non-smooth analysis, and singular vanishing capillarity-viscosity limits.

Simplification or approximation (JC-LeFloch [1998]–JC [2017])

of conservation laws (no strong spacial, temporal inhomogeneities media), e.g.,

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div}\left(\varepsilon\, b_j\big(u, \nabla u\big) + \delta\, g(u) \sum_{k=1}^{d} \partial_{x_k} c_{jk}\big(g(u)\, \nabla u\big)\right)_{1 \le j \le d}$$

by hyperbolic models $(\varepsilon, \delta \searrow 0$ if $\delta \le K\varepsilon^{\gamma})$

$$\partial_t u + \operatorname{div} f(u) = 0.$$

Real-life mechanisms...

A good bad-idea!:

Hilbert's 6th problem
(Slemrod [2013], from Boltzmann to Euler)

4. Implications of Gorban and Karlin's summation for Hilbert's 6th problem

The implication of the exact summation of C–E by Gorban and Karlin now becomes clear. The whole issue may be seen in Eq. (11), the energy balance. If we put the Knudsen number scaling into (11), the coefficient α is actually a term $\alpha_0\varepsilon^2$ and to recover the classical balance of energy of the Euler equation would require the sequence

$$\boxed{\varepsilon^2\rho^\varepsilon\partial_i\rho^\varepsilon\partial_i\rho^\varepsilon \to 0}$$

in the sense of distributions as $\varepsilon \to 0$. This would require a strong interaction with viscous dissipation. The natural analogy is given by the use of the KdV–Burgers equation:

$$\boxed{u_t + uu_x = \varepsilon u_{xx} - K\varepsilon^2 u_{xxx}} \tag{12}$$

where at a more elementary level we see the competition between viscosity and capillarity. The result in (12) is known but far from trivial. Specifically in the absence of viscosity we have the KdV equation

$$u_t + uu_x = -K\varepsilon^2 u_{xxx} \tag{13}$$

and we know from the results of Lax and Levermore [9] that as $\varepsilon \to 0$ the solution of (13) will not approach the solution of the conservation law

$$u_t + uu_x = 0 \tag{14}$$

after the breakdown time of smooth solutions of (14). On the other hand, addition of viscosity which is sufficiently strong, i.e. K sufficiently small in (12), will allow passage as $\varepsilon \to 0$ to a solution of (14). This has been proven in the paper of Schonbek [10]. So, the next question is whether we are in the Lax–Levermore case (13) or the Schonbek case (12) with K sufficiently small. In my paper [11] I noted the C–E summation of Gorban and Karlin for the Grad 10-moment system leads to a rather weak viscous dissipation, i.e. Eqs. (5.10), (5.11) of [11]. At the moment, this is all we have to go on and I can only conclude that things are not looking too promising for a possible resolution of Hilbert's 6th problem. It appears that in the competition between viscosity and capillarity (mathematically, dissipation of oscillation versus generation of oscillation), capillarity has become a very dogged opponent, and the capillarity energy will not vanish in the limit as $\varepsilon \to 0$. Hilbert's hope may have been justified in 1900, but as a result of the work of Gorban, Karlin, Lax, Levermore, and Schonbek, I think that serious doubts are now apparent.

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Seminal work (Schonbek [1982])

Generalized KdV-B equation

$$u_t + f(u)_x = \varepsilon\, u_{xx} - \delta\, u_{xxx}$$

solutions converges as $\varepsilon, \delta \searrow 0$ if $\delta \leq K\varepsilon^\gamma$ to a weak (non necessarily entropic) solution of

$$u_t + f(u)_x = 0$$

- Kružkov solutions only for convex fluxes $f(\cdot)$;
- viscosity and capillarity are linear[1];
- many restrictions: 1-D, specific L^p and m growth flux[2];
- $\delta = o(\varepsilon^2)$ if $f(u) = u^2/2$ but $\mathcal{O}(\varepsilon^3)$ for general quadratic flux.

[1] Bedjaoui-JC-Mammeri [2015]
[2] JC-LeFloch [1998]

Paradigm breaking (Perthame-Ryzhic [2007])

Sharp balance in KdV-B could be $\delta = o(\varepsilon^{1})$?
(True for the Riemann problem considering the ε, δ-limit of travelling waves.)

Paradigm breaking (Truskinovsky [1993])

Examples in phase transition problems of physical nonclassical solutions (considering dispersive terms).

Singular limits

KdV-Burgers equation

$$u_t + (u^2/2)_x = \varepsilon\, u_{xx} - \delta\, u_{xxx}$$

- ($\delta = 0$) Burgers' equation ("vanishing viscosity method"; Kružkov [1970])

  $$u_t + (u^2/2)_x = \varepsilon\, u_{xx}$$

  solutions converge in a strong topology

- ($\varepsilon = 0$) KdV equation ("zero-dispersion limit"; Lax-Levermore [1983])

  $$u_t + (u^2/2)_x = -\delta\, u_{xxx}$$

  solutions do not converge in a strong topology (oscillatory effect of capillarity).

Simplification or approximation

$$u_t + f(u)_x = \varepsilon\, b(u_x)_x - \delta\, g(u_{xx})_x$$

- simplified (as $\varepsilon, \delta \ll 1$) neglecting small scale mechanisms ("ε and δ terms are spurious")

or

- taking the limit (as $\varepsilon, \delta \searrow 0$)

$$\partial_t u + f(u)_x = 0.$$

Can we do it?

Yes, if dissipation dominates dispersion, but otherwise. . .

Singular limits

And, can we recover as such a limit the physical but nonclassical solutions (in phase transition regime)?

Because of 'singular limits' and of 'nonuniqueness'[3]

- classification is a practical problem (about failure, reliability and integrity domains);
- theoretical/applied points of view of approaching/approached equations conduct us to a dilemma (as 'spurious terms' are not modelled).

[3] Numerics is hopeless.

We must be concerned with

- a proof/criteria for a "vanishing viscosity-capillarity method" relying on
  - the well-posedness of the genKdV-B equations (dispersive techniques),
  - the convergence of their solutions (DiPerna's measure-valued solution techniques);
- the behaviour/selection of the
  - right models,
  - right solutions.

Paradigm breaking (Brenier-Levy [1999])

The "pure-dispersive" equation

$$u_t + (u^2/2)_x = -\delta(u_{xx}^2)_x$$

exhibits a dissipative behaviour.

Conjecture: there exist equations of dissipative KdV-type (open[4]).

[4] Bedjaoui-JC [2012] & Bedjaoui-JC-Mammeri [2014, 2015, 2016]

Real-life problems

Dissipative or dispersive linear terms in the equations becomes from the assumption of small gradients and hessians when the continuum equations are derived from the physical microscopic systems and the flux functional expansion in polynomials is truncated.

Linear flux gradient or hessian relations provides infinite flux responses to sharp interfaces, while the physical response is clearly finite.

Consider the saturating mechanism of dissipation at infinity[5] given by

$$b(u_x) = \frac{u_x}{\sqrt{1+u_x^2}}.$$

[5] Introduced by Rosenau; references within Goodman-Kurganov-Rosenau [1999].

In between 2011 and 2015, several authors showed that

- the associated elliptic problem has, for small values of ε, uncountable discontinuous equilibria (which are numerically stable);
- the pure-dissipative ($\delta = 0$) Burger-Rosenau's equation blows up in finite time in u_x.

Issues:

- What is a good definition of weak solution here?
- Still, those solutions converge to the hyperbolic solution?

(Work in progress...)

Dhanyawad!!!