# On the Restriction of the Optimal Transportation Problem to the set of Martingale Measures with Uniform Marginals

Daryl Allen Saddi[a], Jose Maria L. Escaner IV[a] and Jorge Salazar[b]
[a]University of the Philippines Diliman, [b]Universidade de Evora

**Abstract**

One of the fundamental problems in mathematical finance is the pricing of derivative assets such as options. In practice, pricing an exotic option, whose value depends on the price evolution of an underlying risky asset, requires a model and then numerical simulations. Having no a priori model for the risky asset, but only the knowledge of its distribution at certain times, we instead look for a lower bound for the option price using the Monge-Kantorovich transportation theory. In this paper, we consider the Monge-Kantorovich problem that is restricted over the set of martingale measure. In order to solve such problem, we first look at sufficient conditions for the existence of an optimal martingale measure. Next, we focus our attention on problems with transports which are two-dimensional real martingale measures with uniform marginals. We then come up with some characterization of the optimizer, using measure-quantization approach.

**Keywords:** martingale measure, $U_n$-quantization, uniform marginals, bistochastic matrices

## 1 Introduction

The origins of theory of optimal transportation can be traced back around 1780's, in France, when Gaspard Monge proposed a problem, which in modern terms can be stated as follows: Given two densities, $f$ and $g$ in $\mathbb{R}^n$, we want to find a map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\int_A g(y)\,dy = \int_{T^{-1}(A)} f(x)\,dx \qquad \text{for any Borel set } A \subset \mathbb{R}^n. \tag{1}$$

We then want to minimize the quantity

$$\int_{\mathbb{R}^n} |T(x) - x| f(x)\,dx$$

among all maps $T$ that satisfy (1).

One can think of $x$ as a mass particle, while $T$ describes where this particle is being transported (that we choose in an optimal way). The integrand above can be thought of as the cost of transporting $x$ to $T(x)$. One can further generalize the above problem to: Given two measure spaces $(\mathscr{X}, \mu)$ and $(\mathscr{Y}, \nu)$, we are interested on a measurable map $T : \mathscr{X} \to \mathscr{Y}$ that sort of transforms the measure $\mu$ into $\nu$ in this manner

$$\nu(B) = \mu(T^{-1}(B)) \qquad \text{for any measurable set } B \subset \mathscr{Y}$$

$$\text{or equivalently,} \int_{\mathscr{Y}} \phi d\nu = \int_{\mathscr{X}} (\phi \circ T) d\mu \qquad \text{for every measurable function } \phi : \mathscr{X} \to \mathbb{R}.$$

Such maps that satisfy the above condition are called *transport maps.* So, given a cost function $c : \mathscr{X} \times \mathscr{Y} \to \mathbb{R} \cup \{\infty\}$, we have the Monge problem given by

$$\min_{\{T:\ T_{\#}\mu = \nu\}} \left\{ \int_{\mathscr{X}} c(x, T(x)) d\mu \right\} \tag{2}$$

If a minimizer $T^*$ exists, then we call such map an *optimal transport map.*

Unfortunately, this problem of Monge is hard to solve due to a number of reasons - including the non-linearity of the problem, as well as the possibility of the non-compacteness of the set of admissible transport maps. Furthermore, an easy example for the non-existence of a transport map is when we need to map a space with finite number of mass points to a space with a continuous measure. It is possible that due to these reasons, Monge's problem remained unsolved for decades.

Then in 1942, a Russian mathematician by the name of Leonid Kantorovich proposed, unknowingly, a relaxation of Monge's transport problem, wherein masses are allowed to be split. He proposed on working with measures on the product space having appropriate marginals, which he called as *transference* or *transport plans*. Unlike with the transport maps of Monge's problem, transference plans always exist, and optimality can be achieved with just mild conditions. In 1975, Kantorovich won the Nobel Prize in Economics for his contribution on the development of linear programming.

In the field of mathematical finance and economics, martingale processes are often used in pricing derivative assets. Normally, in modeling an arbitrage-free financial asset, one uses a martingale or a random process that can be transformed into a martingale via a change of measure. This new measure is usually referred to as a *risk-neutral measure*. Martingales are vital in non-arbitrage pricing since the martingale property of an asset is equivalent to not being able to get or create arbitrage through trades in that asset.

The study on the set of martingale measures first came up as an application of the Theory of Optimal Transportation [10]. From then on, authors ([1] [3]) have adapted it to exotic option pricing. Furthermore, such papers are focused on the existence and characterization of the martingale measure that optimizes their desired pay-off function. Through the use of Riesz-Kakutani theorem, one can see that the space of measures over $\mathbb{R}^2$ is infinitely dimensional, thus in order to investigate the properties of the set of martingale measures having fixed marginals, we shall first apply a discrete approximation on the marginal measures, where we shall use the so called $U_n$-quantization, proposed by David Baker on his PhD dissertation [4]. The novelty of this paper is we shall transforming an optimization problem over an infinite dimensional set into a linear programming problem via the use of such quantization.

# 2 Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $X$, a random variable on that space. $X$ induces a new the probability space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X)$, with $\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B))$ for all $B \in \mathscr{B}(\mathbb{R})$. Here, $\mathbb{P}_X$ is called as the *law* or *distribution* of $X$. We shall use the notation law$(X)$ to denote the law of $X$ or write $X \sim \mathbb{P}_X$. Moreover, the function $F_X : \mathbb{R} \to [0,1]$, such that $F_X(x) = \mathbb{P}_X((-\infty, x])$ is called the *(cumulative) distribution function* of $X$ (and of $\mathbb{P}_X$).

## 2.1 Martingale Measures

Let $X$ and $Y$ be random variables defined over $(\Omega, \mathscr{F}, \mathbb{P})$, such that law$(X) = \mu$ and law$(Y) = \nu$. If $Z$ is a coupling (bivariate random variable) of $X$ and $Y$ then the law$(Z)$ is called a *transport plan*. Clearly, the marginal measures of law$(Z)$ are $\mu$ and $\nu$. The set of transport plans with marginals $\mu$ and $\nu$ is denoted by $\Pi(\mu, \nu)$. The next theorem gives one of the best property of the set of transport plans, which is useful on optimization.

**Lemma 2.1** ([10])**.** *Let $\mathscr{X}$ and $\mathscr{Y}$ be compact. Take $\mu \in \mathscr{P}(\mathscr{X})$ and $\nu \in \mathscr{P}(\mathscr{Y})$. The set $\Pi(\mu, \nu)$ is weakly compact.*

*Proof.* Take a sequence $\gamma_n \subset \Pi(\mu, \nu)$. By the Banach-Alaoglu theorem, the closed Ball$(C(\mathscr{X} \times \mathscr{Y})^*)$ is weak-$^*$ compact, thus there exists a subsequence $\gamma_{k_n}$ that converges weakly to $\gamma \in C(\mathscr{X} \times \mathscr{Y})^*$. Furthermore, by this weak convergence, $\gamma \in \mathscr{P}(\mathscr{X} \times \mathscr{Y})$. What remains to be shown is that $\gamma \in \Pi(\mu, \nu)$.

Indeed, for any $\phi \in C(\mathscr{X})$ and $\psi \in C(\mathscr{Y})$, we have $\displaystyle\int_{\mathscr{X} \times \mathscr{Y}} \phi \, d\gamma_{k_n} = \int_{\mathscr{X}} \phi \, d\mu$ and $\displaystyle\int_{\mathscr{X} \times \mathscr{Y}} \psi \, d\gamma_{k_n} = \int_{\mathscr{X}} \psi \, d\nu$.
Passing to the limit, we have $\displaystyle\int_{\mathscr{X} \times \mathscr{Y}} \phi \, d\gamma = \int_{\mathscr{X}} \phi \, d\mu$ and $\displaystyle\int_{\mathscr{X} \times \mathscr{Y}} \psi \, d\gamma = \int_{\mathscr{X}} \psi \, d\nu$. Which implies that the marginals of $\gamma$ are $\mu$ and $\nu$, respectively. □

We are interested on a specific subset of transport plans. We define the set $\mathscr{M}(\mu, \nu)$ to be a subset of $\Pi(\mu, \nu)$ such that any $\pi \in \mathscr{M}(\mu, \nu)$ satisfies $\mathbb{E}_\pi[Y|X] = X$, $\mu$-almost surely. Any such measure is a called a *martingale measure*.

**Theorem 2.2** ([3])**.** *Let $X$ and $Y$ be random varibles defined over the same space. If law$X = \mu$, law$Y = \nu$ and $Q \in \Pi(\mu,\nu)$. Then, the following are equivalent.*

1. $\mathbb{E}_Q[Y|X] = X$, *$\mu$-almost surely.*
2. $\mathbb{E}_Q[(Y-X)\Delta(X)] = 0$ *for any continuous, bounded* $\Delta$.

**Example 2.3.** *Let $\Omega_1 = \{H, T\}$. Take $\Omega = \Omega_1 \times \Omega_1$, $\mathscr{F} = 2^{\Omega}$ and $P$ be the uniform (discrete) measure on $\Omega$. $X \sim Uniform\{1,3\}$ and $\frac{Y}{2} \sim Binomial(2,0.5)$. $Z = (X,Y) \sim Unif\{(1,0),(1,2),(3,2),(3,4)\}$.*

$$\begin{aligned} So, \quad P(Y=4|X=3) = P(Y=2|X=3) &= 0.5 \\ P(Y=2|X=1) = P(Y=0|X=1) &= 0.5 \\ P(Y=4|X=1) = P(Y=0|X=3) &= 0 \end{aligned}$$

$$\begin{aligned} Then, \quad \mathbb{E}[Y|X=3] &= 4\cdot\frac{1}{2} + 2\cdot\frac{1}{2} = 3 \\ \mathbb{E}[Y|X=1] &= 2\cdot\frac{1}{2} + 0\cdot\frac{1}{4} = 1 \end{aligned}$$

*Hence, $\mathbb{E}[Y|X] = X$ and the distribution of $Z$ is a martingale measure.*

## 2.2 Convex Ordering

The set $\Pi(\mu,\nu)$ is non-empty since it always contains the product measure $\mu\times\nu$. However, the set $\mathscr{M}(\mu,\nu)$ can be empty as demonstrated in the next example.

**Example 2.4.** *Let $X \sim \mathscr{N}(0,\sigma^2)$ and $Y \sim \mathscr{N}(1,\sigma^2)$. Take $P \in \Pi(\mathrm{law}(X), \mathrm{law}(Y))$. Then,*

$$\mathbb{E}_P[(Y-X)\cdot 1] = \int_{\mathbb{R}} (y-x)\,dP(x,y) = \int_{\mathbb{R}} y\,d\,\mathrm{law}(Y) - \int_{\mathbb{R}} x\,d\,\mathrm{law}(X) = 1 \neq 0.$$

*So, any transport plan can not be a martingale measure. Thus, $\mathscr{M}(\mathrm{law}(X), \mathrm{law}(Y)) = \phi$.*

As it turns out, having an additional assumption on the marginal measures leads to the existence of a martingale measure. Let $\mu,\nu \in \mathscr{P}(\mathbb{R})$. We say that $\mu$ is dominated by $\nu$ in (stochastic) *convex order*, denoted by $\mu \preceq_c \nu$, if for all convex function $\phi(x)$,

$$\mathbb{E}_\mu[\phi] \le \mathbb{E}_\nu[\phi]$$

Moreover, two random variables $X$ and $Y$ are said to be in convex order, denoted by $X \preceq_c Y$, if $\mathrm{law}(X) \preceq_c \mathrm{law}(Y)$. The next important theorem, shows that convex ordering in the distributions of the marginal random variables is both sufficient and necessary for the existence of a martingale measure.

**Theorem 2.5** (Strassen [7])**.** *Let $\mu,\nu \in \mathscr{P}(\mathbb{R})$. The set $\mathscr{M}(\mu,\nu)$ is non-empty if and only if $\mu \preceq_c \nu$.*

The next two theorems, in conjuction with Strassen's Theorem, give sufficient conditions for the existence of martingale measures and will be used in the latter sections.

**Lemma 2.6** (Ohlin [8])**.** *Suppose $X$ and $Y$ are random variables with finite and equal means. Let $F$ and $G$ be the distribution functions of $X$ and $Y$, respectively. Then $X \preceq_c Y$ whenever there exists $x_0 \in \mathbb{R}$ such that*

$$F(x) \le G(x),\ \forall x \le x_0 \quad and \quad F(x) \ge G(x),\ \forall x \ge x_0.$$

**Theorem 2.7** ([6])**.** *Let $X$ and $Y$ be random variables defined over the same probability space, such that $\mathrm{law}(X) = \mu$ and $\mathrm{law}(Y) = \nu$. Let $F_X$ and $F_Y$ be the cumulative functions of $X$ and $Y$, respectively, and let $F_X^{-1}$ and $F_Y^{-1}$ be the quantile functions of $X$ and $Y$, respectively. Then, the following are equivalent.*

1. $\mu \preceq_c \nu$

2. $\begin{cases} \mathbb{E}[X] = \mathbb{E}[Y] \\ \int_{-\infty}^{x} F_X(t)dt \leq \int_{-\infty}^{x} F_Y(t)dt\,, \;\; \forall x \in \mathbb{R} \end{cases}$

3. $\begin{cases} \mathbb{E}[X] = \mathbb{E}[Y] \\ \int_{0}^{k} F_X^{-1}(u)du \geq \int_{0}^{k} F_Y^{-1}(u)du,\,, \;\; \forall k \in [0,1] \end{cases}$

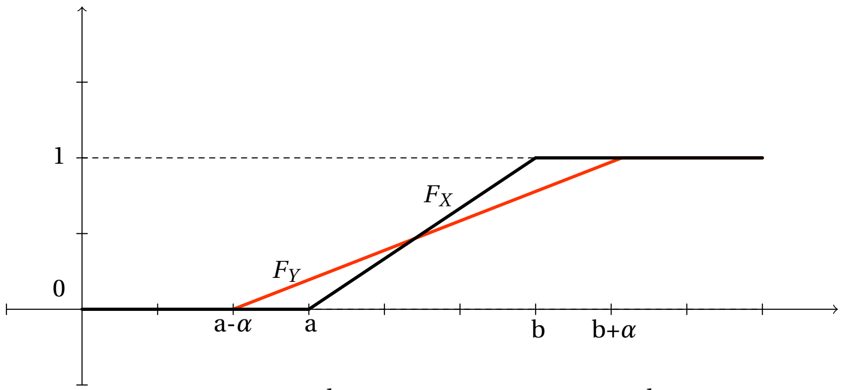
Using these previous results, we then come up with sufficient and necessary conditions for the existence of martingale measures having continuous, uniform marginals.

**Proposition 2.1.** *Suppose $X \sim Unif([a,b])$ and $Y \sim Unif([c,d])$. Then $X \leq_c Y$ if and only if there exists $\alpha \geq 0$ such that $c = a - \alpha$ and $d = b + \alpha$.*

*Proof.* ($\Rightarrow$) By definition of convex ordering, $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathbb{E}[X^2] \leq \mathbb{E}[Y^2]$. Moreover, $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \leq \mathbb{E}[Y^2] - (\mathbb{E}[X])^2 = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \text{Var}(Y)$.

WLOG, assume that $a = 0$. Using the expected value and variance of both $X$ and $Y$, we get that $b = c + d$ and $b^2 \leq (d-c)^2$. If $d \in (0, \hat{b})$, then so does $c$, and thus, $d - c < b$, which is a contradiction to the required variances of $X$ and $Y$. Therefore $d \geq b$. Take $\alpha = d - b$ and so $c = b - d = -\alpha$. Translating $X$ along the real line gives us the desired conclusion.
($\Leftarrow$) We have the following distribution functions:

$$\begin{aligned} F_X(x) &= \frac{x-a}{b-a}\mathbb{1}_{[a,b]}(x) + \mathbb{1}_{(b,\infty)}(x) \\ F_Y(y) &= \frac{y-a+\alpha}{b-a+2\alpha}\mathbb{1}_{[a-\alpha,b+\alpha]}(x) + \mathbb{1}_{(b+\alpha,\infty)}(y) \end{aligned}$$




Note that $F_X \leq F_Y$ for all $x \leq \dfrac{a+b}{2}$ and $F_X \geq F_Y$ for all $x \geq \dfrac{a+b}{2}$. So by Ohlin's Lemma, $X \leq_c Y$. $\square$

## 2.3 Quadratic Cost Function

Consider the classical Monge-Kantorovich Problem

$$\inf_{\gamma \in \Pi(\mu,\nu)} \left\{ \int_{\mathbb{R}^2} c(x,y)d\gamma \right\} \tag{3}$$

If $\gamma^* \in \Pi(\mu,\nu)$ minimizes the integral, then it is called an *optimal transport plan.*

Having $c$ to be the quadratic cost funtion, gives us some interesting results for both the classical and martingale settings.

**Theorem 2.8** (Brenier [2])**.** *Let $X$ and $Y$ be compact subsets of $\mathbb{R}$, and $c(x,y) = \dfrac{(x-y)^2}{2}$. If $\mu$ is absolutely continuous with respect to $\lambda$, then there exist a unique optimal transport plan that solves (3), which is induced by a map $T$. Moreover, this map is given by $T = F_\nu^{-1} \circ F_\mu$.*

For the martingale setting, if $Q \in \mathcal{M}(\mu,\nu)$, then

$$\begin{aligned}
\mathbb{E}_Q\left[(Y-X)^2\right] &= \mathbb{E}_Q\left[Y^2-2XY+X^2\right] \\
&= \mathbb{E}_Q\left[Y^2\right]+\mathbb{E}_Q\left[X^2\right]-2(\mathbb{E}_Q[X])^2-2\mathbb{E}_Q[XY]+2(\mathbb{E}_Q[X])^2 \\
&= Var(X)+Var(Y)-2Cov_Q(X,Y)
\end{aligned}$$

So minimizing the left hand side is equivalent to maximizing $\mathrm{Cov}(X,Y)$.
Using this fact, together with Prop. (2.1) we obtain the following result.

**Proposition 2.2.** *Let $X, Y$ be uniform distributed such that $X \preceq_c Y$. Then*

$$\min_{Q\in\mathcal{M}(\mu,\nu)} \mathbb{E}_Q\left[(Y-X)^2\right] = \left(\sqrt{Var(Y)}-\sqrt{Var(X)}\right)^2 = \frac{\alpha^2}{3}$$

*and that the optimal martingale measure is induced by the map $T(x) = x-\alpha+2\alpha\left(\frac{x-a}{b-a}\right)$.*

# 3 Quantization of Measure

From here on, we only focus on martingale measures over $\mathbb{R}^2$. Even with this streamlining, working with the set of martingale measures with arbitrary marginal measures over $\mathbb{R}$ still proves to be difficult, so we instead approximate the marginal measures and from there form a set of approximate martingale measures.

## 3.1 $U_n$-quantization

Approximating a probability measure by a discrete probability measuring is called quantization of the original measure. For our purposes, we shall be using the $U_n$-quantization proposed by [4]. It is called $U_n$-quantization due to the fact that it produces a uniformly discrete distributed random variable with at most $n$ support points.

Let $X$ be a random variable with law $\mu$, distribution function $F$ and quantile function $F^{-1}$. Given $n \in \mathbb{N}$, the *$U_n$-quantization* of $X$ is the discrete random variable with law $\mu_n$, which is a (discrete) uniform measure with mass points $a_1, a_2, \ldots a_n$, where

$$a_i = n\int_{\frac{i-1}{n}}^{\frac{i}{n}} F^{-1}(u)\,du. \tag{4}$$

Here $\mu_n$ is also referred as the $U_n$-quantization of $\mu$.

**Example 3.1.** *Take $X \sim Exp(0.5)$, so $F(x) = (1-e^{-0.5x})\mathbb{1}_{[0,\infty)}(x)$. Thus $F^{-1}(u) = -2\ln(1-u)\mathbb{1}_{[0,1]}(u)$. Take $n = 4$. Then,*

$$a_1 = 4\int_0^{1/4} -2\ln(1-u)du = 2+4\ln\left(\frac{3\sqrt{3}}{8}\right) \qquad a_3 = 4\int_{1/2}^{3/4} -2\ln(1-u)du = 2$$

$$a_2 = 4\int_{1/4}^{1/2} -2\ln(1-u)du = 2+4\ln\left(\frac{4}{3\sqrt{3}}\right) \qquad a_4 = 4\int_{3/4}^{1} -2\ln(1-u)du = 2+4\ln 2$$

*So $U_4 \sim Uniform\{a_1, a_2, a_3, a_4\}$.*

**Proposition 3.1.** *Suppose $X \sim Unif([a,b])$ then its quantization $U_n(a_1,\ldots,a_n)$ has mass points given by*

$$a_i = a+\frac{b-a}{2n}+\frac{(b-a)(i-1)}{n}, \quad i = 1,\ldots,n.$$

The proof of the above theorem uses the fact that if $X \sim Unif([a,b])$, then its cumulative function is given by

$$F_X(x) = \frac{x-a}{b-a}$$

while its quantile function is given by

$$F_X^{-1}(u) = a+(b-a)u.$$

Then applying the formula from (4), gives the desired mass points.
Next, we present some of the properties of the $U_n$-quantization, which show that this type of approximation is an ideal tool for the purposes of maintaining the existence of marginal measures.

**Lemma 3.2** ($U_n$-quantization preserves the mean)**.** *Let $X \sim \mu$ and has distribution function $F$. If $U(a_1,\ldots,a_n)$ is the $U_n$ - quantization of $X$, then $X$ and $U(a_1,\ldots,a_n)$ have the same mean.*

*Proof.*

$$\begin{aligned} \mathbb{E}\left[U(a_1,\ldots,a_n)\right] &= \sum_{i=1}^{n} a_i \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{n} \cdot n \int_{i-1/n}^{i/n} F^{-1}(u)\, du \\ &= \int_0^1 F^{-1}(u)\, du = \mathbb{E}\left[F^{-1}(U)\right] = \mathbb{E}[X]. \end{aligned}$$

□

**Theorem 3.3.** *$U_n$-quantization preserves convex ordering, that is, if $X \preceq_c Y$ with $U$ and $V$ to be the $U_n$-quantization of $X$ and $Y$, respectively, then $U \preceq_c V$.*

**Theorem 3.4.** *Let $U(a_1,\ldots,a_n)$ be the $U_n$ - quantization of $X$. Then $U(a_1,\ldots,a_n)$ converges to $X$ in the sense of distribution.*

The above two theorems are consequences of the definition of $U_n$-quantization together with (2.7) and (3.2).

### 3.2 Matrix Representation

After applying the $U_n$-quantization to any measure, we can then represent the quantized measure as an $n$-dimensional vector. It would also be of help, if there is an easy way to represent a martingale measure with quantized marginals.

**Theorem 3.5** (Disintegration Theorem)**.** *A measure $Q \in \mathscr{P}(\mathbb{R}\times\mathbb{R})$ can be represented by a measure $\mu \in \mathscr{P}(\mathbb{R})$ and a transition kernel $K(\cdot,\cdot)$ given by*

$$Q(A\times B) = \int_A K(x,B)\, d\mu(x)$$

*for all measurable sets $A, B$ in $\mathbb{R}$.*

In the discrete case, the above simplifies to

$$Q(A\times B) = \sum_{a\in A} \mathbb{P}(Y\in B|X=a)\,\mathbb{P}(X=a).$$

Since we are to fix the marginal measures to be discrete uniform distributions, knowledge of the transport plan is equivalent to the knowledge of its transition probability. Moreover, similar to what is usually done with Markov chains, we can associate every transport plan with a bistochastic matrix $B = (b_{ij})$ that represents the transition kernel, where $b_{ij} = \mathbb{P}(Y = b_j | X = a_i)$.

A non-negative matrix $B = (b_{ij}) \in M_n(\mathbb{R})$ is said to be an $n \times n$ *bistochastic matrix* if

$$\begin{aligned} \sum_{k=1}^{n} b_{ik} &= 1 \quad i = 1,2,\ldots,n \\ \sum_{k=1}^{n} b_{kj} &= 1 \quad j = 1,2,\ldots,n \end{aligned}$$

In general,the an $n$ by $n$ bistochastic matrices would look like

$$\begin{bmatrix} \sum_{i=2}^{n}\sum_{j=2}^{n} a_{ij} - (n-2) & 1-\sum_{i=2}^{n} a_{i2} & 1-\sum_{i=2}^{n} a_{i3} & \dots & 1-\sum_{i=2}^{n} a_{in} \\ 1-\sum_{j=2}^{n} a_{2j} & a_{22} & a_{23} & \dots & a_{2n} \\ 1-\sum_{j=2}^{n} a_{3j} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1-\sum_{j=2}^{n} a_{nj} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix},$$

with each of its entries to be in $[0,1]$.

# 4 Results

Restating (3) for the martingale case, we have

$$\inf_{Q\in\mathcal{M}(\mu,\nu)} \left\{\mathbb{E}_Q\left[c(X,Y)\right]\right\}. \tag{5}$$

If $Q^* \in \mathcal{M}(\mu,\nu)$ minimizes the above expected value, then it is called an *optimal martingale measure*. Solving such problem can be seen as equivalent to solving for the lower bound of an option price, where the pay-off function $c$ depends on the current price of the underlying asset at two different times $t_1$ and $t_2$. $X$ and $Y$ can be seen as the random variables that model the prices of the asset at times $t_1$ and $t_2$, respectively.

We now present our results for the martingale case.

## 4.1 Existence of a Minimizer

**Proposition 4.1.** *Let $X$ and $Y$ be real random variables such that law($X$) $=\mu$, law$Y=\nu$ and that $\mu,\nu$ have compact support. Then the set $\mathcal{M}(\mu,\nu)$ is weakly compact.*

*Proof.* First, let $f$ be a continuous, bounded function and construct the functional $L_f : \Pi(\mu,\nu) \to \mathbb{R}\cup\{\infty\}$ given by

$$L_f(\pi) = \int_{\mathbb{R}^2} f(x)(y-x)d\pi.$$

By the weak-* convergence of in $\Pi(\mu,\nu)$, $L_f$ is continuous.

Thus, by the continuity of $L_f$, the set $L_f^{-1}(\{0\}) = \left\{\pi\in\Pi(\mu,\nu) : \int_{\mathbb{R}^2} f(x)(y-x)d\pi = 0\right\}$ is closed. The set $\mathcal{M}(\mu,\nu)$ is the intersection of all the sets $L_f^{-1}(\{0\})$ with $f$ varying across all continuous and bounded functions. So, $\mathcal{M}(\mu,\nu)$ is closed. By lemma 2.1, the set $\Pi(\mu,\nu)$ is weakly compact, which by definition contains $\mathcal{M}(\mu,\nu)$. Thus, $\mathcal{M}(\mu,\nu)$ is also weakly compact. □

If $\mu \preceq_c \nu$ then it guarantees that $\mathcal{M}(\mu,\nu)$ is non-empty and convex, thus the minimizer is always guaranteed under a mild condition.

**Theorem 4.1** (Weierstrass Theorem)**.** *If $f : \mathcal{X} \to \mathbb{R}\cup\{+\infty\}$ is lower semi-continuous and $\mathcal{X}$ is compact, then there exists $\bar{x}\in\mathcal{X}$ such that $f(\bar{x}) = \min\{f(x) : x\in\mathcal{X}\}$.*

**Proposition 4.2.** *Let $\mu,\nu\in\mathcal{P}(\mathbb{R})$ such that $\mu\preceq_c\nu$ and $c:\mathbb{R}^2\to\mathbb{R}$ be a continuous function. Then an optimal martingale measure exist that solves (5).*

*Proof.* Consider the mapping $K : \mathcal{M}(\mu,\nu) \to \mathbb{R}\cup\{\infty\}$ by $K(\gamma) = \int c\,d\gamma$. This mapping is continuous due to the continuity of $c$ and the weak convergence of probability measures. By Theorem 2.1 and Weierstrass Theorem, there exists $\bar{\gamma}\in\mathcal{M}(\mu,\nu)$ such that $K(\bar{\gamma}) = \min\{K(\gamma) : \gamma\in\mathcal{M}(\mu,\nu)\}$. □

Even if an optimizer is guaranteed for a wide class of cost functions and marginal measures, characterizing such optimal martingale measure proves to be difficult. So to have a nice characterization, we shall first restrict the marginal measures to have uniform distributions and then apply some quantization.

## 4.2 Quantized Marginals

**Proposition 4.3.** *Let $X$ and $Y$ be distinct, continuous uniformly distributed random variables such that law$(X) = \mu$, law$(Y) = \nu$ and $X \preceq_c Y$. Let $n \in \mathbb{N}$, and take $\mu_n$ and $\nu_n$ to be the $U_n$-quantization of $\mu$ and $\nu$, respectively. Then, the set of transition kernels corresponding to the elements of $\mathscr{M}(\mu_n, \nu_n)$ is an $(n-1)(n-2)$-dimensional polytope.*

*Proof.* Let $\mu_n$ have mass points $a_1, \ldots, a_n$ while $\nu_n$ have mass points $b_1, \ldots, b_n$, obtained using formula (4). Note that these mass points which are arranged in increasing order. Take $Q \in \mathscr{M}(\mu_n, \nu_n)$. By the disintegration theorem, there exists an $n \times n$ bistochastic matrix $M = (m_{ij})$ corresponding to $Q$ such that $m_{ij} = \mathbb{P}[Y = b_j | X = a_i]$. We then have the following equations satisfied by the entries of $M$.

$$\begin{aligned} \sum_{k=1}^{n} m_{ik} = 1 \quad & i = 1, 2, \ldots, n \quad Eqn's(1) - (n) \\ \sum_{k=1}^{n} m_{kj} = 1 \quad & j = 1, 2, \ldots, n \quad Eqn's(n+1) - (2n) \\ \sum_{k=1}^{n} b_k m_{ik} = a_i \quad & i = 1, 2, \ldots, n \quad Eqn's(2n+1) - (3n) \end{aligned}$$

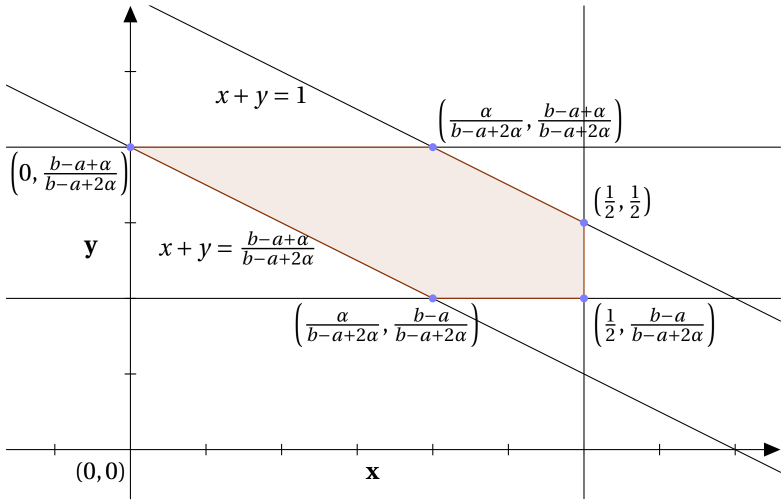
We can show that equation $(2n)$ can be obtained using equations $(1)$ to $(2n-1)$. Moreover, equation $(n+1)$ can also be obtained but now using equations $(n+2), (n+3), \ldots (2n-1), (2n+1), \ldots (3n)$. Due to the linear form of the $a_i$'s and the $b_j$'s, the remaining $3n-2$equations can be shown to be all linearly independent. Hence the dimension of the solution space is $n^2 - (3n-2) = (n-2)(n-1)$. Due to the fact that all the entries of $Q$ are non-negative, the resulting set $\mathscr{M}(\mu_n, \nu_n)$ will be the intersection of the above solution space and of all flat regions having the form $\{e \in \mathbb{R}^{n^2} : e(i) \in [0,1],$ for some $i \leq n^2\}$. □

We further restrict the marginal measures by applying only $U_3$-quantization. By this restriction, together with the previous proposition, we can then simplify (5) from an optimization problem on an infinite dimensional set to a linear optimization problem over a subset of $\mathbb{R}^2$.

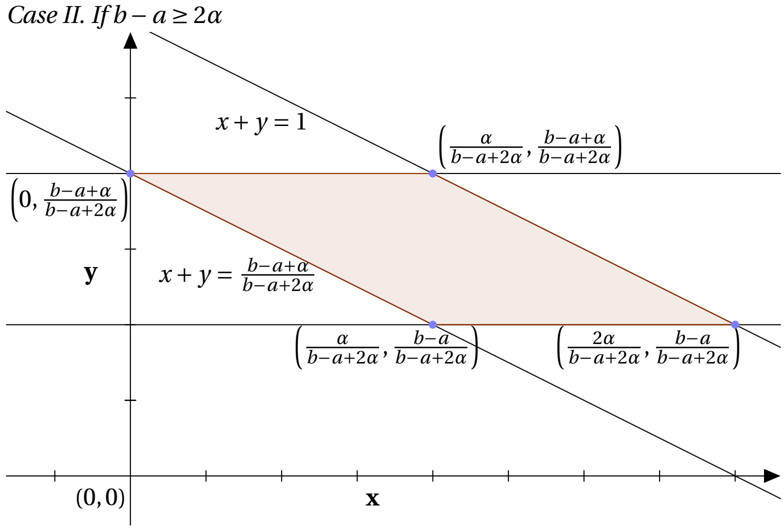
**Proposition 4.4.** *Let $X$ and $Y$ be continuous uniformly distributed random variables such that $X \preceq_c Y$. Moreover, let law$(X) = \mu$ and law$(Y) = \nu$. If $\mu_3$ and $\nu_3$ are the $U_3$-quantization of $\mu$ and $\nu$, respectively, then any martingale measure in $\mathscr{M}(\mu_3, \nu_3)$ can be associated to a matrix of the form*

$$\begin{bmatrix} \dfrac{2b-2a+2\alpha}{b-a+2\alpha} - x - y & 2x + 2y - \dfrac{2b-2a+2\alpha}{b-a+2\alpha} & 1 - x - y \\ x & 1-2x & x \\ y - \dfrac{b-a}{b-a+2\alpha} & \dfrac{2b-2a+2\alpha}{b-a+2\alpha} - 2y & y \end{bmatrix}$$

*where $a, b$ and $\alpha$ are the same as with Proposition (2.1) and that $(x, y)$ lies on the following feasible regions:*
*Case I. If $b - a < 2\alpha$*




*Case II. If* $b-a \geq 2\alpha$




**Proposition 4.5.** *Let $X$ and $Y$ be uniformly continuous random variables such that $X \preceq_c Y$ and $c$ is continuous. Then an optimal martingale measure $Q^*$ for*

$$\min_{Q\in\mathcal{M}(\mu_3,\nu_3)} \mathbb{E}_Q[c(X,Y)]$$

*lies on the boundary of $\mathcal{M}(\mu_3,\nu_3)$.*

**Example 4.2.** *Suppose $X \sim Unif([1,2])$ and $Y \sim Unif([0,3])$. By Proposition (2.1), $X \preceq_c Y$. Let $n=3$ and take $\mu$ and $\nu$ to be the $U_3$-quantization of* $\mathrm{law}(X)$ *and* $\mathrm{law}(Y)$*, resp. Then*

$$\mu = \frac{1}{3}\left[\delta_{7/6} + \delta_{3/2} + \delta_{11/6}\right]$$

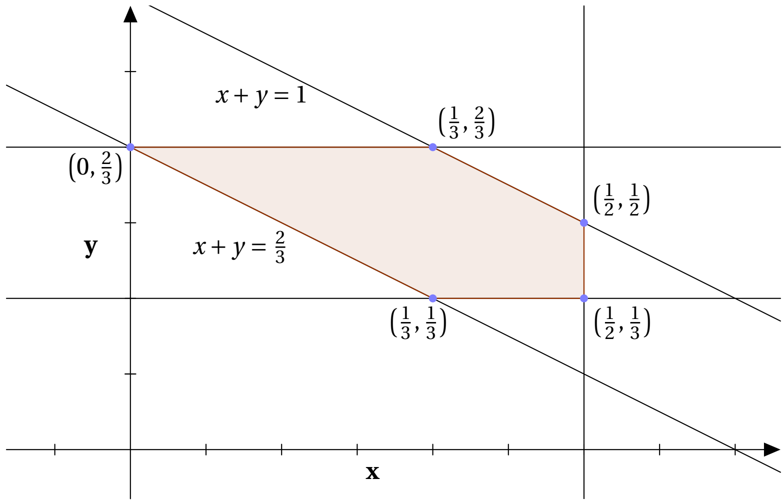
*While,*

$$\nu = \frac{1}{3}\left[\delta_{1/3} + \delta_{3/2} + \delta_{8/3}\right].$$

*Any martingale measure in $\mathcal{M}(\mu,\nu)$ can be represented as*

$$\begin{bmatrix} \frac{4}{3}-x-y & 2x+2x-\frac{4}{3} & 1-x-y \\ x & 1-2x & x \\ y-\frac{1}{3} & \frac{4}{3}-2y & y \end{bmatrix}$$

*Where, $(x,y)$ lies on the region given below:*




*So, for the problem*

$$\min_{Q\in\mathcal{M}(\mu,\nu)} \mathbb{E}_Q\left[(X-Y)^3\right]$$

*we have*

$$\mathbb{E}_Q\left[(X-Y)^3\right]=\sum_{i=1}^{3}\sum_{j=1}^{3}(a_i-b_j)^3Q(\{a_i,b_j\})=\frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3}(a_i-b_j)^3m_{ij}=\frac{2x}{3}+\frac{4y}{3}-\frac{8}{9}$$

*minimizing the right hand side over the feasible region, we shall get the optimal martingale measure $Q^*$ to be associated to the matrix*

$$\begin{bmatrix}\frac{2}{3} & 0 & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ 0 & \frac{2}{3} & \frac{1}{3}\end{bmatrix}$$

*with* $\mathbb{E}_{Q^*}\left[(X-Y)^3\right]=\frac{-2}{9}$.
*Whereas, for the problem*

$$\min_{Q\in\mathcal{M}(\mu,\nu)} \mathbb{E}_Q\left[|X-Y|^3\right]$$

*we have*

$$\mathbb{E}_Q\left[|X-Y|^3\right]=\sum_{i=1}^{3}\sum_{j=1}^{3}|a_i-b_j|^3Q(\{a_i,b_j\})=\frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3}|a_i-b_j|^3m_{ij}=\frac{160}{243}-\frac{70x}{81}$$

*and again, minimizing the right hand side over the feasible region, we get that any martingale measure $Q^*$ that is associated to the matrix*

$$\begin{bmatrix}\frac{5}{6}-y & 2y-\frac{1}{3} & \frac{1}{2}-y\\ \frac{1}{2} & 0 & \frac{1}{2}\\ y-\frac{1}{3} & \frac{4}{3}-2y & y\end{bmatrix}$$

*with* $y\in\left[\frac{1}{3},\frac{1}{2}\right]$ *will be optimal, and that* $\mathbb{E}_{Q^*}\left[(X-Y)^3\right]=\frac{55}{243}$.

# 5 Summary

Under the assumption that the marginal measures are in convex order, we have shown the existence of an optimal martingale measure for (5). Furthermore, using the $U_n$-quantization on continuously uniform random variables, we can came up with discrete marginal measures having $n$ points as support. Since this type of quantization preserves the convex ordering of the original measures, we also came up with a set of martingale measures which can be represented as bistochatic matrices. It was then shown that for this scenario, the set of $n\times n$ matrices which represent martingale measures is $(n-2)(n-1)$-dimensional. Lastly, we used $U_3$-quantization and simplified (5) into a linear programming problem in two variables.

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