UNIVERSIDADE DE ÉVORA

# Sistemas de equações diferenciais não lineares de ordem superior em domínios limitados ou não limitados

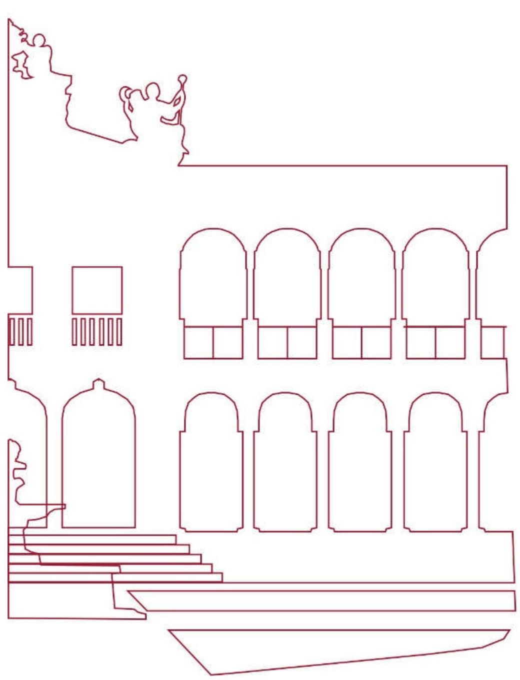

Robert de Sousa

Orientador | Feliz Manuel Barrão Minhós


Tese apresentada à Universidade de Évora para obtenção do Grau de Doutor em Matemática

Especialidade: Matemática e Aplicações

Évora, Fevereiro 2019



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*To Ray, my first son*
*and the memory of Robert Junior, my son*
*(2016 – 2016).*

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To Professor Feliz Minhós, my mentor and above all a great friend. Thank you without your guidance, suggestions, work and rigor, i would have never followed alone.

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To my friends, a big thank you.

[1] UNIVERSIDADE CABO VERDE uni

[2] FUNDAÇÃO CALOUSTE GULBENKIAN

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# *Acronyms and notations*

| | |
|---|---|
| a.e.t | Almost every t |
| BCs | Boundary conditions |
| BVPs | Boundary Value Problems |
| sup | Supremum |
| inf | Infimum |
| NLS | Nonlinear Schrödinger System |
| 2-DOF | Two degrees of freedom |
| ${}^{c}D^{\alpha}u(t)$ | Caputo fractional derivative of order $\alpha$ |
| $\mathbb{R}_{+}$ | Positive real numbers, $]0, +\infty)$ |
| $\nabla W(t, u(t))$ | Gradient of $W$ with respect to $u$ |
| $C^{0}\big(\mathbb{R}, [0, +\infty)\big)$ | Set of positive continuous functions in $\mathbb{R}$ |
| $C^{1}(\mathbb{R})$ | Set of continuous differentiable functions in $\mathbb{R}$ |
| $C^{2}[a, b]$ | Set of two times continuous differentiable functions defined in $[a, b]$ |
| $C([0, 1] \times [0, +\infty)^{2}, [0, +\infty))$ | Set of nonnegative continuous functions defined in $[0, 1] \times [0, +\infty) \times [0, +\infty)$ |
| $\big(C^{3}[0, 1], (0, +\infty)\big)^{2}$ | Set of $(u, v)$ positive three times continuous differentiable functions in $[0, 1]$ |
| $L^{1}(\mathbb{R})$ | Lebesgue integrable functions in $\mathbb{R}$ |
| $L^{\infty}(\mathbb{R})$ | Bounded measurable functions in $\mathbb{R}$ |
| $W^{1,1}(\mathbb{R})$ | Functions with integrable weak derivative in $\mathbb{R}$ |

**Nonlinear higher order systems of differential equations on bounded and unbounded domains**

## *Abstract*

The Boundary value problems on bounded or unbounded intervals, involving two or more coupled systems of the nonlinear differential equations with full nonlinearities are scarce and have gap in literature. The present work modestly try to fill this gap.

The systems covered in the work are essentially of the second-order (except for the first chapter of the first part) with boundary constraints either in bounded or unbounded intervals presented in several forms and conditions (three points, mixed, with functional dependence, homoclinic and heteroclinic).

The existence, and in some cases, the localization of the solutions is carried out in of Banach space and norms considered, following arguments and approaches such as: Schauder's fixed-point theorem or of Guo–Krasnosel'skiĭ fixed-point theorem in cones, allied to Green's function or its estimates, lower and upper solutions, convenient truncatures, the Nagumo condition presented in different forms, concept of equiconvergence, Carathéodory functions and sequences.

On the other hand, parallel to the theoretical explanation of this work, there is a range of practical examples and applications involving real phenomena, focusing on the physics, mechanics, biology, forestry, and dynamical systems.

**Keywords:** Coupled systems, Bounded and unbounded intervals, Lower and upper solutions, Nagumo condition, Green's functions, Fixed point theory.

**Sistemas de equações diferenciais não lineares de ordem superior em domínios limitados ou não limitados**

# *Resumo*


A falta ou a raridade de problemas de valor fronteira na literatura, quer em domínios limitados ou ilimitados, envolvendo sistemas de duas ou mais equações não lineares acopladas com todas as não linearidades completas, levou à elaboração do presente trabalho.

Os sistemas abordados no trabalho são essencialmente de segunda ordem (exceto o primeiro capítulo da primeira parte) com condições de fronteira em domínios limitados ou ilimitados, de diversos tipos (três pontos, mistas, com condicões funcionais, homoclínicas e heteroclínicas).

A existência e em alguns casos a localização das soluções dos sistemas é considerada em espaços de Banach, seguindo vários argumentos e abordagens: o teorema de ponto fixo de Schauder ou de Guo–Krasnosel'skiĭ em cones, aliados a funções de Green ou suas estimativas, sub e sobre-soluções, truncaturas convenientes, a condição de Nagumo apresentada sob várias formas, o conceito de equiconvergência e funções e sucessões de Carathéodory.

Por outro lado, paralelamente à componente teórica do trabalho, encontra-se um leque de aplicações e exemplos práticos envolvendo fenómenos reais, com enfoque na física, mecânica, biologia, exploração florestal e sistemas dinâmicos.



**Palavras-chave:** Sistemas acoplados, Domínios limitados e ilimitados, Sub e sobre-soluções, Condição de Nagumo, Funções de Green, Teoria do ponto fixo.

# *Introduction*

A system of differential equations is a set of equations, where each one relates some values of the function and its derivatives. When there is on interaction and dependence between different variables on a system of differential equations, it is said that the system is coupled. The phenomena, laws and systems that command the universe are neither independent nor isolated. The interaction or coupling is a fundamental characteristic of everything that surrounds us in the universe.

Natural phenomena are generally nonlinear and are modeled with systems of nonlinear higher order differential equations. In addition, these systems also serve to study and to explain various and important problems in science and engineering, that are not possible to analyze with linear systems.

Like this, the present work focuses on systems of nonlinear higher order differential equations with boundary value problems, where the systems considered are coupled.

The main precursors in the study of the differential equations, according to the history of Mathematics, were Gottfried Leibniz[3], Isaac Newton[4] and the brothers, James Bernoulli [5] and John Bernoulli[6] [57, 104].

In 1675, Leibniz was the first to study differential equations

[3] Gottfried Leibniz (1646–1716), was a German polymath, philosopher, scientist, mathematician, diplomat and librarian.

[4] Isaac Newton (1643–1727) was a English mathematician, astronomer, alchemist, natural philosopher, theologian and English scientist, most recognized as physicist and mathematician.

[5] James Bernoulli (1654 – 1705), from Switzerland, was the first mathematician to develop infinitesimal calculus beyond what had been done by Newton and Leibniz, applying it to new problems.

[6] John Bernoulli (1667–1748) was a Swiss mathematician. Its field of action included variational calculus, physical, physics, chemistry, astronomy, optics, theory of navigation and mathematics.

when considering and solving the trivial equation

$$\int y dy = \frac{1}{2}y^2,$$

thus providing tools such as the signal of integral and inverse problem of tangents. It was Leibniz who discovered the technique of separating variables by studying the solution of the equation $f(x)dx = g(y)dy$, developed the technique to solve the homogeneous equation $dy = f\left(\frac{x}{y}\right)dx$, as well as numerous contributions and applications.

Parallel to Leibniz, Newton had great impact as a contributor and in the history of differential equations. One of the most remarkable and significant Newton's contributions in this area were in the study of fluxions and their applications. It was in the study of fluxions that he established: given $f(x, y) = 0$, with $x$ and $y$ functions of $t$,

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0.$$

At the same time, Bernouli brothers created several methods (used to date) to solve and boost the development of the differential equations. As example, we refer the study of the famous brachistochrone.

The determination of solutions for nonlinear higher order systems of differential equation with boundary or initial value problems is very complicated and, in most cases, impossible to determine.

In the literature, discretization methods are usually presented, together with numerical methods for the determination of approximate solutions. Even using numerical methods, there is no guarantee that all equations or systems of nonlinear equations will present stable solutions [32].

Thus, in this work, we present sufficient conditions of existence and, in some cases of localization of solutions for systems of differential equations with several full nonlinearities with boundary values, finite or infinite intervals, with generalized impulses, with multipoint conditions, with Phi-Laplacians,.... Depending on the type of the system considered, we will be able to prove the existence of several solutions, especially in bounded and unbounded domains, homoclinics and heteroclinics, and several applications, either to theoretical problems or to real-life phenomena.

The existence, uniqueness or multiplicity of solutions, and their structure depend on the nonlinearity involved and on the type of boundary conditions considered. Thus, the study of the solvability of systems of nonlinear differential equations of higher order considers these two arguments: nonlinearity and boundary conditions. In the literature, the study has been mostly developed in bounded domains with incomplete nonlinearities, that is, where there is no dependence on all variables and all their derivatives, as can be seen, for example, in [35, 36, 38, 78, 79, 80, 81, 82, 83, 84, 91, 92, 94, 102].

This thesis intends to contribute to the literature presenting results with complete nonlinearities.

In bounded intervals, a wide variety of methods and techniques can be applied, generally based on compact or completely continuous operators.

In the last decade the theory and the application of the Boundary Value Problems (BVP) have been developed in bounded and unbounded intervals, obtaining sufficient conditions to guarantee, in these cases, the existence and possible multiplicity of bounded or unbounded solutions.

As an example, we refer [2], where some of existence conditions in infinite intervals are presented for the problem

$$\begin{cases} x''\phi(t)f(t,x,x') = 0, \ 0 < t < \infty, \\ x(0) = 0, \ \ x(t) \text{ bounded on } \ [0, \infty)\,, \end{cases}$$

with physically reasonable assumptions on $\phi$ and $f$, or

$$\begin{cases} \frac{1}{p(t)}\,(p(t)x')' = \phi(t)f(t,x,p(t)x'), \ 0 < t < \infty, \\ -\alpha x(0) + \beta \lim\limits_{t\to 0^+} p(t)x'(t) = c, \ \alpha \geq 0, \, \beta \geq 0, \, c \geq 0 \\ x(t) \text{ bounded on } \ [0, \infty)\,. \end{cases}$$

We can find many more important contributions in [2, 19, 26, 68, 73, 77, 102, 119, 120, 131, 143, 169, 170, 190, 199].

The previous references contain several and varied examples and applications that illustrate the importance of BVP on unbounded domains, such as the study of symmetrical radial solutions in elliptic nonlinear equations, drain flux theory, physical properties of plasmas, flux study unstable gases through porous media, in the determination of the electric potential in an isolated neutral atom, among others.

However, the theoretical framework for higher order BVP on unbounded domains has not yet been established for a large number of boundary conditions.

Besides, the study of systems of impulsive differential equations in both, bounded and unbounded domains (see [132]), is too scarce and has very important accessible applications in phenomena studied in physics, chemistry, population dynamics, biotechnology and economics. For these and other reasons, we have devoted some time to these systems in this work.

The qualitative analysis of the BVP in the real line, namely the existence of homoclinic and heteroclinic solutions, has been restricted to differential equations, where in some cases it is possible to obtain the phase portrait or the graphic of the homoclinic or heteroclinic solution. However, in a system of nonlinear differential equations the graphical or geometric component are lost and the study of the solution is scarce, as far as we know. In this respect the results obtained in this thesis are completely innovative.

In order to elaborate all the work, the arguments used are Banach spaces and the corresponding norms, based on Green's function, Guo–Krasnosel'skiĭ fixed-point theorem of compression-expansion cones, Schauder's fixed-point theorem, lower and upper solutions method, truncature technique, Nagumo conditions, $L^1$Carathéodory functions and sequences and equiconvergent of the associated operators (see, [2, 7, 99, 113, 185, 197])

The present work is structured in three parts.

**Part I - Boundary value problems on bounded domains**, consists of two chapters:

**Chapter 1** – *Third-order three point systems with dependence on the first derivative.* Systems where the nonlinearities can depend on the first derivatives are scarce. This chapter contributes to fill that gap, applying cones theory to the third order three point boundary value problem. A key point in our method is the fact that the Green's function associated to the linear problem, and its first derivatives, are nonnegative and verify some adequate estimates. The existence of a positive and increasing solution of the third-order three point systems of differential equations with dependence on the first derivative, is obtained by the well-known Guo–Krasnosel'skiĭ theorem on cones compression-expansion. The dependence on the first derivatives is overcome by the construction of an adequate cone and suitable conditions of superlinearity/sublinearity near 0 and $+\infty$. In last section an example illustrates the applicability of the theorem.

**Chapter 2** – *Functional coupled systems with full nonlinear terms.* In this chapter we consider the boundary value problem composed by the coupled system constituted by second order differential equations with full nonlinearities, together with the functional boundary conditions. More precisely, we explained an existence and localization result and an example to show the applicability of the main theorem. An application to a real phenomenon is shown in the last section: a coupled mass-spring system together with functional behavior at the final instant.

**Part II - Coupled systems on unbounded domains**, is organized as it follows:

**Chapter 3** – *Second order coupled systems on the half-line.* Second order coupled systems on the half-line have many applications in physics, biology, mechanics and among other areas. We use lower and upper solutions method combined with a Nagumo type growth condition, the equiconvergence at infinity to establish an existence and location result for the solutions of the coupled system in the half-line. An application for the existence result is applied to a real phenomena: a predator-prey model. An example is used to show the applicability of the localization part.

**Chapter 4** – *Homoclinic solutions for second-order coupled systems.* In this chapter, we apply the fixed point theory, lower and upper solutions method combined with an adequate growth assumptions on the nonlinearities, to obtain sufficient conditions for the existence of homoclinic solutions of the coupled system. An application to a real phenomenon is shown in the last section: coupled nonlinear Schrödinger system (NLS) modeling spatial solitons in crystals.

**Chapter 5 –** *Heteroclinic solutions with phi-Laplacians.* In this chapter, we apply the fixed point theory, to obtain sufficient conditions for the existence of heteroclinic solutions of the coupled system involving phi-Laplacians, assuming some adequate conditions on their inverse and on the asymptotic values $A, B, C, D \in \mathbb{R}$.

**Part III - Coupled impulsive systems**, is divided into three chapters:

**Chapter 6 –** *Impulsive coupled systems with generalized jump conditions.* We consider a second order impulsive coupled system with full nonlinearities, mixed boundary conditions and generalized impulsive conditions with dependence on the first derivative. This chapter will establish existence solution of this problem, illustrated an example and an applications to a real phenomena: transverse vibrations system of elastically coupled double-string model.

**Chapter 7 –** *Impulsive coupled systems on the half-line.* Here, we consider the second order impulsive coupled system in half-line composed by the differential equations. It stands out, Carathéodory functions and sequences, the equiconvergence at each impulsive moment and at infinity to prove an existence result for the impulsive coupled systems with generalized jump conditions in half-line and with full nonlinearities, that depend on the unknown functions and their first derivatives. Finally, is presented an application to a real phenomena: a model of the motion of a spring pendulum.

**Chapter 8 –** *Localization results for impulsive second order coupled systems on the half-line.* Two localization results are obtained for coupled systems on the half-line, with fully differential equations, together with generalized impulsive conditions considered on infinite impulsive moments. For the result of more general impulsive conditions, is applied a real phenomena: logging timber model for removal trees in cables attached too a helicopter.

# Part I

# Boundary value problems on bounded intervals

# *Introduction*

This work will consider Boundary Value Problem (BVP) composed by a system of differential equations subject to a set of admissibility restrictions that must be respected by the solution set of the system.

Systems of nonlinear equations with boundary conditions in finite intervals are more common in the literature and are generally easier to address in determining or guaranteeing the existence of solutions.

The first boundary value problem at the bounded domain is associated with Dirichlet problem which is formulated as follows: given a closed region $\Omega$ with the boundary $\partial\Omega$ and the boundary condition

$$\phi = f(x), \quad x \in \partial\Omega$$

where $f(x)$ is a continuous function, we are asked to find an harmonic function $\phi(x)$ (for a historical remarks see, [44]).

In this Part we study problems composed of nonlinear second order and three order systems, with complete nonlinearities, and with boundary conditions on bounded intervals, more precisely we seek to guarantee the existence and location of solutions on bounded domains with multi-point conditions, functional boundary conditions and mixed boundary conditions.

The arguments used in this Part are based mainly on [17, 68, 128]. We underline:

- The research will focus the search for sufficient conditions to require to the non-linearities in order to guarantee the existence of a non-negative solution for three order systems. The arguments are based on cones theory for third order three point boundary value problem.

- Combination of the lower and upper solutions method with fixed point theory.

This first Part consists of two chapters which cover the existence and location of coupled systems on bounded intervals.

- In the *first chapter* we consider existence of solutions for nonlinear three-order coupled systems with boundary conditions on the bounded interval $[a, b],\ a, b \in \mathbb{R},\ a < b$;

- Int the *second chapter*, we study the existence and location of solutions for nonlinear second-order coupled systems on $[a, b],\ a, b \in \mathbb{R},\ a < b$, with mixed boundary conditions.

# 1

# *Third-order three point systems with dependence on the first derivative*

The solvability of systems of differential equations of second and higher order, with different types of boundary conditions has received an increasing interest in last years. See, for instance, [17, 46, 85, 86, 91, 93, 101, 114, 135] and references therein.

Guezane–Lakoud and Zenkoufi, [71], using the Laray–Schauder nonlinear alternative, the Banach contraction theorem and Guo–Krasnosel'skiĭ theorem, study the existence, uniqueness and positivity of solution to the third-order three-point nonhomogeneous boundary value problem

$$u''' + f(t, u(t), u'(t)) = 0,$$

$$\alpha u'(1) = \beta u'(\eta),\ u(0) = u'(0) = 0,$$

where

$$\alpha,\ \beta \in \mathbb{R}_+, 0 < \eta < 1 \text{ and } f \in C\left([0,\, 1) \times [0,\, \infty) \times [0,\, \infty),\ [0,\, \infty)\right).$$

In [128], the authors consider the existence of positive solution for boundary value problem

$$\begin{cases} -u''' = a(t)f(t,\, v) \\ -v''' = b(t)h(t,\, u) \\ u(0) = u'(0) = 0,\ u'(1) = \alpha u'(\eta) \\ v(0) = v'(0) = 0, v'(1) = \alpha v'(\eta), \end{cases}$$

where $f,\, h \in C\left([0,\, 1] \times [0,\, \infty),\ [0,\, \infty)\right)$, $0 < \eta < 1$, $1 < \alpha < \frac{1}{\eta}$, $a(t), b(t) \in C\left([0,\, 1],\ [0,\, \infty)\right)$ and are not identically zero on $\left[\frac{\eta}{\alpha},\, \eta\right]$.

However systems where the nonlinearities can depend on the first derivatives are scarce (see [96]). Motivated by the works referred above, this chapter contributes to fill that gap, applying cones theory to the third order three point boundary value problem

$$\begin{cases} -u'''(t) = f(t,\, v(t),\, v'(t)) \\ -v'''(t) = h(t,\, u(t),\, u'(t)) \\ u(0) = u'(0) = 0,\ u'(1) = \alpha u'(\eta) \\ v(0) = v'(0) = 0,\ v'(1) = \alpha v'(\eta). \end{cases} \tag{1.1}$$

The non-negative continuous functions $f,\ h \in C\left([0,\ 1] \times [0,\ +\infty)^2,\ [0,\ +\infty)\right)$ verify adequate superlinear and sublinear conditions, $0 < \eta < 1$ and the parameter $\alpha$ is such that $1 < \alpha < \frac{1}{\eta}$. Moreover, this chapter is based in [157].

Third order differential equations can model various phenomena in physics, biology or physiology such as the flow of a thin film of viscous fluid over a solid surface (see [30, 192]), the solitary waves solution of the Korteweg–de Vries equation ([133]), or the thyroid-pituitary interaction ([48]).

A key point in our method is the fact that the Green's function associated to the linear problem and its first derivative are non-negative and verify some adequate estimates. The existence of a positive and increasing solution of the system (1.1), is obtained by the well-known Guo–Krasnosel'skiĭ theorem on cones compression-expansion. The dependence on the first derivatives is overcome by the construction of an adequate cone and suitable conditions of superlinearity/sublinearity near 0 and $+\infty$.

## 1.1 Preliminary results

It is clear that the pair of functions $(u(t), v(t)) \in \left(C^3[0,\ 1],\ (0,\ +\infty)\right)^2$ is a solution of problem (1.1) if and only if $(u(t), v(t))$ verify the following system of integral equations

$$\begin{cases} u(t) = \int_0^1 G(t,s) f(s,\ v(s),\ v'(s))ds \\ \\ v(t) = \int_0^1 G(t,s) h(s,\ u(s),\ u'(s))ds, \end{cases} \tag{1.2}$$

where $G(t,s)$ is the Green's function associated to problem (1.1), defined by

$$G(t,s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} (2ts - s^2)(1-\alpha\eta) + t^2 s(\alpha - 1) & s \le \min\{\eta, t\}, \\ t^2(1-\alpha\eta) + t^2 s(\alpha-1) & t \le s \le \eta, \\ (2ts - s^2)(1-\alpha\eta) + t^2(\alpha\eta - s) & \eta \le s \le t, \\ t^2(1-s) & \max\{\eta, t\} \le s. \end{cases} \tag{1.3}$$

Next Lemmas provide some properties of the Green's functions and its derivative.

**Lemma 1.1.1** *([129]) Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then for any $(t,s) \in [0,\,1] \times [0,\,1]$, we have $0 \leq G(t,s) \leq g_0(s)$, where*

$$g_0(s) = \frac{1+\alpha}{1-\alpha\eta} s(1-s).$$

**Lemma 1.1.2** *([129]) Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then for any $(t,s) \in [\frac{\eta}{\alpha},\, \eta] \times [0,\,1]$, the Green function $G(t,s)$ verifies $G(t,s) \geq k_0 g_0(s)$, where*

$$0 < k_0 := \frac{\eta^2}{2\alpha^2(1+\alpha)} \min\{\alpha - 1,\, 1\} < 1. \tag{1.4}$$

The derivative of $G$ is given by

$$\frac{\partial G}{\partial t}(t,s) = \frac{1}{(1-\alpha\eta)} \begin{cases} s(1-\alpha\eta) + ts(\alpha-1) & s \leq \min\{\eta, t\}, \\ t(1-\alpha\eta) + ts(\alpha-1) & t \leq s \leq \eta, \\ s(1-\alpha\eta) + t(\alpha\eta - s) & \eta \leq s \leq t, \\ t(1-s) & \max\{\eta, t\} \leq s, \end{cases}$$

and verifies the following lemmas:

**Lemma 1.1.3** *For $0 < \eta < 1$, $1 < \alpha < \frac{1}{\eta}$ and any $(t,s) \in [0,\,1] \times [0,\,1]$, we have $0 \leq \frac{\partial G}{\partial t}(t,s) \leq g_1(s)$, where*

$$g_1(s) = \frac{(1-s)}{(1-\alpha\eta)}.$$

**Proof** For $s \leq \min\{\eta, t\}$, we have

$$\begin{aligned} \frac{s(1-\alpha\eta) + ts(\alpha-1)}{(1-\alpha\eta)} &\leq \frac{s(1-\alpha\eta) + s(\alpha-1)}{(1-\alpha\eta)} = \frac{s(\alpha - \alpha\eta)}{(1-\alpha\eta)} \\ &= \frac{s\alpha(1-\eta)}{(1-\alpha\eta)} \leq \frac{s\alpha(1-s)}{(1-\alpha\eta)} \leq \frac{(1-s)}{(1-\alpha\eta)}. \end{aligned}$$

If $t \leq s \leq \eta$,

$$\begin{aligned} \frac{t(1-\alpha\eta) + ts(\alpha-1)}{(1-\alpha\eta)} = \frac{t(1-\alpha\eta + s\alpha - s)}{(1-\alpha\eta)} &\leq \frac{(1-\alpha\eta + \eta\alpha - s)}{(1-\alpha\eta)} \\ &= \frac{(1-s)}{(1-\alpha\eta)}. \end{aligned}$$

For $\eta \le s \le t$,

$$\begin{aligned}\frac{s(1-\alpha\eta)+t(\alpha\eta-s)}{(1-\alpha\eta)} &\le \frac{s(1-\alpha\eta)+(\alpha\eta-s)}{(1-\alpha\eta)} = \frac{\alpha\eta(1-s)}{(1-\alpha\eta)}\\ &\le \frac{\alpha s(1-s)}{(1-\alpha\eta)} \le \frac{(1-s)}{(1-\alpha\eta)}.\end{aligned}$$

If $\max\{\eta, t\} \le s$, then

$$\frac{t(1-s)}{(1-\alpha\eta)} \le \frac{s(1-s)}{(1-\alpha\eta)} \le \frac{(1-s)}{(1-\alpha\eta)}.$$

So,

$$\frac{\partial G}{\partial t}(t,s) \le g_1(s) := \frac{(1-s)}{(1-\alpha\eta)}, \text{ for } (t,s) \in [0,\ 1] \times [0,\ 1].$$

■

**Lemma 1.1.4** *For $0 < \eta < 1$, $1 < \alpha < \frac{1}{\eta}$ and any $(t,s) \in [\frac{\eta}{\alpha},\ \eta] \times [\frac{\eta}{\alpha},\ \eta]$, the derivative of the Green function $\frac{\partial G}{\partial t}(t,s)$ verifies $\frac{\partial G}{\partial t}(t,s) \ge k_1 g_1(s)$, with*

$$0 < k_1 := \min\left\{ \begin{array}{c} \left[\frac{\eta}{\alpha^2}(1-\alpha\eta) + \frac{\eta^2}{\alpha^3}(\alpha-1)\right](\alpha-\eta), \\ \frac{\eta(\alpha-\eta)(1-\alpha\eta)}{\alpha^2} \end{array} \right\}. \quad (1.5)$$

**Proof** To find $k_1$ such that

$$k_1 g_1(s) \le \frac{\partial G}{\partial t}(t,s),$$

we evaluate it in each branch of $\frac{\partial G}{\partial t}(t,s)$ for $(t,s) \in [\frac{\eta}{\alpha},\ \eta] \times [\frac{\eta}{\alpha},\ \eta]$,

**(i)** For $s \le \min\{\eta, t\}$, we must prove that

$$k_1 \frac{1-s}{1-\alpha\eta} \le \frac{s(1-\alpha\eta)+ts(\alpha-1)}{1-\alpha\eta},$$

that is

$$k_1 \le \frac{s(1-\alpha\eta)+ts(\alpha-1)}{1-s}.$$

In fact,

$$\begin{aligned}\frac{s(1-\alpha\eta)+ts(\alpha-1)}{1-s} &\geq \frac{\frac{\eta}{\alpha}(1-\alpha\eta)+\frac{\eta^2}{\alpha^2}(\alpha-1)}{1-s}\\ &\geq \frac{\frac{\eta}{\alpha}(1-\alpha\eta)+\frac{\eta^2}{\alpha^2}(\alpha-1)}{\frac{\alpha}{\alpha-\eta}}\\ &\geq \left[\frac{\eta}{\alpha^2}(1-\alpha\eta)+\frac{\eta^2}{\alpha^3}(\alpha-1)\right](\alpha-\eta)\geq k_1>0.\end{aligned}$$

**(ii)** If $t \leq s \leq \eta$, the inequality

$$k_1\frac{1-s}{1-\alpha\eta} \leq \frac{t(1-\alpha\eta+s\alpha-s)}{1-\alpha\eta}$$

holds for

$$k_1 \leq \frac{t(1-\alpha\eta+s\alpha-s)}{1-s}.$$

Therefore, we can take

$$\begin{aligned}\frac{t(1-\alpha\eta+s\alpha-s)}{1-s} &\geq \frac{\frac{\eta}{\alpha}\left(1-\alpha\eta+\frac{\eta\alpha}{\alpha}-\eta\right)}{\frac{\alpha}{\alpha-\eta}}\\ &= \frac{\eta(\alpha-\eta)(1-\alpha\eta)}{\alpha^2}\geq k_1>0.\end{aligned}$$

So, for

$$0<k_1:=\min\left\{\begin{array}{c}\left[\frac{\eta}{\alpha^2}(1-\alpha\eta)+\frac{\eta^2}{\alpha^3}(\alpha-1)\right](\alpha-\eta),\\ \frac{\eta(\alpha-\eta)(1-\alpha\eta)}{\alpha^2}\end{array}\right\},$$

we have

$$\frac{\partial G}{\partial t}(t,s)\geq k_1 g_1(s).$$

■

The existence tool will be the well known Guo-Krasnoselskii result in expansive and compressive cones theory:

**Lemma 1.1.5** *([72]) Let $(E, \|\cdot\|)$ be a Banach space, and $P \subset E$ be a cone in $E$. Assume that $\Omega_1$ and $\Omega_2$ are open subsets of $E$ such that $0 \in \Omega_1,\ \overline{\Omega_1} \subset \Omega_2$.*

*If $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

*(i) $\|Tu\| \leq \|u\|,\ u \in P \cap \partial\Omega_1,$ and $\|Tu\| \geq \|u\|,\ u \in P \cap \partial\Omega_2,$ or*

*(ii) $\|Tu\| \geq \|u\|,\ u \in P \cap \partial\Omega_1,$ and $\|Tu\| \leq \|u\|,\ u \in P \cap \partial\Omega_2,$*

*then $T$ has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.*

## 1.2 Main result

Consider the following growth assumptions

$(A1)$ $\limsup\limits_{t\in[0,1],\ \|v\|_{C^1}\to 0} \frac{f(t,v,v')}{|v|+|v'|} = 0$ and $\limsup\limits_{t\in[0,1],\ \|u\|_{C^1}\to 0} \frac{h(t,u,u')}{|u|+|u'|} = 0$;

$(A2)$ $\liminf\limits_{t\in[0,1],\ \|v\|_{C^1}\to +\infty} \frac{f(t,v,v')}{|v|+|v'|} = +\infty$ and $\liminf\limits_{t\in[0,1],\ \|u\|_{C^1}\to +\infty} \frac{h(t,u,u')}{|u|+|u'|} = +\infty$;

$(A3)$ $\liminf\limits_{t\in[0,1],\ \|v\|_{C^1}\to 0} \frac{f(t,v,v')}{|v|+|v'|} = +\infty$ and $\liminf\limits_{t\in[0,1],\ \|u\|_{C^1}\to 0} \frac{h(t,u,u')}{|u|+|u'|} = +\infty$;

$(A4)$ $\limsup\limits_{t\in[0,1],\ \|v\|_{C^1}\to +\infty} \frac{f(t,v,v')}{|v|+|v'|} = 0$ and $\limsup\limits_{t\in[0,1],\ \|u\|_{C^1}\to +\infty} \frac{h(t,u,u')}{|u|+|u'|} = 0$.

The main result is given by next theorem :

**Theorem 1.2.1** *Let $f,\ h : [0,\,1] \times [0,\,+\infty)^2 \to [0,\,+\infty)$ be continuous functions such that assumptions $(A1)$ and $(A2)$, or $(A3)$ and $(A4)$, hold. Then problem (1.1) has at least one positive solution $(u(t),\,v(t)) \in \left(C^3[0,\,1]\right)^2$, that is $u(t) > 0,\ v(t) > 0,\ \forall t \in [0,\,1]$.*

**Proof** Let $E = C^1[0,\,1]$ be the Banach space equipped with the norm $\|\cdot\|_{C^1}$, defined by $\|w\|_{C^1} := \max\{\|w\|, \|w'\|\}$ and $\|y\| := \max\limits_{t\in[0,\,1]} |y(t)|$.

Consider the set

$$K=\left\{w\in E:\ w(t)\geq 0,\ \min_{t\in\left[\frac{\eta}{\alpha},\,\eta\right]} w(t)\geq k_0\|w\|,\ \min_{t\in\left[\frac{\eta}{\alpha},\,\eta\right]} w'(t)\geq k_1\|w'\|\right\},$$

with $k_0$ and $k_1$ given by (1.4) and (1.5), respectively, and the operators $T_1 : K \to K$ and $T_2 : K \to K$ such that

$$\begin{cases} T_1u(t)=\int_0^1 G(t,s)f(s,\,v(s),\,v'(s))ds \\ T_2v(t)=\int_0^1 G(t,s)h(s,\,u(s),\,u'(s))ds. \end{cases} \tag{1.6}$$

By (1.2), the solutions of the initial system (1.1) are fixed points of the operator $T := (T_1, T_2)$.

First we show that $K$ is a cone. By definition of $K$ it is clear that $K$ is not identically zero or empty.

Consider $a,\ b\in\mathbb{R}^+$ and $x,\ y\in K$. Then

$$x\in K\Rightarrow x\in E\ :\ x(t)\geq 0,\ \min_{t\in[0,\,1]} x(t)\geq k_0\|x\|,\ \min_{t\in[0,\,1]} x'(t)\geq k_1\|x'\|,$$

$$y\in K\Rightarrow y\in E\ :\ y(t)\geq 0,\ \min_{t\in[0,\,1]} y(t)\geq k\|y\|,\ \min_{t\in[0,\,1]} y'(t)\geq k_1\|y'\|.$$

As $E$ is a vector space, consider the linear combination $ax+by\in E$.

$$\begin{aligned}\min_{t\in[0,\,1]}\left(ax(t)+by(t)\right) &= a\min_{t\in[0,\,1]} x(t)+b\min_{t\in[0,\,1]} y(t)\\ &\geq ak_0\|x\|+bk_0\|y\|=k_0\left(a\|x\|+b\|y\|\right)\\ &\geq k_0\left\|ax(t)+by(t)\right\|,\end{aligned}$$

and

$$\begin{aligned}\min_{t\in[0,\,1]}\left(ax(t)+by(t)\right)' &= a\min_{t\in[0,\,1]}\left(x(t)\right)'+b\min_{t\in[0,\,1]}\left(y(t)\right)'\\ &\geq ak_1\|x'\|+bk_1\|y'\|=k_1\left(a\|x'\|+b\|y'\|\right)\\ &\geq k_0\left\|\left(ax(t)+by(t)\right)'\right\|.\end{aligned}$$

Therefore $ax+by\in K$, that is $K$ is a cone.

Now we show that $T_1$ and $T_2$ are completely continuous, i.e, $T_1$ and $T_2$ are equicontinuous and uniformly bounded. For the reader's convenience the proof for $T_1$ will follow several steps and claims. The arguments for $T_2$ are analogous.

**Step 1:** *$T_1$ and $T_2$ are well defined in $K$.*

To prove that $T_1 K \subset K$ consider $u \in K$.
As $G(t,s) \geq 0$ for $(t,s) \in [0,\, 1] \times [0,\, 1]$, it is clear that $T_1 u(t) \geq 0$.

By Lemma 1.1.1, the positivity of $f$ and (1.6),

$$\begin{aligned} 0 \leq T_1 u(t) &= \int_0^1 G(t,s) f(s,\, v(s),\, v'(s)) ds \\ &\leq \int_0^1 g_0(s) f(s,\, v(s),\, v'(s)) ds. \end{aligned}$$

So,

$$\|T_1 u\| \leq \int_0^1 g_0(s) f(s,\, v(s),\, v'(s)) ds. \tag{1.7}$$

From Lemma 1.1.2 and (1.7),

$$\begin{aligned} T_1 u(t) &= \int_0^1 G(t,s) f(s,\, v(s),\, v'(s)) ds \\ &\geq k_0 \int_0^1 g_0(s) f(s,\, v(s),\, v'(s)) ds \geq k_0 \|T_1 u\|,\ \text{for } t \in \left[\frac{\eta}{\alpha}, \eta\right], \end{aligned}$$

with $k_0$ given by (1.4). By Lemma 1.1.3,

$$\begin{aligned} (T_1 u(t))' &= \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s,\, v(s),\, v'(s)) ds \\ &\leq \int_0^1 g_1(s) f(s,\, v(s),\, v'(s)) ds, \end{aligned}$$

So,

$$\|(T_1 u)'\| \leq \int_0^1 g_1(s) f(s,\, v(s),\, v'(s)) ds. \tag{1.8}$$

By Lemma 1.1.4 and (1.8), it follows

$$\begin{aligned} (T_1 u(t))' &= \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s,\, v(s),\, v'(s)) ds \\ &\geq k_1 \int_0^1 g_1(s) f(s,\, v(s),\, v'(s) ds \\ &\geq k_1 \|(T_1 u)'\|,\ \text{for } t \in \left[\frac{\eta}{\alpha}, \eta\right], \end{aligned}$$

and $k_1$ as in (1.5).

So $T_1 K \subset K$. Analogously it can be shown that $T_2 K \subset K$.

Assume that $(A1)$ and $(A2)$ hold.

By $(A1)$, there exists $0 < \delta_1 < 1$ such that, for $(t, v, v') \in [0,1] \times [0, \delta_1]^2$ and $(t, u, u') \in [0,1] \times [0, \delta_1]^2$,

$$f(s,\, v(s),\, v'(s)) \leq \varepsilon_1 \left(|v(s)| + |v'(s)|\right) \tag{1.9}$$

and

$$h(s,\, u(s),\, u'(s)) \leq \varepsilon_2 \left(|u(s)| + |u'(s)|\right), \tag{1.10}$$

with $\varepsilon_1$ and $\varepsilon_2$ to be defined forward.

**Step 2:** *$T_1$ and $T_2$ are completely continuous in $C^1[0,\, 1]$.*

$T_1$ is continuous in $C^1[0,\, 1]$ as $G(t,s)$, $\frac{\partial G}{\partial t}(t,s)$ and $f$ are continuous.

Consider the set $B \subset K$, bounded in $C^1$, and let $u, v \in B$. Then there are $M_1, M_2 > 0$ such that $\|u\|_{C^1} < M_1$ and $\|v\|_{C^1} < M_2$.

**Claim 2.1**. *$T_1$ is uniformly bounded in $C^1[0,\, 1]$.*

In fact, taking $\delta_1 := \min\{M_1, M_2\}$ in (1.9), there are $M_3, M_4 > 0$ such that

$$\begin{aligned}
\|T_1 u\| &= \max_{t\in[0,\,1]} |T_1 u(t)| \\
&= \max_{t\in[0,\,1]} \left| \int_0^1 G(t,s) f(s,\, v(s),\, v'(s)) ds \right| \\
&\leq \int_0^1 \max_{t\in[0,\,1]} |G(t,s)| \left| f(s,\, v(s),\, v'(s)) \right| ds \\
&\leq \int_0^1 \max_{t\in[0,\,1]} |G(t,s)|\, \varepsilon_1 \left(|v(s)| + |v'(s)|\right) ds \\
&\leq 2\varepsilon_1 \|v\|_{C^1} \int_0^1 \max_{t\in[0,\,1]} |G(t,s)| < M_3,\ \forall u \in B,
\end{aligned}$$

$$\begin{aligned}
\|\,(T_1u)'\,\| &= \max_{t\in[0,\,1]} \left|(T_1u(t))'\right| \\
&= \max_{t\in[0,\,1]} \left|\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s,\,v(s),\,v'(s)ds\right| \\
&\le \int_0^1 \max_{t\in[0,\,1]} \left|\frac{\partial G}{\partial t}(t,s)\right| |f(s,\,v(s),\,v'(s)|\,ds \\
&\le \int_0^1 \max_{t\in[0,\,1]} \left|\frac{\partial G}{\partial t}(t,s)\right| \varepsilon_1\left(|v(s)|+|v'(s)|\right)ds \\
&\le 2\varepsilon_1\|v\|_{C^1}\int_0^1 \max_{t\in[0,\,1]} \left|\frac{\partial G}{\partial t}(t,s)\right| ds < M_4,\ \forall u\in B.
\end{aligned}$$

Defining $M := \max\{M_3, M_4\}$, then $\|T_1u\|_{C^1} \le M$.

**Claim 2.2**. *$T_1$ is equicontinuous in $C^1[0,\,1]$.*

Let $t_1$ and $t_2 \in [0,1]$. Without loss of generality suppose $t_1 \le t_2$. So

$$\begin{aligned}
|Tu(t_1)-Tu(t_2)| &= \left|\int_0^1 \left[G(t_1,s)-G(t_2,s)\right] f(s,\,v(s),\,v'(s))ds\right| \\
&\le \int_0^1 |G(t_1,s)-G(t_2,s)|\,\varepsilon_1\left(|v|+|v'|\right)ds \\
&\le 2\varepsilon_1\|v\|_{C^1}\int_0^1 |G(t_1,s)-G(t_2,s)|\,ds \to 0,
\end{aligned}$$

as $t_1 \to t_2$ and

$$\begin{aligned}
&\left|(Tu(t_1))'-(Tu(t_2))'\right| \\
=\ &\left|\int_0^1 \left[\frac{\partial G}{\partial t}(t_1,s)-\frac{\partial G}{\partial t}(t_2,s)\right] f(s,\,v(s),\,v'(s))ds\right| \\
\le\ &\int_0^1 \left|\frac{\partial G}{\partial t}(t_1,s)-\frac{\partial G}{\partial t}(t_2,s)\right| \varepsilon_1\left(|v|+|v'|\right)ds \\
\le\ &2\varepsilon_1\|v\|_{C^1}\int_0^1 \left|\frac{\partial G}{\partial t}(t_1,s)-\frac{\partial G}{\partial t}(t_2,s)\right| ds \to 0,
\end{aligned}$$

as $t_1 \to t_2$.

By the Arzèla-Ascoli Theorem, $T_1B$ is relatively compact, that is, $T_1$ is compact.

Applying the same technique, using (1.10), it can be shown that $T_2$ is compact, too. Consequently $T$ is compact.

Next steps will prove that assumptions of Lemma 1.1.5 hold.

**Step 3:** $\|T_1 u\|_{C^1} \leq \|u\|_{C^1}$, *for some* $\rho_1 > 0$ *and* $u \in K \cap \partial\Omega_1$ *with* $\Omega_1 = \{u \in E : \|u\|_{C^1} < \rho_1\}$.

By $(A1)$, define $0 < \rho_1 < 1$ such that $(t, v, v') \in [0, 1] \times [0, \rho_1]^2$ and $(t, u, u') \in [0, 1] \times [0, \rho_1]^2$.

From (1.9) and (1.10), choose $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small such that

$$\max \left\{ \begin{array}{c} \varepsilon_1 \varepsilon_2 \int_0^1 g_0(s)ds \int_0^1 (g_0(r) + g_1(r))\, dr, \\ \varepsilon_1 \varepsilon_2 \int_0^1 g_1(s)ds \int_0^1 (g_0(r) + g_1(r))\, dr \end{array} \right\} < \frac{1}{2}. \qquad (1.11)$$

If $u \in K$ and $\|u\|_{C^1} = \rho_1$, then, by Lemma 1.1.1, (1.2) and (1.11),

$$\begin{aligned} T_1 u(t) &\leq \int_0^1 g_0(s) \varepsilon_1 (|v| + |v'|)\, ds \\ &\leq \int_0^1 g_0(s) \varepsilon_1 \int_0^1 \left( |G(t,r)| + \left| \frac{\partial G}{\partial t}(t,r) \right| \right) |h(r, u(r), u'(r))|\, drds \\ &\leq \int_0^1 g_0(s) \varepsilon_1 \int_0^1 (g_0(r) + g_1(r))\, |h(r, u(r), u'(r))|\, drds \\ &\leq \varepsilon_1 \varepsilon_2 \int_0^1 g_0(s)ds \int_0^1 (g_0(r) + g_1(r))\, (|u(r)| + |u'(r)|)\, dr \\ &\leq 2\varepsilon_1 \varepsilon_2 \|u\|_{C^1} \int_0^1 g_0(s)ds \int_0^1 (g_0(r) + g_1(r))\, dr < \|u\|_{C^1}, \end{aligned}$$

and

$$\begin{aligned} (T_1 u(t))' &= \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s, v(s), v'(s)ds \leq \int_0^1 g_1(s) \varepsilon_1 (|v| + |v'|)\, ds \\ &\leq \int_0^1 g_1(s) \varepsilon_1 \int_0^1 (g_0(r) + g_1(r))\, |h(r, u(r), u'(r))|\, drds \\ &\leq 2\varepsilon_1 \varepsilon_2 \|u\|_{C^1} \int_0^1 g_1(s)ds \int_0^1 (g_0(r) + g_1(r))\, dr < \|u\|_{C^1}, \end{aligned}$$

Therefore $\|T_1 u\|_{C^1} \leq \|u\|_{C^1}$.

**Step 4:** $\|T_1 u\|_{C^1} \geq \|u\|_{C^1}$, *for some* $\rho_2 > 0$ *and* $u \in K \cap \partial\Omega_2$ *with* $\Omega_2 = \{u \in E : \|u\|_{C^1} < \rho_2\}$.

By $(A2)$, $\|v\|_{C^1} \to +\infty$ and $\|u\|_{C^1} \to +\infty$. Therefore there are several cases to be considered:

**Case 4.1.** *Suppose that there exist* $\theta_1, \theta_2 > 0$ *such that* $\|v\| \to +\infty$, $\|v'\| \leq \theta_1$, $\|u\| \to +\infty$ *and* $\|u'\| \leq \theta_2$.

Consider $\rho > 0$ such that for $(t, v, v') \in [0,\, 1] \times [\rho,\, +\infty) \times [0, \theta_1]$ and $(t,\, u, u') \in [0,\, 1] \times [\rho,\, +\infty) \times [0, \theta_2]$, we have

$$f(t, v(t), v'(t)) \geq \xi_1 \left(|v(t)| + |v'(t)|\right) \tag{1.12}$$

and

$$h(t, u(t), u'(t)) \geq \xi_2 \left(|u(t)| + |u'(t)|\right), \tag{1.13}$$

with $\xi_1$, $\xi_2$ such that

$$\min \left\{ \begin{array}{c} (k_0)^2\, \xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right) dr, \\ \xi_1\xi_2 k_0 k_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right) dr, \\ \xi_1\xi_2 k_0 k_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right) dr, \\ (k_1)^2\, \xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right)\ dr \\ k_0\left(k_0 + k_1\right) \xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right)\ dr, \\ k_0\left(k_0 + k_1\right) \xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right)\ dr \end{array} \right\} > 1, \tag{1.14}$$

with $k_0, k_1$ as in (1.4) and (1.5).

Let $u, v \in K$ such that $\|u\|_{C^1} = \rho_2$, where $\rho_2 := \max\left\{2\rho_1,\, \frac{\rho}{k_0},\, \frac{\rho}{k_1}\right\}$.

Then $\|u\|_{C^1} = \|u\| = \rho_2$ and $u(t) \geq k_0\|u\|_{C^1} = k_0\rho_2 \geq \rho$, $t \in [0,\, 1]$. Similarly, $\|v\|_{C^1} = \|v\| = \rho_2$ and $v(t) \geq k_1\|v\|_{C^1} = k_0\rho_2 \geq \rho$.

By Lemma 1.1.2, (1.2) and (1.14),

$$\begin{aligned} & T_1 u(t) \geq \int_{\frac{\eta}{\alpha}}^{\eta} G(t,s) f(s,\, v(s),\, v'(s)) ds \\ \geq\ & k_0 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ f(s,\, v(s),\, v'(s)) ds \geq k_0\xi_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ \left(|v(s)| + |v'(s)|\right) ds \\ =\ & k_0\xi_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(|G(t,r)| + \left|\frac{\partial G}{\partial t}(t,r)\right|\right) |h(r,\, u(r),\, u'(r))|\, dr \\ \geq\ & k_0\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right)\ \left(|u(r)| + |u'(r)|\right) dr \end{aligned}$$

$$
\begin{aligned}
&= k_0\xi_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left( |G(t,r)| + \left| \frac{\partial G}{\partial t}(t,r) \right| \right) |h(r,\, u(r),\, u'(r))|\, dr \\
&\geq k_0\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0 g_0(r) + k_1 g_1(r)) \ \ (|u(r)| + |u'(r)|)\, dr \\
&\geq k_0\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s) ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0 g_0(r) + k_1 g_1(r)) \begin{pmatrix} \min\limits_{r\in\left[\frac{\eta}{\alpha},\eta\right]} u(r) \\ + \min\limits_{r\in\left[\frac{\eta}{\alpha},\eta\right]} u'(r) \end{pmatrix} dr \\
&\geq k_0\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s) ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0 g_0(r) + k_1 g_1(r)) \left(k_0 \|u\| + k_1 \|u'\|\right) dr \\
&\geq k_0\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0 g_0(r) + k_1 g_1(r)) \ \ k_0 \|u\|_{C^1} dr \\
&= k_0^2 \|u\|_{C^1} \xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0 g_0(r) + k_1 g_1(r)) \ \ dr > \|u\|_{C^1},
\end{aligned}
$$

and, analogously,

$$
\begin{aligned}
&(T_1 u(t))' \geq \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial G}{\partial t}(t,s) f(s,\, v(s),\, v'(s)) ds \\
&\geq k_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ f(s,\, v(s),\, v'(s)) ds \\
&\geq k_1\xi_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ \ (|v(s)| + |v'(s)|)\, ds \\
&\geq k_1\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0 g_0(r) + k_1 g_1(r)) \ \ k_0 \|u\|_{C^1} dr \\
&= k_1 k_0 \|u\|_{C^1} \xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0 g_0(r) + k_1 g_1(r)) \ \ dr > \|u\|_{C^1}.
\end{aligned}
$$

Therefore $\|T_1 u\|_{C^1} \geq \|u\|_{C^1}$.

**Case 4.2.** *Suppose that there exist* $\theta_3, \theta_4 > 0$ *such that* $\|v'\| \to +\infty$, $\|v\| \leq \theta_3$, $\|u'\| \to +\infty$ *and* $\|u\| \leq \theta_4$.

Consider $\rho > 0$ such that for $(t, v, v') \in [0,\, 1] \times [0, \theta_3] \times [\rho,\, +\infty)$ and $(t,\, u, u') \in [0,\, 1] \times [0, \theta_4] \times [\rho,\, +\infty)$, conditions (1.12), (1.13) and (1.14) hold.

Let $u, v \in K$ such that $\|u\|_{C^1} = \rho_2$, where $\rho_2 := \max\left\{2\rho_1, \frac{\rho}{k_0}, \frac{\rho}{k_1}\right\}$.

Then $\|u\|_{C^1} = \|u'\| = \rho_2$ and $u'(t) \geq k_1\|u'\| = k_1\rho_2 \geq \rho$, $t \in [0, 1]$. Similarly, $\|v\|_{C^1} = \|v'\| = \rho_2$ and $v'(t) \geq k_1\|v'\| = k_1\rho_2 \geq \rho$.

As in the previous case, by Lemma 1.1.2, (1.2) and (1.14)

$$
\begin{aligned}
& T_1u(t) \geq \int_{\frac{\eta}{\alpha}}^{\eta} G(t,s)f(s,\, v(s),\, v'(s))ds \\
\geq \; & k_0 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\; f(s,\, v(s),\, v'(s))ds \geq k_0\xi_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_0\; \left(|v(s)| + |v'(s)|\right) ds \\
= \; & k_0\xi_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\; ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(|G(t,r)| + \left|\frac{\partial G}{\partial t}(t,r)\right|\right) |h(r,\, u(r),\, u'(r))|\, dr \\
= \; & k_1k_0\|u\|_{C^1}\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\; ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0g_0(r) + k_1g_1(r)\right)\; dr > \|u\|_{C^1},
\end{aligned}
$$

and

$$
\begin{aligned}
& (T_1u(t))' \geq \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial G}{\partial t}(t,s)f(s,\, v(s),\, v'(s))ds \\
\geq \; & k_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\; f(s,\, v(s),\, v'(s))ds \\
\geq \; & k_1\xi_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\; \left(|v(s)| + |v'(s)|\right) ds \\
\geq \; & k_1\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\; ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0g_0(r) + k_1g_1(r)\right)\; k_1\|u\|_{C^1} dr \\
= \; & (k_1)^2\, \|u\|_{C^1}\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\; ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0g_0(r) + k_1g_1(r)\right)\; dr > \|u\|_{C^1}.
\end{aligned}
$$

**Case 4.3.** *Suppose that* $\|v\| \to +\infty$, $\|v'\| \to +\infty$, $\|u\| \to +\infty$ *and* $\|u'\| \to +\infty$.

Consider $\rho > 0$ such that for $(t, v, v') \in [0,\, 1] \times [\rho,\, +\infty)^2$ and $(t,\, u, u') \in [0,\, 1] \times [\rho,\, +\infty)^2$, conditions (1.12), (1.13) and (1.14) hold.

Let $u, v \in K$ such that $\|u\|_{C^1} = \rho_2$, where $\rho_2 := \max\left\{2\rho_1, \frac{\rho}{k_0}, \frac{\rho}{k_1}\right\}$.

Then $\|u\|_{C^1} = \|u\| = \|u'\| = \rho_2$ and $u(t) \geq k_0\|u\| = k_0\rho_2 \geq$

$\rho$, $u'(t) \geq k_1\|u\| = k_1\rho_2 \geq \rho$, $t \in [0, 1]$. Similarly, $\|v\|_{C^1} = \|v\| = \|v'\| = \rho_2$ , $v(t) \geq k_0\|v\| = k_0\rho_2 \geq \rho$ and $v'(t) \geq k_1\|v'\| = k_1\rho_2 \geq \rho$.

As before,

$$\begin{aligned}
& T_1u(t) \\
\geq\ & k_0\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0g_0(r) + k_1g_1(r))\,(|u(r)| + |u'(r)|)\,dr \\
\geq\ & k_0\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0g_0(r) + k_1g_1(r))\ \ (k_0 + k_1)\,\|u\|_{C^1}dr \\
=\ & k_0\,(k_0 + k_1)\,\|u\|_{C^1}\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0g_0(r) + k_1g_1(r))\ \ dr \\
>\ & \|u\|_{C^1},
\end{aligned}$$

and

$$\begin{aligned}
& (T_1u(t))' \geq k_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ f(s,\,v(s),\,v'(s))ds \\
\geq\ & k_1\xi_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ \ (|v(s)| + |v'(s)|)\,ds \\
=\ & k_1\xi_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(|G(t,r)| + \left|\frac{\partial G}{\partial t}(t,r)\right|\right) |h(r,\,u(r),\,u'(r))|\,dr \\
\geq\ & k_1\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0g_0(r) + k_1g_1(r))\ \ (k_0 + k_1)\,\|u\|_{C^1}dr \\
=\ & k_1\,(k_0 + k_1)\,\|u\|_{C^1}\xi_1\xi_2 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0g_0(r) + k_1g_1(r))\ \ dr \\
>\ & \|u\|_{C^1}.
\end{aligned}$$

The other cases follow the same arguments.

Therefore $\|T_1u\|_{C^1} \geq \|u\|_{C^1}$.

Then, by Lemma 1.1.5, $T_1$ has a fixed point in $K \cap (\overline{\Omega_2}\backslash\Omega_1)$.

By the same steps it can be proved that $T_2$ has a fixed point in $K \cap (\overline{\Omega_2}\backslash\Omega_1)$, too.

Assume that $(A3)$ and $(A4)$ are verified.

**Step 5:** $\|T_1u\|_{C^1} \geq \|u\|_{C^1}$, *for some* $\rho_3 > 0$ *and* $u \in K \cap \partial\Omega_3$ *with* $\Omega_3 = \{u \in E\ :\ \|u\|_{C^1} < \rho_3\}$.

By $(A3)$, it can be chosen $\rho_3 > 0$ such that $(t,\, v, v') \in [0,\, 1] \times [0,\, \rho_3]^2$, $(t,\, u, u') \in [0,\, 1] \times [0,\, \rho_3]^2$, and there are $\xi_3,\, \xi_4 > 0$ with

$$\begin{array}{rcl} f(t, v(t), v(t)) & \geq & \xi_3 \left(|v(t)| + |v'(t)||\right), \\ h(t, u(t), u'(t)) & \geq & \xi_4 \left(|u(t)| + |u'(t)||\right) \end{array}$$

and

$$\min \left\{ \begin{array}{c} (k_0)^2 \xi_3\xi_4 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s) ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right) dr, \\ k_0 k_1 \xi_3 \xi_4 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right)\ dr \\ \xi_3\xi_4 k_0 k_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right) dr \\ (k_1)^2 \xi_3\xi_4 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right)\ dr \\ \xi_3\xi_4 k_0 \left(k_0 + k_1\right) \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right) dr \\ k_1 \left(k_0 + k_1\right) \xi_3\xi_4 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right)\ dr \end{array} \right\} > 1. \tag{1.15}$$

Let $u \in K$ and $\|u\|_{C^1} = \rho_3$.

**Case 5.1.** *Suppose* $\|u\|_{C^1} = \|u\| = \rho_3$.

By Lemma 1.1.2, (1.2) and (1.15),

$$\begin{array}{rl} & T_1 u(t) \geq \xi_3 \displaystyle\int_{\frac{\eta}{\alpha}}^{\eta} G(t, s) \left(|v(s)| + |v'(s)|\right) ds \\ \geq & \xi_3\xi_4 k_0 \displaystyle\int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right)\ k_0 \|u\|_{C^1} dr \\ = & \xi_3\xi_4 \left(k_0\right)^2 \|u\|_{C^1} \displaystyle\int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0 g_0(r) + k_1 g_1(r)\right) dr \\ > & \|u\|_{C^1}, \end{array}$$

and

$$(T_1u(t))' \geq k_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ f(s,\, v(s),\, v'(s))ds$$

$$\begin{aligned}
&\geq\ k_1\xi_3 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ \left(|v(s)| + |v'(s)|\right) ds \\
&\geq\ k_1\xi_3\xi_4 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0g_0(r) + k_1g_1(r)\right)\ k_0\|u\|_{C^1} dr \\
&=\ k_0k_1\|u\|_{C^1}\xi_3\xi_4 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0g_0(r) + k_1g_1(r)\right)\ dr \\
&>\ \|u\|_{C^1}.
\end{aligned}$$

**Case 5.2.** *Suppose* $\|u\|_{C^1} = \|u'\| = \rho_3$.

By Lemma 1.1.2, (1.2) and (1.15)

$$T_1u(t) \geq \xi_3 \int_{\frac{\eta}{\alpha}}^{\eta} G(t,s) \left(|v(s)| + |v'(s)|\right) ds$$

$$\begin{aligned}
&\geq\ \xi_3k_0 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s) \int_{\frac{\eta}{\alpha}}^{\eta} \begin{pmatrix} |G(t,r)| \\ +\left|\frac{\partial G}{\partial t}(t,r)\right| \end{pmatrix} |h(r,\, u(r),\, u'(r))|\, drds \\
&\geq\ \xi_3\xi_4k_0 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0g_0(r) + k_1g_1(r)\right)\ k_1\|u\|_{C^1} dr \\
&=\ \xi_3\xi_4k_0k_1\|u\|_{C^1} \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0g_0(r) + k_1g_1(r)\right) dr > \|u\|_{C^1},
\end{aligned}$$

and

$$\begin{aligned}
&(T_1u(t))' \\
&\geq\ k_1\xi_3\xi_4 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0g_0(r) + k_1g_1(r)\right)\ \left(|u(r)| + |u'(r)|\right) dr \\
&\geq\ k_1\xi_3\xi_4 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0g_0(r) + k_1g_1(r)\right)\ \begin{pmatrix} \min\limits_{r\in\left[\frac{\eta}{\alpha},\eta\right]} u(r) \\ +\min\limits_{r\in\left[\frac{\eta}{\alpha},\eta\right]} u'(r) \end{pmatrix} dr \\
&=\ (k_1)^2\, \|u\|_{C^1}\xi_3\xi_4 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} \left(k_0g_0(r) + k_1g_1(r)\right)\ dr \\
&>\ \|u\|_{C^1}.
\end{aligned}$$

**Case 5.3.** *Suppose* $\|u\|_{C^1} = \|u\| = \|u'\| = \rho_3$.

By Lemma 1.1.2, (1.2) and (1.15)

$$\begin{aligned}
T_1u(t) &\geq \int_{\frac{\eta}{\alpha}}^{\eta} G(t,s)f(s,\, v(s),\, v'(s))ds \\
&\geq \xi_3 k_0 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s) \int_{\frac{\eta}{\alpha}}^{\eta} \begin{pmatrix} |G(t,r)| \\ +\left|\frac{\partial G}{\partial t}(t,r)\right| \end{pmatrix} |h(r,\, u(r),\, u'(r))|\, dr ds \\
&\geq \xi_3\xi_4 k_0 \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0 g_0(r) + k_1 g_1(r)) \ \ (k_0\|u\| + k_1\|u'\|)\, dr \\
&= \xi_3\xi_4 k_0 \left(k_0 + k_1\right) \|u\|_{C^1} \int_{\frac{\eta}{\alpha}}^{\eta} g_0(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0 g_0(r) + k_1 g_1(r))\, dr \\
&> \|u\|_{C^1},
\end{aligned}$$

and

$$\begin{aligned}
(T_1u(t))' &\geq k_1 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ f(s,\, v(s),\, v'(s))ds \\
&\geq k_1\xi_3 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ \ (|v(s)| + |v'(s)|)\, ds \\
&\geq k_1\xi_3\xi_4 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0 g_0(r) + k_1 g_1(r)) \begin{pmatrix} \min\limits_{r\in\left[\frac{\eta}{\alpha},\eta\right]} u(r) \\ +\min\limits_{r\in\left[\frac{\eta}{\alpha},\eta\right]} u'(r) \end{pmatrix} dr \\
&\geq k_1 \left(k_0 + k_1\right) \|u\|_{C^1} \xi_3\xi_4 \int_{\frac{\eta}{\alpha}}^{\eta} g_1(s)\ ds \int_{\frac{\eta}{\alpha}}^{\eta} (k_0 g_0(r) + k_1 g_1(r))\, dr \\
&> \|u\|_{C^1}.
\end{aligned}$$

In any case, $\|T_1u\|_{C^1} \geq \|u\|_{C^1}$.

**Step 6:** $\|T_1u\|_{C^1} \leq \|u\|_{C^1}$, *for some* $\rho_4 > 0$ *and* $u \in K \cap \partial\Omega_4$ *with* $\Omega_4 = \{u \in E : \|u\|_{C^1} < \rho_4\}$.

Let $u \in K$ and $\|u\|_{C^1} = \rho_4$.

**Case 6.1.** Suppose that $f$ and $h$ are bounded.

Then there is $N > 0$ such that $f(t, v(t), v'(t)) \leq N$, $h(t, u(t), u'(t)) \leq N$, $\forall u,\, v \in [0, \infty)$.

Choose

$$\rho_4 = \max\left\{2\rho_3,\ N\int_0^1 g_0(s)ds,\ N\int_0^1 g_1(s)ds\right\}.$$

Then

$$\begin{aligned} T_1u(t) &= \int_0^1 G(t,s)f(s,\ v(s),\ v'(s))ds \\ &\le\ N\int_0^1 g_0(s)ds \le \rho_4,\ \text{for } t\in[0,\ 1], \end{aligned}$$

and

$$\begin{aligned} (T_1u(t))' &= \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s,\ v(s),\ v'(s))ds \\ &\le\ N\int_0^1 g_1(s)ds \le \rho_4,\ \text{for } t\in[0,\ 1]. \end{aligned}$$

Thus, $\|T_1u\|_{C^1} \le \|u\|_{C^1}$. Similarly $\|T_2v\|_{C^1} \le \|v\|_{C^1}$ for any $v\in K$ and $\|v\|_{C^1} = \rho_4$.

**Case 6.2.** Consider that $f$ is bounded and $h$ is unbounded.

So, there is $N > 0$ such that $f(t,v(t),v'(t)) \le N$, $\forall\,(v,v') \in [0,\ +\infty)^2$.

By $(A4)$, there exists $M > 0$ such that $h(t,u(t),u'(t)) \le \mu\left(|u(t)|+|u'(t)|\right)$, whenever $|u(t)|+|u'(t)| \ge M$, with $\mu$ verifying

$$\max\left\{\mu\int_0^1 g_0(s)ds, \mu\int_0^1 g_1(s)ds\right\} < \frac{1}{2}. \tag{1.16}$$

Setting

$$p(r) := \max\{h(t,u(t),u'(t))\ :\ t\in[0,\ 1],\ 0\le u\le r,\ 0\le u'\le r\},$$

we have

$$\lim_{r\to\infty} p(r) = +\infty.$$

Define

$$\rho_4 = \max\left\{2\rho_3,\ M,\ N\int_0^1 g_0(s)ds,\ N\int_0^1 g_1(s)ds\right\}. \tag{1.17}$$

such that $p(\rho_4) \geq p(r),\ 0 \leq r \leq \rho_4$. Then

$$T_1u(t) = \int_0^1 G(t,s) f(s,\, v(s),\, v'(s)) ds \leq N \int_0^1 g_0(s) ds \leq \rho_4,$$

and

$$\begin{aligned} (T_1u(t))' &= \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s,\, v(s),\, v'(s)) ds \\ &\leq \ N \int_0^1 g_1(s) ds \leq \rho_4, \text{ for } t \in [0,\, 1]. \end{aligned}$$

So, $\|T_1u\|_{C^1} \leq \|u\|_{C^1}$ for $\|u\|_{C^1} = \rho_4$.

Moreover, if $v \in K$ such that $\|v\|_{C^1} = \rho_4$, we have $|u(t)|+|u'(t)| \geq \rho_4 \geq M$,

$$h(t, u(t), u'(t)) \leq \mu \left( |u(t)| + |u'(t)| \right) \leq 2\mu\rho_4, \tag{1.18}$$

and $p(\rho_4) \leq 2\mu\rho_4$. Therefore

$$\begin{aligned} T_2v(t) &= \int_0^1 G(t,s) h(s,\, u(s),\, u'(s)) ds \leq \int_0^1 g_0(s) p(\rho_4) ds \\ &\leq \ p(\rho_4) \int_0^1 g_0(s) ds \leq 2\mu\rho_4 \int_0^1 g_0(s) ds \leq \rho_4, \end{aligned}$$

and

$$\begin{aligned} (T_2v(t))' &= \int_0^1 \frac{\partial G}{\partial t}(t,s) h(s,\, u(s),\, u'(s)) ds \leq \int_0^1 g_1(s) p(\rho_4) ds \\ &\leq \ p(\rho_4) \int_0^1 g_1(s) ds \leq 2\mu\rho_4 \int_0^1 g_1(s) ds \leq \rho_4. \end{aligned}$$

Consequently $\|T_2v\|_{C^1} \leq \|v\|_{C^1}$ for $\|v\|_{C^1} = \rho_4$.

**Case 6.3.** Suppose that $f$ is unbounded and $h$ is bounded.

Then, there is $N > 0$ such that $h(t, u(t), u'(t)) \leq N$, $\forall\, (u, u') \in [0,\, +\infty)^2$., and, by $(A4)$, there exists $M > 0$ such that $f(t, v(t), v'(t)) \leq \mu \left( |v| + |v'| \right)$, for $(|v| + |v'|) \geq M$ with $\mu$ satisfying (1.16).

Choosing $\rho_4$ as in (1.17), the arguments follow like in the previous case.

**Case 6.4.** Consider that $f$ and $h$ are unbounded.

By $(A4)$, there is $M > 0$ such that $f(t, v(t), v'(t)) \leq \mu(|v| + |v'|)$, $h(t, u(t), u'(t)) \leq \mu(|u| + |u'|)$ for $|v| + |v'| \geq M$ and $|u| + |u'| \geq M$ with $\mu$ as in (1.16).

Setting

$$\begin{aligned} p(r) &: = \max\{h(t, u(t), u'(t)) : t \in [0, 1], 0 \leq u \leq r, \ 0 \leq u' \leq r\}, \\ q(r) &: = \max\{f(t, v(t), v'(t)) : t \in [0, 1], 0 \leq v \leq r, \ 0 \leq v' \leq r\} \end{aligned}$$

we have

$$\lim_{r\to\infty} p(r) = +\infty \text{ and } \lim_{r\to\infty} q(r) = +\infty.$$

Choose

$$\rho_4 = \max\{2\rho_3, \, M\}$$

such that $p(\rho_4) \geq p(r)$ and $q(\rho_4) \geq q(r)$ for $0 \leq r \leq \rho_4$.

Let $u, \, v \in K$ and $\|u\|_{C^1} = \|v\|_{C^1} = \rho_4$.

Arguing as in (1.18) it can be easily shown that $\|T_1 u\|_{C^1} \leq \|u\|_{C^1}$, $\|T_2 v\|_{C^1} \leq \|v\|_{C^1}$.

By Lemma 1.1.5 the operators $T_1, \ T_2$ has a fixed point in $K \cap (\overline{\Omega_4} \backslash \Omega_3)$, therefore $T = (T_1, T_2)$ has a fixed point $(u, v)$ which is a positive solution of the initial problem.

Moreover these functions $u$ and $v$ are given by

$$\begin{cases} u(t) = \int_0^1 G(t,s) f(s, \, v(s), \, v'(s)) ds \\ \\ v(t) = \int_0^1 G(t,s) h(s, \, u(s), \, u'(s)) ds. \end{cases}$$

and are both increasing functions.

■

## 1.3 Example

Consider the following third order nonlinear system

$$\begin{cases} -u'''(t) = (t^2+1)\left(e^{-v(t)} + \sqrt{|v'(t)|}\right) \\ \\ -v'''(t) = (\, u(t)+1)^2 \arctan(|u'(t)|+1) \\ \\ u(0) = u'(0) = 0, u'(1) = \frac{3}{2} u'\left(\frac{1}{2}\right) \\ \\ v(0) = v'(0) = 0, v'(1) = \frac{3}{2} v'\left(\frac{1}{2}\right). \end{cases} \tag{1.19}$$

In fact this problem is a particular case of system (1.1) with

$$\begin{aligned} f(t,\, v(t),\, v'(t)) \quad &: \quad = \left(t^2+1\right)\left(e^{-v(t)}+\sqrt{|v'(t)|}\right) \\ h(t,\, u(t),\, u'(t)) \quad &: \quad = \left(\, u(t)+1\right)^2 \arctan\left(|u'(t)|+1\right), \\ \eta \quad &= \quad \frac{1}{2} \text{ and } \alpha = \frac{3}{2}. \end{aligned}$$

It can be easily check that the above functions are non-negative and verify the assumptions $(A3)$ and $(A4)$.

The functions $f$ and $h$ are non-negative, because they are product of non negative functions. Note that, $\forall t \in [0,\, 1]$ and $\forall (u(t),\ v(t)) \in (C^3[0,\, 1],\, (0,\ +\infty))$, $(t^2+1) \geq 1$, $e^{-v(t)} = \frac{1}{e^{v(t)}} \geq 0$, $\sqrt{|v'(t)|} \ \geq \ 0$, $\left(\, u(t)+1\right)^2 \ \geq \ 1$ and for definition of $\arctan\, :\, \mathbb{R} \to \left]-\frac{\pi}{2},\ \frac{\pi}{2}\right[$ is such that $\arctan(x) \to \frac{\pi}{2}$, as $x \to +\infty$ and $\arctan(x) \to -\frac{\pi}{2}$, as $x \to -\infty$. So, $f \geq 0$, $h \geq 0$.

Finally, as

$$\begin{aligned} \liminf_{t\in[0,1],\ \|v\|_{C^1}\to 0} \frac{(t^2+1)\left(e^{-v(t)}+\sqrt{|v'(t)|}\right)}{|v|+|v'|} \quad &= \quad +\infty, \\ \liminf_{t\in[0,1],\ \|u\|_{C^1}\to 0} \frac{\left(\, u(t)+1\right)^2 \arctan\left(|u'(t)|+1\right)}{|u|+|u'|} \quad &= \quad +\infty, \end{aligned}$$

condition $(A3)$ holds and

$$\begin{aligned} \limsup_{t\in[0,1],\ \|v\|_{C^1}\to +\infty} \frac{(t^2+1)\left(e^{-v(t)}+\sqrt{|v'(t)|}\right)}{|v|+|v'|} \quad &= \quad 0, \\ \limsup_{t\in[0,1],\ \|u\|_{C^1}\to +\infty} \frac{\left(\, u(t)+1\right)^2 \arctan\left(|u'(t)|+1\right)}{|u|+|u'|} \quad &= \quad 0, \end{aligned}$$

assumption $(A4)$ is satisfied.

Therefore, by Theorem 1.2.1, problem (1.19) has at least a positive solution $(u(t),\, v(t)) \in \left(C^3[0,\, 1]\right)^2$, that is $u(t) > 0$, $v(t) > 0$, $\forall t \in [0,\, 1]$.

# 2

## *Functional coupled systems with full nonlinear terms*

In this chapter we consider the boundary value problem composed by the coupled system of the second order differential equations with full nonlinearities

$$\begin{cases} u''(t) = f(t, u(t), v(t), u'(t), v'(t)), \ t \in [a, b], \\ v''(t) = h(t, u(t), v(t), u'(t), v'(t)) \end{cases} \tag{2.1}$$

where $f, h : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ continuous functions, and the functional boundary conditions

$$\begin{cases} u(a) = v(a) = 0 \\ L_1(u, u(b), u'(b)) = 0 \\ L_2(v, v(b), v'(b)) = 0, \end{cases} \tag{2.2}$$

where $L_1, L_2 : C\,[a, b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions verifying some monotone assumptions.

Ordinary differential systems have been studied by many authors, like, for instance, [10, 17, 85, 86, 91, 96, 101, 128, 139, 157, 158, 195, 210]. In particular, coupled second order ordinary differential systems can be applied to several real phenomena, such as, Lokta-Volterra models, reaction diffusion processes, prey-predator or other interaction systems, Sturm-Liouville problems, mathematical biology, chemical systems (see, for example, [9, 13, 14, 118, 144, 201] and the references therein).

In [189] the authors study the existence of solutions for the nonlinear second order coupled system

$$\begin{cases} -u''(t) = f_1(t, v(t)) \\ -v''(t) = f_2(t, v(t)), \end{cases} \quad t \in [0, 1],$$

with $f_1, f_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ continuous functions, together with the nonlinear boundary conditions

$$\begin{aligned} \varphi\,(u(0), v(0), u'(0), v'(0), u'(1), v'(1)) &= (0, 0), \\ \psi\,(u(0), v(0)) + (u(1), v(1)) &= (0, 0), \end{aligned}$$

where $\varphi : \mathbb{R}^6 \to \mathbb{R}^2$ and $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous functions.

In [46] it is provided some growth conditions on the nonnegative nonlinearities of the system

$$\begin{cases} -x''(t) = f_1(t, x(t), y(t)) \\ -y''(t) = f_2(t, x(t), y(t)), t \in (0,1), \\ x(0) = y(0) = 0, \\ x(1) = \alpha\,[y]\,, \\ y(1) = \beta\,[x]\,, \end{cases}$$

where $f_1, f_2 : (0,1) \times [0,+\infty)^2 \to [0,+\infty)$ are continuous and may be singular at $t = 0,1$, and $\alpha[x]$, $\beta[x]$ are bounded linear functionals on $C[0,1]$ given by

$$\alpha\,[y] = \int_0^1 y(t)dA(t),\ \ \beta\,[x] = \int_0^1 x(t)dB(t),$$

involving Stieltjes integrals, and $A, B$ are functions of bounded variation with positive measures.

Motivated by these works we consider the second order coupled fully differential equations (2.1) together with the functional boundary conditions (2.2). This chapter is based in [161], and to the best of our knowledge, it is the first time where these coupled differential systems embrace functional boundary conditions. Remark that, the functional dependence includes and generalizes the classical boundary conditions such as separated, multi-point, nonlocal, integro-differential, with maximum or minimum arguments,... More details on such conditions and their potentialities can be seen, for instance, in [35, 67, 70, 159, 198] and the references therein. Our main result is applied to a coupled mass-spring systems subject to a new type of global boundary data.

The arguments in this chapter follow lower and upper solutions method and fixed point theory. Therefore, the main result is an existence and localization theorem, as it provides not only the existence of solution, but a strip where the solution varies, as well. Due to an adequate auxiliary problem, including a convenient truncature, there is no need of sign, bound, monotonicity or other growth assumptions on the nonlinearities, besides the Nagumo condition.

## 2.1 Definitions and preliminaries

Let $E = C^1[a,b]$ be the Banach space equipped with the norm $\|\cdot\|_{C^1}$, defined by

$$\|w\|_{C^1} := \max\{\|w\|, \|w'\|\},$$

where

$$\|y\| := \max_{t\in[a,b]} |y(t)|$$

and $E^2 = (C^1[a,b])^2$ with the norm

$$\|(u,v)\|_{E^2} = \|u\|_{C^1} + \|v\|_{C^1}.$$

Forward in this work, we consider the following assumption

**(A)** The functions $L_1, L_2 : C[a,b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, nonincreasing in the first variable and nondecreasing in the second one.

To apply lower and upper solutions method we consider next definition:

**Definition 2.1.1** *A pair of functions* $(\alpha_1, \alpha_2) \in (C^2[a,b])^2$ *is a coupled lower solution of problem (2.1), (2.2) if*

$$\begin{aligned}
\alpha_1''(t) &\leq f(t, \alpha_1(t), \alpha_2(t), \alpha_1'(t), \alpha_2'(t)) \\
\alpha_2''(t) &\leq h(t, \alpha_1(t), \alpha_2(t), \alpha_1'(t), \alpha_2'(t)) \\
\alpha_1(a) &\leq 0 \\
\alpha_2(a) &\leq 0 \\
L_1(\alpha_1, \alpha_1(b), \alpha_1'(b)) &\geq 0 \\
L_2(\alpha_2, \alpha_2(b), \alpha_2'(b)) &\geq 0.
\end{aligned}$$

*A pair of functions* $(\beta_1, \beta_2) \in (C^2[a,b])^2$ *is a coupled upper solution of problem (2.1), (2.2) if it verifies the reverse inequalities.*

The Nagumo-type conditions is useful to obtain *a priori* bounds on the first derivatives of the unknown functions:

**Definition 2.1.2** *Let* $\alpha_1(t), \beta_1(t), \alpha_2(t)$ *and* $\beta_2(t)$ *be continuous functions such that*

$$\alpha_1(t) \leq \beta_1(t),\ \ \alpha_2(t) \leq \beta_2(t),\ \ \forall t \in [a,b]\,.$$

*The continuous functions* $f, h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ *satisfy Nagumo-type conditions relative to intervals* $[\alpha_1(t), \beta_1(t)]$ *and* $[\alpha_2(t), \beta_2(t)]$, *if, there are* $N_1 > r_1,\ N_2 > r_2$, *with*

$$r_1 \ :\ = \max\left\{\frac{\beta_1(b) - \alpha_1(a)}{b-a},\ \frac{\beta_1(a) - \alpha_1(b)}{b-a}\right\}, \tag{2.3}$$

$$r_2 \ :\ = \max\left\{\frac{\beta_2(b) - \alpha_2(a)}{b-a},\ \frac{\beta_2(a) - \alpha_2(b)}{b-a}\right\}, \tag{2.4}$$

*and continuous positive functions* $\varphi, \psi : [0, +\infty) \to (0, +\infty)$, *such that*

$$|f(t,x,y,z,w)| \leq \varphi\left(|z|\right), \tag{2.5}$$

*for*

$$\alpha_1(t) \leq x \leq \beta_1(t), \alpha_2(t) \leq y \leq \beta_2(t),\ \ \forall t \in [a,b], \text{and } w \in \mathbb{R}, \tag{2.6}$$

*and*

$$|h(t,x,y,z,w)| \leq \psi\left(|w|\right), \tag{2.7}$$

*for*

$$\alpha_1(t) \leq x \leq \beta_1(t), \alpha_2(t) \leq y \leq \beta_2(t),\ \ \forall t \in [a,b], \text{and } z \in \mathbb{R}, \tag{2.8}$$

*verifying*

$$\int_{r_1}^{N_1} \frac{ds}{\varphi(s)} > b-a, \int_{r_2}^{N_2} \frac{ds}{\psi(s)} > b-a. \tag{2.9}$$

**Lemma 2.1.1** *Let* $f, h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ *be continuous functions satisfying Nagumo-type conditions to (2.5) in (2.6) and (2.7) in (2.8).*
*Then for every solution* $(u,v) \in \left(C^2\,[a,b]\right)^2$ *verifying (2.6) and (2.8), there are* $N_1, N_2 > 0$, *given by (2.9), such that*

$$\|u'\| \leq N_1 \text{ and } \|v'\| \leq N_2. \tag{2.10}$$

**Proof** Let $(u(t), v(t))$ be a solution of (2.1) satisfying (2.6).

By Lagrange Theorem, there are $t_0, t_1 \in [a, b]$ such that

$$u'(t_0) = \frac{u(b) - u(a)}{b - a} \text{ and } v'(t_1) = \frac{v(b) - v(a)}{b - a}.$$

Suppose, by contradiction, that $|u'(t)| > r_1$, $\forall t \in [a, b]$, with $r_1$ given by (2.3). If $u'(t) > r_1$, $\forall t \in [a, b]$, by (2.6) we obtain the following contradiction with (2.3):

$$u'(t_0) = \frac{u(b) - u(a)}{b - a} \leq \frac{\beta_1(b) - \alpha_1(a)}{b - a} \leq r_1.$$

If $u'(t) < -r_1$, $\forall t \in [a, b]$, the contradiction is similar.

In the case where $|u'(t)| \leq r_1$, $\forall t \in [a, b]$, the proof will be finished.

So, assume that there are $t_2, t_3 \in [a, b]$ such that $t_2 < t_3$, and

$$u'(t_2) < r_1 \text{ and } u'(t_3) > r_1.$$

By continuity, there is $t_4 \in [t_2, t_3]$ such that

$$u'(t_4) = r_1 \text{ and } u'(t_3) > r_1, \ \forall t \in ]t_4, t_3].$$

So, by a convenient change of variable, by (2.5), (2.6), and (2.9), we obtain

$$\begin{aligned} \int_{u'(t_4)}^{u'(t_3)} \frac{ds}{\varphi(|s|)} &= \int_{t_4}^{t_3} \frac{u''(t)}{\varphi(|u'(t)|)} dt \leq \int_a^b \frac{|u''(t)|}{\varphi(|u'(t)|)} dt \\ &= \int_a^b \frac{|f(t, u(t), v(t), u'(t), v'(t)|}{\varphi(|u'(t)|)} dt \leq b - a < \int_{r_1}^{N_1} \frac{ds}{\varphi(|s|)}. \end{aligned}$$

Therefore $u'(t_3) < N_1$, and, as $t_3$ is taken arbitrarily, $u'(t_3) < N_1$, for values of $t$ where $u'(t) > r_1$.

If $t_2 > t_3$, the technique is analogous for $t_4 \in [t_3, t_2]$.

The same conclusion can be achieved if there are $t_2, t_3 \in [a, b]$ such that

$$u'(t_2) > -r_1 \text{ and } u'(t_3) < -r_1.$$

Therefore $\|u'\| \leq N_1$ and, by similar arguments, it can be proved that $\|v'\| \leq N_2$.

■

For the reader's convenience we present Schauder's fixed point theorem:

**Theorem 2.1.1** *([206]) Let $Y$ be a nonempty, closed, bounded and convex subset of a Banach space $X$, and suppose that $P : Y \to Y$ is a compact operator. Then $P$ as at least one fixed point in $Y$.*

## 2.2 Main result

Along this chapter we denote $(a,\ b) \leq (c,\ d)$ meaning that $a \leq c$ and $b \leq d$, for $a, b, c, d \in \mathbb{R}$.

**Theorem 2.2.1** *Let $f, h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous functions, and assume that hypothesis (A) holds. If there are coupled lower and upper solutions of (2.1)-(2.2), $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$, respectively, according Definition 2.1.1, such that*

$$(\alpha_1(t), \alpha_2(t)) \leq (\beta_1(t), \beta_2(t)), \ \forall t \in [a,b], \tag{2.11}$$

*and $f$ and $h$ verify the Nagumo conditions, then there is at least a pair $(u(t), v(t)) \in \left(C^2[a,b],\, \mathbb{R}\right)^2$ solution of (2.1)-(2.2) and, moreover,*

$$\alpha_1(t) \leq u(t) \leq \beta_1(t), \quad \alpha_2(t) \leq v(t) \leq \beta_2(t), \ \forall t \in [a,b],$$

*and*

$$\|u'\| \leq N_1 \text{ and } \|v'\| \leq N_2,$$

*with $N_1$ and $N_2$ given by Lemma 2.1.1.*

**Proof** Consider the auxiliary functions $F(t, x, y, z, w) := F$ defined as

- $f\left(t, \beta_1(t), \beta_2(t), \beta_1'(t), \beta_2'(t)\right) - \frac{x-\beta_1(t)}{1+|x-\beta_1(t)|} - \frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $x > \beta_1(t)$, $y > \beta_2(t)$,

- $f\left(t, u(t), v(t), u'(t), v'(t)\right) - \frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $\alpha_1(t) \leq x \leq \beta_1(t)$, $y > \beta_2(t)$,

- $f\left(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)\right)-\frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|}+\frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $x<\alpha_1(t)$, $y>\beta_2(t)$,
- $f\left(t,\beta_1(t),\beta_2(t),\beta_1'(t),\beta_2'(t)\right)-\frac{x-\beta_1(t)}{1+|x-\beta_1(t)|}$, if $x>\beta_1(t)$, $\alpha_2(t)\le y\le\beta_2(t)$,
- $f\left(t,u(t),v(t),u'(t),v'(t)\right)$ if $\alpha_1(t)\le x\le\beta_1(t)$, $\alpha_2(t)\le y\le\beta_2(t)$,
- $f\left(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)\right)-\frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|}$ if $x<\alpha_1(t)$, $\alpha_2(t)\le y\le\beta_2(t)$,
- $f\left(t,\beta_1(t),\beta_2(t),\beta_1'(t),\beta_2'(t)\right)-\frac{x-\beta_1(t)}{1+|x-\beta_1(t)|}+\frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $x>\beta_1(t)$, $y<\alpha_2(t)$,
- $f\left(t,u(t),v(t),u'(t),v'(t)\right)-\frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $\alpha_1(t)\le x\le\beta_1(t)$, $y<\alpha_2(t)$,
- $f\left(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)\right)-\frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|}-\frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $x<\alpha_1(t)$, $y<\alpha_2(t)$,

and $H(t,x,y,z,w):=H$ given by

- $h\left(t,\beta_1(t),\beta_2(t),\beta_1'(t),\beta_2'(t)\right)-\frac{x-\beta_1(t)}{1+|x-\beta_1(t)|}-\frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $x>\beta_1(t)$, $y>\beta_2(t)$,
- $h\left(t,u(t),v(t),u'(t),v'(t)\right)-\frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $\alpha_1(t)\le x\le\beta_1(t)$, $y>\beta_2(t)$,
- $h\left(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)\right)-\frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|}+\frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $x<\alpha_1(t)$, $y>\beta_2(t)$,
- $h\left(t,\beta_1(t),\beta_2(t),\beta_1'(t),\beta_2'(t)\right)-\frac{x-\beta_1(t)}{1+|x-\beta_1(t)|}$, if $x>\beta_1(t)$, $\alpha_2(t)\le y\le\beta_2(t)$,
- $h\left(t,u(t),v(t),u'(t),v'(t)\right)$ if $\alpha_1(t)\le x\le\beta_1(t)$, $\alpha_2(t)\le y\le\beta_2(t)$.
- $h\left(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)\right)-\frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|}$ if $x<\alpha_1(t)$, $\alpha_2(t)\le y\le\beta_2(t)$,

- $h\left(t,\beta_1(t),\beta_2(t),\beta_1'(t),\beta_2'(t)\right)-\frac{x-\beta_1(t)}{1+|x-\beta_1(t)|}+\frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $x>\beta_1(t)$, $y<\alpha_2(t)$,
- $h\left(t,u(t),v(t),u'(t),v'(t)\right)-\frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $\alpha_1(t)\le x\le\beta_1(t)$, $y<\alpha_2(t)$,
- $h\left(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)\right)-\frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|}-\frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $x<\alpha_1(t)$, $y<\alpha_2(t)$,

and the auxiliary problem

$$\left\{\begin{array}{c} u''(t)=F(t,u(t),v(t),u'(t),v'(t)) \\ v''(t)=H(t,u(t),v(t),u'(t),v'(t)) \\ u(a)=v(a)=0 \\ u(b)=\delta_1\left(b,u\left(b\right)+L_1(u,u(b),u'(b))\right) \\ v(b)=\delta_2\left(b,v\left(b\right)+L_2(v,v(b),v'(b))\right), \end{array}\right. \tag{2.12}$$

where, for each $i=1,2$,

$$\delta_i(t,w)=\left\{\begin{array}{ccc} \beta_i(t) & , & w>\beta_i(t) \\ w & , & \alpha_i(t)\le w\le\beta_i(t) \\ \alpha_i(t) & , & w<\alpha_i(t). \end{array}\right. \tag{2.13}$$

**Claim 1**: *Solutions of problem (2.12) can be written as*

$$\begin{aligned} u(t) &= \frac{t-a}{b-a}\delta_1\left(b,u\left(b\right)+L_1(u,u(b),u'(b))\right) \\ &\quad+\int_a^b G(t,s)F(s,u(s),v(s),u'(s),v'(s))ds \\ v(t) &= \frac{t-a}{b-a}\delta_2\left(b,v\left(b\right)+L_2(v,v(b),v'(b))\right) \\ &\quad+\int_a^b G(t,s)H(s,u(s),v(s),u'(s),v'(s))ds, \end{aligned}$$

*where*

$$G(t,s)=\frac{1}{b-a}\left\{\begin{array}{ll} (a-s)(b-t), & a\le t\le s\le b \\ (a-t)(b-s), & a\le s\le t\le b. \end{array}\right. \tag{2.14}$$

In fact, for the equation $u''(t)=F(t)$, the solution is,

$$u(t)=At+B+\int_a^t(t-s)F(s)ds, \tag{2.15}$$

for some $A, B \in \mathbb{R}$.

By the boundary conditions, it follows that,

$$A = \frac{1}{b-a}\delta_1\left(b, u\left(b\right) + L_1(u, u(b), u'(b))\right) \tag{2.16}$$

$$-\frac{1}{b-a}\int_a^b (b-s)F(s)ds, \tag{2.17}$$

and

$$\begin{aligned} B &= -\frac{a}{b-a}\delta_1\left(b, u\left(b\right) + L_1(u, u(b), u'(b))\right) \\ &\quad + \frac{a}{b-a}\int_a^b (b-s)F(s)ds. \end{aligned}$$

By (2.15) and (2.17), then

$$\begin{aligned} u(t) &= \frac{t-a}{b-a}\delta_1\left(b, u\left(b\right) + L_1(u, u(b), u'(b))\right) \\ &\quad - \frac{t-a}{b-a}\int_a^b (b-s)F(s)ds + \int_a^t (t-s)F(s)ds \\ &= \frac{t-a}{b-a}\delta_1\left(b, u\left(b\right) + L_1(u, u(b), u'(b))\right) \\ &\quad + \int_a^t \left(\frac{(a-t)(b-s)}{b-a} + t - s\right)F(s)ds \\ &\quad + \int_t^b \frac{(a-t)(b-s)}{b-a}F(s)ds \\ &= \frac{t-a}{b-a}\delta_1\left(b, u\left(b\right) + L_1(u, u(b), u'(b))\right) \\ &\quad + \int_a^t \left(\frac{(a-s)(b-t)}{b-a} + t - s\right)F(s)ds \\ &\quad + \int_t^b \frac{(a-t)(b-s)}{b-a}F(s)ds \\ &= \frac{t-a}{b-a}\delta_1\left(b, u\left(b\right) + L_1(u, u(b), u'(b))\right) + \int_a^t G(t,s)F(s)ds, \end{aligned}$$

with $G(t,s)$ given by (2.14).

The integral form of $v(t)$ can be achieved by the same arguments.

**Claim 2**: *Every solution $(u,v)$ of (2.12) satisfies*

$$\|u'\| \leq N_1 \text{ and } \|v'\| \leq N_2,$$

*with $N_1$ and $N_2$ given by Lemma 2.1.1.*

This claim is a direct consequence of Lemma *2.1.1, as* $F(t,x,y,z,w)$ and $H(t,x,y,z,w)$ verify de Nagumo-type conditions.

Define the operators $T_1 : (C^1[a,b])^2 \to C^1[a,b]$ and $T_2 : (C^1[a,b])^2 \to C^1[a,b]$ such that

$$\begin{aligned} T_1(u,v)(t) &= \tfrac{t-a}{b-a}\delta_1(b,u(b)+L_1(u,u(b),u'(b))) \\ &\quad + \textstyle\int_a^b G(t,s)F(s,u(s),v(s),u'(s),v'(s))ds \\ T_2(u,v)(t) &= \tfrac{t-a}{b-a}\delta_2(b,v(b)+L_2(v,v(b),v'(b))) \\ &\quad + \textstyle\int_a^b G(t,s)H(s,u(s),v(s),u'(s),v'(s))ds, \end{aligned} \tag{2.18}$$

where $G(t,s)$ is given by (2.14), and $T : (C^1[a,b])^2 \to (C^1[a,b])^2$ by

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)). \tag{2.19}$$

By Claim 1, fixed points of the operator $T := (T_1,T_2)$ are solutions of problem (2.12) .

**Claim 3**: *The operator $T$, given by (2.19) has a fixed point $(u_0,v_0)$.*

In order to apply Theorem 2.1.1, we will prove the following steps for operator $T_1(u,v)$. The proof for the operator $T_2(u,v)$ is analogous.

(i) $T_1 : (C^1[a,b])^2 \to C^1[a,b]$ *is well defined.*

The function $F$ is bounded by the Nagumo-type conditions, (2.5) and (2.6), and the Green function $G(t,s)$ is continuous in $[a,b]^2$, then the operator $T_1(u,v)$ is continuous. Moreover, as $\frac{\partial G}{\partial t}(t,s)$ is bounded in $[a,b]^2$ and

$$\begin{aligned} (T_1(u,v))'(t) &= \frac{1}{b-a}\delta_1(b,u(b)+L_1(u,u(b),u'(b))) \\ &\quad + \int_a^b \frac{\partial G}{\partial t}(t,s)\ F(s,u(s),v(s),u'(s),v'(s))ds, \end{aligned}$$

with

$$\frac{\partial G}{\partial t}(t,s) = \frac{1}{b-a}\begin{cases} s-a, & a \le t \le s \le b \\ s-b, & a \le s \le t \le b, \end{cases}$$

verifying

$$\left|\frac{\partial G}{\partial t}(t,s)\right| \le 1,\ \forall (t,s) \in [a,b]^2,$$

therefore, $(T_1(u,v))'$ is continuous on $[a,b]$. So, $T_1 \in C^1[a,b]$.

(ii) *$TB$ is uniformly bounded, for $B$ a bounded set in $(C^1[a,b])^2$.*

Let $B$ be a bounded set of $(C^1[a,b])^2$. Then there exists $K > 0$ such that

$$\|(u,v)\|_{E^2} = \|u\|_{C^1} + \|v\|_{C^1} \le K,\ \forall (u,v) \in B.$$

By (2.13), and taking into account that $F$ and $H$ are bounded, then there are $M_1$, $M_2$, $M_3 > 0$ such that

$$\delta_1 \le \max\{\|\alpha_1\|, \|\beta_1\|\} := M_1,$$
$$\int_a^b \max_{t\in[a,b]} |G(t,s)|\,|F(s,u(s),v(s),u'(s),v'(s))|\,ds \le M_2,$$
$$\int_a^b \max_{t\in[a,b]} \left|\frac{\partial G}{\partial t}(t,s)\right| |F(s,u(s),v(s),u'(s),v'(s))|\,ds \le M_3.$$

Moreover,

$$\begin{aligned}
\|T_1(u,v)(t)\| &= \max_{t\in[a,b]}\left|\frac{t-a}{b-a}\delta_1(b,u(b)+L_1(u,u(b),u'(b)))\right. \\
&\quad \left. + \int_a^b G(t,s)F(s,u(s),v(s),u'(s),v'(s))ds\right| \\
&\le \max_{t\in[a,b]}\left|\frac{t-a}{b-a}\right| |\delta_1(b,u(b)+L_1(u,u(b),u'(b)))| \\
&+ \int_a^b \max_{t\in[a,b]} |G(t,s)|\,|F(s,u(s),v(s),u'(s),v'(s))|\,ds \\
&\le M_1 + M_2 < +\infty,\ \forall (u,v) \in B,
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \left(T_1\left(u,v\right)\right)'(t)\right\| = \max_{t\in[a,\,b]}\left|\frac{1}{b-a}\delta_1\left(b, u\left(b\right) + L_1(u,u(b),u'(b))\right)\right. \\
&\left. + \int_a^b \frac{\partial G}{\partial t}(t,s)F(s,u(s),v(s),u'(s),v'(s))ds\right| \\
\leq\ & \frac{M_1}{b-a} + \int_a^b \max_{t\in[a,\,b]}\left|\frac{\partial G}{\partial t}(t,s)\right| \left|F(s,u(s),v(s),u'(s),v'(s))\right| ds \\
\leq\ & \frac{M_1}{b-a} + M_3 < +\infty,\ \forall\left(u,v\right)\in B.
\end{aligned}$$

So, $TB$ is uniformly bounded, for $B$ a bounded set in $\left(C^1\left[a,b\right]\right)^2$ .

(iii) $TB$ *is equicontinuous on* $\left(C^1\left[a,b\right]\right)^2$.

Let $t_1$ and $t_2 \in [a,b]$. Without loss of generality suppose $t_1 \leq t_2$. As $G(t,s)$ is uniformly continuous and $F$ is bounded, then

$$\begin{aligned}
& \left|T_1\left(u,v\right)\left(t_1\right) - T_1\left(u,v\right)\left(t_2\right)\right| \\
=\ & \left|\frac{(t_1-a)-(t_2-a)}{b-a}\delta_1\left(b, u\left(b\right) + L_1(u,u(b),u'(b))\right)\right. \\
& \left. + \int_a^b \left[G(t_1,s) - G(t_2,s)\right] F(s,u(s),v(s),u'(s),v'(s))ds\right| \\
\leq\ & \left|\frac{t_1-t_2}{b-a}\right| M_1 \\
& + \left|\int_a^b \left[G(t_1,s) - G(t_2,s)\right]\right| F(s,u(s),v(s),u'(s),v'(s))ds \\
\rightarrow\ & 0,\ \text{as } t_1 \rightarrow t_2,
\end{aligned}$$

and

$$\begin{aligned} & \left|(T_1(u,v)(t_1))' - (T_1(u,v)(t_2))'\right| \\ = & \left|\int_a^b \left[\frac{\partial G}{\partial t}(t_1,s) - \frac{\partial G}{\partial t}(t_2,s)\right] F(s,u(s),v(s),u'(s),v'(s))ds\right| \\ \leq & \int_a^{t_1} \left|\frac{\partial G}{\partial t}(t_1,s) - \frac{\partial G}{\partial t}(t_2,s)\right| \left|F(s,u(s),v(s),u'(s),v'(s))\right| ds \\ & + \int_{t_1}^{t_2} \left|\frac{\partial G}{\partial t}(t_1,s) - \frac{\partial G}{\partial t}(t_2,s)\right| \left|F(s,u(s),v(s),u'(s),v'(s))\right| ds \\ & + \int_{t_2}^{b} \left|\frac{\partial G}{\partial t}(t_1,s) - \frac{\partial G}{\partial t}(t_2,s)\right| \left|F(s,u(s),v(s),u'(s),v'(s))\right| ds. \end{aligned}$$

As the function $\frac{\partial G}{\partial t}(t,s)$ has only a jump discontinuity at $t = s$, therefore, as previously, the first and third integrals tend to 0, as $t_1 \to t_2$. For the second integral, as the functions $\frac{\partial G}{\partial t}(t_1,s)$ and $\frac{\partial G}{\partial t}(t_2,s)$ are uniformly continuous, for $s \in [a,t_1[\cup]t_1,b]$ and $s \in [a,t_2[\cup]t_2,b]$, respectively, and

$$\left|\frac{\partial G}{\partial t}(t_1,s) - \frac{\partial G}{\partial t}(t_2,s)\right| \left|F(s,u(s),v(s),u'(s),v'(s))\right|$$

is bounded, then

$$\int_{t_1}^{t_2} \left|\frac{\partial G}{\partial t}(t_1,s) - \frac{\partial G}{\partial t}(t_2,s)\right| \left|F(s,u(s),v(s),u'(s),v'(s))\right| ds \to 0,$$

as $t_1 \to t_2$.

By the Arzèla-Ascoli Theorem $T_1(u,v)$ is compact in $\left(C^1[a,b]\right)^2$.

Following similar arguments with $K_1,\ K_2,\ K_3 > 0$ such that

$$\begin{gathered} \delta_2 \leq \max\{\|\alpha_2\|, \|\beta_2\|\} := K_1, \\ \int_a^b \max_{t\in[a,b]} |G(t,s)| \left|H(s,u(s),v(s),u'(s),v'(s))\right| ds \leq K_2, \\ \int_a^b \max_{t\in[a,b]} \left|\frac{\partial G}{\partial t}(t,s)\right| \left|H(s,u(s),v(s),u'(s),v'(s))\right| ds \leq K_3, \end{gathered}$$

it can be shown that $T_2(u,v)$ is compact in $\left(C^1[a,b]\right)^2$, too.

(iv) $TD \subset D$ for some $D \subset \left(C^1[a,b]\right)^2$ *a closed and bounded set.*

Suppose $D \subset \left(C^1[a,b]\right)^2$ defined by

$$D = \left\{(u,v) \in \left(C^1[a,b]\right)^2 : \|(u,v)\|_{E^2} \leq 2\rho\right\},$$

where $\rho$ is such that

$$\rho := \max\left\{M_1 + M_2, \frac{M_1}{b-a} + M_3, K_1 + K_2, \frac{K_1}{b-a} + K_3, N_1, N_2\right\},$$

with $N_1, N_2$ given by (2.10).

Arguing as in Claim 3 (ii), it can be shown that

$$\begin{aligned} \|T_1(u,v)\| &\leq M_1 + M_2 \leq \rho, \\ \left\|(T_1(u,v))'\right\| &\leq \frac{M_1}{b-a} + M_3 \leq \rho \end{aligned}$$

and, therefore, $\|T_1(u,v)\|_{C^1} \leq \rho$.

Analogously $\|T_2(u,v)\|_{C^1} \leq \rho$ and, so,

$$\begin{aligned} \|T(u,v)\|_{E^2} &= \|(T_1(u,v), T_2(u,v))\|_{E^2} \\ &= \|T_1(u,v)\|_{C^1} + \|T_2(u,v)\|_{C^1} \leq 2\rho. \end{aligned}$$

By Theorem 2.1.1, the operator $T$, given by (2.19) has a fixed point $(u_0, v_0)$.

**Claim 4:** *This fixed point* $(u_0, v_0)$ *is also solution of the initial problem (2.1), (2.2), if every solution of (2.12) verifies*

$$\alpha_1(t) \leq u_0(t) \leq \beta_1(t),\ \alpha_2(t) \leq v_0(t) \leq \beta_2(t),\ \forall t \in [a,b] \quad (2.20)$$

$$\alpha_1(b) \leq u_0(b) + L_1(u_0, u_0(b), u_0'(b)) \leq \beta_1(b), \quad (2.21)$$

$$\alpha_2(b) \leq v_0(b) + L_2(v_0, v_0(b), v_0'(b)) \leq \beta_2(b). \quad (2.22)$$

Let $(u_0, v_0)$ be a fixed point of $T$, that is $(u_0, v_0)$ is a fixed point of $T_1$ and $T_2$.

By Claim 1, $(u_0, v_0)$ is solution of problem (2.12).

In the following we will prove the estimations for $u_0$, as for $v_0$ the procedure is analogous.

Suppose, by contradiction, that the first inequality of 2.20 is not true. So, there exists $t \in [a,b]$ such that $\alpha_1(t) > u_0(t)$ and it can be defined

$$\max_{t\in[a,b]} (\alpha_1(t) - u_0(t)) := \alpha_1(t_0) - u_0(t_0) > 0. \quad (2.23)$$

Remark that, by (2.12), Definition 2.1.1 and (2.13), $t_0 \neq a$, as $\alpha_1(a) - u_0(a) \leq 0$, and $t_0 \neq b$, because

$$\alpha_1(b) - u_0(b) = \alpha_1(b) - \delta_1\left(b, u_0(b) + L_1(u_0, u_0(b), u_0'(b))\right) \leq 0.$$

Then $t_0 \in ]a, b[$,

$$\alpha_1'(t_0) - u_0'(t_0) = 0 \text{ and } \alpha_1''(t_0) - u_0''(t_0) \leq 0. \tag{2.24}$$

There are three possibilities for the value of $v_0(t_0)$ :

- If $v_0(t_0) > \beta_2(t_0)$, then, by (2.12) and Definition 2.1.1, the following contradiction with (2.24) is obtained

$$\begin{aligned} u_0''(t_0) &= F(t_0, u_0(t_0), v_0(t_0), u_0'(t_0), v_0'(t_0)) \\ &= f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) \\ &\quad - \frac{u_0(t_0) - \alpha_1(t_0)}{1 + |u_0(t_0) - \alpha_1(t_0)|} + \frac{v_0(t_0) - \beta_2(t_0)}{1 + |v_0(t_0) - \beta_2(t_0)|} \\ &> f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) \geq \alpha_1''(t_0). \end{aligned}$$

- If $\alpha_2(t_0) \leq v_0(t_0) \leq \beta_2(t_0)$, then

$$\begin{aligned} u_0''(t_0) &= F(t_0, u_0(t_0), v_0(t_0), u_0'(t_0), v_0'(t_0)) \\ &= f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) \\ &\quad - \frac{u_0(t_0) - \alpha_1(t_0)}{1 + |u_0(t_0) - \alpha_1(t_0)|} \\ &> f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) \geq \alpha_1''(t_0). \end{aligned}$$

- If $\alpha_2(t_0) < v_0(t_0)$, the contradiction is

$$\begin{aligned} u_0''(t_0) &= F(t_0, u_0(t_0), v_0(t_0), u_0'(t_0), v_0'(t_0)) \\ &= f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) \\ &\quad - \frac{u_0(t_0) - \alpha_1(t_0)}{1 + |u_0(t_0) - \alpha_1(t_0)|} + \frac{v_0(t_0) - \alpha_2(t_0)}{1 + |v_0(t_0) - \alpha_2(t_0)|} \\ &> f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) \geq \alpha_1''(t_0). \end{aligned}$$

Therefore $\alpha_1(t) \leq u_0(t)$, $\forall t \in [a, b]$.

By a similar technique it can be shown that $u_0(t) \leq \beta_1(t)$, $\forall t \in [a, b]$, and so,

$$\alpha_1(t) \leq u_0(t) \leq \beta_1(t), \forall t \in [a, b]. \tag{2.25}$$

Assume now that, to prove the first inequality of (2.21),

$$\alpha_1(b) > u_0\,(b) + L_1(u_0, u_0(b), u_0'(b)). \tag{2.26}$$

Then, by (2.12) and (2.13),

$$u_0\,(b) = \delta_1\,(b, u_0\,(b) + L_1(u_0, u_0(b), u_0'(b))) = \alpha_1(b),$$

and

$$u_0'(b) \le \alpha_1'(b).$$

By (2.26), the assumption (A) and Definition 2.1.1, we have the contradiction

$$\begin{aligned} 0 &> L_1(u_0, u_0(b), u_0'(b)) + u_0\,(b) - \alpha_1(b) \\ &\ge L_1(u_0, u_0(b), u_0'(b)) \ge L_1(\alpha_1, \alpha_1(b), \alpha_1'(b)) \ge 0. \end{aligned}$$

Then $\alpha_1(b) \le u_0\,(b) + L_1(u_0, u_0(b), u_0'(b))$.

To prove the second inequality of (2.21), assume that

$$u_0\,(b) + L_1(u_0, u_0(b), u_0'(b)) > \beta_1(b). \tag{2.27}$$

Then, by (2.12) and (2.13),

$$u_0\,(b) = \delta_1\,(b, u_0\,(b) + L_1(u_0, u_0(b), u_0'(b))) = \beta_1(b). \tag{2.28}$$

By (2.27), (2.28) and Definition 2.1.1, we have a contradiction with (2.27).

So, $u_0\,(b) + L_1(u_0, u_0(b), u_0'(b)) \le \beta_1(b)$, and, therefore, (2.21) holds.

To prove (2.22) the technique is analogous.

So, the fixed point $(u_0, v_0)$ of $T$, solution of problem (2.12), is a solution of problem (2.1), (2.2), too.

■

## 2.3 Example

Consider the boundary value problem composed by the coupled system constituted by the second order differential equations with full nonlinearities

$$\begin{cases} u''(t) = -u'(t)\ v(t) + \arctan(u(t)\ v'(t)) + t, \\ v''(t) = t^2\left[-e^{-|u'(t)|}\ v(t) + u(t)\,(v'(t)-2)\right] \end{cases} \tag{2.29}$$

with $t \in [0,\, 1]$, and the functional boundary conditions

$$\begin{cases} u(0) = v(0) = 0 \\ u(1) = \max\limits_{t\in[0,1]} u(t) - (u'(1))^2 \\ v(1) = 2\int\limits_0^1 v(s)ds - \frac{(v'(1))^3}{8}. \end{cases} \tag{2.30}$$

This problem is a particular case of system (2.1)-(2.2) with

$$\begin{aligned} f(t,x,y,z,w) &= -z\ y + \arctan(x\ w) + t, \\ h(t,x,y,z,w) &= t^2\left[-e^{-|z|}\ y + x\,(w-2)\right], \end{aligned} \tag{2.31}$$

continuous functions, $t \in [0,\, 1]$, $a = 0$, $b = 1$, and

$$\begin{aligned} L_1(w,x,y) &= x - \max_{t\in[0,1]} w(t) + y^2, \\ L_2(w,x,y) &= x - 2\int\limits_0^1 w(s)ds + \frac{y^3}{8}. \end{aligned} \tag{2.32}$$

Remark that, $L_1$ and $L_2$ are continuous functions, verifying (A).

The functions given by

$$(\alpha_1(t),\, \alpha_2(t)) = (-1,\, -1) \text{ and } (\beta_1(t),\, \beta_2(t)) = (3+t,\, 2t+2)$$

are, respectively, lower and upper solutions of problem (2.29)-(2.30), satisfying (2.11), as, by Definition 2.1.1, we have

$$\begin{aligned} &\alpha_1''(t) = 0 \le f(t,-1,-1,0,0) = t,\ t \in [0,\, 1] \\ &\alpha_2''(t) = 0 \le h(t,-1,-1,0,0) = 3t^2,\ t \in [0,\, 1] \\ &L_1(\alpha_1, \alpha_1(1), \alpha_1'(1)) = 0 \ge 0,\ L_2(\alpha_2, \alpha_2(b), \alpha_2'(b)) = \frac{7}{8} \ge 0, \end{aligned}$$

and

$$\beta_1(t)''(t) = 0 \geq f(t, 3+t, 2t+2, 1, 2)$$
$$= -(2t+2) + \arctan(6t+2) + t,\ t \in [0,\ 1]$$
$$\beta_2''(t) = 0 \geq h(t, 3+t, 2t+2, 1, 2) = -\frac{2t^3+2t^2}{e},\ t \in [0,\ 1]$$
$$L_1(\beta_1, \beta_1(1), \beta_1'(1)) = 0 \leq 0,\ L_2(\beta_2, \beta_2(b), \beta_2'(b)) = -1 \leq 0.$$

Furthermore, the functions $f$ and $h$, given by (2.31), satisfy Nagumo conditions relative to the intervals $[-1, 2+t]$ and $[-1, 2t+1]$, for $t \in [0,\ 1]$, with $r_1 = r_2 = 4$,

$$|f(t,x,y,z,w)| \leq 3|z| + \frac{\pi}{2} + 1 := \varphi(|z|),$$

$$|h(t,x,y,z,w)| \leq 3(1 + |w-2|) := \psi(|w|),$$

and

$$\int_4^{N_1} \frac{ds}{\varphi(|s|)} = \int_4^{N_1} \frac{1}{3s + \frac{\pi}{2} + 1} ds > 1,$$

for $N_1 \geq 100$ and

$$\int_4^{N_2} \frac{ds}{\psi(|s|)} = \int_4^{N_2} \frac{1}{3(1 + |s-2|)\, ds} > 1,$$

for $N_2 \geq 80$.

Then, by Theorem 2.2.1, there is at least a pair $(u(t), v(t)) \in (C^2[0,1], \mathbb{R})^2$ solution of (2.29)-(2.30) and, moreover,

$$-1 \leq u(t) \leq 2+t, \quad -1 \leq v(t) \leq 2t+1,\ \forall t \in [0,1],$$
$$\|u'\| \leq N_1 \text{ and } \|v'\| \leq N_2.$$

## 2.4 Coupled mass-spring system

Consider the mass-spring system composed by two springs with constants of proportionality $k_1$ and $k_2$, and two weights of mass $m_1$ and $m_2$. The mass $m_1$ is suspended vertically from a fixed support

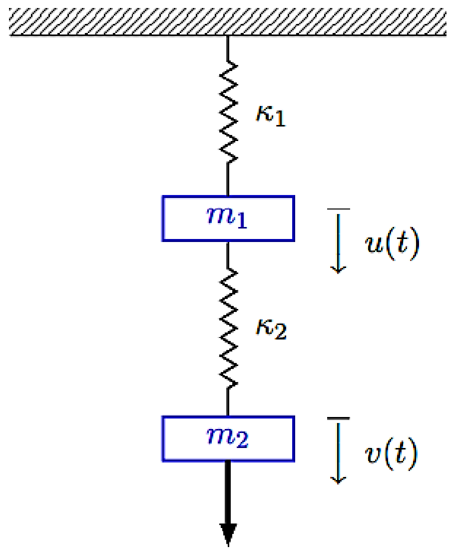


**FIGURE 2.1**
The coupled springs

by a spring with constant $k_1$ and the mass $m_2$ is attached to the first weight by a spring with constant $k_2$. The system described is illustrated in the Figure 2.1.

Let us call $u(t)$ and $v(t)$ the displacements of the weights of mass $m_1$ and $m_2$, respectively, in relation to their respective equilibrium positions. Thus, at time $t$, the position of the displacement of the mass $m_1$ is $u(t)$ and the displacement of mass $m_2$ is $v(t)$.

For simplicity we consider $t \in [0, 1]$, and, therefore, $u(0)$ and $u(1)$ are the initial and final displacement of mass $m_1$, and $v(0)$ and $v(1)$ are the similar displacements of mass $m_2$.

As it can be seen in [60], the above system is modelled by the second order nonlinear system of differential equations forced and with friction

$$\begin{cases} m_1 u''(t) = -\delta_1 u'(t) - \kappa_1 u(t) + \mu_1 (u(t))^3 - \kappa_2 (u(t) - v(t)) \\ \qquad\qquad\qquad + \mu_2 (u(t) - v(t))^3 + F_1 \cos(\omega_1 t), \\ \\ m_2 v''(t) = -\delta_2 v'(t) - \kappa_2 (v(t) - u(t)) + \mu_2 (v(t) - u(t))^3 \\ \qquad\qquad\qquad + F_2 \cos(\omega_2 t) \end{cases} \tag{2.33}$$

where $t \in [0,\ 1]$,

- $\delta_1$, $\delta_2$ are the damping coefficients;

- $\mu_1$, $\mu_2$ are the coefficients of the nonlinear terms of each system equation;
- $\kappa_2(u(t)-v(t))+\mu_2(u(t)-v(t))^3$ and $\kappa_2(v(t)-u(t))+\mu_2(v(t)-u(t))^3$ are the nonlinear restoring forces;
- $F_1$, $F_2$ are the forcing amplitudes of the sinusoidal forces $F_1\cos(\omega_1 t)$ and $F_2\cos(\omega_2 t)$, where $\omega_1$, $\omega_2$ are the forcing frequencies.

In this work we add to the system the functional boundary conditions

$$\begin{cases} u(0)=v(0)=0 \\ u(1)=\max\limits_{t\in[0,1]} u(t)+2u'(1) \\ v(1)=\max\limits_{t\in[0,1]} v(t)+2\left(v'(1)\right)^3 . \end{cases} \tag{2.34}$$

The functional conditions (2.34) can have a physical meaning such as, for example, the first one can be seen as the displacement of mass 1 at the final moment given by the sum of the maximum displacement in this period of time, with the double of the velocity of the displacement at the end point.

Clearly, the above model (2.33), (2.34) is a particular case of system (2.1)-(2.2) with

$$\begin{aligned} f(t,x,y,z,w) &= \tfrac{1}{m_1}\left[\begin{array}{c} -\delta_1 z-\kappa_1 x+\mu_1 x^3-\kappa_2(x-y)+\mu_2(x-y)^3 \\ +F_1\cos(\omega_1 t) \end{array}\right], \\ h(t,x,y,z,w) &= \tfrac{1}{m_2}\left[-\delta_2 w-\kappa_2(y-x)+\mu_2(y-x)^3+F_2\cos(\omega_2 t)\right]. \end{aligned} \tag{2.35}$$

These functions are continuous in $[0,1]\times\mathbb{R}^4$, and

$$\begin{aligned} L_1(w,x,y) &= x-\max_{t\in[0,1]} w(t)-2y, \\ L_2(w,x,y) &= x-\max_{t\in[0,1]} w(t)-2y^3 \end{aligned} \tag{2.36}$$

verify (A).

The functions given by

$$(\alpha_1(t),\, \alpha_2(t)) = (-t,\, -t) \text{ and } (\beta_1(t),\, \beta_2(t)) = (t,\, t)$$

are, respectively, lower and upper solutions of problem (2.33)-(2.34), satisfying (2.11), for every positive $m_1$, $m_2$, non negative $\delta_1$, $\delta_2$, $F_1$, $F_2$, $\omega_1$, $\omega_2$, and any real $\kappa_1$, $\kappa_2$, $\mu_1$, $\mu_2$, such that

$$\begin{array}{rcl} F_1 & \leq & \delta_1, \\ F_2 & \leq & \delta_2, \\ \mu_1 & \leq & 0. \end{array} \tag{2.37}$$

Indeed, Definition 2.1.1 holds because $\forall t \in [0,\, 1]$,

$$\begin{array}{rcl} \alpha_1''(t) = 0 & \leq & \dfrac{1}{m_1}\left(\delta_1 - F_1\right) \\ & \leq & \dfrac{1}{m_1}\left[\delta_1 - \mu_1 t^3 + F_1 \cos(\omega_1 t)\right] \\ & \leq & \dfrac{1}{m_1}\left[\delta_1 + \kappa_1 t - \mu_1 t^3 + F_1 \cos(\omega_1 t)\right] \\ \alpha_2''(t) = 0 & \leq & \dfrac{1}{m_2}\left(\delta_2 - F_2\right) \leq \dfrac{1}{m_2}\left[\delta_2 + F_2 \cos(\omega_2 t)\right] \\ L_1(\alpha_1, \alpha_1(1), \alpha_1'(1)) & = & \alpha_1(1) - \max\limits_{t\in[0,1]} \alpha_1(t) - 2\alpha_1'(1) = 1 \geq 0 \\ L_2(\alpha_2, \alpha_2(b), \alpha_2'(b)) & = & \alpha_2(1) - \max\limits_{t\in[0,1]} \alpha_2(t) - 2(\alpha_2'(1))^3 = 1 \geq 0, \end{array}$$

and

$$\begin{array}{rcl} \beta_1''(t) = 0 & \geq & \dfrac{1}{m_1}\left(-\delta_1 + F_1\right) \\ & \geq & \dfrac{1}{m_1}\left[-\delta_1 - \kappa_1 t + \mu_1 t^3 + F_1 \cos(\omega_1 t)\right] \\ \beta_2''(t) = 0 & \geq & \dfrac{1}{m_2}\left(-\delta_2 + F_2\right) \geq \dfrac{1}{m_2}\left[-\delta_2 + F_2 \cos(\omega_2 t)\right] \\ L_1(\beta_1, \beta_1(1), \beta_1'(1)) & = & \beta_1(1) - \max\limits_{t\in[0,1]} \beta_1(t) - 2\beta_1'(1) = -2 \leq 0 \\ L_2(\beta_2, \beta_2(b), \beta_2'(b)) & = & \beta_2(1) - \max\limits_{t\in[0,1]} \beta_2(t) - 2(\beta_2'(1))^3 = -2 \leq 0. \end{array}$$

Furthermore, $f$ and $h$, given by (2.35), verify Nagumo-type conditions relative to the interval $[-t, t]$, for $t \in [0,\, 1]$, with $r_1 =$

$r_2 = 1$, for

$$\begin{array}{rcl} -t & \leq & x \leq t, \quad -t \leq y \leq t, \forall t \in [0,1], \\ |f(t,x,y,z,w)| & \leq & \frac{1}{m_1}\left(\delta_1 |z| + \kappa_1 + |\mu_1| + 2\kappa_2 + 8|\mu_2| + F_1\right) \\ & := & \varphi(|z|), \\ |h(t,x,y,z,w)| & \leq & \frac{1}{m_2}\left(\delta_2 |w| + 2\kappa_2 + 8|\mu_2| + F_2\right) := \psi(|w|), \end{array}$$

and for $N_1$ and $N_2$ positive, and large enough such that

$$\int_1^{N_1} \frac{ds}{\varphi(s)} = \int_1^{N_1} \left( \frac{m_1}{\delta_1 s + \kappa_1 + |\mu_1| + 2\kappa_2 + 8|\mu_2| + F_1} \right) ds > 1,$$

and

$$\int_1^{N_2} \frac{ds}{\psi(s)} = \int_1^{N_2} \left( \frac{m_2}{\delta_2 s + 2\kappa_2 + 8|\mu_2| + F_2} \right) ds > 1.$$

So, by Theorem 2.2.1, there is a solution $(u(t), v(t))$ of the mass-spring system (2.33), (2.34), for the values of coefficients verifying (2.37), such that

$$\begin{array}{rcl} -t & \leq & u(t) \leq t, \\ -t & \leq & v(t) \leq t, \forall t \in [0,1], \end{array}$$

that is, both with values in the strip given by Figure 2.2 and

$$\|u'\| \leq N_1 \text{ and } \|v'\| \leq N_2.$$

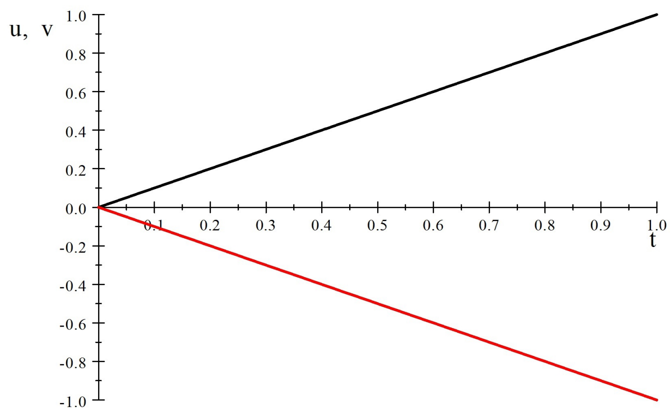


**FIGURE 2.2**
The functions $u(t)$ and $v(t)$ lie in the strip

# Part II

# Coupled systems on unbounded intervals

# Introduction

The theory and techniques used in the previous chapter can not be applied to systems defined on unbounded domains, mainly due to the lack of compactness of the associated operator, cases in which the literature for differential equation systems is scarce. Compared to boundary value problems in bounded intervals, we can say that in the literature there is a lack of results and publications that guarantee the existence of solutions for nonlinear coupled systems on unbounded intervals. To the best of our knowledge, so far there has been no work on nonlinear coupled systems with homoclinic or heteroclinic solutions, which will be addressed in this Part.

The first appearance of boundary value problems on unbounded intervals is related to A. Kneser [107], where its discussed monotone solutions and their derivatives on $[0, \infty)$ for the second-order ordinary differential equation

$$\frac{d^2y}{dx^2} = y(x).$$

Assuming some conditions, Krasnosel'skii in [112], was one of the first authors to treat fixed-point problems in Banach spaces on unbounded domains, where considered the translation operator and periodic solutions of differential equations in Banach space. In particular he its studied

$$\frac{dx}{dt} = A(t)x + \phi(t, x),$$

where $A(t)$ is a linear continuous operator that depends continuously (with respect to the norm of the operators) on $t$ and where $\phi(t, x)$ is completely continuous in the sense that it is continuous with respect to the variables $t \in (-\infty, +\infty)$, $x$ belongs to a Banach space $E$ and maps the Cartesian product of each finite interval and an arbitrary ball in the space $E$ into a compact subset of $E$.

In [11] and the references therein, we can find historical remarks about BVPs at infinite intervals, from the early 1970s, as well as the main techniques used to address these problems. Recently, in [149], Minhós and Carrasco addressed several nonlinear higher order problems in unbounded domains with applications to various real phenomena.

In this Part we study problems composed of nonlinear second order systems with complete nonlinearities and with boundary conditions at half-line or real line. More precisely, we seek to guarantee the existence and also the localization of solutions on unbounded domain using different approaches, depending on the problem being studied and on the boundary conditions.

The research methodology followed in this Part is based essentially in [150, 153, 156] and:

- Applying a *Nagumo-type growth* condition of nonlinearities and the concept of *equiconvergence* at $\infty$, to recover the compactness of the associated operators;

- Choice of an *adequate functional* context and the consideration of *asymptotic conditions* and *growth assumptions* on the nonlinearities;

- Based on *phi-Laplacians* and on the consideration of *growth and asymptotic conditions for homeomorphisms* and nonlinearities.

This second Part consists of three chapters which cover the existence and location of coupled systems on unbounded domains, where:

- In the *third chapter* we consider existence and localizations solutions for nonlinear second-order coupled systems with boundary conditions on the semi-infinite interval $[0, \infty)$;

- Int the *fourth chapter*, we consider the existence and location of homoclinic solutions for nonlinear second-order coupled systems on real line;

- In the *fifth chapter* we apply previous arguments to the existence of heteroclinic solutions for nonlinear second-order coupled systems on $(-\infty, +\infty)$.

# 3

# Second order coupled systems on the half-line

In the present chapter it is studied the second order system defined in a non-compact domain,

$$\begin{cases} u''(t) = f(t, u(t), v(t), u'(t), v'(t)), \ t \in [0, +\infty[\,, \\ v''(t) = h(t, u(t), v(t), u'(t), v'(t)), \end{cases} \tag{3.1}$$

with $f, h : [0, +\infty[ \times \mathbb{R}^4 \to \mathbb{R}$ be $L^1$-Carathéodory functions, together with the boundary conditions

$$\begin{cases} u(0) = A_1, \ v(0) = A_2, \\ u'(+\infty) = B_1, \ v'(+\infty) = B_2, \end{cases} \tag{3.2}$$

with $A_1, A_2, B_1, B_2 \in \mathbb{R}$,

$$u'(+\infty) := \lim_{t \to +\infty} u'(t) \text{ and } v'(+\infty) := \lim_{t \to +\infty} v'(t).$$

The study of second order differential equations on the half-line, can be seen in [19, 26, 38, 65, 68, 75, 121, 122, 143, 157, 193, 194, 201].

These type of equations have many applications in physics, biology, mechanics and among other areas. Examples of models of specific phenomena can be found on dynamical rotations, gravitational and electrostatic interactions, [47], damped and undamped oscillations of a rigid pendulum, circuit analogue and spring mass-systems, [211], motion of a piston inside a cylinder and micro-electro-mechanical systems, [191], cross diffusion epidemiology and thermographic tumor detection, [181].

However, coupled systems where the nonlinearities may depend on different and all of the unknown functions are scarce in the literature.

In [132] it is studied the BVP for second-order singular differential system on the whole line with impulse effects, i.e., consisting of the differential system

$$\begin{aligned} [\phi_p(\rho(t)x'(t))]' &= f(t, x(t), y(t)), \quad a.e.\, t \in \mathbb{R}, \\ [\phi_q(\varrho(t)y'(t))]' &= g(t, x(t), y(t)), \quad a.e.\, t \in \mathbb{R} \end{aligned}$$

subjected to the boundary conditions

$$\lim_{t\to\pm\infty} x(s) = 0,$$
$$\lim_{t\to\pm\infty} y(s) = 0$$

and the impulse effects

$$\Delta x(t_k) = I_k(t_k, x(t_k), y(t_k)), \quad k \in \mathbb{Z}$$
$$\Delta y(t_k) = J_k(t_k, x(t_k), y(t_k)), \quad k \in \mathbb{Z},$$

where

- $(a)$ $\rho, \varrho \in C^0(\mathbb{R}, [0, \infty))$, $\rho(t), \varrho(t) > 0$ for all $t \in \mathbb{R}$ with $\int_{-\infty}^{+\infty} \frac{ds}{\rho(s)} < +\infty$ and $\int_{-\infty}^{+\infty} \frac{ds}{\varrho(s)} < +\infty$,
- $(b)$) $\phi_p(x) = x|x|^{p-2}$, $\phi_q(x) = x|x|^{q-2}$ with $p > 1$ and $q > 1$ are p-Laplacian and q-Laplacian operators;
- $(c)$ $f, g$ on $\mathbb{R}^3$ are Carathéodory functions;
- $(d)$ $t_k$ is an increasing sequence, $k \in \mathbb{Z}$, $\cdots < t_k < t_{k+1} < t_{k+2} < \cdots$ with $\lim_{k\to-\infty} t_k = -\infty$ and $\lim_{k\to+\infty} t_k = +\infty$, $\Delta x(t_k) = u(t_k^+) - x(t_k^-)$ and $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ and $\mathbb{Z}$ is the set of all integers;
- $(c)$ $\{I_k\}, \{J_k\}$, with $I_k, J_k : \mathbb{R}^3 \to \mathbb{R}$ are Carathéodory sequences.

In [166], the authors study the existence of nontrivial solutions to the boundary value problem

$$\begin{cases} -u'' + cu' + \lambda u = f(x, u), \ -\infty < x < +\infty, \\ u(-\infty) = u(+\infty) = 0, \end{cases}$$

and to the system

$$\begin{cases} u'' + c_1 u' + \lambda_1 u = f(x, u, v), \ -\infty < x < +\infty, \\ v'' + c_2 v' + \lambda_2 v = g(x, u, v), \ -\infty < x < +\infty, \\ u(-\infty) = u(+\infty) = 0, \ \ v(-\infty) = v(+\infty) = 0, \end{cases}$$

where $c, c_1, c_2, \lambda, \lambda_1, \lambda_2$ are real positive constants and the nonlinearities $f$ and $g$ satisfy suitable conditions.

Motivated by these works, our method follows arguments applied in [156], for differential equations in the half-line, and in [161], for coupled systems. Note that, our technique is based on

[154], where it was the first time where coupled systems of differential equations are considered, namely with first derivative dependence on both unknown variables in unbounded domains. To be more precise, we use lower and upper solutions method combined with a Nagumo type growth condition. The equiconvergence at infinity plays a key role to recover the compactness of the correspondent operators.

## 3.1 Definitions and preliminary results

Define de space

$$X=\left\{\ x : x\in C^1([0,+\infty[) : \lim_{t\to+\infty}\tfrac{|x(t)|}{1+t}\in\mathbb{R},\ \lim_{t\to+\infty}|x'(t)|\in\mathbb{R}\ \right\},$$

and the norm $\|w\|_X=\max\{\|w\|_0,\|w'\|_1\}$, where

$$\|\Upsilon\|_0:=\sup_{t\in[0,+\infty[}\frac{|\Upsilon(t)|}{1+t}\ \text{ and }\ \|\Upsilon\|_1:=\sup_{t\in[0,+\infty[}|\Upsilon(t)|.$$

Denoting $E:=X\times X$ with the norm

$$\begin{aligned}\|(u,v)\|_E &= \max\{\|u\|_X,\|v\|_X\}\\ &= \max\{\|u\|_0,\|u'\|_1,\|v\|_0,\|v'\|_1\}.\end{aligned}$$

So, $(E,\|.\|_E)$ is a Banach space.

$L^1-$ Carathéodory functions are considered in the space $X$:

**Definition 3.1.1** *A function* $g:[0,+\infty[\times\mathbb{R}^4\to\mathbb{R}$ *is a* $L^1-$ *Carathéodory if*

*i) for each* $(x,y,z,w)\in\mathbb{R}^4$, $t\mapsto g(t,x,y,z,w)$ *is measurable on* $[0,+\infty[$;

*ii) for almost every* $t\in[0,+\infty[$, $(x,y,z,w)\mapsto g(t,x,y,z,w)$ *is continuous on* $\mathbb{R}^4$;

*iii) for each* $\rho>0$, *there exists a positive function* $\phi_\rho\in L^1([0,+\infty[)$ *such that, for* $(x,y,z,w)\in\mathbb{R}^4$

$$\sup_{t\in[0,+\infty[}\left\{\frac{|x|}{1+t},\frac{|y|}{1+t},|z|,|w|\right\}<\rho, \qquad (3.3)$$

*one has*

$$|g(t,x,y,z,w)| \leq \phi_\rho(t), \ \ a.e. \ \ t \in [0,+\infty[\,.$$

Take $\alpha_1, \beta_1, \alpha_2, \beta_2 \in C([0,+\infty[)$ such that, for $t \in [0,+\infty[$, $\alpha_1(t) \leq \beta_1(t)$, $\alpha_2(t) \leq \beta_2(t)$ and define the following set,

$$S = \{(t,x,y,z,w) \in [0,+\infty[\times\mathbb{R}^4 : \alpha_1(t) \leq x \leq \beta_1(t), \ \alpha_2(t) \leq y \leq \beta_2(t)\}.$$

**Definition 3.1.2** *The functions $f, \ h : S \to \mathbb{R}$, satisfy a Nagumo-type growth condition in $S$, if for some positive continuous functions $\phi, \varphi, l_1, l_2$ and some $\varepsilon > 1$, $R_1, R_2 > 0$, such that*

$$\sup_{t\in[0,+\infty[} \phi(t)(1+t)^\varepsilon < R_1, \quad \int_0^{+\infty} \frac{s}{l_1(s)} ds = +\infty, \tag{3.4}$$

$$\sup_{t\in[0,+\infty[} \varphi(t)(1+t)^\varepsilon < R_2, \quad \int_0^{+\infty} \frac{s}{l_2(s)} ds = +\infty, \tag{3.5}$$

$$|f(t,x,y,z,w)| \leq \phi(t) l_1(|z|) \ \ \forall (t,x,y,z,w) \in S \tag{3.6}$$

*and*

$$|h(t,x,y,z,w)| \leq \varphi(t) l_2(|w|) \ \ \forall (t,x,y,z,w) \in S. \tag{3.7}$$

**Lemma 3.1.1** *Let $f, h : [0,+\infty[\, \times \mathbb{R}^4 \to \mathbb{R}$ be $L^1-$ Carathéodory functions, where $f$ and $h$ satisfy (3.6), (3.7), and $\varepsilon > 1$. Then for every solution $(u,v) \in \left(C^2[0,+\infty[\right)^2 \cap E$, verifying the inequalities,*

$$\alpha_1(t) \leq u(t) \leq \beta_1(t), \ \ \alpha_2(t) \leq v(t) \leq \beta_2(t), \ \forall t \in [0,+\infty[\,, \tag{3.8}$$

*there are $N_1, N_2 > 0$ such that*

$$\|u'\|_1 \leq N_1 \ \ and \ \ \|v'\|_1 \leq N_2. \tag{3.9}$$

**Proof** Let $(u(t), v(t))$ be a solution of (3.1), (3.2) and consider $r > 0$ such that $r > \max\{|B_1|, |B_2|\}$.

If $|u'(t)| \leq r$, $\forall t \in [0,+\infty[$, then taking $N_1 \geq r$ the proof would be concluded.

As, by (3.2), $|u'(t)| > r$, $\forall t \in [0,+\infty[$, can not happen. Then assume that there is $t \in [0,+\infty[$ such that $u'(t) > r$. Therefore, there are $t_0, t_1 \in [0,+\infty[$ such that $t_0 < t_1$, $u'(t_1) = r$ and, for $t \in [t_0, t_1[$, we have $u'(t) > r$.

Take $N_i > r$, $i = 1, 2$, such that

$$\int_r^{N_i} \frac{s}{l_i(s)} ds > R_i \max \left\{ \begin{array}{c} M_i + \sup_{t\in[0,+\infty[} \frac{\beta_i(t)}{1+t} \frac{\varepsilon}{\varepsilon-1}, \\ M_i - \inf_{t\in[0,+\infty[} \frac{\alpha_i(t)}{1+t} \frac{\varepsilon}{\varepsilon-1} \end{array} \right\},$$

for $i = 1, 2$, $R_i$ given by (3.4) and (3.5), and

$$M_i := \sup_{t\in[0,+\infty[} \frac{\beta_i(t)}{(1+t)^\varepsilon} - \inf_{t\in[0,+\infty[} \frac{\alpha_i(t)}{(1+t)^\varepsilon}.$$

Thus, by a convenient change of variable and (3.6),

$$\begin{aligned}
& \int_{u'(t_0)}^{u'(t_1)} \frac{s}{l_1(s)} ds = \int_{t_0}^{t_1} \frac{u'(s)}{l_1(u'(s))} u''(s) ds \\
\leq & \int_{t_0}^{t_1} \frac{u'(s)}{l_1(u'(s))} |f(u(s), v(s), u'(s), v'(s))| ds \leq \int_{t_0}^{t_1} \phi(s) u'(s) ds \\
\leq & \int_{t_0}^{t_1} \frac{R_1 u'(s)}{(1+s)^\varepsilon} ds = R_1 \left[ \int_{t_0}^{t_1} \left( \frac{u(s)}{(1+s)^\varepsilon} \right)' + \frac{\varepsilon u(s)}{(1+s)^{1+\varepsilon}} \right] ds \\
\leq & R_1 \left( \sup_{t\in[0,+\infty[} \frac{\beta_1(t)}{(1+t)^\varepsilon} - \inf_{t\in[0,+\infty[} \frac{\alpha_1(t)}{(1+t)^\varepsilon} + \int_{t_0}^{t_1} \frac{\varepsilon u(s)}{(1+s)^{1+\varepsilon}} ds \right) \\
= & R_1 \left( M_1 + \int_{t_0}^{t_1} \frac{\varepsilon u(s)}{(1+s)^{1+\varepsilon}} ds \right) \leq R_1 \left( M_1 + \int_{t_0}^{t_1} \frac{\beta_1(s)}{(1+s)^{1+\varepsilon}} \varepsilon ds \right)
\end{aligned}$$

$$\leq R_1 \left( M_1 + \sup_{t\in[0,+\infty[} \frac{\beta_1(t)}{(1+t)} \int_0^{+\infty} \frac{\varepsilon}{(1+s)^\varepsilon} ds \right) \leq \int_r^{N_1} \frac{s}{l_1(s)} ds.$$

Since we can take arbitrarily $t_0 \in [0, +\infty[$ , such that $u'(t_0) > r$, therefore $u'(t_0) < N_1$.

If $u'(t) < -r$ the technique is analogous. The same conclusion can be achieved if there are $t_{-1}, t_2 \in [0, +\infty[$ such that $t_{-1} < t_2$, $u'(t_2) = -r$ and $u'(t) < -r$, $\forall t \in [t_{-1}, t_2[$.

Therefore $\|u'\| \leq N_1$. By similar arguments, it can be proved that $\|v'\| \leq N_2$.

■

**Lemma 3.1.2** *Let $f, h : [0, +\infty[ \times \mathbb{R}^4 \to \mathbb{R}$ be $L^1-$ Carathéodory functions. Then the coupled system*

$$\begin{cases} u''(t) = f(t, u(t), v(t), u'(t), v'(t)), \ t \in [0, +\infty[\,, \\ v''(t) = h(t, u(t), v(t), u'(t), v'(t)), \end{cases}$$

*with boundary conditions*

$$u(0) = A_1, \ v(0) = A_2, \ u'(+\infty) = B_1, \ v'(+\infty) = B_2,$$

*with $A_1, A_2, B_1, B_2 \in \mathbb{R}$, has a solution expressed by*

$$\begin{aligned} u(t) &= A_1 + B_1 t + \int_0^{+\infty} G(t, s) f(s, u(s), v(s), u'(s), v'(s)) ds \\ v(t) &= A_2 + B_2 t + \int_0^{+\infty} G(t, s) h(s, u(s), v(s), u'(s), v'(s)) ds, \end{aligned}$$

*where*

$$G(t, s) = \begin{cases} -t, & 0 \leq t \leq s \leq +\infty \\ -s, & 0 \leq s \leq t \leq +\infty. \end{cases} \tag{3.10}$$

Next lemma gives a convenient criterion to guarantee the compacity on unbounded domains, and can be easily obtained from [2], Theorem 4.3.1, or [153], Theorem 2.3.

**Theorem 3.1.1** *A set $M \subset X$ is relatively compact if the following conditions hold:*

*i) both $\{t \to x(t) : x \in M\}$ and $\{t \to x'(t) : x \in M\}$ are uniformly bounded;*

*ii) both $\{t \to x(t) : x \in M\}$ and $\{t \to x'(t) : x \in M\}$ are equicontinuous on any compact interval of $\mathbb{R}$;*

*iii) both* $\{t \to x(t) : x \in M\}$ *and* $\{t \to x'(t) : x \in M\}$ *are equiconvergent at* $\pm\infty$*, that is, for any given* $\epsilon > 0$*, there exists* $t_\epsilon > 0$ *such that*

$$\left|\frac{x(t)}{1+t} - \lim_{t\to+\infty}\frac{x(t)}{1+t}\right| < \epsilon, \left|x'(t) - \lim_{t\to+\infty} x'(t)\right| < \epsilon, \forall t > t_\epsilon, x \in M.$$

The existence tool will be given by Schauder's fixed point theorem (Theorem 2.1.1).

## 3.2 Existence result

In this section we prove the existence of solution for the problem (3.1)-(3.2).

**Theorem 3.2.1** *Let* $f, h \ : \ [0, +\infty[ \ \times \ \mathbb{R}^4 \ \to \ \mathbb{R}$ *be* $L^1-$ *Carathéodory functions and assume that there is* $R > 0$ *such that*

$$R > \max\left\{\max\{K_1,\ K_2\} + \int_0^{+\infty} K_3(s)\phi_R(s)ds, \right.$$
$$\left. \max\{|B_1|,\ |B_2|\} + \int_0^{+\infty} \phi_R(s)ds\right\}, \qquad (3.11)$$

*where*

$$K_i := \sup_{t\in[0,+\infty[}\left(\frac{|A_i| + |B_i t|}{1+t}\right),\ \ i = 1, 2,\ \ K_3(s) := \sup_{t\in[0,+\infty[}\frac{|G(t,s)|}{1+t}. \qquad (3.12)$$

*Then there is at least a pair* $(u, v) \in \left(C^2\left[0, +\infty\right[\right)^2 \cap E$*, solution of (3.1)-(3.2).*

**Proof** Define the operators $T_1 : E \to X,\ \ T_2 : E \to X$ and $T : E \to E$ given by

$$T(u, v) = \left(T_1(u, v), T_2(u, v)\right), \qquad (3.13)$$

with

$$\begin{aligned}(T_1(u,v))(t) &= A_1 + B_1 t \\ &\quad + \int_0^{+\infty} G(t,s) f(s,u(s),v(s),u'(s),v'(s))ds, \\ (T_2(u,v))(t) &= A_2 + B_2 t \\ &\quad + \int_0^{+\infty} G(t,s) h(s,u(s),v(s),u'(s),v'(s))ds,\end{aligned}$$

where $G(t,s)$ is defined in (3.10).

The proof will follow several steps for clearness, only for operator $T_1(u,v)$. The technique for operator $T_2(u,v)$ is similar.

**Step 1**: $T$ *is well defined and continuous.*

Let be $(u,v) \in E$. Therefore, it must be proved that $T(u,v) \in E$, that is $T_1(u,v) \in X$ and $T_2(u,v) \in X$.

As $(u,v) \in E$, then there exists some $r > 0$ such that $\|(u,v)\|_E \leq r$.

By the Lebesgue Dominated Theorem and Lemma 3.1.2,

$$\begin{aligned}&\lim_{t\to+\infty} \frac{T_1(u,v)(t)}{1+t} = \lim_{t\to+\infty} \frac{A_1 + B_1 t}{1+t} \\ &\quad + \int_0^{+\infty} \lim_{t\to+\infty} \frac{G(t,s)}{1+t} f(s,u(s),v(s),u'(s),v'(s))ds \\ \leq&\ B_1 + \int_0^{+\infty} |f(s,u(s),v(s),u'(s),v'(s))|ds \\ \leq&\ B_1 + \int_0^{+\infty} \phi_r(s)ds < +\infty,\end{aligned}$$

and

$$\begin{aligned}&\lim_{t\to+\infty} T_1(u,v)'(t) \\ =&\ B_1 - \lim_{t\to+\infty} \int_t^{+\infty} f(s,u(s),v(s),u'(s),v'(s))ds \\ \leq&\ B_1 + \lim_{t\to+\infty} \int_t^{+\infty} |f(s,u(s),v(s),u'(s),v'(s))|ds \\ \leq&\ B_1 + \int_0^{+\infty} |f(s,u(s),v(s),u'(s),v'(s))|ds \\ \leq&\ B_2 + \int_0^{+\infty} \phi_r(s)ds < +\infty.\end{aligned}$$

Therefore $T_1 \in X$. Analogously, $T_2 \in X$. So, $T$ is well defined in $E$ and, as $f$, $h$ are $L^1-$ Carathéodory functions, $T$ is continuous.

**Step 2**: *$TD$ is uniformly bounded, for $D$ a bounded set in $E$.*

Let $D$ be a bounded subset on $E$. Thus, there is $\rho_1 > 0$ such that

$$\|(u,v)\|_E = \max\{\|u\|_X, \|v\|_X\} = \max\left\{ \begin{array}{c} \|u\|_0, \|u'\|_1, \\ \|v\|_0, \|v'\|_1 \end{array} \right\} < \rho_1. \tag{3.14}$$

As $0 \leq K_3(s) \leq 1$, $\forall s \in [0,+\infty[$ and $f$ is a $L^1$-Carathéodory function, then

$$\begin{aligned}
\|T_1(u,v)\|_0 &= \sup_{t\in[0,+\infty[} \frac{|T(u,v)(t)|}{1+t} \leq \sup_{t\in[0,+\infty[} \left( \frac{|A_1|+|B_1 t|}{1+t} \right) \\
&+ \int_0^{+\infty} \sup_{t\in[0,+\infty[} \frac{|G(t,s)|}{1+t} |f(s,u(s),v(s),u'(s),v'(s))| ds \\
&\leq K_1 + \int_0^{+\infty} K_3(s)\phi_{\rho_1}(s) ds < +\infty, \forall (u,v) \in D,
\end{aligned}$$

and

$$\begin{aligned}
\|(T_1(u,v))'\|_1 &= \sup_{t\in[0,+\infty[} |(T_1(u,v))'(t)| \\
&\leq |B_1| + \int_t^{+\infty} |f(s,u(s),v(s),u'(s),v'(s))| ds \\
&\leq |B_1| + \int_0^{+\infty} \phi_{\rho_1}(s) ds < +\infty, \forall (u,v) \in D.
\end{aligned}$$

Taking into account these arguments, $T_2$ verifies similar bounds and $\|T(u,v)\|_E < \rho_1$, that is $TD$ is uniformly bounded.

**Step 3**: *$TD$ is equicontinuous in $E$.*

Let $t_1, t_2 \in [0,+\infty[$ and suppose, without loss of generality, that $t_1 \leq t_2$. So, by the continuity of $G(t,s)$,

$$\begin{aligned}
&\lim_{t_1\to t_2} \left| \frac{T_1(u,v)(t_1)}{1+t_1} - \frac{T_1(u,v)(t_2)}{1+t_2} \right| \leq \lim_{t_1\to t_2} \left| \left( \frac{A_1+B_1 t_1}{1+t_1} - \frac{A_1+B_1 t_2}{1+t_2} \right) \right| \\
&+ \int_0^{+\infty} \lim_{t_1\to t_2} \left| \frac{G(t_1,s)}{1+t_1} - \frac{G(t_2,s)}{1+t_2} \right| \left| f(s,u(s),v(s),u'(s),v'(s)) \right| ds = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t_1\to t_2}\left|\left(T_1\left(u,v\right)\left(t_1\right)\right)'-\left(T_1\left(u,v\right)\left(t_2\right)\right)'\right|\\
=&\lim_{t_1\to t_2}\left|-\int_{t_1}^{+\infty}f(s,u(s),v(s),u'(s),v'(s))ds\right.\\
&\left.+\int_{t_2}^{+\infty}f(s,u(s),v(s),u'(s),v'(s))ds\ \right|\\
=&\lim_{t_1\to t_2}\left|-\int_{t_1}^{t_2}f(s,u(s),v(s),u'(s),v'(s))ds\right|\\
\leq&\lim_{t_1\to t_2}\int_{t_1}^{t_2}\phi_{\rho_1}(s)ds=0.
\end{aligned}$$

Therefore, $T_1D$ is equicontinuous on $E$. In the same way it can be proved that $T_2D$ is equicontinuous on $E$, too. Thus, $TD$ is equicontinuous on $E$.

**Step 4**: *$TD$ is equiconvergent at infinity* .

For the operator $T_1$, we have

$$\begin{aligned}
&\left|\frac{T_1\left(u,v\right)\left(t\right)}{1+t}-\lim_{t\to+\infty}\frac{T_1\left(u,v\right)\left(t\right)}{1+t}\right|\leq\left|\frac{A_1+B_1t}{1+t}-B_1\right|\\
+&\int_0^{+\infty}\left|\frac{G(t,s)}{1+t}-\lim_{t\to+\infty}\frac{G(t,s)}{1+t}\right||f(s,u(s),v(s),u'(s),v'(s))ds|
\end{aligned}$$

$\to 0$, as $t\to+\infty$.

Similarly,

$$\begin{aligned}
&\left|\left(T_1\left(u,v\right)\left(t\right)\right)'-\lim_{t\to+\infty}\left(T_1\left(u,v\right)\left(t\right)\right)'\right|\\
=&\left|-\int_t^{+\infty}f(s,u(s),v(s),u'(s),v'(s))ds\right|\\
\leq&\int_t^{+\infty}|f(s,u(s),v(s),u'(s),v'(s))|ds\to 0,
\end{aligned}$$

as $t\to+\infty$.

$T_1D$ is equiconvergent at $+\infty$ and, following a similar technique, we can prove that $T_2D$ is equiconvergent at $+\infty$, too. So,

$TD$ is equiconvergent at $+\infty$. By Theorem 3.1.1, $TD$ is relatively compact and, consequently, $T$ is completely continuous.

**Step 5**: $T\Omega \subset \Omega$ for some $\Omega \subset E$ *a closed and bounded set.*

Consider

$$\Omega := \left\{ (u,v) \in E : \|(u,v)\|_E \leq \rho_2 \right\},$$

with $\rho_2 > 0$ such that

$$\rho_2 := \max \left\{ \begin{array}{c} \rho_1, K_1 + \int_0^{+\infty} K_3(s)\phi_{\rho_2}(s)ds, K_2 + \int_0^{+\infty} K_3(s)\phi_{\rho_2}(s)ds, \\ \\ |B_1| + \int_0^{+\infty} \phi_{\rho_2}(s)ds, \ |B_2| + \int_0^{+\infty} \phi_{\rho_2}(s)ds \end{array} \right\},$$

with $\rho_1$ given by (3.14), according to Step 2 and $K_1$, $K_2$ and $K_3(s)$ given by (3.12), we have

$$\begin{aligned} \|T(u,v)\|_E &= \|(T_1(u,v), T_2(u,v))\|_E \\ &= \max \left\{ \|T_1(u,v)\|_X , \ \|T_2(u,v)\|_X \right\} \\ &= \max \left\{ \|T_1(u,v)\|_0 , \ \left\|(T_1(u,v))'\right\|_1 , \right. \\ & \qquad \left. \|T_2(u,v)\|_0 , \ \left\|(T_2(u,v))'\right\|_1 \right\} \leq \rho_2. \end{aligned}$$

So, $T\Omega \subset \Omega$, and by Schauder's Theorem (Theorem 2.1.1), the operator $T(u,v) = (T_1(u,v), T_2(u,v))$, has a fixed point $(u,v)$. By standard techniques it can be shown that this fixed point is a solution of problem (3.1)-(3.2).

■

## 3.3 Application to a predator-prey model

Mathematical models have been considered to study the dynamics of some species and organisms living in environments where there are strong unidirectional flows, such as a river or a stream. An important goal is to understand how populations and ecosystems can survive in such media.

Some examples: predator-prey dynamics can be studied via reaction-diffusion- advection systems (for details see [208]); the effect of network structures and discrete diffusion rates on predator-prey-subsidy dynamics of Stepping-Stone models, [183]; a predator-prey model with switching between a traditional model (the free system) and a model with a nonlinear harvesting regime for the predator population (the harvesting system), [202]; the spatially heterogeneous subsidy distributions, focusing on the potential they have for destabilizing coexistence equilibria between predator-prey interactions, leading to ecological cascade effects, [25].

In this section we consider an application to a stationary predator-prey model in a very long domain, given by the second order nonlinear coupled system on the half-line, composed by the differential equations

$$\begin{cases} u''(x) = \left(\frac{b_1}{\gamma_1}u'(x) - \frac{1}{\gamma_1}u(x)\left[r_1 - b_{11}u(x) - b_{12}v(x)\right]\right)\frac{1}{1+x^4} \\ \\ v''(x) = \left(\frac{b_2}{\gamma_2}v'(x) - \frac{1}{\gamma_2}v(x)\left[r_2 - b_{21}v(x) - b_{22}u(x)\right]\right)\frac{1}{1+x^4}, \end{cases} \tag{3.15}$$

where:

- $u(x)$ and $v(x)$ denote the densities of the prey and the predator populations, respectively, at position $x \in [0, +\infty[$;
- $\gamma_1, \gamma_2 > 0$, are diffusion constants of the prey and the predator, respectively;
- $b_1, b_2 \in \mathbb{R}$, are the advection rates of the prey and the predator, respectively;
- $r_1 > 0, r_2 \in \mathbb{R}$, are the growth rates of the prey and the predator, respectively;
- $b_{11}, b_{21} > 0$, are the density-dependent constants of the prey and the predator, respectively;
- $b_{12}, b_{22} > 0$, are is the predation and conversion rates, respectively;

together the boundary conditions

$$\begin{cases} u(0) = A_1, \;\; v(0) = A_2, \\ u'(+\infty) = B_1, \;\; v'(+\infty) = B_2, \end{cases} \tag{3.16}$$

with $A_1, A_2 > 0$, $B_1, B_2 \in \mathbb{R}$.

Remark that:

1. $B_i$ are real numbers, for $i = 1, 2$. Therefore, this model may consider cases of growth, decay or constant populations of preys and predators;

2. The derivatives at infinity provide the variation of preys and predators with increasing distance, this is, $x \to +\infty$. The possibility of having infinite space is important, especially when the population size of prey becomes small or when populations lose spatial contact;

3. $u(x) = 0$, meansing the absence of prey, will lead to the extinction of the predator population due to lack of food;

4. $v(x) = 0$, means the absence of the predator. The population of prey will increase, without any type of obstacle, till an eventually resource deflection.

The system (3.15)-(3.16) is a particular case of the problem (3.1)-(3.2), with

$$f(x, m, y, z, w) = \left( \frac{b_1}{\gamma_1} z - \frac{1}{\gamma_1} m \left[ r_1 - b_{11} m - b_{12} y \right] \right) \frac{1}{1 + x^4},$$

and

$$h(x, m, y, z, w) = \left( \frac{b_2}{\gamma_2} w - \frac{1}{\gamma_2} y \left[ r_2 - b_{21} y - b_{22} m \right] \right) \frac{1}{1 + x^4}.$$

In fact, these functions are $L^1$-Carathéodory functions, with

$$\begin{aligned} & f(x, m, y, z, w) \\ \leq\ & \left( \frac{b_1}{\gamma_1} \rho + \frac{1}{\gamma_1} \rho^2 (1 + x)^2 (b_{11} + b_{12}) + \rho(1 + x) r_1 \right) \frac{1}{1 + x^4} \\ :=\ & \phi_\rho(x), \end{aligned}$$

and

$$\begin{aligned} & h(x,m,y,z,w) \\ \leq\ & \left( \frac{b_2}{\gamma_2}\rho + \frac{1}{\gamma_2}\rho^2(1+x)^2(b_{21}+b_{22}) + \rho(1+x)r_2 \right) \frac{1}{1+x^4} \\ :=\ & \varphi_\rho(x), \end{aligned}$$

for some $\rho > 0$, such that

$$\sup_{x\in[0,+\infty[} \left\{ \frac{|m|}{1+x}, \frac{|y|}{1+x}, |z|, |w| \right\} < \rho.$$

So, by Theorem 3.2.1, there is at least a pair $(u,v) \in (C^2\,[0,+\infty[)^2 \cap E$, solution of (3.15)-(3.16).

## 3.4 Existence and localization result

In this section we shall prove the existence and localization of a solution for problem (3.1)-(3.2).

Let $A_1, A_2, B_1, B_2 \in \mathbb{R}$. Lower and upper functions are defined in the following way:

**Definition 3.4.1** *A pair of functions $(\alpha_1, \alpha_2) \in (C^2\,[0,+\infty[)^2 \cap E$ is a lower solution of problem (3.1), (3.2) if*

$$\begin{aligned} \alpha_1''(t) &\leq f(t, \alpha_1(t), \alpha_2(t), \alpha_1'(t), w), \forall w \in \mathbb{R} \\ \alpha_2''(t) &\leq h(t, \alpha_1(t), \alpha_2(t), z, \alpha_2'(t)), \forall z \in \mathbb{R} \\ \alpha_1(0) &\leq A_1 \\ \alpha_2(0) &\leq A_2 \\ \alpha_1'(+\infty) &< B_1 \\ \alpha_2'(+\infty) &< B_2, \end{aligned}$$

*A pair of functions $(\beta_1, \beta_2) \in (C^2\,[0,+\infty[)^2 \cap E$ is an upper solution of problem (3.1), (3.2) if it verifies the reverse inequalities.*

**Theorem 3.4.1** *Let $f, h : [0,+\infty[ \times \mathbb{R}^4 \to \mathbb{R}$ be $L^1-$Carathéodory functions, the assumptions of Theorem 3.2.1 and the Nagumo-type conditions given by Definition 3.1.2, hold.*

*If there are coupled lower and upper solutions of (3.1)-(3.2), $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$, respectively, such that*

$$\alpha_1(t) \leq \beta_1(t), \;\; \alpha_2(t) \leq \beta_2, \; \forall t \in [0, +\infty[\,,$$

*then there is at least a pair $(u(t), v(t)) \in \left(C^2\,[0, +\infty[\right)^2 \cap E$ solution of (3.1)-(3.2), such that*

$$\alpha_1(t) \leq u(t) \leq \beta_1(t), \quad \alpha_2(t) \leq v(t) \leq \beta_2(t), \;\; \forall t\,[0, +\infty[\,, \quad (3.17)$$

*and*

$$\|u'\| \leq N_1 \text{ and } \|v'\| \leq N_2,$$

*with $N_1$ and $N_2$ given by Lemma 3.1.1.*

**Proof** Consider the operator $T$ given by (3.13). By Theorem 3.2.1, there is a fixed point of $T$, $(u, v)$, which is a solution of problem (3.1)-(3.2).

To prove the localization part given by (3.17), consider the auxiliary functions $\delta_i : [0, +\infty[ \times \mathbb{R} \to \mathbb{R}$, i=1,2, defined as

$$\delta_i(t, w) = \begin{cases} \beta_i(t) & , \quad w > \beta_i(t) \\ w & , \quad \alpha_i(t) \leq w \leq \beta_i(t) \\ \alpha_i(t) & , \quad w < \alpha_i(t), \end{cases}$$

and the truncated and perturbed coupled system

$$\begin{cases} u''(t) = f(t, \delta_1(t, u(t)), \delta_2(t, v(t)), u'(t), v'(t)) + \frac{1}{1+t} \frac{u(t) - \delta_1(t, u(t))}{|u(t) - \delta_1(t, u(t))|} \\ \\ v''(t) = h(t, \delta_1(t, u(t)), \delta_2(t, v(t)), u'(t), v'(t)) + \frac{1}{1+t} \frac{v(t) - \delta_2(t, v(t))}{|v(t) - \delta_2(t, v(t))|}. \end{cases}$$

Suppose, by contradiction, that there is $t \in [0, +\infty[$, such that $\alpha_1(t) > u(t)$ and define

$$\inf_{t \in [0, +\infty[} (u(t) - \alpha_1(t)) := u(t_0) - \alpha_1(t_0) < 0.$$

Then, $t_0 \neq 0$ and $t_0 \neq +\infty$, as, by Definition 3.4.1 and (3.2),

$$u(0) - \alpha_1(0) = A_1 - \alpha_1(0) \geq 0,$$

and

$$u'(+\infty) - \alpha_1'(+\infty) = B_1 - \alpha_1'(+\infty) > 0.$$

Therefore, $t_0 \in \,]0, +\infty[$, $u'(t_0) = \alpha_1'(t_0)$, $u''(t_0) - \alpha_1''(t_0) \geq 0$. So, we deduce the follow contradiction

$$\begin{aligned}
0 &\leq u''(t_0) - \alpha_1''(t_0) \\
&= f(t_0, \delta_1(t_0, u(t_0)), \delta_2(t_0, v(t_0)), u'(t_0), v'(t_0)) \\
&\quad + \frac{1}{1+t_0} \frac{u(t_0) - \delta_1(t, u(t_0))}{|u(t_0) - \delta_1(t, u(t_0))| + 1} - \alpha_1''(t_0) \\
&\leq f(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), v'(t_0)) \\
&\quad + \frac{1}{1+t_0} \frac{u(t_0) - \alpha_1(t_0)}{|u(t_0) - \alpha_1(t_0)| + 1} - \alpha_1''(t_0) \\
&\leq \frac{1}{1+t_0} \frac{u(t_0) - \alpha_1(t_0)}{|u(t_0) - \alpha_1(t_0)| + 1} < 0.
\end{aligned}$$

So, $\alpha_1(t) \leq u(t)$, $\forall t \in [0, +\infty[$, and the remaining inequalities $u(t) \leq \beta_1(t)$, $\forall t \in [0, +\infty[$, can be proved by same technique.

Applying the method above, it may be shown that $\alpha_2(t) \leq v(t) \leq \beta_2(t)$, $\forall t \in [0, +\infty[$.

■

## 3.5 Example

Consider the second order system

$$\begin{cases} u''(t) = \dfrac{\pi u'(t)\left[u(t) - 3\sqrt[3]{v(t)} + \arctan(v'(t))\right]}{t^4+1}, \ t \in [0, +\infty[\,, \\ v''(t) = \dfrac{v'(t)\left[2e^{-|u'(t)|} - (u(t))^3 - \sqrt[3]{v(t)}\right]}{1+t^6}, \ t \in [0, +\infty[ \end{cases} \tag{3.18}$$

and the boundary conditions

$$u(0) = v(0) = 0, \ u'(+\infty) = 1, \ v'(+\infty) = 1, \tag{3.19}$$

In fact, this problem is a particular case of system (3.1)-(3.2), with

$$\begin{aligned} f(t,x,y,z,w) &= \frac{\pi z\left[x-3\sqrt[3]{y}+\arctan(w)\right]}{t^4+1} \\ &\leq \frac{\pi\rho\left[(t+1)\rho+3\sqrt[3]{\rho(1+t)}+\frac{\pi}{2}\right]}{t^4+1} := \phi_\rho(t), \end{aligned}$$

and

$$\begin{aligned} h(t,x,y,z,w) &= \frac{w\left[2e^{-|z|}+x^3+\sqrt[3]{y}\right]}{1+t^6} \\ &\leq \frac{\rho\left[2+(1+t)^3\rho^3+\sqrt[3]{(1+t)\rho}\right]}{1+t^6} := \phi_\rho(t), \end{aligned}$$

for some $\rho > 0$, such that

$$\sup_{t\in[0,+\infty[}\left\{\frac{|x(t)|}{1+t}, \frac{|y(t)|}{1+t}, |z(t)|, |w(t)|\right\} < \rho.$$

Therefore, $f, h : [0,+\infty[\times\mathbb{R}^4 \to \mathbb{R}$ are $L^1$-Carathéodory functions.

The functions given by

$$(\alpha_1(t),\, \alpha_2(t)) = (-2-t,\, -1+t) \text{ and } (\beta_1(t),\, \beta_2(t)) = (1+t,\, 2+t)$$

are, respectively, lower and upper solutions of problem (3.18)-(3.19), with $\alpha_1(t) = -2-t \leq \beta_1(t) = 1+t$ and $\alpha_2(t) = -1+t \leq \beta_2(t) = 2+t$, $\forall t \in [0,+\infty[$.

Furthermore, the functions

$$f(t,x,y,z,w) \leq \frac{\pi|z|\left[(t+1)+3\sqrt[3]{2+t}+\frac{\pi}{2}\right]}{t^4+1}$$

and

$$\begin{aligned} h(t,x,y,z,w) &= \frac{|w|\left[2e^{-|z|}+(1+t)^3+\sqrt[3]{(2+t)}\right]}{1+t^6} \\ &\leq \frac{|w|\left[2+(1+t)^3+\sqrt[3]{2+t}\right]}{1+t^6}, \end{aligned}$$

satisfy a Nagumo condition relative to the set

$S=\{(t,x,y,z,w)\in[0,+\infty[\times\mathbb{R}^4 : -2-t\le x\le 1+t,\ -1+t\le y\le 2+t\}$, with

$$\phi(t):=\frac{(t+1)+3\sqrt[3]{2+t}+\frac{\pi}{2}}{t^4+1},\quad l_1(|z|)=\pi|z|,$$
$$\varphi(t):=\frac{2+(1+t)^3+\sqrt[3]{2+t}}{1+t^6},\quad l_2(|w|)=|w|.$$

We can also see that,

$$\begin{aligned}\int_0^{+\infty}\phi(s)ds &= \int_0^{+\infty}\frac{(s+1)+3\sqrt[3]{2+s}+\frac{\pi}{2}}{s^4+1}ds\\ &\simeq 8.26013<+\infty,\end{aligned}$$

$$\begin{aligned}\int_0^{+\infty}\varphi(s)ds &= \int_0^{+\infty}\frac{2+(1+s)^3+\sqrt[3]{2+s}}{1+s^6}ds\\ &\simeq 8.56284<+\infty,\end{aligned}$$

$$\int_0^{+\infty}\frac{s}{l_1(s)}ds=\int_0^{+\infty}\frac{s}{\pi s}ds=+\infty;\quad \int_0^{+\infty}\frac{s}{l_2(s)}ds=\int_0^{+\infty}ds=+\infty,$$

and, for $1<\varepsilon\le 3$,

$$\begin{aligned}\sup_{t\in[0,+\infty[}\phi(t)(1+t)^\varepsilon &= \sup_{t\in[0,+\infty[}\frac{(t+1)+3\sqrt[3]{2+t}+\frac{\pi}{2}}{t^4+1}(1+t)^\varepsilon\\ &\simeq 31.59018\le R_1,\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in[0,+\infty[}\varphi(t)(1+t)^\varepsilon &= \sup_{t\in[0,+\infty[}\frac{2+(1+t)^3+\sqrt[3]{2+t}}{1+t^6}(1+t)^\varepsilon\\ &\simeq 45.76900\le R_2.\end{aligned}$$

Thereby, by Theorem 3.2.1, there is at least a pair $(u(t), v(t)) \in (C^2([0, +\infty[), \mathbb{R})^2$ solution of (3.18)-(3.19) and, moreover,

$$-2 - t \leq u(t) \leq 1 + t, \quad -1 + t \leq v(t) \leq 2 + t, \ \forall t \in [0, +\infty[\,,$$

and there are positive $N_1$, $N_2$, such that $\|u'\|_1 \leq N_1$ and $\|v'\|_1 \leq N_2$.

# 4

# *Homoclinic solutions for second-order coupled systems*

In this chapter, we consider the second order coupled system on the real line

$$\begin{cases} u''(t) - k_1^2 u(t) = f(t, u(t), v(t), u'(t), v'(t)), \ t \in \mathbb{R}, \\ v''(t) - k_2^2 v(t) = h(t, u(t), v(t), u'(t), v'(t)), \ t \in \mathbb{R}, \end{cases} \tag{4.1}$$

with $f, h : \mathbb{R}^5 \to \mathbb{R}$ $L^1$-Carathéodory functions, $k_1, k_2 > 0$, and the boundary conditions

$$\begin{cases} u(\pm\infty) = 0, \ v(\pm\infty) = 0, \\ u'(\pm\infty) = 0, \ v'(\pm\infty) = 0, \end{cases} \tag{4.2}$$

where,

$$u(\pm\infty) := \lim_{t\to\pm\infty} u(t), \quad v(\pm\infty) := \lim_{t\to\pm\infty} v(t),$$

$$u'(\pm\infty) := \lim_{t\to\pm\infty} u'(t), \quad v'(\pm\infty) := \lim_{t\to\pm\infty} v'(t).$$

Moreover, we present sufficient conditions for the existence of homoclinic solutions, based on [155].

An homoclinic orbit is a flow trajectory of a dynamical system that joins a saddle equilibrium point into itself, that is, the homoclinic trajectory converges an equilibrium point as $t \to \pm\infty$. So, by an homoclinic solution of system (4.1), we mean a nontrivial solution of (4.1) verifying (4.2).

The importance of homoclinics is described in [3, 59, 138, 140, 173], with applications involving dynamic systems, namely, dynamics of coupled cell networks, bifurcation theory, chaos and homoclinic orbits near resonances in reversible systems.

In [41, 42], Champneys and Lord study homoclinic solutions of periodic orbits in a reduced water-wave problem, in reversible systems and their applications in mechanics, fluids and optics. Algaba et al., in [6], deal with an application to Rössler system. In [61], Fečkan presents homoclinic orbits with Melnikov Mappings, difference equation and planar perturbations.

Results about the existence of homoclinic multiple solutions can be seen, for example: in [204], where Wilczak shows the existence of infinitely many homoclinic and heteroclinic connections between two odd periodic solutions of the Michelson system $x''' + x' + 0.5x^2 = 1$; in [188], Sun and Wu present some new results of homoclinic solutions and the existence of two different homoclinic solutions for second-order Hamiltonian systems; in [187], by establishing a compactness lemma and using variational methods, the author proves the existence of multiple homoclinic solutions for a class of fourth order differential equations with a perturbation; in [110], based on the fountain theorem in combination with variational technique, Kong obtained a sufficient condition for the existence of infinitely many homoclinic solutions of higher order difference equation with p-Laplacian and containing both advances and delays.

In [43], it is proved the existence and multiplicity for infinitely many homoclinic orbits from 0 of the second-order Hamiltonian system

$$\ddot{\mathrm{u}} - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0,$$

where $p \geq 2$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $a : \mathbb{R} \to \mathbb{R}$ and $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$.

In [137], Liu, Guo and Zhang, study the existence and also the multiplicity of homoclinic orbits for the following second-order Hamiltonian systems

$$\ddot{u} - L(t)u + \nabla W(t, u) = 0,$$

where $L \in C(\mathbb{R}, \mathbb{R}^{\mathbb{N}^2})$ is a symmetric and positive-definite matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $\nabla W(t, u)$ is the gradient of $W$ with respect to $u$.

Motivated by the works above and based in the arguments used in [153], we apply the fixed point theory, the lower and upper solutions method combined with an adequate growth assumptions on the nonlinearities, to obtain sufficient conditions for the existence of homoclinic solutions of the coupled system (4.1). Moreover, our technique provides a strip where both homoclinics lie.

On the other hand, there are many phenomena that are not modeled by differential equations, but by systems. The family of second order nonlinear coupled systems of type (4.1) can model various phenomena. For example, in [179], the authors analyze the peak solutions for a reaction-diffusion model; to study the binary fluid convection model, (see, [177]); example from the geophysical morphodynamics (see, [108]).

We emphasize that the vast area of applications for this family of second order nonlinear coupled systems certainly warrants a study of the coupled system (4.1).

Note that it is the first time where homoclinic solutions for second order coupled differential systems is considered with system having full nonlinearities on both unknown functions. We point out that although the arguments are similar to [153], for coupled systems the technique is more delicate as it requires stronger definitions of lower and upper solutions due to the dependence on both unknown functions and their fist derivative (see Definition 4.1.1).

## 4.1 Preliminary results

Consider the following spaces

$$X_1 := \left\{ x \in C^1(\mathbb{R}) : \lim_{|t| \to \infty} x^{(i)}(t) \in \mathbb{R},\ i = 0, 1 \right\},$$

and $X := X_1 \times X_1$, equipped with the norm

$$\|x\|_{X_1} = \max\{\|x\|_\infty, \|x'\|_\infty\},$$

where

$$\|w\|_\infty := \sup_{t \in \mathbb{R}} |w(t)|,$$

and

$$\|(u,v)\|_X = \max\left\{\|u\|_{X_1}, \|v\|_{X_1}\right\}.$$

It can be proved that, $(X_1, \|\cdot\|_{X_1})$ and $(X, \|\cdot\|_X)$ are Banach spaces.

For the reader's convenience we precise $L^1-$ Carathéodory functions are given by, Definition 3.1.1, now for a function $g : \mathbb{R}^5 \to \mathbb{R}$ and condition *(iii)* replaced by:

for each $\rho > 0$, there exists a positive function $\phi_\rho \in L^1(\mathbb{R})$ such that, whenever $x, y, z, w \in [-\rho, \rho]$, then

$$|g(t, x, y, z, w)| \leq \phi_\rho(t),\ a.e.\ t \in \mathbb{R}. \tag{4.3}$$

Next lemma guarantees the existence of solution of problem (4.1), (4.2) in $L^1(\mathbb{R})$.

**Lemma 4.1.1** *Let $f^*, h^* \in L^1(\mathbb{R})$. Then the system*

$$\begin{cases} u''(t) - k_1^2 u(t) = f^*(t),\ a.e. t \in \mathbb{R} \\ v''(t) - k_2^2 v(t) = h^*(t), \end{cases} \tag{4.4}$$

*with conditions (4.2), has a unique solution expressed by*

$$\begin{aligned} u(t) &= \int_{-\infty}^{+\infty} G_1(t,s) f^*(s)ds, \\ v(t) &= \int_{-\infty}^{+\infty} G_2(t,s) h^*(s)ds, \end{aligned}$$

*where*

$$G_1(t,s) = \begin{cases} -\frac{1}{2k_1} e^{k_1(s-t)}, & -\infty \le s \le t \le +\infty \\ -\frac{1}{2k_1} e^{k_1(t-s)}, & -\infty \le t \le s \le +\infty, \end{cases} \tag{4.5}$$

*and*

$$G_2(t,s) = \begin{cases} -\frac{1}{2k_2} e^{k_2(s-t)}, & -\infty \le s \le t \le +\infty \\ -\frac{1}{2k_2} e^{k_2(t-s)}, & -\infty \le t \le s \le +\infty. \end{cases} \tag{4.6}$$

**Proof** The solution of the homogeneous differential system

$$\begin{cases} u''(t) - k_1^2 u(t) = 0, \\ v''(t) - k_2^2 v(t) = 0, \end{cases}$$

is

$$(u(t),\ v(t)) = \left( c_1 e^{k_1^2 t} + c_2 e^{-k_1^2 t},\ c_3 e^{k_2^2 t} + c_4 e^{-k_2^2 t} \right),$$

with $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

By the method of variation of parameters, the general solution of (4.4), (4.2) is given by

$$\begin{aligned} u(t) &= -\int_{-\infty}^{t} \frac{e^{k_1(s-t)}}{2k_1} f^*(s)ds - \int_{t}^{+\infty} \frac{e^{k_1(t-s)}}{2k_1} f^*(s)ds, \\ v(t) &= -\int_{-\infty}^{t} \frac{e^{k_2(s-t)}}{2k_2} h^*(s)ds - \int_{t}^{+\infty} \frac{e^{k_2(t-s)}}{2k_2} h^*(s)ds, \end{aligned}$$

and, therefore,

$$\begin{aligned} u(t) &= \int_{-\infty}^{+\infty} G_1(t,s) f^*(s)ds, \\ v(t) &= \int_{-\infty}^{+\infty} G_2(t,s) h^*(s)ds, \end{aligned}$$

with $G_1$ and $G_2$ given by (4.5) and (4.6), respectively.

■

To guarantee a convenient criterion for compacity, we consider Theorem 3.1.1, with condition *(iii)* replaced by:

$$|f(t) - f(\pm\infty)| < \epsilon,\ |f'(t) - f'(\pm\infty)| < \epsilon, \forall |t| > t_\epsilon, f \in M.$$

Strict lower and upper solutions of problem (4.1), (4.2) are defined in the following way:

**Definition 4.1.1** *A pair of functions* $(\alpha_1, \alpha_2) \in X$ *is a strict lower solution of problem (4.1), (4.2) if*

$$\begin{array}{rcl} \alpha_1''(t) - k_1^2\alpha_1(t) & > & f(t, \alpha_1(t), \alpha_2(t), \alpha_1'(t), w),\ \ t,\, w \in \mathbb{R}, \\ \alpha_2''(t) - k_2^2\alpha_2(t) & > & h(t, \alpha_1(t), \alpha_2(t), z, \alpha_2'(t)),\ \ t,\, z \in \mathbb{R}, \\ \alpha_1(\pm\infty) & \leq & 0, \\ \alpha_2(\pm\infty) & \leq & 0. \end{array} \tag{4.7}$$

*A pair of functions* $(\beta_1, \beta_2) \in X$ *is a strict upper solution of problem (4.1), (4.2) if it verifies the reverse inequalities.*

The existence tool will be given by Theorem 2.1.1.

Moreover, along this work we denote $(a,\ b) \leq (c,\ d)$ meaning that $a \leq c$ and $b \leq d$, for $a, b, c, d \in \mathbb{R}$.

## 4.2 Existence and localization of homoclinics

In this section we prove the existence and localization for a solution of the problem (4.1)-(4.2). Moreover, by homoclinic solutions, we mean a non trivial solution of (4.1)-(4.2).

**Theorem 4.2.1** *Let* $f, h : \mathbb{R}^5 \to \mathbb{R}$ *be* $L^1-$ *Carathéodory functions with* $f(t, 0, 0, 0, 0) \neq 0$*,* $h(t, 0, 0, 0, 0) \neq 0$*, for some* $t \in \mathbb{R}$*, and there is* $R > 0$ *such that*

$$\max\left\{\int_{-\infty}^{+\infty} M_1(s)\psi_R(s)ds < +\infty, \int_{-\infty}^{+\infty} M_2(s)\psi_R^*(s)ds < +\infty, \right\} < R$$

*where*

$$M_i(s) := \max\left\{ \sup_{t\in\mathbb{R}} |G_i(t,s)|\,,\, \sup_{t\in\mathbb{R}} \left|\frac{\partial G_i(t,s)}{\partial t}\right|,\ i=1,2, \right\},$$

*and, for* $\rho > 0$, $x,y,z,w \in [-\rho,\,\rho]$,

$$|f(t,x,y,z,w)| \le \psi_\rho(t),\ \text{a.e. } t\in\mathbb{R}, \tag{4.8}$$

$$|h(t,x,y,z,w)| \le \psi^*_\rho(t),\ \text{a.e. } t\in\mathbb{R}. \tag{4.9}$$

*Moreover if* $(\alpha_1,\alpha_2)\,,\,(\beta_1,\beta_2)\in X$ *are lower and upper solutions of problem (4.1)-(4.2), respectively, such that*

$$(\alpha_1(t),\alpha_2(t)) \le (\beta_1(t),\beta_2(t))\,,\ \forall t\in\mathbb{R}, \tag{4.10}$$

*and, if* $f(t,x,y,z,w)$ *is nonincreasing on* $y$ *and monotone (nonincreasing or nondecreasing) on* $z$*, for fixed* $(t,x,w)\in\mathbb{R}^3$*, and* $h(t,x,y,z,w)$ *is nonincreasing in* $x$ *and monotone (nonincreasing or nondecreasing) on* $w$*, for fixed* $(t,y,z)\in\mathbb{R}^3$*, then the problem (4.1)-(4.2) has a homoclinic solution* $(u,v)\in X$ *such that*

$$\alpha_1(t)\le u(t)\le\beta_1(t),\quad \alpha_2(t)\le v(t)\le\beta_2(t),\ \forall t\in\mathbb{R}.$$

**Proof** Consider the auxiliary functions $\delta_i:\mathbb{R}^2\to\mathbb{R}$, for $i=1,2$, given by

$$\delta_i(t,w) = \begin{cases} \beta_i(t) & , \quad w>\beta_i(t) \\ w & , \quad \alpha_i(t)\le w\le\beta_i(t) \\ \alpha_i(t) & , \quad w<\alpha_i(t), \end{cases} \tag{4.11}$$

and the truncated and perturbed coupled system

$$\begin{cases} u''(t)-k_1^2u(t) = f(t,\delta_1(t,u(t)),\delta_2(t,v(t)),u'(t),v'(t)) \\ \\ v''(t)-k_2^2v(t) = h(t,\delta_1(t,u(t)),\delta_2(t,v(t)),u'(t),v'(t)). \end{cases} \tag{4.12}$$

Define the operators $T_1,T_2: X\to X_1$, and $T: X\to X$ by

$$T\,(u,v) = (T_1\,(u,v)\,,T_2\,(u,v))\,, \tag{4.13}$$

with

$$\begin{aligned} & (T_1\,(u,v))\,(t) \\ = & \int_{-\infty}^{+\infty} G_1(t,s)f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))ds, \\ & (T_2\,(u,v))\,(t) \\ = & \int_{-\infty}^{+\infty} G_2(t,s)h(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))ds, \end{aligned}$$

where $G_1(t,s)$, $G_2(t,s)$ is defined by (4.5) and (4.6), respectively.

By Lemma 4.1.1, the fixed points of $T$ are solutions of (4.1)-(4.2) and from Theorem 2.1.1, we need to show that the operator $T$ is compact, in order to prove that $T(u,v)$ has a fixed point.

For clearness, we divide this proof into several steps. The technique for operator $T_2(u,v)$ is similar.

**(i)** *$T$ is well defined and continuous in $X$.*

Let $(u,v) \in X$. As $f, h$ are $L^1-$ Carathéodory functions, $T$ is continuous and by the Lebesgue dominated theorem,

$$\begin{aligned}
& \lim_{t\to\pm\infty} T_1(u,v)(t) \\
= \ & \lim_{t\to\pm\infty} \int_{-\infty}^{+\infty} G_1(t,s) f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))ds \\
= \ & \lim_{t\to-\infty} \int_{-\infty}^{t} \frac{-1}{2k_1} e^{k_1(s-t)} f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))ds \\
+ \ & \lim_{t\to+\infty} \int_{t}^{+\infty} \frac{-1}{2k_1} e^{k_1(t-s)} f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))ds \\
= \ & 0.
\end{aligned}$$

As

$$\frac{\partial G_1(t,s)}{\partial t} = \begin{cases} \frac{1}{2} e^{k_1(s-t)}, & -\infty \le s < t \le +\infty \\ -\frac{1}{2} e^{k_1(t-s)}, & -\infty \le t < s \le +\infty, \end{cases} \tag{4.14}$$

then

$$\begin{aligned}
& \lim_{t\to\pm\infty} (T_1(u,v)(t))' \\
= \ & \lim_{t\to\pm\infty} \int_{-\infty}^{+\infty} \frac{\partial G_1(t,s)}{\partial t} f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))ds \\
= \ & \lim_{t\to-\infty} \int_{-\infty}^{t} \frac{-1}{2} e^{k_1(s-t)} f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))ds \\
+ \ & \lim_{t\to+\infty} \int_{t}^{+\infty} \frac{1}{2} e^{k_1(t-s)} f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))ds \\
= \ & 0.
\end{aligned}$$

Therefore, $T_1(u,v) \in X_1$. For $T_2(u,v) \in X_1$ the arguments are similar and, therefore, $T(u,v) \in X$.

**(ii)** *$TB$ is uniformly bounded on $B \subseteq X$, for some bounded $B$.*

As $f$, $h$ are a $L^1$-Carathéodory functions, there exist $\psi_\rho$, $\psi_\rho^* \in L^1(\mathbb{R})$, with $\rho > 0$, such that,

$$\rho > \max\Big\{\|\alpha_i\|_\infty,\ \|\beta_i\|_\infty,\ i = 1,2\Big\},$$

where

$$|f(t,\delta_1(t,u(t)),\delta_2(t,v(t)),u'(t),v'(t))| \leq \psi_\rho(t),\ \ a.e.t \in \mathbb{R}, \quad (4.15)$$

$$|h(t,\delta_1(t,u(t)),\delta_2(t,v(t)),u'(t),v'(t))| \leq \psi_\rho^*(t),\ \ a.e.t \in \mathbb{R}, \quad (4.16)$$

Define

$$M_i(s) := \max\left\{\sup_{t\in\mathbb{R}} |G_i(x,s)|\,,\ \sup_{t\in\mathbb{R}} \left|\frac{\partial G_i(t,s)}{\partial t}\right|,\ i = 1,2,\right\} \quad (4.17)$$

and remark that

$$\Big|M_i(s)\Big| \leq \max\left\{\frac{1}{2},\ \frac{1}{2k_i},\ i = 1,2\right\},\ \forall s \in \mathbb{R}. \quad (4.18)$$

For some

$$\rho_1 > \max\left\{\rho,\ \int_{-\infty}^{+\infty} M_1(s)\psi_\rho(s)ds,\ \int_{-\infty}^{+\infty} M_2(s)\psi_\rho^*(s)ds\right\}, \quad (4.19)$$

let $B$ be a bounded set of $X$, defined by

$$B := \{(u,v) \in X : \max\{\|u\|_\infty, \|u'\|_\infty, \|v\|_\infty, \|v'\|_\infty\} \leq \rho_1\}. \quad (4.20)$$

So,

$$\begin{aligned} & \|T_1(u,v)(t)\|_\infty \\ = & \sup_{t\in\mathbb{R}}\left(\left|\int_{-\infty}^{+\infty} G_1(t,s)f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))ds\right|\right) \\ \leq & \int_{-\infty}^{+\infty} M_1(s)\psi_\rho(s)ds < +\infty, \end{aligned}$$

and

$$
\begin{aligned}
& \left\| \left(T_1(u,v)\right)'(t) \right\|_\infty \\
= & \sup_{t\in\mathbb{R}} \left( \left| \int_{-\infty}^{+\infty} \frac{\partial G_1(t,s)}{\partial t} f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))ds \right| \right) \\
\leq & \int_{-\infty}^{+\infty} M_1(s)\psi_\rho(s)ds < +\infty.
\end{aligned}
$$

So, $T_1B$ is uniformly bounded on $X_1$. By similar arguments, we conclude that $T_2$ is also uniformly bounded on $X_1$. Therefore $TB$ is uniformly bounded on $X$.

**(iii)** *$TB$ is equicontinuous on $X$.*

Let $t_1, t_2 \in \mathbb{R}$ and suppose, without loss of generality, that $t_1 \leq t_2$. By the continuity of $G_1(t,s)$

$$
\begin{aligned}
& \lim_{t_1\to t_2} \left| T_1(u,v)(t_1) - T_1(u,v)(t_2) \right| \\
\leq & \int_{-\infty}^{+\infty} \left( \begin{array}{c} \lim_{t_1\to t_2} \left| G_1(t_1,s) - G_1(t_2,s) \right| \\ \left| f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s)) \right| \end{array} \right) ds = 0.
\end{aligned}
$$

As $\frac{\partial G_1(t,s)}{\partial t}$ is bounded on $\mathbb{R}^2$, therefore

$$
\begin{aligned}
& \lim_{t_1\to t_2} \left| \left(T_1(u,v)\right)'(t_1) - \left(T_1(u,v)\right)'(t_2) \right| \\
\leq & \lim_{t_1\to t_2} \int_{-\infty}^{+\infty} \left( \begin{array}{c} \left| \frac{\partial G_1(t_1,s)}{\partial t} - \frac{\partial G_1(t_2,s)}{\partial t} \right| \\ \left| f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s)) \right| \end{array} \right) ds \\
= & \lim_{t_1\to t_2} \int_{-\infty}^{t_1} \left( \begin{array}{c} \left| \frac{\partial G_1(t_1,s)}{\partial t} - \frac{\partial G_1(t_2,s)}{\partial t} \right| \\ \left| f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s)) \right| \end{array} \right) ds \\
& + \lim_{t_1\to t_2} \int_{t_1}^{t_2} \left( \begin{array}{c} \left| \frac{\partial G_1(t_1,s)}{\partial t} - \frac{\partial G_1(t_2,s)}{\partial t} \right| \\ \left| f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s)) \right| \end{array} \right) ds \\
& + \lim_{t_1\to t_2} \int_{t_2}^{+\infty} \left( \begin{array}{c} \left| \frac{\partial G_1(t_1,s)}{\partial t} - \frac{\partial G_1(t_2,s)}{\partial t} \right| \\ \left| f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s)) \right| \end{array} \right) ds.
\end{aligned}
$$

By the continuity of $\frac{\partial G_1(t_1,s)}{\partial t}$, for $s \in ]-\infty, t_1[$ and the continuity of $\frac{\partial G_1(t_2,s)}{\partial t}$ on $s \in ]t_2, +\infty[$, then the first and the third integrals tend to 0.

As the second one we have, by (4.15) and (4.18)

$$\lim_{t_1\to t_2}\int_{t_1}^{t_2}\left(\frac{\left|\frac{\partial G_1(t_1,s)}{\partial t}-\frac{\partial G_1(t_2,s)}{\partial t}\right|}{|f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))|}\right)ds=0.$$

Therefore, $T_1B$ is equicontinuous on $X_1$. Analogously, it can be proved that $T_2B$ is equicontinuous on $X_1$ and so, $TB$ is equicontinuous on $X$.

**(iv)** *$TD$ is equiconvergent at $t=\pm\infty$.*

For the operator $T_1$, we have, by (i) and (4.15)

$$\begin{aligned}
&\left|T_1(u,v)(t)-\lim_{t\to\pm\infty}T_1(u,v)(t)\right|\\
\leq\ &\int_{-\infty}^{+\infty}|G_1(t,s)|\,|f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))|\,ds\\
\leq\ &\int_{-\infty}^{+\infty}|G_1(t,s)|\,|\psi_\rho(s)|\,ds\to 0,
\end{aligned}$$

as $t\to\pm\infty$ and, by (4.18),

$$\begin{aligned}
&\left|(T_1(u,v))'(t)-\lim_{t\to\pm\infty}(T_1(u,v))'(t)\right|\\
\leq\ &\int_{-\infty}^{+\infty}\left|\frac{\partial G_1(t,s)}{\partial t}\right|\,|f(s,\delta_1(s,u(s)),\delta_2(s,v(s)),u'(s),v'(s))|\,ds\\
\leq\ &\int_{-\infty}^{+\infty}\left|\frac{\partial G_1(t,s)}{\partial t}\right|\,|\psi_\rho(s)|\,ds\to 0,
\end{aligned}$$

as $t\to\pm\infty$.

Therefore, $T_1D$ is equiconvergent at $\pm\infty$ and, following a similar technique, we can prove that $T_2D$ is equiconvergent at $\pm\infty$, too. So, $TD$ is equiconvergent at $\pm\infty$.

**(v)** *For $D\subset X$ a nonempty, closed, bounded and convex set, we have $T(D)\subset D$.*

Let $\rho_2>\rho_1$, with $\rho_1$ given by (4.19), and consider

$$D:=\{(u,v)\in X:\|(u,v)\|_X\leq\rho_2\}.$$

Applying the same method as in Step **(ii)**, we have $\|T_1(u,v)\|_X \le \rho_2$ and $\|T_2(u,v)\|_X \le \rho_2$ and therefore $\|T(u,v)\|_X \le \rho_2$.

By Theorem 3.1.1, $TD$ is relatively compact, therefore, by Theorem 2.1.1, $T$ as at least one fixed point $(u,v) \in X$, which is a solution of problem (4.12), (4.2).

**(vi)** *This solution of (4.12), (4.2),* $(u,v) \in X$, *is a solution of the initial problem (4.1), (4.2) if*

$$\alpha_1(t) \le u(t) \le \beta_1(t), \quad \alpha_2(t) \le v(t) \le \beta_2(t), \ \forall t \in \mathbb{R}.$$

Let $(u(t), v(t))$ be a solution of problem (4.12), (4.2) and suppose, by contradiction, that there is $t \in \mathbb{R}$, such that $\alpha_1(t) > u(t)$. Define

$$\inf_{t\in\mathbb{R}} (u(t) - \alpha_1(t)) := u(t_0) - \alpha_1(t_0) < 0.$$

As, by (4.2) and Definition 4.1.1,

$$u(\pm\infty) - \alpha_1(\pm\infty) = -\alpha_1(\pm\infty) \ge 0,$$

therefore, $t_0 \ne \pm\infty$.

Assume that $t_0 \in \mathbb{R}$, such that

$$\min_{t\in\mathbb{R}} (u(t) - \alpha_1(t)) := u(t_0) - \alpha_1(t_0) < 0.$$

So, there exists an interval, $[t_1, t_2]$ such that $t_0 \in [t_1, t_2]$, and

$$u(t) - \alpha_1(t) < 0, \ u''(t) - \alpha_1''(t) \le 0, \ a.e. t \in [t_1, t_2],$$

$$u'(t) - \alpha_1'(t) \le 0, \ t \in [t_1, t_0] \text{ and } u'(t) - \alpha_1'(t) \ge 0, \ t \in [t_0, t_2].$$

If $f(t, x, y, z, w)$ is nonincreasing on $z$, for $t \in [t_0, t_2]$, we have the following contradiction

$$\begin{aligned}
0 &\le \int_{t_0}^{t} \Big( u''(s) - \alpha_1''(s) \Big) ds \\
&= \int_{t_0}^{t} \left[ \begin{array}{c} (s, \delta_1(s, u(s)), \delta_2(s, v(s)), u'(s), v'(s)) \\ +k_1^2 u(s) - \alpha_1''(s) \end{array} \right] ds \\
&\le \int_{t_0}^{t} \Big[ f(s, \alpha_1(s), \alpha_2(s), \alpha_1'(s), v'(s)) + k_1^2 u(s) - \alpha_1''(s) \Big] ds \\
&\le k_1^2 \int_{t_0}^{t} \Big[ u(s) - \alpha_1(s) \Big] ds < 0.
\end{aligned}$$

If $f(t,x,y,z,w)$ is nondecreasing on $z$, we consider $t \in [t_1, t_0]$, and, from the previous arguments, we obtain a similar contradiction. Therefore, $\alpha_1(t) \le u(t)$, $\forall t \in \mathbb{R}$.

Applying the method above for function $h(t,x,y,z,w)$, it may be shown that $\alpha_2(t) \le v(t) \le \beta_2(t)$, $\forall t \in \mathbb{R}$.

■

In Theorem 4.2.1, the guarantee that the homoclinic solution is non trivial is given by the assumptions

$$f(t,0,0,0,0) \neq 0, h(t,0,0,0,0) \neq 0, \text{ for some } t \in \mathbb{R}.$$

However, these conditions can be dropped, as it is shown in the following theorem, which proof is analogous to Theorem 4.2.1:

**Theorem 4.2.2** *Let $f, h : \mathbb{R}^5 \to \mathbb{R}$ be $L^1-$ Carathéodory functions verifying the assumptions of Theorem 4.2.1. Assume that there are $(\alpha_1, \alpha_2)$, $(\beta_1, \beta_2) \in X$ lower and upper solutions of problem (4.1)-(4.2), respectively, such that*

$$0 < \alpha_i(t) \le \beta_i(t), \text{ for } i = 1,2 \text{ for some } t \in \mathbb{R},$$

*or*

$$\alpha_i(t) \le \beta_i(t) < 0, \text{ for } i = 1,2 \text{ for some } t \in \mathbb{R}.$$

*If $f(t,x,y,z,w)$ is nonincreasing on $y$ and monotone (nonincreasing or nondecreasing) on $z$, for fixed $(t,x,w) \in \mathbb{R}^3$, and $h(t,x,y,z,w)$ is nonincreasing in $x$ and monotone (nonincreasing or nondecreasing) on $w$, for fixed $(t,y,z) \in \mathbb{R}^3$, then the problem (4.1)-(4.2) has a homoclinic solution $(u,v) \in X$ such that*

$$\alpha_1(t) \le u(t) \le \beta_1(t), \quad \alpha_2(t) \le v(t) \le \beta_2(t), \ \forall t \in \mathbb{R}.$$

## 4.3 Application to a coupled nonlinear of real Schrödinger system type

In this section we consider an application to a family of a coupled stationary nonlinear Schrödinger system (NLS).

The Schrödinger equations are physically linked, with, for example, waves and solitons, [88, 106], nonlinear optics and photons, [109, 172].

In [27], the authors prove the existence of two different kinds of homoclinic solutions to the origin, describing solitary waves of physical relevance, using a system of two coupled nonlinear Schrödinger equations with inhomogeneous parameters, including a linear coupling.

In [42], (see also [33, 34]), the author considers the system of two coupled Schrödinger equations, modelling spatial solitons in crystals, or modeling pulse propagation in fibres, [172],

$$\begin{cases} i\frac{\partial u}{\partial t} + r\frac{\partial^2 u}{\partial x^2} - u + uv = 0 \\ i\frac{\partial v}{\partial t} + s\frac{\partial^2 v}{\partial x^2} - \alpha v + \frac{1}{2}u^2 = 0, \end{cases} \tag{4.21}$$

where:

- $u(t,x)$ and $v(t,x)$ represent the first and second harmonics of the amplitude envelope of an optical pulse, respectively;
- $t$ and $x$ are the time and space, respectively;
- $\alpha$ is the wave-vector mismatch between the two harmonics;
- $r, s = \pm 1$ and their signs are determined by the signs of the dispersions/diffractions (temporal/spatial cases, respectively).

In this application, we consider a real stationary family of system (4.21) (see, [34]), with $r = s = \alpha = 1$ and $u(t)$, $v(t)$ are considered real,

$$\begin{cases} u''(t) - u(t) = -\frac{1}{9(t^2+1)}|u(t) - 1|\,(v(t) + 1) \\ v''(t) - v(t) = -\frac{1}{9(2t^2+1)}\,(u(t) + 1)\,|v(t) - 1|, \end{cases} \tag{4.22}$$

together with the boundary conditions (4.2).

Notice that, the system (4.22), (4.2) is a particular case of problem (4.1), (4.2) with $k_1 = k_2 = 1$, and

$$\begin{aligned} f(t,x,y,z,w) &= -\frac{1}{9(t^2+1)}|x-1|\,(y+1)\,, \\ h(t,x,y,z,w) &= -\frac{1}{9(2t^2+1)}\,(x+1)\,|y-1|. \end{aligned}$$

Moreover, $f$ and $h$ verify the monotone assumptions of Theorem

4.2.1, and for $|x|$, $|y| < \rho$, both are $L^1$-Carathéodory functions, with

$$\phi_\rho(t) = \frac{1}{9(t^2+1)} (\rho+1)^2, \quad \varphi_\rho(t) = \frac{1}{9(2t^2+1)} (\rho+1)^2 .$$

The functions $\alpha_i$, $\beta_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2$, given by $\alpha_i(t) = -1$ and $\beta_i(t) = 1$ are, respectively, lower and upper solutions of problem (4.22), (4.2), satisfying (4.10), once that

$$\begin{array}{rcl}
\alpha_1''(t) - \alpha_1(t) & = & 1 > f(t,-1,-1,0,w) = 0, \\
\alpha_2''(t) - \alpha_2(t) & = & 1 > h(t,-1,-1,z,0) = 0, \\
\beta_1''(t) - \beta_1(t) & = & -1 < f(t,1,1,0,w) = 0, \\
\beta_2''(t) - \beta_2(t) & = & -1 < h(t,1,1,z,0) = 0.
\end{array}$$

As assumptions of Theorem 4.2.1 are verified for $\rho \in [1,\ 3.4]$, then the problem (4.22), (4.2) has a homoclinic solution $(u,v) \in X$ such that, for $i = 1,2$,

$$-1 \leq u(t) \leq 1 \text{ and } -1 \leq v(t) \leq 1, \forall t \in \mathbb{R}.$$

Remark that this solution is not the trivial one, as the null function is not a solution of (4.22).

# 5

# *Heteroclinic solutions with phi-Laplacians*

In this chapter, we consider the second order coupled system on the real line

$$\begin{cases} \big(a(t)\phi\big(u'(t)\big)\big)' = f(t,u(t),v(t),u'(t),v'(t)), \\ \\ \big(b(t)\psi\big(v'(t)\big)\big)' = h(t,u(t),v(t),u'(t),v'(t)), \ t \in \mathbb{R}, \end{cases} \tag{5.1}$$

with $\phi$ and $\psi$ increasing homeomorphisms verifying some adequate relations on their inverses, $a, b : \mathbb{R} \to (0, +\infty[$ are continuous functions, $f, h : \mathbb{R}^5 \to \mathbb{R}$ are $L^1$-Carathéodory functions, together with asymptotic conditions

$$\begin{cases} u(-\infty) = A, \ u(+\infty) = B, \\ v(-\infty) = C, \ v(+\infty) = D, \end{cases} \tag{5.2}$$

for $A, B, C, D \in \mathbb{R}$, satisfying some relations, and where

$$u(\pm\infty) := \lim_{t\to\pm\infty} u(t), \quad v(\pm\infty) := \lim_{t\to\pm\infty} v(t).$$

Heteroclinic trajectories play an important role in geometrical analysis of dynamical systems, connecting unstable and stable equilibria having two or more equilibrium points, [95]. In fact, the homoclinic or heteroclinic orbits are a kind of spiral structures, which are general phenomena in nature, [212]. Graphical illustrations and a very complete explanation on homoclinics and heteroclinics bifurcations can be seen in [87]. A planar homoclinic theorem and heteroclinic orbits, to analyze fluid models, is studied in [31]. Applications of dynamic systems techniques to the problem of heteroclinic connections and resonance transitions, are treated in [111], on planar circular domains. To prove the existence of heteroclinic solutions, for a class of non-autonomous second-order equations, see [8, 53, 142]. Topological, variational and minimization methods to find heteroclinic connections can be found in [207].

On heteroclinic coupled systems, among many published works, we highlight some of them:

In [3], Aguiar et al. consider the dynamics of small networks of coupled cells, with one of the points, analyzed as invariant subsets, can support robust heteroclinic attractors;

In [15], Ashwin and Karabacak study coupled phase oscillators and discuss heteroclinic cycles and networks between partially synchronized states and in [103], they analyze coupled phase oscillators, highlighting a dynamic mechanism, nothing more than a heteroclinic network;

In [16], the authors investigate such heteroclinic network between partially synchronized states, where the phases cluster are divided into three groups;

Moreover, in [62], the authors present some applications, results, methods and problems that have been recently reported and, in addition, they suggest some possible research directions, and some problems for further studies on homoclinics and heteroclinics.

Cabada and Cid, in [37], study the following boundary value problem on the real line

$$\begin{cases} \big(\phi\big(u'(t)\big)\big)' = f(t, u(t), u'(t)), \text{ on } \mathbb{R}, \\ u(-\infty) = -1, \quad u(+\infty) = 1, \end{cases}$$

with a singular $\phi$-Laplacian operator where $f$ is a continuous function that satisfies suitable symmetric conditions.

In [39], Calamai discusses the solvability of the following strongly nonlinear problem:

$$\begin{cases} \big(a(x(t))\phi\big(x'(t)\big)\big)' = f(t, x(t), x'(t)), \quad t \in \mathbb{R}, \\ x(-\infty) = \alpha, \quad x(+\infty) = \beta, \end{cases}$$

where $\alpha < \beta$, $\phi : (-r, r) \to \mathbb{R}$ is a general increasing homeomorphism with bounded domain (singular $\phi$-Laplacian), $a$ is a positive, continuous function and $f$ is a Carathéodory nonlinear function.

Recently, in [100], Kajiwara proved the existence of a heteroclinic solution of the FitzHugh-Nagumo type reaction-diffusion system, under certain conditions on the heterogeneity.

Motivated by these works, this chapter is based on [165], and using the techniques suggested in [134, 148, 153, 155], we apply the fixed point theory, to obtain sufficient conditions for the existence of heteroclinic solutions of the coupled system (5.1), (5.2), assuming some adequate conditions on $\phi^{-1}$, $\psi^{-1}$.

We emphasize that it is the first time where heteroclinic solutions for second order coupled differential systems are considered for systems with full nonlinearities depending on both unknown functions and their first derivatives. An example illustrates the potentialities of our main result, and an application to coupled nonlinear systems of two degrees of freedom (2-DOF), shows the applicability of the main theorem.

## 5.1 Notations and preliminary results

Consider the following spaces

$$X := \left\{ x \in C^1(\mathbb{R}) : \lim_{|t|\to\infty} x^{(i)}(t) \in \mathbb{R},\ i = 0,1 \right\},$$

equipped with the norm

$$\|x\|_X = \max\left\{\|x\|_\infty, \|x'\|_\infty\right\},$$

where

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} |x(t)|,$$

and $X^2 := X \times X$ with

$$\|(u,v)\|_{X^2} = \max\left\{\|u\|_X, \|v\|_X\right\}.$$

It can be proved that $(X, \|\cdot\|_X)$ and $(X^2, \|\cdot\|_{X^2})$ are Banach spaces.

By solution of problem (5.1), (5.2) we mean a pair $(u,v) \in X^2$ such that

$$a(t)\phi(u'(t)) \in W^{1,1}(\mathbb{R}) \text{ and } b(t)\psi(v'(t)) \in W^{1,1}(\mathbb{R}),$$

verifying (5.1), (5.2).

For the reader's convenience we consider $L^1-$ Carathéodory functions given by Definition 3.1.1, with a function $g : \mathbb{R}^5 \to \mathbb{R}$ and condition *(iii)* replaced by:

*(iii)* for each $\rho > 0$, there exists a positive function $\vartheta_\rho \in L^1(\mathbb{R})$ such that, whenever $x, y, z, w \in [-\rho, \rho]$, then

$$|g(t,x,y,z,w)| \le \vartheta_\rho(t), \quad \text{a.e. } t \in \mathbb{R}. \tag{5.3}$$

Along this chapter we assume that

(H1) $\phi$,$\psi : \mathbb{R} \longrightarrow \mathbb{R}$ are increasing homeomorphisms such that

**a)** $\phi(\mathbb{R}) = \mathbb{R}, \quad \phi(0) = 0, \quad \psi(\mathbb{R}) = \mathbb{R}, \quad \psi(0) = 0;$

**b)** $|\phi^{-1}(x)| \le \phi^{-1}(|x|), \quad |\psi^{-1}(x)| \le \psi^{-1}(|x|).$

(H2) $a, b : \mathbb{R} \to (0, +\infty[$ are positive continuous functions such that

$$\lim_{t\to\pm\infty} \frac{1}{a(t)} \in \mathbb{R} \text{ and } \lim_{t\to\pm\infty} \frac{1}{b(t)} \in \mathbb{R}.$$

Next lemma presents the integral form of solutions for problem (5.1), (5.2).

**Lemma 5.1.1** *Let $\phi$, $\psi : \mathbb{R} \to \mathbb{R}$ be increasing homeomorphisms, $f, h : \mathbb{R}^5 \to \mathbb{R}$, $L^1-$Carathéodory functions such that (H1) and (H2) hold. Then $(u, v) \in X^2$ is a pair of heteroclinic solutions of the system (5.1), (5.2) if, and only if,*

$$u(t) = \int_{-\infty}^{t} \phi^{-1}\left(\frac{\mu_{(u,v)} + \int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r))dr}{a(s)}\right) ds + A,$$

$$v(t) = \int_{-\infty}^{t} \psi^{-1}\left(\frac{\nu_{(u,v)} + \int_{-\infty}^{s} h(r, u(r), v(r), u'(r), v'(r))dr}{b(s)}\right) ds + C$$

*with $\mu_{(u,v)}$ and $\nu_{(u,v)}$ the unique solutions of*

$$B - A = \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\mu_{(u,v)} + \int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r))dr}{a(s)}\right) ds \tag{5.4}$$

*and*

$$D - C = \int_{-\infty}^{+\infty} \psi^{-1}\left(\frac{\nu_{(u,v)} + \int_{-\infty}^{s} h(r, u(r), v(r), u'(r), v'(r))dr}{b(s)}\right) ds, \tag{5.5}$$

*respectively. Moreover*

$$\mu_{(u,v)} \in \left[-\int_{-\infty}^{+\infty} |f(t, u(r), v(r), u'(r), v'(r))|dr, \int_{-\infty}^{+\infty} |f(t, u(r), v(r), u'(r), v'(r))|dr\right] \tag{5.6}$$

*and*

$$\nu_{(u,v)} \in \left[-\int_{-\infty}^{+\infty} |h(t, u(r), v(r), u'(r), v'(r))|dr, \int_{-\infty}^{+\infty} |h(t, u(r), v(r), u'(r), v'(r))|dr\right]. \tag{5.7}$$

**Proof** Let $(u,\, v) \in X^2$ be a solution of (5.1), (5.2). By integration and (5.2),

$$\phi\big(u'(t)\big) = \frac{\mu_{(u,v)} + \int_{-\infty}^{t} f(r, u(r), v(r), u'(r), v'(r))dr}{a(t)}, \;\; t \in \mathbb{R},$$

and

$$u'(t) = \phi^{-1}\left(\frac{\mu_{(u,v)} + \int_{-\infty}^{t} f(r, u(r), v(r), u'(r), v'(r))dr}{a(t)}\right).$$

So,

$$u(t) = \int_{-\infty}^{t} \phi^{-1}\left(\frac{\mu_{(u,v)} + \int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r))dr}{a(s)}\right) ds + A.$$

To prove the uniqueness of $\mu_{(u,v)}$ let

$$F(y) = \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{y + \int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r))dr}{a(s)}\right) ds + B - A$$

be a continuous and increasing function, and

$$m_1 = -\int_{-\infty}^{+\infty} |f(t, u(r), v(r), u'(r), v'(r))|dr,$$

$$m_2 = \int_{-\infty}^{+\infty} |f(t, u(r), v(r), u'(r), v'(r))|dr.$$

Therefore

$$\begin{aligned} F(m_1) = \quad & \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{m_1 + \int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r))dr}{a(s)}\right) ds \\ & + B - A \le B - A, \end{aligned}$$

and

$$\begin{aligned} F(m_2) = \quad & \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{m_2 + \int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r))dr}{a(s)}\right) ds \\ & + B - A \ge B - A. \end{aligned}$$

Consequently, by Bolzano's theorem and the monotonicity of $F$, there is

$$\mu_{(u,v)} \in \left[-\int_{-\infty}^{+\infty} |f(t,u(r),v(r),u'(r),v'(r))|dr, \int_{-\infty}^{+\infty} |f(t,u(r),v(r),u'(r),v'(r))|dr\right],$$

which is the unique solution of (5.4).

Applying the same arguments, it can be proved a similar integral expression for the function $v(t)$ and $\nu_v$ the unique solution of (5.5) verifying

$$\nu_{(u,v)} \in \left[-\int_{-\infty}^{+\infty} |h(t,u(r),v(r),u'(r),v'(r))|dr, \int_{-\infty}^{+\infty} |h(t,u(r),v(r),u'(r),v'(r))|dr\right].$$

■

**Remark 5.1.1** *If $A = B$ and $C = D$, then this pair of functions $(u, v) \in X^2$ will be a pair of homoclinic solutions of the system (5.1).*

A convenient criterion for the compacity of the operators, is given by Theorem 3.1.1 with the adaptation considered in the previous chapter and the existence tool will be given by Schauder's fixed point theorem (Theorem 2.1.1).

## 5.2 Existence of heteroclinics

In this section we prove the existence for a pair of heteroclinic solutions to the coupled system (5.1), (5.2), for some constants $A, B, C, D \in \mathbb{R}$.

**Theorem 5.2.1** *Let $\phi$, $\psi : \mathbb{R} \to \mathbb{R}$ be increasing homeomorphisms and $a, b : \mathbb{R} \to (0, +\infty[$ continuous functions satisfying (H2).*

*Assume that* $f, h : \mathbb{R}^5 \to \mathbb{R}$ *are* $L^1-$*Carathéodory functions verifying (H1), such that*

$$\begin{aligned} B - A &= \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\mu_{(u,v)} + \int_{-\infty}^{s} f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)}\right) ds, \\ D - C &= \int_{-\infty}^{+\infty} \psi^{-1}\left(\frac{\nu_{(u,v)} + \int_{-\infty}^{s} h(r,u(r),v(r),u'(r),v'(r))dr}{b(s)}\right) ds, \end{aligned}$$

*and there is* $R > 0$ *and* $\vartheta_R,\ \theta_R \in L^1(\mathbb{R})$ *such that*

$$\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \vartheta_R(r)dr}{a(s)}\right) ds < +\infty, \tag{5.8}$$

$$\int_{-\infty}^{+\infty} \psi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \theta_R(r)dr}{b(s)}\right) ds < +\infty, \tag{5.9}$$

*with*

$$\sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \vartheta_R(r)dr}{a(t)}\right) ds < +\infty,$$

$$\sup_{t\in\mathbb{R}} \psi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \theta_R(r)dr}{b(t)}\right) ds < +\infty,$$

$$|f(t,u(t),v(t),u'(t),v'(t))| \le \vartheta_R(t), \tag{5.10}$$

$$|h(t,u(t),v(t),u'(t),v'(t))| \le \theta_R(t). \tag{5.11}$$

*Then problem (5.1), (5.2) has, at least, a pair of heteroclinic solutions* $(u,v) \in X^2$*.*

**Proof** Define the operators $T_1 : X^2 \to X$, $T_2 : X^2 \to X$ and $T : X^2 \to X^2$ by

$$T(u,v) = (T_1(u,v), T_2(u,v)), \tag{5.12}$$

with

$$\begin{aligned} &(T_1(u,v))(t) \\ = &\int_{-\infty}^{t} \phi^{-1}\left(\frac{\mu_{(u,v)} + \int_{-\infty}^{s} f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)}\right) ds + A, \\ &(T_2(u,v))(t) \\ = &\int_{-\infty}^{t} \psi^{-1}\left(\frac{\nu_{(u,v)} + \int_{-\infty}^{s} h(r,u(r),v(r),u'(r),v'(r))dr}{b(s)}\right) ds + C. \end{aligned}$$

In order to apply Lemma 5.1.1 and Theorem 2.1.1, we shall prove that $T$ is compact and has a fixed point.

To simplify the proof, we detail the arguments only for $T_1(u,v)$, as for the operator $T_2(u,v)$ the technique is similar.

To be clear, we divide the proof into claims **(i)-(v)**.

**(i)** *$T$ is well defined and continuous in $X^2$.*

Let $(u,v) \in X^2$ and take $\rho > 0$ such that $\|(u,v)\|_{X^2} < \rho$. As $f$ is a $L^1-$ Carathéodory function, there exists a positive function $\vartheta_\rho \in L^1(\mathbb{R})$ verifying (5.10). So,

$$\begin{aligned}&\int_{-\infty}^{t} |f(r,u(r),v(r),u'(r),v'(r))|dr \\ \leq &\int_{-\infty}^{+\infty} |f(r,u(r),v(r),u'(r),v'(r))|dr \leq \int_{-\infty}^{+\infty} \vartheta_\rho(t)dt < +\infty.\end{aligned}$$

Now we will show that $\mu_{(u,v)}, \nu_{(u,v)}$ are continuous.

Consider the sequences $(u_n,v_n)_{n\in\mathbb{N}}$ such that $(u_n,v_n) \to (u_0,v_0)$, as $n \to \infty$. Then for $n = 0,1,2,...,$

$$\exists K > 0: \ ||(u_n,v_n)|| \leq K$$

and by (5.4) and (5.5),

$$B-A = \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\mu_{(u_n,v_n)} + \int_{-\infty}^{s} f(r,u_n(r),v_n(r),u_n'(r),v_n'(r))dr}{a(s)}\right) ds,$$

$$D-C = \int_{-\infty}^{+\infty} \psi^{-1}\left(\frac{\nu_{(u_n,v_n)} + \int_{-\infty}^{s} h(r,u_n(r),v_n(r),u_n'(r),v_n'(r))dr}{b(s)}\right) ds.$$

To prove that

$$\mu_{(u_n,v_n)} \longrightarrow \mu_{(u_0,v_0)} \text{ and } \nu_{(u_n,v_n)} \longrightarrow \nu_{(u_0,v_0)},$$

we will assume that

$$\mu_{(u_n,v_n)} \nrightarrow \mu_{(u_0,v_0)} \text{ or } \nu_{(u_n,v_n)} \nrightarrow \nu_{(u_0,v_0)}.$$

By (5.6) and (5.10),

$$\begin{aligned}|\mu_{(u_n,v_n)}| &\leq \int_{-\infty}^{+\infty} |f(r,u_n(r),v_n(r),u_n'(r),v_n'(r))|dr \\ &\leq \int_{-\infty}^{+\infty} \vartheta_\rho(t)dt < +\infty.\end{aligned}$$

So $\mu_{(u_n,v_n)}$ is uniformly bounded. Analogously, by (5.7) and (5.11), it can be proved that $\nu_{(u_n,v_n)}$ is uniformly bounded, too.

Therefore, assume that there are two subsequences $(u^{(1)}_{n_k},\, v^{(1)}_{n_k})$, $(u^{(2)}_{n_k},\, v^{(2)}_{n_k}) \in X^2$ such that

$$(\mu_{u^{(1)}_{n_k}},\, \nu_{v^{(1)}_{n_k}}) \longrightarrow (\mu_{u_1},\, \nu_{v_1}) \text{ and } (\mu_{u^{(2)}_{n_k}},\, \nu_{v^{(2)}_{n_k}}) \longrightarrow (\mu_{u_2},\, \nu_{v_2}),$$

as $n_k \longrightarrow +\infty$ with

$$(\mu_{u_1},\, \nu_{v_1}) \neq (\mu_{u_2},\, \nu_{v_2}). \tag{5.13}$$

By (5.4) and, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} & B-A \\ = & \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\mu_{\left(u^{(1)}_{n_k},v^{(1)}_{n_k}\right)}+\int_{-\infty}^{s} f(r,u^{(1)}_{n_k}(r),v^{(1)}_{n_k}(r),\left(u^{(1)}_{n_k}\right)'(r),\left(v^{(1)}_{n_k}\right)'(r))dr}{a(s)}\right)ds \\ \to & \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\mu_{(u_1,v_1)}+\int_{-\infty}^{s} f(r,u_1(r),v_1(r),{u_1}'(r),{v_1}'(r))dr}{a(s)}\right)ds \\ = & B-A, \end{aligned}$$

as $n_k \to \infty$, and

$$\begin{aligned} & B-A \\ = & \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\mu_{\left(u^{(2)}_{n_k},v^{(2)}_{n_k}\right)}+\int_{-\infty}^{s} f(r,u^{(2)}_{n_k}(r),v^{(2)}_{n_k}(r),\left(u^{(2)}_{n_k}\right)'(r),\left(v^{(2)}_{n_k}\right)'(r))dr}{a(s)}\right)ds \\ \to & \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\mu_{(u_2,v_2)}+\int_{-\infty}^{s} f(r,u_2(r),v_2(r),{u_2}'(r),{v_2}'(r))dr}{a(s)}\right)ds \\ = & B-A \end{aligned}$$

as $n_k \to \infty$.

Then by Lemma 5.1.1, $\mu_{(u_1,v_1)} = \mu_{(u_2,v_2)}$. Applying the same arguments and (5.5) we can prove that $\nu_{(u_1,v_1)} = \nu_{(u_2,v_2)}$. So,

$$(\mu_{(u_1,v_1)},\, \nu_{(u_1,v_1)}) = (\mu_{(u_2,u_2)},\, \nu_{(u_2,v_2)}),$$

which contradicts (5.13). Thereby,

$$\mu_{(u_n,v_n)} \longrightarrow \mu_{(u_0,v_0)} \text{ and } \nu_{(u_n,v_n)} \longrightarrow \nu_{(u_0,v_0)}.$$

So, $T_1$ is continuous on $X$. Furthermore,

$$\left(T_1(u,v)\right)'(t)=\phi^{-1}\left(\frac{\mu_{(u,v)}+\int_{-\infty}^{t}f(r,u(r),v(r),u'(r),v'(r))dr}{a(t)}\right)$$

is also continuous on $X$ and, therefore, $T_1(u,v)\in C^1(\mathbb{R})$.

By (5.2) and (5.4),

$$\begin{aligned}&\lim_{t\to-\infty}T_1(u,v)(t)\\&=\lim_{t\to-\infty}\int_{-\infty}^{t}\phi^{-1}\left(\tfrac{\mu_{(u,v)}+\int_{-\infty}^{s}f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)}\right)ds+A=A,\end{aligned}$$

$$\begin{aligned}&\lim_{t\to+\infty}T_1(u,v)(t)\\&=\int_{-\infty}^{+\infty}\phi^{-1}\left(\tfrac{\mu_{(u,v)}+\int_{-\infty}^{s}f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)}\right)ds+A=B,\end{aligned}$$

and, by (5.6), (5.10) and (H2),

$$\begin{aligned}\lim_{t\to\pm\infty}\left(T_1(u,v)(t)\right)'&=\lim_{t\to\pm\infty}\phi^{-1}\left(\tfrac{\mu_{(u,v)}+\int_{-\infty}^{t}f(r,u(r),v(r),u'(r),v'(r))dr}{a(t)}\right)\\&\le\lim_{t\to\pm\infty}\phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty}\vartheta_\rho(r)dr}{a(t)}\right)<+\infty.\end{aligned}$$

Therefore, $T_1(u,v)\in X$, and, by the same arguments, $T_2(u,v)\in X$. So, $T(u,v)\in X^2$.

**(ii)** *$TM$ is uniformly bounded on $M\subseteq X^2$, for some bounded $M$.*

Let $M$ be a bounded set of $X^2$, defined by

$$M:=\{(u,v)\in X^2:\max\{\|u\|_\infty,\|u'\|_\infty,\|v\|_\infty,\|v'\|_\infty\}\le\rho_1\},\tag{5.14}$$

for some $\rho_1>0$.

By (5.6), (5.8), (5.10), (H1) and (H2), we have

$$\begin{aligned}
&\|T_1(u,v)(t)\|_\infty \\
=\ & \sup_{t\in\mathbb{R}} \left( \begin{array}{c} \left| \int_{-\infty}^{t} \phi^{-1}\left(\frac{\mu_{(u,v)}+\int_{-\infty}^{s} f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)}\right) ds \right. \\ \left. +A \right| \end{array} \right) \\
\le\ & \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \left( \begin{array}{c} \left| \phi^{-1}\left(\frac{\mu_{(u,v)}+\int_{-\infty}^{s} f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)}\right) \right| \\ ds + |A| \end{array} \right) \\
\le\ & \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \left( \begin{array}{c} \phi^{-1}\left(\frac{|\mu_{(u,v)}|+\int_{-\infty}^{s} \left|f(r,u(r),v(r),u'(r),v'(r))\right|dr}{a(s)}\right) ds \\ +|A| \end{array} \right) \\
\le\ & \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \vartheta_{\rho_1}(r)dr}{a(s)}\right) ds + |A| < +\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\|\left(T_1(u,v)\right)'(t)\|_\infty \\
=\ & \sup_{t\in\mathbb{R}} \left| \phi^{-1}\left(\frac{\mu_{(u,v)}+\int_{-\infty}^{t} f(r,u(r),v(r),u'(r),v'(r))dr}{a(t)}\right) \right| \\
\le\ & \sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{|\mu_{(u,v)}|+\int_{-\infty}^{t} |f(r,u(r),v(r),u'(r),v'(r))|\, dr}{a(t)}\right) \\
\le\ & \sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \vartheta_{\rho_1}(r)dr}{a(t)}\right) < +\infty.
\end{aligned}$$

So, $\|T_1(u,v)(t)\|_X < +\infty$, that is, $T_1M$ is uniformly bounded on $X$.

By similar arguments, $T_2$ is uniformly bounded on $X$. Therefore $TM$ is uniformly bounded on $X^2$.

**(iii)** *$TM$ is equicontinuous on $X^2$.*

Let $t_1, t_2 \in [-K, K] \subseteq \mathbb{R}$ for some $K > 0$, and suppose, without loss of generality, that $t_1 \le t_2$. Thus, by (5.6), (5.8), (5.10) and (H1),

$$
\begin{aligned}
&\left|T_1(u,v)(t_1) - T_1(u,v)(t_2)\right| \\
= &\left|\int_{-\infty}^{t_1} \phi^{-1}\left(\frac{\mu_{(u,v)} + \int_{-\infty}^{s} f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)}\right) ds\right. \\
&\left. - \int_{-\infty}^{t_2} \phi^{-1}\left(\frac{\mu_{(u,v)} + \int_{-\infty}^{s} f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)}\right) ds\right| \\
\leq &\int_{t_1}^{t_2} \phi^{-1}\left(\frac{|\mu_{(u,v)}| + \int_{-\infty}^{s} |f(r,u(r),v(r),u'(r),v'(r))dr|}{a(s)}\right) ds \\
\leq &\int_{t_1}^{t_2} \phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \vartheta_\rho(r)dr}{a(s)}\right) ds \to 0,
\end{aligned}
$$

uniformly for $(u,v) \in M$, as $t_1 \to t_2$, and

$$
\begin{aligned}
&\left|(T_1(u,v))'(t_1) - (T_1(u,v))'(t_2)\right| \\
= &\left|\phi^{-1}\left(\frac{\mu_{(u,v)} + \int_{-\infty}^{t_1} f(r,u(r),v(r),u'(r),v'(r))dr}{a(t_1)}\right)\right. \\
&\left. -\phi^{-1}\left(\frac{\mu_{(u,v)} + \int_{-\infty}^{t_2} f(r,u(r),v(r),u'(r),v'(r))dr}{a(t_2)}\right)\right| \to 0,
\end{aligned}
$$

uniformly for $(u,v) \in M$, as $t_1 \to t_2$.

Therefore, $T_1M$ is equicontinuous on $X$. Analogously, it can be proved that $T_2M$ is equicontinuous on $X$. So, $TM$ is equicontinuous on $X^2$.

**(iv)** *$TM$ is equiconvergent at $t = \pm\infty$.*

Let $(u,v) \in M$. For the operator $T_1$, we have, by (5.6), (5.8), (5.10) and (H1),

$$
\begin{aligned}
&\left|T_1(u,v)(t) - \lim_{t\to-\infty} T_1(u,v)(t)\right| \\
= &\left|\int_{-\infty}^{t} \phi^{-1}\left(\frac{\mu_{(u,v)} + \int_{-\infty}^{s} f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)}\right) ds\right| \\
\leq &\int_{-\infty}^{t} \phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \vartheta_\rho(r)dr}{a(s)}\right) ds \to 0,
\end{aligned}
$$

uniformly in $(u,v) \in M$, as $t \to -\infty$, and,

$$\begin{aligned}
&\left| T_1(u,v)(t) - \lim_{t\to+\infty} T_1(u,v)(t) \right| \\
= &\left| \int_{-\infty}^{t} \phi^{-1}\left( \frac{\mu_{(u,v)} + \int_{-\infty}^{s} f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)} \right) ds \right. \\
- &\left. \int_{-\infty}^{+\infty} \phi^{-1}\left( \frac{\mu_{(u,v)} + \int_{-\infty}^{s} f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)} \right) ds \right| \\
= &\left| \int_{t}^{+\infty} \phi^{-1}\left( \frac{\mu_{(u,v)} + \int_{-\infty}^{s} f(r,u(r),v(r),u'(r),v'(r))dr}{a(s)} \right) ds \right| \\
\leq &\int_{t}^{+\infty} \phi^{-1}\left( \frac{2\int_{-\infty}^{+\infty} \vartheta_\rho(r)dr}{a(s)} \right) ds \to 0,
\end{aligned}$$

uniformly in $(u,v) \in M$, as $t \to +\infty$.

For the derivative it follows that,

$$\begin{aligned}
&\left| (T_1(u,v))'(t) - \lim_{t\to+\infty} (T_1(u,v))'(t) \right| \\
= &\left| \phi^{-1}\left( \frac{\mu_{(u,v)} + \int_{-\infty}^{t} f(r,u(r),v(r),u'(r),v'(r))dr}{a(t)} \right) \right. \\
&\left. -\phi^{-1}\left( \lim_{t\to+\infty} \frac{\mu_{(u,v)} + \int_{-\infty}^{+\infty} f(r,u(r),v(r),u'(r),v'(r))dr}{a(t)} \right) \right|
\end{aligned}$$

$\to 0$, uniformly in $(u,v) \in M$, as $t \to +\infty$, and

$$\begin{aligned}
&\left| (T_1(u,v))'(t) - \lim_{t\to-\infty} (T_1(u,v))'(t) \right| \\
= &\left| \phi^{-1}\left( \tfrac{\mu_{(u,v)} + \int_{-\infty}^{t} f(r,u(r),v(r),u'(r),v'(r))dr}{a(t)} \right) - \phi^{-1}\left( \lim_{t\to-\infty} \tfrac{\mu_{(u,v)}}{a(t)} \right) \right| \\
\leq &\left| \phi^{-1}\left( \frac{\mu_{(u,v)}}{a(t)} \right) - \phi^{-1}\left( \lim_{t\to-\infty} \frac{\mu_{(u,v)}}{a(t)} \right) \right| \to 0
\end{aligned}$$

uniformly in $(u,v) \in M$, as $t \to -\infty$.

Therefore, $T_1M$ is equiconvergent at $\pm\infty$ and, following a similar technique, we can prove that $T_2M$ is equiconvergent at $\pm\infty$, too. So, $TM$ is equiconvergent at $\pm\infty$.

By Theorem 3.1.1, $TM$ is relatively compact.

**(v)** *$T : X \to X$ has a fixed point.*

In order to apply Schauder's fixed point theorem for operator $T(u,v)$, we need to prove that $TD \subset D$, for some closed, bounded and convex $D \subset X^2$.

Consider

$$D := \left\{ (u,v) \in X^2 : \|(u,v)\|_{X^2} \leq \rho_2 \right\},$$

with $\rho_2 > 0$ such that

$$\rho_2 := \max \left\{ \begin{array}{c} \rho_1, \ \int_{-\infty}^{+\infty} \phi^{-1}\left( \frac{2\int_{-\infty}^{+\infty} \vartheta_{\rho_2}(r)dr}{a(s)} \right) ds + |A|, \\ \\ \int_{-\infty}^{+\infty} \psi^{-1}\left( \frac{2\int_{-\infty}^{+\infty} \theta_{\rho_2}(r)dr}{b(s)} \right) ds + |C|, \\ \\ \sup_{t\in\mathbb{R}} \phi^{-1}\left( \frac{2\int_{-\infty}^{+\infty} \vartheta_{\rho_2}(r)dr}{a(t)} \right), \\ \\ \sup_{t\in\mathbb{R}} \psi^{-1}\left( \frac{2\int_{-\infty}^{+\infty} \theta_{\rho_2}(r)dr}{b(t)} \right) \end{array} \right\}$$

with $\rho_1$ given by (5.14).

Following similar arguments as in **(ii)**, we have, for $(u,v) \in D$,

$$\begin{aligned} \|T(u,v)\|_{X^2} &= \|(T_1(u,v), T_2(u,v))\|_{X^2} \\ &= \max\left\{ \|T_1(u,v)\|_X, \ \|T_2(u,v)\|_X \right\} \\ &= \max\Big\{ \|T_1(u,v)\|_\infty, \ \left\|(T_1(u,v))'\right\|_\infty, \\ &\quad \|T_2(u,v)\|_\infty, \ \left\|(T_2(u,v))'\right\|_\infty \Big\} \leq \rho_2, \end{aligned}$$

and $TD \subset D$.

By Theorem 2.1.1, the operator $T(u,v) = (T_1(u,v), T_2(u,v))$ has a fixed point $(u,v) \in X^2$, which is, by Lemma 5.1.1, a pair of heteroclinic solutions of problem (5.1), (5.2).

■

## 5.3 Application to coupled systems of nonlinear 2-DOF model

Generic nonlinear coupled systems of two degrees of freedom (2-DOF), are especially important in Physics and Mechanics. For example in [145], the authors use this type of system to investigate the transient in a system containing a linear oscillator, linearly coupled to an essentially nonlinear attachment with a comparatively small mass. The family of coupled non-linear systems of 2-DOF is used to study the global bifurcations in the motion of an externally forced coupled nonlinear oscillatory system or for the nonlinear vibration absorber subjected to periodic excitation, see [141]. Moreover, in [12], the authors deal with the stochastic moment stability of such systems.

Motivated by these works, in this section we consider an application of system (5.1), (5.2), to a family of coupled non-linear systems of 2-DOF model, given by the nonlinear coupled system (see [141])

$$\left\{\begin{array}{l} \left((1+t^4)\left(q_1'(t)\right)^3\right)' = \frac{t^4}{(1+t^6)^2}\left[2\zeta\omega_0\left(q_1'(t)\right)^3 + \omega_0^2 q_1(t)\right. \\ \qquad\qquad \left. +\gamma((q_1(t))^3 - 3d^2 q_1(t)q_2(t)) + \cos(t)\right], \\ \\ \tau^2\left((1+t^4)\left(q_2'(t)\right)^3\right)' = \frac{t^4}{(1+t^6)^2}\left[2\zeta\omega_0\left(q_2'(t)\right)^3 + \omega_0^2 q_2(t)\right. \\ \qquad\qquad \left. +\gamma(d^2\left(q_2(t)\right)^3 - 3\left(q_1(t)\right)^2 q_2(t))\right], \ t\in\mathbb{R}, \end{array}\right. \tag{5.15}$$

where

- $q_1(t)$ and $q_2(t)$ represent the generalized coordinates;
- $d,\ \tau,\ \gamma$ are positive constant coefficients which depend on the characteristics of the physical or mechanical system under consideration;
- $\cos(t)$ is related to the type of excitation of the system under consideration;
- $\zeta, \omega_0$, are the damping coefficient and the frequency, respectively.

As the asymptotic conditions we consider

$$\left\{\begin{array}{l} q_1(-\infty) = A,\ \ q_1(+\infty) = B, \\ q_2(-\infty) = C,\ \ q_2(+\infty) = D, \end{array}\right. \tag{5.16}$$

with $A, B, C, D \in \mathbb{R}$, such that

$$B - A = \int_{-\infty}^{+\infty} \left( \sqrt[3]{\frac{\int_{-\infty}^{s} \frac{r^4}{(1+r^6)^2} \begin{bmatrix} 2\zeta\omega_0 \left(q_1'(r)\right)^3 + \omega_0^2 q_1(r) + \gamma((q_1(r))^3 \\ -3d^2 q_1(r) q_2(r)) + \cos(r) \end{bmatrix} dr}{1+s^4}} \right) ds \tag{5.17}$$

and

$$D - C = \int_{-\infty}^{+\infty} \left( \sqrt[3]{\frac{\int_{-\infty}^{s} \frac{r^4}{\tau^2(1+r^6)^2} \begin{bmatrix} 2\zeta\omega_0 \left(q_2'(r)\right)^3 + \omega_0^2 q_2(r) + \gamma(d^2(q_2(r))^3 \\ -3\left(q_1(r)\right)^2 q_2(r)) \end{bmatrix} dr}{1+s^4}} \right) ds. \tag{5.18}$$

It is clear that (5.15) is a particular case of (5.1) with:

$$\phi(z) = \psi(z) = z^3, \;\; a(t) = b(t) = 1 + t^4,$$

$f, h : \mathbb{R}^5 \to \mathbb{R}$ are $L^1$-Carathéodory functions where

$$\begin{aligned} f(t,x,y,z,w) &= \frac{t^4}{(1+t^6)^2} \left(2\zeta\omega_0 z^3 + \omega_0^2 x + \gamma x^3 - 3d^2 xy + \cos(t)\right) \\ &\leq \frac{t^4}{\left(t^6+1\right)^2} \left(2\left|\zeta\omega_0\right| \rho^3 + \omega_0^2 \rho + \gamma\rho^3 + 3d^2\rho^2 + 1\right) \\ &:= \delta_\rho(t) \end{aligned}$$

$$\begin{aligned} h(t,x,y,z,w) &= \frac{t^4}{\tau^2(1+t^6)^2} \left(2\zeta\omega_0 w^3 + \omega_0^2 y + \gamma d^2 y^3 - 3x^2 y\right) \\ &\leq \frac{t^4}{\tau^2(t^6+1)^2} \left(2\left|\zeta\omega_0\right| \rho^3 + \omega_0^2 \rho + \gamma d^2 \rho^3 + 3\rho^3\right) \\ &:= \varepsilon_\rho(t) \end{aligned}$$

where $\delta_\rho(t)$ and $\varepsilon_\rho(t)$ are functions in $L^1(\mathbb{R})$, for $\rho > 0$ such that

$$\rho := \max\left\{|x|, |y|, |z|, |w|\right\}. \tag{5.19}$$

Moreover, conditions (H1) and (H2) hold as,

- $\phi(\mathbb{R}) = \psi(w) = \mathbb{R}$ and $\phi(0) = \psi(0) = 0$;

- $|\phi^{-1}(z)| = |\sqrt[3]{z}| = \phi^{-1}(|z|) = \sqrt[3]{|z|}$ and $|\psi^{-1}(w)| = |\sqrt[3]{w}| = \psi^{-1}(|w|) = \sqrt[3]{|w|}$;
- $\lim\limits_{t\to\pm\infty} \frac{1}{a(t)} = \lim\limits_{t\to\pm\infty} \frac{1}{1+t^4} = \lim\limits_{t\to\pm\infty} \frac{1}{b(t)} = 0.$

For $\rho > 0$ such that

$$\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty}\delta_\rho(r)dr}{a(s)}\right) ds = \int_{-\infty}^{+\infty}\left(\sqrt[3]{\frac{2\int_{-\infty}^{+\infty} r^4 \frac{\left(2|\zeta\omega_0|\rho^3+\omega_0^2\rho+\gamma\rho^3+3d^2\rho^2+1\right)}{(1+r^6)^2}dr}{1+s^4}}\right) ds < \rho \quad (5.20)$$

and

$$\int_{-\infty}^{+\infty} \psi^{-1}\left(\frac{2\int_{-\infty}^{+\infty}\varepsilon_\rho(r)dr}{b(s)}\right) ds = \int_{-\infty}^{+\infty}\left(\sqrt[3]{\frac{2\int_{-\infty}^{+\infty} r^4 \frac{\left(2|\zeta\omega_0|\rho^3+\omega_0^2\rho+\gamma d^2\rho^3+3\rho^3\right)}{\tau^2(1+r^6)^2}dr}{1+s^4}}\right) ds < \rho, \quad (5.21)$$

by Theorem 5.2.1, the system (5.15) together with the asymptotic conditions (5.16), has at least a pair $(q_1, q_2) \in X^2$ of heteroclinic solutions with $A, B, C, D \in \mathbb{R}$ verifying (5.17) and (5.18). As example, in particular, for

$$|\zeta| = \frac{1}{2\sqrt{1000}},\ |\omega_0| = \frac{1}{\sqrt{1000}},\ |\gamma| = \frac{1}{1000},\ d^2 = \frac{1}{3000},$$

the conditions (5.20) and (5.21) hold for $\rho \geq 8.35172$.

Remark that, if $A = B$ and $C = D$, the system (5.15) has a pair of homoclinic solutions $(q_1, q_2) \in X^2$.

# Part III

# Coupled impulsive systems

# *Introduction*

Impulsive differential equations describe processes in which a sudden change of state occurs at certain moments. Usually one of the characteristics of these processes is the existence of instantaneous disturbances and of very short time in relation to the process itself.

These situations arise naturally, for example, in phenomena studied in physics, chemistry, population dynamics, biotechnology, economics, optimal control, medicine and others [20, 23, 40, 52, 89, 186, 196]. However, the study of systems involving two or more differential impulsive equations is scarce in both bounded and unbounded domains (see, [132]).

The first appearance of equations or systems involving impulses arises in the early 20th century and appears to be related to the Dirac delta distribution, [24, 45, 51].

To the best of our knowledge, the first paper to make reference on impulses is [147], about the stability of motion with the presence of impulses.

Since then, several different approaches and applications on systems, boundary value problems and numerical methods involving impulses were developed, [21, 22, 29, 49, 56, 58, 66, 146, 175, 180].

In this Part we study problems composed of second order systems with complete nonlinearities together mixed boundary conditions at two points, and subject to generalized impulse conditions which allow jumps in the unknown functions and its derivative. Our goal, is to determine sufficient conditions on the nonlinearities to ensure the solvability of the problem.

To investigate this Part, we highlight the arguments used in [151, 152, 160], combined mainly with:

- ***Carathéodory sequences*** to control the behavior of the infinite moments of impulse;
- ***Equiconvergence*** at each impulsive moment and at $\pm\infty$ to recover the compacity on unbounded domains;
- Similar methods and techniques to those ones used in Part I and II.

This third Part consists of three chapters which cover the existence and location of solutions for impulsive coupled systems on bounded and unbounded domains. More precisely:

- In *Chapter 6*, we consider impulsive coupled system with mixed boundary conditions, together with generalized impulsive conditions, with dependence on the first derivative on bounded intervals;
- Int *Chapter 7*, we study the existence of solutions for impulsive coupled system with boundary conditions on unbounded intervals;
- Finally, in *Chapter 8*, we present some techniques to localize the solution, given in the previous chapter, applying the method of lower and upper solutions.

It is pointed out that, each chapter contain an application to real phenomena, to illustrate the applicability of our results in impulsive environments.

# 6

# *Impulsive coupled systems with generalized jump conditions*

In this chapter we consider the second order impulsive coupled system with mixed boundary conditions

$$\begin{cases} u''(x) = f(x, u(x), u'(x), v(x), v'(x)) \\ v''(x) = h(x, u(x), u'(x), v(x), v'(x)) \\ u(a) = A_1, \ u'(b) = B_1, \\ v(a) = A_2, \ v(b) = B_2, \end{cases} \tag{6.1}$$

with $f, h : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ $L^1$-Carathéodory functions, $A_1, A_2, B_1, B_2 \in \mathbb{R}$, with the generalized impulsive conditions

$$\begin{cases} \Delta u(x_k) = I_{0k}(x_k, u(x_k), u'(x_k)), \ k = 1, 2, ..., n, \\ \Delta u'(x_k) = I_{1k}(x_k, u(x_k), u'(x_k)), \\ \Delta v(\tau_j) = J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)), \ j = 1, 2, ..., m, \\ \Delta v'(\tau_j) = J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)), \end{cases} \tag{6.2}$$

where, for $i = 0, 1$, $\Delta u^{(i)}(x_k) = u^{(i)}(x_k^+) - u^{(i)}(x_k^-)$, $\Delta v^{(i)}(\tau_j) = v^{(i)}(\tau_j^+) - v^{(i)}(\tau_j^-)$, and being $I_{ik}, J_{ij} \in C([a, b] \times \mathbb{R}^2, \mathbb{R})$, with $x_k, \tau_j$ fixed points such that $a < x_1 < x_2 < \cdots < x_n < b$ and $a < \tau_1 < \tau_2 < \cdots < \tau_m < b$.

This chapter is based on [164].

The theory of impulsive differential equations describes processes in which a sudden change of state occurs at certain moments. Several authors (see for example, [1, 28, 63, 64, 73, 94, 93, 97, 124, 125, 136, 151, 159]) have dealt with impulsive differential equations, from different points of view and using many techniques.

There are many phenomena and applications related to impulsive differential systems, for example, we can find biological models, population dynamics, neural networks, models in economics, on time scales, on state-dependent delays, on delay-dependent impulsive control, on electrochemical communication between cells in the brain, (see for instance, [29, 40, 60, 91, 126, 127, 139, 158, 157, 161, 168, 178, 186, 189]), among others.

In [182], the author considers a sufficient conditions for the existence and uniqueness of solutions to the following complex dynamical network in the form of a coupled system of $m+2$ point boundary conditions for impulsive fractional differential equations

$$\begin{cases} {}^cD^\alpha u(t) = \phi(t, u(t), v(t)), \ \ t \in [0,1], \ \ t \neq t_j, \ \ j = 1, ..., m, \\ {}^cD^\beta v(t) = \psi(t, u(t), v(t)), \ \ t \in [0,1], \ \ t \neq t_i, \ \ i = 1, ..., n, \\ u(0) = h(u), \ \ u(1) = g(u) \ \text{ and } \ v(0) = k(v), \ \ v(1) = f(v), \\ \Delta u(t_j) = I_j(u(t_j)), \ \ \Delta u'(t_j) = \bar{\mathrm{I}}_j(u(t_j)), \quad j = 1, ..., m, \\ \Delta v(t_i) = I_i(v(t_i)), \ \ \Delta v'(t_i) = \bar{\mathrm{I}}_i(v(t_i)), \quad i = 1, ..., n, \end{cases}$$

where $1 < \alpha, \beta \leq 2$, $\phi, \psi : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and $g, h : X \to \mathbb{R}$, $f, k : Y \to \mathbb{R}$ are continuous functionals define by

$$g(u) = \sum_{j=1}^{p} \lambda_j u(\xi_j), \quad h(u) = \sum_{j=1}^{p} \lambda_j u(\eta_j),$$

$$f(v) = \sum_{i=1}^{q} \delta_i v(\xi_i), \quad k(v) = \sum_{i=1}^{q} \delta_i v(\eta_i),$$

$\xi_i, \eta_i, \xi_j, \eta_j \in (0,1)$ for $i = 1, ..., q$ and $j = 1, ..., q$.

In [132], it is studied the BVP composed by the second-order singular differential system on the whole line, with impulse effects, i.e., consisting of the differential system

$$\begin{array}{c} [\phi_p(\rho(t)x'(t))]' = f(t, x(t), y(t)), \ \ a.e.\, t \in \mathbb{R}, \\ [\phi_q(\varrho(t)y'(t))]' = g(t, x(t), y(t)), \ \ a.e.\, t \in \mathbb{R} \end{array}$$

subjected to the asymptotic conditions

$$\begin{array}{c} \lim_{t\to\pm\infty} x(s) = 0, \\ \lim_{t\to\pm\infty} y(s) = 0 \end{array}$$

and the impulsive effects

$$\begin{array}{c} \Delta x(t_k) = I_k(t_k, x(t_k), y(t_k)), \ \ k \in \mathbb{Z} \\ \Delta y(t_k) = J_k(t_k, x(t_k), y(t_k)), \ \ k \in \mathbb{Z}, \end{array}$$

where

(a) $\rho, \varrho \in C^0(\mathbb{R}, [0, \infty))$, $\rho(t), \varrho(t) > 0$ for all $t \in \mathbb{R}$ with $\int_{-\infty}^{+\infty} \frac{ds}{\rho(s)} < +\infty$ and $\int_{-\infty}^{+\infty} \frac{ds}{\varrho(s)} < +\infty$, (b)) $\phi_p(x) = x|x|^{p-2}$, $\phi_q(x) = x|x|^{q-2}$ with $p > 1$ and $q > 1$ are Laplace operators,

(c) $f, g$ on $\mathbb{R}^3$ are Carathéodory functions,

(d) $\cdots < t_k < t_{k+1} < t_{k+2} < \cdots$ with $\lim_{k\to-\infty} t_k = -\infty$ and $\lim_{k\to+\infty} t_k = +\infty$, $\Delta x(t_k) = u(t_x^+) - x(t_k^-)$ and $\Delta y(t_k) = y(t_x^+) - y(t_k^-)$ $(k \in \mathbb{Z})$, $\mathbb{Z}$ is the set of all integers,

(c) $\{I_k\}, \{J_k\}$, with $I_k, J_k : \mathbb{R}^3 \to \mathbb{R}$ are Carathéodory sequences.

Motivated by these works, we follow arguments applied in [160], to study problem (6.1)-(6.2). We point out, that is the first time where second order coupled differential equations systems include full nonlinearities. That is, they depend on the unknown functions and on their first derivatives, together with generalized impulsive conditions with dependence on the first derivative, too.

## 6.1 Definitions and auxiliary results

Define $u(x_k^\pm) := \lim_{x\to x_k^\pm} u(x)$ and consider the set

$$PC_1([a,b]) = \left\{ \begin{array}{c} u : u \in C([a,b],\mathbb{R}) \text{ continuous for} \\ x \neq x_k, u(x_k) = u(x_k^-),\ u(x_k^+) \\ \text{exists for } k = 1, 2, ..., n \end{array} \right\},$$

and the space $X_1 := PC_1^1([a,b]) = \{u : u'(t) \in PC_1([a,b])\}$ equipped with the norm $\|u\|_{X_1} = \max\{\|u\|, \|u'\|\}$, where

$$\|w\| := \sup_{x\in[a,b]} |w(x)|.$$

Analogously, define the set $X_2 := PC_2^1([a,b]) = \{v : v'(x) \in PC_2([a,b])\}$, with

$$PC_2([a,b]) = \left\{ \begin{array}{c} v : v \in C([a,b],\mathbb{R}) \text{ continuous for} \\ \tau \neq \tau_j, v(\tau_j) = v(\tau_j^-),\ v(\tau_j^+) \\ \text{exists for } j = 1, 2, ..., m \end{array} \right\},$$

equipped with the norm $\|v\|_{X_2} = \max\{\|v\|, \|v'\|\}$.

Denoting $X := X_1 \times X_2$ and the norm $\|(u,v)\|_X = \max\left\{\|u\|_{X_1}, \|v\|_{X_2}\right\}$, it is clear that $(X, \|.\|_X)$ is a Banach space.

A pair of functions $(u,v)$ is a solution of problem (6.1)-(6.2) if $(u,v) \in X$ and verifies conditions (6.1) and (6.2).

$L^1-$ Carathéodory functions are applied in the sense of Definition 3.1.1, but adapted now for a functions $g : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ and condition *(iii)* replaced by:

for each $\rho > 0$, there exists a positive function $\phi_\rho \in L^1([a,b])$ and for $(t,y,z,w) \in \mathbb{R}^4$ such that

$$\max\{|t|, |y|, |z|, |w|\} < \rho, \tag{6.3}$$

one has

$$|g(x,t,y,z,w)| \leq \phi_\rho(t), \; a.e. x \in [a,b].$$

**Lemma 6.1.1** *A pair of functions $(u,v) \in X$ is a solution of problem (6.1)-(6.2) if, and only if,*

$$\begin{aligned} & u(x) = A_1 + B_1 (x-a) \\ + & \sum_{x_k < x} [I_{0k}(x_k, u(x_k), u'(x_k)) + I_{1k}(x_k, u(x_k), u'(x_k))(x - x_k)] \\ - & (x-a) \sum_{k=1}^{n} I_{1k}(x_k, u(x_k), u'(x_k)) \\ + & \int_a^b G_1(x,s) \; f(s, u(s), u'(s), v(s), v'(s)) \; ds, \end{aligned}$$

*with $G_1(x,s)$ given by*

$$G_1(x,s) = \begin{cases} a - s, & a \leq x \leq s \leq b, \\ a - x, & a \leq s \leq x \leq b, \end{cases} \tag{6.4}$$

*and*

$$\begin{aligned} & v(x) = A_2 + \frac{B_2 - A_2}{b-a}(x-a) \\ + & \sum_{\tau_j < x} [J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))(x - \tau_j)] \\ - & \frac{x-a}{b-a} \sum_{j=1}^{m} [J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))(x - \tau_j)] \\ + & \int_a^b G_2(x,s) \; h(s, u(s), u'(s), v(s), v'(s)) \; ds. \end{aligned}$$

*with* $G_2(x,s)$ *defined by*

$$G_2(x,s) = \frac{1}{a-b}\begin{cases} (a-s)(b-x) & a \leq x \leq s \leq b, \\ (x-a)(b-s) & a \leq s \leq x \leq b. \end{cases} \tag{6.5}$$

The proof follows standard calculus and it is omitted.

## 6.2 Main theorem

The main result will provide the existence of, at least, a solution for problem (6.1)-(6.2).

**Theorem 6.2.1** *Let* $f, h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ *be* $L^1$*-Carathéodory functions and* $I_{ik}, J_{ij} : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous functions for* $i = 0,1,\ k = 1,2,...,n,,\ j = 1,2,...,m$. *Moreover, assume that there is* $R > 0$, *such that*

$$\max\left\{\begin{array}{c} |A_1| + |B_1|(b-a) + \sum_{k=1}^{n}\left[\varphi_{0k} + 2(b-a)\varphi_{1k}\right] \\ + \int_a^b M_1(s)\phi_R(s)ds, \\ \\ |B_1| + 2\sum_{k=1}^{n}\varphi_{1k} + \int_a^b \phi_R(s)ds, \\ \\ |A_2| + |B_2 - A_2| + 2\sum_{\tau_j < x}\left[\varphi_{0j}^* + \varphi_{1j}^*(b-a)\right] \\ + \int_a^b M_2(s)\psi_R(s)ds, \\ \\ \frac{|B_2 - A_2|}{b-a} + \frac{1}{b-a}\sum_{j=1}^{m}\varphi_{0j}^* + 3\sum_{j=1}^{m}\varphi_{1j}^* \\ + \frac{1}{b-a}\int_a^b \left|\frac{\partial G_2}{\partial x}(x,s)\right|\psi_R(s)ds \end{array}\right\} < R, \tag{6.6}$$

*where* $M_1,\ M_2 \in L^\infty(\mathbb{R})$ *given by,*

$$M_1(s) := \sup_{x\in[a,b]}|G_1(x,s)|,\ M_2(s) := \sup_{x\in[a,b]}|G_2(x,s)|, \tag{6.7}$$

$\phi_R,\ \psi_R \in L^1([a,b])$ *positive functions,* $(t,x,y,z,w) \in \mathbb{R}^5$ *with*

$$|f(t,x,y,z,w)| \leq \phi_R(t),\ a.e.x \in [a,b], \tag{6.8}$$

$$|h(t,x,y,z,w)| \leq \psi_R(t),\ a.e.x \in [a,b], \tag{6.9}$$

*and* $\varphi_{ik}$, $\varphi_{ij}^*$ *positive constants such that with*

$$|I_{ik}(x_k, u(x_k), u'(x_k)| \leq \varphi_{ik} \text{ and } |J_{ij}(\tau_j, v(\tau_j), v'(\tau_j)| \leq \varphi_{ij}^*,$$

*for* $i = 0, 1,\ k = 1, 2, ..., n,$ *and* $j = 1, 2, ..., m,.$

*Then, there is at least a pair of functions* $(u, v) \in X$ *solution of (6.1)-(6.2).*

**Proof** Define the operators $T_1 : X \to X_1$ ,$T_2 : X \to X_2$, and $T : X \to X$ by

$$T(u, v) = (T_1(u, v), T_2(u, v)), \tag{6.10}$$

with

$$\begin{aligned} & (T_1(u,v))(x) = A_1 + B_1(x-a) \\ + & \sum_{x_k<x} [I_{0k}(x_k, u(x_k), u'(x_k)) + I_{1k}(x_k, u(x_k), u'(x_k))(x - x_k)] \\ - & (x-a)\sum_{k=1}^{n} I_{1k}(x_k, u(x_k), u'(x_k)) \\ + & \int_a^b G_1(x,s)\ f(s, u(s), u'(s), v(s), v'(s))\ ds, \end{aligned}$$

$$\begin{aligned} & (T_2(u,v))(x) = A_2 + \frac{B_2 - A_2}{b-a}(x-a) \\ + & \sum_{\tau_j<x} [J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))(x - \tau_j)] \\ - & \frac{x-a}{b-a}\sum_{j=1}^{m} [J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))(x - \tau_j)] \\ + & \int_a^b G_2(x,s)\ h(s, u(s), u'(s), v(s), v'(s))\ ds. \end{aligned}$$

where $G_1(x, s)$ and $G_2(x, s)$ are given by (6.4) and (6.5), respectively.

By Lemma 6.1.1, it is obvious that the fixed points of $T$ are solutions of (6.1)-(6.2), so we shall prove that $T$ has a fixed point, following, for clearness, several steps.

**Step 1**: *$T$ is well defined and continuous in $X$.*

As $f, h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ are $L^1$-Carathéodory functions, then $T_1(u,v) \in PC_1^1$ and $T_2(u,v) \in PC_2^1$. In fact, $T(u,v) = (T_1(u,v), T_2(u,v))$ is continuous and

$$\begin{aligned} & (T_1(u,v))'(x) \\ = \;& B_1 + \sum_{x_k<x} I_{1k}(x_k, u(x_k), u'(x_k)) - \sum_{k=1}^{n} I_{1k}(x_k, u(x_k), u'(x_k)) \\ - \;& \int_a^x f(s, u(s), u'(s), v(s), v'(s))\, ds, \end{aligned}$$

$$\begin{aligned} (T_2(u,v))'(x) &= \frac{B_2 - A_2}{b-a} + \sum_{\tau_j<x} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)) \\ - \;& \frac{1}{b-a}\sum_{j=1}^{m}\left[J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))(x-\tau_j)\right] \\ - \;& \frac{x-a}{b-a}\sum_{j=1}^{m} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)) \\ + \;& \int_a^b \frac{\partial G_2}{\partial x}(x,s)\, h(s, u(s), u'(s), v(s), v'(s))\, ds, \end{aligned}$$

with

$$\frac{\partial G_2}{\partial x}(x,s) = \frac{1}{b-a}\begin{cases} s-a, & a \le x \le s \le b \\ b-s, & a \le s \le x \le b. \end{cases} \tag{6.11}$$

Therefore, $T_1(u,v) \in X_1$, $T_2(u,v) \in X_2$ and $T(u,v) \in X$.

**Step 2**: *$TB$ is uniformly bounded in $B \subset X$.*

Let $B$ be a bounded set of $X$. Then, there is $\rho_1 > 0$ such that

$$\max\left\{\|u\|_{X_1}, \|v\|_{X_2}\right\} < \rho_1. \tag{6.12}$$

Moreover,

$$\begin{aligned}
& \|T_1(u,v)(x)\| \\
& \leq \sup_{x\in[a,b]} (|A_1| + |B_1||x-a| \\
& + \sum_{x_k<x} |\left[I_{0k}(x_k,u(x_k),u'(x_k)) + I_{1k}(x_k,u(x_k),u'(x_k))\,(x-x_k)\right]| \\
& + |\,(x-a)\,| \sum_{k=1}^{n} |I_{1k}(x_k,u(x_k),u'(x_k))| \\
& + \int_a^b |G_1(x,s)|\ |f(s,u(s),u'(s),v(s),v'(s))|\ ds \\
\leq\ & |A_1| + |B_1|\,(b-a) + \sum_{k=1}^{n} \left[\varphi_{0k} + 2\,(b-a)\,\varphi_{1k}\right] \\
& + \int_a^b M_1(s)\phi_{\rho_1}(s)ds < +\infty,
\end{aligned}$$

$$\begin{aligned}
& \|\,(T_1(u,v))'(x)\| \leq \sup_{x\in[a,b]} \Bigg( |B_1| + \sum_{x_k<x} |I_{1k}(x_k,u(x_k),u'(x_k))| \\
& + \sum_{k=1}^{n} |I_{1k}(x_k,u(x_k),u'(x_k))| \\
& + \int_a^b\ |f(s,u(s),u'(s),v(s),v'(s))|\ ds \Bigg) \\
\leq\ & |B_1| + 2\sum_{k=1}^{n} \varphi_{1k} + \int_a^b \phi_{\rho_1}(s)ds < +\infty,
\end{aligned}$$

$$\begin{aligned}
& \|T_2(u,v)(x)\| \leq \sup_{x\in[a,b]} \Bigg( |A_2| + \frac{|B_2-A_2|}{b-a}|x-a| \\
& + \sum_{\tau_j<x} \left[|J_{0j}(\tau_j,v(\tau_j),v'(\tau_j)) + J_{1j}(\tau_j,v(\tau_j),v'(\tau_j))\,(x-\tau_j)\,|\right] \\
& + \frac{|x-a|}{b-a} \sum_{j=1}^{m} |J_{0j}(\tau_j,v(\tau_j),v'(\tau_j)) + J_{1j}(\tau_j,v(\tau_j),v'(\tau_j))\,(x-\tau_j)
\end{aligned}$$

$$
\begin{aligned}
+ \quad & \int_a^b |G_2(x,s)| \; |h(s,u(s),u'(s),v(s),v'(s))| \; ds\Big) \\
& \le |A_2| + |B_2 - A_2| + 2\sum_{\tau_j<x} \left[\varphi_{0j}^* + \varphi_{1j}^*\,(b-a)\right] \\
& + \int_a^b M_2(s)\psi_{\rho_1}(s)ds < +\infty,
\end{aligned}
$$

and, by (6.11),

$$
\begin{aligned}
& \| \left(T_2\,(u,v)\right)'(x)\| \\
& \le \sup_{x\in[a,b]} \Bigg( \frac{|B_2 - A_2|}{b-a} + \sum_{\tau_j<x} |J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))| \\
& + \frac{1}{b-a} \sum_{j=1}^m |J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))\,(x-\tau_j)\,| \\
& + \frac{|x-a|}{b-a} \sum_{j=1}^m |J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))| \\
& + \frac{1}{b-a} \int_a^b \left| \frac{\partial G_2}{\partial x}(x,s) \right| |h(s,u(s),u'(s),v(s),v'(s))| ds \Bigg) \\
\le \quad & \frac{|B_2 - A_2|}{b-a} + \frac{1}{b-a} \sum_{j=1}^m \varphi_{0j}^* + 3 \sum_{j=1}^m \varphi_{1j}^* \\
& + \frac{1}{b-a} \int_a^b \left| \frac{\partial G_2}{\partial x}(x,s) \right| \psi_{\rho_1}(s)ds < +\infty.
\end{aligned}
$$

So, $TB$ is uniformly bounded on $X$.

**Step 3**: *$T$ is equicontinuous on each interval $]x_k, x_{k+1}]\times]\tau_j, \tau_{j+1}]$, that is, $T_1B$ is equicontinuous on each interval $]x_k, x_{k+1}]$, for $k = 0, 1, ..., n$, with $x_0 = a$ and $x_{n+1} = b$, and $T_2B$ is equicontinuous on each interval $]\tau_j, \tau_{j+1}]$, for $j = 0, 1, ..., m$, with $\tau_0 = a$ and $\tau_{m+1} = b$.*

Consider $J \subseteq ]x_k, x_{k+1}]$ and $\iota_1, \iota_2 \in J$ such that $\iota_1 \le \iota_2$.

So, by the continuity of $G_1$

$$\begin{aligned}
& \left|T_1(u,v)(\iota_1) - T_1(u,v)(\iota_2)\right| \\
= & \left|B_1(\iota_1 - \iota_2)\right. \\
& + \sum_{x_k < \iota_1} \left[I_{0k}(x_k, u(x_k), u'(x_k)) + I_{1k}(x_k, u(x_k), u'(x_k))(\iota_1 - x_k)\right] \\
& - (\iota_1 - \iota_2) \sum_{k=1}^{n} I_{1k}(x_k, u(x_k), u'(x_k))
\end{aligned}$$

$$\begin{aligned}
& - \sum_{x_k < \iota_2} \left[I_{0k}(x_k, u(x_k), u'(x_k)) + I_{1k}(x_k, u(x_k), u'(x_k))(\iota_2 - x_k)\right] \\
& \left. + \int_a^b \left[G_1(\iota_1, s) - G_1(\iota_2, s)\right] f(s, u(s), u'(s), v(s), v'(s)\ ds \right| \to 0,
\end{aligned}$$

as $\iota_1 \to \iota_2$,

$$\begin{aligned}
& \left|(T_1(u,v)(\iota_1))' - (T_1(u,v)(\iota_2))'\right| \\
= & \left|\sum_{x_k < \iota_1} I_{1k}(x_k, u(x_k), u'(x_k)) - \sum_{x_k < \iota_2} I_{1k}(x_k, u(x_k), u'(x_k))\right. \\
& \left. - \int_{\iota_1}^{\iota_2} f(s, u(s), u'(s), v(s), v'(s)\ ds \right| \\
\to & \ 0, \ \text{as } \iota_1 \to \iota_2,
\end{aligned}$$

$$\begin{aligned}
& \left|T_2(u,v)(\iota_1) - T_2(u,v)(\iota_2)\right| \\
= & \left|\frac{B_2 - A_2}{b-a}(\iota_1 - \iota_2)\right. \\
& + \sum_{x_k < \iota_1} \left[J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))(\iota_1 - \tau_j)\right] \\
& + \frac{\iota_2 - \iota_1}{b-a} \sum_{j=1}^{m} J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) \\
& - \frac{\iota_1 - a}{b-a} \sum_{j=1}^{m} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))(\iota_1 - \tau_j) \\
& - \sum_{x_k < \iota_2} \left[J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))(\iota_2 - \tau_j)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\iota_2-a}{b-a}\sum_{j=1}^{m}J_{1j}(\tau_j,v(\tau_j),v'(\tau_j))\,(\iota_2-\tau_j)\\
&+\int_a^b\left[G_2(\iota_1,s)-G_2(\iota_2,s)\right]h(s,u(s),u'(s),v(s),v'(s)\ ds\Big|\\
\to\ & 0,\ \text{as } \iota_1\to\iota_2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|(T_2\,(u,v)\,(\iota_1))'-(T_2\,(u,v)\,(\iota_2))'\right|\\
=\ &\Bigg|\sum_{\tau_j<\iota_1}J_{1j}(\tau_j,v(\tau_j),v'(\tau_j))\\
&-\frac{1}{b-a}\sum_{j=1}^{m}J_{1j}(\tau_j,v(\tau_j),v'(\tau_j))\,(\iota_1-\iota_2)\\
&+\frac{(\iota_1-\iota_2)}{b-a}\sum_{j=1}^{m}J_{1j}(\tau_j,v(\tau_j),v'(\tau_j))-\sum_{\tau_j<\iota_2}J_{1j}(\tau_j,v(\tau_j),v'(\tau_j))\\
&+\frac{1}{b-a}\int_{\iota_1}^{\iota_2}\frac{\partial G_2}{\partial x}(x,s)h(s,u(s),u'(s),v(s),v'(s)\ ds\Bigg|\to 0,
\end{aligned}
$$

as $\iota_1\to\iota_2$, and $\frac{\partial G_2}{\partial x}$ given by (6.11).

**Step 4**: *$TB$ is equiconvergent, that is, $T_1B$ is equiconvergent at $x=x_k^+$ for $k=0,1,...,n$, and $T_2B$ is equiconvergent at $\tau=\tau_j^+$ for $\tau=1,...,m$.*

In fact,

$$
\begin{aligned}
&\left|T_1\,(u,v)\,(x)-T_1\,(u,v)\,(x_k^+)\right|\\
&=\Big|B_1(x-x_k^+)\\
&+\sum_{x_k<x}\left[I_{0k}(x_k,u(x_k),u'(x_k))+I_{1k}(x_k,u(x_k),u'(x_k))\,(x-x_k)\right]\\
&-\left(x-x_k^+\right)\sum_{k=1}^{n}I_{1k}(x_k,u(x_k),u'(x_k))\\
&-\sum_{x_k<x_k^+}\left[I_{0k}(x_k,u(x_k),u'(x_k))+I_{1k}(x_k,u(x_k),u'(x_k))\left(x_k^+-x_k\right)\right]\\
&+\int_a^b\left[G_1(x,s)-G_1(x_k^+,s)\right]f(s,u(s),u'(s),v(s),v'(s)\ ds\Big|\to 0,
\end{aligned}
$$

uniformly as $x \to x_k^+$ and

$$\begin{aligned}
&\left|(T_1(u,v)(x)' - \left(T_1(u,v)(x_k^+)\right)'\right| \\
= &\left|\sum_{x_k<x} I_{1k}(x_k, u(x_k), u'(x_k)) - \sum_{x_k<x_k^+} I_{1k}(x_k, u(x_k), u'(x_k))\right. \\
&\left. - \int_{x_k^+}^{x} f(s, u(s), u'(s), v(s), v'(s)\ ds\right| \to 0,
\end{aligned}$$

uniformly as $x \to x_k^+$. So, $T_1 B$ is equiconvergent at $x = x_{k^+}$ for $k = 0, 1, ..., n$.

Similarly,

$$\begin{aligned}
&\left|T_2(u,v)(\tau) - T_2(u,v)(\tau_j^+)\right| \\
&= \left|\frac{B_2 - A_2}{b-a}(\tau - \tau_j^+)\right. \\
&+ \sum_{\tau_j<\tau} \left[J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))\,(\tau - \tau_j)\right] \\
&+ \frac{\tau_j^+ - \tau}{b-a} \sum_{j=1}^{m} J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) \\
&- \frac{\tau - a}{b-a} \sum_{j=1}^{m} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))\,(\tau - \tau_j) \\
&- \sum_{\tau_j<\tau_j^+} \left[J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))\left(\tau_j^+ - \tau_j\right)\right] \\
&+ \frac{\tau_j^+ - a}{b-a} \sum_{j=1}^{m} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))\left(\tau_j^+ - \tau_j\right) \\
&+ \left.\int_a^b \left[G_2(\tau, s) - G_2(\tau_j^+, s)\right] h(s, u(s), u'(s), v(s), v'(s)\ ds\right| \to 0,
\end{aligned}$$

uniformly as $\tau \to \tau_j^+$, and

$$
\begin{aligned}
&\left|\left(T_2(u,v)(\tau)\right)' - \left(T_2(u,v)(\tau_j^+)\right)'\right| \\
&= \left| \sum_{\tau_j<\tau} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)) \right. \\
&-\frac{1}{b-a}\sum_{j=1}^{m} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))\left(\tau-\tau_j^+\right) \\
&+\frac{\left(\tau-\tau_j^+\right)}{b-a}\sum_{j=1}^{m} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)) - \sum_{\tau_j<\tau_j^+} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)) \\
&+\frac{1}{b-a}\left(\int_{\tau_j^+}^{\tau}(b-s)h(s,u(s),u'(s),v(s),v'(s)ds\right. \\
&\left.\left.+\int_{\tau_j^+}^{\tau}(s-a)h(s,u(s),u'(s),v(s),v'(s)ds\right)\ \right| \to 0,
\end{aligned}
$$

uniformly as $\tau \to \tau_j^+$. Then, $T_2B$ is equiconvergent at $\tau = \tau_j^+$ for $\tau = 1,...,m$.

Therefore, $T_1$ and $T_2$ maps bounded sets into relatively compact sets, that is, $T_1 : X \to X_1$ and $T_2 : X \to X_2$ are compacts. Therefore, $T : X \to X$ is compact (for details see [132], Lemma 2.4).

**Step 5**: *$T : X \to X$ has a fixed point.*

In order to apply Schauder's fixed point theorem for operator $T(u,v)$, we need to prove that $TD \subset D$, for some closed, bounded and convex $D \subset X$.

Consider

$$D := \left\{(u,v) \in X : \|(u,v)\|_X \le \rho_2\right\},$$

with $\rho_2 > 0$ such that

$$\rho_2 := \max \left\{ \begin{array}{c} \rho_1, \\ |A_1| + |B_1|\,(b-a) + \sum_{k=1}^{n} \left[\varphi_{0k} + 2\,(b-a)\,\varphi_{1k}\right] \\ + \int_a^b M_1(s)\phi_{\rho_2}(s)ds, \\ |B_1| + 2\sum_{k=1}^{n} \varphi_{1k} + \int_a^b \phi_{\rho_2}(s)ds, \\ |A_2| + |B_2 - A_2| + 2\sum_{\tau_j<x} \left[\varphi_{0j}^{*} + \varphi_{1j}^{*}\,(b-a)\right] \\ + \int_a^b M_2(s)\psi_{\rho_2}(s)ds, \\ \frac{|B_2-A_2|}{b-a} + \frac{1}{b-a}\sum_{j=1}^{m} \varphi_{0j}^{*} + 3\sum_{j=1}^{m} \varphi_{1j}^{*} \\ + \frac{1}{b-a}\int_a^b \left|\frac{\partial G_2}{\partial x}(x,s)\right| \psi_{\rho_2}(s)ds \end{array} \right\}, \tag{6.13}$$

with $\rho_1$ given by (6.12), according to Step 2 and $M_1$, $M_2$ are given by (6.7).

Following similar arguments as in Step 2, pursue

$$\begin{aligned} \|T(u,v)\|_X &= \|(T_1(u,v), T_2(u,v))\|_X \\ &= \max\left\{\|T_1(u,v)\|_{X_1},\ \|T_2(u,v)\|_{X_2}\right\} \\ &= \max\left\{\|T_1(u,v)\|,\ \left\|(T_1(u,v))'\right\|,\right. \\ &\qquad \left.\|T_2(u,v)\|,\ \left\|(T_2(u,v))'\right\|\right\} \\ &\leq \rho_2, \end{aligned}$$

and $TD \subset D$.

By Schauder's fixed point theorem, the operator $T$, given by (6.10) has a fixed point $(u_0, v_0)$. Thus, by Lemma 6.1.1, the problem (6.1)-(6.2) has at least a pair solution $(u,\, v) \in X$.

■

## 6.3 Example

Consider the coupled system composed by the second order differential equations with the mixed boundary conditions

$$\begin{cases} u''(x) = \frac{sgn\left(x-\frac{1}{2}\right)u(x)v(x)+(u'(x))^2v'(x)}{100}, \ x \in ]0,\ 1[ \\ v''(x) = \frac{-(x+1)(v'(x))^2u(x)+(u'(x))^3e^{-v(x)}}{500} \\ u(0)=1, \ u'(1)=\frac{1}{2}, \\ v(0)=1, \ v(1)=2, \end{cases} \tag{6.14}$$

and the generalized impulsive conditions

$$\begin{cases} \Delta u(x_k) = \frac{(u(x_k))^2(1-x_k)+u'(x_k)}{100}, \ \ k=1,2,3, \\ \Delta u'(x_k) = \frac{\sum_{k=1}^{3} x_k u(x_k)u'(x_k)}{100}, \\ \Delta v(\tau_j) = \frac{\tau_j |v(\tau_j)| v'(\tau_j)}{500}, \ \ j=1,2, \\ \Delta v'(\tau_j) = \frac{\sum_{j=1}^{2}(1-\tau_j)\frac{v(\tau_j)}{2}(v'(\tau_j))^2}{500}, \end{cases} \tag{6.15}$$

keep on $0 < x_1 < x_2 < x_3 < 1$, $0 < \tau_1 < \tau_2 < 1$.

This problem is a particular case of system (6.1)-(6.2) with

$$\begin{aligned} f(x,\alpha,\beta,\gamma,\delta) &= \frac{sgn\left(x-\frac{1}{2}\right)\alpha\beta+\gamma^2\delta}{100}, \\ h(x,\alpha,\beta,\gamma,\delta) &= \frac{-(x+1)\delta^2\alpha+\gamma^3e^{-\beta}}{500}, \\ A_1 &= 1, \ B_1=\frac{1}{2}, \ \ A_2=1, \ B_2=2, \\ I_{0k}(x_k,\alpha,\beta) &= \frac{\alpha^2(1-x_k)+\gamma}{100}, \ \ I_{1k}(x_k,\alpha,\beta)=\frac{\sum_{k=1}^{3}x_k\alpha\beta}{100}, \\ J_{0j}(\tau_j,\gamma,\delta) &= \frac{\tau_j|\beta|\delta}{500}, \ \ J_{1j}(\tau_j,\gamma,\delta)=\frac{\sum_{j=1}^{2}(1-\tau_j)\frac{\beta}{2}\delta^2}{500}. \end{aligned}$$

In fact, $f$, $h$ are $L^1$-Carathéodory functions in $[0,\ 1]$, with $\rho > 0$ such that

$$\max\{|\alpha|, |\beta|, |\gamma|, |\delta|\} < \rho,$$

we have

$$|f(x,\alpha,\beta,\gamma,\delta)| \le \frac{\rho^2+\rho^3}{100} := \phi_\rho(x),$$

$$|h(x,\alpha,\beta,\gamma,\delta)| \le \frac{(x+1)\rho^3+\rho^4}{500} := \psi_\rho(x).$$

Moreover, for $\rho \in [2.0348,\ 14.805]$, the assumptions o Theorem 6.2.1 hold and, therefore problem (6.14)-(6.15) has at least a solution $(u, v) \in X$.

## 6.4 The transverse vibration system of elastically coupled double-string

Consider the transverse vibration system of elastically coupled double-string with damping. The strings have the same length L, are attached by a viscoelastic element and stretched at a constant tension, according to Figure 1.

By [168], the system of elastically coupled double-string stationary model is given by the second order nonlinear system of differential equations

$$\begin{cases} S_1 u''(x) - K\left(u(x) - v(x)\right) = -l_1(x), \\ S_2 v''(x) - K\left(v(x) - u(x)\right) = -l_2(x), \end{cases} \tag{6.16}$$

where $x \in [0,\ L]$,

- $u(x)$, $v(x)$ are the transverse deflections of strings $u$ and $v$, respectively;
- $l_1(x)$ and $l_2(x)$ are the exciting distributed load;
- $K$ is the modulus of Kelvin-Voigt viscoelastic;
- $S_1$, $S_2$ are the string tensions of $u$ and $v$, respectively.

Adding to the system (6.16) the boundary conditions,

$$\begin{cases} u(0) = 0,\ u'(L) = B_1, \\ v(0) = 0,\ v(L) = 0. \end{cases} \tag{6.17}$$

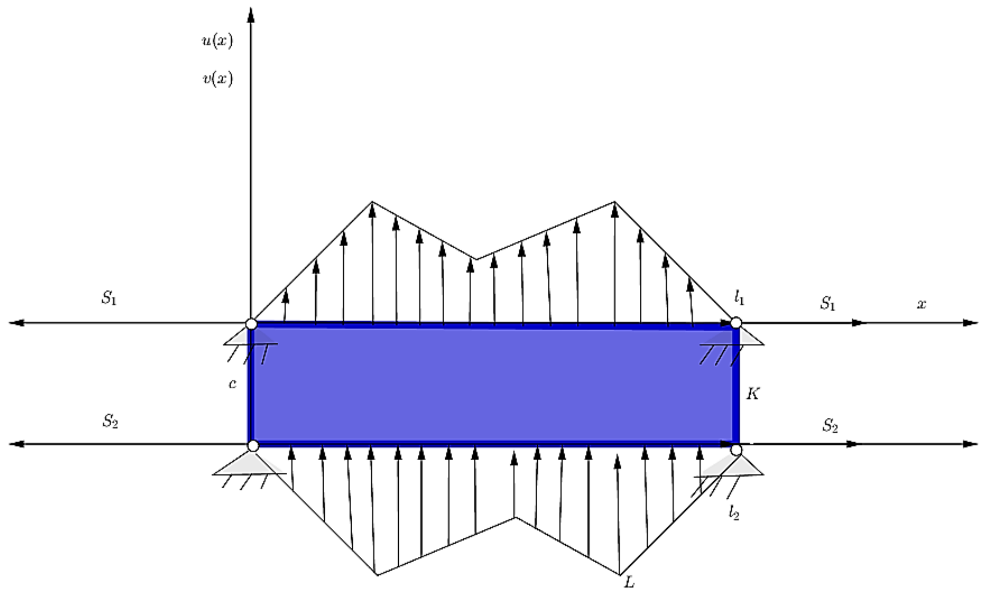


**FIGURE 6.1**
Elastically coupled double-string

We remark that strings $u$ and $v$ have different behaviors at the end points. Moreover, we consider impulsive conditions that may depend on the string deflections and on the slope of the corresponding deflections,

$$\begin{cases} I_{0k}(x_k, u(x_k), u'(x_k)) = \eta_1 u(x_k) + \eta_2 u'(x_k) + x_k, \\ I_{1k}(x_k, u(x_k), u'(x_k)) = \eta_3 u(x_k) + \eta_4 u'(x_k) + x_k, \\ J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) = \eta_5 v(\tau_j) + \eta_6 v'(\tau_j) + \tau_j, \\ J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)) = \eta_7 v(\tau_j) + \eta_8 v'(\tau_j) + \tau_j, \end{cases} \tag{6.18}$$

with $B_1, \eta_i \in \mathbb{R}$, $i = 1, .., 8$, $k = 1, .., n$ and $j = 1, .., m$. In this way, we have a particular case of (6.1)-(6.2).

In the system (6.16)-(6.18), it follows that,

$$f(x, \alpha, \beta, \gamma, \delta) = \tfrac{1}{S_1}\left[K(\alpha - \gamma) - l_1(x)\right],$$

$$h(x, \alpha, \beta, \gamma, \delta) = \tfrac{1}{S_2}\left[+K(\gamma - \alpha) - l_2(x)\right],$$

and

$$I_{0k}(x_k, \alpha, \gamma) = \eta_1\alpha + \eta_2\gamma,\ I_{1k}(x_k, \alpha, \gamma) = \eta_3\alpha + \eta_4\gamma + x_k,$$

$$J_{0j}(\tau_j, \beta, \delta) = \eta_5\beta + \eta_6\delta,\ J_{1j}(\tau_j, \beta, \delta) = \eta_7\beta + \eta_8\delta + \tau_j,$$

with $k = 1, .., n$ and $j = 1, .., m$.

Notice that, $f$, $h$ are $L^1$-Carathéodory functions in $[0, L] \times \mathbb{R}^4$, with for $\rho > 0$ and

$$\max\{|\alpha|, |\beta|, |\gamma|, |\delta|\} < \rho,$$

$$|f(x, \alpha, \beta, \gamma, \delta)| \leq \frac{1}{|S_1|}[2|K|\rho + |l_1(x)|] := \phi_\rho(x),$$

$$|h(x, \alpha, \beta, \gamma, \delta)| \leq \frac{1}{|S_2|}[2|K|\rho + |l_2(x)|] := \psi_\rho(x).$$

As the impulsive conditions, $I_{ik}$, for $i = 1, 2$, $k = 1, .., n$ and $J_{ij}$, for $j = 1, .., m$, are continuous functions on $[0, L] \times \mathbb{R}^2$. Then, for the values of $S_1$, $S_2$, $K$, $l_1(x)$, $l_2(x)$ and $\rho$ such that the assumptions of Theorem 6.2.1 are satisfied, the problem (6.16)-(6.18) has at least solution $(u, v) \in X$.

# 7

# *Impulsive coupled systems on the half-line*

Boundary-value problems in unbounded domains, can be applied to a large variety of contexts, (see for instance, [19, 54, 77, 119, 131, 132, 143, 170, 199]).

Some examples of impulsive effects can be found in [50], where Dishliev et al., make a very complete explanation of these equations, and show also some applications about pharmacokinetic model, logistic model, Gompertz model (mathematical model for a time series), Lotka-Volterra model and population dynamics. In [73], Guo uses the fixed point theory to investigate the existence and uniqueness of solutions of two-point boundary value problems for second order non-linear impulsive integro-differential equations on infinite intervals in a Banach space. The same author, in [74], by a comparison result, obtains the existence of maximal and minimal solutions of initial value problems for a class of second-order impulsive integro-differential equations in a Banach space. In [117], Lee and Liu study the existence of extremal solutions for a class of singular boundary value problems of second order impulsive differential equations. In [152], Minhós, and Carapinha study separated impulsive problems with a fully third order differential equation, including an increasing homeomorphism, and impulsive conditions given by generalized functions. In [171], Pang et al., consider a second-order impulsive differential equation with integral boundary conditions, where they proposed some sufficient conditions for the existence of solutions, by using the method of upper and lower solutions and Leray-Schauder degree theory. In [116], Lee and Lee combine the method of upper and lower solutions with fixed point index theorems on a cone to study the existence of positive solutions for a singular two point boundary value problem of second order impulsive equation with fixed moments.

In [184], Shen and Wang investigate the boundary value problem with impulse effect

$$\begin{array}{l} x''(t) = f(t, x(t), x'(t)), \quad t \in J, \ t \neq t_k \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, ...p, \\ \Delta x'(t_k) = J_k(x(t_k), x'(t_k)), \quad k = 1, 2, ...p, \\ g(x(0), x'(0)) = 0, \quad h(x(1), x'(1)) = 0, \end{array}$$

where $J = [0, 1]$, $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $I_k \in C(\mathbb{R})$,

$J_k \in C(\mathbb{R}^2)$ for $1 \le k \le p$, $0 = t_0 < t_1 < ... < t_p < t_{p+1} = 1$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, denotes the jump of $x(t)$ at $t = t_k$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively, $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$ where

$$x'(t_k^-) = \lim_{h\to 0^-} h^{-1}[x(t_k+h)-x(t_k)], \ \ x'(t_k^+) = \lim_{h\to 0^+} h^{-1}[x(t_k+h)-x(t_k)],$$

and $g, h : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

In [200], Wang, Zhang and Liang, consider the initial value problem for second order impulsive integro-differential equations, which nonlinearity depend on the first derivative, in a Banach space $E$:

$$\begin{cases} x''(t) = f(t, x(t), x'(t), Tx(t), Sx(t)), \ \ t \neq t_k, \ k = 1, 2, ...m, \\ \Delta x(t_k) = I_k(x(t_k), x'(t_k)), \ \ k = 1, 2, ...m, \\ \Delta x'(t_k) = \overline{I_k}(x(t_k), x'(t_k)), \ \ k = 1, 2, ...m, \\ x(0) = x_0, \ \ x'(0) = x_0^*, \end{cases}$$

where $f \in C[J \times E^4, E]$, $J = [0, 1]$, $0 < t_0 < t_1 < ... < t_k < ... < t_m < 1$. $I_k$, $\overline{I_k} \in C[E^2, E]$, $k = 1, 2, ...m$, $x_0, x_0^* \in E$, $\theta$ denotes de zero element of $E$, $J' = J \setminus \{t_1, t_2, ..., t_m\}$ and $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, ...m$, $t_{m+1} = 1$,

$$Tx(t) = \int_0^t k(t, s)x(s)ds, \quad Sx(t) = \int_0^1 h(t, s)x(s)ds, \ \ \forall t \in J,$$

where $k \in C[D, \mathbb{R}_+]$, $D = \{(t, s) : J \times J | t \ge s\}$, $h \in C[J \times J, \mathbb{R}_+]$, $\mathbb{R}_+ = [0, +\infty)$.

In [130], the authors study the existence of multiple and single positive solutions of two-point boundary value problems for the systems of nonlinear second-order singular and impulsive differential equations:

$$\begin{cases} -u''(t) = h_1(t)f_1(t, u, v), \ \ t \in J', \\ -v''(t) = h_2(t)f_2(t, u, v), \ \ t \in J' \\ -\Delta u' \mid_{t=t_k} = I_{1,k}(u(t_k)), \ \ k = 1, 2, ...m, \\ -\Delta v' \mid_{t=t_k} = I_{2,k}(v(t_k)), \ \ k = 1, 2, ...m, \\ \alpha u(0) - \beta u'(0) = 0, \ \ \alpha v(0) - \beta v'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \ \ \gamma v(1) + \delta v'(1) = 0, \end{cases}$$

where $\alpha, \beta, \gamma, \delta \ge 0$, $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$, $J = (0, 1)$, $0 < t_1 < ... < t_m < 1$, $J' = J \setminus \{t_1, t_2, ..., t_m\}$, $\overline{J} = [0, 1]$, $f_i \in C(\overline{J} \times$

$(\mathbb{R}^+)^2, \mathbb{R}^+)$, $I_{i,k} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $h_i \in (J, (0, +\infty))$, $(i = 1, 2)$, and may be singular at $t = 0$ or $t = 1$, $\mathbb{R}^+ = [0, +\infty)$.

In [105], we found the study of second-order nonlinear differential equation

$$(p(t)u'(t))' = f(t, u(t)), \quad t \in (0,\ \infty)\backslash\{t_1, t_2, ..., t_n\},$$

where $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, $p \in [0, +\infty) \cap C(0, +\infty)$ and $p(t) \geq 0$ for all $t > 0$, with the impulsive conditions

$$\Delta u'(t_k) = I_k(u(t_k)), \quad k = 1, ...n,$$

where $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1, ..., n$, are Lipschitz continuous, $n \geq 1$, and the boundary conditions

$$\begin{gathered} \alpha u(0) - \beta \lim_{t \to 0^+} p(t)u'(t) = 0, \\ \gamma \lim_{t \to \infty} u(t) + \delta \lim_{t \to \infty} p(t)u'(t) = 0. \end{gathered}$$

In order to ensure that the non-resonant scenario is considered, the condition

$$\rho = \gamma\beta + \alpha\delta + \alpha\gamma \int_0^\infty \frac{d\tau}{p(\tau)} \neq 0$$

is imposed.

In [115], the authors prove the existence of multiple positive solutions for a singular Gelfand type boundary value problem with the following second-order impulsive differential system:

$$\begin{aligned} & u''(t) + \lambda h_1(t) f(u(t), v(t)) = 0, \quad t \in (0,\ 1), \ \ t \neq t_1, \\ & v''(t) + \mu h_2(t) g(u(t), v(t)) = 0, \quad t \in (0,\ 1), \ \ t \neq t_1, \\ & \qquad \Delta u \mid_{t=t_1} = I_u(u(t_1)), \ \ \Delta v \mid_{t=t_1} = I_v(v(t_1)), \\ & \qquad \Delta u' \mid_{t=t_1} = N_u(u(t_1)), \ \ \Delta v' \mid_{t=t_1} = N_v(v(t_1)), \\ & u(0) = a \geq 0, \ \ v(0) = b \geq 0, \ \ u(1) = c \geq 0, \ \ v(1) = d \geq 0, \end{aligned}$$

where $\lambda$, $\mu$ are positive real parameters, $\Delta u \mid_{t=t_1} = u(t_1^+) - u(t_1^-)$, $\Delta u' \mid_{t=t_1} = u'(t_1^+) - u'(t_1^-)$, $f, g \in C(\mathbb{R}^2, (0, \infty))$, $I_u$, $I_v \in C(\mathbb{R}, \mathbb{R})$ satisfying $I_u(0) = 0 = I_v(0)$, $N_u$, $N_v \in C(\mathbb{R}, (-\infty, 0])$, and $h_1$, $h_2 \in C((0,\ 1), (0,\ \infty))$.

Inspired by these works, we follow arguments and techniques considered in [151] and [154], in particular, about impulsive problems on the half-line and second order coupled systems on the half-line, respectively. However, it is the first time where the existence of solutions is obtained for impulsive coupled systems, with generalized jump conditions in half-line and with full nonlinearities, that depend on the unknown functions and their first derivatives.

In particular, in the present chapter, we consider the second order impulsive coupled system in half-line composed by the differential equations, for $t \in [0, +\infty[$,

$$\begin{cases} u''(t) = f(t, u(t), v(t), u'(t), v'(t)), \ t \neq t_k, \\ v''(t) = h(t, u(t), v(t), u'(t), v'(t)), \ t \neq \tau_j, \end{cases} \tag{7.1}$$

where $f, h : [0, +\infty[ \times \mathbb{R}^4 \to \mathbb{R}$ are $L^1$-Carathéodory functions, the boundary conditions

$$\begin{cases} u(0) = A_1, \ v(0) = A_2, \\ u'(+\infty) = B_1, \ v'(+\infty) = B_2, \end{cases} \tag{7.2}$$

for $A_1, A_2, B_1, B_2 \in \mathbb{R}$ and the generalized impulsive conditions

$$\begin{cases} \Delta u(t_k) = I_{0k}(t_k, u(t_k), u'(t_k)), \\ \Delta u'(t_k) = I_{1k}(t_k, u(t_k), u'(t_k)), \\ \Delta v(\tau_j) = J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)), \\ \Delta v'(\tau_j) = J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)), \end{cases} \tag{7.3}$$

where, $k, j \in \mathbb{N}$,

$$\Delta u^{(i)}(t_k) = u^{(i)}(t_k^+) - u^{(i)}(t_k^-), \ \ \Delta v^{(i)}(\tau_j) = v^{(i)}(\tau_j^+) - v^{(i)}(\tau_j^-),$$

$I_{ik}, J_{ij} \in C([0, +\infty[ \times \mathbb{R}^2, \mathbb{R})$, $i = 0, 1$, with $t_k, \tau_j$ fixed points such that $0 < t_1 < \cdots < t_k < \cdots$, $0 < \tau_1 < \cdots < \tau_j < \cdots$ and

$$\lim_{k \to +\infty} t_k = +\infty, \ \ \lim_{j \to +\infty} \tau_j = +\infty.$$

Some arguments, based on [162], play a key role, such as: Carathéodory functions and sequences, the equiconvergence at each impulsive moment and at infinity, Banach spaces with weighted norms, and Schauder's fixed point theorem, to prove the existence of solutions.

## 7.1 Definitions and preliminary results

Define $u(t_k^{\pm}) := \lim_{t\to t_k^{\pm}} u(t)$, $v(\tau_j^{\pm}) := \lim_{t\to \tau_j^{\pm}} v(t)$, and consider the set

$$PC_1\left([0,+\infty[\right) = \left\{ \begin{array}{c} u : u \in C([0,+\infty[ \setminus \{t_k\}, \mathbb{R}), u(t_k) = u(t_k^-), \\ u(t_k^+) \text{ exists for } k \in \mathbb{N} \end{array} \right\},$$

$$PC_1^n\left([0,+\infty[\right) = \left\{u : u^{(n)} \in PC_1\left([0,+\infty[\right)\right\},\ n = 1, 2,$$

$$PC_2\left([0,+\infty[\right) = \left\{ \begin{array}{c} v : v \in C([0,+\infty[ \setminus \{\tau_j\}, \mathbb{R}), v(\tau_j) = v(\tau_j^-), \\ v(\tau_j^+) \text{ exists for } j \in \mathbb{N} \end{array} \right\},$$

and $PC_2^n\left([0,+\infty[\right) = \left\{v : v^{(n)} \in PC_2\left([0,+\infty[\right)\right\},\ n = 1, 2.$

Denote the space

$$X_1 := \left\{x : x \in PC_1^1\left([0,+\infty[\right) : \lim_{t\to+\infty} \frac{x(t)}{1+t} \in \mathbb{R},\ \lim_{t\to+\infty} x'(t) \in \mathbb{R}\right\},$$

$$X_2 := \left\{y : y \in PC_2^1\left([0,+\infty[\right) : \lim_{t\to+\infty} \frac{y(t)}{1+t} \in \mathbb{R},\ \lim_{t\to+\infty} y'(t) \in \mathbb{R}\right\},$$

and $X := X_1 \times X_2$.

In fact, $X_1$, $X_2$ and $X$ are Banach spaces with the norms

$$\|u\|_{X_1} = \max\left\{\|u\|_0, \|u'\|_1\right\},\quad \|v\|_{X_2} = \max\left\{\|v\|_0, \|v'\|_1\right\},$$

and

$$\|(u,v)\|_X = \max\left\{\|u\|_{X_1}, \|v\|_{X_2}\right\},$$

respectively, where

$$\|w\|_0 := \sup_{t\in[0,+\infty[} \frac{|w(t)|}{1+t} \text{ and } \|w\|_1 := \sup_{t\in[0,+\infty[} |w(t)|.$$

In this chapter we consider $L^1-$ Carathéodory functions mentioned in Definition 3.1.1, now adapted a function $g : [0,+\infty[\times\mathbb{R}^4 \to \mathbb{R}$ and condition *(iii)* adapted by:

for each $\rho > 0$, there exists a positive function $\phi_\rho \in L^1([0,+\infty[)$ such that, for $a.e.t \in [0,+\infty[$, $(x,y,z,w) \in \mathbb{R}^4$ with

$$\sup_{t\in[0,+\infty[} \left\{ \frac{|x|}{1+t}, \frac{|y|}{1+t}, |z|, |w| \right\} < \rho, \tag{7.4}$$

one has

$$|g(t,x,y,z,w)| \le \phi_\rho(t). \tag{7.5}$$

**Definition 7.1.1** *A sequence $(c_n)_{n\in\mathbb{N}} : [0,+\infty[\, \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory sequence if it verifies*

> *i) for each $(a,b) \in \mathbb{R}^2$, $(a,b) \longmapsto c_n(t_n,a,b)$ is continuous for all $n \in \mathbb{N}$;*
>
> *ii) for each $\rho > 0$, there are nonnegative constants $\chi_{n,\rho} \ge 0$ with $\sum_{n=1}^{+\infty} \chi_{n,\rho} < +\infty$ such that for $|a| < \rho(1+t)$, $t \in [0,+\infty[$ and $|b| < \rho$ we have $|c_n(t,a,b)| \le \chi_{n,\rho}$, for every $n \in \mathbb{N}$, $t \in [0,+\infty[$.*

**Lemma 7.1.1** *Assume that $f, h : [0,+\infty[\, \times \mathbb{R}^4 \to \mathbb{R}$ are $L^1-$Carathéodory functions and $I_{0k}, I_{1k}, J_{0j}, J_{1j} : [0,+\infty[\, \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory sequences, for $k, j \in \mathbb{N}$. Then the system (7.1) with conditions (7.2), (7.3), has a solution $(u, v) \in X$ expressed by*

$$\begin{aligned}
u(t) &= A_1 + B_1 t \\
&+ \sum_{0<t_k<t<+\infty} \left[ I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k))\,(t-t_k) \right] \\
&- t \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) \\
&+ \int_0^{+\infty} G(t,s) f(s, u(s), v(s), u'(s), v'(s))\,ds, \\
v(t) &= A_2 + B_2 t \\
&+ \sum_{0<\tau_j<t<+\infty} \left[ J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))\,(t-\tau_j) \right] \\
&- t \sum_{j=1}^{+\infty} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)) \\
&+ \int_0^{+\infty} G(t,s) h(s, u(s), v(s), u'(s), v'(s))\,ds,
\end{aligned}$$

*where*

$$G(t,s) = \begin{cases} -t, & 0 \le t \le s \le +\infty, \\ -s, & 0 \le s \le t \le +\infty. \end{cases} \tag{7.6}$$

The proof follows standard techniques and it is omitted.

**Definition 7.1.2** *The operator $T : X \to X$ is said to be compact if $T(D)$ is relatively compact, for $D \subseteq X$.*

The Schauder's fixed point theorem give the existence solutions (see, Theorem 2.1.1).

## 7.2 Existence result

In this section we prove the existence of solution for the problem (7.1)-(7.3).

**Theorem 7.2.1** *Let $f, h : [0,+\infty[ \times \mathbb{R}^4 \to \mathbb{R}$ be $L^1-$Carathéodory functions and $I_{0k}, I_{1k}, J_{0j}, J_{1j} : [0,+\infty[ \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory sequences such that there is $R > 0$ verifying*

$$\max \left\{ \begin{array}{c} K_1 + \sum_{k=1}^{+\infty} \varphi_{k,R} + 2\sum_{k=1}^{+\infty} \psi_{k,R} + \int_0^{+\infty} \Phi_R(s)ds, \\ K_2 + \sum_{k=1}^{+\infty} \phi_{j,R} + 2\sum_{k=1}^{+\infty} \vartheta_{j,R} + \int_0^{+\infty} \Psi_R(s)ds, \\ |B_1| + 2\sum_{k=1}^{+\infty} \psi_{k,R} + \int_0^{+\infty} \Phi_R(s)ds, \\ |B_2| + 2\sum_{k=1}^{+\infty} \vartheta_{j,R} + \int_0^{+\infty} \Psi_R(s)ds \end{array} \right\} < R,$$

*where $\varphi_{k,R}$, $\psi_{k,R}$, $\phi_{j,R}$, $\vartheta_{j,R}$ are nonnegative constants such that*

$$\sum_{k=1}^{+\infty} \varphi_{k,R} < +\infty, \ \sum_{k=1}^{+\infty} \psi_{k,R} < +\infty, \ \sum_{j=1}^{+\infty} \phi_{j,R} < +\infty, \ \sum_{j=1}^{+\infty} \vartheta_{j,R} < +\infty, \tag{7.7}$$

$$\begin{aligned} &|I_{0k}(t_k, x, y)| \le \varphi_{k,R}, \ \ |I_{1k}(t_k, x, y)| \le \psi_{k,R}, \\ &\qquad \textit{for } |x| < R(1+t_k), \ |y| < R, \ k \in \mathbb{N}, \end{aligned} \tag{7.8}$$

$$|J_{0j}(\tau_j, x, y)| \leq \phi_{j,R}, \quad |J_{1j}(\tau_j, x, y)| \leq \vartheta_{j,R},$$
$$\textit{for } |x| < R(1+\tau_j),\ |y| < R,\ j \in \mathbb{N}, \tag{7.9}$$

*and*

$$K_i := \sup_{t\in[0,+\infty[} \left( \frac{|A_i| + |B_i t|}{1+t} \right), \quad i = 1, 2. \tag{7.10}$$

*Then there is at least a pair* $(u, v) \in \Big(PC_1^2\left([0, +\infty[\right) \times PC_2^2\left([0, +\infty[\right)\Big) \cap X$*, solution of (7.1)-(7.3).*

**Proof** Define the operators $T_1 : X \to X_1$ , $T_2 : X \to X_2$, and $T : X \to X$ by

$$T(u, v) = (T_1(u, v), T_2(u, v)), \tag{7.11}$$

with

$$\begin{aligned}
&(T_1(u, v))(t) = A_1 + B_1 t \\
&+ \sum_{0<t_k<t<+\infty} \left[ I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k))\,(t - t_k) \right] \\
&-t \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) \\
&+ \int_0^{+\infty} G(t, s) f(s, u(s), v(s), u'(s), v'(s)) ds,
\end{aligned}$$

$$\begin{aligned}
&(T_2(u, v))(t) = A_2 + B_2 t \\
&+ \sum_{0<\tau_j<t<+\infty} \left[ J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))\,(t - \tau_j) \right] \\
&-t \sum_{j=1}^{+\infty} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)) \\
&+ \int_0^{+\infty} G(t, s) h(s, u(s), v(s), u'(s), v'(s)) ds,
\end{aligned}$$

where $G(t, s)$ is defined in (7.6).

The proof will follow several steps which, for clearness, are detailed for operator $T_1(u, v)$. The technique for operator $T_2(u, v)$ is similar.

**Step 1**: $T$ *is well defined and continuous on* $X$.

Let $(u,v)\in X$. By the Lebesgue dominated convergence theorem, (7.7) and (7.8),

$$\begin{aligned}
&\lim_{t\to+\infty}\frac{T_1(u,v)(t)}{1+t}=\lim_{t\to+\infty}\left(\frac{A_1+B_1t}{1+t}\right.\\
&+\frac{1}{1+t}\sum_{0<t_k<t<+\infty}[I_{0k}(t_k,u(t_k),u'(t_k))\\
&\left.+\,I_{1k}(t_k,u(t_k),u'(t_k))\,(t-t_k)]-\frac{t}{1+t}\sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k))\right)\\
&+\int_0^{+\infty}\lim_{t\to+\infty}\frac{G(t,s)}{1+t}f(s,u(s),v(s),u'(s),v'(s))ds\\
\leq\;& B_1+\sum_{0<t_k<t<+\infty}I_{1k}(t_k,u(t_k),u'(t_k))-\sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k))\\
&+\int_t^{+\infty}|f(s,u(s),v(s),u'(s),v'(s))|ds\\
\leq\;& B_1+2\sum_{k=1}^{+\infty}\psi_{k,\rho}+\int_0^{+\infty}\Phi_\rho(s)ds<+\infty,
\end{aligned}$$

for $\rho>0$ given by (7.4) and

$$\begin{aligned}
&\lim_{t\to+\infty}\left(T_1(u,v)\right)'(t)=B_1+\sum_{0<t_k<t<+\infty}I_{1k}(t_k,u(t_k),u'(t_k))\\
&-\sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k))\\
&-\lim_{t\to+\infty}\int_t^{+\infty}f(s,u(s),v(s),u'(s),v'(s))ds\\
\leq\;& B_1+2\sum_{k=1}^{+\infty}\psi_{k,\rho}+\int_0^{+\infty}\Phi_\rho(s)ds<+\infty.
\end{aligned}$$

So, $T_1X\subset X_1$. Analogously, $T_2X\subset X_2$. Therefore, $T$ is well defined in $X$ and, as $f$ and $h$ are $L^1-$ Carathéodory functions, by

Definition 7.1.1, $T$ is continuous.

To prove that $TD$ is relatively compact, for $D \subseteq X$ a bounded subset , it is enough to show that:

*i)* $TD$ is uniformly bounded, for $D$ a bounded set in $X$;

*ii)* $TD$ is equicontinuous on each interval $]t_k, t_{k+1}] \times ]\tau_j, \tau_{j+1}]$, for $k, j = 1, 2, ...$;

*iii)* $TD$ is equiconvergent at each impulsive point and at infinity.

**Step 2**: *$TD$ is uniformly bounded, for $D$ a bounded set in $X$.*

Let $D \subset X$ be a bounded subset. Thus, there is $\rho_1 > 0$ such that, for $(u, v) \in D$,

$$\begin{aligned} \|(u,v)\|_X &= \max\left\{\|u\|_{X_1}, \|v\|_{X_2}\right\} \\ &= \max\left\{\|u\|_0, \|u'\|_1, \|v\|_0, \|v'\|_1\right\} < \rho_1. \quad (7.12) \end{aligned}$$

As, $f$ is a $L^1$-Carathéodory function, then

$$\begin{aligned} &\|T_1(u,v)\|_0 = \sup_{t\in[0,+\infty[} \frac{|T_1(u,v)(t)|}{1+t} \\ &\le \sup_{t\in[0,+\infty[} \left( \frac{|A_1| + |B_1 t|}{1+t} \right. \\ &+ \frac{1}{1+t} \sum_{0<t_k<t<+\infty} |I_{0k}(t_k, u(t_k), u'(t_k)) \\ &\left. + I_{1k}(t_k, u(t_k), u'(t_k))(t-t_k)| + \frac{t}{1+t} \sum_{k=1}^{+\infty} |I_{1k}(t_k, u(t_k), u'(t_k))| \right) \\ &+ \int_0^{+\infty} \sup_{t\in[0,+\infty[} \frac{|G(t,s)|}{1+t} |f(s, u(s), v(s), u'(s), v'(s))| ds \\ \le\ & K_1 + \sup_{t\in[0,+\infty[} \left( \frac{1}{1+t} \sum_{0<t_k<t<+\infty} [\varphi_{k,\rho_1} + \psi_{k,\rho_1}(t-t_k)] \right) \\ &+ \sup_{t\in[0,+\infty[} \left( \frac{t}{1+t} \sum_{k=1}^{+\infty} \psi_{k,\rho_1} \right) + \int_0^{+\infty} \Phi_{\rho_1}(s) ds \\ &\le K_1 + \sum_{k=1}^{+\infty} \varphi_{k,\rho_1} + 2 \sum_{k=1}^{+\infty} \psi_{k,\rho_1} \end{aligned}$$

$$+\int_0^{+\infty}\Phi_{\rho_1}(s)ds<+\infty, \forall\,(u,v)\in D,$$

and

$$\begin{aligned}
\|\,(T_1\,(u,v))'\,\|_1 &= \sup_{t\in[0,+\infty[}|\,(T_1\,(u,v))'\,(t)| \\
&\le |B_1|+\sup_{t\in[0,+\infty[}\sum_{0<t_k<t<+\infty}|I_{1k}(t_k,u(t_k),u'(t_k))| \\
&\quad +\sum_{k=1}^{+\infty}|I_{1k}(t_k,u(t_k),u'(t_k))| \\
&\quad +\sup_{t\in[0,+\infty[}\int_t^{+\infty}|f(s,u(s),v(s),u'(s),v'(s))|ds \\
&\le |B_1|+2\sum_{k=1}^{+\infty}\psi_{k,\rho_1}+\int_0^{+\infty}\Phi_{\rho_1}(s)ds<+\infty.
\end{aligned}$$

Therefore, $T_1D$ is bounded and by similar arguments, $T_2D$ is also bounded. Furthermore, $\|T(u,v)\|_X<+\infty$, that is $TD$ is uniformly bounded on $X$.

**Step 3**: *$TD$ is equicontinuous on each interval $]t_k,t_{k+1}]\times]\tau_j,\tau_{j+1}]$, that is, $T_1D$ is equicontinuous on each interval $]t_k,t_{k+1}]$, for $k\in\mathbb{N}$, $0<t_1<\cdots<t_k<\cdots$, and $T_2D$ is equicontinuous on each interval $]\tau_j,\tau_{j+1}]$, for $j\in\mathbb{N}$ and $0<\tau_1<\cdots<\tau_j<\cdots$.*

Consider $I\subseteq]t_k,t_{k+1}]$ and $\iota_1,\iota_2\in I$ such that $\iota_1\le\iota_2$. For $(u,v)\in D$, we have

$$\begin{aligned}
&\lim_{\iota_1\to\iota_2}\left|\frac{T_1\,(u,v)\,(\iota_1)}{1+\iota_1}-\frac{T_1\,(u,v)\,(\iota_2)}{1+\iota_2}\right|\le\lim_{\iota_1\to\iota_2}\left|\frac{A_1+B_1\iota_1}{1+\iota_1}-\frac{A_1+B_1\iota_2}{1+\iota_2}\right| \\
&+\left|\frac{1}{1+\iota_1}\sum_{0<t_k<\iota_1}[I_{0k}(t_k,u(t_k),u'(t_k))+I_{1k}(t_k,u(t_k),u'(t_k))\,(\iota_1-t_k)]\right. \\
&-\frac{1}{1+\iota_2}\sum_{0<t_k<\iota_2}[I_{0k}(t_k,u(t_k),u'(t_k))+I_{1k}(t_k,u(t_k),u'(t_k))\,(\iota_2-t_k)] \\
&\left.-\frac{\iota_1}{1+\iota_1}\sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k))+\frac{\iota_2}{1+\iota_2}\sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k))\right| \\
&+\int_0^{+\infty}\lim_{\iota_1\to\iota_2}\left|\frac{G(\iota_1,s)}{1+\iota_1}-\frac{G(\iota_2,s)}{1+\iota_2}\right|\left|f(s,u(s),v(s),u'(s),v'(s))\right|ds=0,
\end{aligned}$$

as $\iota_1 \to \iota_2$, and

$$\begin{aligned}
&\lim_{\iota_1\to\iota_2}\left|\left(T_1\left(u,v\right)\left(\iota_1\right)\right)' - \left(T_1\left(u,v\right)\left(\iota_2\right)\right)'\right| \\
&\leq \lim_{\iota_1\to\iota_2}\left|\sum_{0<t_k<\iota_1} I_{1k}(t_k,u(t_k),u'(t_k)) - \sum_{0<t_k<\iota_2} I_{1k}(t_k,u(t_k),u'(t_k))\right. \\
&- \int_{\iota_1}^{+\infty} f(s,u(s),v(s),u'(s),v'(s))ds \\
&\left. + \int_{\iota_2}^{+\infty} f(s,u(s),v(s),u'(s),v'(s))ds\right| \\
&\leq \lim_{\iota_1\to\iota_2}\sum_{\iota_1<t_k<\iota_2} |I_{1k}(t_k,u(t_k),u'(t_k))| \\
&+ \int_{\iota_1}^{\iota_2} |f(s,u(s),v(s),u'(s),v'(s))|ds \\
\leq\ & \lim_{\iota_1\to\iota_2}\sum_{\iota_1<t_k<\iota_2} \psi_{k,\rho_1} + \int_{\iota_1}^{\iota_2}\Phi_{\rho_1}(s)ds = 0.
\end{aligned}$$

Therefore, $T_1D$ is equicontinuous on $X_1$. Similarly, we can show that $T_2D$ is equicontinuous on $X_2$, too. Thus, $TD$ is equicontinuous on $X$.

**Step 4**: *$TD$ is equiconvergent at each impulsive point and at infinity, that is $T_1D$, is equiconvergent at $t = t_i^+$, $i = 1,2,...$, and at infinity*, and $T_2D$, *is equiconvergent at $\tau = \tau_l^+$, $l = 1,2,...,$and at infinity* .

First, let us prove the equiconvergence at $t = t_i^+$, for $i = 1,2,....$ The proof for the equiconvergence at $\tau = \tau_l^+$, for $l = 1,2,...$, is analogous.

Thus, it follows

$$\begin{aligned}
&\left|\frac{T_1\left(u,v\right)\left(t\right)}{1+t} - \lim_{t\to t_i^+}\frac{T_1\left(u,v\right)\left(t\right)}{1+t}\right| \leq \left|\frac{A_1+B_1t}{1+t} - \frac{A_1+B_1t_i}{1+t_i}\right| \\
&+\left|\frac{1}{1+t}\sum_{0<t_k<t<+\infty}\left[I_{0k}(t_k,u(t_k),u'(t_k))\right.\right. \\
&+ I_{1k}(t_k,u(t_k),u'(t_k))\left(t-t_k\right)]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{1+t_i}\sum_{0<t_k<t_i^+}[I_{0k}(t_k,u(t_k),u'(t_k)) \\
&+\ I_{1k}(t_k,u(t_k),u'(t_k))\,(t_i-t_k)]| \\
&+\left|-\frac{t}{1+t}\sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k))+\frac{t_i}{1+t_i}\sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k))\right| \\
&+\int_0^{+\infty}\left|\frac{G(t,s)}{1+t}-\frac{G(t,s)}{1+t_i}\right|\Phi_{\rho_1}(s)ds\to 0,
\end{aligned}
$$

uniformly on $(u,v)\in D$, as $t\to t_i^+$, for $i=1,2,...$ and

$$
\begin{aligned}
&\left|\left(T_1\left(u,v\right)(t)\right)'-\lim_{t\to t_i^+}\left(T_1\left(u,v\right)(t)\right)'\right| \\
&=\Bigg|\sum_{0<t_k<t<+\infty}I_{1k}(t_k,u(t_k),u'(t_k))-\sum_{0<t_k<t_i^+}I_{1k}(t_k,u(t_k),u'(t_k)) \\
&-\ \int_t^{+\infty}f(s,u(s),v(s),u'(s),v'(s))ds \\
&+\ \int_{t_i}^{+\infty}f(s,u(s),v(s),u'(s),v'(s))ds\Bigg| \\
&\le\ \left|\sum_{0<t_k<t<+\infty}I_{1k}(t_k,u(t_k),u'(t_k))-\sum_{0<t_k<t_i^+}I_{1k}(t_k,u(t_k),u'(t_k))\right| \\
&\quad+\Bigg|-\int_t^{+\infty}f(s,u(s),v(s),u'(s),v'(s))ds \\
&\quad+\int_{t_i}^{+\infty}f(s,u(s),v(s),u'(s),v'(s))ds\Bigg| \\
&\quad\le\left|\sum_{0<t_k<t<+\infty}I_{1k}(t_k,u(t_k),u'(t_k))-\sum_{0<t_k<t_i^+}I_{1k}(t_k,u(t_k),u'(t_k))\right| \\
&\quad+\int_{t_i}^{t}\phi_{\rho_1}(s)ds\to 0,
\end{aligned}
$$

uniformly on $(u,v)\in D$, as $t\to t_i^+$, for $i=1,2,....$

Therefore, $T_1D$ is equiconvergent at each point $t = t_i^+$, for $i = 1, 2, ....$ Analogously, it can be proved that $T_2D$ is equiconvergent at each point $\tau = \tau_l^+$, for $l = 1, 2, ...,.$

So, $TD$ is equiconvergent at each impulsive point.

To prove the equiconvergence at infinity, for the operator $T_1$, we have

$$\begin{aligned}
&\left|\frac{T_1(u,v)(t)}{1+t} - \lim_{t\to+\infty}\frac{T_1(u,v)(t)}{1+t}\right| \leq \left|\frac{A_1+B_1t}{1+t} - B_1\right| \\
&+ \left|\frac{1}{1+t}\sum_{0<t_k<t<+\infty}[I_{0k}(t_k,u(t_k),u'(t_k))\right. \\
&\quad +I_{1k}(t_k,u(t_k),u'(t_k))(t-t_k)] \\
&\quad -\lim_{t\to+\infty}\frac{1}{1+t}\sum_{0<t_k<t<+\infty}[I_{0k}(t_k,u(t_k),u'(t_k)) \\
&\quad \left.+I_{1k}(t_k,u(t_k),u'(t_k))(t-t_k)]\right| \\
&\quad +\left|-\frac{t}{1+t}\sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k))\right. \\
&\quad \left.+\lim_{t\to+\infty}\frac{t}{1+t}\sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k))\right| \\
&+ \int_0^{+\infty}\left|\frac{G(t,s)}{1+t} - \lim_{t\to+\infty}\frac{G(t,s)}{1+t}\right||f(s,u(s),v(s),u'(s),v'(s))ds| \\
&\leq \left|\frac{A_1+B_1t}{1+t} - B_1\right| \\
&+ \left|\frac{1}{1+t}\sum_{0<t_k<t<+\infty}[I_{0k}(t_k,u(t_k),u'(t_k))\right. \\
&+ \left.I_{1k}(t_k,u(t_k),u'(t_k))(t-t_k)] - \sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k))\right| \\
&\quad +\left|-\frac{t}{1+t}\sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k)) + \sum_{k=1}^{+\infty}I_{1k}(t_k,u(t_k),u'(t_k))\right| \\
&\quad +\int_0^{+\infty}\left|\frac{G(t,s)}{1+t} - \lim_{t\to+\infty}\frac{G(t,s)}{1+t}\right|\Phi_{\rho_1}(s)ds \to 0,
\end{aligned}$$

uniformly on $(u,v) \in D$, as $t \to +\infty$.

Analogously,

$$\begin{aligned}
&\left|(T_1(u,v)(t))' - \lim_{t\to+\infty}(T_1(u,v)(t))'\right| \\
= &\left|\sum_{0<t_k<t<+\infty} I_{1k}(t_k,u(t_k),u'(t_k)) - \sum_{k=1}^{+\infty} I_{1k}(t_k,u(t_k),u'(t_k))\right. \\
&- \int_t^{+\infty} f(s,u(s),v(s),u'(s),v'(s))ds \\
&- \lim_{t\to+\infty} \sum_{0<t_k<t<+\infty} I_{1k}(t_k,u(t_k),u'(t_k)) \\
&+ \lim_{t\to+\infty} \sum_{k=1}^{+\infty} I_{1k}(t_k,u(t_k),u'(t_k)) \\
&\left.+ \lim_{t\to+\infty} \int_t^{+\infty} f(s,u(s),v(s),u'(s),v'(s))ds\right| \\
\le &\left|\sum_{k=0}^{+\infty} I_{1k}(t_k,u(t_k),u'(t_k)) - \sum_{k=1}^{+\infty} I_{1k}(t_k,u(t_k),u'(t_k))\right| \\
&+ \int_t^{+\infty} |f(s,u(s),v(s),u'(s),v'(s))|ds \\
\le &\left|\sum_{0<t_k<t<+\infty} I_{1k}(t_k,u(t_k),u'(t_k)) - \sum_{k=1}^{+\infty} I_{1k}(t_k,u(t_k),u'(t_k))\right| \\
&+ \int_t^{+\infty} \Phi_{\rho_1}(s)ds \to 0,
\end{aligned}$$

uniformly on $(u,v) \in D$, as $t \to +\infty$.

So, $T_1D$ is equiconvergent at $+\infty$. Following the same arguments, $T_2D$ is equiconvergent at $+\infty$, too. Therefore, $TD$ is equiconvergent at $+\infty$.

Therefore, $TD$ is relatively compact and, by Definition 7.1.2, $T$ is compact.

In order to apply Theorem 7.2.1, we need the next step:

**Step 5**: *$T\Omega \subset \Omega$ for some $\Omega \subset X$ a closed and bounded set.*

Consider

$$\Omega := \{(u,v) \in E : \|(u,v)\|_X \leq \rho_2\},$$

with $\rho_2 > 0$ such that

$$\rho_2 := \max \left\{ \begin{array}{c} \rho_1, \;\; K_1 + \sum_{k=1}^{+\infty} \varphi_{k,\rho_2} + 2\sum_{k=1}^{+\infty} \psi_{k,\rho_2} + \int_0^{+\infty} \Phi_{\rho_2}(s)ds, \\ \\ K_2 + \sum_{k=1}^{+\infty} \phi_{j,\rho_2} + 2\sum_{k=1}^{+\infty} \vartheta_{j,\rho_2} + \int_0^{+\infty} \Psi_{\rho_2}(s)ds, \\ \\ |B_1| + 2\sum_{k=1}^{+\infty} \psi_{k,\rho_2} + \int_0^{+\infty} \Phi_{\rho_2}(s)ds, \\ \\ |B_2| + 2\sum_{k=1}^{+\infty} \vartheta_{j,\rho_2} + \int_0^{+\infty} \Psi_{\rho_2}(s)ds, \end{array} \right\}, \tag{7.13}$$

with $\rho_1 > 0$ given by (7.12). According to Step 2 and $K_1$, $K_2$ given by (7.10), we have

$$\begin{aligned} \|T(u,v)\|_X &= \|(T_1(u,v), T_2(u,v))\|_X \\ &= \max\left\{\|T_1(u,v)\|_{X_1}, \; \|T_2(u,v)\|_{X_2}\right\} \leq \rho_2. \end{aligned}$$

So, $T\Omega \subset \Omega$, and by Theorem 2.1.1, the operator $T(u,v) = (T_1(u,v), T_2(u,v))$, has a fixed point $(u,v)$.

By standard techniques, and Lemma 7.1.1, it can be shown that this fixed point is a solution of problem (7.1)-(7.3).

■

## 7.3 Motion of a spring pendulum

Consider the motion of the spring pendulum of a mass attached to one end of a spring and the other end attached to the "ceiling. By [167], this motion can be represented by the modified Bessel equations in the system,

$$\left\{ \begin{array}{l} (t^3+1)\, l''(t) = l(t)\theta'(t) - g\cos(\theta(t)) - \frac{k}{m}(l(t) - l_0), \; t \in [0,+\infty[ \\ \\ (t^3+1)\, \theta''(t) = \frac{-gl(t)\sin(\theta(t)) - 2l(t)l'(t)\theta'(t)}{l^2(t)}, \end{array} \right. \tag{7.14}$$

where:

- $l(t)$, $l_0$ are the length at time $t$ and the natural length of the spring, respectively;
- $\theta(t)$ is the angle between the pendulum and the vertical;
- $m$, $k$, $g$ are the mass, the spring constant and gravitational force, respectively;

together with the boundary conditions

$$\begin{cases} l(0)=0, \;\; \theta(0)=0, \\ l'(+\infty)=B_1, \;\; \theta'(+\infty)=B_2, \end{cases} \tag{7.15}$$

with $B_1, B_2 \in [0, \pi]$, and the generalized impulsive conditions

$$\begin{cases} \Delta l(t_k) = \frac{1}{k^3}(\alpha_1 l(t_k) + \alpha_2 l'(t_k)), \\ \Delta l'(t_k) = \frac{1}{k^4}(\alpha_3 l(t_k) + \alpha_4 l'(t_k)), \\ \Delta \theta(\tau_j) = \frac{1}{j^5}(\alpha_5 \theta(\tau_j) + \alpha_6 \theta'(\tau_j)), \\ \Delta \theta'(\tau_j) = \frac{1}{j^3}(\alpha_7 \theta(\tau_j) + \alpha_8 \theta'(\tau_j)), \end{cases} \tag{7.16}$$

with $\alpha_i \in \mathbb{R}$, $i=1,2,...,8$ and for $k, j \in \mathbb{N}$, $0 < t_1 < \cdots < t_k < \cdots$, $0 < \tau_1 < \cdots \tau_j < \cdots$.

The system (7.14)-(7.16) is a particular case of the problem (7.1)-(7.3), with

$$\begin{aligned} f(t,x,y,z,w) &= \frac{1}{t^3+1}\left(xw - g\cos(y) - \frac{k}{m}(x-l_0)\right), \\ h(t,x,y,z,w) &= \frac{1}{t^3+1}\left(\frac{-gx\sin(y) - 2xzw}{x^2}\right), \\ I_{0k}(t_k,x,z) &= \frac{1}{k^3}(\alpha_1 x + \alpha_2 z), \;\; I_{1k}(t_k,x,z) = \frac{1}{k^4}(\alpha_3 x + \alpha_4 z), \\ J_{0j}(\tau_j,y,w) &= \frac{1}{j^5}(\alpha_5 y + \alpha_6 w), \;\; J_{1j}(\tau_j,y,w) = \frac{1}{j^3}(\alpha_7 y + \alpha_8 w), \end{aligned}$$

with $t_k = k$, $\tau_j = j$, $k, j \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, $i = 1,2,...,8$.

Choose $\rho > 0$ such that, for adequate $m$, $k$, $g$, $\phi_\rho(t)$, $\varphi_\rho(t)$, $\alpha_i \in \mathbb{R}$, $i=1,2,...,8$, its holds the following relations

$$f(t,x,y,z,w) \leq \frac{1}{t^3+1}\left(\rho^2(1+t) + g + \frac{k}{m}(\rho(1+t)+l_0)\right) := \phi_\rho(t),$$

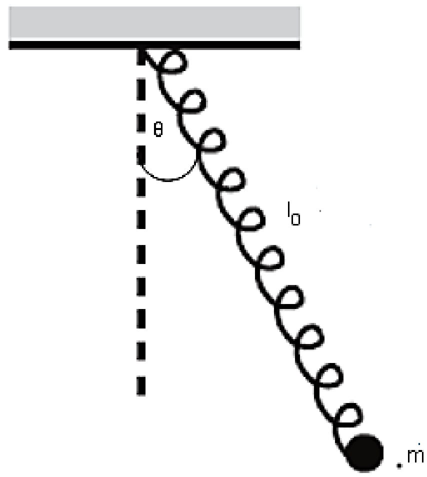


**FIGURE 7.1**
Motion of the spring pendulum.

$$\begin{aligned} h(t,x,y,z,w) &\leq \frac{1}{t^3+1}\left(\frac{g\rho(1+t)+2\rho^3(1+t)}{l^2(t)}\right) \\ &\leq \frac{1}{t^3+1}\left(\frac{g\rho(1+t)+2\rho^3(1+t)}{l^2(t)}\right) \\ &\leq \frac{1}{t^3+1}\cdot\frac{1}{\left(\min_{t\in[0,+\infty[} l(t)\right)^2}\left(g\rho(1+t)+2\rho^3(1+t)\right) \\ &:= \varphi_\rho(t), \end{aligned}$$

$$\begin{aligned} I_{0k}(t_k,x,z) &\leq \frac{\rho[(\alpha_1(1+k)+\alpha_2)]}{k^3},\ \ I_{1k}(t_k,x,z) \leq \frac{\rho[(\alpha_3(1+k)+\alpha_4)]}{k^4}, \\ J_{0j}(\tau_j,y,w) &\leq \frac{\rho[(\alpha_5(1+k)+\alpha_6)]}{j^5},\ \ J_{1j}(\tau_j,y,w) \leq \frac{\rho[(\alpha_7(1+k)+\alpha_8)]}{j^3}. \end{aligned}$$

So, by Theorem 7.2.1, there is at least a pair $(l,\theta) \in \left(PC_1^2([0,+\infty[) \times PC_2^2([0,+\infty[)\right) \cap X$, solution of problem (7.14), (7.15) and (7.16).

# 8

# *Localization results for impulsive second order coupled systems on the half-line*

Impulsive differential equations are a suitable mathematical tool for modeling the processes and phenomena, which are subjected to some short term external effects during their development. In fact, the duration of these effects is negligible, compared with the total duration of the processes or phenomena. The effects are instantaneous and they take form of impulses.

Some examples of what has already developed involving impulses, can be seen in [4, 5, 50, 90, 132, 151, 152, 164, 209], and for state impulse problems in [174]. Let us look with more detail some examples. In [29], we can found various theories and techniques on impulsive differential equations and inclusions, as for instance: impulsive functional, neutral and semilinear functional, nonlocal impulsive semilinear differential inclusions, existence results for impulsive functional semilinear, double positive solutions for impulsive boundary value problems and so forth. In [186], the importance of impulsive differential equations, is emphasized with several applications (impulsive biological models, impulsive models in population dynamics, impulsive neural networks, impulsive models in economics).

In [171], Pang and Cai employ the method of upper and lower solutions together with Leray-Schauder degree theory to study the existence of a solution of the impulsive BVP

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, \ \ t \in J^*, \\ \Delta x(t_k) = I_k(x(t_k)), \ \ k = 1, 2, ...p, \\ \Delta x'(t_k) = J_k(x(t_k), x'(t_k)), \ \ k = 1, 2, ...p, \\ x(0) = x(1) = \int_0^1 g(s)x(s)ds, \end{cases}$$

where $J = [0, 1]$, $J^* = J \setminus \{t_1, t_2, ..., t_p\}$, $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $I_k$, $J_k \in C(\mathbb{R})$ for $1 \leq k \leq p$, $g \in L^1[0, 1]$ is nonnegative, $0 = t_0 < t_1 < ... < t_p < t_{p+1} = 1$.

Motivated by the works above and applying the method developed in [154, 162], we obtain localization results, via lower and upper solutions for the problem (7.1), (7.2) with two types of impulsive conditions:

First we consider the impulsive conditions given by the generalized functions

$$\begin{cases}\Delta u(t_k) = I_{0k}(t_k, u(t_k)),\\ \Delta v(\tau_j) = J_{0j}(\tau_j, v(\tau_j))\\ \Delta u'(t_k) = I_{1k}(t_k, u(t_k), u'(t_k)),\\ \Delta v'(\tau_j) = J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)),\end{cases} \tag{8.1}$$

where, $k, j \in \mathbb{N}$,

$$\Delta u^{(i)}(t_k) = u^{(i)}(t_k^+) - u^{(i)}(t_k^-), \;\; \Delta v^{(i)}(\tau_j) = v^{(i)}(\tau_j^+) - v^{(i)}(\tau_j^-),$$

for $i = 0, 1$, $I_{0k}, J_{0j} : \mathbb{R}^2 \to \mathbb{R}$ and $I_{1k}, J_{1j} : \mathbb{R}^3 \to \mathbb{R}$ Carathéodory sequences verifying some monotone conditions, with $t_k, \tau_j$ fixed points such that $0 = t_0 < t_1 < \cdots < t_k < \cdots$, $0 = \tau_0 < \tau_1 < \cdots < \tau_j < \cdots$ and

$$\lim_{k \to +\infty} t_k = +\infty, \;\; \lim_{j \to +\infty} \tau_j = +\infty.$$

Secondly we will have the impulsive effects

$$\begin{cases}\Delta u'(t_k) = I^*_{0k}(t_k, u(t_k), u'(t_k)),\\ \Delta v'(\tau_j) = J^*_{0j}(\tau_j, v(\tau_j), v'(\tau_j)),\\ \Delta u'(t_k) = I^*_{1k}(t_k, u(t_k), u'(t_k)),\\ \Delta v'(\tau_j) = J^*_{1j}(\tau_j, v(\tau_j), v'(\tau_j)),\end{cases} \tag{8.2}$$

with $I^*_{ik}, J^*_{ij} : \mathbb{R}^3 \to \mathbb{R}$, for $i = 0, 1$, are Carathéodory sequences satisfying some growth assumptions.

The method and techniques follow [163].

It should be noted that it is the first time where localization results are considered for coupled systems on the half-line with full nonlinearities, together with generalized infinite impulsive effects.

The main techniques in this chapter make use of Carathéodory functions and sequences, equiconvergence at each impulsive moment and at infinity, following the method suggested in [154], and lower and upper solutions technique combined with Nagumo type condition.

## 8.1 Preliminary results

In this chapter we adopt the same spaces and norms used in the previous chapter (see Section 7.1). We, also consider similar definition of $L^1-$ Carathéodory functions and Carathéodory sequences (Definition 7.1.1) as in chapter 7.

Next two lemmas provide an integral form for solution of problem (7.1), (7.2), (8.1) or problem (7.1), (7.2), (8.2), as in the previous section.

**Lemma 8.1.1** *Let $f, h : [0, +\infty[ \times \mathbb{R}^4 \to \mathbb{R}$ be $L^1-$ Carathéodory functions. Then the system (7.1) with conditions (7.2), (8.1), has a solution $(u(t),\, v(t))$ expressed by*

$$\begin{aligned}
u(t) &= A_1 + B_1 t \\
&+ \sum_{0<t_k<t<+\infty} \left[ I_{0k}(t_k, u(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k))\,(t - t_k) \right] \\
&- t \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) \\
&+ \int_0^{+\infty} G(t,s) f(s, u(s), v(s), u'(s), v'(s))ds, \\
v(t) &= A_2 + B_2 t \\
&+ \sum_{0<\tau_j<t<+\infty} \left[ J_{0j}(\tau_j, v(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))\,(t - \tau_j) \right] \\
&- t \sum_{j=1}^{+\infty} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)) \\
&+ \int_0^{+\infty} G(t,s) h(s, u(s), v(s), u'(s), v'(s))ds,
\end{aligned}$$

*where*

$$G(t,s) = \begin{cases} -t, & 0 \le t \le s \le +\infty \\ -s, & 0 \le s \le t \le +\infty. \end{cases} \tag{8.3}$$

**Lemma 8.1.2** *Let $f, h : [0, +\infty[ \times \mathbb{R}^4 \to \mathbb{R}$ be $L^1-$ Carathéodory functions. Then the problem (7.1) with conditions (7.2), (8.2), has a solution $(u(t),\, v(t))$ given by*

$$\begin{aligned}
u(t) &= A_1 + B_1 t \\
&+ \sum_{0<t_k<t<+\infty} \left[ I_{0k}^*(t_k, u(t_k), u'(t_k)) + I_{1k}^*(t_k, u(t_k), u'(t_k))\,(t - t_k) \right] \\
&- t \sum_{k=1}^{+\infty} I_{1k}^*(t_k, u(t_k), u'(t_k))
\end{aligned}$$

$$+\int_0^{+\infty} G(t,s)f(s,u(s),v(s),u'(s),v'(s))ds,$$

$$v(t) = A_2 + B_2 t$$

$$+\sum_{0<\tau_j<t<+\infty} \left[J_{0j}^*(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}^*(\tau_j, v(\tau_j), v'(\tau_j))\,(t-\tau_j)\right]$$

$$-t\sum_{j=1}^{+\infty} J_{1j}^*(\tau_j, v(\tau_j), v'(\tau_j))$$

$$+\int_0^{+\infty} G(t,s)h(s,u(s),v(s),u'(s),v'(s))ds,$$

*where $G(t,s)$ is as in (8.3).*

## 8.2 Localization results

The problem (7.1), (7.2), (8.1) is a particular case of the impulsive problem (7.1), (7.2), (7.3) considered in [162], so, in this section, we only prove the localization of solution for problem (7.1), (7.2), (8.1), applying lower and upper solutions method, according the following definition:

**Definition 8.2.1** *A pair of functions $(\alpha_1, \alpha_2) \in \Big(PC_1^2\,([0,+\infty[) \times PC_2^2\,([0,+\infty[)\Big) \cap X$ is a lower solution of problem (7.1), (7.2), (8.1) if*

$$\begin{array}{rcl}
\alpha_1''(t) & \geq & f(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),w),\ \ \forall w\in\mathbb{R},\\
\alpha_2''(t) & \geq & h(t,\alpha_1(t),\alpha_2(t),z,\alpha_2'(t)),\ \ \forall z\in\mathbb{R},\\
\alpha_1(0) & \leq & A_1,\quad \alpha_2(0)\leq A_2,\\
\alpha_1'(+\infty) & \leq & B_1,\quad \alpha_2'(+\infty)\leq B_2,\\
\Delta\alpha_1(t_k) & = & I_{0k}(t_k,\alpha_1(t_k)),\\
\Delta\alpha_2(\tau_j) & = & J_{0j}(\tau_j,\alpha_2(\tau_j)),\\
\Delta\alpha_1'(t_k) & > & I_{1k}(t_k,\alpha_1(t_k),\alpha_1'(t_k)),\\
\Delta\alpha_2'(\tau_j) & > & J_{1j}(\tau_j,\alpha_2(\tau_j),\alpha_2'(\tau_j)),
\end{array}$$

*where $A_1, A_2, B_1, B_2 \in \mathbb{R}$, $k, j \in \mathbb{N}$.*

*A pair of functions* $(\beta_1, \beta_2) \in \Big(PC_1^2\left([0,+\infty[\right) \times PC_2^2\left([0,+\infty[\right)\Big) \cap X$ *is an upper solution of problem (7.1), (7.2), (8.1) if it verifies the reverse inequalities.*

Consider the following assumptions

**(A1)** $f$, $h$ verify the Nagumo conditions in $S_1$ and $S_2$, respectively, with

$$f(t,x,\alpha_2(t),z,w) \geq f(t,x,y,z,w) \geq f(t,x,\beta_2(t),z,w), \quad (8.4)$$

for fixed $(t,x,z,w) \in [0,+\infty[\, \times \mathbb{R}^3$, and

$$h(t,\alpha_1(t),y,z,w) \geq h(t,x,y,z,w) \geq h(t,\beta_1(t),y,z,w) \quad (8.5)$$

for fixed $(t,y,z,w) \in [0,+\infty[\, \times \mathbb{R}^3$;

**(A2)** For $i = 1, 2$,

$$N_i^* = \max\Big\{|B_i|, \|\alpha_i'\|_1, \|\beta_i'\|_1\Big\}, \quad (8.6)$$

$I_{1k}(t_k, x, y)$ and $J_{1j}(\tau_j, x, w)$ are nondecreasing on $y \in [-N_1^*, N_1^*]$ and on $w \in [-N_2^*, N_2^*]$, for all $k, j \in \mathbb{N}$, and fixed $x \in \mathbb{R}$.

The main theorem is an existence and localization result:

**Theorem 8.2.1** *Consider* $A_1, A_2, B_1, B_2 \in \mathbb{R}$. *Let* $f, h : [0,+\infty[\times \mathbb{R}^4 \to \mathbb{R}$ *be* $L^1-$ *Carathéodory functions verifying the assumptions of Theorem 7.2.1, (A1) and (A2), respectively.*

*Suppose that* $I_{0k}, J_{0j} : \mathbb{R}^2 \to \mathbb{R}$ *and* $I_{1k}, J_{1j} : \mathbb{R}^3 \to \mathbb{R}$ *are Carathéodory sequences, for* $k, j \in \mathbb{N}$, *verifying the assumptions of Theorem 7.2.1, (A1) and (A2), respectively.*

*Assume that there are* $(\alpha_1, \alpha_2)$ *and* $(\beta_1, \beta_2)$ *coupled lower and upper solutions of problem (7.1), (7.2), (8.1), respectively, such that*

$$\alpha_1(t) \leq \beta_1(t), \;\; \alpha_2(t) \leq \beta_2(t), \;\; t \in [0,+\infty[\,, \quad (8.7)$$

*Then there is at least a pair* $(u(t), v(t)) \in \Big(PC_1^2\left([0,+\infty[\right) \times PC_2^2\left([0,+\infty[\right)\Big) \cap X$ *solution of (7.1), (7.2), (8.1), such that*

$$\alpha_1(t) \leq u(t) \leq \beta_1(t), \quad \alpha_2(t) \leq v(t) \leq \beta_2(t), \;\; \forall t\, [0,+\infty[\,. \quad (8.8)$$

**Proof** The existence solution for problem (7.1), (7.2), (8.1), is guaranteed by Theorem 7.2.1.

To prove the localization part, consider the auxiliary functions $\delta_i : [0,+\infty[ \times \mathbb{R} \to \mathbb{R}$, given by

$$\delta_i(t,w) = \begin{cases} \beta_i(t) & , \quad w > \beta_i(t) \\ w & , \quad \alpha_i(t) \le w \le \beta_i(t) \\ \alpha_i(t) & , \quad w < \alpha_i(t), \end{cases}$$

for $i = 1,2$. The truncated and perturbed auxiliary coupled system is composed by

$$\begin{cases} u''(t) = f(t, \delta_1(t,u(t)), \delta_2(t,v(t)), u'(t), v'(t)) \\ \qquad + \frac{1}{1+t} \frac{u(t)-\delta_1(t,u(t))}{|u(t)-\delta_1(t,u(t))|+1}, \; t \neq t_k, \\ \\ v''(t) = h(t, \delta_1(t,u(t)), \delta_2(t,v(t)), u'(t), v'(t)) \\ \qquad + \frac{1}{1+t} \frac{v(t)-\delta_2(t,v(t))}{|v(t)-\delta_2(t,v(t))|+1}, \; t \neq \tau_j, \end{cases} \tag{8.9}$$

together with conditions (7.2) and the truncated impulsive conditions

$$\begin{cases} \Delta u(t_k) = I_{0k}(t_k, \delta_1(t_k, u(t_k))) \\ \Delta v(\tau_j) = J_{0j}(\tau_j, \delta_2(\tau_j, v(\tau_j))), \\ \Delta u'(t_k) = I_{1k}(t_k, \delta_1(t_k, u(t_k)), u'(t_k)) \\ \Delta v'(\tau_j) = J_{1j}(\tau_j, \delta_2(\tau_j, v(\tau_j)), v'(\tau_j)). \end{cases} \tag{8.10}$$

Let $(u(t), v(t))$ be a solution o problem (8.9), (7.2), (8.10). Suppose, by contradiction, that there is $t \in [0,+\infty[$, such that $\alpha_1(t) > u(t)$ and define

$$\inf_{t\in[0,+\infty[} (u(t) - \alpha_1(t)) := u(t_0) - \alpha_1(t_0) < 0.$$

Then, $t_0 \neq 0$ and $t_0 \neq +\infty$, by Definition 8.2.1 and (7.2),

$$u(0) - \alpha_1(0) = A_1 - \alpha_1(0) \ge 0,$$

and

$$u'(+\infty) - \alpha_1'(+\infty) = B_1 - \alpha_1'(+\infty) \ge 0.$$

Therefore, $t_0 \in ]0,+\infty[$, and next two cases can happen:

**Case 1:** Suppose that there is $p \in \{0,1,2,...\}$, such that $t_0 \in ]t_p, t_{p+1}[$. Then,

$$u'(t_0) = \alpha_1'(t_0), \;\; u''(t_0) - \alpha_1''(t_0) \ge 0, \tag{8.11}$$

and we deduce the following contradiction, by (8.4) and Definition 8.2.1

$$\begin{aligned}
0 &\leq u''(t_0) - \alpha_1''(t_0) \\
&= f(t_0, \delta_1(t_0, u(t_0)), \delta_2(t_0, v(t_0)), u'(t_0), v'(t_0)) \\
&\quad + \frac{1}{1+t_0} \frac{u(t_0) - \delta_1(t_0, u(t_0))}{|u(t_0) - \delta_1(t_0, u(t_0))| + 1} - \alpha_1''(t_0) \\
&= f(t_0, \alpha_1(t_0), \delta_2(t_0, v(t_0)), \alpha_1'(t_0), v'(t_0)) \\
&\quad + \frac{1}{1+t_0} \frac{u(t_0) - \alpha_1(t_0)}{|u(t_0) - \alpha_1(t_0)| + 1} - \alpha_1''(t_0) \\
&\leq f(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), v'(t_0)) \\
&\quad + \frac{1}{1+t_0} \frac{u(t_0) - \alpha_1(t_0)}{|u(t_0) - \alpha_1(t_0)| + 1} - \alpha_1''(t_0) \\
&< f(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), v'(t_0)) - \alpha_1''(t_0) \leq 0.
\end{aligned}$$

**Case 2:** Assume that there is $p \in \{1, 2, ...\}$ such that

$$\min_{t \in [0,+\infty[} (u(t) - \alpha_1(t)) := u(t_p) - \alpha_1(t_p) < 0. \tag{8.12}$$

Then by (8.12), we have

$$u'(t_p) \leq \alpha_1'(t_p). \tag{8.13}$$

As,

$$\begin{aligned}
\Delta(u - \alpha_1)(t_p) &= \Delta u(t_p) - \Delta\alpha_1(t_p) \\
&= I_{0p}(t_p, \delta_1(t_p, u(t_p))) - I_{0p}(t_p, \alpha_1(t_p)) \\
&= I_{0p}(t_p, \alpha_1(t_p)) - I_{0p}(t_p, \alpha_1(t_p)) = 0,
\end{aligned}$$

by (8.6), (8.10) and (8.13) we have the following contradiction with (8.12),

$$\begin{aligned}
u'(t_p^+) - \alpha_1'(t_p^+) &< I_{1p}(t_p, \delta_1(t_p, u(t_p)), u'(t_p)) + u'(t_p) \\
&\quad - I_{1p}(t_p, \alpha_1(t_p), \alpha_1'(t_p)) - \alpha_1'(t_p) \\
&= I_{1p}(t_p, \alpha_1(t_p), u'(t_p)) - I_{1p}(t_p, \alpha_1(t_p), \alpha_1'(t_p)) \leq 0.
\end{aligned}$$

So, $\alpha_1(t) \leq u(t)$, $\forall t \in [0, +\infty[$, and the remaining inequality $u(t) \leq \beta_1(t)$, $\forall t \in [0, +\infty[$, can be proved by same technique.

Applying the method above, it can be shown that $\alpha_2(t) \leq v(t) \leq \beta_2(t)$, $\forall t \in [0, +\infty[$, and, therefore, the problems (8.9), (7.2), (8.10) and (7.1), (7.2), (8.1) are equivalent.

■

## 8.3 Localization result for more general impulsive conditions

The solvability of problem (7.1), (7.2), (8.2) under the adequate assumptions is guaranteed by Theorem 7.2.1. To show the localization part, we define lower and upper solutions in a more general way than Definition 8.2.1, as it follows:

**Definition 8.3.1** *A pair of functions* $(\alpha_1, \alpha_2) \in \Big(PC_1^2\left([0,+\infty[\right) \times PC_2^2\left([0,+\infty[\right)\Big) \cap X$ *is a lower solution of problem (7.1), (7.2), (8.2) if*

$$\begin{array}{rcl} \alpha_1''(t) & \geq & f(t, \alpha_1(t), \alpha_2(t), \alpha_1'(t), \alpha_2'(t)), \\ \alpha_2''(t) & \geq & h(t, \alpha_1(t), \alpha_2(t), \alpha_1'(t), \alpha_2'(t)), \\ \alpha_1(0) & \leq & A_1, \quad \alpha_2(0) \leq A_2, \\ \alpha_1'(+\infty) & \leq & B_1, \quad \alpha_2'(+\infty) \leq B_2, \\ \Delta\alpha_1(t_k) & \leq & I_{0k}^*(t_k, \alpha_1(t_k), \alpha_1'(t_k)), \\ \Delta\alpha_2(\tau_j) & \leq & J_{0j}^*(\tau_j, \alpha_2(\tau_j), \alpha_2'(\tau)), \\ \Delta\alpha_1'(t_k) & > & I_{1k}^*(t_k, \alpha_1(t_k), \alpha_1'(t_k)), \\ \Delta\alpha_2'(\tau_j) & > & J_{1j}^*(\tau_j, \alpha_2(\tau_j), \alpha_2'(\tau)), \end{array}$$

*where* $A_1, A_2, B_1, B_2 \in \mathbb{R}$.
*A pair of functions* $(\beta_1, \beta_2) \in \Big(PC_1^2\left([0,+\infty[\right) \times PC_2^2\left([0,+\infty[\right)\Big) \cap X$ *is an upper solution of problem (7.1), (7.2), (8.2) if it verifies the reverse inequalities.*

**Theorem 8.3.1** *Consider* $A_i, B_i \in \mathbb{R}$, *for* $i = 1, 2$. *Let* $f, h : [0,+\infty[ \times \mathbb{R}^4 \to \mathbb{R}$ *be* $L^1-$ *Carathéodory functions verifying the assumptions of Theorem 7.2.1 and* $I_{ik}^*, J_{ij}^* : \mathbb{R}^3 \to \mathbb{R}$ *be Carathéodory sequences, for* $i = 0, 1$.

*Assume that:*

- *there are* $(\alpha_1, \alpha_2)$ *and* $(\beta_1, \beta_2)$ *coupled lower and upper solutions of problem (7.1), (7.2), (8.2), respectively, such that*

$$\alpha_1'(t) \leq \beta_1'(t), \;\; \alpha_2'(t) \leq \beta_2'(t), \;\; t \in [0,+\infty[\,; \tag{8.14}$$

- *$f$ and $h$ verify*

$$f(t,\alpha_1(t),\alpha_2(t),z,\alpha_2'(t)) \geq f(t,x,y,z,w) \geq f(t,\beta_1(t),\beta_2(t),z,\beta_2'(t)), \tag{8.15}$$

*for fixed* $(t,z) \in [0,+\infty[ \times \mathbb{R}$ *and*

$$h(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),w) \geq h(t,x,y,z,w) \geq h(t,\beta_1(t),\beta_2(t),\beta_1'(t),w), \tag{8.16}$$

*for fixed* $(t,w) \in [0,+\infty[ \times \mathbb{R}$*;*

*the impulsive conditions satisfy*

$$I_{0k}^*(t_k,\alpha_1(t_k),\alpha_1'(t_k)) \leq I_{0k}^*(t_k,x,z) \leq I_{0k}^*(t_k,\beta_1(t_k),\beta_1'(t_k)), \tag{8.17}$$

*for* $\alpha_1(t_k) \leq x \leq \beta_1(t_k)$*,* $\alpha_1'(t_k) \leq z \leq \beta_1'(t_k)$ *and* $k \in \mathbb{N}$*,*

$$J_{0j}^*(\tau_j,\alpha_2(\tau_j),\alpha_2'(\tau_j)) \leq J_{0j}^*(\tau_j,y,w) \leq J_{0j}^*(\tau_j,\beta_2(\tau_j),\beta_2'(\tau_j)), \tag{8.18}$$

*for* $\alpha_2(\tau_j) \leq y \leq \beta_2(\tau_j)$*,* $\alpha_2'(\tau_j) \leq w \leq \beta_2'(\tau_j)$ *and* $\tau_j \in \mathbb{N}$*,*

$$I_{1k}^*(t_k,\alpha_1(t_k),z) \geq I_{1k}^*(t_k,x,z) \geq I_{1k}^*(t_k,\beta_1(t_k),z), \tag{8.19}$$

*for* $\alpha_1(t) \leq x \leq \beta_1(t)$*,* $k \in \mathbb{N}$ *and fixed* $z \in \mathbb{R}$*,*

$$J_{1j}^*(\tau_j,\alpha_2(\tau_j),w) \geq J_{1j}^*(\tau_j,y,w) \geq J_{j}^*(\tau_j,\beta_2(\tau_j),w), \tag{8.20}$$

*for* $\alpha_2(t) \leq y \leq \beta_2(t)$*,* $j \in \mathbb{N}$ *and fixed* $w \in \mathbb{R}$*.*

Then there is at least a pair $(u(t),v(t)) \in \left(PC_1^2([0,+\infty[) \times PC_2^2([0,+\infty[)\right) \cap X$ solution of (7.1), (7.2), (8.2), such that

$$\alpha_1^{(i)}(t) \leq u^{(i)}(t) \leq \beta_1^{(i)}(t), \quad \alpha_2^{(i)}(t) \leq v^{(i)}(t) \leq \beta_2^{(i)}(t), \quad i=0,1, \tag{8.21}$$

for all $t \in [0,+\infty[$.

**Proof** To prove the localization part given by (8.21), consider the auxiliary functions $\delta_i^j : [0,+\infty[ \times \mathbb{R} \to \mathbb{R}$, given by

$$\delta_i^j(t,w) = \begin{cases} \beta_i^{(j)}(t) & , \quad w > \beta_i^{(j)}(t) \\ w & , \quad \alpha_i^{(j)}(t) \leq w \leq \beta_i^{(j)}(t) \\ \alpha_i^{(j)}(t) & , \quad w < \alpha_i^{(j)}(t), \end{cases}$$

for $i = 1, 2$, $j = 0, 1$, and the truncated and perturbed coupled system

$$\left\{\begin{array}{l} u''(t) = f(t, \delta_1^0(t, u(t)), \delta_2^0(t, v(t)), \delta_1^1(t, u'(t)), \delta_2^1(t, v'(t))) \\ \qquad + \frac{1}{1+t} \frac{u'(t) - \delta_1^1(t, u'(t))}{|u'(t) - \delta_1^1(t, u'(t))| + 1} \\ \\ v''(t) = h(t, \delta_1^0(t, u(t)), \delta_2^0(t, v(t)), \delta_1^1(t, u'(t)), \delta_2^1(t, v'(t))) \\ \qquad + \frac{1}{1+t} \frac{v'(t) - \delta_2^1(t, v'(t))}{|v'(t) - \delta_2^1(t, v'(t))| + 1}, \end{array}\right. \tag{8.22}$$

with conditions (7.2) and the truncated impulsive conditions

$$\begin{cases} \Delta u(t_k) = I_{0k}^*(t_k, \delta_1^0(t_k, u(t_k)), u'(t_k)), \\ \Delta v(\tau_j) = J_{0j}^*(\tau_j, \delta_2^0(\tau_j, v(\tau_j)), v'(\tau_j)), \\ \Delta u'(t_k) = I_{1k}^*(t_k, \delta_1^0(t_k, u(t_k)), \delta_1^1(t_k, u'(t_k))), \\ \Delta v'(\tau_j) = J_{1j}^*(\tau_j, \delta_2^0(\tau_j, v(\tau_j)), \delta_2^1(\tau_j, v'(\tau_j)). \end{cases} \tag{8.23}$$

Let $(u(t), v(t))$ be a solution o problem (8.22), (7.2), (8.23).

Suppose, by contradiction, that there is $t \in [0, +\infty[$, such that $\alpha_1'(t) > u'(t)$ and define

$$\inf_{t \in [0, +\infty[} (u'(t) - \alpha_1'(t)) := u'(t_0) - \alpha_1'(t_0) < 0. \tag{8.24}$$

By (7.2) and Definition 8.3.1, $t_0 \neq +\infty$, as

$$u'(+\infty) - \alpha_1'(+\infty) = B_1 - \alpha_1'(+\infty) \geq 0.$$

If $t_0 = 0$, then $u''(0) - \alpha_1''(0) \geq 0$ and the following contradiction holds, by Definition 8.3.1, (8.15) and (8.22),

$$\begin{aligned} 0 &\leq u''(0) - \alpha_1''(0) \\ &= f(0, \delta_1^0(0, u(0)), \delta_2^0(0, v(0)), \delta_1^1(0, u'(0)), \delta_2^1(0, v'(0))) \\ &\quad + \frac{u'(0) - \delta_1^1(0, u'(0))}{|u'(0) - \delta_1^1(0, u'(0))| + 1} - \alpha_1''(0) \\ &\leq f(0, \delta_1^0(0, u(0)), \delta_2^0(0, v(0)), \alpha_1'(t_0), \delta_2^1(0, v'(0))) \\ &\quad + \frac{u'(0) - \alpha_1'(0)}{|u'(0) - -\alpha_1'(0)| + 1} - \alpha_1''(0) \\ &< f(0, \delta_1^0(0, u(0)), \delta_2^0(0, v(0)), \alpha_1'(0), \delta_2^1(0, v'(0))) - \alpha_1''(0) \\ &\leq f(0, \alpha_1(0), \alpha_2(0), \alpha_1'(0), \alpha_2'(0)) - \alpha_1''(0) \leq 0. \end{aligned}$$

Therefore, $t_0 \in ]0, +\infty[$ and we have three possible cases:

- Assume that there is $p \in \{0,1,2,...\}$, such that $t_0 \in \, ]t_p, t_{p+1}[$. Then,
$$u''(t_0) = \alpha_1''(t_0),$$
and by the arguments above, it follows a similar contradiction.

- If there is $p \in \{1,2,...\}$, such that
$$\min_{t\in[0,+\infty[} (u'(t) - \alpha_1'(t)) := u'(t_p) - \alpha_1'(t_p) < 0. \tag{8.25}$$

  So, by Definition 8.3.1, (8.19) and (8.23), the following contradiction holds,
$$\begin{array}{rcl} 0 & \leq & \Delta(u-\alpha_1)'(t_p) = \Delta u'(t_p) - \Delta\alpha_1'(t_p) \\ & < & I_{1p}^*(t_p, \delta_1^0(t_p, u(t_p)), \delta_1^1(t_p, u'(t_p)) - I_{1p}^*(t_p, \alpha_1(t_p), \alpha_1'(t_p)) \\ & = & I_{1p}^*(t_p, \delta_1^0(t_p, u(t_p)), \alpha_1'(t_p)) - I_{1p}^*(t_p, \alpha_1(t_p), \alpha_1'(t_p)) \\ & = & I_{1p}^*(t_p, \alpha_1(t_p), \alpha_1'(t_p)) - I_{1p}^*(t_p, \alpha_1(t_p), \alpha_1'(t_p)) \leq 0. \end{array}$$

- Assume that there is $p \in \{1,2,...\}$, such that
$$\inf_{t\in[0,+\infty[} (u'(t) - \alpha_1'(t)) := u'(t_p^+) - \alpha_1'(t_p^+) < 0.$$

  So, there is $\varepsilon > 0$ sufficiently small such that
$$u'(t) - \alpha_1'(t) < 0, \; u''(t) - \alpha_1''(t) \geq 0, \; \forall t \in ]t_p, \, t_p + \varepsilon[.$$

  So, for $t \in ]t_p, \, t_p + \varepsilon[$, we have the contradiction, given by,
$$\begin{array}{rcl} 0 & \leq & u''(t) - \alpha_1''(t) \\ & = & f(t, \delta_1^0(t, u(t)), \delta_2^0(t, v(t)), \delta_1^1(t, u'(t)), \delta_2^1(t, v'(t))) \\ & & + \dfrac{u'(t) - \delta_1^1(t, u'(t))}{|u'(t) - \delta_1^1(t, u'(t))| + 1} - \alpha_1''(t) \\ & = & f(t, \delta_1^0(t, u(t)), \delta_2^0(t, v(t)), \alpha_1'(t), \delta_2^1(t, v'(t))) \\ & & + \dfrac{u'(t) - \alpha_1'(t)}{|u'(t) - \alpha_1'(t)| + 1} - \alpha_1''(t) \\ & < & f(t, \delta_1^0(t, u(t)), \delta_2^0(t, v(t)), \alpha_1'(t), \delta_2^1(t, v'(t))) - \alpha_1''(t) \\ & \leq & f(t, \alpha_1(t), \alpha_2(t), \alpha_1'(t), \alpha_2'(t)) - \alpha_1''(t) \leq 0. \end{array}$$

Therefore, $\alpha_1'(t) \leq u'(t)$, $\forall t \in [0,+\infty[$. Analogously it can be proved that $u(t)' \leq \beta_1'(t)$, $\forall t \in [0,+\infty[$. By the analogous method, we can show that $\alpha_2'(t) \leq v'(t) \leq \beta_2'(t)$, $\forall t \in [0,+\infty[$.

From the integration of the inequality

$$\alpha_1'(t) \leq u'(t) \leq \beta_1'(t),\ \forall t \in [0,+\infty[\,,$$

for $t \in [0,\, t_1[$, we have, by (7.2) and Definition 8.3.1,

$$\alpha_1(t) \leq u(t) + \alpha_1(0) - u(0) = u(t) - \alpha_1(0) - A_1 \leq u(t).$$

So,

$$\alpha_1(t) \leq u(t),\ \forall t \in [0, t_1]\,. \tag{8.26}$$

Repeating the above process, for $t \in [t_1,\, t_2[$, it follow that, by (8.23), (8.17), Definition (8.3.1) and (8.26),

$$\begin{aligned}
\alpha_1(t) &\leq u(t) + \alpha_1(t_1^+) - u(t_1^+) \\
&= u(t) + \alpha_1(t_1^+) - I_{01}^*(t_1, \delta_1^0(t_1, u(t_1)), u'(t_1)) - u(t_1^-) \\
&\leq u(t) + I_{01}^*(t_1, \alpha_1(t_1), \alpha_1'(t_1)) - I_{01}^*(t_1, \delta_1(t_1, u(t_1)), u'(t_1)) \\
&\quad -u(t_1^-) + \alpha_1(t_1^-) \\
&\leq u(t) - \big(u(t_1^-) - \alpha_1(t_1^-)\big) \leq u(t).
\end{aligned}$$

So, $\alpha_1(t) \leq u(t)$, $\forall t \in\, ]t_1, t_2[$. By iteration, we obtain that $\alpha_1(t) \leq u(t)$, $\forall t \in [0,+\infty[$.

Similarly, it can be show that, $u(t) \leq \beta_1(t)$, $\forall t \in [0,+\infty[$. Applying the arguments above, we can prove that $\alpha_2(t) \leq v(t) \leq \beta_2(t)$, $\forall t \in [0,+\infty[$.

■

## 8.4 Logging timber by helicopter

Helicopter logging is a system for removal trees in cables attached to a helicopter, of felled and bucked logs from areas where some, or all, of the tree have been felled, (see [205]).

In [176], the authors compare the impact of helicopter and rubber-tired skidder extraction of timber after harvesting on the structure and function of a blackwater forested wetland. In [98], Jones et al., studies the removal of logs via helicopters advocated to minimize soil damage and facilitate rapid revegetation. Moreover, they also test the impacts of the helicopter compared to skidder harvesting systems in regeneration, community structure of woody plants and biomass growth in three blackwater streams floodplain in southern Alabama.

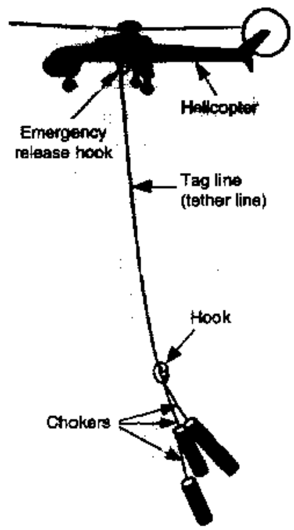


**FIGURE 8.1**
Logging timber by helicopter.

Helicopter logging is considered practical for harvesting high value timber from inaccessible sites and is a preferred alternative, when harvesting timber from environmentally timber salvage sites, or inaccessible sites, [203].

In [69], the authors cite countless advantages of the logging helicopter, among which: is best suited for fast removals of timber, especially in places that require it, to reduce fire risk, to limit the spread of pests, where these are prioritized against cost of the operations and is practice in the valorization of timber of the inaccessible places. So, it is an economically viable alternative compared to other forms of timber extraction.

Motivated by these works, we consider a logging timber by helicopter model (see [76]), represented by the second linear system of differential equations

$$\begin{cases} (m_1+m_2)L_1^2\theta_1''(t)+m_2L_1L_2\theta_2''(t)+(m_1+m_2)L_1g\theta_1(t)=0, \\ m_2L_1L_2\theta_1''(t)+m_2L_2^2\theta_2''(t)+m_2L_2g\theta_2(t)=0, \end{cases}$$

where:

- $\theta_1$ and $\theta_2$ denote the angles of oscillation of the two connecting cables, measured from the gravity vector direction;
- $g$ is the gravitation constant;
- $m_1$, $m_2$ denote the masses of the two trees;
- $L_1$, $L_2$ are the cable lengths.

The trees are hung in the helicopter in cables and the load for two trees approaching a double pendulum, which oscillates during flight.

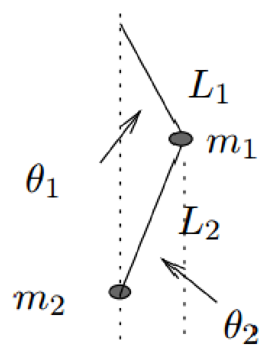


**FIGURE 8.2**
Representation of the cable logging timber by helicopter.

From the above linear system, we can obtain a nonlinear system valid for a long time helicopters flights under forcing terms, given by the second order nonlinear coupled system on the half-line

$$\begin{cases} \theta_1''(t) = \frac{g}{L_1 m_1 (t^3+1)} \left[ +(m_1+m_2)\theta_1(t) - m_2\theta_2(t) \right] \\ \qquad +g_1(t, \theta_1(t), \theta_2(t), \theta_1'(t), \theta_2'(t)), \\ \\ \theta_2''(t) = \frac{g(m_1+m_2)}{L_2 m_1 (t^3+1)} (-\theta_1(t) + \theta_2(t)) + g_2(t, \theta_1(t), \theta_2(t), \theta_1'(t), \theta_2'(t)), \end{cases} \tag{8.27}$$

for $t \in [0, +\infty[$, $g_1, g_2 : [0, +\infty[ \times \mathbb{R}^4 \to \mathbb{R}$ are $L^1$-Carathéodory forcing functions, to be defined forward, such that $g_1(t, x, y, z, w)$ is nonincreasing in $y$ and $g_2(t, x, y, z, w)$ is nonincreasing in $x$, together with the boundary conditions

$$\begin{cases} \theta_1(0) = 0, \ \ \theta_2(0) = 0, \\ \theta_1'(+\infty) = 0, \ \ \theta_2'(+\infty) = 0. \end{cases} \tag{8.28}$$

Moreover, we consider the impulsive conditions

$$\begin{cases} \Delta\theta_1(k) = \frac{3k^2+3k+1}{k^3(k+1)^3}; \quad \Delta\theta_2(j) = \frac{3j^2+3j+1}{j^3(j+1)^3}, \\ \Delta\theta_1'(k) = \frac{1}{k^3}(-\alpha_1(k)+\alpha_1'(k)); \quad \Delta\theta_2'(j) = \frac{1}{k^j}(-\alpha_2(j)+\alpha_2'(j)), \end{cases} \tag{8.29}$$

for $k, j \in \mathbb{N}$, $0 < t_1 < \cdots < t_k < \cdots$, $0 < \tau_1 < \cdots \tau_j < \cdots$.

The system (8.27)-(8.29) is a particular case of the problem (7.1), (7.2), (8.1), with

$$\begin{aligned} f(t,x,y,z,w) &= \frac{g}{L_1 m_1(t^3+1)}\left[(m_1+m_2)x - m_2 y\right] + g_1(t,x,y,z,w), \\ h(t,x,y,z,w) &= \frac{g(m_1+m_2)}{L_2 m_1(t^3+1)}(-x+y) + g_2(t,x,y,z,w), \\ I_{0k}(k,x) &= \frac{3k^2+3k+1}{k^3(k+1)^3}, \quad J_{0j}(j,y) = \frac{3j^2+3j+1}{j^3(j+1)^3}, \\ I_{1k}(k,x,z) &= \frac{1}{k^3}(-x+z), \quad J_{1j}(j,y,w) = \frac{1}{j^3}(-y+w), \end{aligned}$$

with $k, j \in \mathbb{N}$, $A_1 = A_2 = B_1 = B_2 = 0$. Choose $\rho > 0$ such that, for adequate $m_1$, $m_2$, $L_1$, $L_2$, $g_1$ and $g_2$, the following relations are satisfied

$$\begin{aligned} |f(t,x,y,z,w)| &\leq \frac{g}{L_1 m_1(t^3+1)}(m_1+m_2)\rho(1+t) + m_2\rho + \Phi_{1_\rho}(t) \\ &:= \Psi_{1\rho}(t), \end{aligned}$$

$$\begin{aligned} |h(t,x,y,z,w)| &\leq \frac{g(m_1+m_2)}{L_2 m_1(t^3+1)} 2\rho(1+t) + \Phi_{2_\rho}(t) \\ &:= \Psi_{2\rho}(t), \end{aligned}$$

where $|g_1(t,x,y,z,w)| \leq \Phi_{1_\rho}(t)$, $|g_2(t,x,y,z,w)| \leq \Phi_{2_\rho}(t)$, for some $\rho > 0$, such that

$$\sup_{t\in[0,+\infty[} \left\{ \frac{|x|}{1+t}, \frac{|y|}{1+t}, |z|, |w| \right\} < \rho,$$

and $\Phi_{i_\rho}$, $\Psi_{i\rho}$, $i = 1, 2$, are positive functions such that $\Phi_{i_\rho}, \Psi_{i\rho} \in L^1([0,+\infty[)$, and

$$I_{1k}(k,x,z) \leq \frac{1}{k^3}\rho(2+k); \quad J_{1j}(j,y,w) \leq \frac{1}{j^3}\rho(2+j).$$

The functions $\alpha_i : [0, +\infty[ \to \mathbb{R}$, $i = 1, 2$, given by

$$\alpha_i(t) = \begin{cases} -1, & t \in [0,\ 1] \\ -\frac{1}{(k+1)^3}, & t \in ]k,\ k+1],\ k > 1, \end{cases}$$

and $\beta_i : [0, +\infty[ \to \mathbb{R}$, given by

$$\beta_i(t) = \begin{cases} t, & t \in [0,\ 1] \\ \frac{1}{(k+1)^3}, & t \in ]k,\ k+1],\ k > 1, \end{cases}$$

are, respectively, lower and upper solutions of problem (8.27)-(8.29), satisfying (8.7), assuming that, for $i = 1, 2$,

$$g_i(t, -1, -1, 0, 0) \leq 0,\ \ g_i(t, 1, 1, 0, 0) \geq 0,\ \ ,\ \forall t \in [0, 1],$$

$$g_i\left(t, -\frac{1}{(k+1)^3}, -\frac{1}{(k+1)^3}, 0, 0\right) \leq 0,\ \forall t \in ]k, k+1],\ k > 1,$$

and

$$g_i\left(t, \frac{1}{(k+1)^3}, \frac{1}{(k+1)^3}, 0, 0\right) \geq 0,\ \forall t \in ]k, k+1],\ k > 1.$$

In fact, for the lower solution $(\alpha_1(t),\ \alpha_2(t))$and $t \in [0, 1]$, we have

$$\begin{aligned} 0 &\geq \frac{-gm_1}{L_1 m_1(t^3+1)} + g_1(t, -1, -1, 0, 0) \\ 0 &\geq g_2(t, -1, -1, 0, 0), \end{aligned}$$

and for $t \in ]k, k+1],\ k > 1$,

$$\begin{aligned} 0 &\geq \frac{-gm_1}{L_1 m_1(t^3+1)(k+1)^3} + g_1\left(t, -\frac{1}{(k+1)^3}, -\frac{1}{(k+1)^3}, 0, 0\right) \\ 0 &\geq \frac{-2g(m_1+m_2)}{L_2 m_1(t^3+1)(k+1)^3} + g_2\left(t, -\frac{1}{(k+1)^3}, -\frac{1}{(k+1)^3}, 0, 0\right). \end{aligned}$$

For the upper solution $(\beta_1(t),\ \beta_2(t))$ and $t \in [0, 1]$,

$$\begin{aligned} 0 &\leq \frac{gm_1}{L_1 m_1(t^3+1)} + g_1(t, 1, 1, 0, 0) \\ 0 &\leq g_2(t, 1, 1, 0, 0), \end{aligned}$$

and for $t \in ]k, k+1], \; k > 1$,

$$\begin{aligned} 0 &\leq \frac{gm_1}{L_1 m_1 (t^3+1)(k+1)^3} + g_2\left(t, \frac{1}{(k+1)^3}, \frac{1}{(k+1)^3}, 0, 0\right) \\ 0 &\leq g_2\left(t, \frac{1}{(k+1)^3}, \frac{1}{(k+1)^3}, 0, 0\right). \end{aligned}$$

Moreover, remark that, for $i = 1, 2$,

$$\lim_{t\to+\infty} \alpha_i'(t) = \lim_{k\to+\infty} \alpha_i'(t) = 0 \text{ and } \lim_{t\to+\infty} \beta_i'(t) = \lim_{k\to+\infty} \beta_i'(t) = 0.$$

So, by Theorem 8.2.1, there is at least a pair $(\theta_1, \theta_2) \in \left(PC_1^2([0,+\infty[) \times PC_2^2([0,+\infty[)\right) \cap X$, solution of problem (8.27)-(8.29) and, moreover,

$$-1 \leq \theta_1(t) \leq 1, \quad -1 \leq \theta_2(t) \leq 1, \; \forall t \in [0, 1],$$

$$-\frac{1}{(k+1)^3} \leq \theta_1(t) \leq \frac{1}{(k+1)^3}, \text{ for } t \in ]k, k+1], \; k = 2, 3, ...$$

and

$$-\frac{1}{(j+1)^3} \leq \theta_2(t) \leq \frac{1}{(j+1)^3}, \text{ for } j \in ]j, j+1], \; j = 2, 3, ...$$

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