Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

Strong generalized synchronization with a particular relationship *R* between the coupled systems

To cite this article: Clara Grácio *et al* 2018 *J. Phys.: Conf. Ser.* **990** 012005

Strong generalized synchronization with a particular relationship R between the coupled systems

Clara Grácio[1], **Sara Fernandes**[2] **and Luís Mário Lopes**[3]

[1]Department of Mathematics, ECT - Universidade de Évora, CIMA-UE
Rua Romão Ramalho, 59, 7000-585 Évora, Portugal
[2]Department of Mathematics, ECT - Universidade de Évora, CIMA-UE
Rua Romão Ramalho, 59, 7000-585 Évora, Portugal
[3]Department of Mathematics, ISEL
Rua Conselheiro Emídio Navarro, 1, 1959-007 Lisboa, Portugal

E-mail: [1]mgracio@uevora.pt, [2]saf@uevora.pt, [3]llopes@adm.isel.pt

Abstract. The question of the chaotic synchronization of two coupled dynamical systems is an issue that interests researchers in many fields, from biology to psychology, through economics, chemistry, physics, and many others. The different forms of couplings and the different types of synchronization, give rise to many problems, most of them little studied. In this paper we deal with general couplings of two dynamical systems and we study strong generalized synchronization with a particular relationship R between them. Our results include the definition of a window in the domain of the coupling strength, where there is an exponentially stable solution, and the explicit determination of this window. In the case of unidirectional or symmetric couplings, this window is presented in terms of the maximum Lyapunov exponent of the systems. Examples of applications to chaotic systems of dimension one and two are presented.

1. Introduction

Chaotic synchronization is a very important phenomenon in many fields involving mathematical, physical, sociological, physiological, biological or other systems [1, 2, 3, 4]. Several different types of synchronization have been studied: complete synchronization [5, 6], phase synchronization [1, 7], lag synchronization [8, 9, 10], generalized synchronization [11, 12], projective synchronization [13, 14], etc. The complete synchronization is the simplest type of synchronization and a lot of work have been produced about it. Much less study have been devoted to one of the most interesting types of synchronous chaotic behavior, the generalized synchronization. When there is generalized synchronization in a coupling of two dynamical systems x_t and y_t, it is established a relationship $y_t = R(x_t)$ between the system states after the transient is finished. Depending on the properties of the relationship R, the synchronization is said to be either strong or weak. Namely, strong synchronization corresponds to a smooth relationship R, while weak synchronization corresponds to a fractal one [17]. Several methods have been suggested to detect generalized synchronization: the method of the mutual false nearest neighbors [11, 15], the auxiliary system method [16], using the conditional Lyapunov exponents [17], the modified system approach [18, 19], the phase tube approach [20, 21]. Nevertheless, most of the studies done focuses on continuous-time systems. Few of them are related to discrete-time systems

[22, 23, 24, 25, 21] and even fewer to bidirectional couplings [20, 26, 21] since generalized synchronization is traditionally introduced for two unidirectionally coupled systems [11, 17, 27].

In this paper we discuss the design of a c-family of couplings in order to achieve generalized synchronization with a desired relationship R between the coupled systems. Similar discussions have already been done for unidirectional couplings or for couplings of continuous-time systems [28, 29, 30], but now we are considering those less studied situations: bidirectional couplings of discrete-time systems. We consider conditions for which a c-family of couplings achieves the referred synchronization and for which we will be able to calculate the R-synchronization window. In fact, only some values of the coupling strength will provide exponentially stable synchronized solutions, defining the referred window.

2. A c-family of couplings that admit generalized synchronization with a particular relationship R

We consider the general coupling of two discrete n-dimensional chaotic dynamical systems x_t and y_t that is widely used in this context [28, 29, 30, 26], given by

$$\begin{cases} x_{t+1} = f(x_t) + c \cdot [F_1(x_t) + F_2(y_t)] \\ y_{t+1} = g(y_t) + c \cdot [G_1(x_t) + G_2(y_t)] \end{cases}, \tag{1}$$

where c is the coupling strength, with values in $[0, 1]$ and f, g, F_1, F_2, G_1 and G_2 are appropriate functions.

The ability of the systems to synchronize depends, not only on the functions F_1, F_2, G_1 and G_2, but also on the coupling strength c. In order to consider couplings where the values of the coupling strength c for which the coupling achieves generalized synchronization with a particular relationship R is not a discrete set, we want to consider couplings that admit this synchronization for all values of c (even if it is not an exponentially stable one). That will allow us to define "synchronization windows" in a similar way that was often done for complete synchronization (either in couplings or in networks [1]). So, we consider the following c-family of couplings that provide a similar frame of analysis for the generalized synchronization with a particular relationship R:

$$\begin{cases} x_{t+1} = f(x_t) + c \cdot [-F_2(R(x_t) + F_2(y_t)] \\ y_{t+1} = g(y_t) + c \cdot [-G_2(R(x_t)) + G_2(y_t)] \end{cases}, \tag{2}$$

with $R \circ f = g \circ R$. In fact, $(x_t, y_t) = (s_t, R(s_t))$ with $s_{t+1} = f(s_t)$ is a solution of (2) for all values of c, since (2) reduces to

$$\begin{cases} s_{t+1} = f(s_t) \\ R(s_{t+1}) = g(R(s_t)) \end{cases}$$

and both equations are verified because $R \circ f = g \circ R$ and $s_{t+1} = f(s_t)$.

We note that the complete synchronization corresponds to $R(u) = u$ and the lag synchronization corresponds to $R(u) = f^{(\Delta t)}(u)$ [17, 19, 31]. Further, if R is a diffeomorphism, then f and g are topologically conjugate maps by the topological conjugacy R. So, a coupling of type (2) of two dynamical systems described by maps that are topologically conjugate by a topological conjugacy R, admits generalized synchronization with a relationship R between the coupled systems.

3. R-synchronization window

Even if a coupling admits generalized synchronization with a particular relationship R, only some values of the coupling strength (or even none) correspond to a coupling that admits a function s_t such that $(x_t, y_t) = (s_t, R(s_t))$ is an exponentially stable solution. So, we consider the following definition.

Definition 1. *For a c-family of couplings that admit generalized synchronization with a particular relationship R, we call R-synchronization window, RSW, the set of values of the coupling strength c for which there is a function s_t such that $(x_t,\, y_t) = (s_t,\, R(s_t))$ is an exponentially stable solution of (1).*

The following proposition establishes conditions for a value of c to belong to the RSW.

Proposition 1. *The R-synchronization window, RSW, of the coupling (2) satisfies*

$$\{c \in [0,1] : \mu_{rs} < 0\} \subset RSW \subset \{c \in [0,1] : \mu_{rs} \leq 0\}$$

where

$$\mu_{rs} = \max_{u_0} \lim_{T\to+\infty} \frac{1}{T} \ln \left| DM^T(s_0) \cdot \frac{u_0}{\|u_0\|} \right|,$$

$$M = g \circ R + c \cdot (G_2 \circ R - R \circ f \cdot F_2 \circ R),$$

$$DM^T(s_0) = DM(s_{T-1}) \cdot DM(s_{T-2}) \cdot \ldots \cdot DM(s_0)$$

and $s_{t+1} = f(s_t)$.

Proof. Considering $u_t = y_t - R(x_t)$, or equivalently, $y_t = R(x_t) + u_t$,

$$\begin{aligned} u_{t+1} = y_{t+1} - R(x_{t+1}) = \\ = g(R(x_t) + u_t) + c \cdot [-G_2(R(x_t)) + G_2(R(x_t) + u_t)] - \\ -R(f(x_t) + c \cdot [-F_2(R(x_t)) + F_2(R(x_t) + u_t)]) \end{aligned}$$

and, near the synchronized solution, that corresponds to $u_t = 0$ and $x_{t+1} = f(x_t)$, we have

$$\begin{aligned} u_{t+1} \simeq g(R(x_t)) + Dg(R(x_t)) \cdot u_t + c \cdot DG_2(R(x_t)) \cdot u_t - \\ -R(f(x_t)) - c \cdot DR(f(x_t)) \cdot DF_2(R(x_t)) \cdot u_t \end{aligned}$$

Since $R \circ f = g \circ R$, we obtain

$$u_{t+1} \simeq Dg(R(x_t)) \cdot u_t + c \cdot DG_2(R(x_t)) \cdot u_t - c \cdot DR(f(x_t)) \cdot DF_2(R(x_t)) \cdot u_t$$

Then, the linearization of the u_t evolution is given by

$$u_{t+1} = DM(s_t) \cdot u_t \tag{3}$$

with

$$DM(s_t) = Dg(R(s_t)) + c \cdot [DG_2(R(s_t)) - DR(f(s_t)) \cdot DF_2(R(s_t))] \tag{4}$$

and $s_{t+1} = f(s_t)$, and if, for a particular value of c, this system is exponentially stable, then the synchronized state $(x_t,\, y_t) = (s_t,\, R(s_t))$ of the coupling (2) is also exponentially stable, i.e. c belongs to its RSW.

Further,

$$u_T = DM(s_{T-1}) \cdot DM(s_{T-2}) \cdot \ldots \cdot DM(s_0) \cdot u_0 = DM^T(s_0) \cdot u_0,$$

and if $\lim_{T\to+\infty} \frac{1}{T} \ln \left| DM^T(s_0) \cdot u_0 \right|$ is negative for all u_0 then (3) is exponentially stable. So we conclude that $\{c \in [0,1] : \mu_{rs} < 0\} \subset RSW$.

On the contrary, if μ_{rs} is positive, then (3) is unstable and we conclude that $RSW \subset \{c \in [0,1] : \mu_{rs} \leq 0\}$. □

We note that this proposition is valid for dynamical systems of any dimension. In the next section, we use it to confirm numerical examples of dimensions one and two.

For some couplings it is possible to explicit the RSW as a function of the maximal Lyapunov exponent of the coupled dynamical systems. In fact, if R is a diffeomorphism and $G_2 = -g$ we have for unidirectional couplings (i.e. couplings with $F_2 = 0$) or for the ones that $F_2 = R^{-1} \circ g$ (we will call them "symmetric couplings") the following propositions.

Remark 1. *If $G_2 = -g$ and $F_2 = R^{-1} \circ g$, the coupling is symmetric since (2) reduces to*

$$\begin{cases} x_{t+1} = (1-c) \cdot f(x_t) + c \cdot \left(f \circ R^{-1}\right)(y_t) \\ y_{t+1} = c \cdot (R \circ f)(x_t) + (1-c) \cdot \left(R \circ f \circ R^{-1}\right)(y_t) \end{cases},$$

Proposition 2. *For an unidirectional coupling (2) with $G_2 = -g$,*

$$\left]1 - e^{-\mu_0}, 1\right] \subset RSW \subset \left[1 - e^{-\mu_0}, 1\right],$$

where μ_0 is the maximal Lyapunov exponent of $s_{t+1} = f(s_t)$.

Proof. For such a coupling, since $G_2 = -g$ and $F_2 = 0$, (4) reduces to

$$DM(s_t) = Dg(R(s_t)) - c \cdot Dg(R(s_t)) = (1-c) \cdot Dg(\widetilde{s}_t),$$

with $\widetilde{s}_{t+1} = R(s_t)$ satisfying $\widetilde{s}_{t+1} = g(\widetilde{s}_t)$. In fact, when $s_{t+1} = f(s_t)$, we have $\widetilde{s}_{t+1} = R(s_{t+1}) = R(f(s_t)) = g(R(s_t)) = g(\widetilde{s}_t)$.

Using $Dg^T(\widetilde{s}_0)$ to stand for $Dg(\widetilde{s}_{T-1}) \cdot Dg(\widetilde{s}_{T-2}) \cdot ... \cdot Dg(\widetilde{s}_0)$, we obtain

$$\mu_{rs} = \max_{u_0} \lim_{T \to +\infty} \frac{1}{T} \ln \left| (1-c)^T \cdot Dg^T(\widetilde{s}_0) \cdot \frac{u_0}{\|u_0\|} \right| = \ln|1-c| + \widetilde{\mu}_0,$$

where $\widetilde{\mu}_0$ is the maximal Lyapunov exponent of $\widetilde{s}_{t+1} = g(\widetilde{s}_t)$. Since f and g are diffeomorphically conjugate, $\widetilde{\mu}_0 = \mu_0$ [32] and we obtain

$$\mu_{rs} < 0 \Leftrightarrow 1 - e^{-\mu_0} < c < 1 + e^{-\mu_0}$$

So, we conclude that $]1 - e^{-\mu_0}, 1] \subset RSW \subset [1 - e^{-\mu_0}, 1]$. □

Proposition 3. *For a symmetric coupling (2) with $G_2 = -g$,*

$$\left]\frac{1-e^{-\mu_0}}{2}, \frac{1+e^{-\mu_0}}{2}\right[\subset RSW \subset \left[\frac{1-e^{-\mu_0}}{2}, \frac{1+e^{-\mu_0}}{2}\right],$$

where μ_0 is the maximal Lyapunov exponent of $s_{t+1} = f(s_t)$.

Proof. For such a coupling, since $G_2 = -g$ and $F_2 = R^{-1} \circ g$, (4) reduces to

$$\begin{aligned} DM(s_t) &= Dg(R(s_t)) + c \cdot \left[-Dg(R(s_t)) - DR(f(s_t)) \cdot D\left(R^{-1} \circ g\right)(R(s_t))\right] \\ &= Dg(R(s_t)) + c \cdot \left[-Dg(R(s_t)) - DR(f(s_t)) \cdot (DR)^{-1}\left(R^{-1}(g(R(s_t)))\right) \cdot Dg(R(s_t))\right] \end{aligned}$$

and, since $R \circ f = g \circ R$, we obtain,

$$\begin{aligned} DM(s_t) &= Dg(R(s_t)) + c \cdot [-Dg(R(s_t)) - Dg(R(s_t))] \\ &= (1-2c) \cdot Dg(\widetilde{s}_t) \end{aligned}$$

with $\widetilde{s}_{t+1} = R(s_t)$, as in the previous proposition.

Also, in a similar way to the proof of the previous proposition, that leads to

$$\mu_{rs} < 0 \Leftrightarrow \frac{1-e^{-\mu_0}}{2} < c < \frac{1+e^{-\mu_0}}{2}$$

So, we conclude that $\left]\frac{1-e^{-\mu_0}}{2}, \frac{1+e^{-\mu_0}}{2}\right[\subset RSW \subset \left[\frac{1-e^{-\mu_0}}{2}, \frac{1+e^{-\mu_0}}{2}\right]$. □

We note that in both of the previous propositions RSW only depends on f, which means that using an unidirectional or a symmetric coupling in order to obtain an exponentially stable R-synchronization, the values of the coupling strength c that must be used are independent of R. In both cases the width of the RSW is $e^{-\mu_0}$ but for a symmetric coupling an exponentially stable R-synchronization is obtained for smaller values of the coupling strength.

4. Examples of the R-synchronization windows

In this section, in order to consider the general scope of the results obtained in the previous one, we illustrate them in situations that do not reduce neither to the complete synchronization nor to the lag synchronization, since those particularizations of the general problem have already been considered before [5, 8, 6, 9, 10], i.e. we consider $R(u)$ different from u and from $f^{(\Delta t)}(u)$. We consider couplings of one dimensional and two dimensional dynamical systems, using emblematic maps for both situations: for the couplings of one dimensional dynamical systems we consider that f is the logistic map and $R(u) = u^2$ and for the couplings of two dimensional dynamical systems we consider that f is the Hénon map and $R(u_1, u_2) = (u_2, u_1)$.

We begin by the examples of unidirectional couplings, i.e. couplings with $G_2 = -g$ and $F_2 = 0$. In figure 1 we show the computed results for both the one dimensional and two dimensional situations. In order to the RSW appear clearly, we show the graphs of the difference of the iterates y_t and $R(x_t)$ (for the two dimensional situation we show the first component of this difference), namely $y_t - x_t^2$ for the one dimensional situation and $(y_t)_1 - (x_t)_2$ for the two dimensional situation. The RSW corresponds to the values of c for which the ordinates of all the iterates are zero (after the transients died out). The RSW shown in the figures are confirmed by proposition 2, since the maximal Lyapunov exponents for the logistic and Hénon maps ($\ln 2$ and 0.419, respectively) provide $]1 - e^{-\mu_0}, 1]$ equal to $\left]\frac{1}{2}, 1\right]$ and $]0.342, 1]$, respectively. In the same

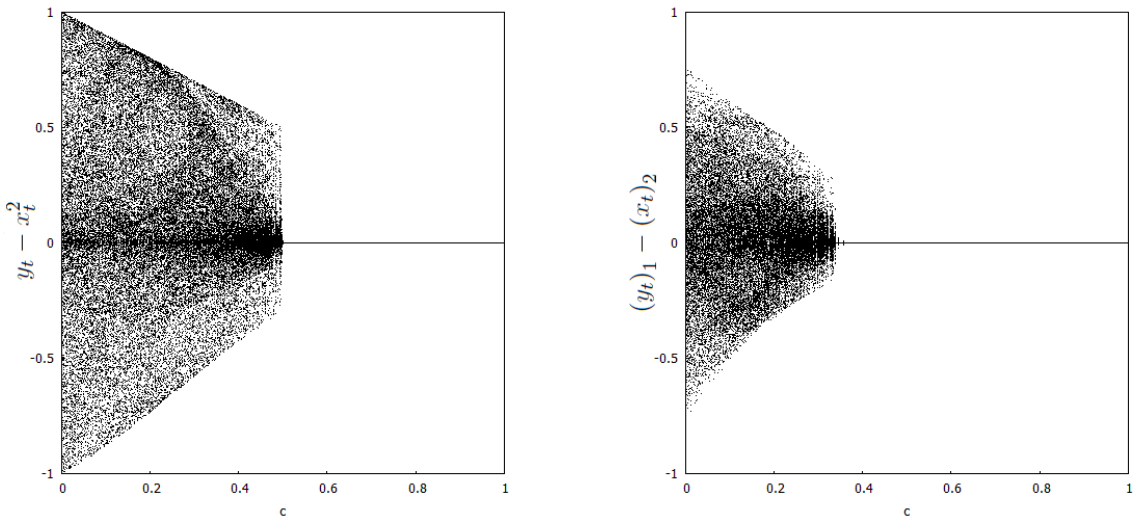

Figure 1. Graphs of the post-transient iterates $y_t - R(x_t)$ as a function of c for an unidirectional coupling with $G_2 = -g$. At the left, we consider that f is the logistic map and $R(u) = u^2$. At the right, we consider that f is the Hénon map and $R(u_1, u_2) = (u_2, u_1)$.

way, we show in figure 2 the computed results for both the one dimensional and two dimensional situations corresponding to symmetric couplings, i.e. couplings with $G_2 = -g$ and $F_2 = R^{-1} \circ g$.

Again, the RSW shown in the graphs are confirmed by the proposition 3, since $\left]\frac{1-e^{-\mu_0}}{2}, \frac{1+e^{-\mu_0}}{2}\right[$ for the one and two dimensional situations considered are $\left]\frac{1}{4}, \frac{3}{4}\right[$ and $]0.171, 0.829[$, respectively.

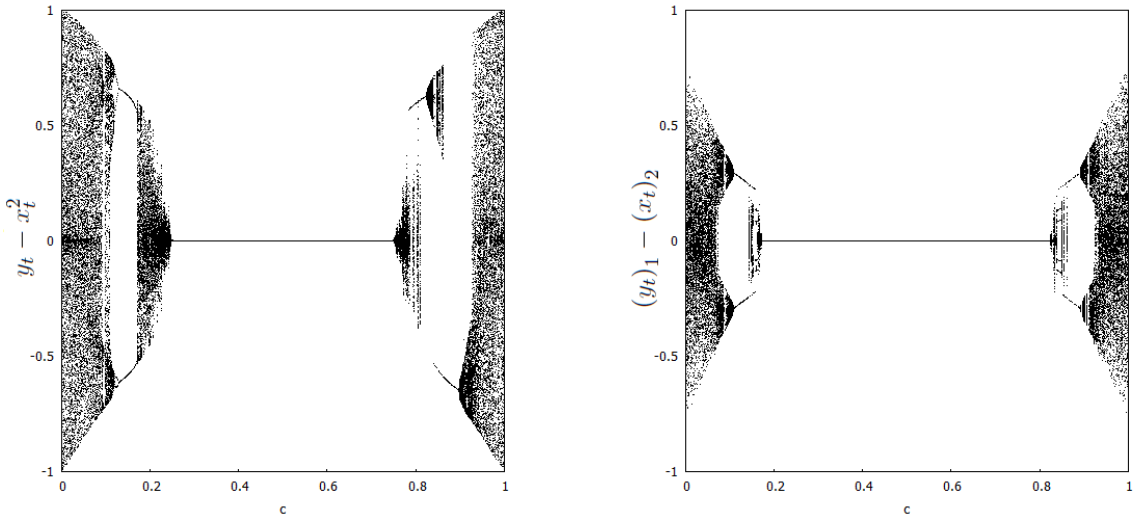

Figure 2. Graphs of the post-transient iterates $y_t - R(x_t)$ as a function of c for a symmetric coupling. At the left, we consider that f is the logistic map and $R(u) = u^2$. At the right, we consider that f is the Hénon map and $R(u_1, u_2) = (u_2, u_1)$.

We consider also two examples of couplings for which the RSW can not be calculated neither by proposition 2 nor by proposition 3. Namely, we consider examples of unidirectional couplings (i.e. couplings with $F_2 = 0$) with $G_2(u) = -g^2(u)$ for the one dimensional situation and $G_2(u_1, u_2) = \left(-g_1^2(u_1, u_2), -g_2^2(u_1, u_2)\right)$ for the two dimensional situation. We show in figure 3 the computed results and we verify that a non-empty RSW exists. In order to show that this window is confirmed by proposition 1 we show in figure 4 the value of μ_{rs}. In fact the values for which μ_{rs} are negative in the figure 4 are the ones for which the ordinates of all the post-transient iterates of $y_t - R(x_t)$ are zero.

5. Conclusions

If some conditions are satisfied, when we couple two discrete chaotic dynamical systems, they achieve generalized synchronization with a particular relationship R between the coupled systems. In fact, if the two systems are topologically conjugate by a topological conjugacy R, a coupling of the type (2) admits that synchronization and there is a synchronized exponentially stable solution for the values of the coupling strength c that belong to the R-synchronization window RSW.

So, if we are dealing with two topologically conjugate dynamical systems, we are able to synchronize them in an exponentially stable way. We just have to couple them with a coupling of type (2) and use a value of c in RSW. Or, the other way round, if we want that a dynamical system R-synchronizes with another, we choose that other system topologically conjugate to the first one by a topological conjugacy R and then we couple them in the referred way.

For couplings of type (2) an analytical expression of the RSW can be calculated. In some situations, as in the unidirectional and symmetric cases considered in propositions 2 and 3, that

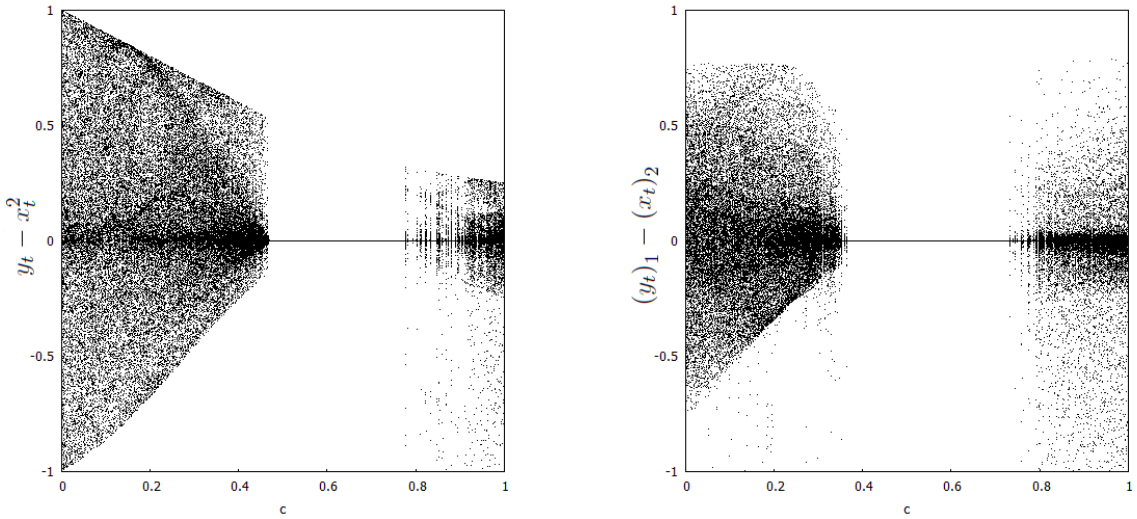

Figure 3. Graphs of the post-transient iterates $y_t - R(x_t)$ as a function of c for an unidirectional coupling. At the left, we consider that f is the logistic map, $R(u) = u^2$ and $G_2(u) = -g^2(u)$. At the right, we consider that f is the Hénon map, $R(u_1, u_2) = (u_2, u_1)$ and $G_2(u_1, u_2) = \left(-g_1^2(u_1,u_2), -g_2^2(u_1,u_2)\right)$.

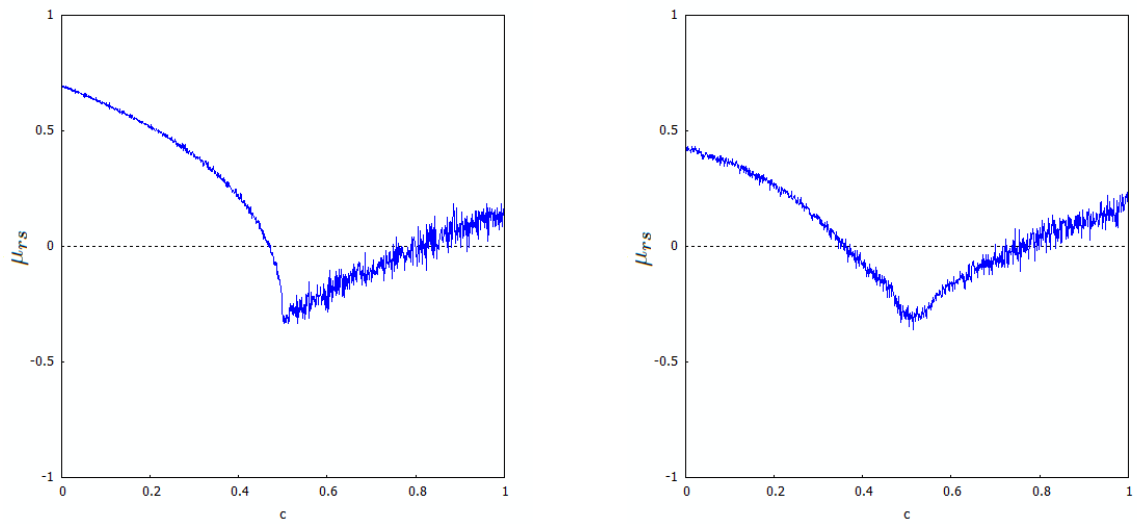

Figure 4. Graphs of μ_{rs} as a function of c for an unidirectional coupling. At the left, we consider that f is the logistic map, $R(u) = u^2$ and $G_2(u) = -g^2(u)$. At the right, we consider that f is the Hénon map, $R(u_1, u_2) = (u_2, u_1)$ and $G_2(u_1, u_2) = \left(-g_1^2(u_1,u_2), -g_2^2(u_1,u_2)\right)$.

expression can be expressed in an easy way as a function of the Lyapunov exponent of the free dynamical systems. The width of the RSW of that unidirectional coupling is the same as the one of the symmetric coupling, but if we choose a symmetric coupling instead of an unidirectional one, the systems synchronize for smaller values of the coupling strength. It is also relevant to

note that those windows are independent of R.

For all the examples that we considered, even the ones that RSW is not expressed as a function of the Lyapunov exponent of the free dynamical systems, the numerical approach used provides the RSW analytically calculated, revealing that the basin of attraction of the exponentially stable synchronized solution is sufficiently large in order to avoid that the RSW is masked by trajectories corresponding to random initial values not belonging to the basin of attraction.

Acknowledgments
Research partially sponsored by national funds through the Fundação Nacional para a Ciência e Tecnologia, Portugal-FCT, under the project UID/MAT/04674/2013 (CIMA)

References

[1] Pikovsky A, Rosenblum M and Kurths J 2001 *Synchronization, a universal concept in nonlinear sciences* (England: Cambridge University Press)
[2] Glass L 2001 *Nature* **410** 277–84
[3] Boccaletti S, Kurths J, Osipov G V, Valladares D L and Zhou C S 2002 *Phys. Reports* **366** 1–101
[4] Koronovskii A A, Moskalenko O I and Hramov A E 2009 *Physics-Uspekhi* **52** 1213–38
[5] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821–4
[6] Luo C and Wang X 2013 *Nonlinear Dyn.* **71** 241–257
[7] Mahmoud G M and Mahmoud E E 2010 *Nonlinear Dyn.* **61** 141–152.
[8] Rosenblum M G, Pikovsky A S and Kurths J 1997 *Phys. Rev. Lett.* **78** 4193–6
[9] Lopes L, Fernandes S and Grácio C 2014 *Nonlinear Dyn.* **79** 1615–24
[10] Zhang F 2015 *Entropy* **17** 4974–85
[11] Rulkov N F, Sushchik M M, Tsimring L S and Abarbanel H D I 1995 *Phys. Rev. E* **51** 980–94
[12] Zhang G, Liu Z R and Ma Z J 2007 *Chaos, Solitons & Fractals* **32** 773–779
[13] Grassi G and Miller D A 2007 *Int. J. Bifurcation Chaos* **17** 1337-42
[14] Nian F, Wan X and Zheng P 2013 *Int. J. Mod. Phys. B* **27** 2252–79
[15] Pecora L M, Carroll T L and Heagy J F 1995 *Phys. Rev. E* **52** 3420–39
[16] Abarbanel H D I, Rulkov N F and Sushchik M M 1996 *Phys. Rev. E* **53** 4528–35.
[17] Pyragas K 1996 *Phys. Rev. E* **54** R4508–11
[18] Hramov A E and Koronovskii A A 2005 *Phys. Rev. E* **71** 067201
[19] Koronovskii A A, Moskalenko O I and Hramov A E 2006 *Technical Physics* **51** 143–150
[20] Koronovskii A A, Moskalenko O I and Hramov A E 2011 *Phys. Rev. E* **84** 037201
[21] Koronovskii A A, Moskalenko O I, Shurygina S A and Hramov A E 2013 *Chaos, Solitons & Fractals* **46** 12-18
[22] Rulkov N F, Afraimovich V S, Lewis C T, Chazottes J R and Cordonet A 2001 *Phys. Rev. E* **64** 016217
[23] Afraimovich V S, Chazottes J R and Cordonet A 2001 *Discrete Contin. Dyn. Syst. B* **1** 21–442
[24] So P, Barreto E, Josic K, Sander E and Schiff S J 2002 *Phys. Rev. E* **65** 046225
[25] Grassi G 2012 *Chin. Phys. B* **21** 050505
[26] Yuan Z, Xu Z and Guo L 2012 *Commun. Nonlinear Sci. Numer. Simulat.* **17** 992-1002
[27] Hramov A E, Koronovskii A A and Moskalenko O I 2005 *Europhys. Lett.* **72** 901–7.
[28] Yang X S and Chen G 2002 *Chaos, Solitons & Fractals* **13**1303–8
[29] Lu J and Xi Y 2003 *Chaos, Solitons & Fractals* **17** 825–31
[30] Jie S, Bollt E M and Nishikawa T 2009 *SIAM Journal on Applied Dynamical Systems* **8** 202–221
[31] Lopes L, Fernandes S and Grácio C 2016 *Journal of Physics: Conference Series* **692** 012008
[32] Bollt E M and Santitissadeekorn N 2013 *Applied and computational measurable dynamics* (Society for Industrial and Applied Mathematics)