UNIVERSIDADE DE ÉVORA

STOCHASTIC DIFFERENTIAL EQUATION HARVESTING MODELS: SUSTAINABLE POLICIES AND PROFIT OPTIMIZATION

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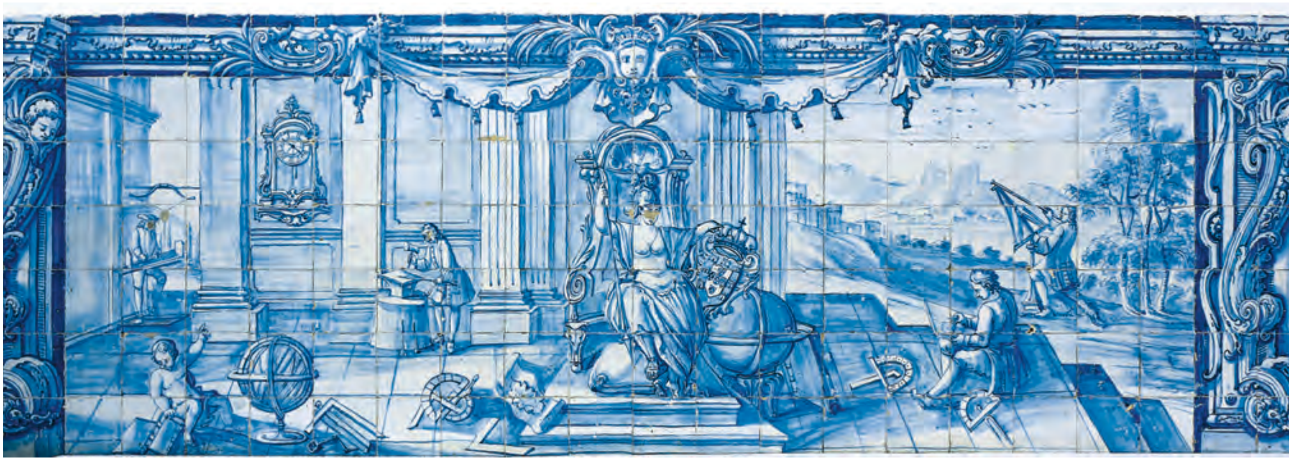

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Abstract

We describe the growth dynamics of a fish or some other harvested population in a random environment using a stochastic differential equation general model, where the harvest term depends on a constant or on a variable fishing effort. We compare the profit obtained by the fishing activity with two types of harvesting policies, one based on variable effort, which is inapplicable, and the other based on a constant effort, which is applicable, sustainable and is socially advantageous. We use real data and consider a logistic and a Gompertz growth models to perform such comparisons. For both optimal policies, profitwise comparisons are also made when considering a logistic-type growth model with weak Allee effects. The mean and variance of the first passage times by a lower and by an upper thresholds are studied and, for a particular threshold value, we estimate the probability density function of the first passage time using the inversion of the Laplace transform.

Keywords: Harvesting Models, Stochastic Differential Equations, Profit Optimization, Sustainable Policies, Constant Effort, Stationary Density

Resumo

MODELOS DE PESCA USANDO EQUAÇÕES DIFERENCIAIS ESTOCÁSTICAS: POLÍTICAS SUSTENTÁVEIS E OTIMIZAÇÃO DO LUCRO

A dinâmica de crescimento de uma população sujeita a pesca em ambiente aleatório é descrita através de modelos de equações diferenciais estocásticas, onde o termo de captura depende de um esforço de pesca constante ou variável. Comparamos o lucro obtido pela atividade de pesca usando dois tipos de políticas de pesca, uma inaplicável e baseada em esforço variável e a outra aplicável, sustentável e socialmente vantajosa, baseada em esforço constante. As comparações são realizadas recorrendo a dados reais e considerando dois modelos de crescimento, o modelo logístico e o modelo de Gompertz. Para ambas as políticas ótimas, as comparações do lucro também são feitas quando se considera um modelo de crescimento do tipo logístico com efeitos de Allee fracos. A média e a variância dos tempos de primeira passagem por um limite inferior e por um limite superior são estudados e, para um determinado valor limite, estimamos a função de densidade do tempo de primeira passagem usando a inversa da transformada de Laplace.

Palavras-chave: Modelos de Pesca, Equações Diferenciais Estocásticas, Optimização do Lucro, Políticas Sustentáveis, Esforço Constante, Densidade Estacionária

Contents

List of Figures

List of Tables

Abbreviations list

ABBREVIATIONS

a.s.	almost sure or almost surely
d.f.	distribution function
KBE	Kolmogorov backward equation
KFE	Kolmogorov forward equation
m.s.	mean square
p.d.f.	probability density function
r.v.	random variable
SDE	stochastic differential equation
SDEs	stochastic differential equations
w.p.1	with probability one

Notation list

NOTATION

$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$s \wedge t$	minimum between s and t
$o(z^p)$	terms of smaller order than z^p, with $p > 0$
$X \sim \mathcal{N}(a, b)$	X is normally distributed with mean a and variance b
$\mathbb{E}[X]$	expectation or mean value of the random variable X
$Var[X]$	variance of the random variable X
$sd[X]$	standard deviation of the random variable X
$\stackrel{d}{=}$	equality in distribution
$\xrightarrow{m.s.}$	mean square convergence
$\xrightarrow{a.s.}$	almost sure convergence
$\xrightarrow{P}$	convergence in probability P
$W(t)$	standard Wiener process

1

Introduction

In this text we will consider the existence of a fish or some other population subject to harvesting. Hence, we will use the terms 'harvesting' and 'fishing' without any difference between them. Also, the terms 'species', 'population' and 'stock' will be used with the same meaning among them. Individuals from one population may be fish, crustaceans or others, and, in terms of aquatic environment, they may be pelagic, demersal or benthonic. We assume a closed population, that is, there is no immigration.

Stochastic differential equations have been studied as a way to explain many physical, biological, economic and social phenomena. A particular case is the application (starting with the pioneering work of Beddington and May (1977)) to the growth dynamics of a harvested population subject to a randomly varying environment, with the purpose of obtaining optimal harvesting policies. Such policies usually are intended to maximize the expected yield or profit over a finite or infinite time horizon T. Since population size depends on fishing effort, it seems natural to consider the effort at the time instant t, $E(t)$, as a control and apply optimal control techniques to achieve either yield or profit optimization, discounted by a social rate.

The profit per unit time can be defined as the difference between sales revenue and fishing costs, i.e.,

$$\Pi(t) \; := \; P(t) - C(t),$$

where $P(t)$ and $C(t)$ are respectively the revenue and cost per unit time. We consider the revenue per unit time to be dependent on the harvesting yield and to have a quadratic form given by

$$P(t) \; = \; p(H(t))H(t), \quad \text{with} \quad p(H(t)) \; = \; p_1 - p_2 H(t),$$

where $p_1 \geq 0$ is the linear price parameter and $p_2 \geq 0$ is the quadratic price parameter. The cost of harvest per unit time is assumed to be dependent on effort and also to have a quadratic form given by

$$C(t) \; = \; c(E(t))E(t), \quad \text{with} \quad c(E(t)) \; = \; c_1 + c_2 E(t),$$

where $c(E(t))$ is the cost per unit effort and $c_1 \geq 0$ is the linear cost parameter and $c_2 > 0$ is the quadratic cost parameter. The quadratic cost structure incorporates the case where the fishermen need to use less efficient vessels and fishing technologies or pay higher overtime wages to implement an extraordinary high effort (see Clark, 1990). However, other more complicated profit structures can be used.

In the deterministic case, there is a quite comprehensive account of optimal harvesting policies regarding yield or profit optimization (Clark, 1990). Under general assumptions, unless we are close to the end of a finite time horizon T, the optimal policy consists in harvesting with maximum intensity (which can be limited to a maximum harvesting effort or be unlimited) when the population is above a critical threshold and stop harvesting (zero effort) when the population is below that threshold. Once the threshold is reached, one just needs to keep the harvesting rate constant at an appropriate value so that the population remains at the threshold size. However, when the population is below the threshold, the fishery should be closed until the threshold is reached, which may take a while.

Stochastic optimal control methods were also applied to derive optimal harvesting strategies in a randomly varying environment (e.g. Alvarez, 2000b,a, Alvarez and Shepp, 1998, Arnason et al., 2004, Hanson and Ryan, 1998, Lande et al., 1994, 1995, Lungu and Øksendal, 1996, Suri, 2008). The optimal policy is similar to the deterministic case, i.e., harvest with maximum intensity when the population is above a critical threshold (not necessarily the same as in the deterministic case) and stop harvesting when below the threshold. However, after the threshold size is attained, due to random fluctuations of the environment, population size will keep varying. In this case, fishing effort must be adjusted at every instant, so that the size of the population does not go above the equilibrium value. Such policies imply that the effort changes frequently and abruptly, according to the random fluctuations of the population. Sudden frequent transitions between quite variable effort levels are not compatible with the logistics of fisheries. Besides, the period of low or no harvesting poses social and economical undesirable implications. In addition to such shortcomings, these optimal policies require the knowledge of the population size at every instant, to define the appropriate level of effort. The estimation of the population size is a difficult, costly, time consuming and inaccurate task and, for these reasons, and the others pointed above, these policies should be considered unacceptable and inapplicable.

In Braumann (1981, 1985, 2008), a constant fishing effort, $E(t) \equiv E$, was assumed, providing an alternative approach to optimal harvesting. For a large class of models (including the logistic and the Gompertz), it was found that, taking a constant fishing effort, there is, under mild conditions, a stochastic sustainable behaviour. Namely, the probability distribution of the population size at time t will converge, as $t \to +\infty$, to an equilibrium probability distribution (the so-called stationary or steady-state distribution) having a probability density function (the so-called stationary density). For the logistic and the Gompertz models, the stationary density function was found, and the effort E that optimizes the steady-state yield was determined. The issue of profit optimization, however, was not addressed.

This study considers the issue of profit optimization for the sustainable constant effort harvesting policy. This policy, rather than switching between large and small or null fishing effort, keeps a constant effort and is therefore compatible with the logistics of fisheries. Furthermore, this alternative policy does not require knowledge of the population size. However, it will result in a reduction of the profit when we compare it with the inapplicable optimal policy. We will examine if such reduction is appreciable or negligible.

Chapter 2 presents a brief review of concepts and results on stochastic processes and on stochastic differential equations. We present the main definitions, properties and theorems necessary for the formulation and resolution of the problems induced by the optimal policies that we intend to develop.

In Chapter 3 we consider a general population growth model with dynamics described by a stochastic differential equation and present an optimal policy problem based on variable effort. The solution of this problem is obtained by applying the stochastic dynamical programming technique, producing a non-linear stochastic partial differential equation which needs to be solved by numerical methods. We apply a discretization scheme to this equation and show how to reach the optimal policy, namely what is the effort that produces maximum profit.

The alternative optimal policy based on constant effort is developed at Chapter 4. This approach is, to the best of our knowledge, the first attempt to obtain an optimal sustainable harvesting policy based on profit per unit time optimization. As in Chapter 3, we consider a general population growth model with harvesting, described by a stochastic differential equation, but with a constant effort harvesting term. Using the theory of stochastic differential equations presented in Chapter 2, we will determine, at the steady-state, the optimal sustainable effort and the optimal sustainable profit per unit time.

Chapter 5 treats the numerical comparisons between the optimal policy with variable effort and the optimal sustainable policy with constant effort. We compare the performance of the two policies in terms of the expected profit earned by the harvester and in terms of the evolution of the population size and the amount of effort across time. In order to simulate the optimal policies, we consider two classical growth models: the logistic model and the Gompertz model. For each model, we set up a basic scenario with parameter values based on realistic data from a fish or a shrimp population and also consider alternative scenarios corresponding to changes of the different parameter values used in the basic scenario to check the influence of such changes in terms of the profit. In addition to the mentioned optimal policies, we present and compare a sub-optimal and applicable policy based on variable effort but with fixed periods of constant effort.

We will also present, for a logistic-like model, the comparison between the optimal variable effort policy and the optimal sustainable constant effort policy when the population is under weak Allee effects. Strong Allee effects should not be considered since they drive the population to extinction, even in the absence of harvesting. This study is on Chapter 6.

Chapter 7 refers to the study of first passage times by a lower and by an upper threshold, which may be of interest in evaluating recovery times or the risk of the population size reaching a dangerous level. We study directly the mean and standard deviation of the first passage times. We also show how to estimate the first passage time probability density function by using the numerical inversion of its Laplace transform.

We end up, in Chapter 8, with the main conclusions.

Computations were carried out with R (`http://r-project.org`) and with Matlab. The R code is presented in Appendix B and the Matlab codes are presented in Appendix C and in Appendix D.

Part of this work has already been accepted for publication in the Fisheries Research journal:

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2

Review on stochastic processes

The evolution of population size is subject to random perturbations of the surrounding environment. Therefore, it makes sense to study this evolution using models that incorporate these perturbations, whenever this is treatable from the mathematical point of view. This chapter precedes the presentation of these models and has a brief review of concepts and results on stochastic processes and on stochastic differential equations. We present the main definitions, properties and theorems necessary for the formulation and resolution of the problems induced by the optimal policies that we intend to develop. Properties and proofs of theorems can be found in reference books of stochastic processes and stochastic differential equations (e.g. Arnold, 1974, Gihman and Skorohod, 1979, Karlin and Taylor, 1981, Øksendal, 1998, Braumann, 2005).

2.1 Stochastic processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\mathcal{B}$ the Borel σ-algebra (the smallest σ-algebra generated by the intervals of the real line) and L^p $(p > 0)$ the space of random variables X such that $\mathbb{E}[|X|^p] = \int_\Omega |X|^p dP$ exists and is finite. If we identify, as is usually done, a random variable (r.v.) X with all others in the same equivalence class, i.e., all r.v. Y such that $P[X = Y] = 1$, and consider L^p to be the space of equivalence classes rather the the r.v. themselves, then

$$||X||_p = \left(\mathbb{E}\big[|X|^p\big]\right)^{\frac{1}{p}}$$

is a norm, called the L^p-norm. This is a Banach space and, in the case of $p = 2$, is even a Hilbert space with inner product $< X, Y >= \mathbb{E}[XY]$. If $X \in L^2$, we say that it is square integrable. The convergence in the L^2-norm is also called mean square convergence, and, given a sequence $X_n \in L^2$, we write

$$X_n \xrightarrow{m.s.} X \quad \text{or} \quad \underset{n\to+\infty}{l.i.m.} X_n = X$$

if $||X_n - X||_2 \to 0$ as $n \to +\infty$, or equivalently, if $\mathbb{E}[(X_n - X)^2] \to 0$, as $n \to +\infty$.

As usual, when X_n converges to X almost surely (i.e., when the set of $\omega-$values for which there is no convergence has probability zero), we use $X_n \to X$ a.s. or $X_n \xrightarrow{a.s.} X$ or $X_n \to X$ w.p.1 (with probability one). We denote the convergence in probability (i.e., when, for every $\delta > 0$, $P[|X_n - X| \geq \delta] \to 0$ when $n \to +\infty$) by $X_n \xrightarrow{P} X$ or $\lim_{n\to+\infty} -P \ \ X_n = X$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $I = [0, T]$, with $T > 0$, or $I = [0, +\infty)$, with $t \in I$ representing time. Let, for $t \in I$, $X(t)$ be a random variable defined on Ω with values in $(0, +\infty)$. The collection $X := (X(t), t \in I)$ is called a stochastic process. We write X, $X(t)$ or X_t whenever we refer to the stochastic process $X = (X(t), t \in T)$. We will consider only the case where the state space is $(0, +\infty)$ and the indexing set is I, being both continuous sets and, thus, $X(t)$ is a continuous-valued process in continuous time.

A stochastic process is also a function $X(t, \omega)$ defined on $T \times \Omega$. Setting ω as a fixed value results in a non-random function of t, which we call trajectory or sample path of the process. Different ω values produces different trajectories. As usual in the literature, we will omit the random dependence and write

$X(t)$ instead of $X(t, \omega)$.

A filtration $(\mathcal{F}_t, t \in I)$ is a family of sub-σ-algebras of $\mathcal{F}$ such that $s \leq t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t$. A stochastic process $X = (X(t), t \in I)$ is adapted to this filtration if $X(t)$ is $\mathcal{F}_t-$mensurable for all $t \in I$. The process X is obviously adapted to its natural filtration $\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t)$, where $\mathcal{F}_t$ is the σ-algebra generated by the present and the past of X.

We say that X_t is a stochastic process

i) with independent increments if and only if for all $n \in \mathbb{N}_0$ and for all $t_0, \ldots, t_n \in T$ such that $t_0 < \ldots < t_n$, the random variables $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

ii) with stationary increments if and only if for all $s, t \in T$ such that $s < t$, the distribution of $X_t - X_s$ depends only on the duration $t - s$.

A stochastic process X_t is a second order process if and only if for all $t \in T : \mathbb{E}[X_t^2] < +\infty$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, X_t a stochastic process and $\mathcal{F}_t$ a filtration. X_t is a $\mathcal{F}_t$-martingale if:

i) X_t is adapted to the filtration $\mathcal{F}_t$;

ii) $\mathbb{E}[|X_t|] < +\infty$;

iii) for all $s \leq t : \mathbb{E}[X_t|\mathcal{F}_s] = X_s$ a.s., where $\mathbb{E}[X_t|\mathcal{F}_s]$ is a random variable called conditional expectation of X given $\mathcal{F}_s$.

When the considered filtration coincides with the natural one, X_t is simply called martingale.

A stochastic process X_t with state space $(0, +\infty)$ is a Markov process if the following condition is satisfied:

$$\forall n \in \mathbb{N}, \forall t_1 < \ldots < t_n < t_{n+1} \in T, \forall x_1, \ldots, x_n > 0, \quad \forall B \in \mathcal{B} :$$

$$P(X_{t_{n+1}} \in B | X_{t_1} = x_1, \ldots, X_{t_n} = x_n) = P(X_{t_{n+1}} \in B | X_{t_n} = x_n),$$

where $\mathcal{B}$ is the Borel σ-algebra and B is a Borel set. In terms of conditional distributions functions, this is equivalent to

$$F_{X_{t_{n+1}}|X_{t_1}=x_1,\ldots,X_{t_n}=x_n}(x) = F_{X_{t_{n+1}}|X_{t_n}=x_n}(x), \quad \forall x \in \mathbb{R}.$$

In current language we can say that in a Markov process, given the present value, future behaviour is independent of its past.

Let X_t be a Markov process, B a Borel set, and $s \leq t$, $s, t \in I$. For the transition probabilities of X_t we can use the alternative notations

$$P_{s,x}[X_t \in B] = P[t, B|s, x] = P[X_t \in B|X_s = x]$$

and, when it exists, the transition density of X_t is

$$p(t, y|s, x) = f_{X_t|X_s=x}(y) = \frac{\partial}{\partial y} F_{X_t|X_s=x}(y)$$

where $F_{X_t|X_s=x}(y) = P_{s,x}[X_t \leq y]$ is the conditional distribution function (d.f.) of X_t given that $X_s = x$. If the conditional expectation exists, we denote it by $\mathbb{E}_{s,x}[X_t] = \mathbb{E}[X_t|X_s = x]$.

In addition, we say that the Markov process X_t is homogeneous when the transition probabilities $P[t, B|s, x]$ only depend on the duration $\tau = t - s$ and we denote them by

$$P[\tau, B|x] = P[s + \tau, B|s, x], \quad \text{for all} \quad s, s + \tau \in I.$$

If the conditional density exists, we write $p(\tau, y|x) = p(s + \tau, y|s, x)$.

A special stochastic process is the Wiener process. In 1828, the English botanist Robert Brown observed small particles of pollen immersed in a liquid moving completely randomly. Later, in 1905, Albert Einstein justified this movement with the constant collision between the particles and the surrounding liquid molecules and characterized it by a stochastic process that would come to be called Wiener process. Finally, in 1918, the first mathematical definition of the term appeared through the mathematician Norbert Wiener. A very interesting description on the history of the Wiener process can be found in Nelson (1967).

In the stochastic processes theory, the Wiener process is a model for the cumulative effect of the random perturbations in the evolution of a given phenomenon under study. Given the importance of this process, we will highlight its definition and present some of its properties.

Definition 2.1.1 *A standard Wiener process, also called Browninan motion, is a stochastic process denoted by* $W := (W_t, t \geq 0)$ *or* $B := (B_t, t \geq 0)$*, defined on a complete probability space* $(\Omega, \mathcal{F}, P)$ *fulfilling the conditions:*

i) $W_0 = 0$ *a.s.;*

ii) $W_t - W_s \sim \mathcal{N}(0, t-s), \quad 0 \leq s \leq t;$

iii) W_t *has independent increments.*

Here $\mathcal{N}(a, b)$ *denotes the Gaussian distribution with mean* a *and variance* b*. Unless otherwise stated, the process* W_t *will be simply denoted by Wiener process.*

We present now a definition required for the understanding of some properties of the Wiener process.

Definition 2.1.2 *Consider a function* $f : [0, t] \to \mathbb{R}$ *and let* $\mathcal{P}_n = \{t_0^n, t_1^n, \ldots, t_n^n\}$ *be partitions of the interval* $[0, t]$ *with* $0 = t_0^n < t_1^n < \ldots < t_n^n = t \geq 0$, $n \in \mathbb{N}$*, such that* $\delta_n = \max_{0 \leq i \leq n-1} |t_{i+1}^n - t_i^n| \to 0$ *when* $n \to +\infty$.

i) The variation of f *on* $[0, t]$ *is defined by*

$$V_f([0,t]) = V_f(t) := \lim_{n \to +\infty} \sum_{i=0}^{n-1} |f(t_{i+1}^n) - f(t_i^n)|$$

if the limit exists and is independent of the choice of the partitions $\mathcal{P}_n$.

ii) We say that f *has bounded variation on* $[0, t]$ *if* $V_f(t) < \infty$;

iii) We say that f *has quadratic variation on* $[0, t]$ *if the below limit exists and is independent of the choice of the partitions* $\mathcal{P}_n$.

$$V_f^2(t) = \lim_{n \to +\infty} \sum_{i=0}^{n-1} |f(t_{i+1}^n) - f(t_i^n)|^2.$$

Proposition 2.1.1 *The Wiener process defined above has the following properties:*

i) has a version with a.s. continuous paths (versions that we will adopt);

ii) $W_t \sim \mathcal{N}(0, t), \quad 0 \leq t;$

iii) $COV[W_s, W_t] = \mathbb{E}[W_s W_t] = s \wedge t;$

iv) is a homogeneous Markov process;

v) the conditional distribution of $W_{s+\tau}$ given $W_s = x$ is Gaussian with mean x and variance τ;

vi) W_t is a martingale;

vii) is not of bounded variation a.s. on $[0,t]$ $(t \geq 0)$.

viii) has non-differentiable paths a.s.;

ix) has a.s. quadratic variation $b-a$ on $[a,b]$.

Although the Wiener process fulfils properties (vii) and $(viii)$, there exists dW_t/dt in the sense of generalized functions. So, we can define

$$\varepsilon_t := \frac{dW_t}{dt}, \tag{2.1}$$

as a generalized stochastic process. This process is called standard white noise.

We will now define a particular class of stochastic processes named diffusion processes, which are, under certain conditions, solutions of stochastic differential equations.

Definition 2.1.3 *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $X(t)$ a Markov process defined on that space. We say that $X(t)$ is a diffusion process if it admits continuous paths a.s., has finite second order moments and, for all $x \in (0,+\infty)$ and $s \in [0,T)$ $(T>0)$, if the below limits exist uniformly in s:*

$$\begin{aligned}
\lim_{\Delta \to 0^+} \frac{P_{s,x}[|X(s+\Delta)-x| > \varepsilon]}{\Delta} &= 0, \quad \textit{for all } \varepsilon > 0,\\
\lim_{\Delta \to 0^+} \mathbb{E}_{s,x}\left[\frac{X(s+\Delta)-x}{\Delta}\right] &= a(s,x),\\
\lim_{\Delta \to 0^+} \mathbb{E}_{s,x}\left[\frac{(X(s+\Delta)-x)^2}{\Delta}\right] &= b(s,x).
\end{aligned}$$

If $a(s,x)$ and $b(s,x)$ are time-independent, we will write $a(x)$ and $b(x)$, and the process is called a homogeneous diffusion process.

Note that there are other non-equivalent definitions of diffusions processes where the existence of second-order moments is not required, and the expectations on that definitions are replaced by truncated expectations, which always exist. Our more restrictive definition will, however, be sufficient for our purposes.

The functions $a(s,x)$ and $b(s,x)$ are named respectively by drift coefficient and diffusion coefficient. The drift coefficient measures the process mean velocity at s, whereas the diffusion coefficient measures the fluctuations intensity of the process at s.

The Dirac function $\delta(x)$ is a generalized function satisfying the following properties:

i) $\delta(x) = 0, \quad x \neq 0;$

ii) $\delta(0) = +\infty;$

iii) $\int_{-\infty}^{+\infty} \delta(x)dx = 1.$

Proposition 2.1.2 *Let X_t be a diffusion process as defined above with $a(s,x)$ and $b(s,x)$ continuous functions. Assume it has probability density function (p.d.f.) $p(t,y)$ and transition density $p(t,y|s,x)$, and that the partial derivatives in the expressions below exist and are continuous. Then,*

i) $p(t,y)$ and $p(t,y|s,x)$ both satisfy the Kolmogorov forward equation (KFE):

$$\frac{\partial p}{\partial t} + \frac{\partial(a(t,x)p)}{\partial y} - \frac{1}{2}\frac{\partial^2(b(t,x)p)}{\partial y^2} = 0, \tag{2.2}$$

with initial condition $\lim_{t\downarrow s} p(t,y|s,x) = \delta(x-y)$ for $p(t,y|s,x)$ or $\lim_{t\downarrow s} p(t,y) = p(s,y)$ for $p(t,y)$;

ii) $p(t,y|s,x)$ satisfies the Kolmogorov backward equation (KBE):

$$\frac{\partial p}{\partial s} + a(s,x)\frac{\partial p}{\partial x} + \frac{1}{2}b(s,x)\frac{\partial^2 p}{\partial x^2} = 0,$$

for $s < t$ and fixed t, with terminal condition $\lim_{s\uparrow t} p(t,y|s,x) = \delta(x-y)$.

2.2 Stochastic differential equations

Ordinary differential equations have been largely used to model the behaviour of dynamical time-dependent phenomena in many scientific areas. Such dynamics, can often be characterized by the rate of change of a variable $X(t)$ and expressed as

$$dX(t) = f(t, X(t))dt, \quad X(0) = X_0. \tag{2.3}$$

There are, however, small oscillations or perturbations on $X(t)$ due to random environmental fluctuations that are not explained by f. In this work we will admit that the accumulated value, until the time instant t, of such random environmental fluctuations is described by a standard Wiener process $W(t)$, and the magnitude of the fluctuations is measured by a function g. Thus, equation (2.3) should be modified to

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t)dt, \quad X(0) = X_0, \tag{2.4}$$

where we assume that X_0 is independent of $W(t)$ for $t \geq 0$. This is a Stochastic Differential Equation (SDE).

A solution of (2.4), if it exists, is given by the integral equation

$$X(t) = X_0 + \int_0^t f(s, X(s))ds + \int_0^t g(s, X(s))dW(s), \tag{2.5}$$

where the first integral can be interpreted as a Riemann integral (defined for each sample path) but the second integral can not be interpreted as a Riemmann-Stieltjes integral for the sample paths, since the Wiener process has unbounded variation paths a.s.. Hence, the limit of the classical Riemann-Stieltjes sums depends on the choice of the intermediate point where g is calculated. However, the Wiener process has finite quadratic variation and thus one can calculate the second integral through the definition of stochastic integral.

We will define the stochastic integrals of the form

$$\int_0^t g(s, X(s))dW(s)$$

appearing in (2.5). A classical example showing that the classical Riemann-Stieltjes definition does not work for the stochastic integral is

$$\int_0^t W(s)dW(s).$$

The application of ordinary calculus rules would lead us to expect the integral to be $\frac{1}{2}W^2(t)$. However, as we will now see, this solution is incorrect.

Let $\mathcal{P}_n = \{t_0^n, t_1^n, \ldots, t_n^n\}$, $n \in \mathbb{N}$, be a sequence of partitions of $[0, t]$ with $0 = t_0^n < t_1^n < \ldots < t_n^n = t \geq 0$, such that the diameters $\delta_n = \max\limits_{0 \leq i \leq n-1} |t_{i+1}^n - t_i^n| \to 0$ when $n \to +\infty$ and let $\xi_i^n \in [t_i^n, t_{i+1}^n]$ be the intermediate points chosen for the computation of the integral function. The Riemann-Stieltjes sums that would be supposed to approximate the integral $\int_0^t W(u)dW(u)$ are

$$\sum_{i=0}^{n-1} W(\xi_i^n)(W(t_{i+1}^n) - W(t_i^n)).$$

Consider the particular cases of $\xi_i^n = (1-\lambda)t_i^n + \lambda t_{i+1}^n$, with $0 \leq \lambda \leq 1$, and the Riemann-Stieltjes sums

$$S_\lambda(W(t)) = \sum_{i=0}^{n-1} W(\xi_i^n)(W(t_{i+1}^n) - W(t_i^n)).$$

Assuming a fixed λ, the mean square limit when $n \to +\infty$ of these sums is $\frac{W^2(t)}{2} + (\lambda - \frac{1}{2})t$, so it depends on the choice of the intermediate point $\xi_i \in [t_i, t_{i+1}]$. Thus, there is no integral in the Riemann-Stieltjes sense because there is no common limit for all the intermediate point choices. Setting $\lambda = 0$, one obtains as intermediate point the initial interval point, that is, $\xi_i = t_i$, and we verify that

$$\int_0^t W(t)dW(t) = \frac{1}{2}W^2(t) - \frac{1}{2}t,$$

which is different from the one, $\frac{1}{2}W^2(t)$, obtained under ordinary calculus rules, which correspond to the case $\lambda = \frac{1}{2}$.

So, there are several different possible definitions of the stochastic integral depending on the choice of intermediate points. We will adopt, as most authors do, the choice of $\xi_i = t_i$, that is, choosing the initial point. This corresponds to the Itô integral, which has nice properties and allows us to define the integral $\int_0^t G(s)dW(s)$ of very general functions G.

The choice that follows ordinary calculus rules corresponds to $\lambda = \frac{1}{2}$ and leads to the Stratonovich integral. The difference between the choice of these two integrals is explained in Braumann (2007). In this text we will only work with the Itô integral.

We will now present the definition of the Itô integral $\int_0^t G(s)dW(s)$ for quite general non-anticipative functions G.

Let $W(t)$, $t \geq 0$ be a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{M}_s = \sigma(W(u), 0 \leq u \leq s)$ be its natural filtration. Let $\mathcal{M}_s^+ = \sigma(W(u) - W(s), u \geq s)$ be the σ-algebra generated by the future increments of the Wiener process. A filtration $\{\mathcal{A}_s : 0 \leq s \leq t\}$ is called non-anticipative if

i) $\mathcal{A}_s \supset \mathcal{M}_s, \quad 0 \leq s \leq t;$

ii) $\mathcal{A}_s$ is independent of $\mathcal{M}_s^+$, $s \geq 0$.

The choice of the non-anticipative filtration $\mathcal{A}_s$ usually coincides with the natural filtration of the Wiener process $\mathcal{M}_s$, provided that it is unnecessary to include additional information about the process. Otherwise, a larger filtration can be considered in order to include, for instance, an initial condition, provided that the this new filtration is non-anticipative.

A stochastic process $G(t)$ is called non-anticipative, with respect to the filtration $\mathcal{A}_t$, if $G(t)$ is adapted to the filtration, i.e., $\mathcal{A}_t$-mensurable, for all $t \geq 0$. In other words, $G(t)$ depends only on the information available up to and including time t.

We are now in a position to define the Itô integral for a special class of non-anticipative functions.

For $t \in I$ denoted by $H^2[0,t]$ the space of functions $G : [0,t] \times \Omega \to \mathbb{R}$ which satisfy the following conditions:

i) G is jointly measurable with respect to the Lebesgue measure l in $[0,t]$ and to the probability measure P;

ii) G is non-anticipative;

iii) $\int_0^t \mathbb{E}[G^2(s,\omega)]ds < +\infty$.

This is a Hilbert space w.r.t. the norm $||G||_{H^2[0,t]} = \left(\int_0^t \mathbb{E}[G^2(s)]ds\right)^{\frac{1}{2}} = \left(\mathbb{E}\left[\int_0^t G^2(s)ds\right]\right)^{\frac{1}{2}}$, its inner product being $< G_1, G_2 >= \left(\int_0^t \mathbb{E}[G_1(s)G_2(s)]ds\right)^{\frac{1}{2}}$.

Observation 2.2.1 *For this to be a norm and $H_2[0,t]$ to be a Hilbert space, we should work with the equivalence class of G w.r.t. the $l \times P-$ almost equality instead of the functions G themselves, but we make the usual convention of identifying two functions G_1 and G_2 that are almost equal, i.e., G_1 and G_2 are equal with the possible exception of a set N of (s,ω) values with product $l \times P-$measure zero.*

We will start by defining the integral for step functions $G \in H_2[0,t]$.

A function G defined on $H^2[0,t]$ is called a step function if there exists a partition $0 = t_0 < t_1 < \ldots < t_n = t$ on $[0,t]$ such that $G(t) = G(t_i)$, $t_i \leq t < t_{i+1}$, $i = 0, \ldots, n-1$. We denote by $H_E^2[0,t]$ the sub-space of step functions of $H^2[0,t]$. The Itô integral of $G \in H_E^2[0,t]$ on $[0,t]$ is defined by

$$\int_0^t G(s)dW(s) := \sum_{i=0}^{n-1} G(t_i)(W(t_{i+1}) - W(t_i)).$$

This definition is consistent, i.e., choosing a different partition or a $l \times P-$almost equal function G will lead to an almost surely equal integral.

Proposition 2.2.1 *Let F and G be two functions on $H^2_E[0,t]$ and $\alpha, \beta \in \mathbb{R}$ two real constants. The following properties hold:*

i) $\int_0^t (\alpha F(s) + \beta G(s))\, dW(s) = \alpha \int_0^t F(s) dW(s) + \beta \int_0^t G(s) dW(s)$;

ii) $\mathbb{E}\left[\int_0^t F(s) dW(s)\right] = 0$;

iii) Norm preservation:
$\mathbb{E}\left[\left(\int_0^t F(s) dW(s)\right)^2\right] = \mathbb{E}\left[\int_0^t F^2(s) ds\right] = \int_0^t \mathbb{E}\left[F^2(s)\right] ds$, *i.e.,* $||F||_{H_2[0,t]} = \left|\left|\int_0^t \mathbb{E}\left[F^2(s)\right] ds\right|\right|_2$;

iv) $\mathbb{E}\left[\int_0^t F(s) dW(s) \int_0^t G(s) dW(s)\right] = \mathbb{E}\left[\int_0^t F(s) G(s) ds\right]$.

So far we have defined Itô integral for a special class of functions - the step functions defined on $H^2_E[0,t]$. Next we will generalize this integral to a class of general functions on $H^2[0,t]$ through the existence of approximate successions of step functions.

Theorem 2.2.1 *Let $G \in H^2[0,t]$. There exists a sequence of step function $G_n \in H^2_E[0,t]$ such that G_n converges to G in the $H^2[0,t]$ norm.*

Let G and G_n defined as in the previous theorem. The Itô integral of G on $[0,t]$ is defined by

$$\int_0^t G(s) dW(s) := \underset{n \to +\infty}{l.i.m.} \int_0^t G_n(s) dW(s).$$

This definition is consistent, i.e., choosing a different approximating sequence G_n will lead to an almost surely equal integral.

Proposition 2.2.2 *Let F and G be two functions defined on $H^2[0,t]$. Then, properties i) to iv) of Proposition 2.2.1 still hold.*

Proposition 2.2.3 *If $G \in H^2[0,t]$ is deterministic, then*

$$\int_0^t G(s) dW(s) \sim \mathcal{N}\left(0, \int_0^t G^2(s) ds\right).$$

For $G \in H^2[0,T]$, we can study the indefinite integral $Z(t) = \int\limits_0^t G(s)dW(s) = \int\limits_0^t G(s)I_{[0,t]}(s)dW(s)$ for $t \in [0,T]$ as a function of the upper limit t. A detailed study on this subject can be found in Øksendal (1998) and in Braumann (2005).

Proposition 2.2.4 *$Z(t)$ satisfies the following properties:*

i) $Z(t)$ is a $\mathcal{A}_t-$martingale;

ii) $Z(t)$ has continuous paths a.s.;

iii) $Z(t)$ has non-correlated increments.

The Itô integral can be extended to functions G belonging to the wider space $M^2[0,T]$. We say that $G(s,\omega)$ is in the space $M^2[0,t]$ if:

i) G is jointly measurable;

ii) G is non-anticipative w.r.t. the filtration $\mathcal{A}_s$;

iii) the integral $\int_0^t G^2(s)ds$ exists and is finite a.s..

Observation 2.2.2 *The property $\int_0^t G^2(s)ds < +\infty$ is less restrictive than the one presented in the definition of the H^2 space. Thus, $H^2[0,t] \subset M^2[0,t]$.*

Theorem 2.2.2 *Let $G \in M^2[0,t]$. Then, there exists a sequence of step functions $G_n \in H_E^2[0,t]$, such that*

$$\int_0^t (G(s) - G_n(s))^2 ds \to 0 \quad a.s, \quad n \to +\infty.$$

Let G and G_n as is the last theorem. The Itô integral of G on $[0,t]$ is defined by

$$\int_0^t G(s)dW(s) = \lim_{n \to +\infty} - P \int_0^t G_n(s)dW(s).$$

After the presentation of the Itô integral, it is now necessary to introduce the rules for the calculation of these integrals: the Itô calculus.

The Itô calculus differs from the usual calculus by simply introducing a new differentiation rule - the Itô chain rule. Let us present the definition of an Itô process and the Itô Theorem, which describes the Itô chain rule.

Definition 2.2.1 *Let $W(t)$, $t \geq 0$ be a Wiener process, X_0 a r.v. $\mathcal{A}_0$-mensurable (which means that it is independent of the Wiener process), F a non-anticipative function such that $\int_0^T |F(s)|ds < +\infty$ a.s. and $G \in M^2[0,T]$. The process $X(t,\omega)$ is called an Itô process definied for $t \in [0,T]$ if it can be written as*

$$X(t,\omega) = X_0(\omega) + \int_0^t F(s,w)ds + \int_0^t G(s,w)dW(s),$$

or, equivalently in differential form (dropping out the explicit reference to the random dependency),

$$dX(t) = F(t)dt + G(t)dW(t).$$

Theorem 2.2.3 (Itô Theorem) *Let $X(t)$ be an Itô process as stated before and $h(t,x)$ a function continuously differentiable with respect to t and x and twice continuously differentiable with respect to x. Let $Y(t) = h(t, X(t))$. Then:*

i) $Y(t)$ is an Itô process with initial condition $Y_0 = h(0, X_0)$;

ii) the differential form of $Y(t)$ is given by the Itô chain rule (Itô formula)

$$dY(t) = \left(\frac{\partial h(t,X(t))}{\partial t} + \frac{\partial h(t,X(t))}{\partial x}F(t) + \frac{1}{2}\frac{\partial^2 h(t,X(t))}{\partial x^2}G^2(t)\right)dt + \frac{\partial h(t,X(t))}{\partial x}G(t)dW(t),$$

which, written in the integral form reads

$$Y(t) = Y_0 + \int_0^t \left(\frac{\partial h(s,X(s))}{\partial s} + \frac{\partial h(s,X(s))}{\partial x}F(s) + \frac{1}{2}\frac{\partial^2 h(s,X(s))}{\partial x^2}G^2(s)\right)ds + \int_0^t \frac{\partial h(s,X(s))}{\partial x}G(s)dW(s).$$

With the stochastic integrals properly defined, a solution of (2.4), or equivalently of the stochastic integral equation (2.5), is well defined, and the Itô Theorem allows us to use change of variable techniques that, in some cases, will lead us to an explicit expression for the solution.

Definition 2.2.2 *The stochastic process $X(t)$, $t \in [0,T]$, is a solution of*

$$dX(t) = f(t,X(t))dt + g(t,X(t))dW(t), \quad X(0) = X_0, \quad t \in [0,T]$$

if:

i) $X(t)$ is $\mathcal{F}_t$-mensurable;

ii) $X(t)$ is a.s. continuous;

iii) $F(t,\omega) = f(t, X(t,\omega))$ *is non-anticipative and* $\int_0^T |F|dt < +\infty$ *a.s.;*

iv) $G(t,\omega) = g(t, X(t,\omega)) \in M^2[0,T]$*;*

v) $X(t) = X_0 + \int_0^t F(s, X(s))ds + \int_0^t G(s, X(s))dW(t) \quad a.s., \quad t \in [0,T].$

Theorem 2.2.4 *Let L be a positive constant and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ be two functions satisfying the:*

i) Lipschitz condition

$$|f(t,x) - f(t,y)| \leq L|x-y| \quad \textit{and} \quad |g(t,x) - g(t,y)| \leq L|x-y|, \quad 0 \leq t \leq T, \quad x, y \in \mathbb{R}$$

and the

ii) restriction on growth

$$|f(t,x)| \leq L(1+|x|) \quad \textit{and} \quad |g(t,x)| \leq L(1+|x|), \quad 0 \leq t \leq T, \quad x \in \mathbb{R}.$$

Let X_0 be a square integrable random variable independent of the future increments of the Wiener process. There exists a unique a.s. continuous solution $X(t)$ of

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \quad X(0) = X_0, \quad t \in [0,T]. \tag{2.6}$$

The solution $X(t)$ is a Markov process and, if f and g are continuous with respect to t, $X(t)$ is also a diffusion process with drift coefficient $a(t,x) = f(t,x)$ and diffusion coefficient $b(t,x) = g^2(t,x)$. The uniqueness can be understood as: if $X(t)$ and $Y(t)$ are both solutions of equation (2.6), then

$$P\left[\sup_{0 \leq t \leq T} \left|X(t) - Y(t)\right|\right] = 1.$$

Corolary 2.2.1 *If equation (2.6) is autonomous, that is, $f(t,y) \equiv f(y)$ and $g(t,y) \equiv g(y)$, where f and g are continuously differentiable, then there exists a unique solution of equation (2.6) up to a possible explosion time, which is a homogeneous diffussion process with drift coefficient $f(y)$ and diffusion coefficient $g^2(y)$. If, with probability one, this time is infinite, then the solution exists and is unique for all $t \geq 0$.*

2.3 Boundary classification and stationary density

Since we are dealing with population sizes, our state space will be set as $(0, +\infty)$. Let $X(t)$ be an homogeneous diffusion process solution of the

$$dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(0) = x,$$

with state space $(0, +\infty)$ and boundaries $X = 0$ and $X = +\infty$. The drift and diffusion coefficients of $X(t)$ are, respectively, $a(x) = f(x)$ and $b(x) = g^2(x) > 0$. We suppose that a and b are continuous functions with respect to x and also that $X(t)$ is a regular process, that is, all the sates in the interior of $(0, +\infty)$ communicate between them. In other words,

$$P_x[T_y < +\infty] > 0, \quad x < y, \quad x, y \in (0, +\infty),$$

where $T_y := \inf\{t \geq 0 : X(t) = y\}$ denotes the first passage time of $X(t)$ by y, allowing $T_y = +\infty$ when the process does not reach y.

The following functions and definitions are useful for the boundary characterization in terms of attractiveness. In Karlin and Taylor (1981) there is an exhaustive treatment of boundaries classification.

Let us consider two measures defined in $(0, +\infty)$: the scale measure S and the speed measure M, which density functions are given respectively by the scale density $s(z)$ and the speed density $m(z)$:

$$s(z) = \exp\left(-\int_{z_0}^{z} \frac{2a(\theta)}{b(\theta)} d\theta\right), \tag{2.7}$$

$$m(z) = \frac{1}{b(z)s(z)}, \tag{2.8}$$

where $z, z_0 \in (0, +\infty)$ and z_0 is an arbitrary constant. From the density functions, one can define the scale and speed functions

$$S(x) = \int_{x_0}^{x} s(z)dz \tag{2.9}$$

and

$$M(x) = \int_{x_0}^{x} m(z)dz, \tag{2.10}$$

where $x, x_0 \in (0, +\infty)$ and x_0 is an arbitrary constant. These functions are similar to distribution functions in the sense that the speed and scale measures of intervals of the form $(c, d]$ $(c, d \in (0, +\infty))$ are determined by

$$S(c, d] = S(d) - S(c) \quad \text{and} \quad M(c, d] = M(d) - M(c).$$

Definition 2.3.1 *The boundary $X = 0$ is non-attractive if $P_x[T_{0^+} < T_b] = 0$ for all $0 < x < b < +\infty$, and is attractive if $P_x[T_{0^+} \leq T_b] > 0$ for all $0 < x < b < +\infty$. The boundary $X = +\infty$ is non-attractive if $P_x[T_{+\infty} < T_a] = 0$ for all $0 < a < x < +\infty$, and is attractive if $P_x[T_{+\infty} \leq T_a] > 0$ for all $0 < a < x < +\infty$. Here, $T_{0^+} = \lim_{y \to 0^+} T_y$ and $T_{+\infty} = \lim_{y \to +\infty} T_y$.*

When a boundary is classified as non-attractive one can say that, when the process falls into its neighbourhood, it tends to be pushed to the opposite side. The non-attractiveness of $X = 0$ implies that the process is pushed to the right. This implies that it is not possible to have $X(t) = 0$ for some finite t, nor $X(t) \to 0$ as $t \to +\infty$. On the other hand, the non-attractiveness of $X = +\infty$ implies that the process tends to be pushed to the left when it reaches high values. This implies that it is not possible to have $X(t) = +\infty$ for some finite t nor $X(t) \to +\infty$ as $t \to +\infty$. In the first case, we say that there is non-extinction from the mathematical point of view and, in the second case we say that there are no explosions.

Proposition 2.3.1 *Let x be any point in $(0, +\infty)$. The boundary $X = 0$ is attractive if $S(0, x] < +\infty$ and non-attractive if $S(0, x] = +\infty$. The boundary $X = +\infty$ is attractive if $S(x, +\infty) < +\infty$ and non-attractive if $S(x, +\infty) = +\infty$. It suffices to verify the property for some $x > 0$ (see, for instance, Karlin and Taylor, 1981).*

The non-attractiveness of both boundaries implies that trajectories tend to be pushed towards the interior of the state space whenever they approach the boundaries. Given that the process is regular, that is, all the states communicate with each other, the transient distribution may have a probability density given by $p(t, y) = f_{X(t)}(y)$. Let us admit the existence of this transient distribution and also that it converges to a limit distribution having p.d.f. $p(y)$ when t tends to infinity. Then, $p(y)$ is called the stationary density. Let $X_{+\infty}$ be the steady-state random variable with probability density given by $p(y) = f_{X_{+\infty}}(y)$.

The stationary density p, when it exists, is time-invariant and so satisfies the Kolmogorov forward equation (2.2) with $\partial p(y)/\partial t = 0$, i.e.,

$$\frac{d}{dy}\left(p(y)a(y)\right) - \frac{1}{2}\frac{d^2}{dy^2}\left(p(y)b(y)\right) = 0.$$

Therefore, after integration with respect to y,

$$p(y)a(y) - \frac{1}{2}\frac{d}{dy}\left(p(y)b(y)\right) = e_1,$$

where e_1 is a real constant. Multiplying this equation by the integrating factor $s(y)$ yields

$$2s(y)p(y)a(y) - s(y)\frac{d}{dy}\left(p(y)b(y)\right) = 2e_1 s(y),$$

or, taking into account (2.7),

$$\frac{d}{dy}\left(s(y)p(y)b(y)\right) = e_2 s(y),$$

where e_2 is a real constant. Integrating again in order to y results in

$$s(y)p(y)b(y) = e_3 S(y) + d,$$

with e_3 and d real constants. Since the boundaries are non-attractive, $s(y)$ is not integrable around its neighborhood and we have $S(y) \to +\infty$ when $y \uparrow +\infty$ and $S(y) \to -\infty$ when $y \downarrow 0$. Therefore, it is only possible to have $p(y) \geq 0$ for all $y \in (0, +\infty)$ when $e_3 = 0$ and consequently we have

$$s(y)p(y)b(y) = d,$$

or, in a equivalent manner and taking into account (2.8),

$$p(y) = Dm(y).$$

When both boundaries are non-attractive and m is integrable over the state space $(0, +\infty)$, then $p(y)$ becomes a probability density with $\frac{1}{D} = \int_0^{+\infty} m(y)dy$, i.e.,

$$p(y) = \frac{m(y)}{\int\limits_0^{+\infty} m(z)dz}, \quad y \in (0, +\infty).$$

It can be proved (see Gihman and Skorohod, 1979) that, under these conditions, $X(t)$ converges in distribution to $X_{+\infty}$ having $p(y)$ as p.d.f..

One consequence of the stationary density existence and the boundaries non-attractiveness is the ergodicity of $X(t)$ (see Gihman and Skorohod, 1979, Braumann, 2005). The great advantage of this property is the estimation of ensemble moments at the steady-state through the time-average moments of the observed trajectory instead of having to average over a set of trajectories. This is particular useful, since in reality there is only one observed trajectory of a real population.

3

Optimal policy with variable effort

In this chapter we present an optimal policy problem with variable effort and solve it by applying the stochastic dynamical programming technique. We consider a general population growth model with dynamics described by a stochastic differential equation and a profit structure based on quadratic polynomials. The applied technique produces a non-linear stochastic partial differential equation which needs to be solved by numerical methods. We apply a discretization scheme to this equation and show how to reach the optimal policy, namely what is the effort that produces maximum profit. This chapter is organized as follows: in section 3.1 we formulate and solve the optimal policy problem by considering a general growth model. We end up with chapter conclusions at section 3.2.

3.1 General growth model

In a random environment the growth dynamics of a population subject to harvesting can be described by the Stochastic Differential Equation

$$dX(t) = f(X(t))X(t)dt - H(t)dt + \sigma X(t)dW(t), \quad X(0) = x > 0, \tag{3.1}$$

where $X(t)$ is the population size at time t, measured as biomass or as number of individuals, $f(X(t))$ is the average *per capita* natural growth rate, $H(t)$ is the harvesting rate, σ measures the strength of environmental fluctuations, $W(t)$ is a standard Wiener process and $x > 0$ is the population size at time 0, which we assume known.

The harvesting rate $H(t)$ depends on the vessels fishing capacity and, in particular, depends on the fishing gears used, nets mesh size, number and type of vessels, number of hours at sea, among others. We will use the most traditional form for H, which has been (e.g. Clark, 1990, Hanson and Ryan, 1998, Suri, 2008, Li and Wang, 2010, Ewald and Wang, 2010, Li et al., 2011, Kar and Chakraborty, 2011)

$$H(t) = qE(t)X(t), \tag{3.2}$$

where $q > 0$ is a constant representing the fraction of biomass harvested per unit of effort and per unit time and $E(t)$ corresponds to the effort exerted on the population at time instant t. We assume $E(t)$ to be non-anticipating, i.e., it only depends on information available up to time t (included). According to the Food and Agriculture Organization of the United Nations (FAO-CWP, 1990), the effort is defined as 'The amount of fishing gear of a specific type used on the fishing grounds over a given unit of time e.g. hours trawled per day, number of hooks set per day or number of hauls of a beach seine per day'. Thus, fishing effort is always non-negative, that is, $E(t) \geq 0$, and can even have a minimum value E_{min}, related to a fixed effort. On the other hand, the number of gears, hours, vessels and manpower (just to name a few) are finite and limited, so we consider the effort to be constrained as

$$0 \leq E_{min} \leq E(t) \leq E_{max} < \infty. \tag{3.3}$$

The profit per unit time can be defined as the difference between sales revenues and fishing costs, i.e.,

$$\Pi(t) := P(t) - C(t) = p(H(t))H(t) - c(E(t))E(t), \tag{3.4}$$

where $P(t)$ is the sale price per unit time from the harvested fish, $C(t)$ is the cost per unit time derived from fishing with effort $E(t)$, $p(H(t))$ denotes the price per unit yield and $c(E(t))$ is the cost per unit effort. Fish prices per unit yield tend to follow the law of downward-sloping demand (see Samuelson and Nordhaus, 2010), that is, the price per unit yield tends to be higher for low values of yield and tends to be lower when yield increases. The cost per unit effort has an opposite behaviour to the price per unit yield. In fact, as the effort increases so does its unit cost, since, at high harvesting efforts, less efficient vessels and fishing technologies may have to be used or overtime higher wages payments may be required (see Clark, 1990, Suri, 2008).

We will assume that the unit prices and costs have the form

$$p(H(t)) = p_1 - p_2 H(t) \quad \text{and}$$

$$c(E(t)) = c_1 + c_2 E(t),$$

where $p_1 \geq 0$, $p_2 \geq 0$, $c_1 \geq 0$ and $c_2 > 0$ are constants. Thus, taking into account (3.2), the profit per unit time becomes

$$\begin{aligned} \Pi(t) &= (p_1 - p_2 H(t))H(t) - (c_1 + c_2 E(t))E(t) \\ &= (p_1 q X(t) - c_1)E(t) - (p_2 q^2 X^2(t) + c_2)E^2(t) \\ &= \big(p_1 q X(t) - c_1 - (p_2 q^2 X^2(t) + c_2)E(t)\big)E(t). \end{aligned}$$

Given the stochastic nature of $X(t)$, the expected profit per unit time is defined as

$$\mathbb{E}\left[\Pi(t)\right] := \mathbb{E}\Big[\big(p_1 q X(t) - c_1 - (p_2 q^2 X^2(t) + c_2)E(t)\big)E(t)\Big].$$

For our purposes, we assume that harvesting begins at time $t = 0$ and the corresponding population size is $X(0) = x > 0$. Furthermore, harvesting continues up to the time horizon $T < +\infty$ and we work with the profit present value, i.e., future profits are discounted by a rate $\delta > 0$ accounting for interest rate and cost of opportunity losses and for other social rates. For a time t in the time interval $[0, T]$, we define

$$J(y,t) := \mathbb{E}\left[\int_t^T e^{-\delta(\tau - t)}\Pi(\tau)d\tau \middle| X(t) = y\right] := \mathbb{E}_{t,y}\left[\int_t^T e^{-\delta(\tau - t)}\Pi(\tau)d\tau\right], \tag{3.5}$$

which is, at time t, the expected discounted future profits when the population size at that time is y. The optimal policy is to optimize the expected accumulated discounted profit earned by the harvester in the interval $[0, T]$, that is, to optimize

$$\begin{aligned} V := J(x,0) &= \mathbb{E}_x\left[\int_0^T e^{-\delta\tau}\Pi(\tau)d\tau\right] \\ &= \mathbb{E}_x\left[\int_0^T e^{-\delta\tau}\Big(p_1qX(\tau) - c_1 - (p_2q^2X^2(\tau) + c_2)E(\tau)\Big)E(\tau)d\tau\right], \end{aligned} \tag{3.6}$$

where $\mathbb{E}_x[\ldots]$ abbreviates $\mathbb{E}_{0,x}[\ldots] = \mathbb{E}[\ldots|X(0) = x]$.

The state variable is $X(t)$ and $E(t)$ is the control variable. Thus, the optimization is carried out with respect to $E(t)$. The optimization problem can be solved by stochastic dynamic programming, which is a technique for solving sequential optimization problems, that is, it turns a complex problem into a sequence of simpler problems. The solution of the complex problem will be a combination of solutions of the simplest problems.

The expression 'dynamic programming' was created by the pioneering work of Richard Bellman in 1957 (see Bellman, 1957), and was based on the Principle of Optimality whose statement can be written as

From any point of the optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point.

Our problem is to find the effort that maximizes V, subject to the population growth dynamics given by equation (3.1) with harvesting rate $H(t) = qE(t)X(t)$ and to the constraints on effort given by the set of inequalities (3.3). In addition, from (3.5) we get $J(X(T), T) = 0$, which is a boundary condition. Summing up, the stochastic optimal control problem consists in determining

$$V^* := J^*(x,0) = \max_{\substack{E(\tau)\\ 0\leq\tau\leq T}} \mathbb{E}_x\left[\int_0^T e^{-\delta\tau}\Big(p_1qX(\tau) - c_1 - (p_2q^2X^2(\tau) + c_2)E(\tau)\Big)E(\tau)d\tau\right], \tag{3.7}$$

subject to

$$dX(t) = f(X(t))X(t)dt - qE(t)X(t)dt + \sigma X(t)dW(t), \quad X(0) = x, \tag{3.8}$$

$$0 \leq E_{min} \leq E(t) \leq E_{max} < \infty,$$

$$J^*(X(T), T) = 0.$$

The maximizer, i.e., the effort function $E(t)$ that leads to the maximum V^*, will be called the optimal variable effort and will be denoted by $E^*(t)$.

In order to obtain the Hamilton-Jacobi-Bellman equation (see Hanson, 2007), one needs to make appropriate assumptions and use the following approximations for small $\Delta t > 0$, which lead to error terms of $o(\Delta t)$ as $\Delta t \to 0$, being careful to use Itô calculus, i.e., to use a Taylor expansion of second order in x:

(A) Δt is a small positive quantity;

(B) $e^{-\delta \Delta t} \approx 1 - \delta \Delta t$;

(C) $X(t + \Delta t) \approx X(t) + \Delta X(t)$;

(D) $\Delta X(t) \approx f(X(t))X(t)\Delta t - qE(t)X(t)\Delta t + \sigma X(t)\Delta W(t)$;

(E) $J^*(X(t + \Delta t), t + \Delta t)$ is known;

(F) during the time interval $[t, t + \Delta t]$ the control $E(t)$ is constant.

Now, let us consider the current value form of equation (3.7)

$$J^*(X(t), t) = \max_{\substack{E(\tau) \\ t \le \tau \le T}} \mathbb{E}_{t,X(t)} \left[\int_t^T e^{-\delta(\tau - t)} \Big(p_1 q X(\tau) - c_1 - (p_2 q^2 X^2(\tau) + c_2) E(\tau) \Big) E(\tau) d\tau \right], \tag{3.9}$$

and divide the integrand function in two parts as follows, with $t + \Delta t < T$,

$$\begin{aligned} J^*(X(t), t) = \max_{\substack{E(\tau) \\ t \le \tau \le T}} \mathbb{E}_{t,X(t)} \Bigg[& \int_t^{t+\Delta t} e^{-\delta(\tau - t)} \Big(p_1 q X(\tau) - c_1 - (p_2 q^2 X^2(\tau) + c_2) E(\tau) \Big) E(\tau) d\tau \\ & + \int_{t+\Delta t}^T e^{-\delta((\tau - t) + \Delta t - \Delta t)} \Big(p_1 q X(\tau) - c_1 - (p_2 q^2 X^2(\tau) + c_2) E(\tau) \Big) E(\tau) d\tau \Bigg]. \end{aligned}$$

Using assumption (B), one can write

$$\begin{aligned} J^*(X(t), t) = \max_{\substack{E(\tau) \\ t \le \tau \le T}} \mathbb{E}_{t,X(t)} \Bigg[& \int_t^{t+\Delta t} e^{-\delta(\tau - t)} \Big(p_1 q X(\tau) - c_1 - (p_2 q^2 X^2(\tau) + c_2) E(\tau) \Big) E(\tau) d\tau \\ & + (1 - \delta \Delta t) \int_{t+\Delta t}^T e^{-\delta((\tau - (t + \Delta t))} \Big(p_1 q X(\tau) - c_1 - (p_2 q^2 X^2(\tau) + c_2) E(\tau) \Big) E(\tau) d\tau \Bigg] + o(\Delta t). \end{aligned} \tag{3.10}$$

By applying Bellman's principle of optimality (see Bellman, 1957, Chiang, 1992) we get

$$\begin{aligned}J^*(X(t),t) = &\max_{\substack{E(\tau)\\ t\le\tau\le t+\Delta t}} \mathbb{E}_{t,X(t)}\Bigg[\int_t^{t+\Delta t} e^{-\delta(\tau-t)}\Big(p_1 qX(\tau) - c_1 - (p_2q^2X^2(\tau)+c_2)E(\tau)\Big)E(\tau)d\tau \\ &+ (1-\delta\Delta t)\max_{\substack{E(\tau)\\ t+\Delta t\le\tau\le T}}\int_{t+\Delta t}^{T} e^{-\delta((\tau-(t+\Delta t))}\Big(p_1 qX(\tau) - c_1 - (p_2q^2X^2(\tau)+c_2)E(\tau)\Big)E(\tau)d\tau\Bigg] \\ &+ o(\Delta t),\end{aligned}$$

or, equivalently, using assumptions (C) and (F),

$$\begin{aligned}J^*(X(t),t) = \max_{\substack{E(\tau)\\ t\le\tau\le T}} \mathbb{E}_{t,X(t)}\Big[&\big(p_1qX(t) - c_1 - (p_2q^2X^2(t)+c_2)E(t)\big)E(t)\Delta t \\ &+ (1-\delta\Delta t)J^*(X(t)+\Delta X(t), t+\Delta t)\Big] + o(\Delta t).\end{aligned} \tag{3.11}$$

A Taylor expansion of $J^*(X(t)+\Delta X(t), t+\Delta t)$ around $(X(t),t)$ gives

$$\begin{aligned}J^*(X(t)+\Delta X(t), t+\Delta t) = J^*(X(t),t) &+ \frac{\partial J^*(X(t),t)}{\partial t}\Delta t + \frac{\partial J^*(X(t),t)}{\partial X(t)}\Delta X(t) \\ &+ \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}(\Delta X(t))^2 + o(\Delta t).\end{aligned} \tag{3.12}$$

Replacing in (3.12) $\Delta X(t)$ by the approximation in (D), we get

$$\begin{aligned}J^*(X(t)+\Delta X(t), t+\Delta t) = J^*(X(t),t) &+ \frac{\partial J^*(X(t),t)}{\partial t}\Delta t \\ &+ \frac{\partial J^*(X(t),t)}{\partial X(t)}\Big(f(X(t))X(t)\Delta t - qE(t)X(t)\Delta t + \sigma X(t)\Delta W(t)\Big) \\ &+ \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}\Big(f(X(t))X(t)\Delta t - qE(t)X(t)\Delta t + \sigma X(t)\Delta W(t)\Big)^2 \\ &+ o(\Delta t).\end{aligned}$$

After some simplifications we can write

$$
\begin{aligned}
J^*(X(t)+\Delta X(t),t+\Delta t) = J^*(X(t),t) &+ \frac{\partial J^*(X(t),t)}{\partial t}\Delta t \\
&+ \frac{\partial J^*(X(t),t)}{\partial X(t)}\Big(f(X(t)) - qE(t)\Big)X(t)\Delta t \\
&+ \frac{\partial J^*(X(t),t)}{\partial X(t)}\sigma X(t)\Delta W(t) + \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}\sigma^2 X^2(t)(\Delta W(t))^2 \\
&+ \frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}\Big(f(X(t)) - qE(t)\Big)\sigma X^2(t)\Delta W(t)\Delta t + o(\Delta t). \qquad (3.13)
\end{aligned}
$$

Replacing $J^*(X(t)+\Delta X(t),t+\Delta t)$ from (3.13) into (3.11) gives

$$
\begin{aligned}
J^*(X(t),t) = \max_{E(t)} \mathbb{E}_{t,X(t)}\Bigg[&\Big(p_1 qX(t) - c_1 - (p_2 q^2 X^2(t) + c_2)E(t)\Big)E(t)\Delta t \\
&+ (1-\Delta dt)\Bigg(J^*(X(t),t) + \frac{\partial J^*(X(t),t)}{\partial t}\Delta t \\
&+ \frac{\partial J^*(X(t),t)}{\partial X(t)}\Big(f(X(t)) - qE(t)\Big)X(t)\Delta t \\
&+ \frac{\partial J^*(X(t),t)}{\partial X(t)}\sigma X(t)\Delta W(t) + \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}\sigma^2 X^2(t)(\Delta W(t))^2 \\
&+ \frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}\Big(f(X(t)) - qE(t)\Big)\sigma X^2(t)\Delta W(t)\Delta t + o(\Delta t)\Bigg)\Bigg]. \qquad (3.14)
\end{aligned}
$$

From the Wiener process properties we know that $\mathbb{E}_{t,X(t)}[\Delta W(t)] = 0$ and $\mathbb{E}_{t,X(t)}[(\Delta W(t))^2] = \Delta t$. Thus, rearranging the latter equation gives

$$
\begin{aligned}
0 = \max_{E(t)}\Bigg\{&\Big(p_1 qX(t) - c_1 - (p_2 q^2 X^2(t) + c_2)E(t)\Big)E(t)\Delta t - \delta J^*(X(t),t)\Delta t + \frac{\partial J^*(X(t),t)}{\partial t}\Delta t \\
&+ \frac{\partial J^*(X(t),t)}{\partial X(t)}\Big(f(X(t)) - qE(t)\Big)X(t)\Delta t + \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}\sigma^2 X^2(t)\Delta t + o(\Delta t)\Bigg\}. \qquad (3.15)
\end{aligned}
$$

Dividing (3.15) by Δt and letting $\Delta t \to 0$ results in

$$
\begin{aligned}
-\frac{\partial J^*(X(t),t)}{\partial t} = \max_{E(t)}\Bigg\{&\Big(p_1 qX(t) - c_1 - (p_2 q^2 X^2(t) + c_2)E(t)\Big)E(t) - \delta J^*(X(t),t) \\
&+ \frac{\partial J^*(X(t),t)}{\partial X(t)}\Big(f(X(t)) - qE(t)\Big)X(t) + \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}\sigma^2 X^2(t)\Bigg\}, \qquad (3.16)
\end{aligned}
$$

where $J^*(X(t),t)$ represents the expected value of the maximized accumulated discounted profit earned when harvesting is initiated at time t up to the terminal time T.

Equation (3.16) is the so-called Hamilton-Jacobi-Bellman equation (HJB) and it is the solution of the stochastic control problem (3.7). One particularity of this equation is its deterministic nature although resulting from a stochastic optimal control problem. Usually it is a non-linear equation and its resolution resorts to numerical methods, as we will see in the next section.

The optimal variable effort is obtained from the HJB equation (3.16). Let D be a function that represents the control switching term present in (3.16), that is,

$$D(E) = \max_{E(t)} \left\{ \big(p_1 q X(t) - c_1 - (p_2 q^2 X^2(t) + c_2) E(t)\big) E(t) - \frac{\partial J^*(X(t),t)}{\partial X(t)} q E(t) X(t) \right\}, \tag{3.17}$$

and denote by $E^*_{free}(t)$ the unconstrained effort resulting from the maximization carried out in equation (3.17). Thus, $E^*_{free}(t)$ is obtained by solving the equation $dD(E)/dE = 0$ with respect to E, which gives

$$E^*_{free}(t) = \frac{\left(p_1 - \frac{\partial J^*(X(t),t)}{\partial X(t)}\right) q X(t) - c_1}{2\left(p_2 q^2 X^2(t) + c_2\right)}.$$

Representing the constrained optimal effort by $E^*(t)$ and replacing $E(t)$ by $E^*(t)$ in equation (3.16) yields the maximized HJB

$$\begin{aligned} -\frac{\partial J^*(X(t),t)}{\partial t} &= (p_1 q X(t) - c_1) E^*(t) - (p_2 q^2 X^2(t) + c_2) E^{*2}(t) - \delta J^*(X(t),t) \\ &+ \frac{\partial J^*(X(t),t)}{\partial X(t)} \Big(f(X(t)) - q E^*(t) \Big) X(t) + \frac{1}{2} \frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)} \sigma^2 X^2(t), \end{aligned} \tag{3.18}$$

where the effort is given by

$$E^*(t) = \begin{cases} E_{min}, & if \quad E^*_{free}(t) < E_{min} \\ E^*_{free}(t), & if \quad E_{min} \leq E^*_{free}(t) \leq E_{max} \\ E_{max}, & if \quad E^*_{free}(t) > E_{max}, \end{cases} \tag{3.19}$$

with

$$E^*_{free}(t) = \frac{\left(p_1 - \frac{\partial J^*(X(t),t)}{\partial X(t)}\right) q X(t) - c_1}{2\left(p_2 q^2 X(t)^2 + c_2\right)}$$

being the unconstrained effort (see Hanson and Ryan, 1998).

The boundary condition associated with the problem formed by equations (3.18) and (3.19) follows from equation (3.5) and is given by $J^*(X(T),T) = 0$. The initial condition is $x(0) = x > 0$.

To determine the optimal policy with variable effort, that is, to determine the values $J^*(x,0)$ and $E^*(t)$, we need to solve equations (3.18) and (3.19) subject to the growth dynamic given by equation (3.8), and the boundary and initial conditions given above. As pointed before, the solutions of (3.18) and (3.19) are obtained via numerical methods using a discretization scheme. Section 3.1.1 deals with this topic.

3.1.1 Numerical solution of the HJB equation

To discretize equation (3.18), we consider that:

- the optimization starts at time $t = 0$ and ends at the terminal time $t = T < +\infty$;
- the time interval is uniformly partitioned as $0 = t_0 < t_1 < \ldots < t_n = T$, with $t_j = t_0 + j\Delta t$ and $\Delta t = T/n$, $j = 1, \ldots, n$;
- the state variable takes values within the interval $[0, x_{max}]$, which is also uniformly partitioned as $0 = x_0 < x_1 < \ldots < x_m = x_{max}$, with $x_i = x_0 + i\Delta x$ and $\Delta x = x_{max}/m$, $i = 1, \ldots, m$; x_{max} should be chosen such that the probability of $X(t)$ exceeding x_{max} is negligible;
- since we have a boundary condition $J^*(X(T), T) = 0$, which is terminal instead of initial, the computation uses time moving backwards from T to 0;
- the above partitions form a grid of points where $J^*_{i,j} := J^*(x_i, t_j)$ and $E^*_{i,j} := E^*(x_i, t_j)$, with $0 \le i \le m$ and $0 \le j \le n$.

The following derivatives are discretized using a Crank-Nicolson scheme (as in Thomas (1995)):

- For $0 \le i \le m-1$ and $0 \le j \le n-1$,

$$\frac{\partial J^*_{i,j}}{\partial t} \approx \frac{J^*_{i,j+1} - J^*_{i,j}}{\Delta t}.$$

- For $1 \le i \le m-1$ and $0 \le j \le n-1$,

$$\frac{\partial J^*_{i,j}}{\partial x} \approx \frac{1}{2}\left(\frac{J^*_{i+1,j+1} - J^*_{i-1,j+1}}{2\Delta x} + \frac{J^*_{i+1,j} - J^*_{i-1,j}}{2\Delta x}\right).$$

$$\begin{aligned}
\frac{\partial^2 J^*_{i,j}}{\partial x^2} &\approx \frac{1}{2}\left(\frac{\frac{J^*_{i+1,j+1} - J^*_{i,j+1}}{\Delta x} - \frac{J^*_{i,j+1} - J^*_{i-1,j+1}}{\Delta x}}{\Delta x} + \frac{\frac{J^*_{i+1,j} - J^*_{i,j}}{\Delta x} - \frac{J^*_{i,j} - J^*_{i-1,j}}{\Delta x}}{\Delta x}\right) \\
&= \frac{1}{2}\left(\frac{J^*_{i+1,j+1} - 2J^*_{i,j+1} + J^*_{i-1,j+1}}{\Delta x^2} + \frac{J^*_{i+1,j} - 2J^*_{i,j} + J^*_{i-1,j}}{\Delta x^2}\right).
\end{aligned}$$

- For $i = m$ and $0 \leq j \leq n-1$,

$$
\begin{aligned}
\frac{\partial J^*_{m,j}}{\partial x} &\approx \frac{\partial J^*_{m-1,j}}{\partial x} + \Delta x \frac{\partial^2 J^*_{m-1,j}}{\partial x^2} \\
&= \frac{1}{2}\left(\frac{J^*_{m,j+1} - J^*_{m-2,j+1}}{2\Delta x} + \frac{J^*_{m,j} - J^*_{m-2,j}}{2\Delta x}\right) \\
&+ \Delta x \frac{1}{2}\left(\frac{J^*_{m,j+1} - 2J^*_{m-1,j+1} + J^*_{m-2,j+1}}{\Delta x^2} + \frac{J^*_{m,j} - 2J^*_{m-1,j} + J^*_{m-2,j}}{\Delta x^2}\right) \\
&= \frac{1}{2}\left(\frac{3J^*_{m,j+1} - 4J^*_{m-1,j+1} + J^*_{m-2,j+1}}{2\Delta x} + \frac{3J^*_{m,j} - 4J^*_{m-1,j} + J^*_{m-2,j}}{2\Delta x}\right).
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 J^*_{m,j}}{\partial x^2} &\approx \frac{2}{\Delta x}\left(\frac{J^*_{m,j} - J^*_{m-1,j}}{\Delta x} - \frac{\partial J^*_{m-1,j}}{\partial x}\right) \\
&= \frac{1}{2}\left(\frac{2J^*_{m,j+1} - 5J^*_{m-1,j+1} + 4J^*_{m-2,j+1} - J^*_{m-3,j+1}}{\Delta x^2}\right) \\
&+ \frac{1}{2}\left(\frac{2J^*_{m,j} - 5J^*_{m-1,j} + 4J^*_{m-2,j} - J^*_{m-3,j}}{\Delta x^2}\right),
\end{aligned}
$$

where,to obtain an approximation of the second derivative we have performed the second order Taylor approximation of $J^*_{m,j}$

$$
J^*_{m,j} \approx J^*_{m-1,j} + \Delta x \frac{\partial J^*_{m-1,j}}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 J^*_{m,j}}{\partial x^2}.
$$

Therefore, the discretized version of equation (3.18) is:

- For $1 \leq i \leq m-1$ and $0 \leq j \leq n-1$,

$$
\begin{aligned}
-\frac{J^*_{i,j+1} - J^*_{i,j}}{\Delta t} &= (p_1 q x_i - c_1)E^*_{i,j+1} - (p_2 q^2 x_i^2 + c_2)E^{*2}_{i,j+1} - \delta\left(\frac{J^*_{i,j+1}}{2} + \frac{J^*_{i,j}}{2}\right) \\
&+ \frac{1}{2}\left(\frac{J^*_{i+1,j+1} - J^*_{i-1,j+1}}{2\Delta x} + \frac{J^*_{i+1,j} - J^*_{i-1,j}}{2\Delta x}\right)\left(f(x_i) - qE^*_{i,j+1}\right)x_i \\
&+ \frac{1}{4}\left(\frac{J^*_{i+1,j+1} - 2J^*_{i,j+1} + J^*_{i-1,j+1}}{\Delta x^2} + \frac{J^*_{i+1,j} - 2J^*_{i,j} + J^*_{i-1,j}}{\Delta x^2}\right)\sigma^2 x_i^2.
\end{aligned}
$$

- For $i = m$ and $0 \leq j \leq n-1$,

$$
\begin{aligned}
-\frac{J^*_{m,j+1} - J^*_{m,j}}{\Delta t} &= (p_1 q x_m - c_1)E^*_{m,j+1} - (p_2 q^2 x_m^2 + c_2)E^{*2}_{m,j+1} - \delta\left(\frac{J^*_{m,j+1}}{2} + \frac{J^*_{m,j}}{2}\right) \\
&+ \frac{1}{2}\left(\frac{3J^*_{m,j+1} - 4J^*_{m-1,j+1} + J^*_{m-2,j+1}}{2\Delta x} + \frac{3J^*_{m,j} - 4J^*_{m-1,j} + J^*_{m-2,j}}{2\Delta x}\right) \\
&\times \left(f(x_m) - qE^*_{m,j+1}\right)x_m \\
&+ \frac{1}{4}\left(\frac{2J^*_{m,j+1} - 5J^*_{m-1,j+1} + 4J^*_{m-2,j+1} - J^*_{m-3,j+1}}{\Delta x^2}\right. \\
&+ \left.\frac{2J^*_{m,j} - 5J^*_{m-1,j} + 4J^*_{m-2,j} - J^*_{m-3,j}}{\Delta x^2}\right)\sigma^2 x_m^2.
\end{aligned}
$$

The free optimal effort has the following discretization:

- For $1 \leq i \leq m-1$ and $0 \leq j \leq n-1$,

$$
E^*_{free_{i,j}} = \frac{\left(p_1 - \frac{J^*_{i+1,j} - J^*_{i-1,j}}{2\Delta x}\right) q x_i - c_1}{2\left(p_2 q^2 x_i^2 + c_2\right)}.
$$

- For $i = m$ and $0 \leq j \leq n-1$,

$$
E^*_{free_{m,j}} = \frac{\left(p_1 - \frac{3J^*_{m,j} - 4J^*_{m-1,j} + J^*_{m-2,j}}{2\Delta x}\right) q x_m - c_1}{2\left(p_2 q^2 x_m^2 + c_2\right)}.
$$

In each iteration, we then correct the free optimal effort to obtain the constrained optimal effort. Namely, for $1 \leq i \leq m$ and $0 \leq j \leq n-1$:

- If $E^*_{free_{i,j+1}} < 0,$ then $E^*_{i,j+1} = E_{min}$;

- If $E_{min} \leq E^*_{free_{i,j+1}} \leq E_{max},$ then $E^*_{i,j+1} = E^*_{free_{i,j+1}}$;

- If $E^*_{free_{i,j+1}} > E_{max},$ then $E^*_{i,j+1} = E_{max}$.

The discretized version of the HJB equation can be written as a system of m equations:

- For $1 \le i \le m-1$ and $0 \le j \le n-1$,

$$\begin{aligned}
&\left(\frac{(f(x_i)-qE^*_{i,j+1})x_i\Delta t}{4\Delta x}-\frac{\sigma^2 x_i^2\Delta t}{4\Delta x^2}\right)\cdot \boldsymbol{J^*_{i-1,j}} \\
&+\left(1+\frac{\delta\Delta t}{2}+\frac{\sigma^2 x_i^2\Delta t}{2\Delta x^2}\right)\cdot \boldsymbol{J^*_{i,j}} \\
&-\left(\frac{(f(x_i)-qE^*_{i,j+1})x_i\Delta t}{4\Delta x}+\frac{\sigma^2 x_i^2\Delta t}{4\Delta x^2}\right)\cdot \boldsymbol{J^*_{i+1,j}} \\
&=\left(-\frac{(f(x_i)-qE^*_{i,j+1})x_i\Delta t}{4\Delta x}+\frac{\sigma^2 x_i^2\Delta t}{4\Delta x^2}\right)\cdot \boldsymbol{J^*_{i-1,j+1}} \\
&+\left(1-\frac{\delta\Delta t}{2}-\frac{\sigma^2 x_i^2\Delta t}{2\Delta x^2}\right)\cdot \boldsymbol{J^*_{i,j+1}} \\
&+\left(\frac{(f(x_i)-qE^*_{i,j+1})x_i\Delta t}{4\Delta x}+\frac{\sigma^2 x_i^2\Delta t}{4\Delta x^2}\right)\cdot \boldsymbol{J^*_{i+1,j+1}} \\
&+(p_1 q x_i - c_1)E^*_{i,j+1}\Delta t-(p_2 q^2 x_i^2+c_2)E^{*2}_{i,j+1}\Delta t.
\end{aligned}$$

- For $i = m$ and $0 \le j \le n-1$,

$$\begin{aligned}
&\frac{\sigma^2 x_m^2\Delta t}{4\Delta x^2}\cdot \boldsymbol{J^*_{m-3,j}} \\
&-\left(\frac{(f(x_m)-qE^*_{m,j+1})x_m\Delta t}{4\Delta x}+\frac{\sigma^2 x_m^2\Delta t}{4\Delta x^2}\right)\cdot \boldsymbol{J^*_{m-2,j}} \\
&+\left(\frac{(f(x_m)-qE^*_{m,j+1})x_m\Delta t}{\Delta x}+\frac{5\sigma^2 x_m^2\Delta t}{4\Delta x^2}\right)\cdot \boldsymbol{J^*_{m-1,j}} \\
&+\left(1+\frac{\delta\Delta t}{2}-\frac{3(f(x_m)-qE^*_{m,j+1})x_m\Delta t}{4\Delta x}-\frac{\sigma^2 x_m^2\Delta t}{2\Delta x^2}\right)\cdot \boldsymbol{J^*_{m,j}} \\
&=-\frac{\sigma^2 x_m^2\Delta t}{4\Delta x^2}\cdot \boldsymbol{J^*_{m-3,j+1}} \\
&+\left(\frac{(f(x_m)-qE^*_{m,j+1})x_m\Delta t}{4\Delta x}+\frac{\sigma^2 x_m^2\Delta t}{4\Delta x^2}\right)\cdot \boldsymbol{J^*_{m-2,j+1}} \\
&-\left(\frac{(f(x_m)-qE^*_{m,j+1})x_m\Delta t}{\Delta x}+\frac{5\sigma^2 x_m^2\Delta t}{4\Delta x^2}\right)\cdot \boldsymbol{J^*_{m-1,j+1}} \\
&+\left(1-\frac{\delta\Delta t}{2}+\frac{3(f(x_m)-qE^*_{m,j+1})x_m\Delta t}{4\Delta x}-\frac{\sigma^2 x_m^2\Delta t}{2\Delta x^2}\right)\cdot \boldsymbol{J^*_{m,j+1}} \\
&+(p_1 q x_m - c_1)E^*_{m,j+1}\Delta t-(p_2 q^2 x_m^2+c_2)E^{*2}_{m,j+1}\Delta t.
\end{aligned}$$

The system can be written using appropriate matrices A, B and C, in the form

$$A\boldsymbol{J^*_-} = B\boldsymbol{J^*_+} + C, \tag{3.20}$$

with

$$\boldsymbol{J^*_-} = \left[\boldsymbol{J^*_0} \quad | \quad \boldsymbol{J^*_1} \quad | \quad \cdots \quad | \quad \boldsymbol{J^*_{n-1}}\right], \quad \boldsymbol{J^*_+} = \left[\boldsymbol{J^*_1} \quad | \quad \boldsymbol{J^*_2} \quad | \quad \cdots \quad | \quad \boldsymbol{J^*_n}\right], \quad \text{and}$$

$$\boldsymbol{J^*_j} = \left[J^*_{0,j} \qquad J^*_{1,j} \quad \cdots \quad J^*_{m,j}\right]^T, \quad 0 \le j \le n,$$

where T is the transpose operator. Solving the system one obtains the optimal solution for the grid points. The optimal solution when the population is at a given value X at time t_k is obtained by polynomial interpolation, between the values at the neighbouring points of X in the partition $x_0, x_1, \ldots, x_m$.

3.2 Conclusions

In this chapter we have presented the problem of obtaining the expected accumulated discounted profit over some finite time horizon T.

We have considered a population size dynamics given by a SDE and a standard harvesting rate based on the idea that harvesting is proportional to the effort and to the population size.

We have used a profit structure where revenues per unit time are quadratic functions of the yield and costs are quadratic functions on effort. This profit structure is a generalization of the usual structure with linear prices and quadratic costs. The formulated problem lead us to obtain the HJB equation and to solve it numerically by applying a finite-difference method.

Since the HJB equation is a parabolic partial differential equation, we choose to apply a Crank-Nicolson scheme and, at each step, the effort was checked and forced to stay within the prescribed bounds.

4

Optimal sustainable policy with constant effort

In this chapter we present the alternative sustainable approach based on constant effort. This approach is, to the best of our knowledge, the first attempt to obtain an optimal sustainable harvesting policy based on profit per unit time optimization. As in the optimal policy problem formulation, we consider a general population growth model with harvesting, described by a stochastic differential equation. The harvesting term is similar as before but now the harvesting effort is constant. Using the theory of stochastic differential equations presented in Chapter 2, we will determine, at the steady-state, the optimal sustainable effort and the optimal sustainable profit per unit time. In order to simulate trajectories under the optimal sustainable policy, for comparison purposes, we consider two classical growth models: the logistic model and the Gompertz model. For these models, the optimal effort is given by the solution of a polynomial equation or a non-linear equation. In both cases, the computations are very simple to obtain, using numerical methods if necessary. The layout of this chapter is: section 4.1 describes the formulation and properties of the general model and how to obtain the optimal sustainable effort and the optimal sustainable profit per unit time. Sections 4.2 and 4.3 study, respectively, the particular cases of the logistic and the Gompertz growth models. Section 4.4 provides the main conclusions.

4.1 General growth model

To apply a constant effort policy, one considers a particular case of equation (3.1) with $E(t) \equiv E$, that is,

$$dX(t) = f(X(t))X(t)dt - qEX(t)dt + \sigma X(t)dW(t), \quad X(0) = x, \tag{4.1}$$

and the following assumptions hold for $X > 0$:

(A) f is continuously differentiable;

(B) f is strictly decreasing;

(C) $f(+\infty) < 0$;

(D) $F(0^+) = 0$, where $F(X) := f(X)X$;

(E) $f(0^+)$ exists and $f(0^+) - qE > \sigma^2/2$.

Assumption (B) translates the idea that, the larger the population size is, the fewer are the available resources for an individual to survive and reproduce. Assumption (C) prevents large populations to grow and assumption (D) implies the existence of a closed population, that is, there is no immigration. Assumption (E) avoids mathematical extinction, i.e., $X(t) = 0$ for some $t > 0$ or $X(t) \to 0$ as $t \to +\infty$ have zero probability of occurring (see Braumann, 1985, with appropriate adaptations since it uses Stratonovich calculus). Mathematical extinction occurs with probability one if $f(0^+) - qE < \sigma^2/2$ (see Braumann, 1985, with appropriate adaptations to Itô calculus). We recall that $f(0^+)$ represents the net arithmetic average *per capita* growth rate at low population density values. Note that assumption (A) is of a technical nature.

Under these assumptions, from Corollary 2.2.1 (see page 20), the solution of equation (4.1) exists and is unique up to an explosion time and is a homogeneous diffusion process with drift and diffusion coefficients given, respectively, by

$$a(X) = f(X)X - qEX \quad \text{and} \quad b(X) = \sigma^2 X^2. \tag{4.2}$$

The state space of $X(t)$ is $(0, +\infty)$ and the boundaries are $X = 0$ and $X = +\infty$. From Braumann (1985), one can see that, if both boundaries are non-attractive, then the solution $X(t)$ will stay inside the interval $(0, +\infty)$ for all $t \geq 0$. To classify the boundaries and to determine the stationary behaviour of $X(t)$, one needs the scale and speed densities (see (2.7) and (2.8)), given respectively by

$$s(X) = C_1 X^{2qE/\sigma^2} \exp\left(-\frac{2}{\sigma^2} \int_{z_0}^{X} \frac{f(z)}{z} dz \right) \tag{4.3}$$

and

$$m(X) = C_2 X^{-2(1+qE/\sigma^2)} \exp\left(\frac{2}{\sigma^2} \int_{z_0}^{X} \frac{f(z)}{z} dz \right), \tag{4.4}$$

where C_1, C_2 are positive constants and $z_0 > 0$ is an arbitrary (but fixed) point in the interior of the state space. From the above densities, one can write the scale and speed functions (see (2.9) and (2.10)) as

$$S(X) = \int_{x_0}^{X} C_1 Z^{2qE/\sigma^2} \exp\left(-\frac{2}{\sigma^2} \int_{z_0}^{Z} \frac{f(\theta)}{\theta} d\theta \right) dZ \tag{4.5}$$

and

$$M(X) = \int_{x_0}^{X} C_2 Z^{-2(1+qE/\sigma^2)} \exp\left(\frac{2}{\sigma^2} \int_{z_0}^{Z} \frac{f(\theta)}{\theta} d\theta \right) dZ, \tag{4.6}$$

where $x_0 > 0$ is an arbitrary (but fixed) point in the interior of the state space.

From Proposition 2.3.1 (on page 22), necessary and sufficient conditions for the non-attractiveness of $X = 0$ and $X = +\infty$ are, respectively, $S(0, x] = +\infty$ for some $x > 0$ and $S[x, +\infty) = +\infty$ for some $x > 0$. This is equivalent, respectively, to $S(0) = -\infty$ and $S(+\infty) = +\infty$. From (4.5) we have, choosing $x_0 = z_0 > 0$:

$$\begin{aligned} S(0) &= \int_{x_0}^{0} C_1 Z^{2qE/\sigma^2} \exp\left(-\frac{2}{\sigma^2} \int_{x_0}^{Z} \frac{f(\theta)}{\theta} d\theta \right) dZ \\ &= -\int_{0}^{x_0} C_1 Z^{2qE/\sigma^2} \exp\left(\frac{2}{\sigma^2} \int_{Z}^{x_0} \frac{f(\theta)}{\theta} d\theta \right) dZ \end{aligned}$$

and

$$S(+\infty) = \int_{x_0}^{+\infty} C_1 Z^{2qE/\sigma^2} \exp\left(\frac{2}{\sigma^2} \int_{x_0}^{Z} \frac{f(\theta)}{\theta} d\theta \right) dZ.$$

From the assumptions (A) – (E), it can be seen (see Braumann, 2008, adapting to Itô calculus) that

$$S(0) = -\infty \quad \text{and} \quad S(+\infty) = +\infty.$$

Therefore, the boundaries $X = 0$ and $X = +\infty$ are non-attractive. The non-attractiveness of the $X = 0$ boundary insures that there is a zero probability of mathematical extinction. The non-attractiveness of the $X = +\infty$ boundary insures non-explosion and therefore existence and uniqueness of the solution for all $t > 0$. According to section 2.3, it may happen that the transient distribution of $X(t)$ stabilizes and converges, as $t \to +\infty$, to a distribution with probability density function denoted by $p(y)$ and named stationary density. Given that the boundaries are non-attractive, from Gihman and Skorohod (1979) and Karlin and Taylor (1981) one can see that a necessary and sufficient condition for the existence of a stationary density is

$$M(0, \infty) = \int_0^{+\infty} m(z)dz < +\infty. \tag{4.7}$$

In fact, given the assumptions (A) – (E) on f, we can see in Braumann (1999) that condition (4.7) holds and, from section 2.3, we can get an expression for the stationary density $p(y)$,

$$p(y) = \frac{m(y)}{\int\limits_0^{+\infty} m(z)dz}, \quad y \in (0, \infty), \tag{4.8}$$

with $m(y)$ given by (4.4).

Summing up, we can conclude that there is a stationary distribution for the population size, i.e., an equilibrium probability distribution, with probability density function p proportional to the speed density, towards which the distribution of the population size converges as $t \to \infty$. Note that this is a stochastic equilibrium, not a deterministic one. In fact, the population size $X(t)$ does not stabilize into an equilibrium size as in the deterministic case. It is the probability distribution of $X(t)$ that stabilizes into an equilibrium distribution with probability density function given by $p(y)$, the stationary density.

The existence of the stationary density plays a central role when defining the optimal sustainable policy that we are going to study below, allowing us to take a steady-state approach. We denote by X_∞ the random variable with density p, i.e., the random variable exhibiting the steady-state probabilistic behaviour. A good approximation of the expected size of the population $\mathbb{E}[X_t]$, for large t, is the expected value of

X_∞,

$$\mathbb{E}[X_\infty] = \int_0^{+\infty} xp(x)dx = \frac{\int_0^{+\infty} xm(x)dx}{\int_0^{+\infty} m(x)dx} := \frac{M_{01}(E)}{M_{00}(E)}, \tag{4.9}$$

with

$$M_{jk}(E) := \int_0^{+\infty} (\ln z)^j z^k \frac{m(z)}{C_2} dz = \int_0^{+\infty} (\ln z)^j z^{k-2(1+qE/\sigma^2)} \exp\left(\frac{2}{\sigma^2}\int_x^z \frac{f(y)}{y}dy\right) dz,$$

choosing $z_0 = x$.

The steady-state structure of the profit per unit time is similar as in the case of the optimal policy and is given by

$$\begin{aligned} \Pi_\infty &:= P_\infty - C \;=\; p(H_\infty)H_\infty - c(E)E \\ &= (p_1 - p_2 H_\infty)H_\infty - (c_1 + c_2 E)E \\ &= (p_1 q X_\infty - c_1)E - (p_2 q^2 X_\infty^2 + c_2)E^2, \end{aligned}$$

where P_∞ is the sale price, C is the fishing cost and $H_\infty := qEX_\infty$ is the sustainable harvesting rate. The expected sustainable profit per unit time is

$$\begin{aligned} \mathbb{E}\left[\Pi_\infty\right] &= \mathbb{E}\left[(p_1 q X_\infty - c_1)E - (p_2 q^2 X_\infty^2 + c_2)E^2\right] \\ &= (p_1 q \mathbb{E}\left[X_\infty\right] - c_1)E - (p_2 q^2 \mathbb{E}\left[X_\infty^2\right] + c_2)E^2 \\ &= \left(p_1 q \frac{M_{01}(E)}{M_{00}(E)} - c_1\right) E - \left(p_2 q^2 \frac{M_{02}(E)}{M_{00}(E)} + c_2\right) E^2. \end{aligned} \tag{4.10}$$

The steady-state optimization problem consists in maximizing the expected sustainable profit per unit time, should it have a maximum for the admissible range of effort values $0 \le E < \frac{1}{q}\left(f(0^+) - \frac{\sigma^2}{2}\right)$. Note that this is the range of values for which we have a stationary density. For $E > \frac{1}{q}\left(f(0^+) - \frac{\sigma^2}{2}\right)$, extinction occurs with probability one, so $X_\infty = 0$ and the steady-state profit per unit time will be $-c_1 E - c_2 E^2$.

If there is an optimal sustainable effort E^{**}, taking into account that

$$\frac{\partial M_{jk}(E)}{\partial E} = -\frac{2q}{\sigma^2} M_{j+1,k}(E),$$

E^{**} is a solution of equation $d\mathbb{E}[\Pi_\infty]/dE = 0$ such that $d^2\mathbb{E}[\Pi_\infty]/dE^2 < 0$, that is, E^{**} is solution of

$$
\begin{aligned}
E^2\left(\frac{p_2 q}{p_1}\frac{M_{12}(E)}{M_{00}(E)}\left(1-\frac{M_{10}(E)}{M_{00}(E)}\frac{M_{02}(E)}{M_{12}(E)}\right)\right)& \\
-E\left(\frac{M_{11}(E)}{M_{01}(E)}\left(1-\frac{M_{01}(E)}{M_{00}(E)}\frac{M_{10}(E)}{M_{11}(E)}\right)+\sigma^2\frac{p_2}{p_1}\left(\frac{M_{02}(E)}{M_{00}(E)}+\frac{c_2}{p_2 q}\right)\right)& \\
+\frac{\sigma^2}{2q}\left(\frac{M_{01}(E)}{M_{00}(E)}-\frac{c_1}{p_1 q}\right)=0.& \qquad (4.11)
\end{aligned}
$$

Finally, the expected sustainable profit per unit time is given by

$$
\mathbb{E}[\Pi_\infty^{**}]=\left(p_1 q\frac{M_{01}(E^{**})}{M_{00}(E^{**})}-c_1\right)E^{**}-\left(p_2 q^2\frac{M_{02}(E^{**})}{M_{00}(E^{**})}+c_2\right)E^{**2} \qquad (4.12)
$$

4.2 Optimal sustainable policy with a logistic growth model

When the population growth function follows a logistic model, equation (4.1) takes the form

$$
dX(t)=rX(t)\left(1-\frac{X(t)}{K}\right)dt-qEX(t)dt+\sigma X(t)dW(t),\quad X(0)=x, \qquad (4.13)
$$

where the parameter $r>0$ denotes the intrinsic population growth rate and $K>0$ is the carrying capacity of the environment.

The solution of equation (4.13) is (see Appendix A.1)

$$
X(t)=\frac{K\exp\left\{\left(r-qE-\frac{\sigma^2}{2}\right)t+\sigma W(t)\right\}}{\frac{K}{x}+r\int_0^t\exp\left\{\left(r-qE-\frac{\sigma^2}{2}\right)s+\sigma W(s)\right\}ds}. \qquad (4.14)
$$

The *per capita* growth function $f(x)=r(1-x/K)$ is a polynomial function and it is very easy to see that f fulfils assumptions (A) – (D) on section 4.1. For the logistic model, the assumption (E) is equivalent to $E<\frac{1}{q}\left(r-\frac{\sigma^2}{2}\right)$. The drift and diffusion coefficients (4.2) are

$$
a(x)=rx\left(1-\frac{x}{K}\right)-qEx \quad \text{and} \quad b(x)=\sigma^2x^2.
$$

This leads to the scale and speed densities

$$s(X) = CX^{-\rho-1}\exp\{\theta x\} \tag{4.15}$$

and

$$m(X) = DX^{\rho-1}\exp\{-\theta x\}, \tag{4.16}$$

with $\rho = \frac{2(r-qE)}{\sigma^2} - 1$, $\theta = \frac{2r}{K\sigma^2}$ and $C,\ D$ are positive constants.

Beddington and May (1977) obtained the stationary density for the stochastic process $X(t)$. Using (4.4) and (4.6), it is given by

$$p(x) = \frac{1}{\Gamma(\rho)}\theta^{\rho}x^{\rho-1}\exp\{-\theta x\}, \quad 0 < x < +\infty, \tag{4.17}$$

where Γ is the Gamma function.

Notice that p is the probability density function of a random variable, say X_∞, with Gamma distribution and so we can write

$$X_\infty \sim Gamma(\rho, \theta).$$

The mean value of X_∞ could be obtained from (4.9) but, in this case, it comes straightforward from (4.17) as

$$\mathbb{E}[X_\infty] = \frac{\rho}{\theta} = K\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right). \tag{4.18}$$

The expected sustainable profit per unit time, $\mathbb{E}[\Pi_\infty]$, is

$$\mathbb{E}[\Pi_\infty] = \left(p_1 qK\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right) - c_1\right)E - \left(p_2 q^2K^2\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right)\left(1 - \frac{qE}{r}\right) + c_2\right)E^2,$$

and the steady-state optimization problem becomes

$$\max_E \mathbb{E}[\Pi_\infty] = \left(p_1 qK\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right) - c_1\right)E - \left(p_2 q^2K^2\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right)\left(1 - \frac{qE}{r}\right) + c_2\right)E^2.$$

If there is a maximum in the admissible range $0 \le E < \frac{r-\sigma^2/2}{q}$, the optimization problem consists in solving the cubic equation $d\mathbb{E}[\Pi_\infty]/dE = 0$, so that the solution satisfies $d^2\mathbb{E}[\Pi_\infty]/dE^2 < 0$. The resulting optimal sustainable effort, E^{**}, is then solution of the equation

$$
\begin{aligned}
p_1 qK\left(1-\frac{qE}{r}-\frac{\sigma^2}{2r}\right)-c_1-\frac{p_1Kq^2}{r}E \\
-2E\left(p_2q^2K^2\left(1-\frac{qE}{r}\right)\left(1-\frac{qE}{r}-\frac{\sigma^2}{2r}\right)+c_2\right) \\
-E^2\left(p_2q^2K^2\left(-\frac{q}{r}\right)\left(1-\frac{qE}{r}-\frac{\sigma^2}{2r}\right)+p_2q^2K^2\left(1-\frac{qE}{r}\right)\left(-\frac{q}{r}\right)\right)=0.
\end{aligned}
$$

The correspondent optimal expected sustainable profit per unit time, $\mathbb{E}[\Pi_\infty^{**}]$, is

$$
\mathbb{E}[\Pi_\infty^{**}]=\left(p_1qK\left(1-\frac{qE^{**}}{r}-\frac{\sigma^2}{2r}\right)-c_1\right)E^{**}-\left(p_2q^2K^2\left(1-\frac{qE^{**}}{r}-\frac{\sigma^2}{2r}\right)\left(1-\frac{qE^{**}}{r}\right)+c_2\right)E^{**2}.
$$

Finally, at the steady-state, the mean value of the population under the optimal effort E^{**} is

$$
\mathbb{E}[X_\infty^{**}]=K\left(1-\frac{qE^{**}}{r}-\frac{\sigma^2}{2r}\right). \tag{4.19}
$$

The particular case $p_2=0$

This price structure has been widely used (see, for instance, Arnason et al. (2004), Engen et al. (1997), Sandal and Steinshamn (1997), Song et al. (2010) and Suri (2008)). In this case, the above expressions for the optimal sustainable effort, optimal expected sustainable profit per unit time and mean value of the population under the optimal sustainable effort are given, respectively, by

$$
E^{**} = \frac{r}{2q}\frac{1-\dfrac{\sigma^2}{2r}-\dfrac{c_1}{p_1qK}}{1+\dfrac{c_2r}{p_1q^2K}}, \tag{4.20}
$$

$$
\mathbb{E}[\Pi_\infty^{**}] = \frac{rKp_1}{4}\frac{\left(1-\dfrac{\sigma^2}{2r}-\dfrac{c_1}{p_1qK}\right)^2}{1+\dfrac{c_2r}{p_1q^2K}} \quad \text{and} \tag{4.21}
$$

$$
\mathbb{E}[X_\infty^{**}] = \frac{K}{2}\frac{\left(1-\dfrac{\sigma^2}{2r}\right)\left(1+\dfrac{2c_2r}{p_1q^2K}\right)+\dfrac{c_1}{p_1qK}}{1+\dfrac{c_2r}{p_1q^2K}}. \tag{4.22}
$$

From these expressions, we can see that the optimal effort, the steady-state optimal expected profit per unit time and the steady-state average population size decrease when the strength σ of the environmental fluctuations increases. The optimal effort and the expected profit per unit time also decrease when c_1 or

c_2 increases and increase when p_1 increases.

4.3 Optimal sustainable policy with a Gompertz growth model

When the population growth function follows a Gompertz model, equation (4.1) takes the form

$$dX(t) = rX(t)\ln\frac{K}{X(t)}dt - qEX(t)dt + \sigma X(t)dW(t), \quad X(0) = x, \tag{4.23}$$

where the parameter $r > 0$ is an intrinsic growth parameter and $K > 0$ is the carrying capacity of the environment.

The solution of equation (4.23) is (see Appendix A.2)

$$X(t) = \exp\left\{e^{-rt}\ln x\right\}\exp\left\{\left(\ln K - \frac{qE}{r} - \frac{\sigma^2}{2r}\right)\left(1 - e^{-rt}\right)\right\}\exp\left\{\sigma e^{-rt}\int_0^t e^{rs}dW(s)\right\}. \tag{4.24}$$

The *per capita* growth function $f(x) = r\ln(K/x)$ is a logarithmic function and is very easy to see that f fulfils assumptions (A) – (D) on section 4.1. For the Gompertz growth model, the assumption (E) is automatically satisfied since $f(0^+) = +\infty$. The drift and diffusion coefficients (4.2) are

$$a(x) = rx\ln\frac{K}{x} - qEx \quad \text{and} \quad b(x) = \sigma^2x^2.$$

From (4.24) one obtains

$$Y(t) := \ln X(t) = e^{-rt}\ln x + \left(\ln K - \frac{qE}{r} - \frac{\sigma^2}{2r}\right)\left(1 - e^{-rt}\right) + \sigma e^{-rt}\int_0^t e^{rs}dW(s).$$

Since the integrand is deterministic, the stochastic integral $\int_0^t e^{rs}dW(s)$ in this expression is Gaussian with mean 0 and variance $\int_0^t (e^{rs})^2\,ds = \frac{1}{2r}\left(e^{2rt} - 1\right)$. Therefore, $Y(t)$ is Gaussian with mean

$$\mu(t) = e^{-rt}\ln x + \left(\ln K - \frac{qE}{r} - \frac{\sigma^2}{2r}\right)\left(1 - e^{-rt}\right)$$

and variance

$$\theta^2(t) = \frac{\sigma^2}{2r}\left(1 - e^{-2rt}\right).$$

Therefore, $Y_\infty := \ln X_\infty \sim \mathcal{N}(\mu, \theta^2)$ with

$$\mu = \ln K - \frac{qE}{r} - \frac{\sigma^2}{2r} \quad \text{and} \quad \theta = \frac{\sigma}{\sqrt{2r}}.$$

So, the stationary density of X_∞ is

$$p(x) = \frac{1}{\theta\sqrt{2\pi}x} \exp\left\{-\frac{(\ln x - \mu)^2}{2\theta^2}\right\}, \quad 0 < x < +\infty. \tag{4.25}$$

The first and second moments of X_∞ come from the expectation properties:

$$\mathbb{E}[X_\infty] = \mathbb{E}\Big[e^{Y_\infty}\Big] = \exp\left\{\mathbb{E}[Y_\infty] + \frac{1}{2}\mathsf{Var}[Y_\infty]\right\} = K\exp\left\{-\frac{qE}{r} - \frac{\sigma^2}{4r}\right\}$$

and

$$\mathbb{E}[X_\infty^2] = \mathbb{E}\Big[e^{2Y_\infty}\Big] = \exp\left\{\mathbb{E}[2Y_\infty] + \frac{1}{2}\mathsf{Var}[2Y_\infty]\right\} = K^2\exp\left\{-\frac{2qE}{r}\right\}.$$

The expected sustainable profit per unit time, $\mathbb{E}[\Pi_\infty]$, is

$$\mathbb{E}[\Pi_\infty] = \left(p_1 qK\exp\left\{-\frac{qE}{r} - \frac{\sigma^2}{4r}\right\} - c_1\right)E - \left(p_2 q^2 K^2 \exp\left\{-\frac{2qE}{r}\right\} + c_2\right)E^2,$$

and the steady-state optimization problem becomes

$$\max_{E\geq 0} \mathbb{E}[\Pi_\infty] = \left(p_1 qK\exp\left\{-\frac{qE}{r} - \frac{\sigma^2}{4r}\right\} - c_1\right)E - \left(p_2 q^2 K^2 \exp\left\{-\frac{2qE}{r}\right\} + c_2\right)E^2.$$

If there is a maximum in the admissible range $E \geq 0$, the optimization problem consists in solving the non-linear equation $d\mathbb{E}[\Pi_\infty]/dE = 0$, so that $d^2\mathbb{E}[\Pi_\infty]/dE^2 < 0$. The resulting optimal sustainable effort, E^{**}, is solution of the equation

$$\begin{aligned} &\exp\left\{-\frac{q}{r}E + \frac{\sigma^2}{4r}\right\} - \frac{c_1}{p_1 qK} + \frac{2p_2 rK}{p_1}\left(\frac{q}{r}E\right)^2 \exp\left\{-\frac{2q}{r}E\right\} \\ &-\frac{q}{r}E\left(\exp\left\{-\frac{q}{r}E + \frac{\sigma^2}{4r}\right\} + \frac{2p_2 rK}{p_1}\exp\left\{-\frac{2q}{r}E\right\} + \frac{2c_2 r}{p_1 q^2 K}\right) \quad = \quad 0, \end{aligned}$$

and can be obtained by numerical methods with the required care to the check for multiple solutions and for maximality.

The correspondent optimal expected sustainable profit per unit time, $\mathbb{E}[\Pi_\infty^{**}]$, is

$$\mathbb{E}[\Pi_\infty^{**}] = \left(p_1 qK \exp\left\{-\frac{qE^{**}}{r} - \frac{\sigma^2}{4r}\right\} - c_1\right) E^{**} - \left(p_2 q^2 K^2 \exp\left\{-\frac{2qE^{**}}{r}\right\} + c_2\right) E^{**2}.$$

Finally, at the steady-state, the mean value of the population under the optimal effort E^{**} is

$$\mathbb{E}[X_\infty^{**}] = K \exp\left\{-\frac{qE^{**}}{r} - \frac{\sigma^2}{4r}\right\}. \tag{4.26}$$

The particular case $p_2 = 0$

As in the case of the logistic growth model, we present now, for the particular case $p_2 = 0$, the equation to obtain the optimal sustainable effort and the expressions for the optimal expected sustainable profit per unit time and for the mean value of the population under the optimal sustainable effort:

E^{**} is given by the solution of equation

$$\begin{aligned} E &= \frac{r}{q}\left(1 - \frac{c_1 + 2c_2 E}{p_1 qK \exp\left\{-\frac{qE}{r} - \frac{\sigma^2}{4r}\right\}}\right), \\ \mathbb{E}[\Pi_\infty^{**}] &= \left(p_1 qK \exp\left\{-\frac{qE^{**}}{r} - \frac{\sigma^2}{4r}\right\} - c_1\right) E^{**} - c_2 E^{**2} \quad \text{and} \\ \mathbb{E}[X_\infty^{**}] &= K \exp\left\{-\frac{qE^{**}}{r} - \frac{\sigma^2}{4r}\right\}. \end{aligned}$$

4.4 Conclusions

In this chapter we have presented an optimal sustainable harvesting policy based on constant effort. For a general growth model we have presented the necessary conditions to avoid non-attractiveness of boundaries and the existence of a stationary density. Following this, we deduced the expressions for the optimal sustainable effort and the expected sustainable profit per unit time.

Next we studied the above problem with the logistic and Gompertz growth models. In both cases, the equation that leads to the optimal sustainable effort needs to be solved numerically.

For the logistic growth model we have considered the particular case $p_2 = 0$, since this is a very often considered case. Here, the steady-state optimization problem consists in maximizing a second degree polynomial, and the expressions for the optimal sustainable effort, expected sustainable profit per unit time and mean value of the population come in a straightforward manner.

The results presented in this chapter are, to the best of our knowledge, the first attempt to obtain an optimal sustainable harvesting policy based on profit per unit time optimization.

5

Comparison of policies

This chapter concentrates on the numerical comparisons between the optimal policy with variable effort and the optimal sustainable policy with constant effort. We will compare the performance of the two policies in terms of the expected profit earned by the harvester and in terms of the evolution of the population size and the amount of effort across time. In the literature, the two most used models to describe population dynamics are the logistic and the Gompertz models. These are the ones used to conduct the comparisons. For each model, we will set up a basic scenario with parameter values based on realistic data from a fished population. We will also consider alternative scenarios corresponding to changes of the different parameter values used in the basic scenario and we will check the influence of such changes in terms of the profit.

In addition to the mentioned optimal policies, we will present a sub-optimal and applicable policy based on a step variable effort function in which the effort remains constant for fixed periods of time. We denote this policy by stepwise policy and we will compare it with the optimal sustainable policy and with the optimal policy, using both the logistic and Gompertz models. The design of this chapter is as follows. Section 5.1 defines the quantities of interest for the comparisons and how they were computed. Sections 5.2 and 5.3 describe, respectively, the policy comparisons with the logistic and Gompertz models. The comparisons with the stepwise policy are shown in section 5.4 for the logistic model and in section 5.5 for the Gompertz model. The main conclusions appear on section 5.6.

5.1 Comparison description

The comparison between the (variable effort) optimal policy and the (constant effort) optimal sustainable policy can not be done directly, since the optimal policy yields the optimal expected accumulated discounted profit over a finite time horizon, V^*, and the optimal sustainable policy yields the optimal expected profit per unit time, $\mathbb{E}[\Pi^{**}_\infty]$, for a large time horizon $T \to +\infty$. Even so, both policies can be compared if one uses Monte Carlo simulations. Let

$$\begin{aligned}
\Pi^*(t) &:= (p_1 q X(t) - c_1)E^*(t) - (p_2 q^2 X^2(t) + c_2){E^*}^2(t), \\
\Pi^{**}(t) &:= (p_1 q X(t) - c_1)E^{**} - (p_2 q^2 X^2(t) + c_2){E^{**}}^2
\end{aligned}$$

be the profit per unit time for the two policies. We can compute, for both policies, four comparable quantities of interest ($*$ refers to the optimal policy and $**$ refers to the optimal sustainable policy):

1. **Expected accumulated discounted profit** in the interval $[0, T]$:

$$V^* := \mathbb{E}_x\left[\int_0^T e^{-\delta\tau}\Pi^*(\tau)d\tau\right], \quad V^{**} := \mathbb{E}_x\left[\int_0^T e^{-\delta\tau}\Pi^{**}(\tau)d\tau\right]. \tag{5.1}$$

   Since we cannot obtain the expectations $\mathbb{E}[\,\cdot\,]$ analytically nor the integrals $\int_0^T e^{-\delta\tau}\Pi^*(\tau)d\tau$ and $\int_0^T e^{-\delta\tau}\Pi^{**}(\tau)d\tau$, we use numerical methods. We approximate the integrals by discretizing time. The expectations are approximated by the average of 1000 Monte Carlo simulated trajectories using an Euler scheme with the same discretization time step. In the case of V^*, the integral for a trajectory can also be estimated by the value corresponding to $T = 0$ of the numerical solution J^* of the HJB equation. We did not use that method (which gives numerical values almost indistinguishable from the method we use) since we want a full comparability with V^{**}, for which such method is not possible.

2. **Expected accumulated undiscounted profit** in the interval $[0, T]$:

$$V_u^* = \mathbb{E}_x\left[\int_0^T \Pi^*(\tau)d\tau\right], \quad V_u^{**} = \mathbb{E}_x\left[\int_0^T \Pi^{**}(\tau)d\tau\right]. \tag{5.2}$$

As before, we use numerical methods.

3. **Average expected profit per unit time (average weighted by the discount factors)**:

$$\overline{\Pi^*} = \frac{V^*}{\displaystyle\int_0^T e^{-\delta\tau}d\tau}, \quad \overline{\Pi^{**}} = \frac{V^{**}}{\displaystyle\int_0^T e^{-\delta\tau}d\tau}. \tag{5.3}$$

4. **Average expected profit per unit time (unweighted average)**:

$$\overline{\Pi_u^*} = \frac{V_u^*}{T}, \quad \overline{\Pi_u^{**}} = \frac{V_u^{**}}{T}. \tag{5.4}$$

Note, for the constant effort optimal policy, that we determine the constant effort E^{**} that maximizes $\mathbb{E}[\Pi_\infty]$, thus obtaining the optimal expected profit per unit time at the steady-state $\mathbb{E}[\Pi_\infty^{**}]$ given by (4.12). This quantity is, due to the ergodicity of $X(t)$, also the limit as $T \to +\infty$ of both the time-average expected profit $\overline{\Pi_u^{**}} = \mathbb{E}_x[\int_0^T \Pi^{**}(\tau)d\tau]/T$ and (with probability 1) of the observed time-average profit $\int_0^T \Pi^{**}(\tau)d\tau/T$ actually experienced by harvesters.

5.2 Comparison of polices with the logistic growth model

5.2.1 Basic scenario

Instead of arbitrary parameters values, we have decided to set up a basic scenario S_0 using realistic values. We found a quite complete set of parameter values (namely r, K, q, p_1, c_1 and c_2) for the Pacific halibut (*Hippoglossus hippoglossus*) in Clark (1990) and Hanson and Ryan (1998) and these are the ones chosen for S_0. The other parameters, for which we had no information ($E_{min}, E_{max}, \sigma, x,$ p_2 and δ), where chosen at reasonable values and the time horizon was set at $T = 50$ years. The complete set of parameter values is listed in Table 5.1. The determination of the expected profit values (5.1)–(5.4) requires numerical computations. We designed a grid with a time and state space discretization using $n = 150$ intervals for time (corresponding to a time step $\Delta t = 4$ months) and using $m = 75$ intervals for the state space (corresponding to a space step $\Delta x = 21.47 \cdot 10^5$ kg, with $x_{max} = 2K$). Other values for the

Table 5.1: Parameter values used in the simulations of the basic scenario S_0 (logistic model). The Standardized Fishing Unit (SFU) measure is defined in Hanson and Ryan (1998).

Item	Description	Values	Units
r	Intrinsic growth rate	0.71	year^{-1}
K	Carrying capacity	$80.5 \cdot 10^6$	kg
q	Catchability coefficient	$3.30 \cdot 10^{-6}$	SFU^{-1}year^{-1}
E_{min}	Minimum fishing effort	0	SFU
E_{max}	Maximum fishing effort	$0.7r/q$	SFU
σ	Strength of environmental fluctuations	0.2	year$^{-1/2}$
x	Initial population size	$0.5K$	kg
δ	Discount factor	0.05	year^{-1}
p_1	Linear price parameter	1.59	\$kg^{-1}
p_2	Quadratic price parameter	0	\$year · kg^{-2}
c_1	Linear cost parameter	$96 \cdot 10^{-6}$	\$SFU^{-1}year^{-1}
c_2	Quadratic cost parameter	$0.10 \cdot 10^{-6}$	\$SFU^{-2}year^{-1}
T	Time horizon	50	year

combination $n \times m$ were considered but, due to the computer execution time, to the algorithm convergence and algorithm stability, we choose the above values for n and m (the computer execution average time was of about 25 minutes for each scenario). The resulting profit values (5.1)–(5.4) for this basic scenario are shown in Table 5.2, where the first column refers to the optimal variable effort policy, the second column refers to the optimal constant effort policy, and the third column indicates the percent loss in the profit value when using the second policy instead of the first.

The first line of Table 5.2 compares the expected accumulated discounted profit (5.1) over the time horizon $T = 50$ years, V^* or V^{**} according to the policy used. One can see that the second policy implies a reduction in the expected profit of only 4.1% compared to the first policy. If one forgets depreciation and looks at the expected accumulated undiscounted profit (5.2), it shows a 4.9% expected profit reduction when comparing the second policy with the first. Obviously, the percent reductions are the same for the corresponding profits per year (5.3) and (5.4), obtained by taking time averages of these quantities over the 50 year horizon. Therefore, the profit reductions that occur when considering a constant effort instead of a variable effort are quite small and, with a constant effort, the fishery manager does not need to worry about the changes on the number of vessels, number of hooks or number of hours worked (just to name

Table 5.2: Numerical comparison between policies of the expected profits 1. to 4. (see expressions (5.1)–(5.4)) for the basic scenario S_0 (logistic model). The percent relative difference between the two policies is denoted by Δ. Besides the expected values, we also present the standard deviations (in parenthesis). Units are in million dollars for 1. and 2. and in million dollars per year for 3. and 4.

	Optimal policy			Optimal sustainable policy			Δ (%)
1.	$V^* \simeq$	413.586	(sd $= 38.322$)	$V^{**} \simeq$	396.423	(sd $= 34.948$)	-4.1
2.	$V_u^* \simeq$	1129.130	(sd $= 88.631$)	$V_u^{**} \simeq$	1073.867	(sd $= 88.543$)	-4.9
3.	$\overline{\Pi^*} \simeq$	22.529	(sd $= 2.087$)	$\overline{\Pi^{**}} \simeq$	21.594	(sd $= 1.904$)	-4.1
4.	$\overline{\Pi_u^*} \simeq$	22.583	(sd $= 1.773$)	$\overline{\Pi_u^{**}} \simeq$	21.477	(sd $= 1.771$)	-4.9

a few possibilities) required to keep adjusting the effort to its optimal value at each moment. This is extremely advantageous in terms of implementation and avoids out-of-model costs such as the purchase of new equipment to sustain increased effort periods or payment of unemployment benefits during effort reduction periods.

Besides the average profits, it is also interesting to look at their standard deviations, which measure the variability across the simulated trajectories. The variability of both policies is very similar (Table 5.2), with the optimal sustainable policy having a slightly lower variability.

Figure 5.1 shows, for the basic scenario S_0, what will happen when applying the optimal variable effort harvesting policy (left side) and the optimal constant effort sustainable policy (right side), in terms of the evolution, from time $t = 0$ to time $t = T = 50$ years, of the following quantities:

- Population size $X(t)$, on top;
- Optimal effort, in the middle: $E^*(t)$ (left) and E^{**} (right);
- Profit per unit time, at the bottom: $\Pi^*(t)$ (left) and $\Pi^{**}(t)$ (right).

The thin lines of Figure 5.1 show one randomly chosen trajectory, corresponding to a possible particular environmental behaviour. It shows what the harvester would typically observe. The figure also presents the expected values of the variables, which are averages taken over all possible environmental behaviours (the one effectively seen and all the others that might have occurred); dashed lines show the exact values at the steady-state (only available for the constant effort policy) and solid lines show good estimates at time t (based on averaging over a 1000 simulated trajectories). Looking at what the harvester typically experiences (thin lines in Figure 5.1), one can see that the two policies behave quite differently. While for

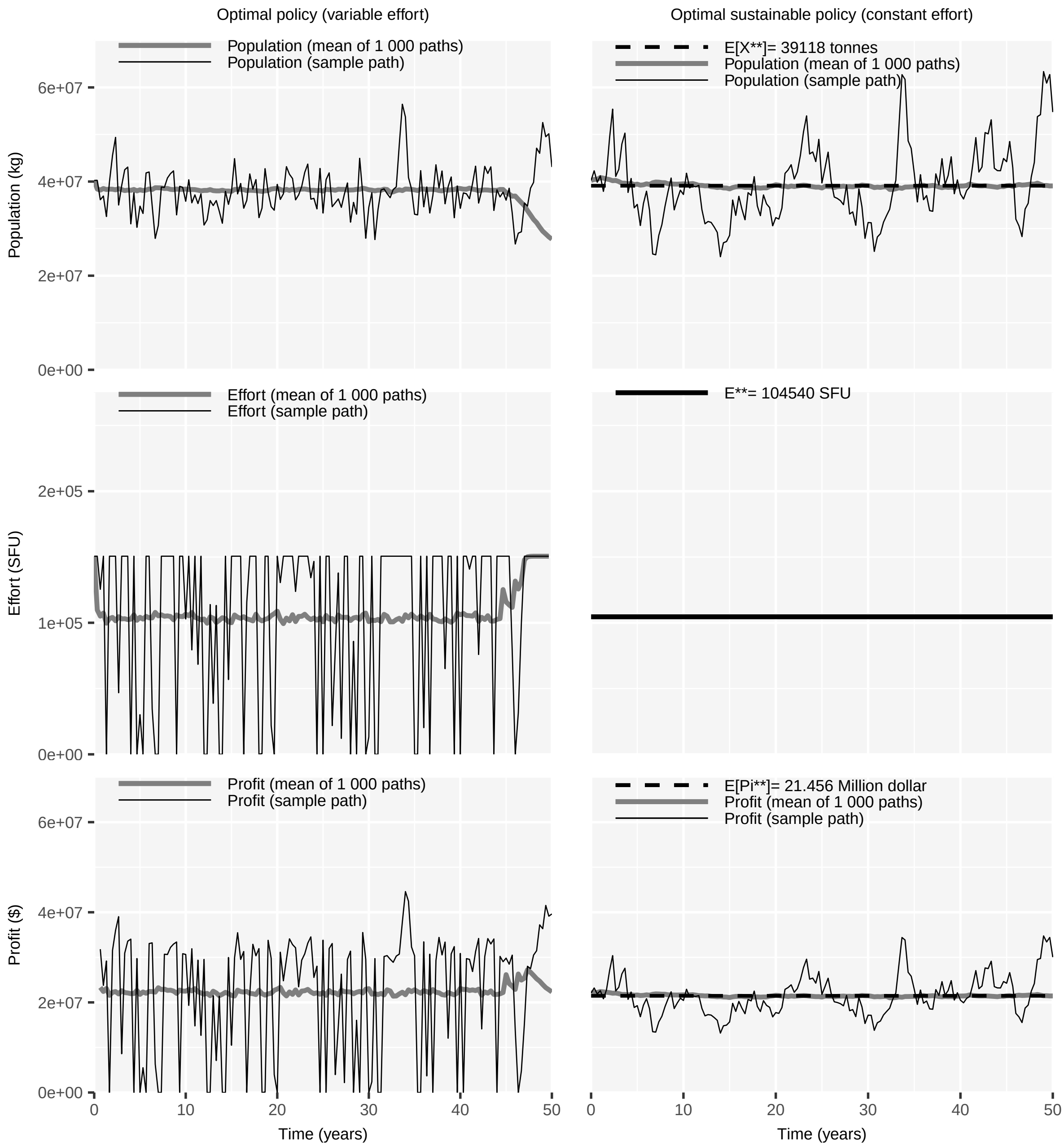


Figure 5.1: Basic scenario S_0 (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

the constant effort policy, we apply the same effort E^{**} irrespective of the population size path and of the environmental conditions (middle right, where, of course, we cannot distinguish between the solid, the thin and the dashed lines), in the optimal policy the effort $E^*(t)$ changes quite frequently and abruptly (thin line on the middle left). This sudden and frequent changes in effort, including frequent closings of the fishery, are not compatible with the fishing activity, since fishermen cannot accommodate frequent changes on the number of vessels, number of gears, number of hours at the sea, among others. Also, the optimal effort values depend on time and on the fish population size (which is influenced by the random fluctuations of environmental conditions), requiring constant evaluation of the fish stock size. Furthermore, it exhibits periods of no and low harvesting, posing social burdens and possible extra costs of unemployment compensation (not considered in our cost structure), and periods of harvesting at the maximum effort E_{max}, which may also involve extra costs (e.g., investment in backup equipment or hiring of extra employees not trained in fishing).

Besides looking at the variation of the effort over time, it is also interesting to look at the time variability experienced by the harvester on the the profit per unit time. For the basic scenario S_0, if we look at the thin lines at the bottom of Figure 5.1 (corresponding to the environmental conditions randomly selected), we see that the optimal policy has frequent periods of zero profit (the periods of zero effort) and a much larger profit variability over time. A good measure of this variability for the chosen trajectory is the sample standard deviation of the profit per unit time values observed at the time instants of the simulations. Such standard deviation is 13.77 million dollars per year for the optimal policy and only 4.38 million dollars per year for the optimal sustainable policy, which provides the harvester with a much steadier profit. Similar results hold when we select other trajectories.

One can also see in Figure 5.1 (left side) that the optimal variable effort policy exhibits a possibly dangerous effect near the time horizon, implying a considerable drop of the average population size (see solid line on top left), corresponding to an increase on the average effort (see solid line on middle left). This final effort increase is quite natural. Since "there is no tomorrow", it is better profitwise to harvest as much as is profitable "now", without worrying about stock preservation for near future fishing.

With the optimal sustainable policy, population size is driven to an equilibrium probability distribution with an average population size higher than the one of the variable effort policy. This expected size at equilibrium is the mean value of the stationary distribution given by (4.22). With the constant effort policy, there is no decay of the expected population size near the end of the time horizon.

Table 5.3: List of alternative scenarios ($S_i, i = 1, \ldots, 20$) and respective changed parameters (with respect to scenario S_0 of the logistic model). Units and unchanged parameters are as in Table 5.1.

Scenario	Changed parameter	Value	Scenario	Changed parameter	Value
S_1	x	$0.25K$	S_{11}	T	10
S_2	x	$0.75K$	S_{12}	T	25
S_3	E_{max}	$0.5r/q$	S_{13}	p_1	1.19
S_4	E_{max}	$0.9r/q$	S_{14}	p_1	1.99
S_5	δ	0.00	S_{15}	c_1	$72 \cdot 10^{-6}$
S_6	δ	0.10	S_{16}	c_1	$120 \cdot 10^{-6}$
S_7	r	0.10	S_{17}	c_2	$0.75 \cdot 10^{-7}$
S_8	r	0.40	S_{18}	c_2	$1.25 \cdot 10^{-7}$
S_9	σ	0.10	S_{19}	p_2	$5 \cdot 10^{-9}$
S_{10}	σ	0.40	S_{20}	p_2	$7.5 \cdot 10^{-9}$

5.2.2 Alternative scenarios

We will now evaluate the influence of the parameters, by considering alternative values, usually one lower and one higher than those in Table 5.1 for basic scenario S_0. For r and T, since S_0 values are high, we consider as alternatives two lower values. The parameters x and K can be scaled together, therefore we choose only to change x. The same applies to r and q, so we did not study changes in q.

This leads to alternative scenarios S_1 to S_{20} (Table 5.3). For each scenario, we have computed the profit values as in Table 5.2 and a similar set of images as in Figure 5.1. The profit values are shown in Tables 5.4 and 5.5 and the Figures are presented at the end of this section.

A general comment concerning Tables 5.4 and 5.5 is that, for almost all the scenarios, the percent reduction of profit Δ incurred by using the optimal constant effort policy instead of the optimal variable effort policy is quite small, of the order of 1.0% to 7.1% for discounted profits and of 1.6% to 7.3% for undiscounted profits. The exceptions are the scenarios S_7, S_{10}, and, in a quite more attenuated way, S_{11}, which exhibit quite higher Δ values. S_7 corresponds to a substantially lower value of r than in the basic scenario. S_{10} corresponds to a quite higher value of the intensity of environmental fluctuations. S_{11} has a very short time horizon ($T = 10$ years), far away from the steady-state for which the constant effort policies were designed. Actually, designing non-sustainable policies, like the optimal variable effort policies considered here, for such short time horizons is ill-advised since these policies care not about future

Table 5.4: Expected discounted profit values for the scenarios presented in Table 5.3 (logistic model). Besides the expected values, we also present the standard deviations (in parenthesis with smaller font size). The percent relative difference between both policies is denoted by Δ. Currency values are in million dollars for V^* and V^{**} and million dollars per year for $\overline{\Pi^*}$ and $\overline{\Pi^{**}}$.

Scenario	V^*		V^{**}		$\overline{\Pi^*}$		$\overline{\Pi^{**}}$		Δ (%)
S_0	413.586	(38.322)	396.423	(34.948)	22.529	(2.087)	21.594	(1.904)	−4.1
S_1	378.677	(37.012)	359.236	(34.198)	20.627	(2.016)	19.568	(1.863)	−5.1
S_2	440.168	(39.440)	418.376	(35.357)	23.977	(2.148)	22.789	(1.926)	−5.0
S_3	400.383	(34.692)	396.423	(34.948)	21.809	(1.890)	21.594	(1.904)	−1.0
S_4	415.555	(39.050)	396.423	(34.948)	22.636	(2.127)	21.594	(1.904)	−4.6
S_5	1130.837	(90.876)	1073.867	(88.543)	22.617	(1.818)	21.477	(1.771)	−5.0
S_6	226.805	(24.645)	215.756	(21.547)	22.834	(2.481)	21.722	(2.169)	−4.9
S_7	62.853	(21.648)	46.477	(15.072)	3.424	(1.179)	2.532	(0.821)	−26.1
S_8	233.516	(35.020)	216.837	(30.820)	12.720	(1.908)	11.811	(1.679)	−7.1
S_9	418.497	(19.555)	412.081	(17.921)	22.796	(1.065)	22.447	(0.976)	−1.5
S_{10}	379.404	(71.883)	334.088	(62.820)	20.667	(3.916)	18.198	(3.422)	−11.9
S_{11}	187.693	(28.181)	172.121	(24.039)	23.851	(3.581)	21.872	(3.055)	−8.3
S_{12}	326.509	(36.801)	309.366	(33.403)	22.881	(2.579)	21.680	(2.341)	−5.3
S_{13}	310.182	(28.741)	297.312	(26.210)	16.896	(1.566)	16.195	(1.428)	−4.1
S_{14}	516.990	(47.904)	495.535	(43.685)	28.161	(2.609)	26.992	(2.380)	−4.1
S_{15}	413.586	(38.322)	396.423	(34.948)	22.529	(2.087)	21.594	(1.904)	−4.1
S_{16}	413.586	(38.322)	396.423	(34.948)	22.529	(2.087)	21.594	(1.904)	−4.1
S_{17}	413.593	(38.323)	396.429	(34.948)	22.529	(2.087)	21.594	(1.904)	−4.1
S_{18}	413.579	(38.322)	396.417	(34.947)	22.528	(2.087)	21.593	(1.904)	−4.1
S_{19}	391.082	(34.396)	378.514	(31.865)	21.303	(1.874)	20.618	(1.736)	−3.2
S_{20}	380.307	(32.566)	369.560	(30.326)	20.716	(1.774)	20.130	(1.652)	−2.8

Table 5.5: Expected undiscounted profit values for the scenarios presented in Table 5.3 (logistic model). Besides the expected values, we also present the standard deviations (in parenthesis with smaller font size). The percent relative difference between both policies is denoted by Δ. Currency values are in million dollars for V_u^* and V_u^{**} and million dollars per year for $\overline{\Pi_u^*}$ and $\overline{\Pi_u^{**}}$.

Scenario	V_u^*		V_u^{**}		$\overline{\Pi_u^*}$		$\overline{\Pi_u^{**}}$		Δ (%)
S_0	1129.130	(88.631)	1073.867	(88.543)	22.583	(1.773)	21.477	(1.771)	−4.9
S_1	1092.856	(87.937)	1030.569	(88.306)	21.857	(1.759)	20.611	(1.766)	−5.7
S_2	1157.066	(89.347)	1098.360	(88.699)	23.141	(1.787)	21.967	(1.774)	−5.1
S_3	1092.263	(83.692)	1073.867	(88.543)	21.845	(1.674)	21.477	(1.771)	−1.7
S_4	1139.690	(89.458)	1073.867	(88.543)	22.794	(1.789)	21.477	(1.771)	−5.8
S_5	1130.837	(90.876)	1073.867	(88.543)	22.617	(1.818)	21.477	(1.771)	−5.0
S_6	1113.768	(84.942)	1073.867	(88.543)	22.275	(1.699)	21.477	(1.771)	−3.6
S_7	148.820	(63.768)	122.197	(47.648)	2.976	(1.275)	2.444	(0.953)	−17.9
S_8	628.749	(83.487)	582.937	(83.202)	12.575	(1.670)	11.659	(1.664)	−7.3
S_9	1138.575	(43.631)	1120.752	(45.206)	22.772	(0.873)	22.415	(0.904)	−1.6
S_{10}	1026.306	(169.107)	887.617	(161.596)	20.526	(3.382)	17.752	(3.232)	−13.5
S_{11}	238.555	(35.192)	218.376	(31.525)	23.855	(3.519)	21.838	(3.153)	−8.5
S_{12}	574.000	(61.020)	539.991	(59.646)	22.960	(2.441)	21.600	(2.386)	−5.9
S_{13}	846.828	(66.472)	805.385	(66.406)	16.937	(1.329)	16.108	(1.328)	−4.9
S_{14}	1411.431	(110.790)	1342.348	(110.679)	28.229	(2.216)	26.847	(2.214)	−4.9
S_{15}	1129.130	(88.631)	1073.867	(88.543)	22.583	(1.773)	21.477	(1.771)	−4.9
S_{16}	1129.130	(88.631)	1073.866	(88.543)	22.583	(1.773)	21.477	(1.771)	−4.9
S_{17}	1129.149	(88.632)	1073.881	(88.544)	22.583	(1.773)	21.478	(1.771)	−4.9
S_{18}	1129.111	(88.630)	1073.852	(88.541)	22.582	(1.773)	21.477	(1.771)	−4.9
S_{19}	1064.048	(80.030)	1025.457	(80.777)	21.281	(1.601)	20.509	(1.616)	−3.6
S_{20}	1034.011	(76.034)	1001.253	(76.899)	20.680	(1.521)	20.025	(1.538)	−3.2

preservation of the stock.

To check the effect of changes in a given parameter, we can compare the results for the basic scenario S_0 with the results of the scenarios corresponding to alternative values of such parameter, using Tables 5.4 and 5.5 and the Figures associated to those scenarios.

Changing the initial population x (scenarios S_1 and S_2) affects, for the optimal variable effort policy, the expected values of population size, harvesting effort and profit per unit time. This happens only at the start of the projection period and is due to the longer time it takes for these expected values to approach their main trends (compare the left side of Figure 5.1 with the left sides of Figures 5.2 and 5.3). As for the constant effort policy, since the effort is designed assuming a steady-state, x has no effect on effort. However, like in the variable effort policy, it has an effect on the expected population size and on the expected profit per unit time at the beginning (before the approach to the mean trend), but, since the process is ergodic, has no effect in the long-term (as $T \to \infty$).

Constraining the maximum effort to $0.5\,r/q$ (scenario S_3), which is very close to E^{**}, almost mimics the behaviour of the constant effort policy. In fact, in this scenario, the difference between V^* and V^{**} is very small. On Figure 5.4 we can confirm the similarities between the two policies. Raising the maximum effort to $0.9\,r/q$ (scenario S_4) gives, for the optimal variable effort policy, similar results to the ones in scenario S_0, although with slightly higher profit due to the fact that the restriction on effort is milder. Obviously, for the optimal constant effort policy, if $E_{max} \geq E^{**}$, the value of E_{max} is irrelevant.

Changing δ (scenarios S_5 and S_6) will, of course, not change the undiscounted profits for the optimal constant effort policy since this policy is designed to optimize the steady-state undiscounted profit per unit time. It has, however, a large effect on the accumulated discounted profit of both constant effort and variable effort policies, but has little effect on average profits per unit time (both discounted and undiscounted). It is also slight the effect, under the variable effort optimal policy, on the time evolution of the expected values of population size, optimal variable effort and profit per unit time (see left sides of Figures 5.1, 5.7 and 5.8).

Intrinsic growth rates lower than 0.71 (scenarios S_7 and S_8) imply lower biomass growth and, consequently, also lower profit values, since the optimal harvesting rates will be smaller for both policy types.

Comparing scenarios S_9 and S_{10} with S_0 shows that a higher intensity of environmental fluctuations reduces the expected profit for both types of policies, although the effect is quite mild. Contrary to the average, the influence on sample trajectories (which is what is experienced) of population size is quite profound. Although averages do not change much, fluctuations of the population size about its average will be more intense when σ is high and will almost fade away as σ approaches zero (deterministic environment). Obviously, sample paths of the profit per unit time will respond to changes in population size and, in the case of the variable effort policy, the same happens to the effort. For the sustainable policy, we had already seen on Section 4.2, from the steady-state expressions (4.20) and (4.21) of the optimal effort E^* and the optimal profit per unit time $\mathbb{E}[\Pi_\infty^{**}]$, that these quantities decrease with a higher σ.

As already mentioned, when the terminal time T decreases (scenarios S_{11} and S_{12}), the differences between the two policies are more pronounced. In fact, the optimal sustainable policy 'needs more time' to get close to the stochastic steady-state. The accumulated profits are, in relation to the base scenario, much smaller since we are talking about shorter periods of time. However, the average profits per unit time are very close to the ones in the S_0 scenario.

A decrease of 25% (scenario S_{13}) or increase of 25% (scenario S_{14}) in the linear price parameter p_1 will have an effect of similar magnitude in profit. This is due to the fact that profit is dominated by the effect of the price parameter p_1, since the cost parameters c_1 and c_2 have, in this case, a low magnitude. For the same reason, variations in the cost parameters, c_1 and c_2, have very little influence on profit values.

Considering a positive value for the quadratic price parameter p_2 (scenarios S_{19} and S_{20}) produces even lower profit differences between the two policies. Comparing these two scenarios with the base scenario S_0 shows also a slight profit reduction. This behaviour is explained by the profit structure, namely by the negative signal of p_2 on expressions (3.6) and (4.10).

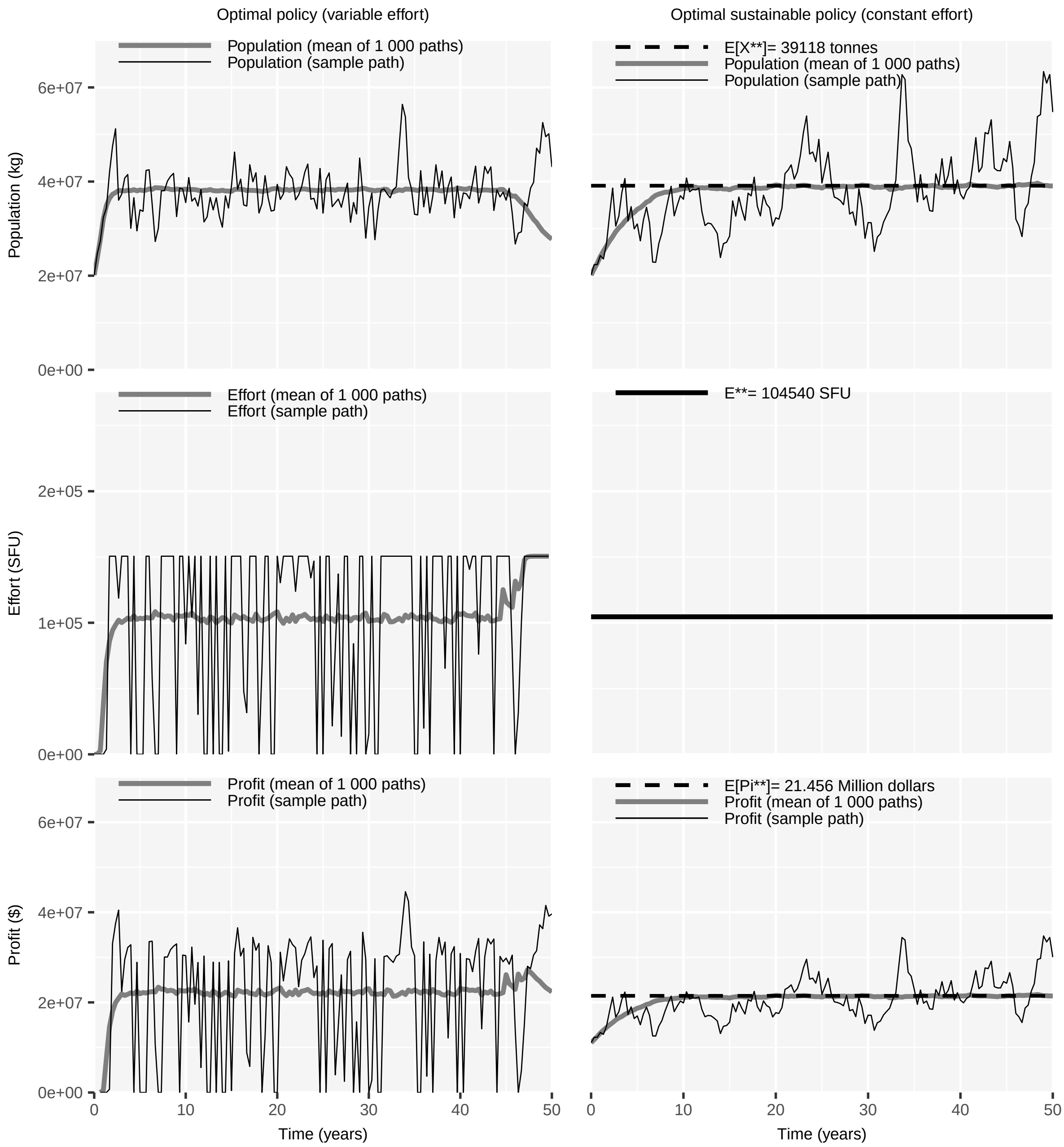


Figure 5.2: Scenario S_1 (logistic model): mean and randomly chosen sample path for the population, the effort and the profit. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

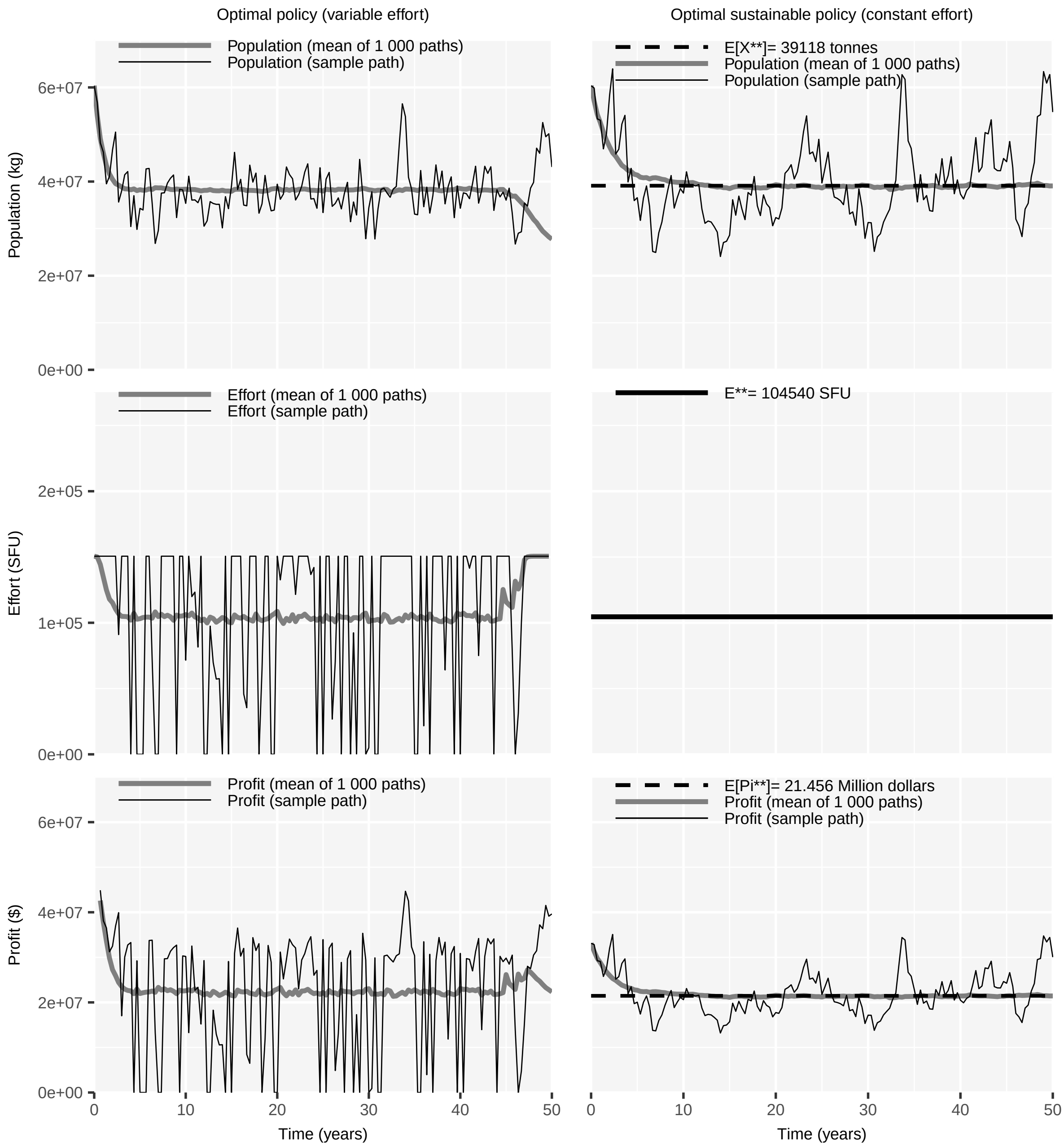


Figure 5.3: Scenario S_2 (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

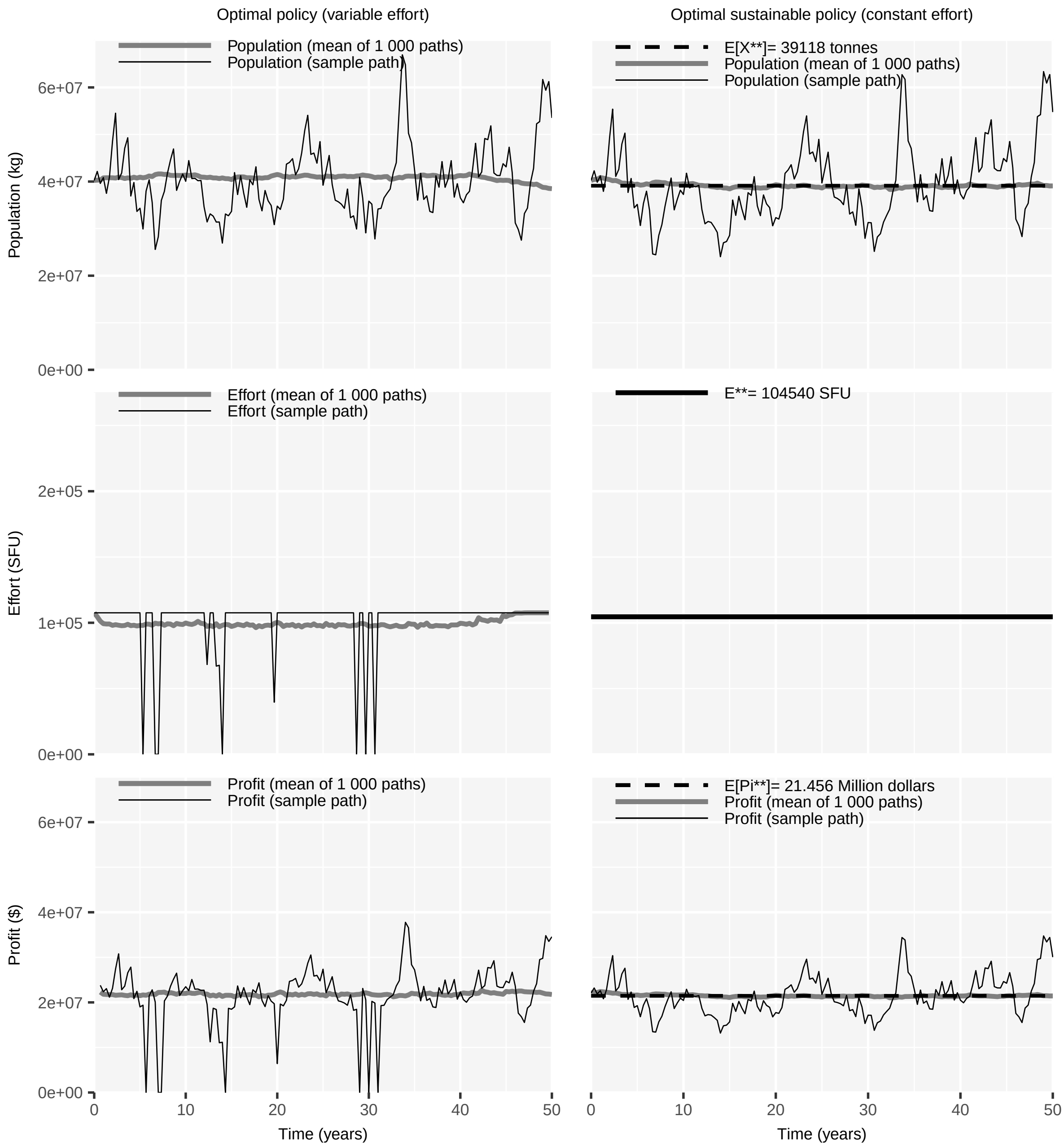


Figure 5.4: Scenario S_3 (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

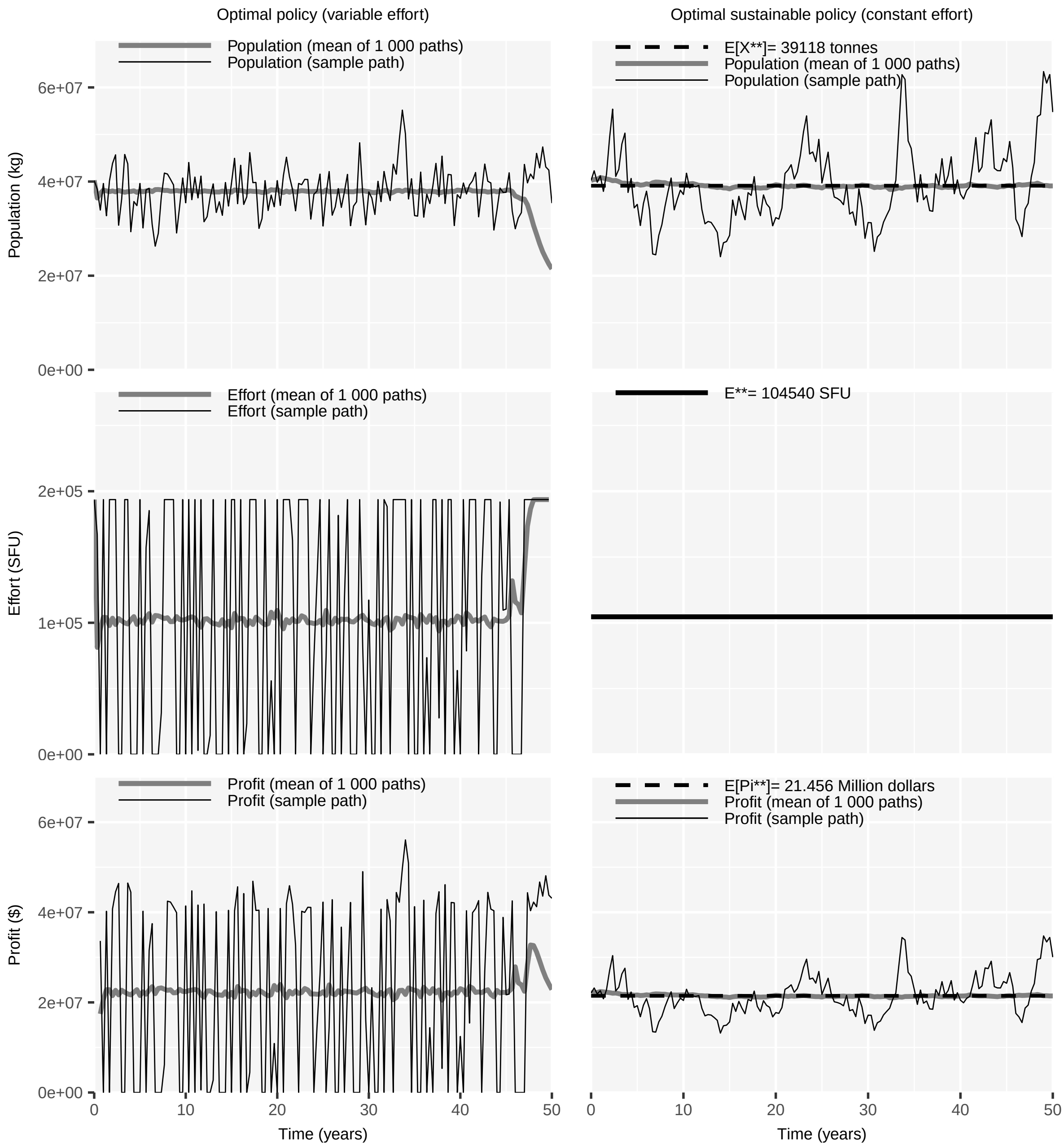


Figure 5.5: Scenario S_4 (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

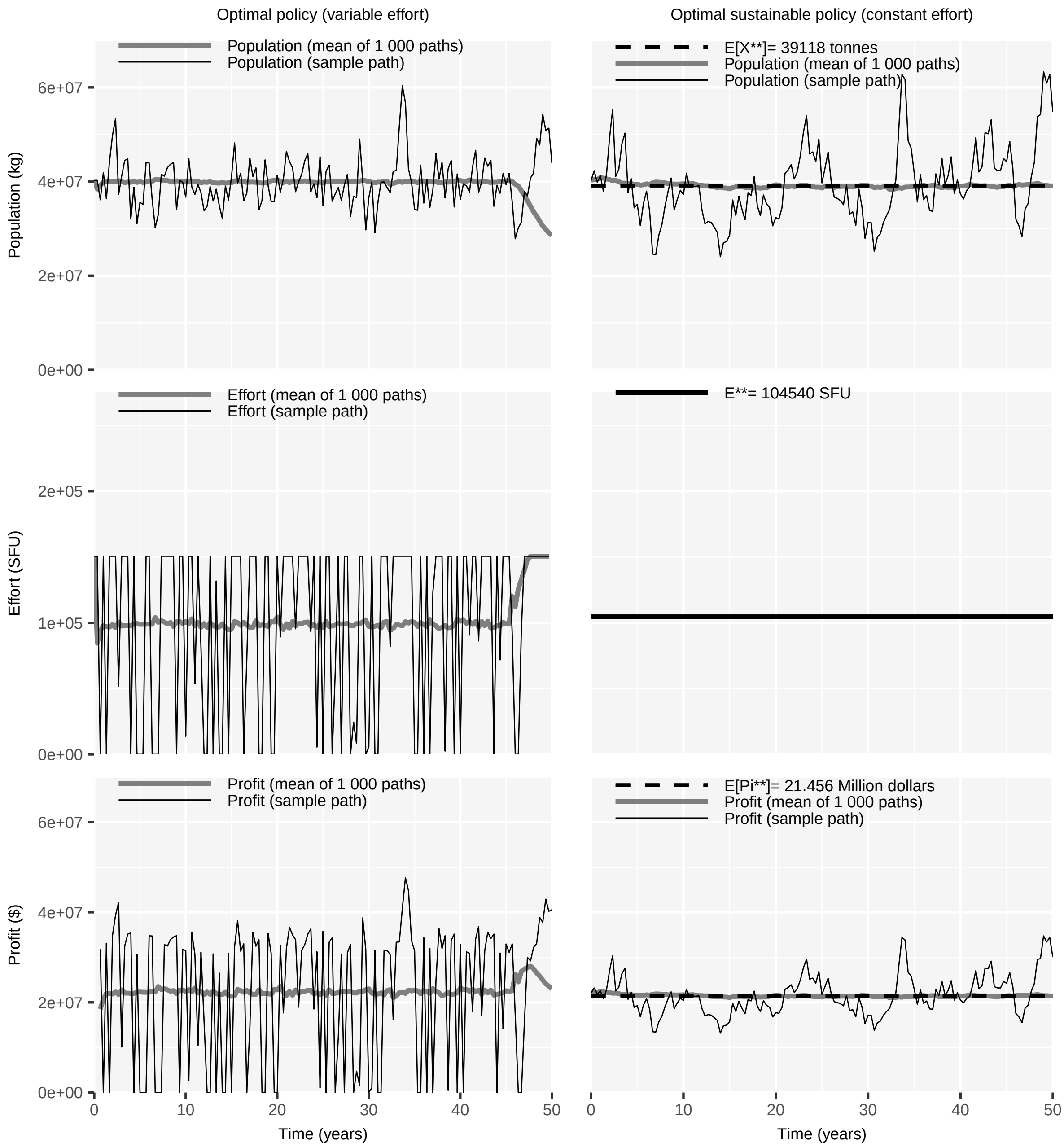


Figure 5.6: Scenario S_5 (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

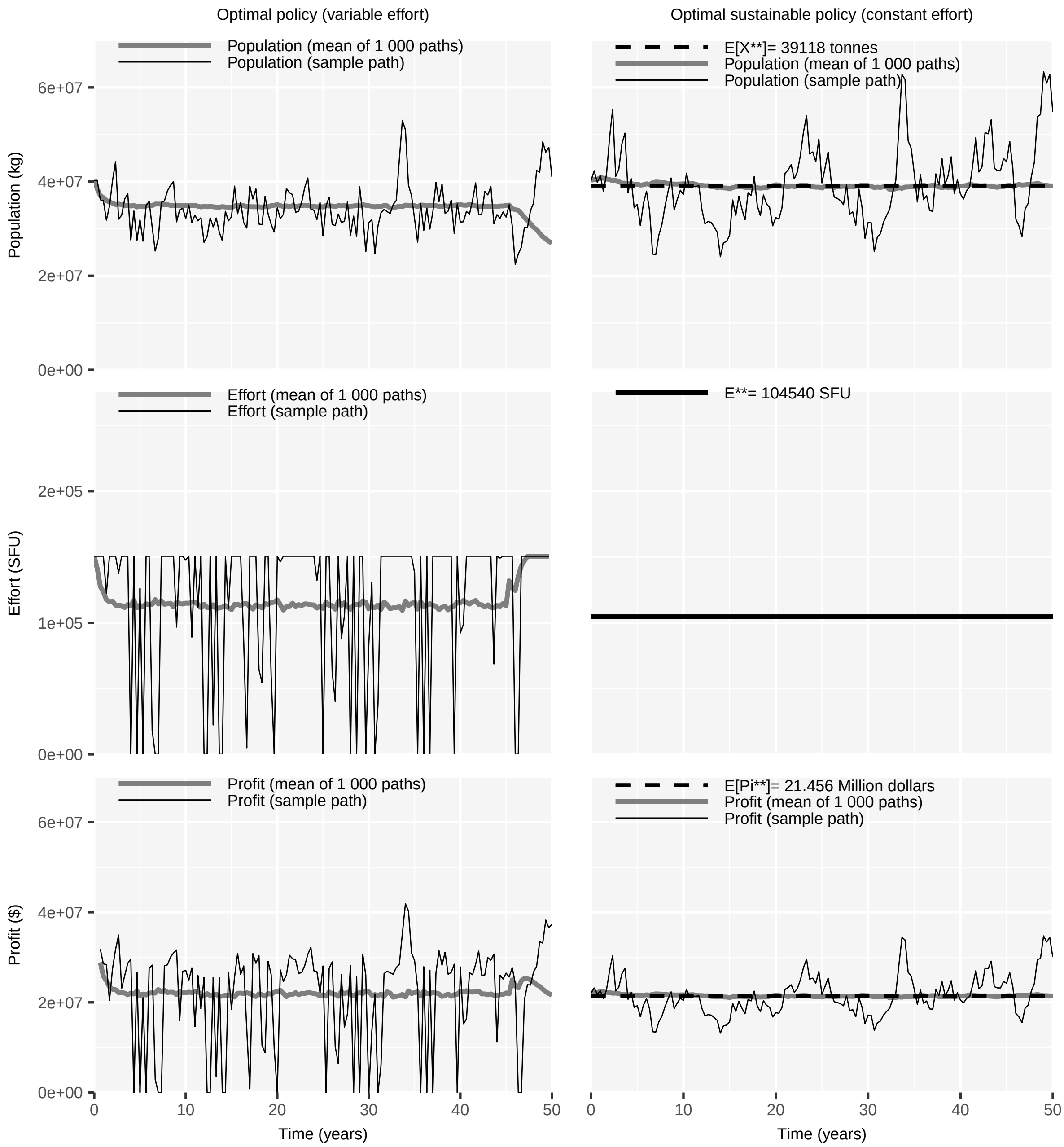


Figure 5.7: Scenario S_6 (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

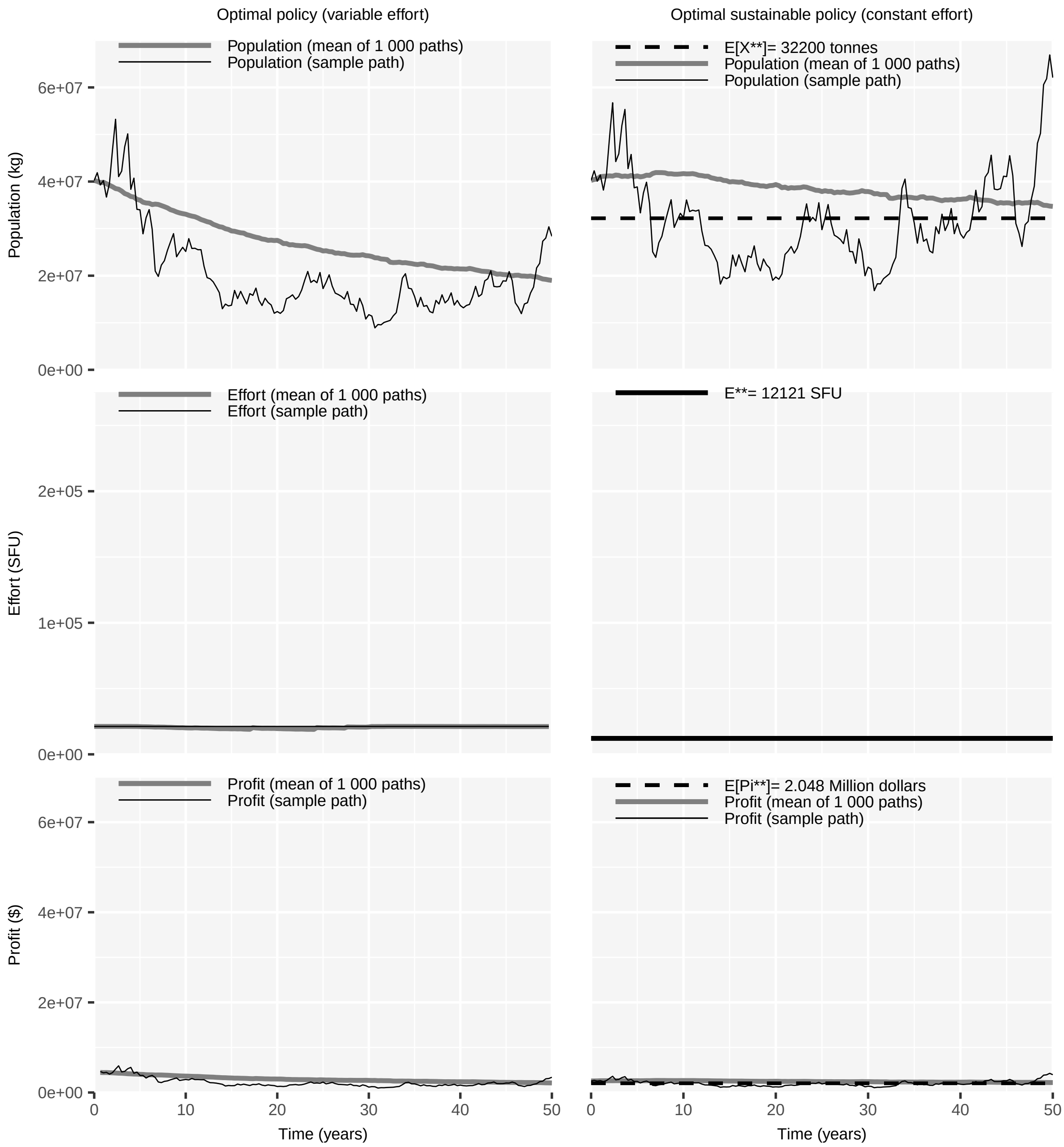


Figure 5.8: Scenario S_7 (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

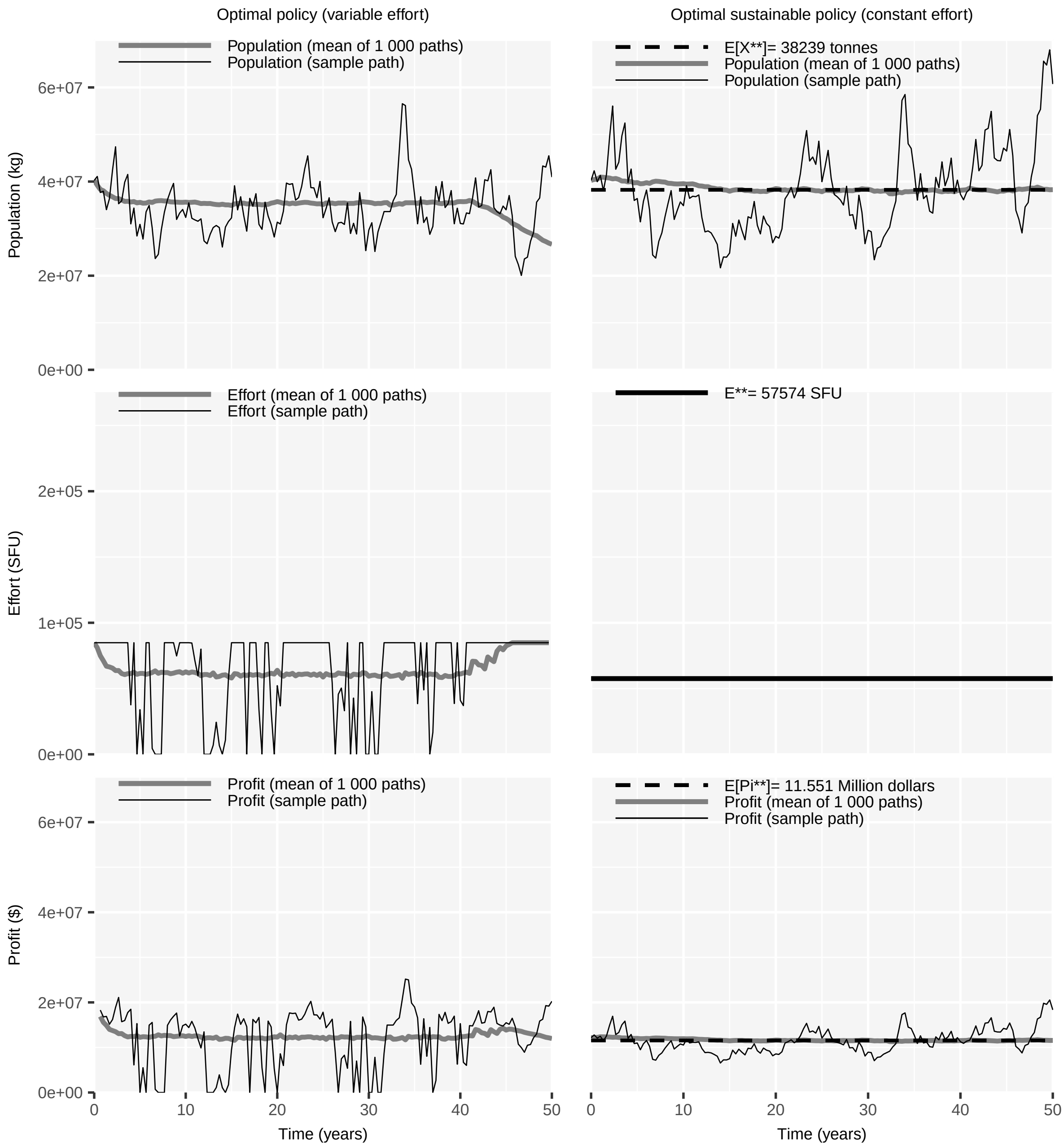


Figure 5.9: Scenario S_8 (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

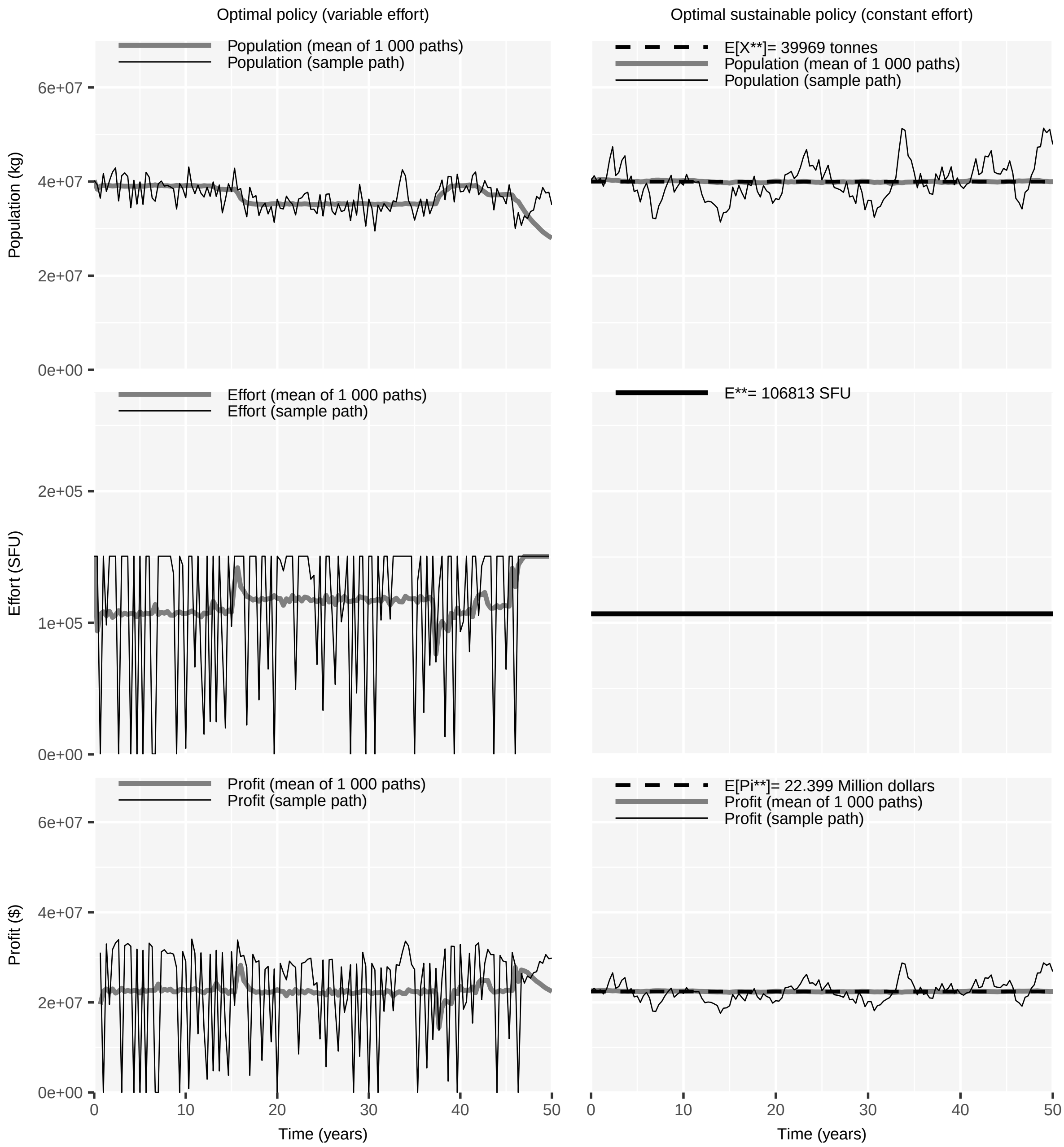


Figure 5.10: Scenario S_9 (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

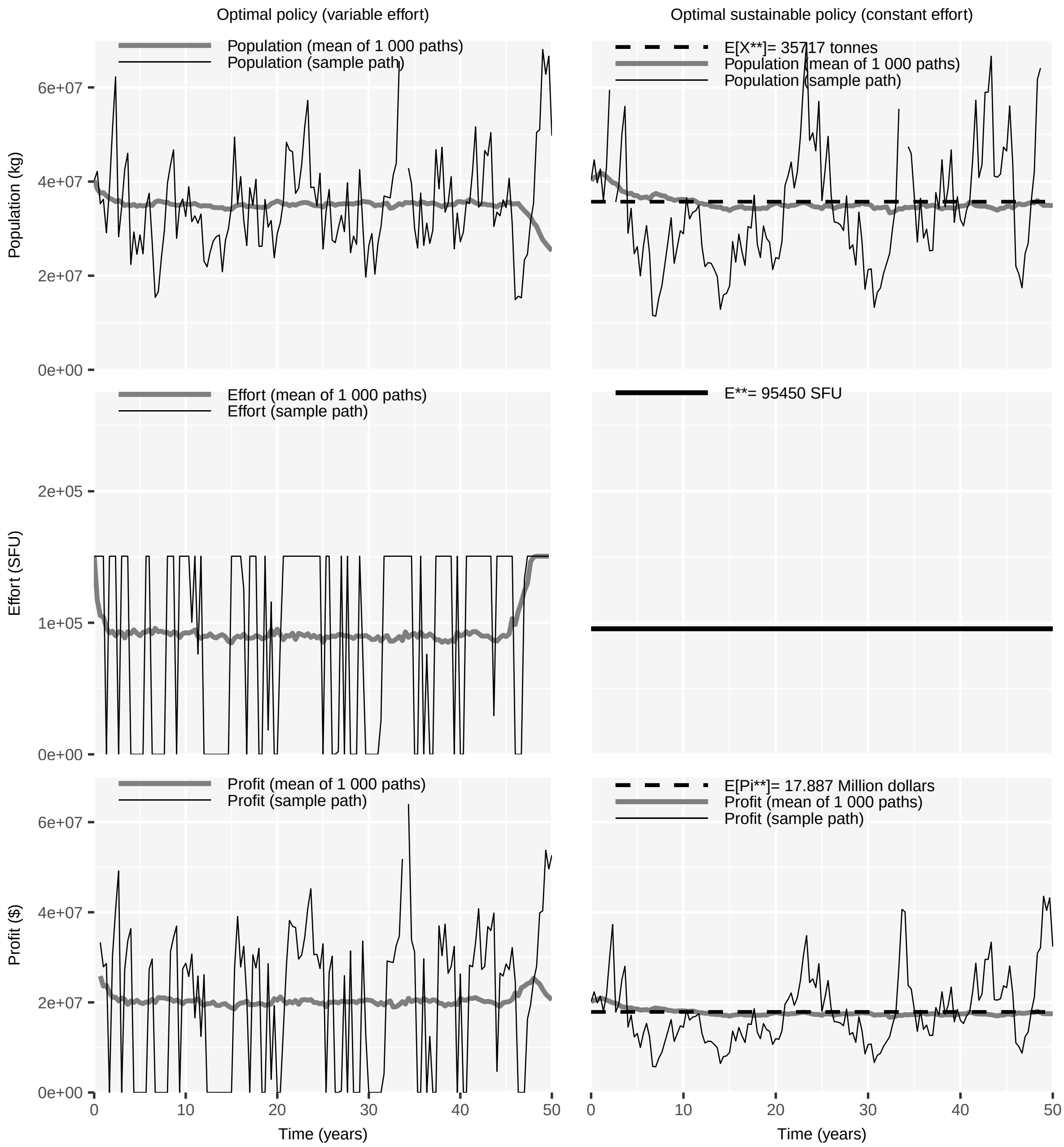


Figure 5.11: Scenario S_{10} (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

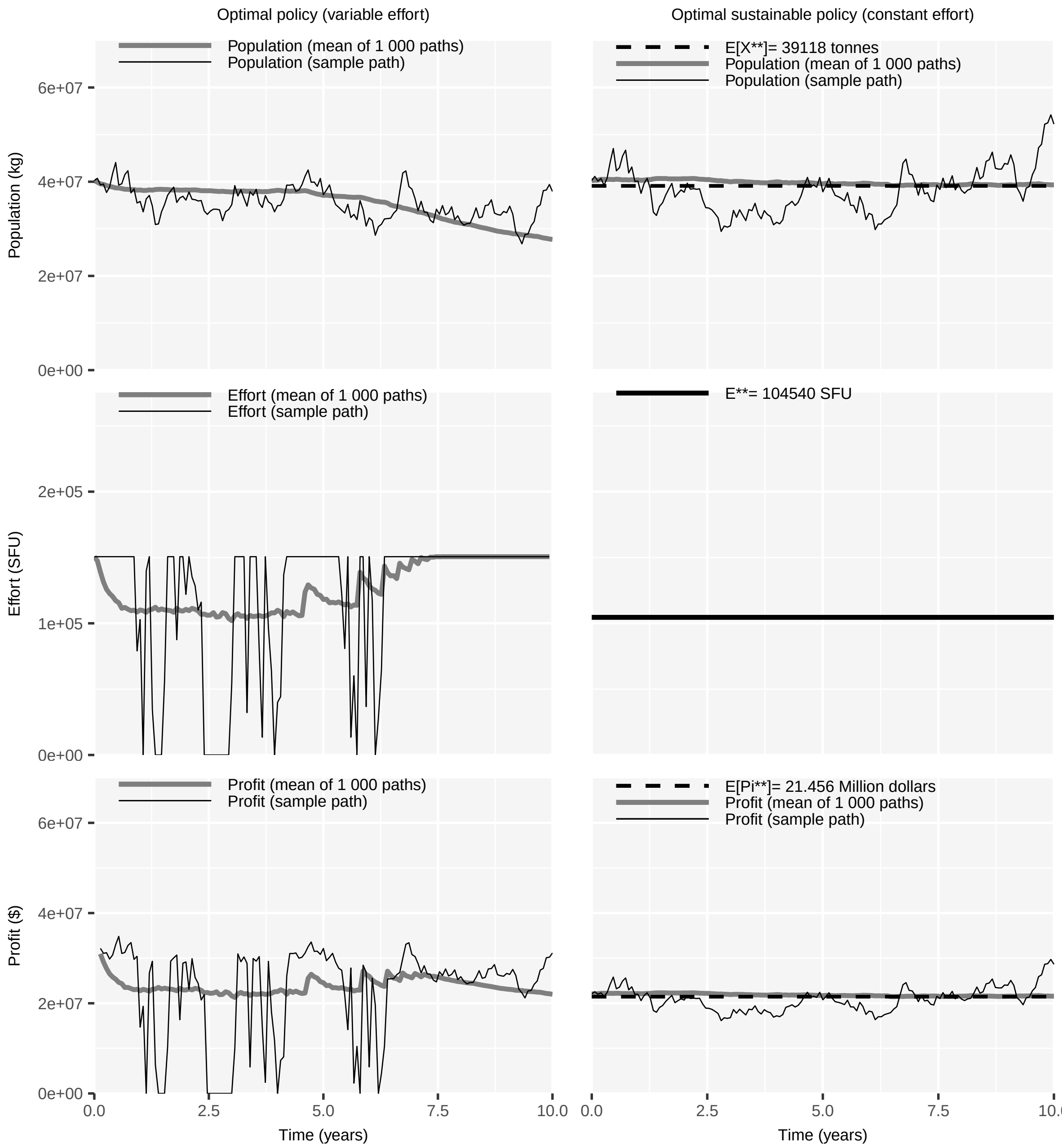


Figure 5.12: Scenario S_{11} (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

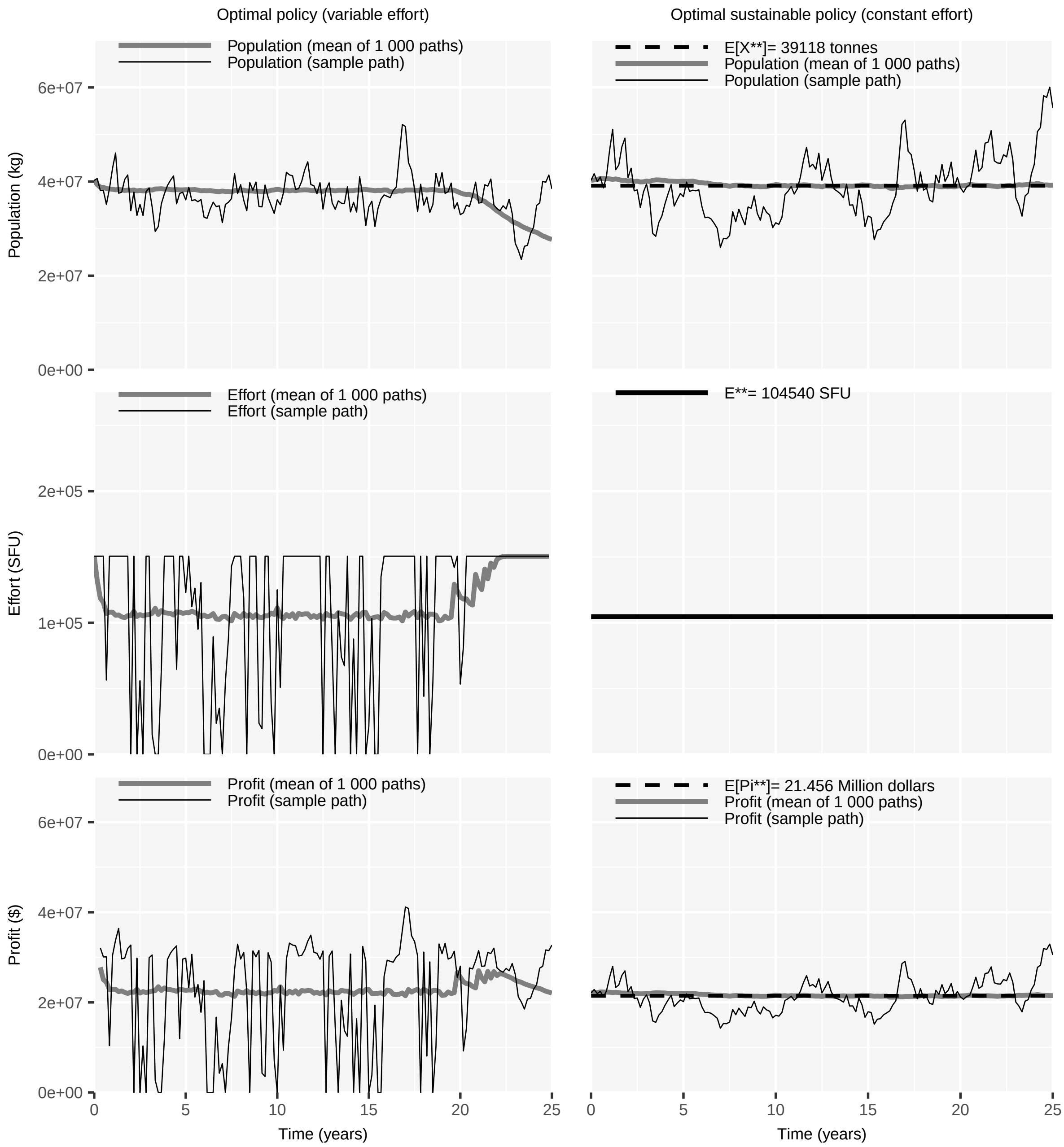


Figure 5.13: Scenario S_{12} (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

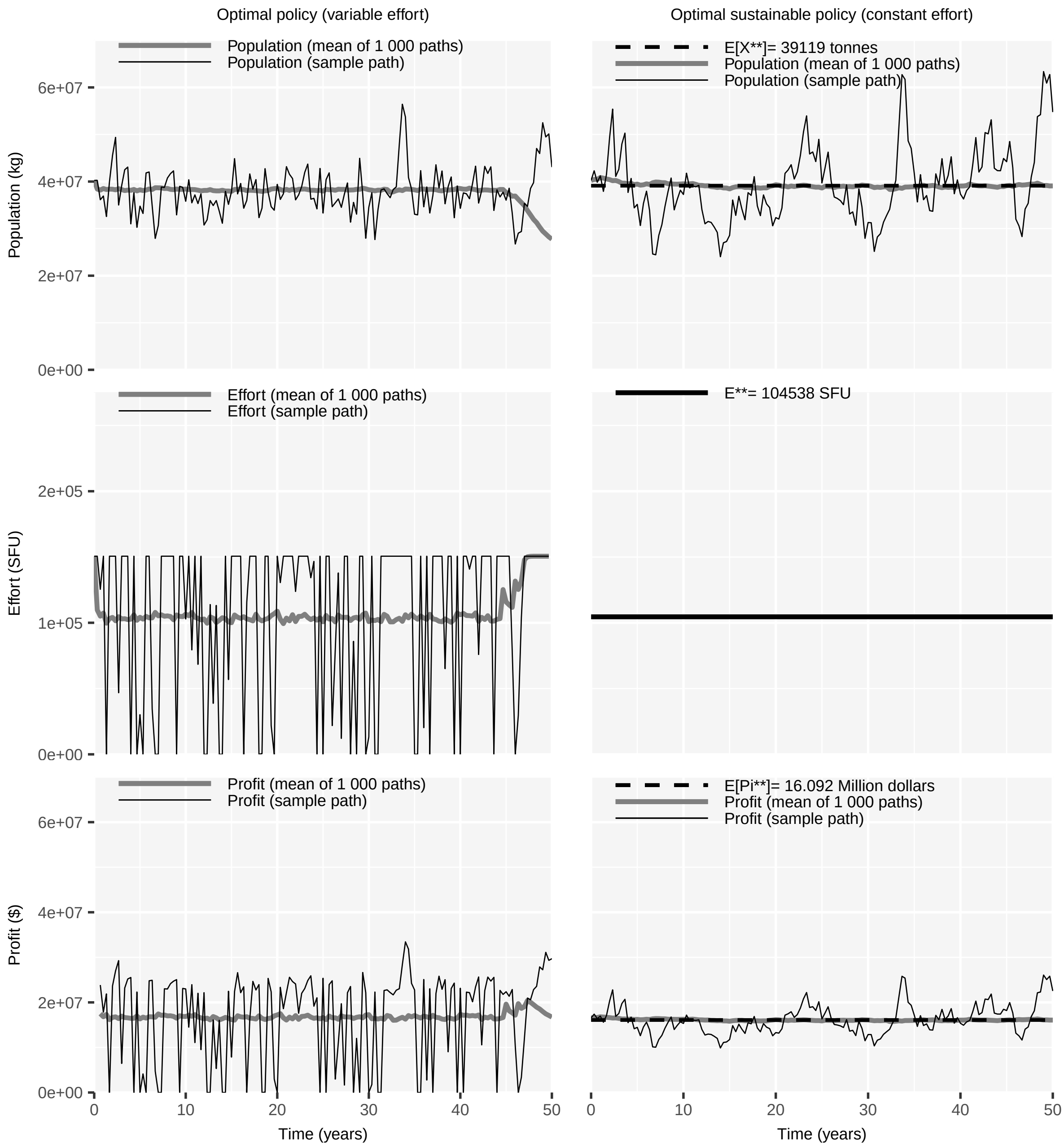


Figure 5.14: Scenario S_{13} (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

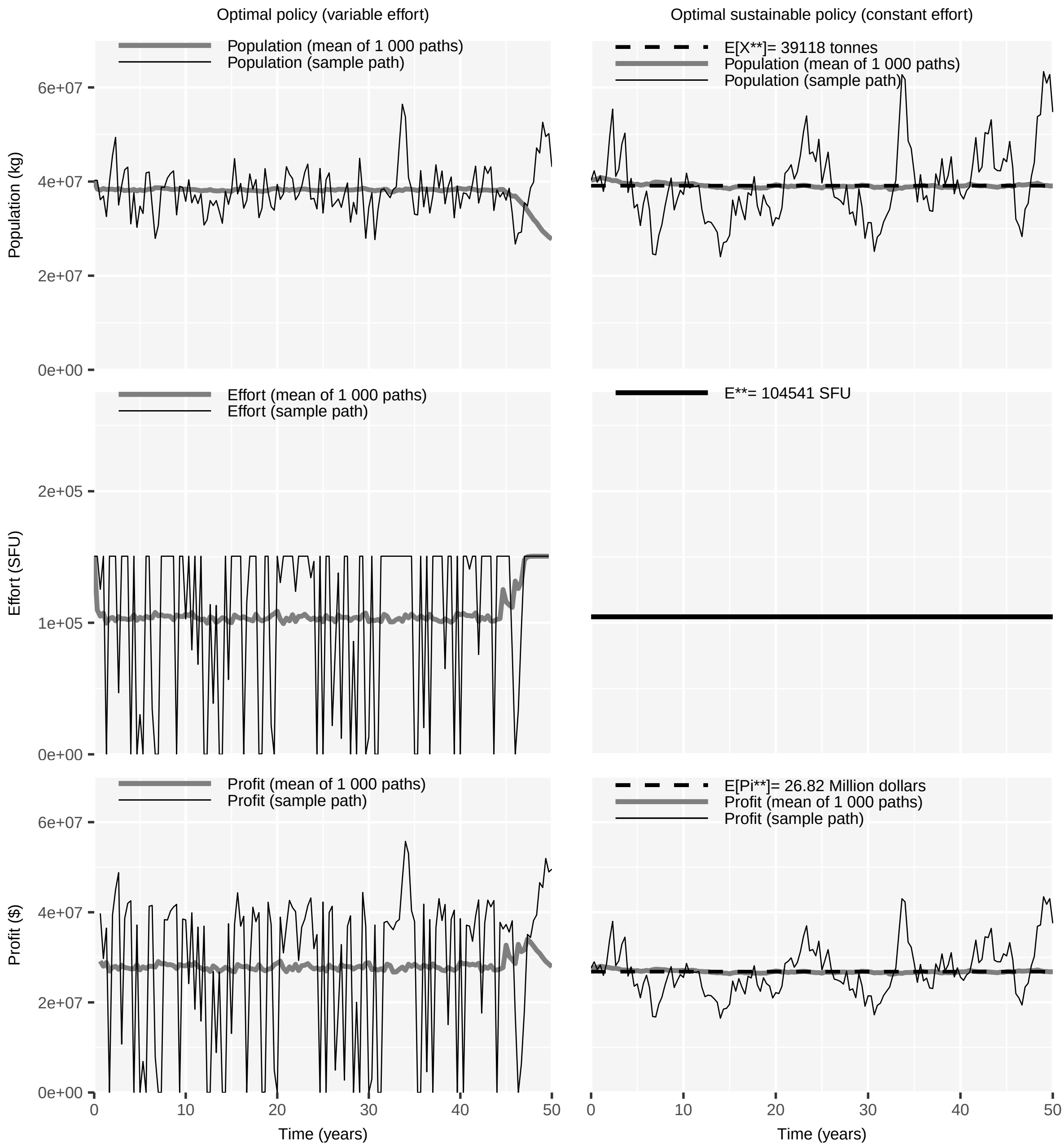


Figure 5.15: Scenario S_{14} (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

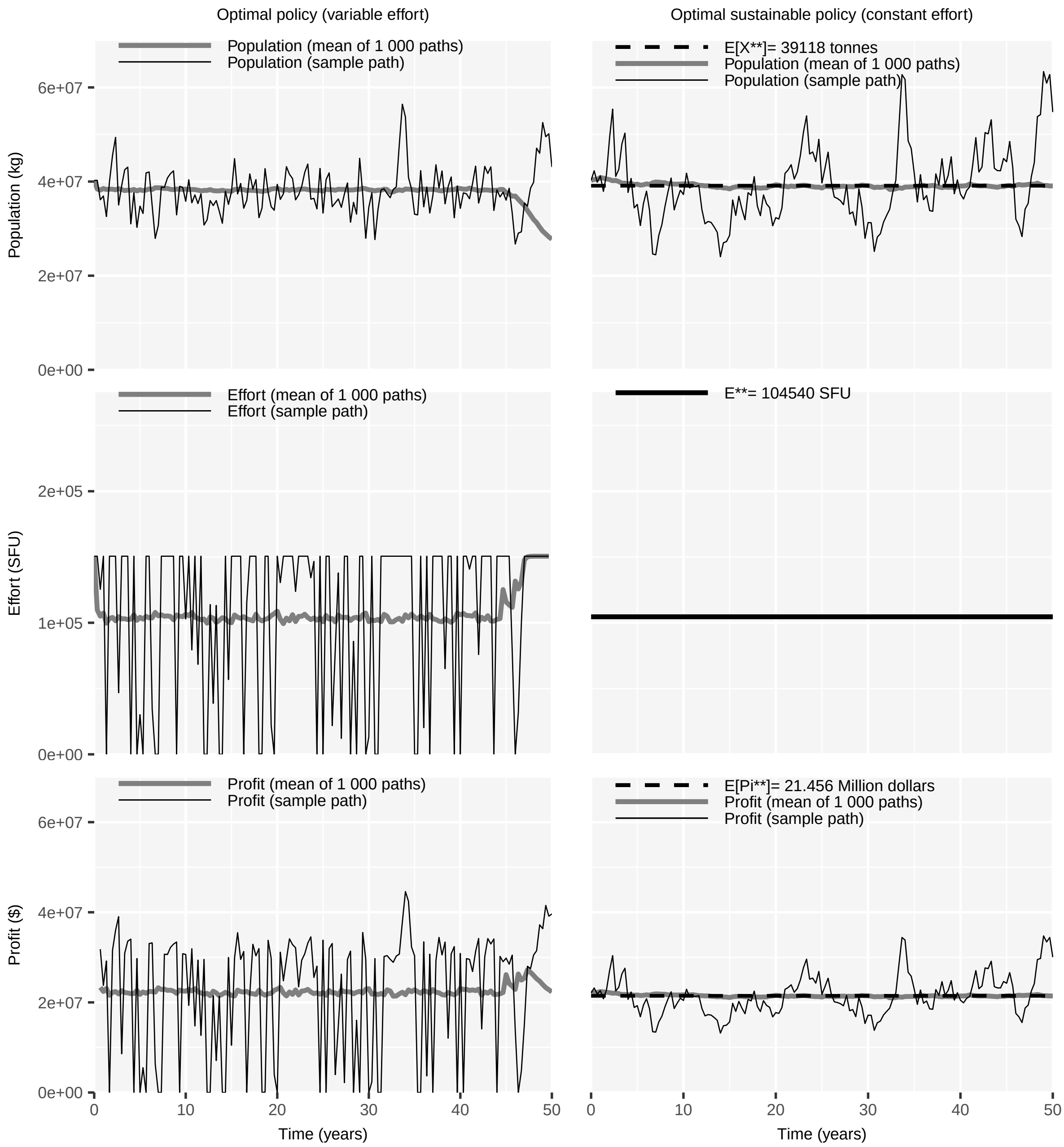


Figure 5.16: Scenario S_{15} (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

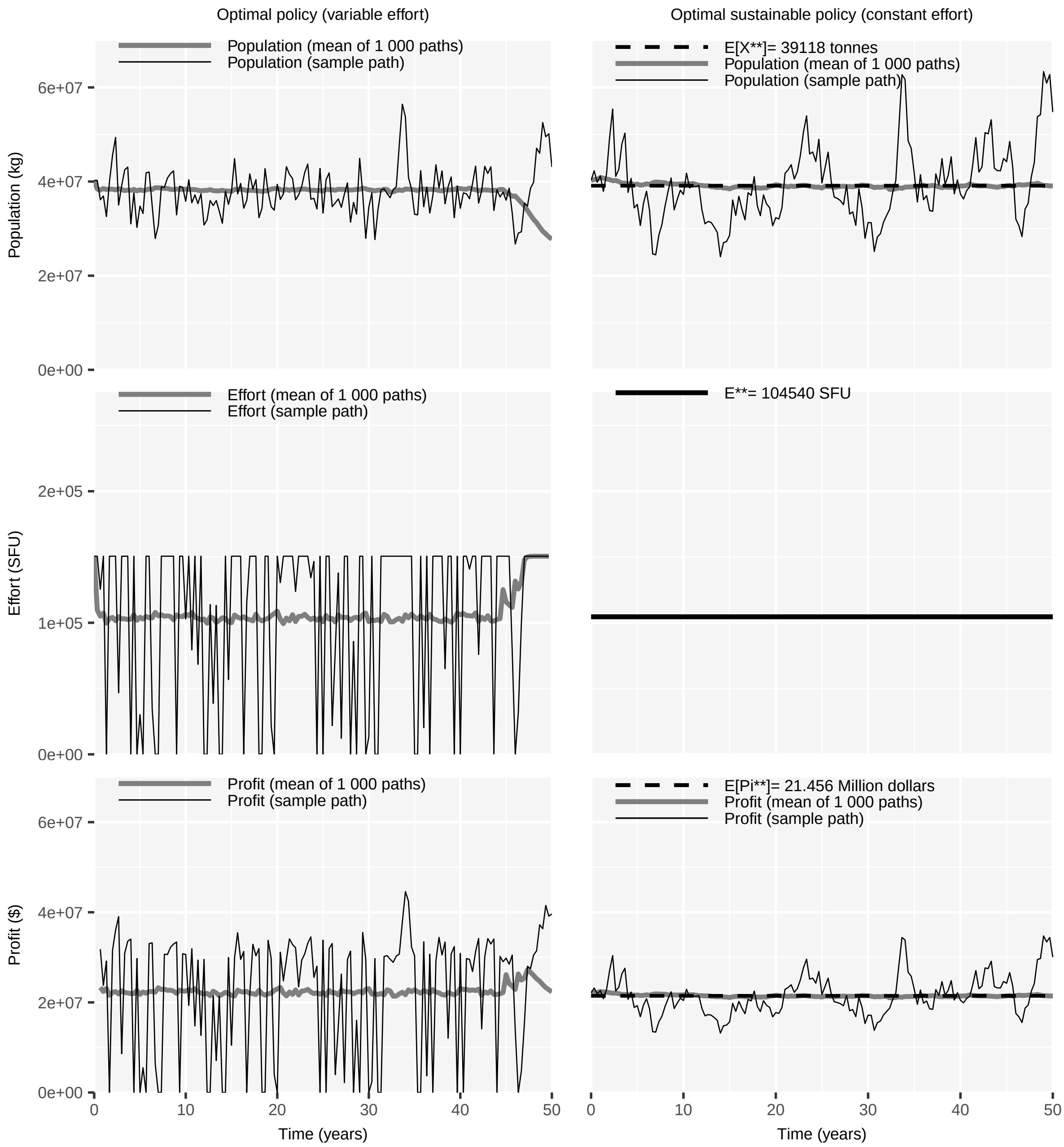


Figure 5.17: Scenario S_{16} (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

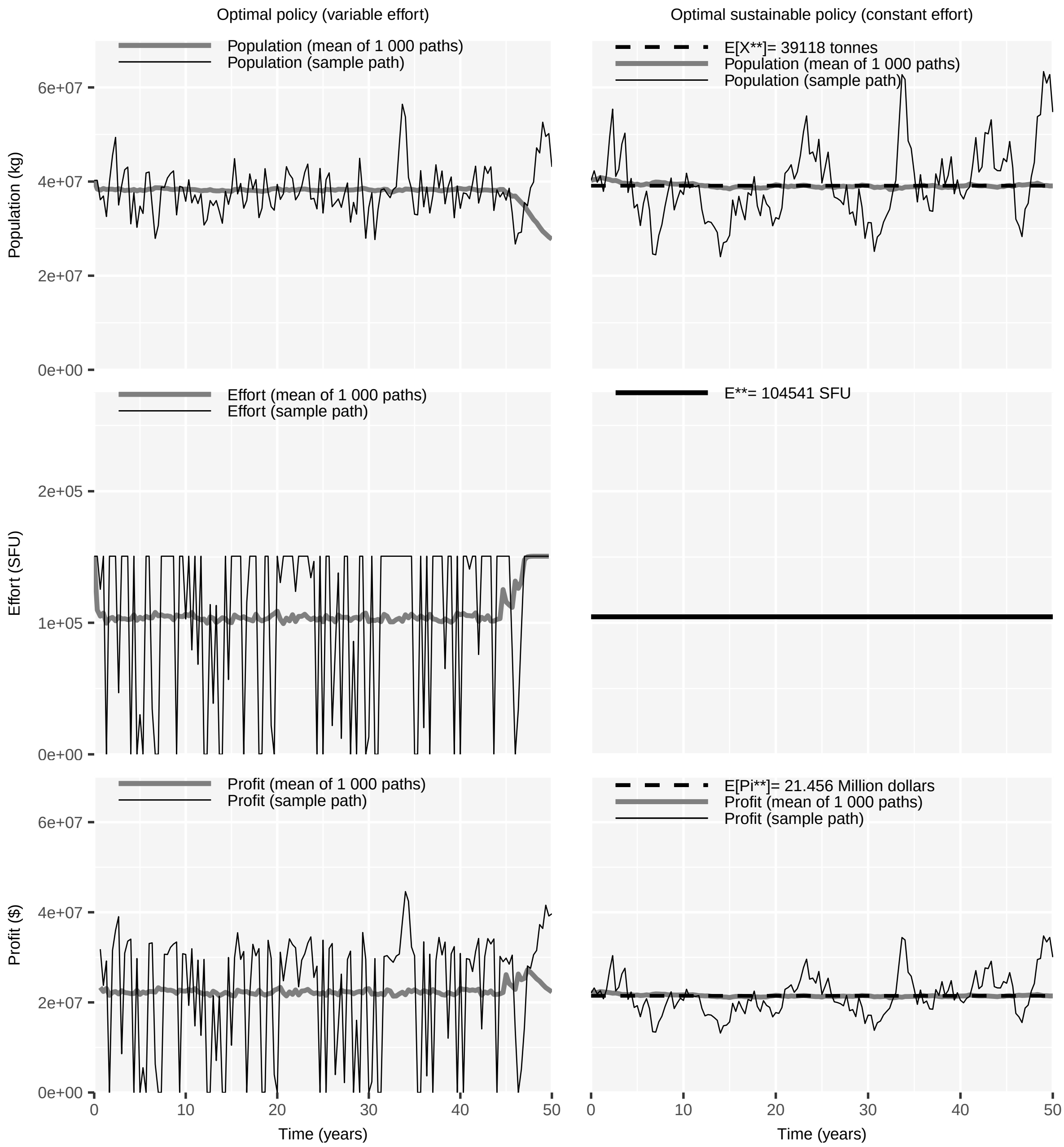


Figure 5.18: Scenario S_{17} (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

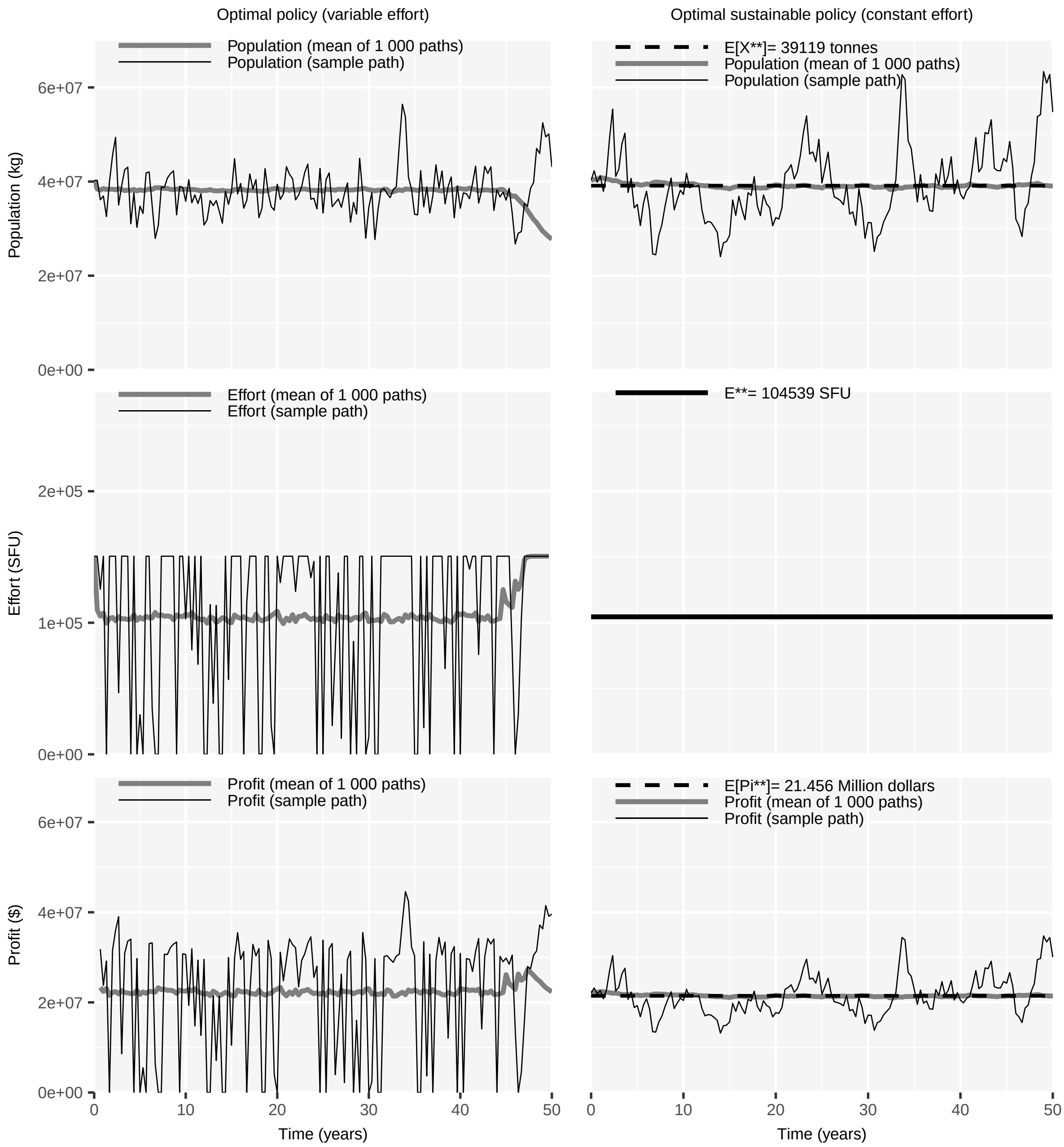


Figure 5.19: Scenario S_{18} (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

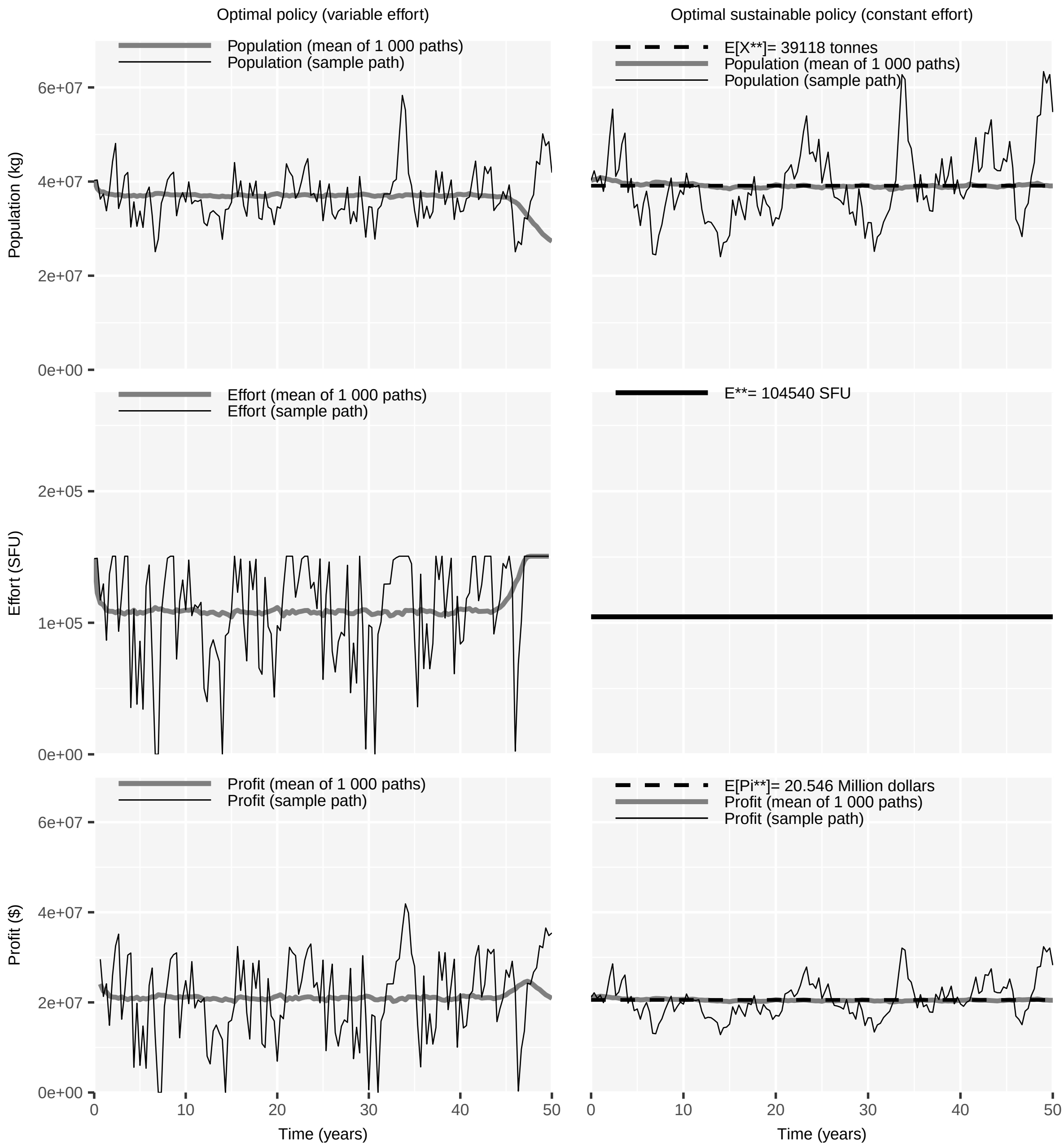


Figure 5.20: Scenario S_{19} (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

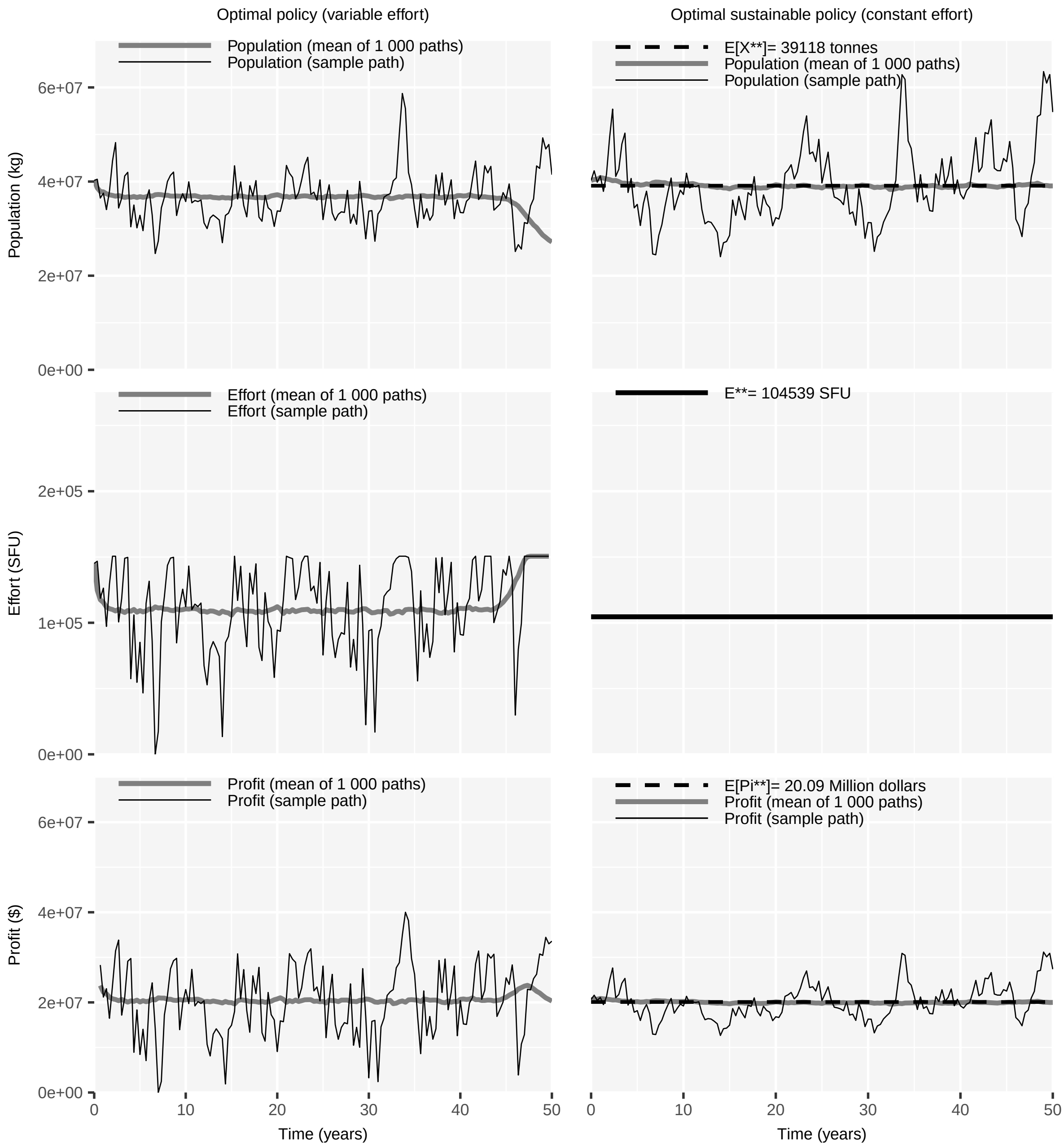


Figure 5.21: Scenario S_{20} (logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

5.3 Comparison of polices with the Gompertz growth model

5.3.1 Basic scenario

To apply the Gompertz model and to set up the basic scenario S_0 we used the parameters values found in Kar and Chakraborty (2011) which refer to the Bangladesh shrimp (*Penaeus monodon*), a kind of species for which the Gompertz model is traditionally considered more suitable (see, for instance, ASMFC, 2009). Other parameters not present in the literature ($E_{min}, E_{max}, \sigma, x,$ p_2 and δ) where chosen at reasonable values and the time horizon was set at $T = 50$ years. The complete set of parameter values is listed in Table 5.6. As in the logistic case, the determination of the expected profit values (5.1) – (5.4) with the Gompertz model also requires numerical computations. We designed a grid with a time and state space discretization using $n = 300$ intervals for time (corresponding to a time step $\Delta t = 2$ months) and using $m = 150$ intervals for the state space (corresponding to a space step $\Delta x = 152$ tonnes, with $x_{max} = 2K$). Other values for the combination $n \times m$ were considered but, due to the computer execution time, to the algorithm convergence and algorithm stability, we choose the above values for n and m (the computer execution average time was of about 1 hour for each scenario).

The resulting profit values (5.1)–(5.4) for the basic scenario are shown in Table 5.7, where the first column refers to the optimal variable effort policy, the second column refers to the optimal constant effort policy, and the third column indicates the percent loss in the profit value when using the second policy instead of the first.

From the first line of Table 5.6 one can see that the optimal sustainable policy implies a reduction in the expected profit of only 1.5% compared to the optimal policy. Looking at the expected accumulated undiscounted profit (5.2), one can see a 1.7% expected profit reduction when comparing the second policy with the first. The percent reductions are the same for the corresponding profits per year (5.3) and (5.4). Conclusions for the Gompertz case are similar to the ones for the logistic case. In fact, profit reductions between the variable and constant policies are quite small and the advantages and disadvantages of both policies hold as in the logistic case. Notice, however, that in this example of the Gompertz case, the profit differences between the optimal policy and the optimal sustainable policy are indeed very small, much smaller than in the case of the logistic model.

The variability across the simulated trajectories, measured by their standard deviations, is very small (Table 5.7), with the optimal sustainable policy having a slightly lower variability.

Table 5.6: Parameter values used in the simulations of the basic scenario S_0 (Gompertz model). The Standardized Fishing Unit (SFU) measure is defined in Kar and Chakraborty (2011).

Item	Description	Values	Units
r	Intrinsic growth rate	1.331	year^{-1}
K	Carrying capacity	11400	tonnes
q	Catchability coefficient	$9.77 \cdot 10^{-5}$	SFU^{-1}year^{-1}
E_{min}	Minimum fishing effort	0	SFU
E_{max}	Maximum fishing effort	r/q	SFU
σ	Strength of environmental fluctuations	0.2	year$^{-1/2}$
x	Initial population size	$0.5K$	tonnes
δ	Discount factor	0.05	year^{-1}
p_1	Linear price parameter	8362.3	Taka · tonnes^{-1}
p_2	Quadratic price parameter	0	Taka · year · tonnes^{-2}
c_1	Linear cost parameter	1156.8	Taka · SFU^{-1}year^{-1}
c_2	Quadratic cost parameter	10^{-2}	Taka · SFU^{-2}year^{-1}
T	Time horizon	50	year

Table 5.7: Numerical comparison between policies of the expected profits 1. to 4. (see expressions (5.1)–(5.4)) for the basic scenario S_0 (Gompertz model). The percent relative difference between the two policies is denoted by Δ. Besides the expected values, we also present the standard deviations (in parenthesis). Units are in million Taka for 1. and 2. and in million Taka per year for 3. and 4.

	Optimal policy			Optimal sustainable policy			Δ (%)
1.	$V^* \simeq$	594.139	(sd $= 21.401$)	$V^{**} \simeq$	584.996	(sd $= 21.107$)	-1.5
2.	$V_u^* \simeq$	1619.672	(sd $= 47.825$)	$V_u^{**} \simeq$	1591.393	(sd $= 48.281$)	-1.7
3.	$\overline{\Pi^*} \simeq$	32.364	(sd $= 1.116$)	$\overline{\Pi^{**}} \simeq$	31.865	(sd $= 1.150$)	-1.5
4.	$\overline{\Pi_u^*} \simeq$	32.393	(sd $= 0.957$)	$\overline{\Pi_u^{**}} \simeq$	31.828	(sd $= 0.966$)	-1.7

Figure 5.22 shows, for the basic scenario S_0 and for both policies, the evolution from time $t = 0$ to time $t = T = 50$ of the population size $X(t)$ (on top), the optimal effort (in the middle) and the profit per unit time (at the bottom).

The definition of all depicted lines in Figure 5.22 are the same as in the logistic case and the conclusions regarding the comparisons between the optimal policy and the optimal sustainable policy are very similar. The optimal effort changes frequently, exhibiting periods of no and low harvesting and periods of harvesting at maximum effort. This behaviour has consequences on the profit per unit time of the optimal policy, which also changes very frequently and abruptly.

The sample standard deviation of the profit per unit time of the depicted trajectory across the time instants of the simulation is 14.33 million Taka per year for the optimal policy and only 5.35 million Taka per year for the optimal sustainable policy, which provides the harvester with a much steadier profit. For other trajectories similar results hold.

One can also see in Figure 5.22 (left side) that the optimal variable effort policy exhibits a possibly dangerous effect near the time horizon, consisting in a considerable drop of the average population size (see solid line on top left), corresponding to a high increase on the average effort (see solid line on middle left). As in the logistic case, this is due to the idea that "there is no tomorrow" and so, it is better profitwise to harvest as much as is profitable "now", without worrying about stock preservation for near future fishing.

With the optimal sustainable policy, population size is driven to an equilibrium probability distribution with an average population size higher than the one of the variable effort policy. This expected size at equilibrium is the mean value of the stationary distribution given by (4.26). With the constant effort policy, there is no decay of the expected population size near the end of the time horizon.

5.3.2 Alternative scenarios

To evaluate the influence of the parameters, we will consider alternative values, usually 25% lower and 25% higher than those in Table 5.6 for S_0. The exceptions are stated as follows. Since the initial population parameter x was set as $0.5K$, we consider as alternatives the lower value $0.25K$ (scenario S_1) and the higher value $0.75K$ (scenario S_2). The δ parameter varies 100% for up and down (scenarios S_5 and S_6, respectively) and, since T is high, we consider two alternative lower values $T = 10$ (scenario S_{11}) and $T = 25$ (scenario S_{12}). Scenario S_{19} corresponds to a positive p_2 value and scenario S_{20} considers an

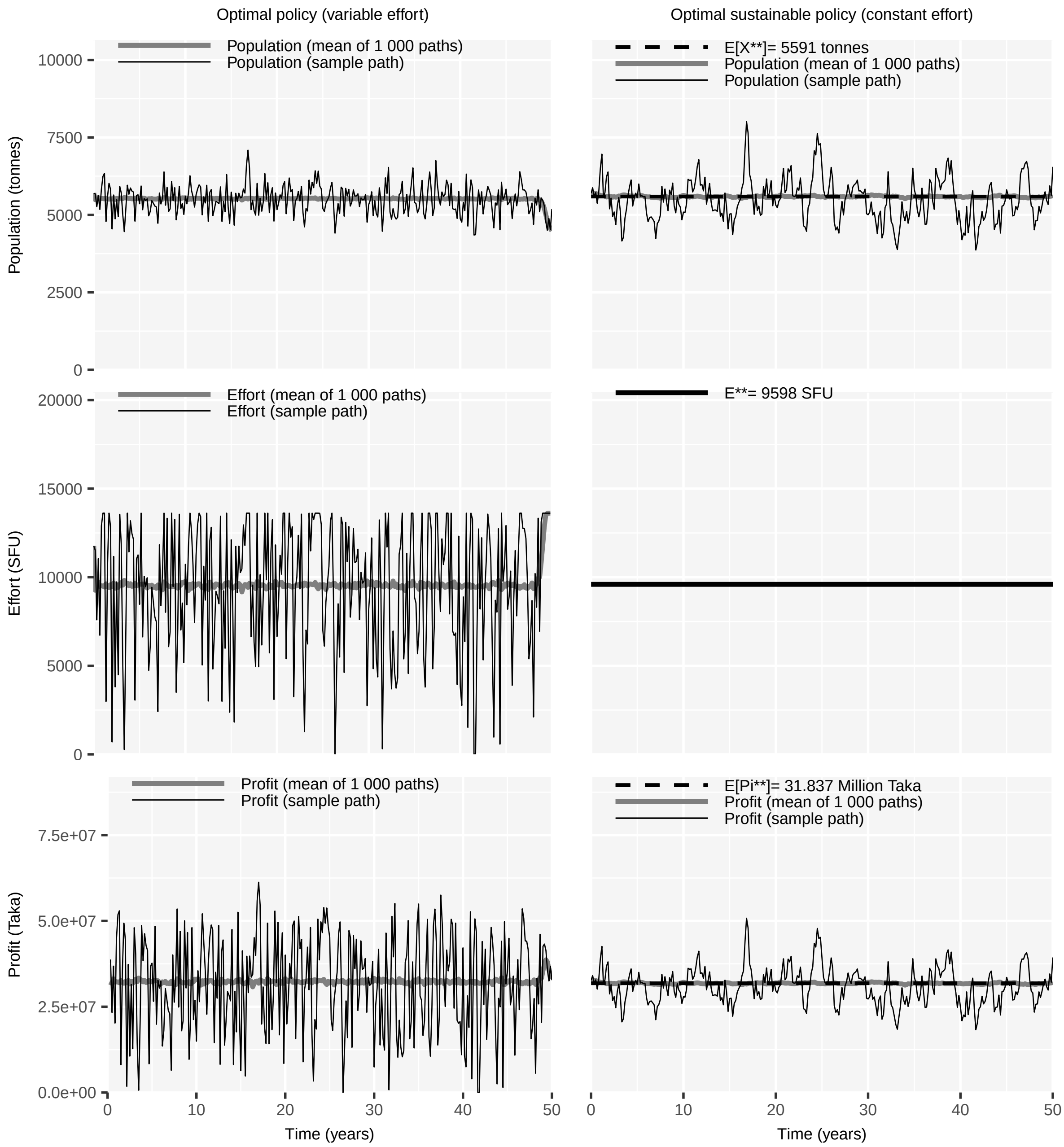


Figure 5.22: Basic scenario S_0 (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

Table 5.8: List of alternative scenarios ($S_i, i = 1, \ldots, 20$) and respective changed parameters (with respect to scenario S_0 of the Gompertz model). Units and unchanged parameters are as in Table 5.6.

Scenario	Changed parameter	Value	Scenario	Changed parameter	Value
S_1	x	$0.25K$	S_{11}	T	10
S_2	x	$0.75K$	S_{12}	T	25
S_3	E_{max}	$0.75r/q$	S_{13}	p_1	6271.7
S_4	E_{max}	$1.25r/q$	S_{14}	p_1	10452.9
S_5	δ	0.00	S_{15}	c_1	867.6
S_6	δ	0.10	S_{16}	c_1	1446.0
S_7	r	0.99	S_{17}	c_2	$0.75 \cdot 10^{-2}$
S_8	r	1.66	S_{18}	c_2	$1.25 \cdot 10^{-2}$
S_9	σ	0.15	S_{19}	p_2	$8 \cdot 10^{-2}$
S_{10}	σ	0.25	S_{20}	p_2	$12 \cdot 10^{-2}$

increase of 50% on this p_2 value.

All the alternative scenarios S_1 to S_{20} are described in Table 5.8. For each one, we have computed the profit values as in Table 5.7 and a similar set of images as in Figure 5.22. The profit values are shown in Tables 5.9 and 5.10 and the Figures are presented at the end of this section.

A general comment concerning Tables 5.4 and 5.5 is that, for almost all the scenarios, the percent reduction of profit Δ incurred by using the optimal constant effort policy instead of the optimal variable effort policy is quite small, and varies from 0.9% to 2.4% for discounted profits and from 0.9% to 2.7% for undiscounted profits. The variability (standard deviation) associated to each profit value is very similar, and in some cases less than 1%, when we compare the optimal policy with the optimal sustainable policy.

To check the effect of changes in a given parameter, we can compare the results for the basic scenario S_0 with the results of the scenarios corresponding to alternative values of such parameter, using Tables 5.9 and 5.10 and the Figures associated to those scenarios. We note that the scenarios with the smaller differences are S_3, S_{19} and S_{20}. For scenario S_3 this happens because the maximum allowed effort, $E_{max} = 0.75r/q = 10212$ is close to the value of $E^{**} = 9598$. On the contrary, the largest difference is exhibited in scenario S_{11} and this is due, as in the logistic case, to the fact that the optimal sustainable policy 'needs more time' to get close to the stochastic steady-state. The qualitative results are very similar to the logistic model, but the differences between the two policies are much smaller for the Gompertz model.

Table 5.9: Expected discounted profit values for the scenarios presented in Table 5.8 (Gompertz model). Besides the expected values, we also present the standard deviations (in parenthesis with smaller font size). The percent relative difference between both policies is denoted by Δ. Currency values are in million Taka for V^* and V^{**} and million Taka per year for $\overline{\Pi^*}$ and $\overline{\Pi^{**}}$.

Scenario	V^*		V^{**}		$\overline{\Pi^*}$		$\overline{\Pi^{**}}$		Δ (%)
S_0	594.139	(21.401)	584.996	(21.107)	32.364	(1.116)	31.865	(1.150)	−1.5
S_1	577.418	(21.122)	565.705	(20.611)	31.453	(1.151)	30.815	(1.123)	−2.0
S_2	610.272	(21.797)	599.356	(21.499)	33.242	(1.187)	32.648	(1.171)	−1.8
S_3	589.537	(21.163)	584.996	(21.107)	32.113	(1.153)	31.865	(1.150)	−0.8
S_4	594.917	(21.363)	584.996	(21.107)	32.406	(1.164)	31.865	(1.150)	−1.7
S_5	1620.276	(48.744)	1591.393	(48.281)	32.406	(0.975)	31.828	(0.966)	−1.8
S_6	321.994	(14.489)	316.917	(14.168)	32.418	(1.459)	31.907	(1.426)	−1.6
S_7	449.992	(21.275)	441.120	(20.899)	24.512	(1.159)	24.028	(1.138)	−2.0
S_8	736.241	(21.434)	726.872	(21.208)	40.104	(1.168)	39.594	(1.155)	−1.3
S_9	593.380	(16.030)	587.705	(15.857)	32.322	(0.873)	32.013	(0.864)	−1.0
S_{10}	594.770	(26.768)	581.474	(26.322)	32.398	(1.458)	31.674	(1.434)	−2.2
S_{11}	257.868	(16.698)	251.617	(16.259)	32.769	(2.122)	31.974	(2.066)	−2.4
S_{12}	462.825	(20.578)	455.265	(20.261)	32.434	(1.442)	31.904	(1.420)	−1.6
S_{13}	392.794	(15.521)	384.897	(15.271)	21.396	(0.845)	20.966	(0.832)	−2.0
S_{14}	800.512	(27.246)	790.232	(26.890)	43.605	(1.484)	43.045	(1.465)	−1.3
S_{15}	646.975	(21.814)	639.310	(21.535)	35.242	(1.188)	34.824	(1.173)	−1.2
S_{16}	545.046	(20.942)	534.271	(20.632)	29.689	(1.141)	29.102	(1.124)	−2.0
S_{17}	599.208	(21.472)	589.528	(21.182)	32.640	(1.170)	32.112	(1.154)	−1.6
S_{18}	589.315	(21.330)	580.568	(21.032)	32.101	(1.162)	31.624	(1.146)	−1.5
S_{19}	548.058	(18.903)	543.591	(18.749)	29.853	(1.030)	29.610	(1.021)	−0.8
S_{20}	526.553	(17.701)	523.266	(17.591)	28.682	(0.964)	28.503	(0.958)	−0.6

Table 5.10: Expected undiscounted profit values for the scenarios presented in Table 5.8 (Gompertz model). Besides the expected values, we also present the standard deviations (in parenthesis with smaller font size). The percent relative difference between both policies is denoted by Δ. Currency values are in million Taka for V_u^* and V_u^{**} and million Taka per year for $\overline{\Pi_u^*}$ and $\overline{\Pi_u^{**}}$.

Scenario	V_u^*		V_u^{**}		$\overline{\Pi_u^*}$		$\overline{\Pi_u^{**}}$		Δ (%)
S_0	1619.672	(47.825)	1591.393	(48.281)	32.393	(0.957)	31.828	(0.966)	−1.7
S_1	1602.745	(47.657)	1571.387	(47.973)	32.055	(0.953)	31.428	(0.959)	−2.0
S_2	1636.020	(48.045)	1606.191	(48.517)	32.720	(0.961)	32.124	(0.970)	−1.8
S_3	1606.002	(47.703)	1591.393	(48.281)	32.120	(0.954)	31.828	(0.966)	−0.9
S_4	1623.333	(47.734)	1591.393	(48.281)	32.467	(0.955)	31.828	(0.966)	−2.0
S_5	1620.276	(48.744)	1591.393	(48.281)	32.406	(0.975)	31.828	(0.966)	−1.8
S_6	1617.717	(46.967)	1591.393	(48.281)	32.354	(0.939)	31.828	(0.966)	−1.6
S_7	1226.065	(47.593)	1198.995	(48.162)	24.521	(0.952)	23.980	(0.963)	−2.2
S_8	2007.483	(47.897)	1978.302	(48.293)	40.150	(0.958)	39.566	(0.966)	−1.5
S_9	1617.663	(35.792)	1599.182	(36.279)	32.353	(0.716)	31.984	(0.726)	−1.1
S_{10}	1621.307	(59.895)	1581.302	(60.195)	32.426	(1.198)	31.626	(1.204)	−2.5
S_{11}	328.241	(20.768)	319.540	(20.602)	32.824	(2.077)	31.954	(2.060)	−2.7
S_{12}	812.046	(33.564)	796.861	(33.724)	32.482	(1.343)	31.874	(1.349)	−1.9
S_{13}	1073.824	(34.723)	1049.558	(35.002)	21.476	(0.694)	20.991	(0.700)	−2.3
S_{14}	2178.384	(60.836)	2146.658	(61.424)	43.568	(1.217)	42.933	(1.228)	−1.5
S_{15}	1760.270	(48.723)	1736.558	(49.189)	35.205	(0.974)	34.731	(0.984)	−1.3
S_{16}	1488.700	(46.817)	1455.653	(47.257)	29.774	(0.936)	29.113	(0.945)	−2.2
S_{17}	1633.104	(47.975)	1603.299	(48.442)	32.662	(0.960)	32.066	(0.969)	−1.8
S_{18}	1606.916	(47.681)	1579.752	(48.121)	32.138	(0.954)	31.595	(0.962)	−1.7
S_{19}	1494.892	(42.472)	1479.957	(42.921)	29.898	(0.849)	29.599	(0.858)	−1.0
S_{20}	1436.799	(39.897)	1425.219	(40.285)	28.736	(0.798)	28.504	(0.806)	−0.8

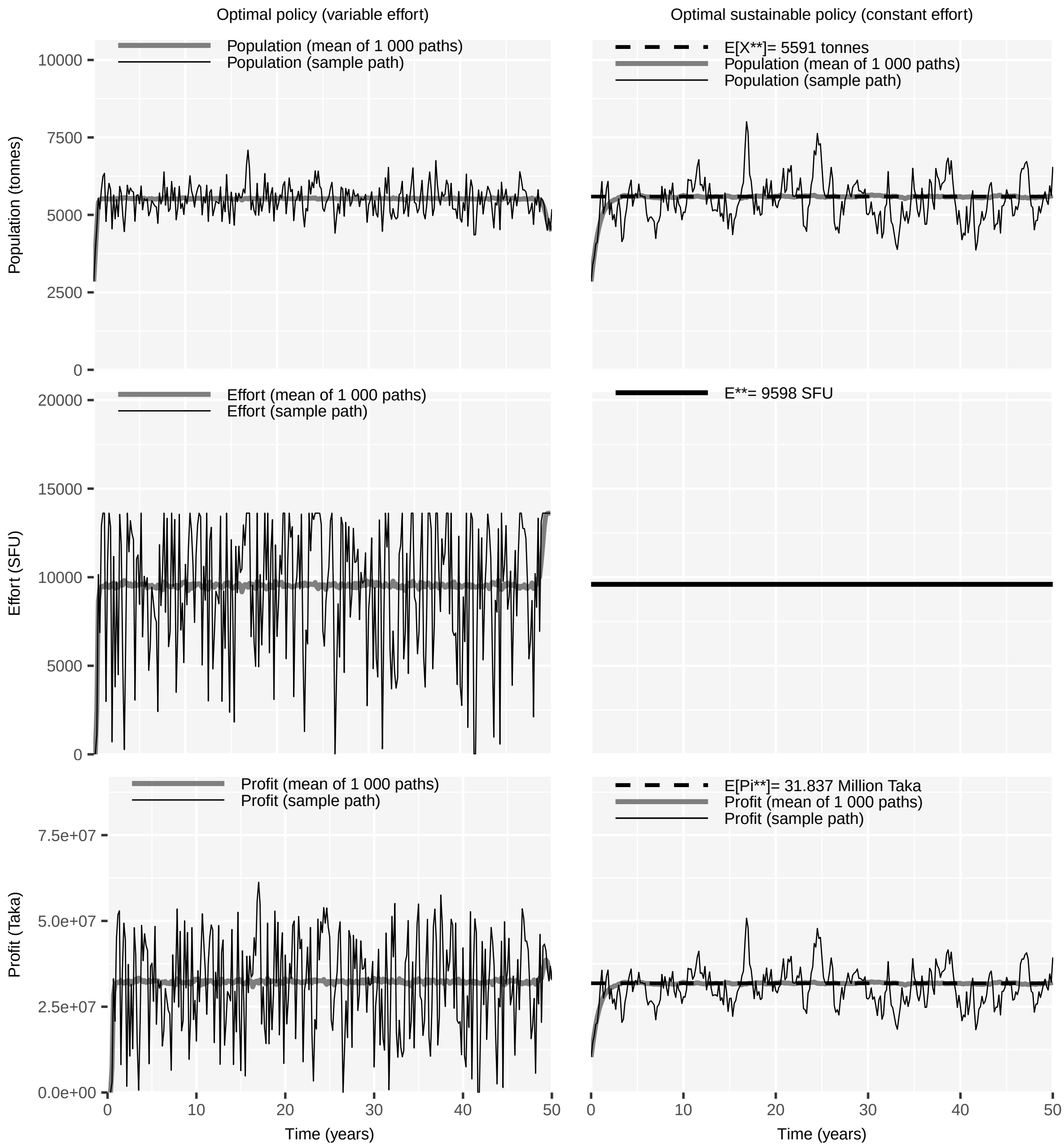


Figure 5.23: Scenario S_1 (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

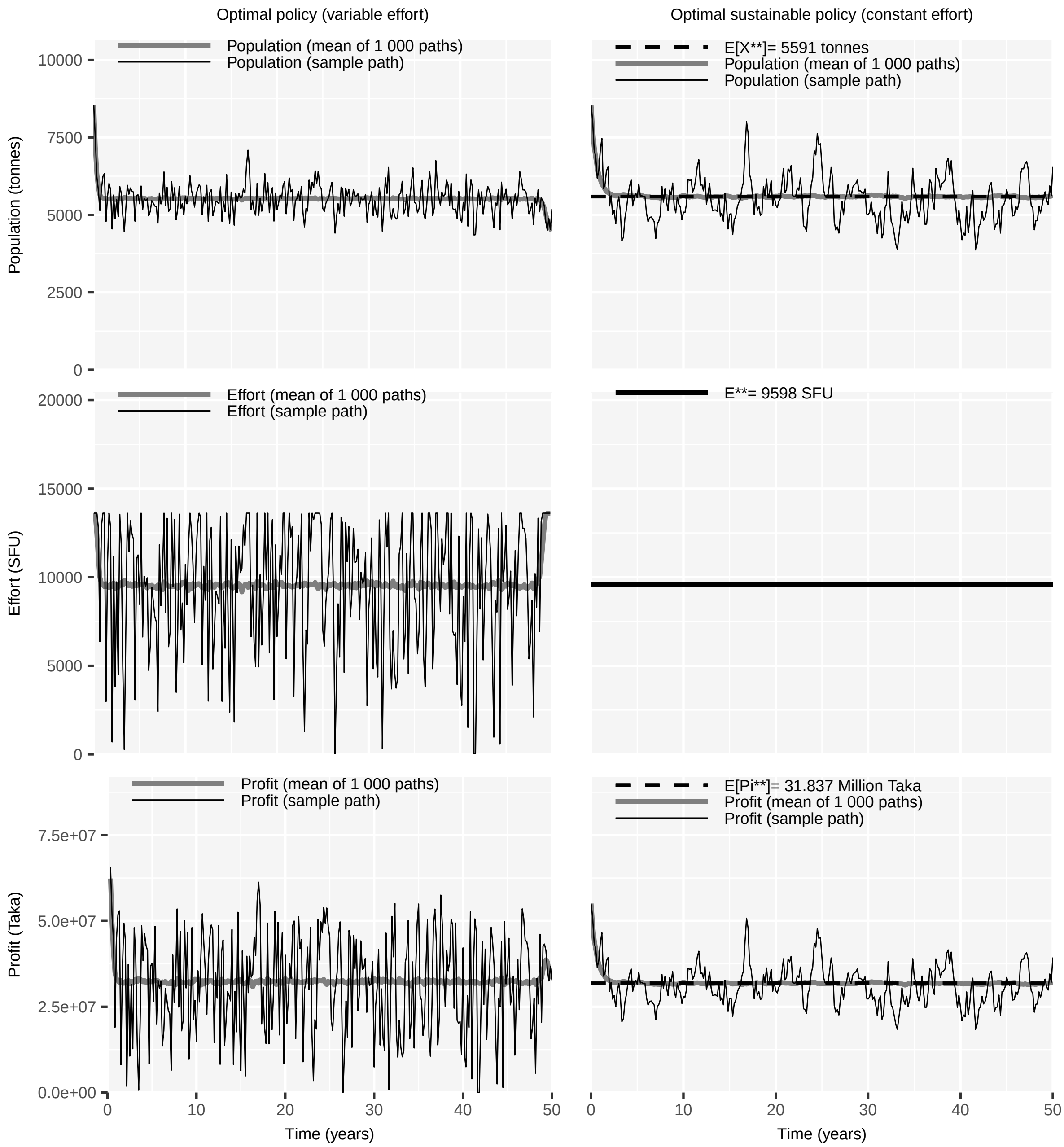


Figure 5.24: Scenario S_2 (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

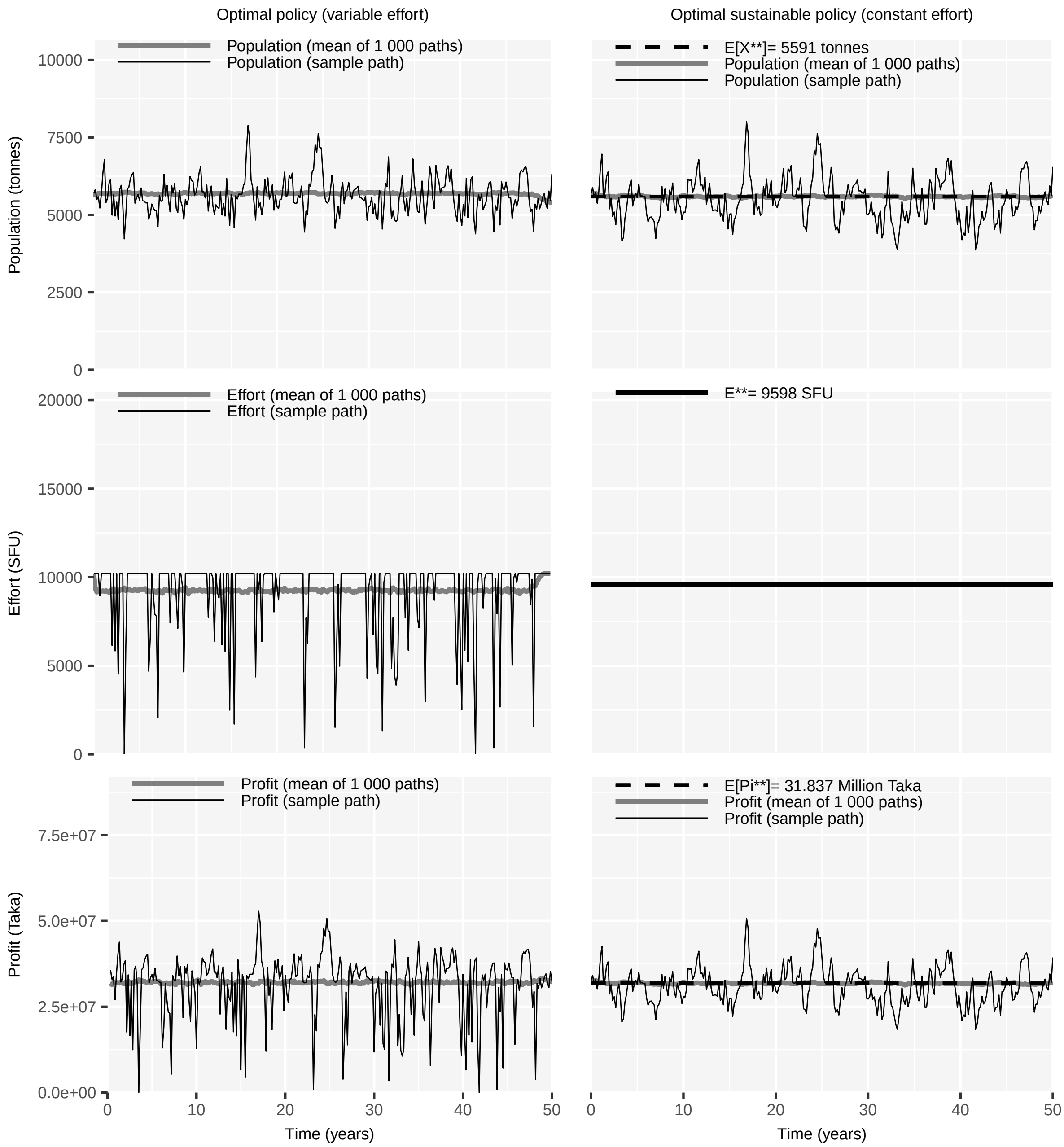


Figure 5.25: Scenario S_3 (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

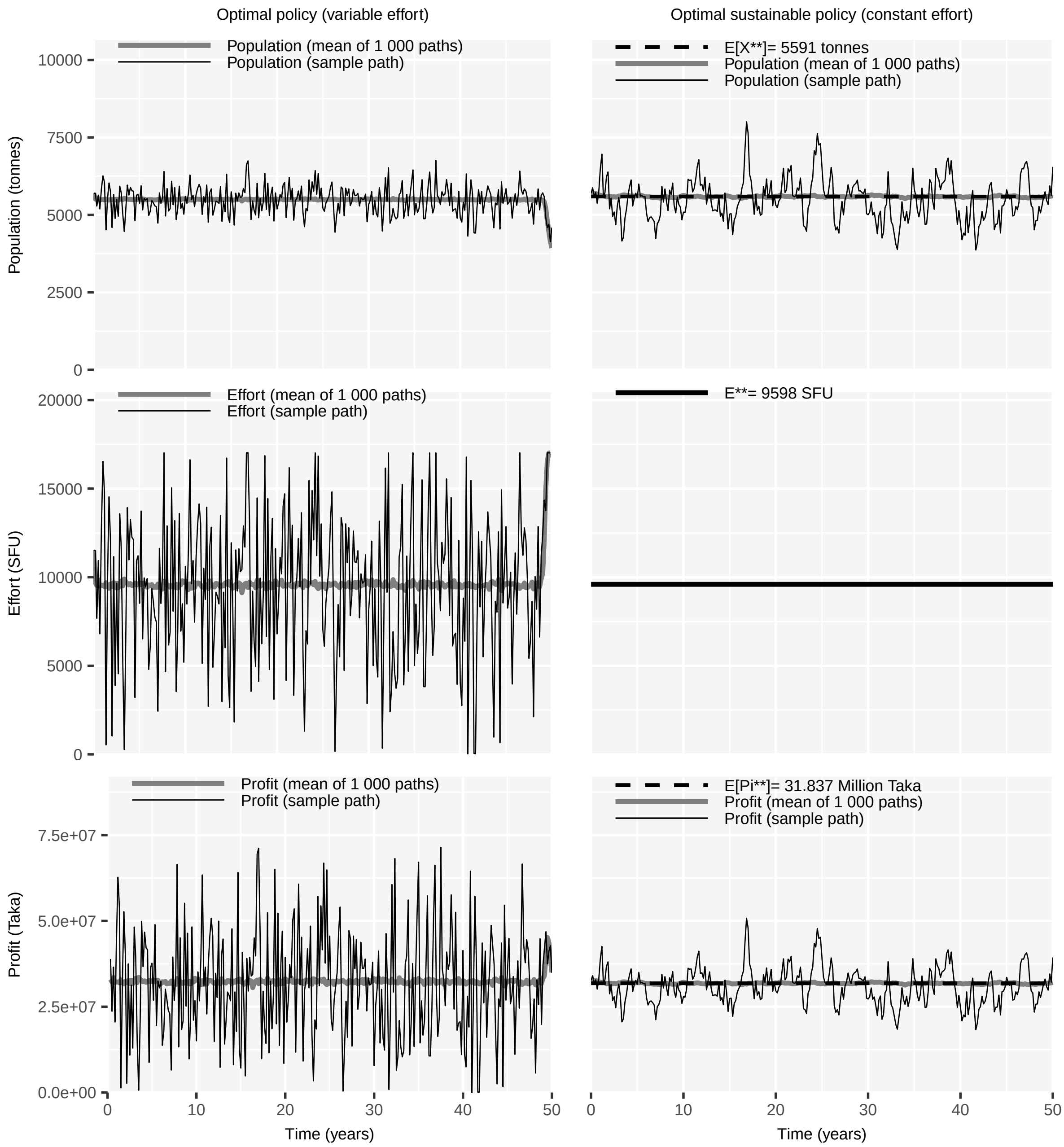


Figure 5.26: Scenario S_4 (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

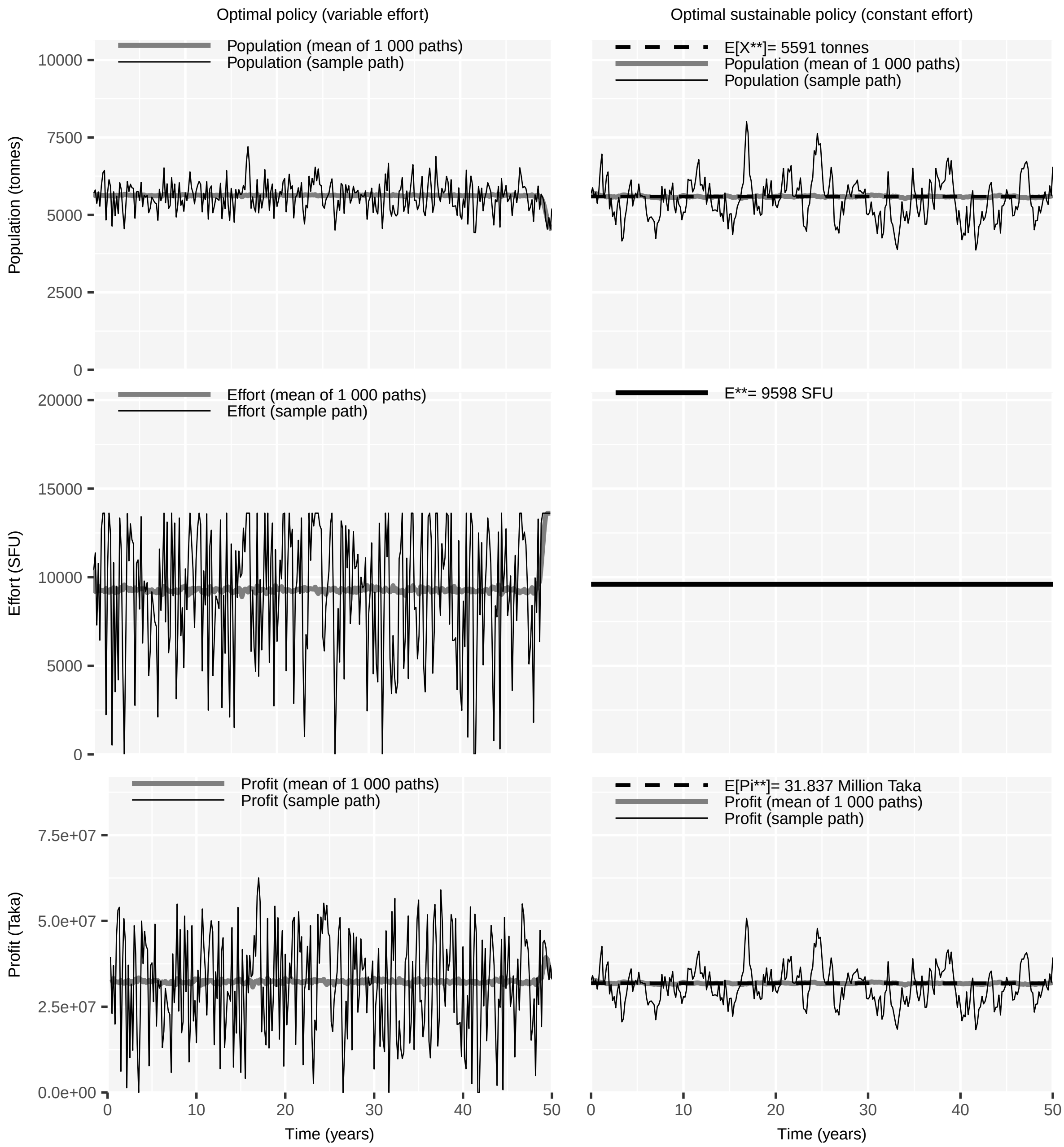


Figure 5.27: Scenario S_5 (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

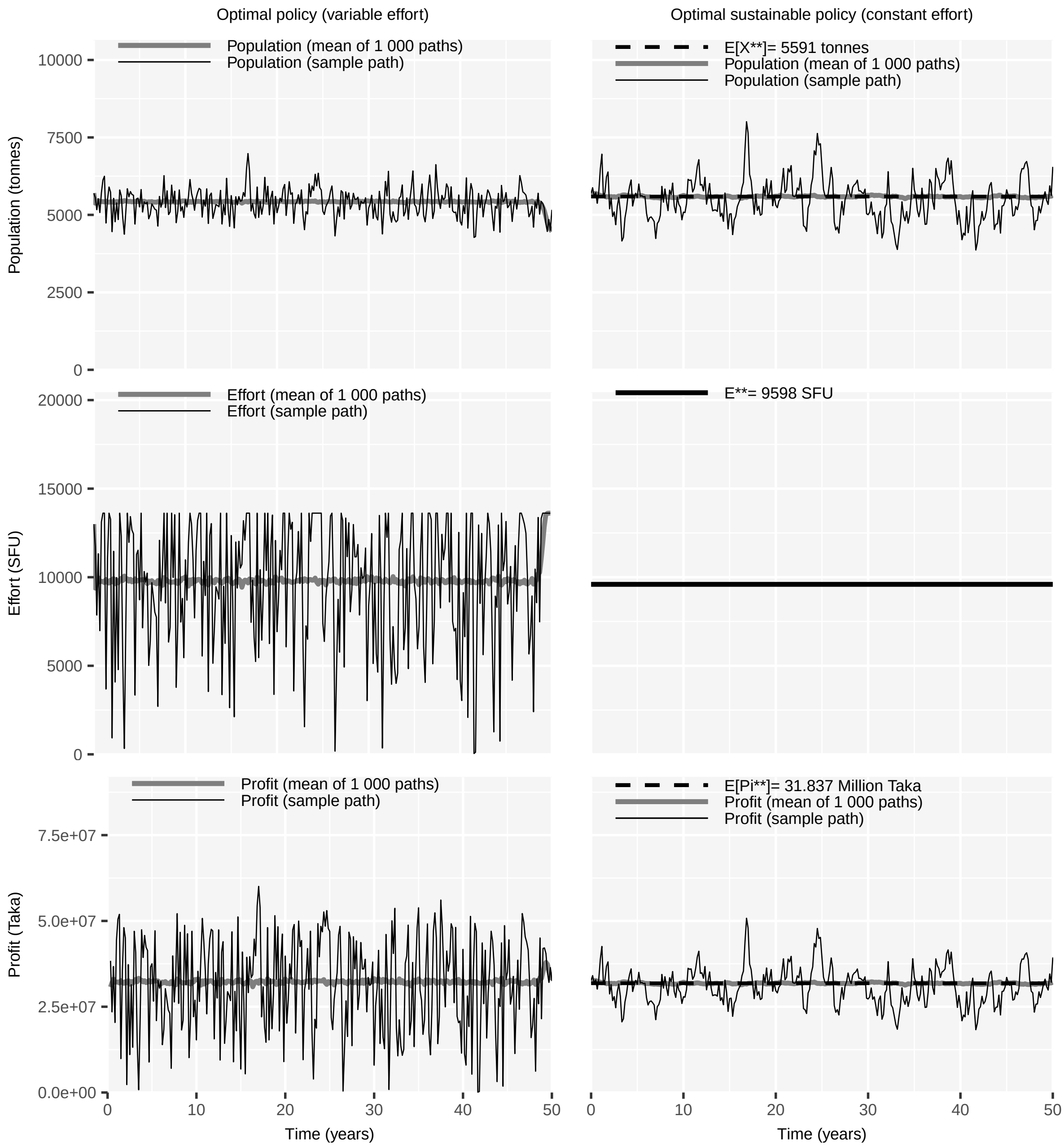


Figure 5.28: Scenario S_6 (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

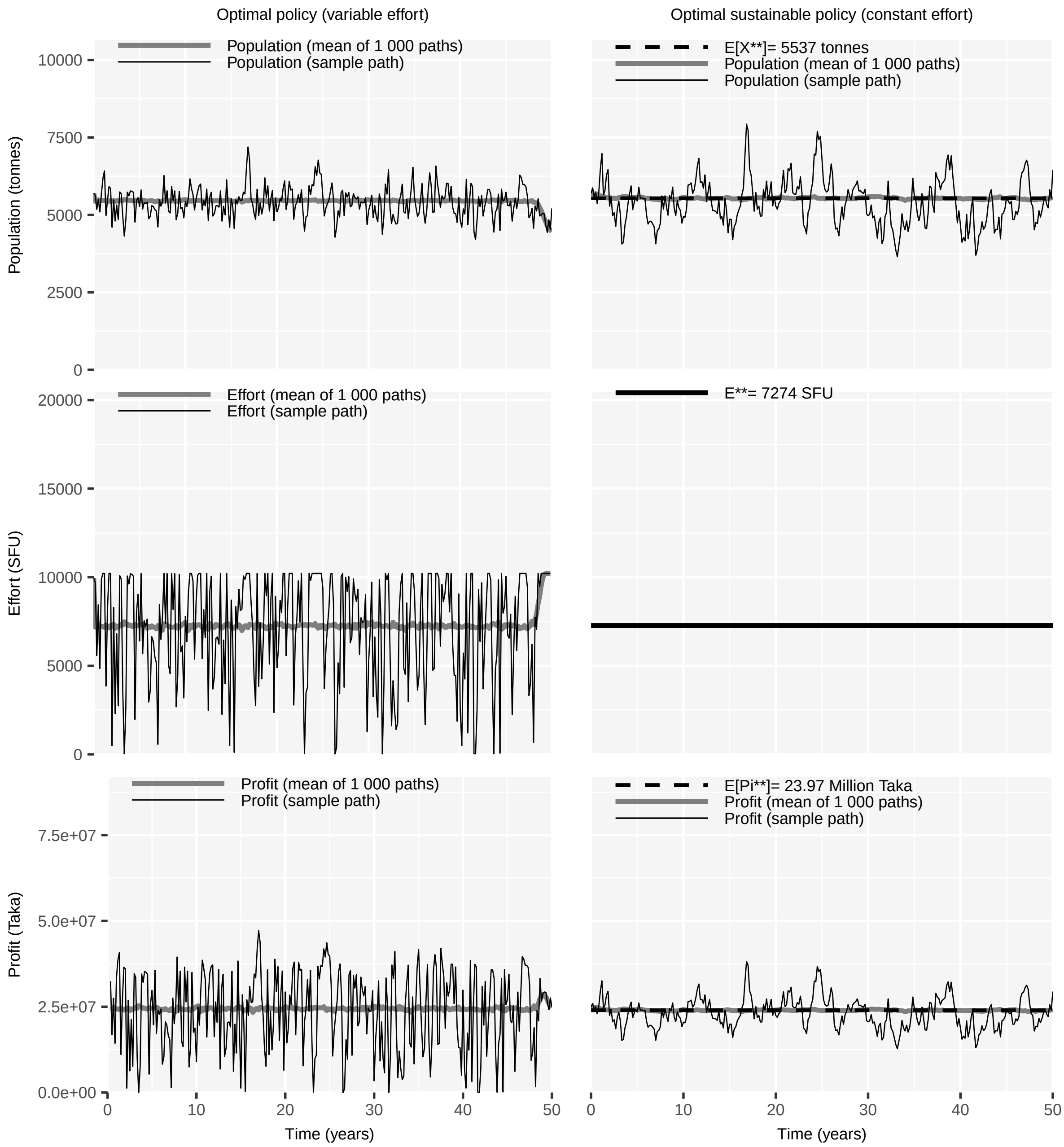


Figure 5.29: Scenario S_7 (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

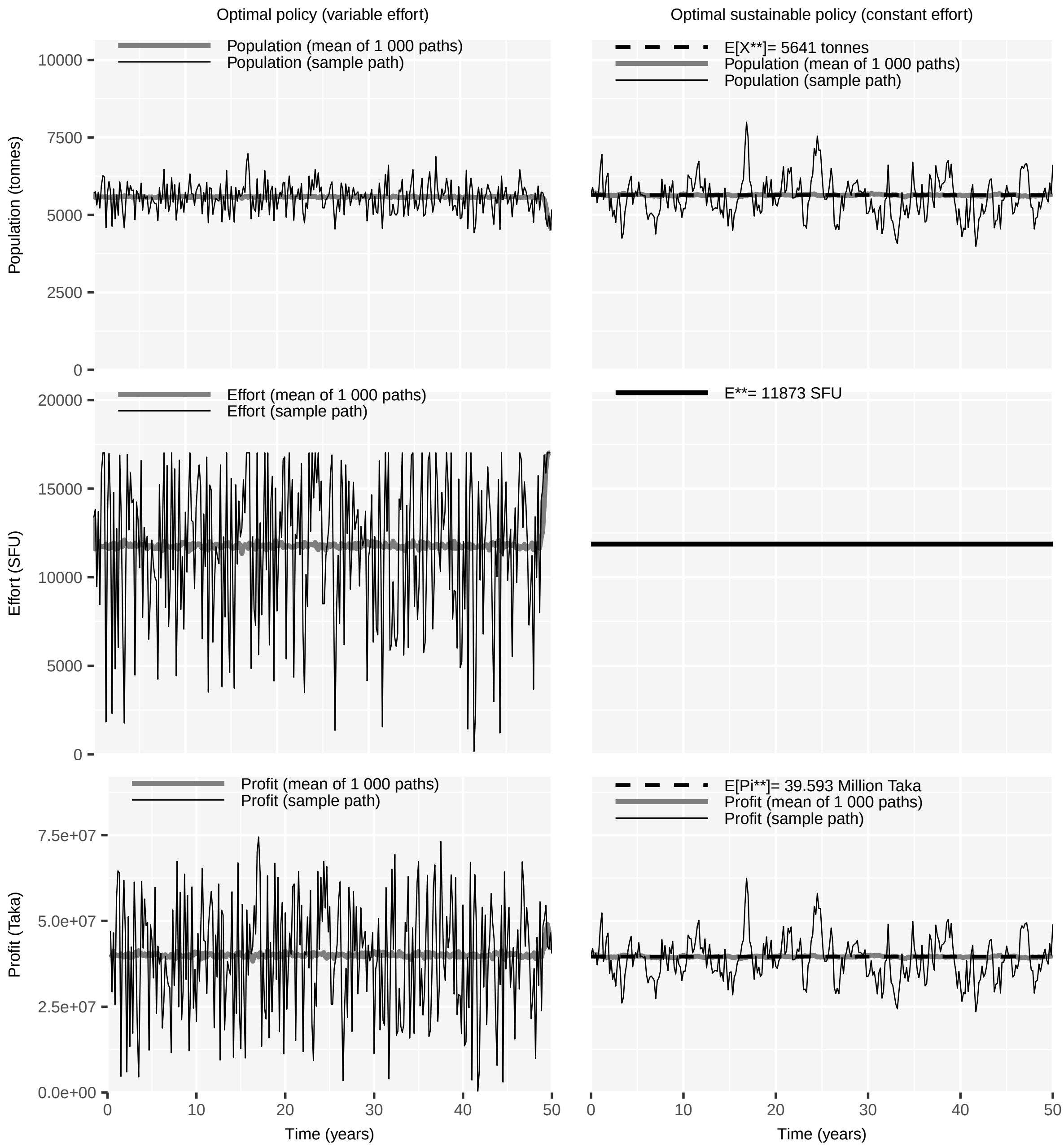


Figure 5.30: Scenario S_8 (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

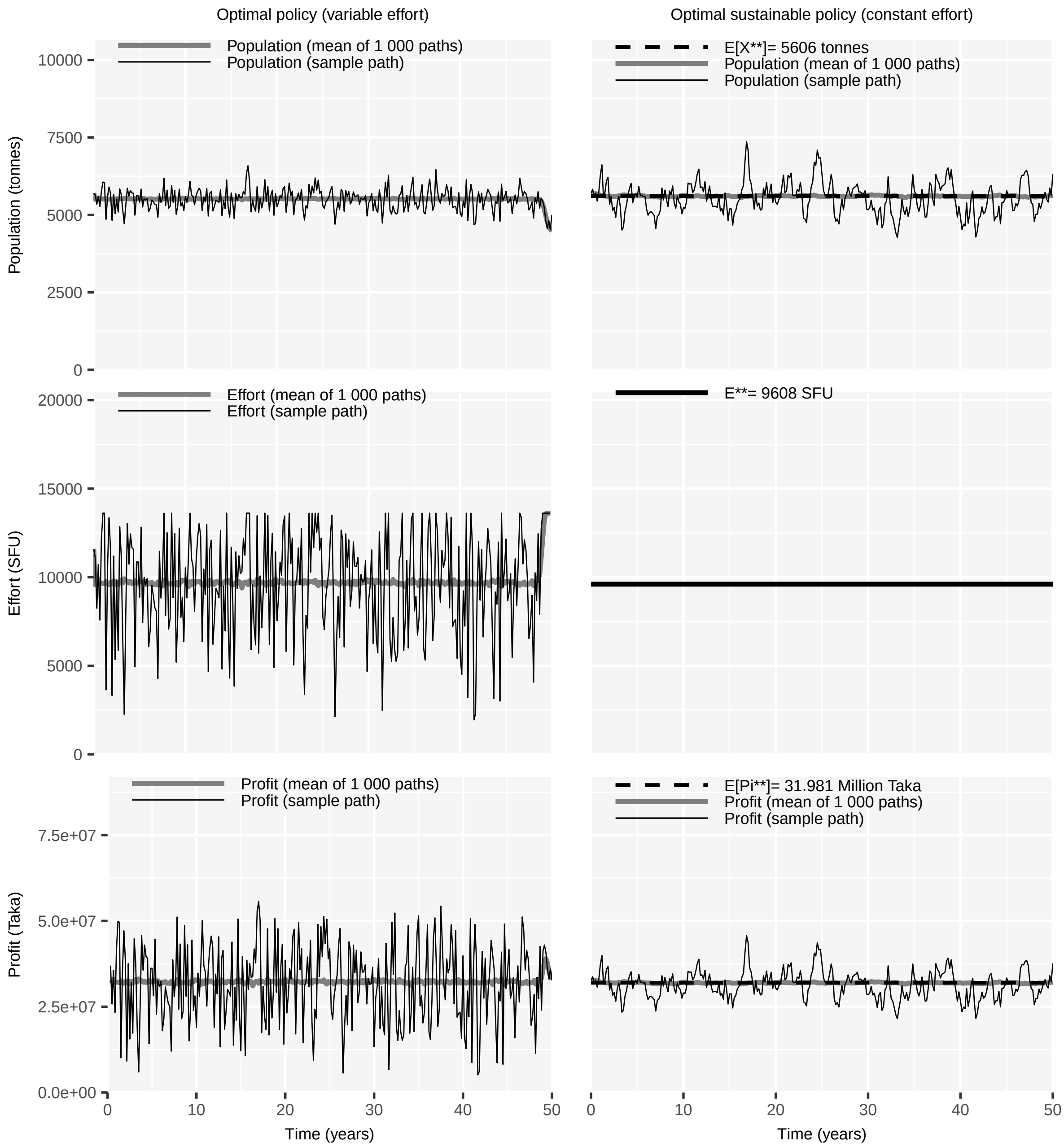


Figure 5.31: Scenario S_9 (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

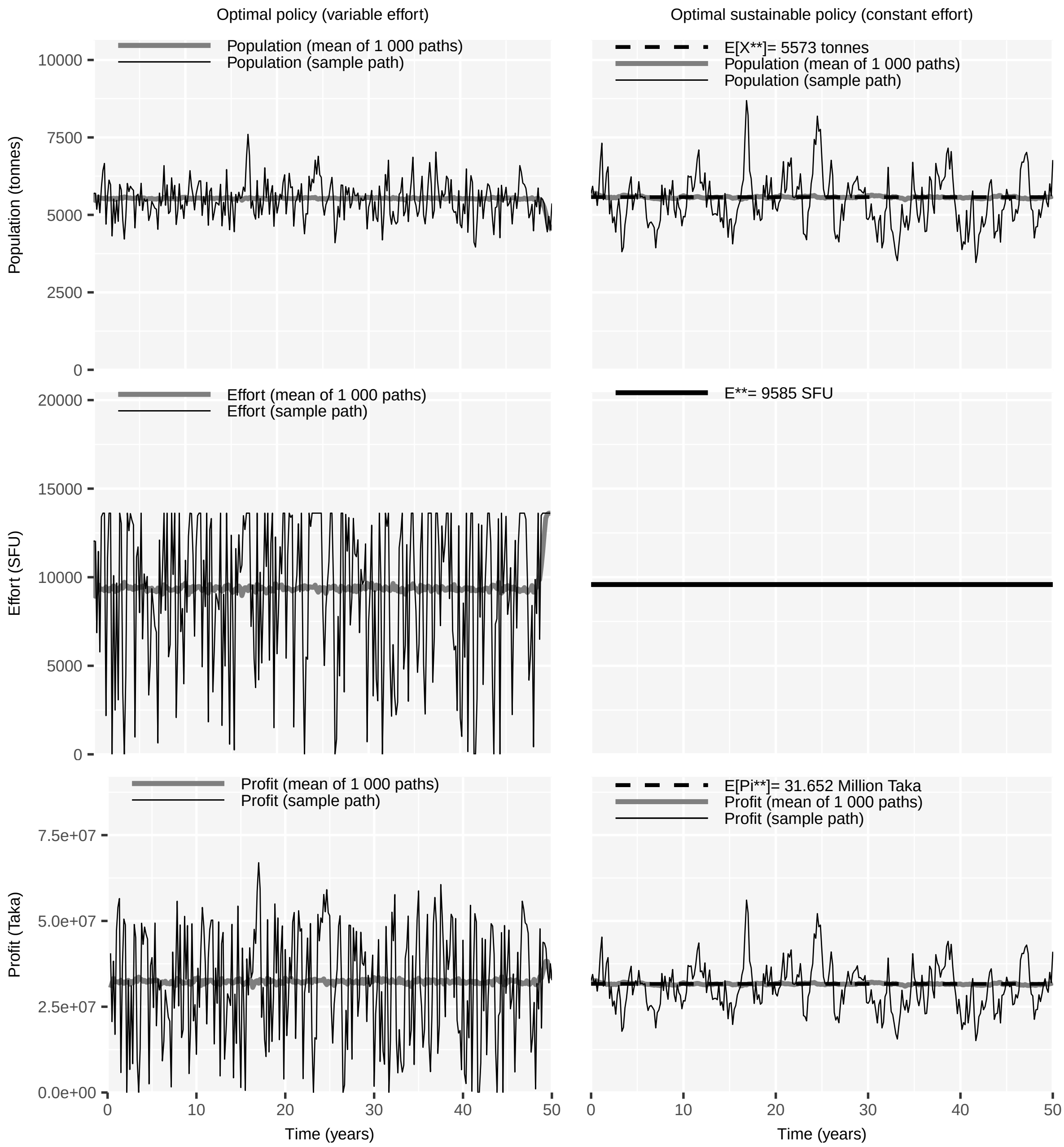


Figure 5.32: Scenario S_{10} (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

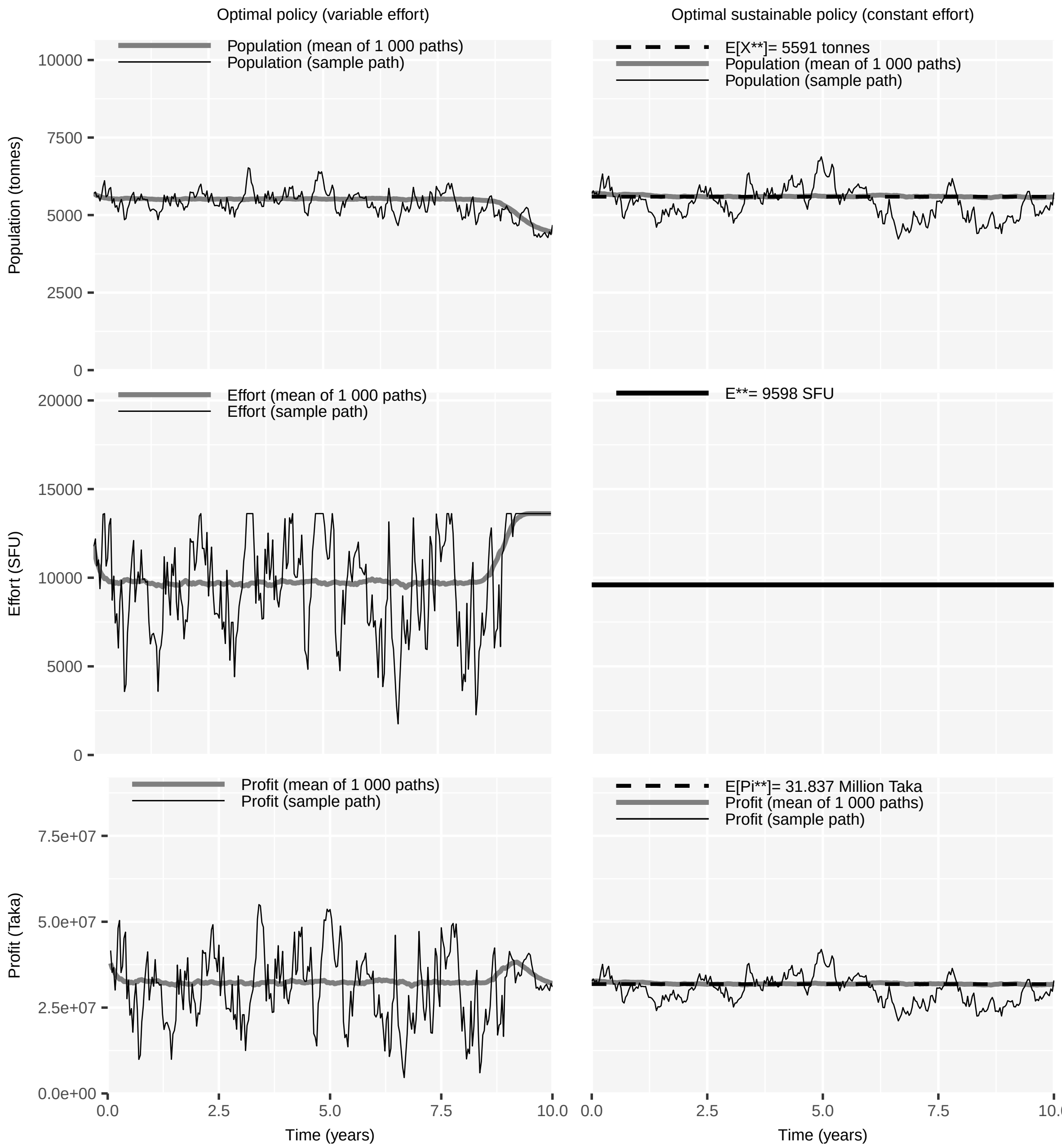


Figure 5.33: Scenario S_{11} (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

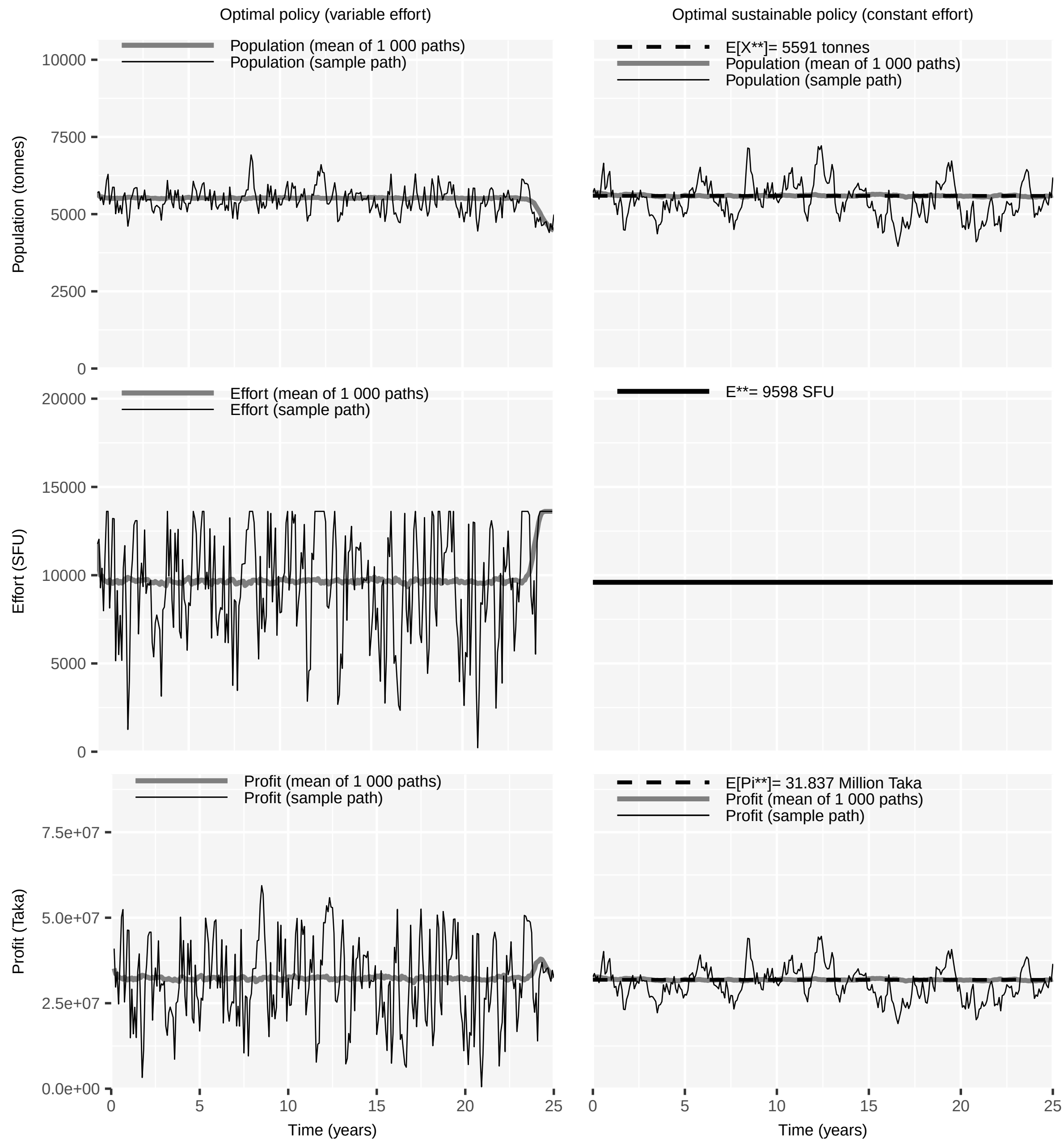


Figure 5.34: Scenario S_{12} (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

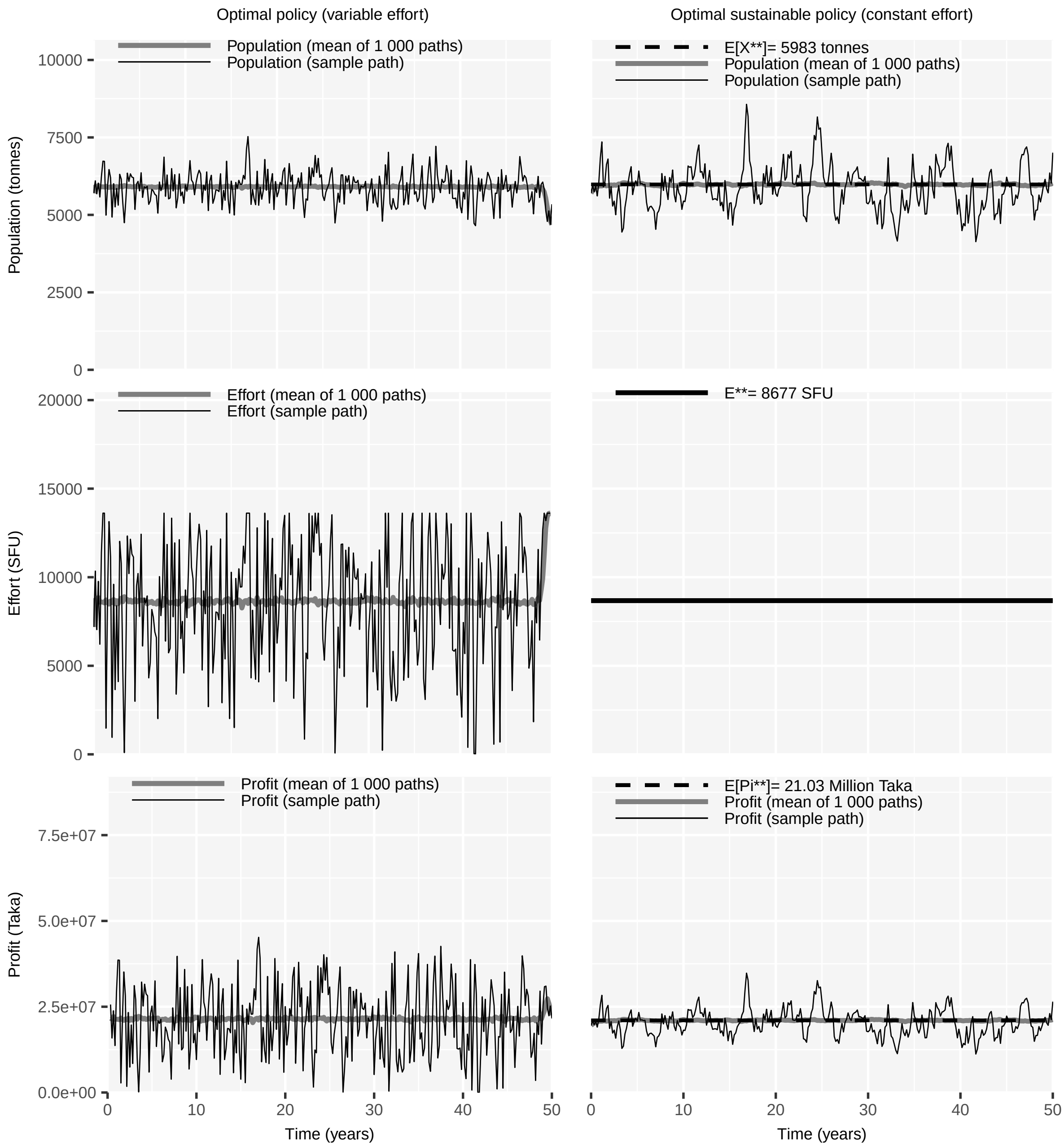


Figure 5.35: Scenario S_{13} (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

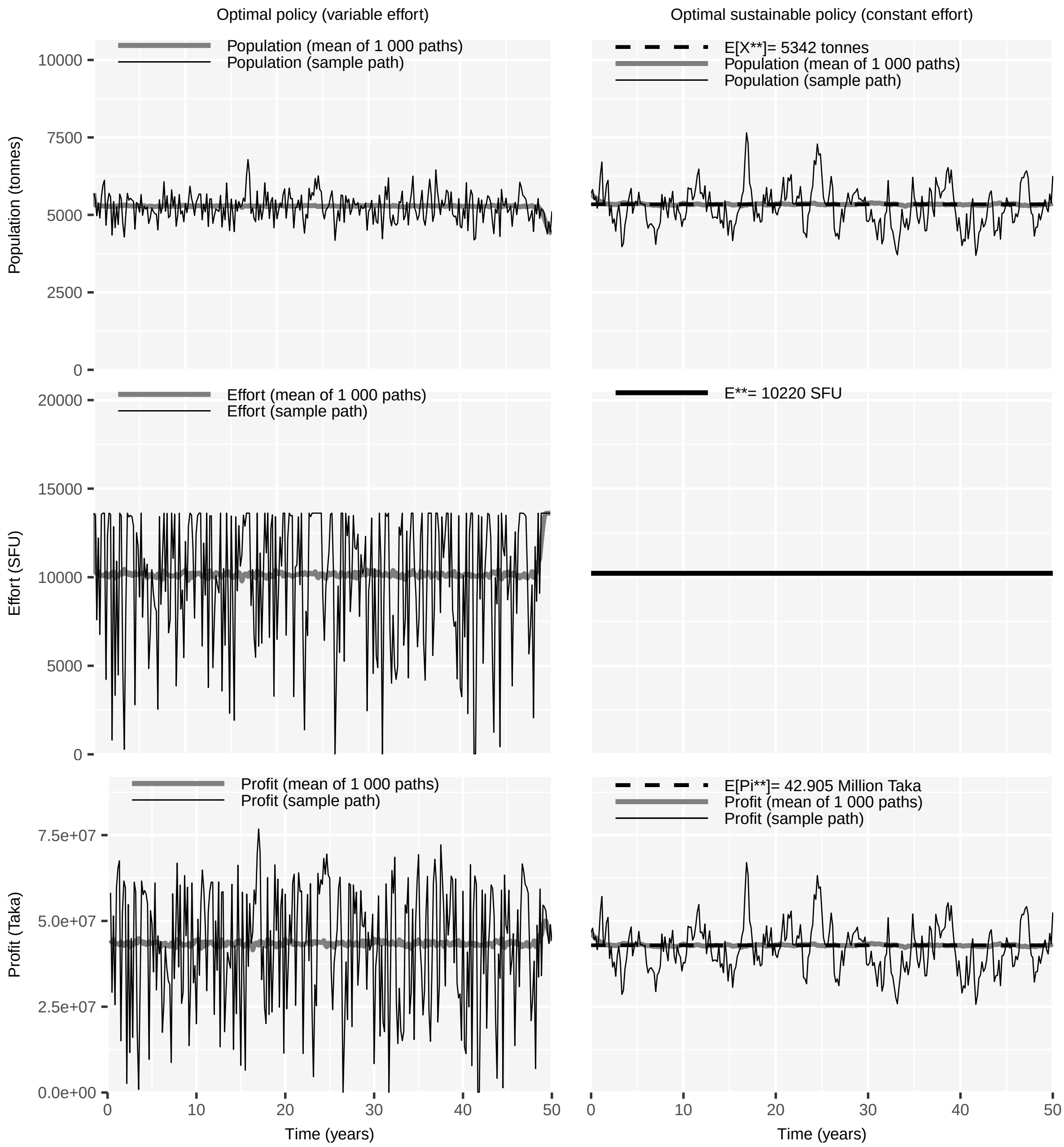


Figure 5.36: Scenario S_{14} (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

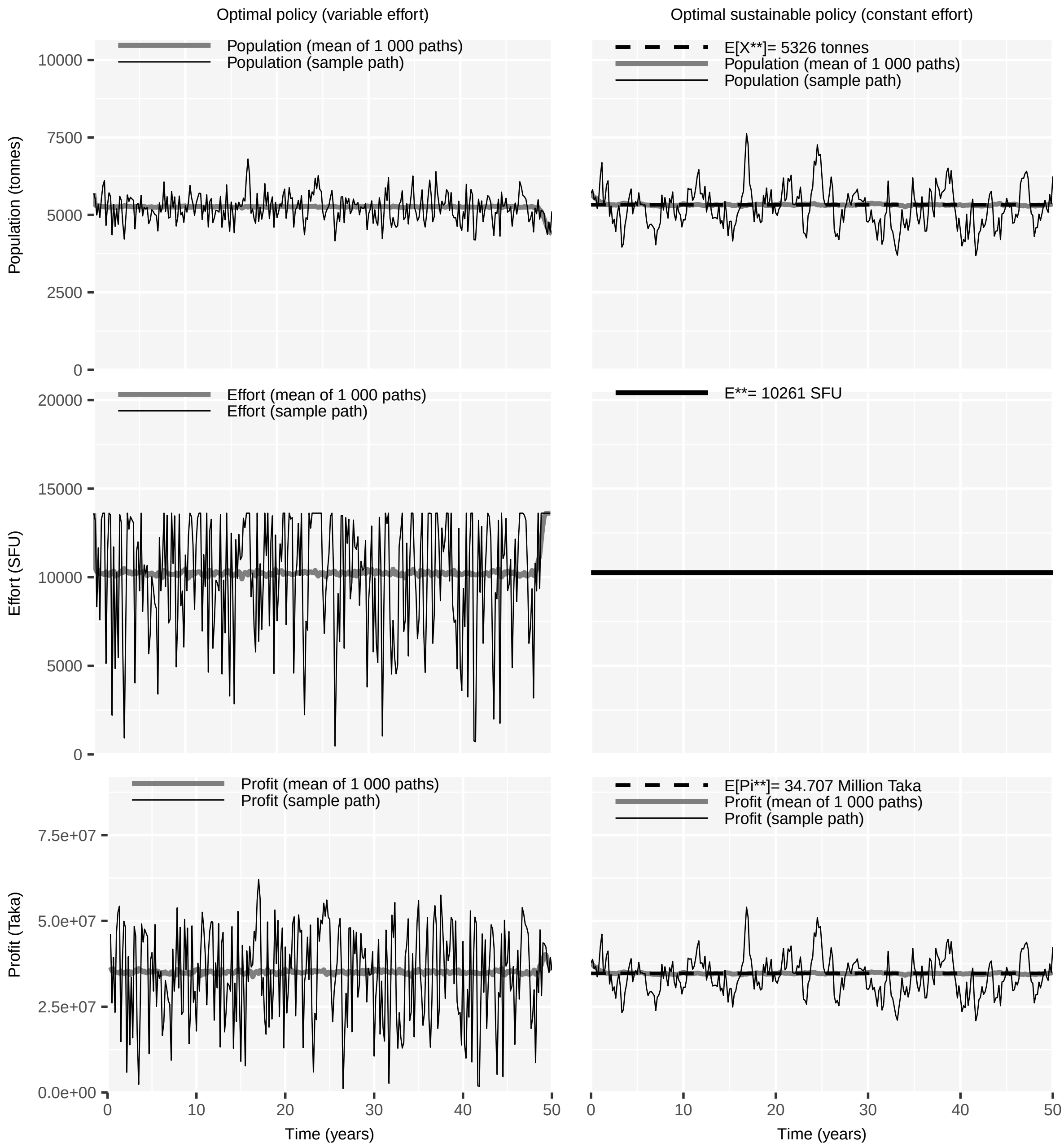


Figure 5.37: Scenario S_{15} (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

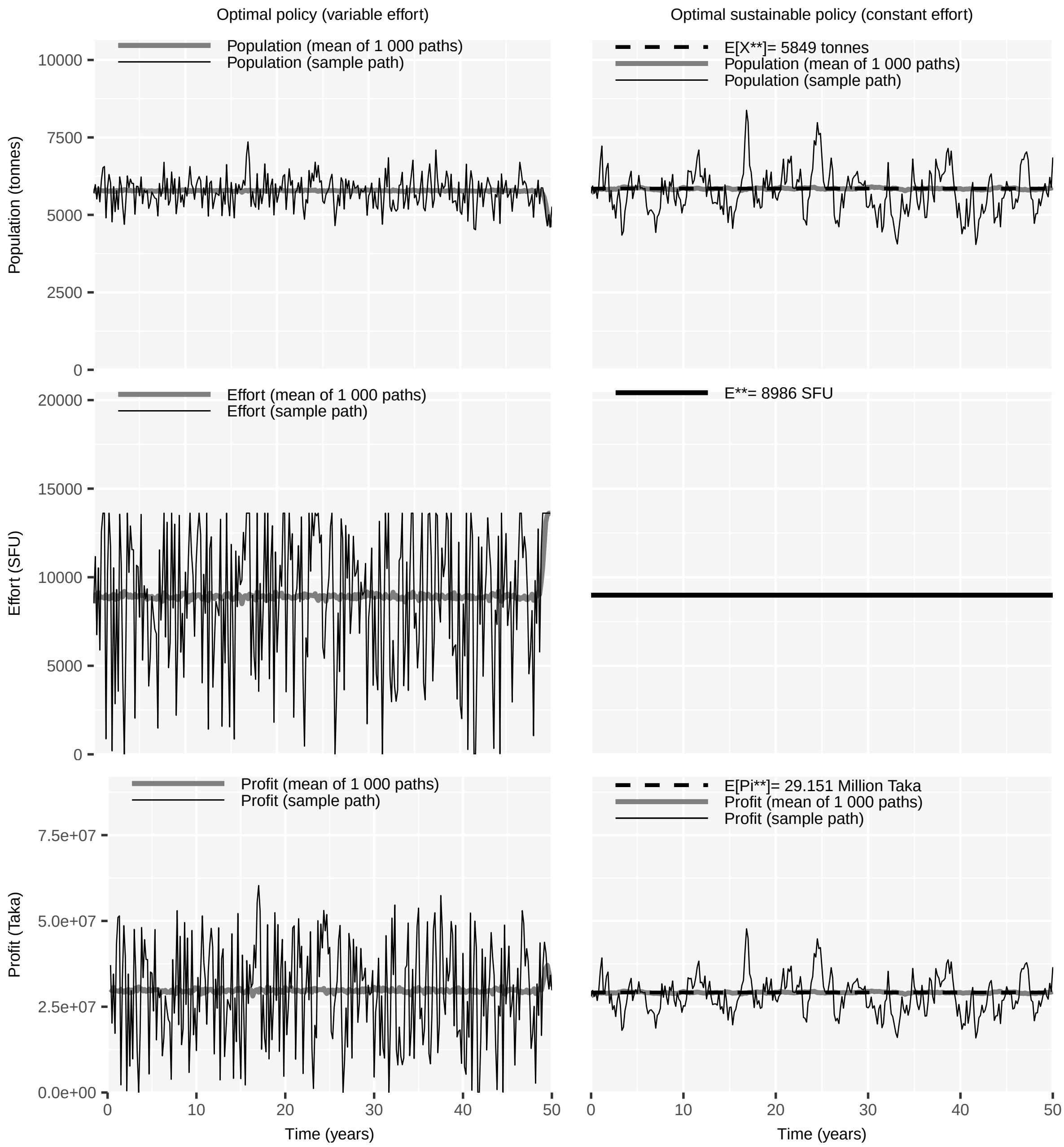


Figure 5.38: Scenario S_{16} (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

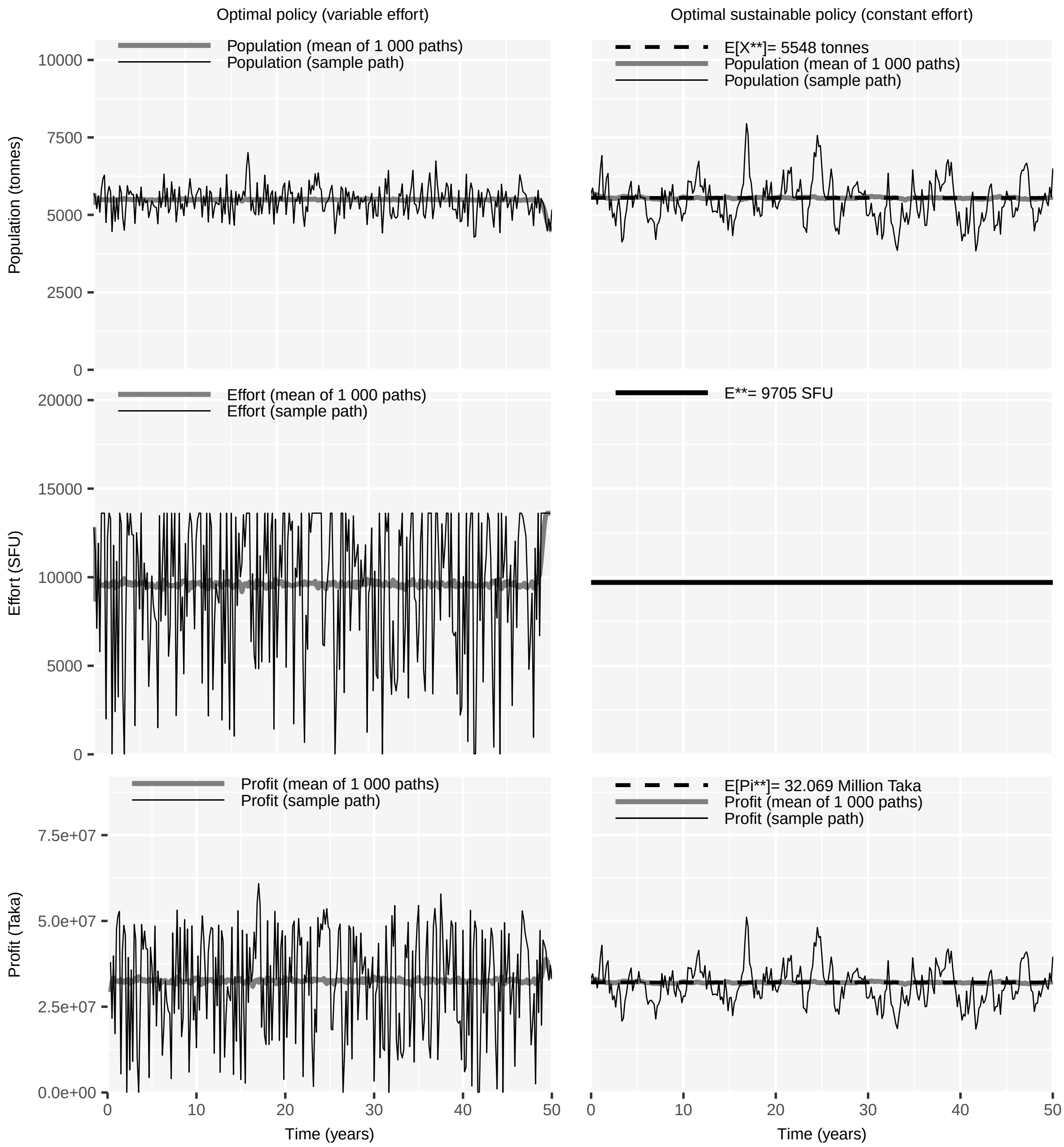


Figure 5.39: Scenario S_{17} (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

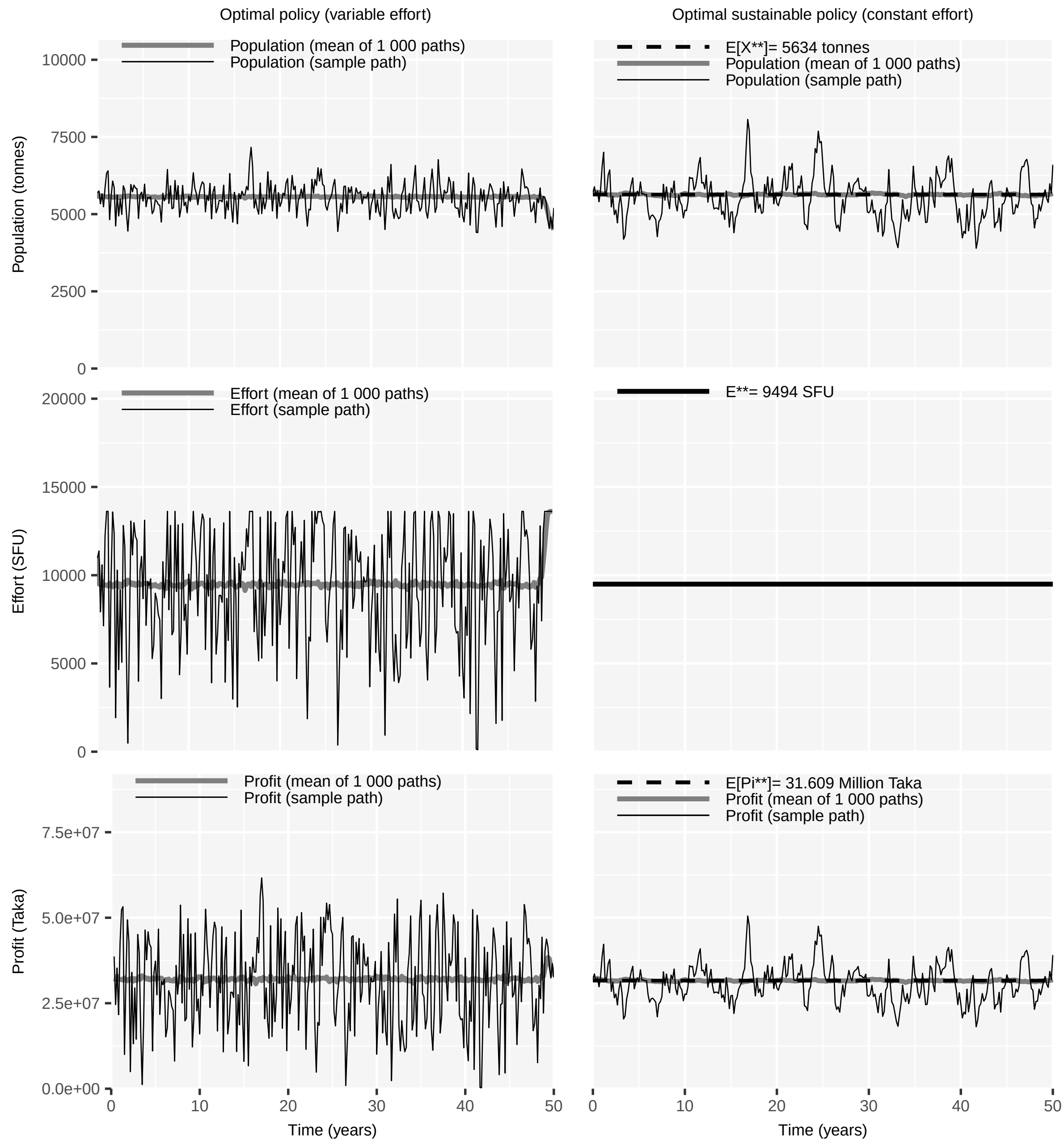


Figure 5.40: Scenario S_{18} (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

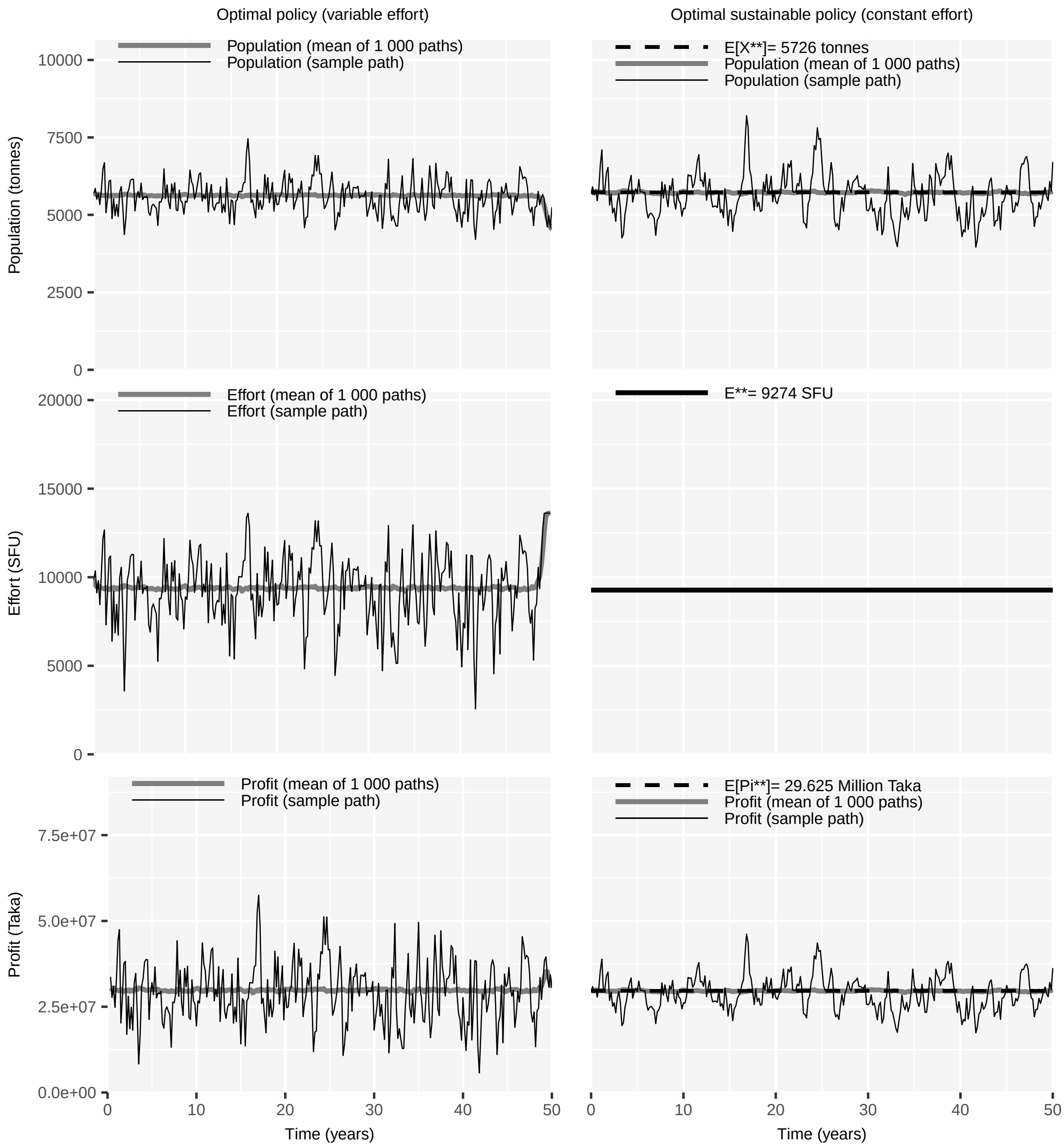


Figure 5.41: Scenario S_{19} (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

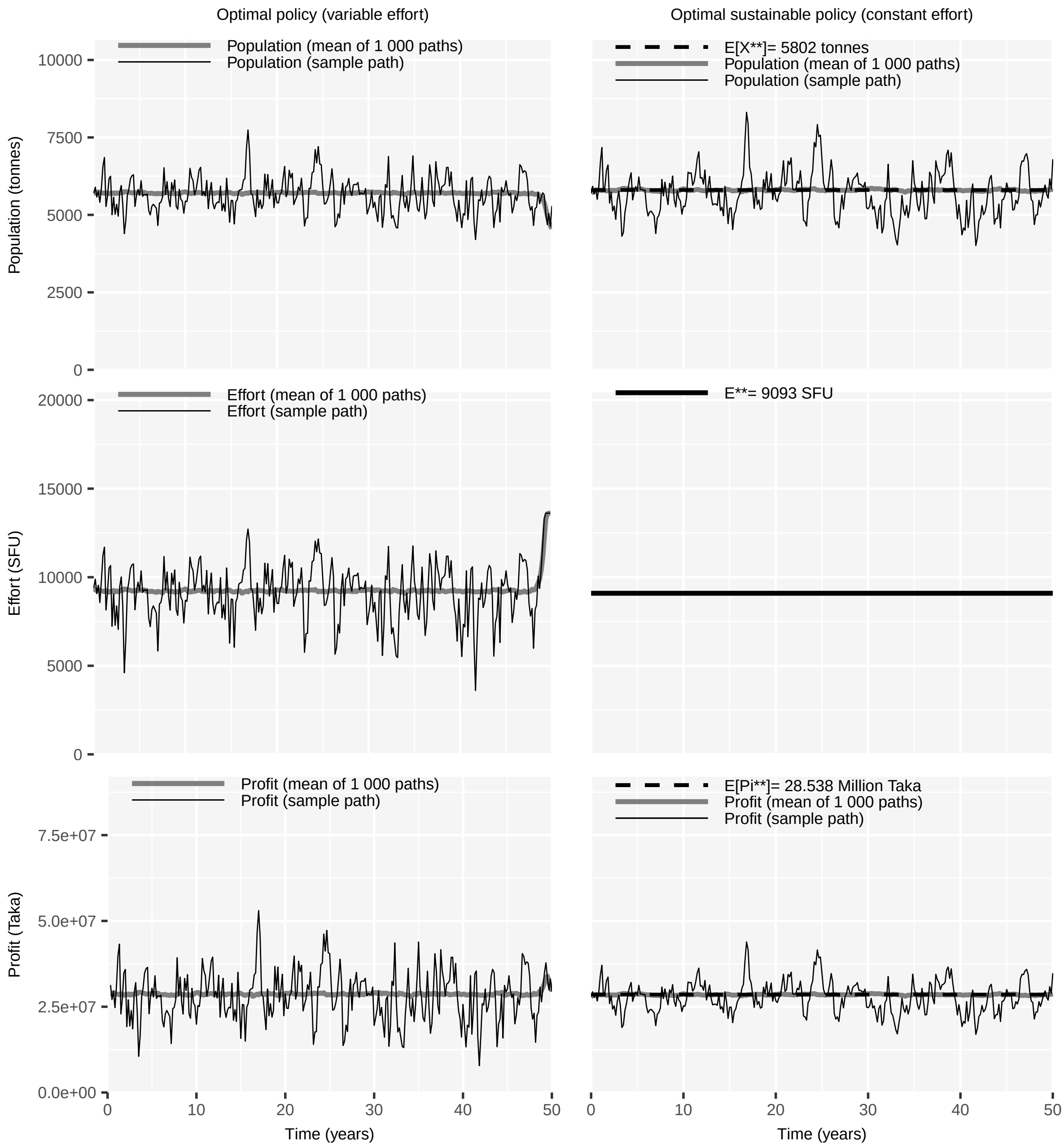


Figure 5.42: Scenario S_{20} (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

5.4 Comparison of policies with stepwise effort and logistic model

The optimal policy leads to a highly variable effort, with likely occurrence of periods of zero effort or periods of harvesting at maximum effort rates. From the point of view of the fishing activity, these frequent and abrupt effort changes would imply also a frequent and abrupt change on the number of vessels and gears, number of working hours and number of fishermen in activity, among others. Thus, the optimal effort policy in not applicable. In this section we study a sub-optimal policy, named stepwise policy, based on the variable effort from the optimal policy, but where the effort is kept constant during periods of duration p, say one or two years. We use $p = v\Delta t$ (v is a positive integer) to be a multiple of the time step Δt used in the numerical computations and in the Monte Carlo simulations. So, in this stepwise policy, for time t in the period $[lp, (l+1)p[= [t_{lv}, t_{(l+1)v}[$, we keep the effort $E^*_{step}(t) = E^*(lp)$ constant and equal to the effort of the optimal policy of the basic scenario S_0 at the beginning of the period. We know that this policy is not optimal nor stepwise optimal, but has however the advantage of being applicable, in contrast with the optimal policy.

We have focused the study on two scenarios: one with constant effort during periods of one year (annual), denoted by S_a scenario, and the other with constant effort during periods of two years (biennial), denoted by S_b scenario. The optimal sustainable effort remains unchanged and it is constant for all time instants, as before.

We keep all the parameters as in the scenario S_0, so the comparisons should be made with respect to S_0.

For scenario S_a, since $\Delta t = 4$ months $= 4/12$ years, we set the effort constant during periods of 1 year, i.e., during 3 consecutive time instants ($v = 3$). Similarly, for scenario S_b, we kept the effort constant during periods of 2 years, i.e., we set the effort constant during 6 time instants ($v = 6$). In reality, the basic scenario S_0 corresponds to using $v = 1$.

The resulting profit values (5.1) – (5.4) for both scenarios are shown in Table 5.11. For comparison purposes we also present the profit values for the basic scenario S_0. From Table 5.11 one can see that, for the scenario S_a, the optimal sustainable policy gives lower profits than the stepwise policy (-2.5%). However, the profit differences are less pronounced when compared to the difference in the basic scenario. On the contrary, when the optimal effort is applied during a longer biennial step-time periods (scenario S_b), the optimal sustainable policy yields slightly higher profits ($+1.5\%$) and, for the undiscouted profits, the difference is almost indistinguishable ($+0.0\%$).

The right side of Figures 5.43 and 5.44 refer to the constant effort sustainable policy, so they are identical to the right side of Figure 5.1 (basic scenario S_0). From the left side of both figures, corresponding to the stepwise policy, we can see an increase of the variability of the population sample path and of the sample profit and, in a more pronounced form, of the mean population and mean profit of the 1000 sample paths. At the middle of both figures one can check the stepwise effort, for both the sample path and mean. Their depicted lines in a form of staircase lend the name to the policy: stepwise policy.

Table 5.11: Expected discounted and undiscounted profit values for the stepwise scenarios S_a (annual periods) and S_b (biennial) for the logistic model. Besides the expected values, we also present the standard deviations (in parenthesis with smaller font size). The percent relative difference between both policies is denoted by Δ. Currency values are in million dollars for V^*_{step}, $V^*_{step,u}$, V^{**} and V^{**}_u, and million dollars per year for $\overline{\Pi^*_{step}}$, $\overline{\Pi^*_{step,u}}$, $\overline{\Pi^{**}}$ and $\overline{\Pi^{**}_u}$. For comparison purposes, we show the information for the basic scenario S_0 of the optimal policy.

Scenario	V^*_{step}		V^{**}		$\overline{\Pi^*_{step}}$		$\overline{\Pi^{**}}$		Δ (%)
S_0	413.586	(38.322)	396.423	(34.948)	22.529	(2.087)	21.594	(1.904)	−4.1
S_a	406.716	(38.792)	396.423	(34.948)	22.154	(2.113)	21.594	(1.904)	−2.5
S_b	390.499	(38.139)	396.423	(34.948)	21.271	(2.077)	21.594	(1.904)	+1.5

	$V^*_{step,u}$		V^{**}_u		$\overline{\Pi^*_{step,u}}$		$\overline{\Pi^{**}_u}$		Δ (%)
S_0	1129.130	(88.631)	1073.867	(88.543)	22.583	(1.773)	21.477	(1.771)	−4.9
S_a	1113.902	(90.775)	1073.867	(88.543)	22.278	(1.816)	21.477	(1.771)	−3.6
S_b	1073.550	(91.169)	1073.867	(88.543)	21.471	(1.823)	21.477	(1.771)	+0.0

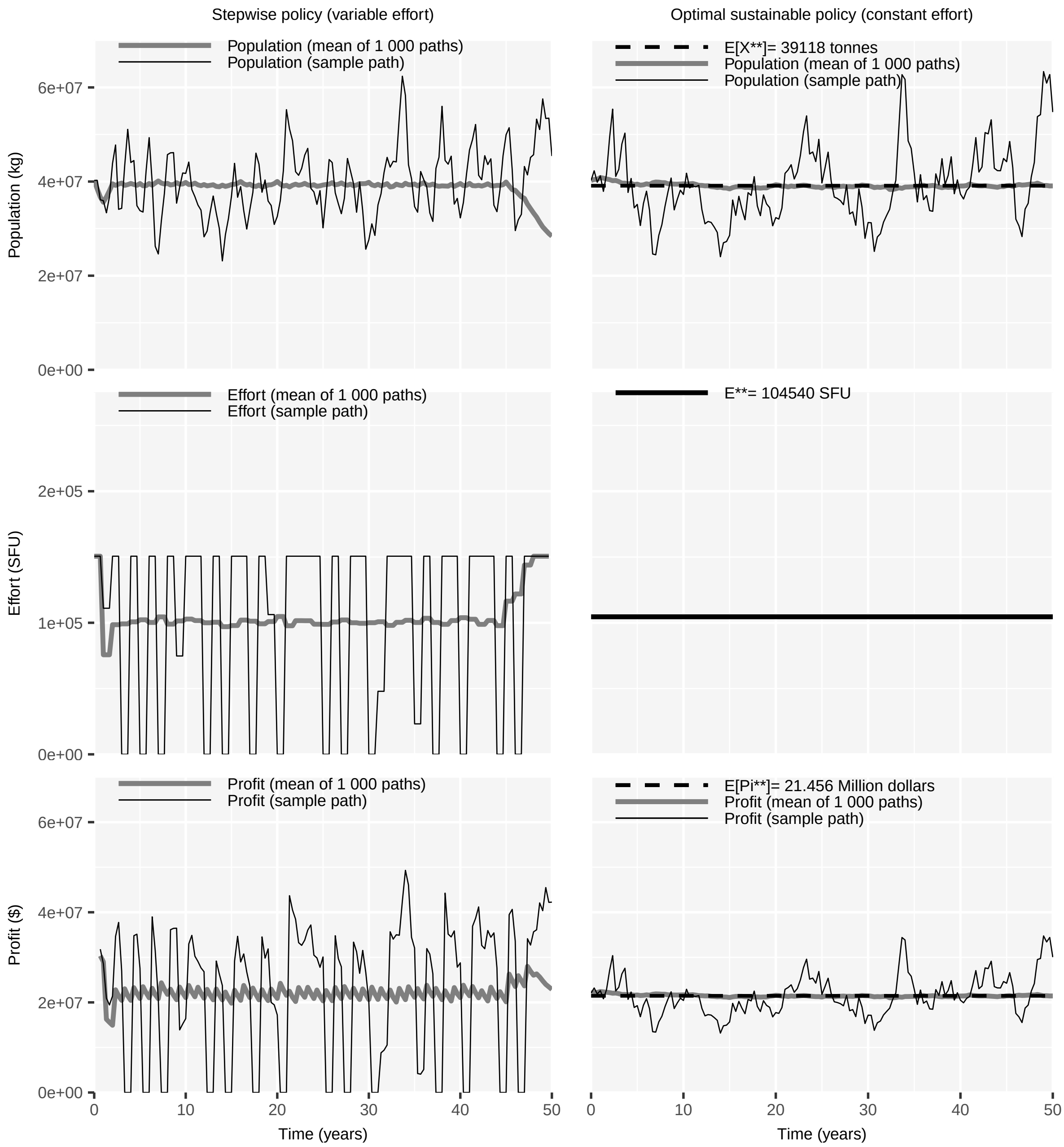


Figure 5.43: Scenario S_a (Logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The stepwise policy (with one year steps) is on the left side and the optimal constant effort sustainable policy is on the right side.

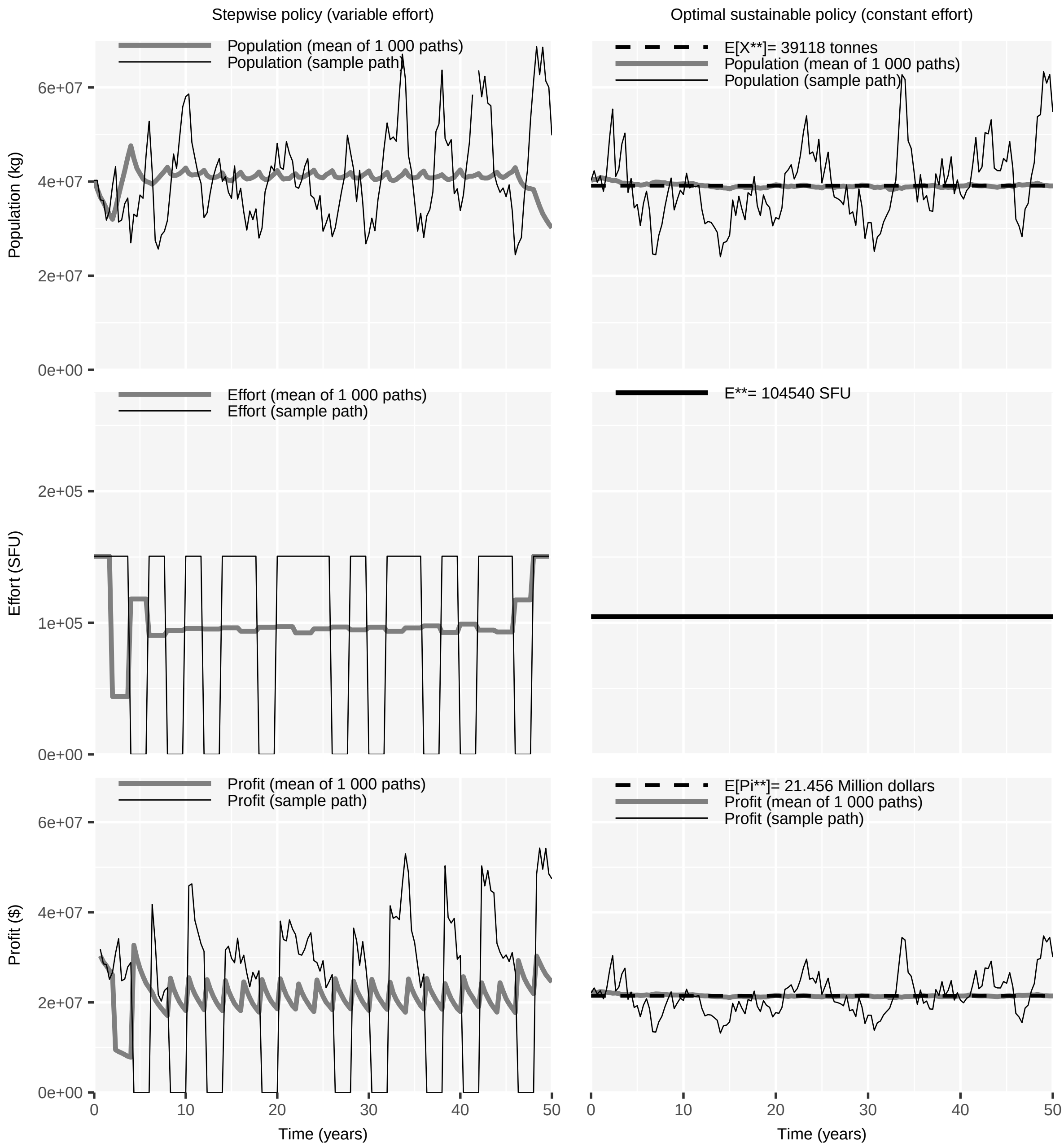


Figure 5.44: Scenario S_b (Logistic model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The stepwise policy (with two year steps) is on the left side and the optimal constant effort sustainable policy is on the right side.

5.5 Comparison of policies with stepwise effort and Gompertz model

For the Gompertz model with stepwise effort, we have also focused on the study of two scenarios: one with constant effort during periods of one year (annual), denoted by S_a scenario, and the other with constant effort during periods of two years (biennial), denoted by S_b scenario. The optimal sustainable effort remains unchanged and it is constant for all time instants, as before. In this case, the discretization scheme is different from the logistic case. We recall that for scenario S_0 we had $\Delta t = 2$ months $= 2/12$ years. So, for scenario S_a, we set the effort constant during periods of one year, i.e., during 6 consecutive time instants $(v = 6)$ and for scenario S_b, we set the effort constant during periods of two years, i.e., during 12 consecutive time instants $(v = 12)$.

The resulting profit values (5.1) – (5.4) for both scenarios are shown in Table 5.12. For comparison purposes we also present the profit values for the basic scenario S_0. Table 5.12 calls our attention directly to the difference between the optimal sustainable policy and the stepwise policy: for both scenarios S_a and S_b, the optimal sustainable policy yields greater profits, being similar for discounted and undiscouted profits.

Figures 5.45 and 5.46 show the behaviour of both policies in terms of population, effort and profit, where we can see the enormous variability of the population and profit sample paths and mean.

Table 5.12: Expected discounted and undiscounted profit values for the stepwise scenarios S_a (annual periods) and S_b (biennial) for the Gompertz model. Besides the expected values, we also present the standard deviations (in parenthesis with smaller font size). The percent relative difference between both policies is denoted by Δ. Currency values are in Taka for V^*_{step}, $V^*_{step,u}$, V^{**} and V^{**}_u, and Taka per year for $\overline{\Pi^*_{step}}$, $\overline{\Pi^*_{step,u}}$, $\overline{\Pi^{**}}$ and $\overline{\Pi^{**}_u}$. For comparison purposes, we show the information for the basic scenario S_0 of the optimal policy.

Scenario	V^*_{step}		V^{**}		$\overline{\Pi^*_{step}}$		$\overline{\Pi^{**}}$		Δ (%)
S_0	594.139	(21.401)	584.996	(21.107)	32.364	(1.116)	31.865	(1.150)	−1.5
S_a	556.611	(24.291)	584.996	(21.107)	30.319	(1.323)	31.865	(1.150)	+5.1
S_b	488.996	(30.006)	584.996	(21.107)	26.636	(1.634)	31.865	(1.150)	+19.6

	$V^*_{step,u}$		V^{**}_u		$\overline{\Pi^*_{step,u}}$		$\overline{\Pi^{**}_u}$		Δ (%)
S_0	1619.672	(47.825)	1591.393	(48.281)	32.393	(0.957)	31.828	(0.966)	−1.7
S_a	1517.928	(56.260)	1591.393	(48.281)	30.359	(1.125)	31.828	(0.966)	+4.8
S_b	1317.646	(65.532)	1591.393	(48.281)	26.353	(1.311)	31.828	(0.966)	+20.8

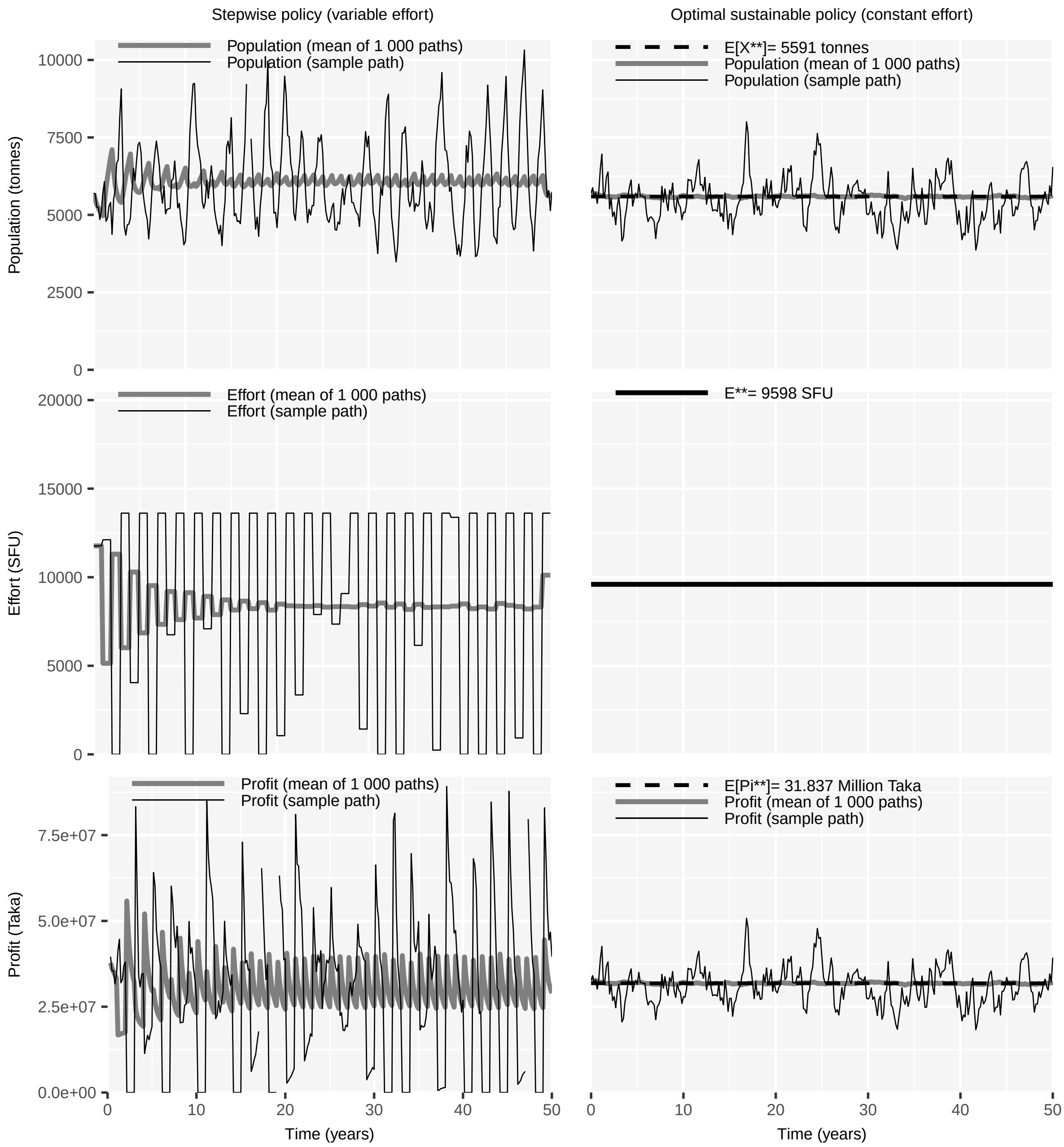


Figure 5.45: Scenario S_a (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The stepwise policy (with one year steps) is on the left side and the optimal constant effort sustainable policy is on the right side.

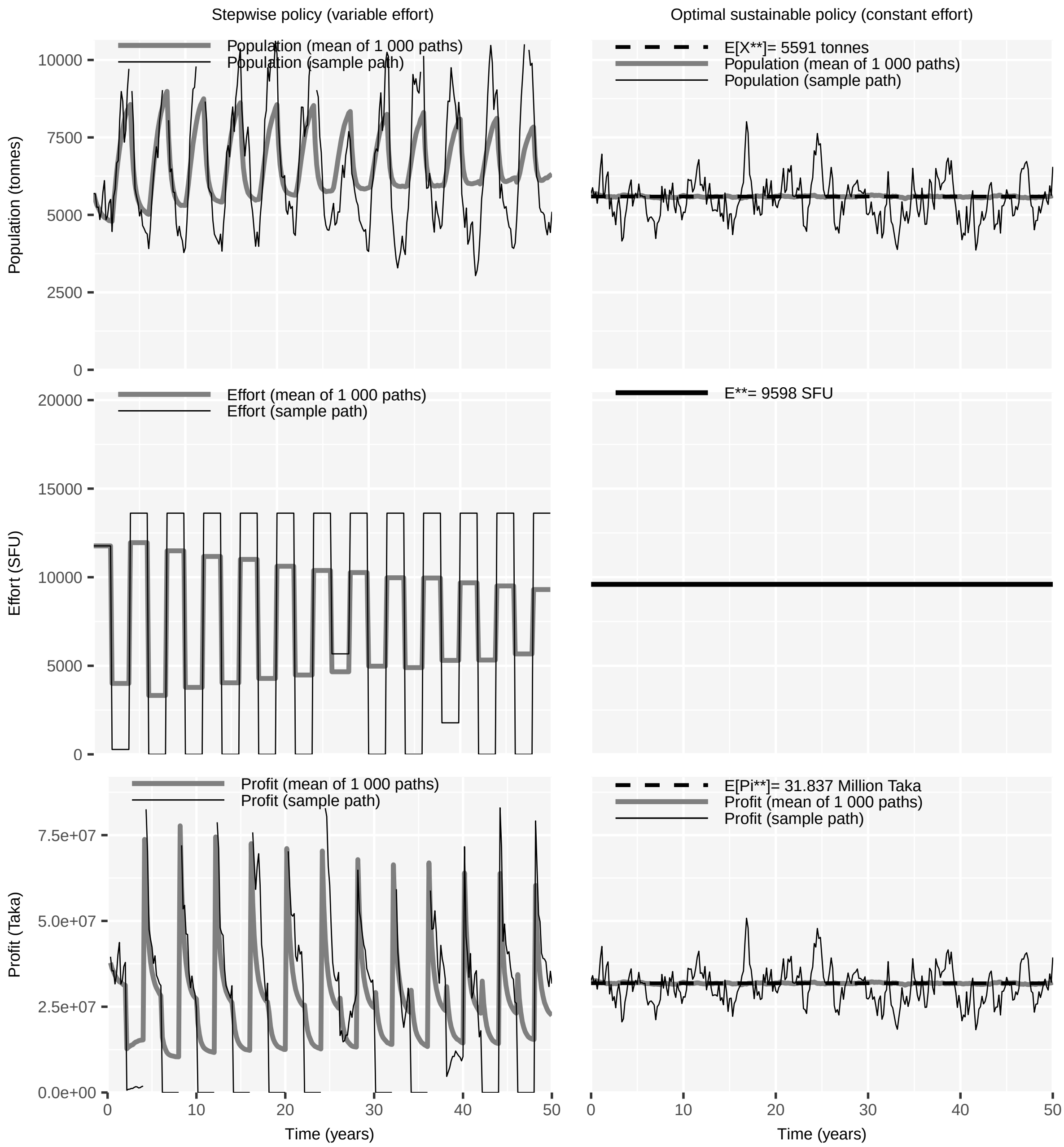


Figure 5.46: Scenario S_b (Gompertz model): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The stepwise policy (with two year steps) is on the left side and the optimal constant effort sustainable policy is on the right side.

5.6 Conclusions

In this chapter we have presented the numerical comparisons between the optimal policy with variable effort (using stochastic control theory) and the optimal sustainable policy with constant effort. The comparisons were realized in terms of four profit quantities: the expected accumulated discounted profit in a finite time interval, the expected accumulated undiscounted profit in a finite time interval, the average expected profit per unit time weighted by the discount factors and the average expected profit per unit time unweighted.

To obtain the profit values we have performed 1000 Monte Carlo simulations using a Crank-Nicolson discretization scheme in time and space of the HJB equation and an Euler scheme for the population paths. To compute the simulations we have applied the logistic and the Gompertz models to realistic data of fished populations and, for each model, we set up a basic scenario with the original data. To study the influence of the parameters on the policies performance, we have considered alternative scenarios by considering changes on the parameter values, usually one lower and one higher than the original value.

For the logistic model, the optimal sustainable policy produces a slight smaller profit in comparison with the optimal policy. It provides, however, a much steadier profit. For the Gompertz model, the profit differences between the two optimal policies are even smaller than in the logistic case. For both models we have seen that the optimal policy have frequent strong changes in effort, including frequent closings of the fishery, posing serious logistic applicability problems, producing social burdens and out-of-model costs (such as unemployment compensations) and leading to a great instability in the profit earned by the harvester. Furthermore, unlike the optimal variable effort policy, in the optimal constant effort policy there is no need to keep adjusting the effort to the randomly varying population size, and so there is no need to determine the size of the population at all times. This is a great advantage, since the estimation of the population size is a difficult, costly, time consuming and inaccurate task. The optimal policy also creates a possibly dangerous effect near the time horizon, implying a considerable drop on the population size. On the contrary, the optimal sustainable policy does not have these shortcomings, is very easy to implement and drives the population to a stochastic equilibrium. With a few exceptions, the alternative scenarios share the same behaviour as the basic scenario.

Since the optimal policy in not applicable, we have presented, for the logistic and the Gompertz models, a sub-optimal policy, named stepwise policy, based on variable effort but with periods of constant effort. This policy is not optimal, but has the advantage of being applicable, since the changes on effort are not so frequent and can be compatible with the fishing activity. Furthermore, although we still need to keep estimating the fish stock size, we do not need to do it so often. Replacing the optimal variable effort policy by these stepwise policy has the advantage of applicability but, at best, considerably reduces the already small advantages they have over the optimal sustainable policy. In some cases, the much easier to implement optimal sustainable policy even outperforms these stepwise policy in terms of profit.

6

Comparison of policies in the presence of weak Allee effects

In this chapter we will study, for a logistic-like model, the comparison between the optimal variable effort policy and the optimal sustainable constant effort policy when the population is under weak Allee effects. In addition, these policies will be compared with the previous optimal policies without Allee effects. This chapter is organized as follows: in section 6.1 we formulate the model for the optimal policy and for the optimal sustainable policy and, for the latter policy, we will prove the existence of a stationary density for the population size under weak Allee effects. Section 6.2 presents the comparisons in terms of profit and in terms of the evolution of the population and effort along a finite time interval. We end up with chapter conclusions at section 6.3.

6.1 Optimal policies with a logistic-like model under weak Allee effects

In this chapter we assume that the population is under the influence of weak Allee effects, that is, at very low values of population size, we observe lower *per capita* growth rates instead of the higher rates one would expected considering the higher availability of resources per individual. The presence of weak Allee effects when population size is low may be due to the difficulty in finding mating partners or in constructing a strong enough group defence against predators. The study of population growth models without harvesting and under Allee effects (weak and strong) can be seen in Carlos and Braumann (2017) and references therein. Considering strong Allee effects would lead the population to extinction, even in the absence of harvesting, since the average natural growth rate would be negative for low population sizes (see Carlos and Braumann, 2017). Therefore, we will consider only weak Allee effects.

Here, we will consider a similar study but when the population is subject to harvesting according to an optimal policy approach. The resultant model is represented by the SDE

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t) - A}{K - A}\right)dt - qE(t)X(t)dt + \sigma X(t)dW(t), \quad X(0) = x, \tag{6.1}$$

where $A \in (-K, 0)$ represents the Allee parameter which measures the strength of the weak Alee effects. The closer to 0 is A, the more intense is the Allee effect. On the contrary, the closer to $-K$ is A, the less intense is the Allee effect. Making $A \to -\infty$ leads to the logistic model. Strong Allee effects occur when $A \in (0, K)$ and they will not be here considered.

Equation (6.1) assumes that the natural growth rate follows a logistic-like model inspired by a similar deterministic model (see, for instance, Dennis, 2002). However, the parametrization here considered is different from the usual in the literature, since it allows comparisons with the logistic model without Allee effects (see, for instance, Carlos and Braumann, 2017). In particular, the logistic model and the logistic-like model here considered have in common the same carrying capacity K and the same slope of the natural growth rate at K.

Figure 6.1 shows, for the deterministic case in the absence of fishing, two examples of the logistic-like model with weak and strong Allee effects. On the top is depicted the total population growth curve and, at the bottom, the *per capita* growth curve. We also show, for comparison purposes, similar curves for the logistic case without Allee effects. For the model with weak Allee effects it is easily seen that, at very low population sizes, the *per capita* growth rates are not at their maximum value as they would ordinarily be based on the *per capita* resource abundance.

For values of $A \leq -K$, the *per capita* growth rates at low population sizes are at their maximum value and so, technically, we do not speak of having Allee effects. However, those rates are still depressed when compared to the logistic model, which is only reached when $A \to -\infty$.

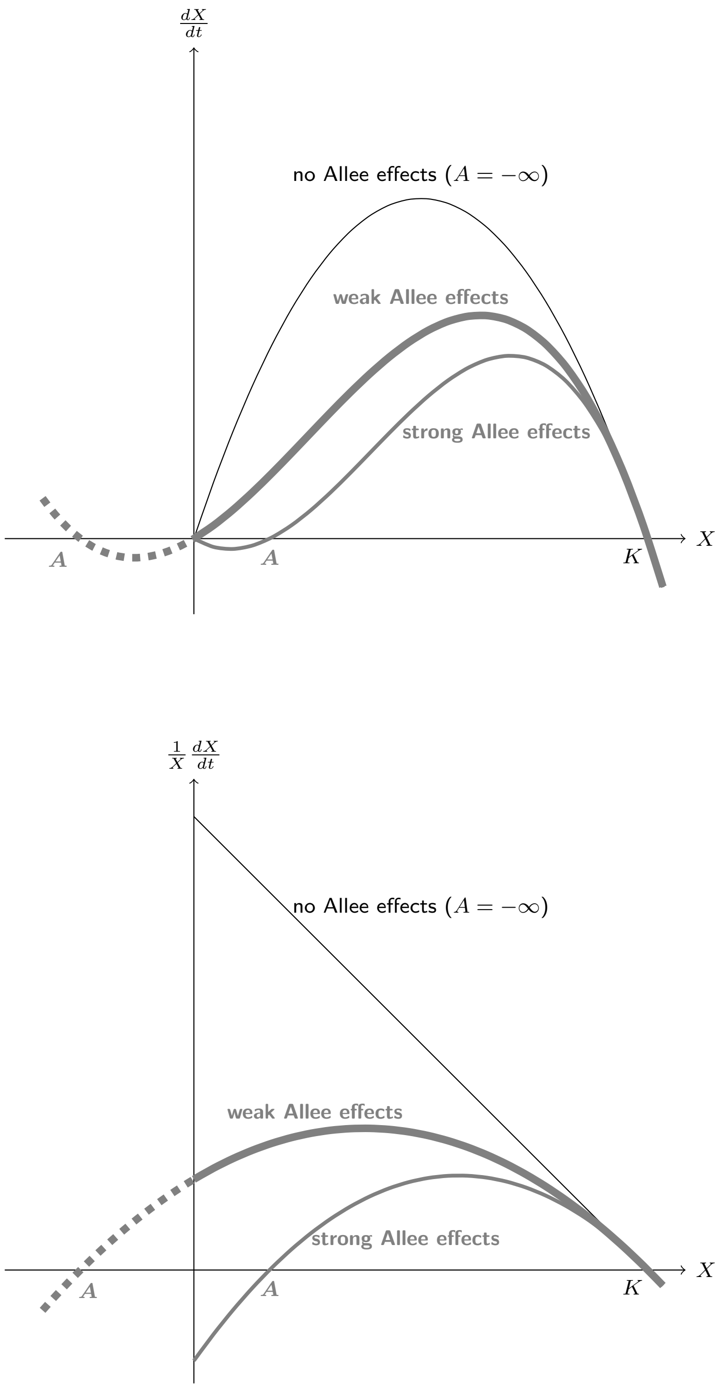


Figure 6.1: Total population growth and *per capita* growth in the absence of fishing for the deterministic logistic model without Allee effects (thin lines) and for the deterministic logistic-like growth model under weak and strong Allee effects (thick lines).

6.1.1 Optimal policy formulation

In the case of the optimal variable effort policy, the optimization problem presented in section 3.1, takes now the form

$$V^* := J^*(x,0) = \max_{\substack{E(\tau) \\ 0\leq\tau\leq T}} \mathbb{E}_x \left[\int_0^T e^{-\delta\tau} \Big(p_1 q X(\tau) - c_1 - (p_2 q^2 X^2(\tau) + c_2)E(\tau) \Big) E(\tau) d\tau \right],$$

subject to equation (6.1), to the effort constraints $0 \leq E_{min} \leq E(t) \leq E_{max} < \infty$, and to the terminal condition $J^*(X(T),T) = 0$.

The solution of the previous optimal control problem is obtained as indicated in Chapter 3 and also resorts to solve a HJB equation numerically. The computations are similar as in Chapter 5 with the required and very easy adaptations, namely considering $f(X(t)) = r\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t)-A}{K-A}\right)$ in equation (3.18).

6.1.2 Optimal sustainable policy formulation

For the logistic-like model with weak Allee effects and a constant effort fishing policy $E(t) \equiv E$, the dynamics of the population are described by the SDE model

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t)-A}{K-A}\right)dt - qEX(t)dt + \sigma X(t)dW(t), \quad X(0) = x. \tag{6.2}$$

Again, we assume weak Allee effects, i.e., we assume $-K < A < 0$.

Since the drift and diffusion coefficients

$$a(X) = rX\left(1 - \frac{X}{K}\right)\left(\frac{X-A}{K-A}\right) - qEX$$

and

$$b^2(X) = \sigma^2 X^2,$$

are of class C^1, equation (6.2) has a unique solution up to a possible explosion time, which is a homogeneous diffusion process.

The scale and speed densities come from expressions (2.7) and (2.8) as

$$s(X) = CX^{-\alpha+\beta E-1} \exp\left\{\gamma(X - (K+A))^2\right\} \tag{6.3}$$

and

$$m(X) = DX^{\alpha-\beta E-1}\exp\left\{-\gamma(X-(K+A))^2\right\}, \tag{6.4}$$

where $\alpha = \frac{2rA}{\sigma^2(A-K)} - 1$, $\beta = \frac{2q}{\sigma^2}$, $\gamma = \frac{r}{\sigma^2 K(K-A)}$ and $C,\ D$ are positive constants.

We are going to assume that $\alpha - \beta E > 0$. This means that

$$E < \frac{r}{q}\left(\frac{A}{A-K} - \frac{\sigma^2}{2r}\right). \tag{6.5}$$

In fact, if $\alpha - \beta E < 0$, the scale measure $S(0,z]$ of a small neighbourhood of the zero boundary is, assuming that $0 < z < K+A$ (note that $K+A > 0$),

$$\begin{aligned} S(0,z] &= \int_0^z CX^{-\alpha+\beta E-1}\exp\left\{\gamma(X-(K+A))^2\right\}dX \\ &\leq C\exp\left\{\gamma(K+A)^2\right\}\int_0^z X^{-\alpha+\beta E-1}dX < +\infty, \end{aligned}$$

and the zero boundary would be attractive, with mathematical extinction occurring with probability one.

If $\alpha - \beta E = 0$, then it is easy to see that the speed measure is not finite ($M(0,+\infty) = +\infty$) since $M(0,z] = +\infty$, and there is no stationary density.

Assuming, from now on, $\alpha - \beta E > 0$, we can see that the zero boundary is non-attractive since

$$S(0,z] \geq C\exp\left\{\gamma(z-(K+A))^2\right\}\int_0^z X^{-\alpha+\beta E-1}dX = +\infty.$$

Notice that the infinite boundary is non-attractive, and so there are no explosions. In fact, considering $z > 0$, the scale measure of a neighbourhood $(z,+\infty)$ of $+\infty$ is

$$\begin{aligned} S(z,+\infty) &= \int_z^{+\infty} CX^{-\alpha+\beta E-1}\exp\left\{\gamma(X-(K+A))^2\right\}dX \\ &= C\int_z^{+\infty}\exp\left\{-(\alpha-\beta E-1)\ln X + \gamma(X-(K+A))^2\right\}dX = +\infty, \end{aligned}$$

since $\gamma > 0$ and so the last integrand tends to $+\infty$ as $X \to +\infty$.

So, assuming $\alpha - \beta E > 0$, mathematical extinction has zero probability of occurring and the solution of equation (6.2) exists and is unique. Furthermore, we will now show that $M(0,+\infty) < +\infty$, which proves that there exists a stationary density for the population size. In fact,

$$M(0,\infty) = \int_0^{+\infty} m(z)dz \tag{6.6}$$

can be written as

$$M_1 + M_2 + M_3 = \int_0^{L_1} m(z)dz + \int_{L_1}^{L_2} m(z)dz + \int_{L_2}^{+\infty} m(z)dz, \tag{6.7}$$

with $0 < L_1 < L_2 < L_3 < +\infty$ and $L_3 > \max\{1, K+A\}$. We just need to show that each of the three integrals is finite. Since $\alpha - \beta E > 0$, we have

$$\begin{aligned} M_1 &= \int_0^{L_1} DX^{\alpha-\beta E-1} \exp\left\{-\gamma(X-(K+A))^2\right\} dX \\ &\leq D\int_0^{L_1} X^{\alpha-\beta E-1} dX = +\infty. \end{aligned}$$

Let η be positive and larger than $\alpha-\beta E-1$. Then, for $X > L_3$, we have $(\alpha-\beta E-1)\ln X < \eta \ln X < \eta X$. Putting $\mu = K + A + \frac{\eta}{2\gamma}$ and $\nu = \frac{1}{\sqrt{2\gamma}}$, we have

$$\begin{aligned} M_3 &= \int_{L_3}^{+\infty} DX^{\alpha-\beta E-1} \exp\left\{-\gamma(X-(K+A))^2\right\} dX \\ &= D\int_{L_3}^{+\infty} \exp\left\{(\alpha-\beta E-1)\ln X - \gamma(X-(K+A))^2\right\} dX \\ &\leq D\int_{L_3}^{+\infty} \exp\left\{\eta X - \gamma(X-(K+A))^2\right\} dX \\ &= D\exp\left\{\eta\left(K+A+\frac{\eta}{4\gamma}\right)\right\}\sqrt{\frac{\pi}{\gamma}}\int_{L_3}^{+\infty} \frac{1}{\nu\sqrt{2\pi}}\exp\left\{-\frac{(X-\mu)^2}{2\nu^2}\right\} dX \\ &\leq D\exp\left\{\eta\left(K+A+\frac{\eta}{4\gamma}\right)\right\}\sqrt{\frac{\pi}{\gamma}} < +\infty, \end{aligned}$$

because the integrand of the last integral is the p.d.f. of a Gaussian random variable with mean μ and variance ν^2.

As for

$$M_2 = \int_{L_1}^{L_2} m(X)dX,$$

it is finite since m is a continuous function on the close interval $[L_1, L_2]$.

From the above and, as in the case of the logistic model without Allee effects (see chapter 4), one can conclude that there exists a stationary density $p(X)$ for the size of the population. In other words, $p(X)$ is the probability density function of a random variable X_∞ and $X(t)$ converges in distribution to X_∞. Furthermore, the stationary density is proportional to the speed density, i.e.,

$$\begin{aligned} p(X) &= \frac{m(X)}{\int\limits_0^{+\infty} m(z)dz}, \quad 0 < X < +\infty \\ &= \frac{X^{\alpha-\beta E-1}\exp\{-\gamma(X-(K+A))^2\}}{\int_0^{+\infty} z^{\alpha-\beta E-1}\exp\{-\gamma(z-(K+A))^2\}dz}, \quad 0 < X < +\infty. \end{aligned}$$

The expected value of X_∞ is obtained as

$$\mathbb{E}[X_\infty] = \int_0^{+\infty} xp(x)dx = \frac{I_1(E)}{I_0(E)},$$

where

$$I_j(E) = \int\limits_0^{+\infty} z^{\alpha-\beta E+j-1}\exp\left\{-\gamma(z-(K+A))^2\right\}dz.$$

The steady-state optimization problem is similar to the logistic case without Allee effects and consists in maximizing the expected sustainable profit per unit time, that is, to determine

$$\max_{E\geq 0}\mathbb{E}[\Pi_\infty] = \max_{E\geq 0}\left\{\left(p_1 q\frac{I_1(E)}{I_0(E)} - c_1\right)E - \left(p_2 q^2\frac{I_2(E)}{I_0(E)} + c_2\right)E^2\right\},$$

in case there is a maximum in the admissible range $0 \leq E < \frac{r}{q}\left(\frac{A}{A-K} - \frac{\sigma^2}{2r}\right)$.

The optimal sustainable effort E^{**} can be obtained by solving the equation $d\mathbb{E}[\Pi_\infty]/dE = 0$ such that the solution satisfies $d^2\mathbb{E}[\Pi_\infty]/dE^2 < 0$, which requires numerical methods. Finally, the expected sustainable profit per unit time is given by

$$\mathbb{E}[\Pi_\infty^{**}] = \left(p_1 q\frac{I_1(E^{**})}{I_0(E^{**})} - c_1\right)E^{**} - \left(p_2 q^2\frac{I_2(E^{**})}{I_0(E^{**})} + c_2\right)E^{**2},$$

and the sustainable expected population is given by

$$\mathbb{E}[X_\infty^{**}] = \frac{\int\limits_0^{+\infty} z^{\alpha-\beta E^{**}}\exp\left\{-\gamma(z-(K+A))^2\right\}dz}{\int\limits_0^{+\infty} z^{\alpha-\beta E^{**}-1}\exp\left\{-\gamma(z-(K+A))^2\right\}dz}.$$

6.2 Comparison of policies

The comparisons between the optimal variable effort policy and the optimal sustainable policy, both with a growth model with weak Allee effects, are very similar to the comparisons made in the case of the logistic model without Allee effects. The data and algorithms are the ones used previously with the necessary adjustments for the inclusion of the Allee effects expressions. We kept the basic scenario S_0 and considered 5 alternative scenarios with varying Allee effects. The list of scenarios is in Table 6.1.

The resulting profit values (5.1) – (5.4) for the considered scenarios are shown in Table 6.2 and the corresponding graphics are in Figures 6.2 – 6.6.

From Table 6.2 one can see that scenario S_{A1}, corresponding to a quite extreme $A = -0.10K$, has a catastrophic behaviour in terms of profit when we apply the optimal sustainable policy. In fact, the discounted profit is reduced by 61.9% in comparison with the optimal policy, and the undiscounted profit is reduced by 63.6%. Figure 6.2 shows that the variability in effort and profit still remains (when compared to the models without Allee effects), having long periods of no harvesting, i.e., long periods of zero effort and also long periods of zero profit. The very low value of the optimal sustainable effort causes high values for the population size (right top).

The others scenarios S_{A2} to S_{A5} correspond to considering weaker Allee effects and we might see, from Table 6.2 and from Figures 6.3 – 6.6, that the profit values under weak Allee effects are lower than in the model without Allee effects, corresponding to the base scenario S_0, but are approaching the S_0 values as A decreases. Of course, in the limit, one would actually reach S_0, corresponding to $A = -\infty$. The same happens with the profit differences between the optimal variable policy and the optimal sustainable policy

Table 6.1: List of alternative scenarios ($S_{Ai}, i = 1, \ldots, 5$) and respective changed Allee parameter. The values used in simulations are in Table 5.1.

Scenario	Allee parameter value A
S_0 (logistic model)	$-\infty$
S_{A1}	$-0.10K$
S_{A2}	$-0.25K$
S_{A3}	$-0.50K$
S_{A4}	$-0.75K$
S_{A5}	$-0.95K$

(Δ values). We notice that, with a few exceptions, the standard deviations have very small variations across the various scenarios.

Table 6.2: Expected discounted and undiscounted profit values for the scenarios $S_{Ai},\ i = 1, \ldots, 5$. Besides the expected values, we also present the standard deviations (in parenthesis with smaller font size). The percent relative difference between both policies is denoted by Δ. Currency values are in million dollars for V^*, V_u^*, V^{**} and V_u^{**}, and million dollars per year for $\overline{\Pi^*}$, $\overline{\Pi_u^*}$, $\overline{\Pi^{**}}$ and $\overline{\Pi_u^{**}}$. For comparison purposes, we show the information for the basic scenario S_0 of the logistic model without Allee effects.

Scenario	V^*		V^{**}		$\overline{\Pi^*}$		$\overline{\Pi^{**}}$		Δ (%)
S_0	413.586	(38.322)	396.424	(34.948)	22.529	(2.087)	21.594	(1.904)	−4.1
S_{A1}	218.79	(44.700)	83.405	(7.397)	11.918	(2.435)	4.543	(0.403)	−61.9
S_{A2}	248.307	(45.209)	186.924	(25.281)	13.526	(2.463)	10.182	(1.377)	−24.7
S_{A3}	277.986	(46.088)	237.896	(35.734)	15.142	(2.510)	12.959	(1.946)	−14.4
S_{A4}	296.141	(44.655)	261.851	(36.193)	16.131	(2.432)	14.263	(1.971)	−11.6
S_{A5}	307.457	(43.234)	276.037	(36.206)	16.748	(2.355)	15.036	(1.972)	−10.2

	V_u^*		V_u^{**}		$\overline{\Pi_u^*}$		$\overline{\Pi_u^{**}}$		Δ (%)
S_0	1129.130	(88.631)	1073.867	(88.543)	22.583	(1.773)	21.477	(1.771)	−4.9
S_{A1}	649.127	(113.860)	236.590	(17.470)	12.983	(2.277)	4.732	(0.349)	−63.6
S_{A2}	716.193	(110.432)	522.531	(68.249)	14.324	(2.209)	10.451	(1.365)	−27.0
S_{A3}	788.676	(107.116)	652.393	(100.028)	15.774	(2.142)	13.048	(2.001)	−17.3
S_{A4}	832.398	(103.334)	716.042	(98.888)	16.648	(2.067)	14.321	(1.978)	−14.0
S_{A5}	857.006	(100.022)	753.825	(97.703)	17.140	(2.000)	15.076	(1.954)	−12.0

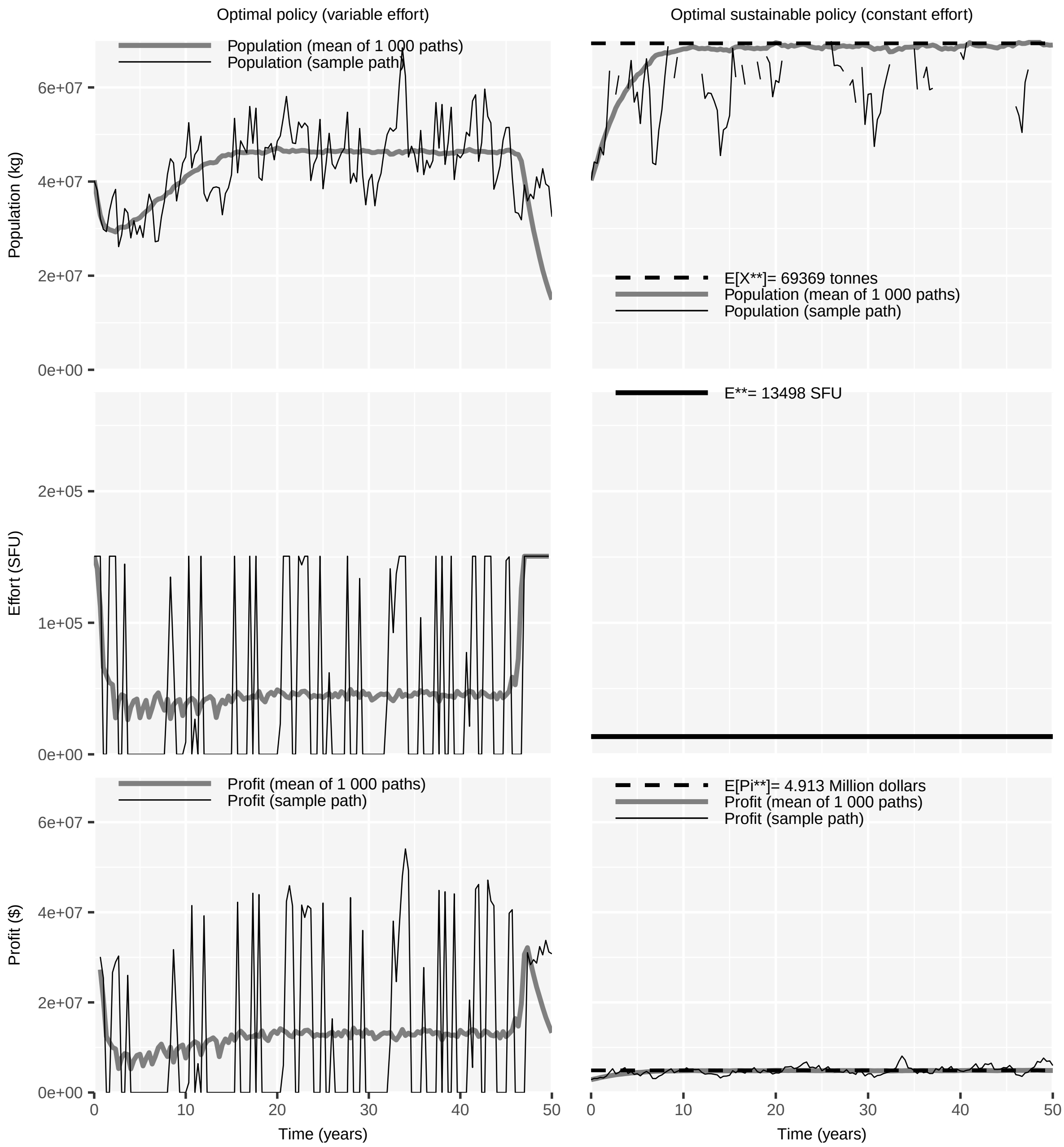


Figure 6.2: Scenario S_{A1} (logistic-like model with weak Allee effects): mean and randomly chosen sample path for the population, the effort and the profit. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

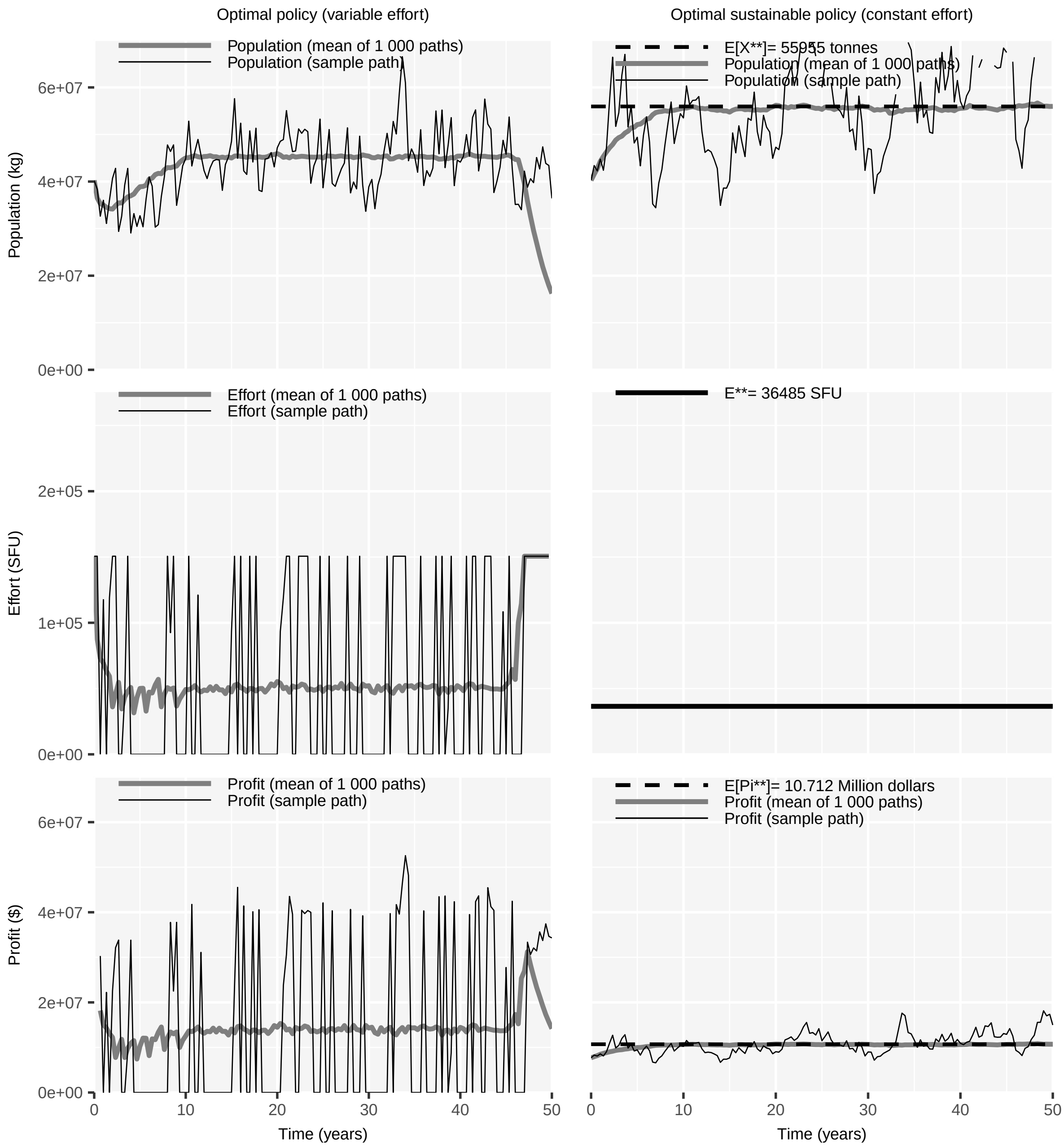


Figure 6.3: Scenario S_{A2} (logistic-like model with weak Allee effects): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The stepwise policy (with two year steps) is on the left side and the optimal constant effort sustainable policy is on the right side.

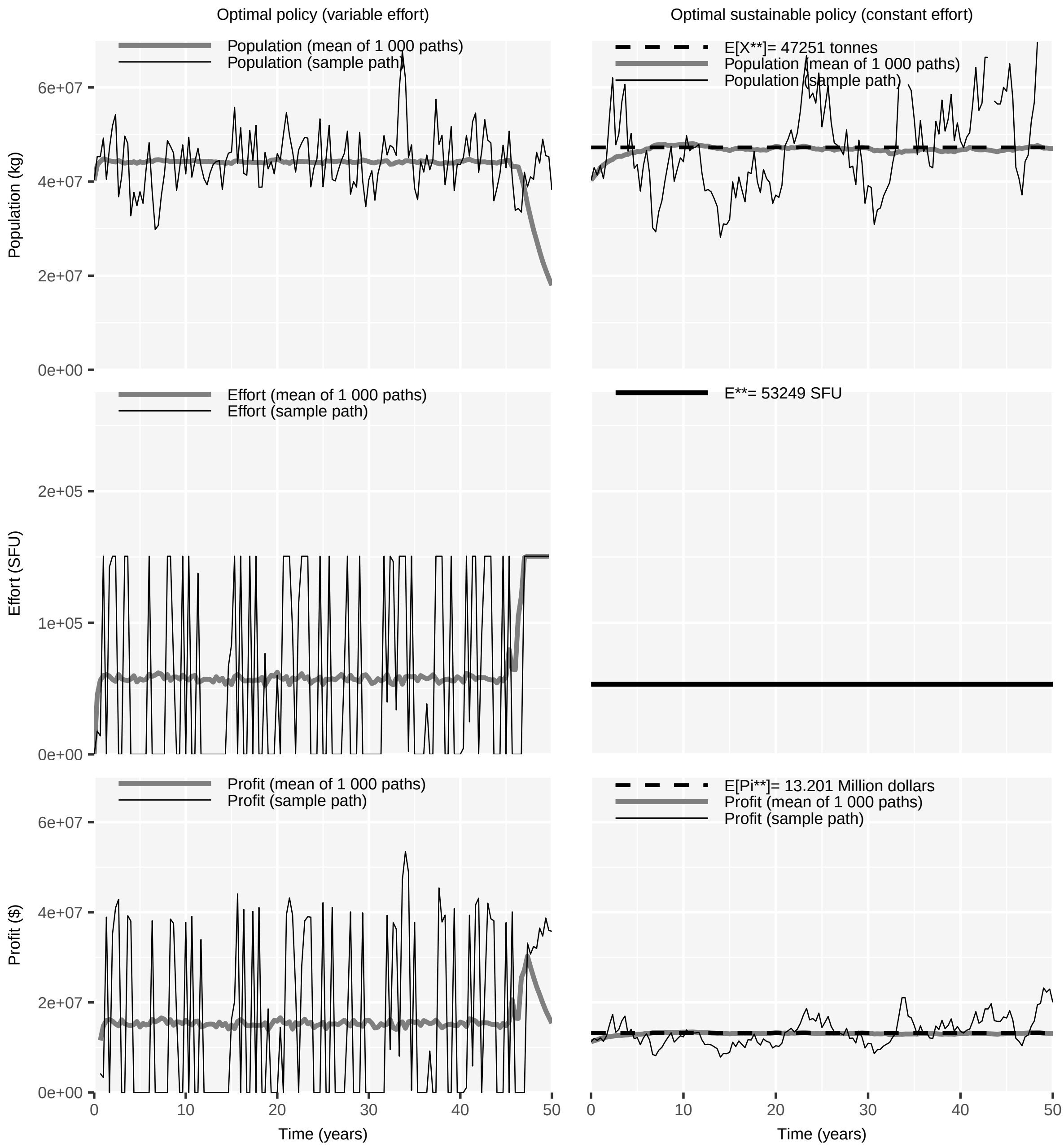


Figure 6.4: Scenario S_{A3} (logistic-like model with weak Allee effects): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The stepwise policy (with one year steps) is on the left side and the optimal constant effort sustainable policy is on the right side.

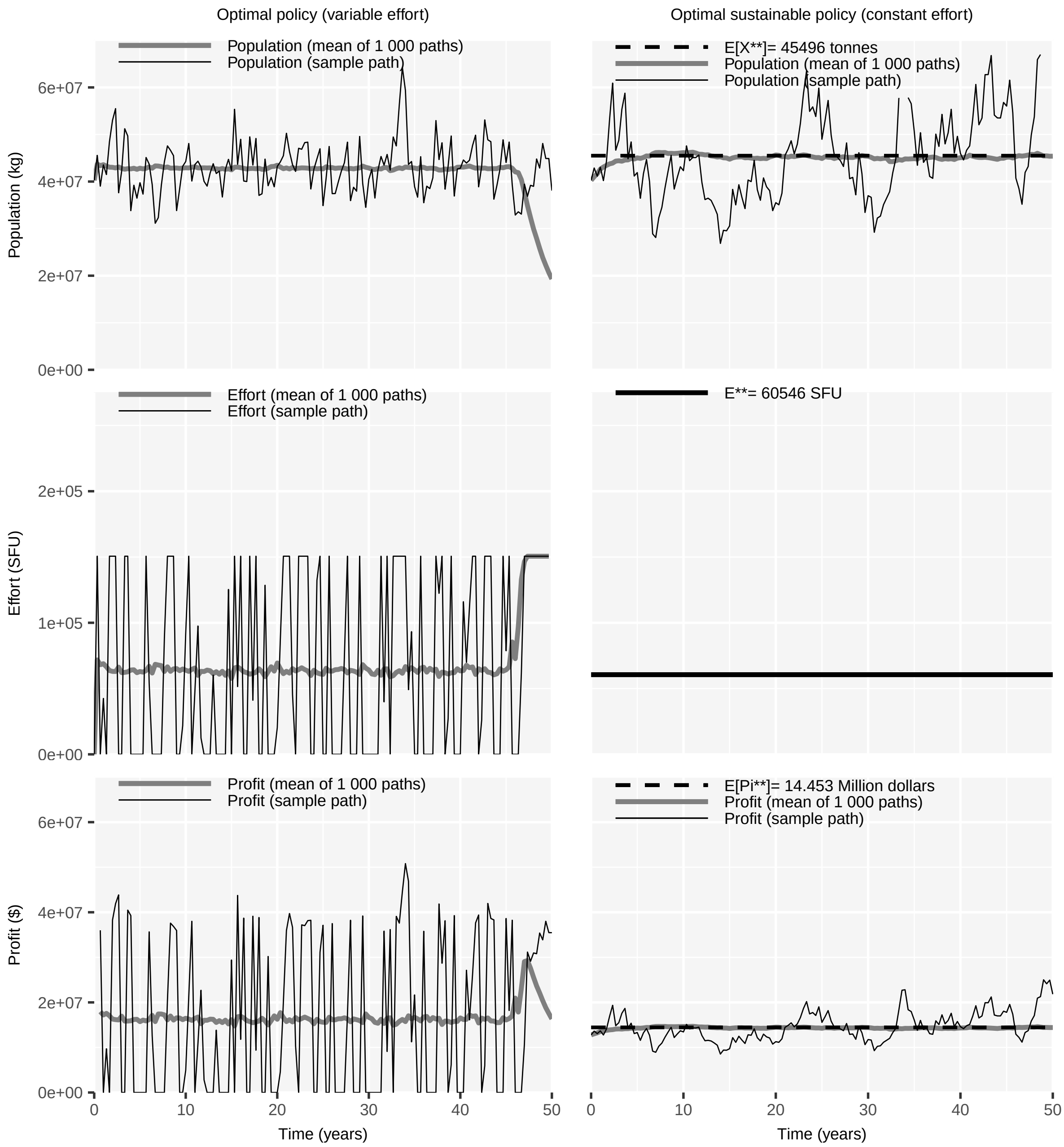


Figure 6.5: Scenario S_{A4} (logistic-like model with weak Allee effects): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The stepwise policy (with two year steps) is on the left side and the optimal constant effort sustainable policy is on the right side.

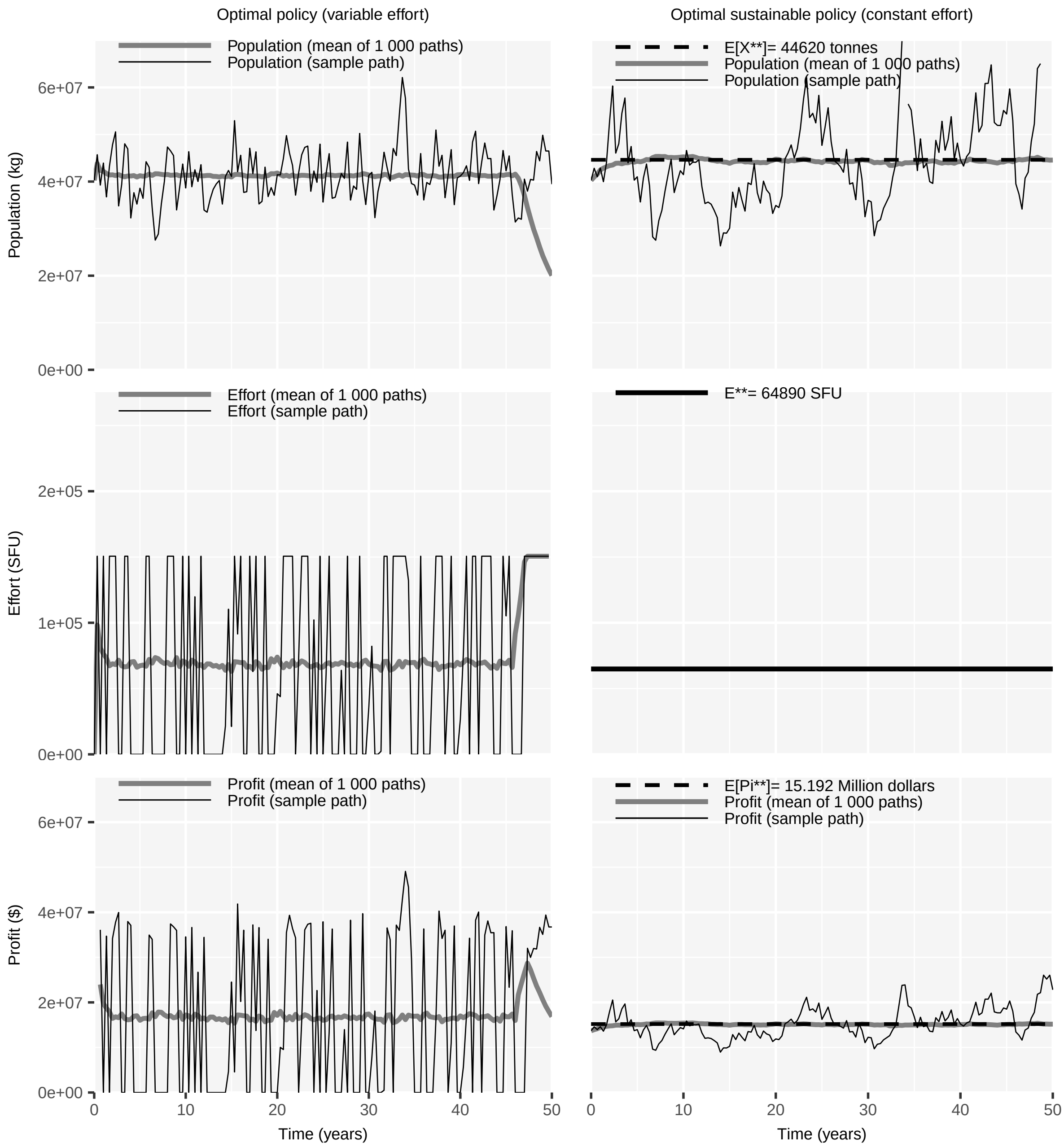


Figure 6.6: Scenario S_{A5} (logistic-like model with weak Allee effects): mean and randomly chosen sample path for the population, the effort and the profit per unit time. The stepwise policy (with one year steps) is on the left side and the optimal constant effort sustainable policy is on the right side.

6.3 Conclusions

In this chapter we have worked with a logistic-type population growth model under the influence of weak Allee effects. For this logistic-type population growth model, we have formulated the problems of the optimal variable effort policy and of the optimal constant effort sustainable policy. For the constant effort model we showed that, if the effort is not too high (in fact, the effort has to fulfil the condition $E < \frac{r}{q}\left(\frac{A}{A-K} - \frac{\sigma^2}{2r}\right)$, the state space boundaries are non-attractive, that there is a stationary density for the population size for which we have found an expression.

Both optimal policies were applied for the parameters values used in the basic scenario S_0 (the basic scenario of the logistic model without Allee effects). To see the influence of the weak Alee effects when comparing both policies, we have simulated 5 scenarios with variations on the Allee parameter. We have seen that, as the Allee parameter becomes smaller, the Allee effects have less influence on both policies and, therefore, the policies tend to behave as in the scenario without Allee effects. When the Allee parameter approaches zero, the Allee effects become more pronounced and imply huge differences in terms of profit values when comparing both harvesting policies; the profit becomes, for both type of policies, substantially lower than in the model without Allee effects.

7

First passage times

In previous chapters we have seen that, for the class of models with constant fishing effort, mathematical extinction occurs with zero probability. However, since we work with ergodic processes, all the states in the interior of $(0, +\infty)$ are attainable with probability one in finite time. In particular, we can define a threshold in the interior of the state space, $y \in (0, +\infty)$, and study how long it takes for the process $X(t)$ to reach y for the first time. This threshold can be seen, for instance, as a biological reference point, i.e., a minimum biomass value which should not be reached, otherwise it is considered that the population self-renewable capacity is endangered. In addition, we can consider the case where the threshold is an upper limit and study the first passage time by this limit. In fisheries, this case is also important because it allows to establish high levels of biomass that could affect, for example, the survival of another species, or warn us that the optimal fishing effort is not being correctly applied.

Based on general expressions for the mean and standard deviation of first passage times by lower and upper thresholds, we will compute such values for the particular cases of the logistic and the logistic-like models with weak Allee effects and for several lower and upper threshold values.

We will also present a way to estimate, for a fixed threshold value, the probability density function of the time to reach that threshold. Using numerical methods we obtain the inverse of the Laplace transform of the probability density function. For this particular case, we will compare the mean values obtained by this technique and by the direct expression of the mean value.

This chapter is organized as follows: Section 7.1 presents the expressions to compute the mean and standard deviation of first passage times, and respective application to the logistic model and to the logistic-like model with weak Allee effects. In section 7.2 we estimate the probability density function of the first passage time using the inversion of the Laplace transform. Section 7.3 presents the main chapter conclusions.

7.1 Moments of the first passage times

We recall that, in this chapter, the models under study are the logistic harvesting stochastic model with constant fishing effort and the logistic-like harvesting stochastic model with weak Allee effects and constant fishing effort, represented by the SDEs:

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)dt - qEX(t)dt + \sigma X(t)dW(t), \quad X(0) = x$$

and

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t) - A}{K - A}\right)dt - qEX(t)dt + \sigma X(t)dW(t), \quad X(0) = x,$$

respectively.

For the first model, the scale and speed densities (see (4.15) and (4.16)) are given, respectively, by

$$\begin{aligned} s(X) &= C_1 X^{-\rho-1}\exp\{\theta X\}, \\ m(X) &= C_2 X^{\rho-1}\exp\{-\theta X\}, \end{aligned}$$

with $\rho = \dfrac{2(r-qE)}{\sigma^2} - 1$, $\theta = \dfrac{2r}{K\sigma^2}$ and $C_1,\ C_2$ are positive constants.

For the second model, the scale and speed densities (see (6.3) and (6.4)) are given, respectively, by

$$\begin{aligned} s(X) &= C_3 X^{-(\alpha+1)+\beta E} \exp\left\{\gamma(X-(K+A))^2\right\}, \\ m(X) &= C_4 X^{\alpha-\beta E-1} \exp\left\{-\gamma(X-(K+A))^2\right\}, \end{aligned}$$

with $\alpha = \dfrac{2rA}{(A-K)\sigma^2} - 1$, $\beta = \dfrac{2q}{\sigma^2}$, $\gamma = \dfrac{r}{K(K-A)\sigma^2}$ and $C_3,\ C_4$ are positive constants.

The definitions of the first passage time by a threshold are as follows:

- the first passage time of $X(t)$ by a lower threshold $L \quad (0 < L < x < +\infty)$ is

$$T_L := \inf\{t \geq 0 : X(t) = L\};$$

- the first passage time of $X(t)$ by an upper threshold $U \quad (0 < x < U < +\infty)$ is

$$T_U := \inf\{t \geq 0 : X(t) = U\}.$$

Our main interest is to study the mean of the first passage by the lower and the upper thresholds. This values represent, on average, the amount of time that the process needs to attain L or U. In Carlos (2013) and Carlos et al. (2013) one can see, for a general class of stochastic processes (where our models can be included with minor adaptations), the expressions for the mean and variance of T_L and T_U. They are given by

- Mean of T_L:
$$\mathbb{E}\big[T_L \big| X(0) = x\big] = 2\int_L^x s(y)\int_y^{+\infty} m(z)dzdy; \tag{7.1}$$

- Variance of T_L:
$$Var\big[T_L | X(0) = x\big] = 8\int_L^x s(y)\int_y^{+\infty} s(z)\left(\int_z^{+\infty} m(\theta)d\theta\right)^2 dzdy; \tag{7.2}$$

- Mean of T_U:
$$\mathbb{E}\big[T_U \big| X(0) = x\big] = 2\int_x^U s(y)\int_0^y m(z)dzdy; \tag{7.3}$$

- Variance of T_U:
$$Var\big[T_U | X(0) = x\big] = 8\int_x^U s(y)\int_0^y s(z)\left(\int_0^z m(\theta)d\theta\right)^2 dzdy \tag{7.4}$$

In these expressions, we have used upper limits of integration $+\infty$ in (7.1) and (7.2), and lower limits of integration 0 in (7.3) and (7.4) because $+\infty$ and 0 are, respectively, the upper and lower boundaries of the state space. If they have different values, one should use such values instead. From these expressions, since the integrands are positive, we see that, when L and U are closer to the initial population size x, the mean and variance of the first passage time decrease, as expected.

These expressions can be simplified for each particular model but, at least for our models, we cannot obtain them in a completely explicit form. For the logistic model, we have used the same data as for the basic scenario S_0 and, for the logistic-like model with weak Allee effects, we have used the data corresponding to scenario S_{A4}, which considers $A = -0.75K$. In both cases, the effort E was set as the optimal effort E^{**} obtained for each scenario in Chapter 5 and Chapter 6, respectively. For both models, the mean and variance of the first passage time by T_L and T_U were computed for the following values for L and U:

$$L = (0.05, 0.10, 0.15, \ldots, 0.90, 0.95, 1.00) \times x, \tag{7.5}$$

and

$$U = (1.00, 1.05, 1.10, \ldots, 3.90, 3.95, 4.00) \times x, \tag{7.6}$$

where x is, in both cases, the initial population size.

Figures 7.1 and 7.2, for the logistic model, and Figures 7.3 and 7.4, for the logistic-like model with weak Allee effects, show, for the L and U values presented in (7.5) and (7.6), the mean value and the standard deviation of T_L and T_U using expressions (7.1) to (7.4). In each Figure, we present on top the values in natural scale and, on bottom and in order to give a better visual discrimination, the values in logarithmic scale.

From Figures 7.1 and 7.3 one can see that, as L increases towards the initial population level x (the end of the X-axis), the mean and standard deviation of T_L are decreasing, taking obviously the value zero at $L = x$ (which is depicted in natural scale, but not in log scale since $\log(0) = -\infty$). From these Figures it is also clear that the mean and the standard deviation of T_L have the same order of magnitude, except for a very few values of L near x, where the standard deviation is slightly greater, being this difference more pronounced in the case of the logistic-type model with weak Allee effects. We also conclude that the results for the mean and standard deviation have a qualitatively similar behaviour for both models but, when there are weak Allee effects, it takes much less time to reach the lower thresholds, thus increasing

Table 7.1: Logistic-like model: Alternative scenarios with approximate values for the mean and standard deviation of T_L and T_U when varying the parameters L, U, x and E.

Scenario	L (tonnes)	x (tonnes)	E (SFU)	$\mathbb{E}[T_L]$ (years)	sd$[T_L]$ (years)
S_{L_1}	$0.10x = 0.40 \times 10^4$	4.03×10^4	E^{**}	1.57×10^{15}	1.57×10^{15}
S_{L_2}	$0.50x = 2.01 \times 10^4$	4.03×10^4	E^{**}	7.02×10^4	7.02×10^4
S_{L_3}	$0.75x = 3.02 \times 10^4$	4.03×10^4	E^{**}	734.86	800.66
S_{L_4}	$\mathbb{E}[X_\infty^{**}] = 3.91 \times 10^4$	4.03×10^4	E^{**}	16.78	38.11
S_{L_5}	0.40×10^4	2.01×10^4	E^{**}	2.67×10^{10}	2.67×10^{15}
S_{L_6}	$0.10x = 0.40 \times 10^4$	4.03×10^4	$1.10 \times E^{**}$	5.42×10^{13}	5.42×10^{13}
S_{L_7}	$0.10x = 0.40 \times 10^4$	4.03×10^4	$1.50 \times E^{**}$	1.01×10^8	1.00×10^8
S_{L_8}	$0.10x = 0.40 \times 10^4$	4.03×10^4	$2.00 \times E^{**}$	1.95×10^2	1.72×10^2

Scenario	U (tonnes)	x (tonnes)	E (SFU)	$\mathbb{E}[T_U]$ (years)	sd$[T_U]$ (years)
S_{U_1}	$\mathbb{E}[X_\infty^{**}] = 3.91 \times 10^4$	2.01×10^4	E^{**}	8.78	4.07

the risk of the population reaching low dangerous levels (notice that, for both models, the values of K and x are the same).

Figures 7.2 and 7.4 present an opposite behaviour to the lower threshold case. One can see that the mean time and standard deviation are increasing as the threshold U increases. Of course, when $U = x$, the mean and standard deviation are zero (not depicted in the log scale graphics). Again, the standard deviation has the same order of magnitude as the mean. In the presence of weak Allee effects, the mean time (and the respective standard deviation) needed to reach any value of U is quite larger then less for the logistic model, thus increasing the time to recovery.

Table 7.1 presents, for the logistic model, a list of scenarios with variations in the thresholds L and U, in the initial population size x and in the effort. For the first 3 scenarios (S_{L_1}, S_{L_2} and S_{L_3}) the mean and standard variation values of the first passage time by L are computed when the initial population size is x, and the lower threshold to reach is 10% of x, 50% of x and 75% of x. These 3 cases are directly observed from Figure 7.1 and we can conclude, as before, that there exists a decrease of the mean and the standard deviation of T_L when L increases towards x. Scenario S_{L_4} shows the case where the lower threshold is the value of the expected sustainable population size $\mathbb{E}[X_\infty^{**}]$ (see (4.19)). Since this value is very close to $K/2 = x$, it is not a surprise to see that the mean time to reach the sustainable average threshold is only about 17 years.

Scenario S_{L_5} considers a low initial population size ($x = 0.25K$). Comparing this Scenario with S_{L_1}, which has a higher initial population size ($x = 0.5K$), we expect to have lower values for the mean and standard deviation of the first passage time. Indeed that is what one can observe from Table 7.1.

Scenarios S_{L_6}, S_{L_7} and S_{L_8} consider the cases where the effort is greater than E^{**}. In the first one, the applied effort is 10% greater than E^{**}. This will produce a higher population catch and, consequently, the population size will decay being closer of L. Hence the mean time to reach L will be lower than in scenario S_{L_1}. Scenario S_{L_7} is very similar to scenario S_{L_6} but with a higher effort, although still sustainable in the sense that the population will have a stationary density. Scenario S_{L_8} is a clear case of heavy overfishing and the effort used almost reaches the value $\frac{r-\sigma^2/2}{q}$ beyond which mathematical extinction occurs with probability one and there is no stationary density; in this case reaching L will happen much faster.

Until now all the thresholds were smaller than the initial value. Scenario S_{U_1} considers the opposite case, i.e., we estimate the time that the population takes, on average, to reach the steady-state expected size but starting with an initial population of $0.25K$ (see scenario S_1 of Section 5.2.2). Since the population tends to become close to $\mathbb{E}[X_\infty^{**}]$, it seems natural that it took only about 9 years, on average, to reach the steady-state average value. So, even if we, by overfishing or other reasons, have depleted the population to a low size of $0.25K$, applying the optimal effort E^{**} from then on, the recovery will take on average only a few years.

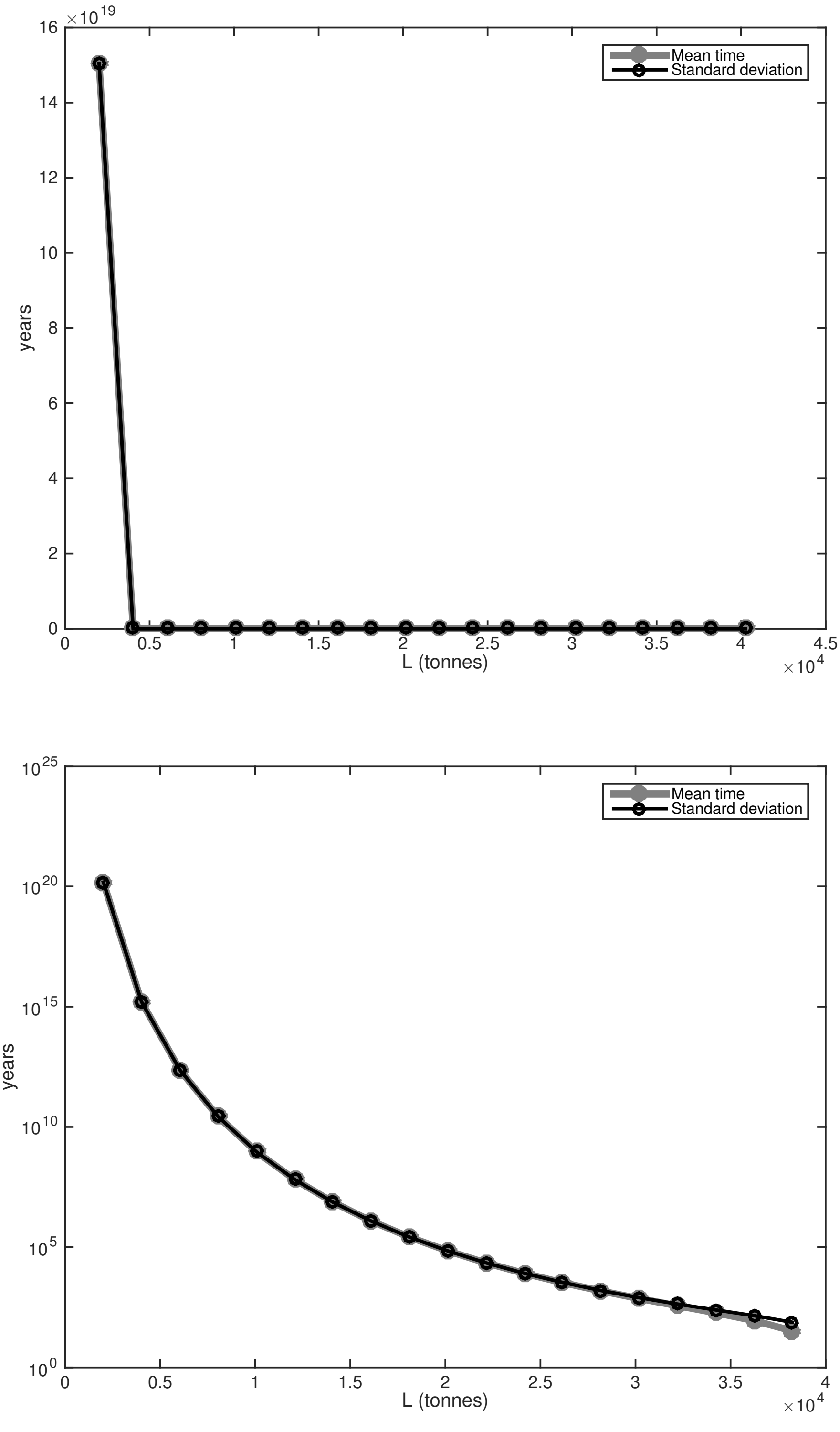
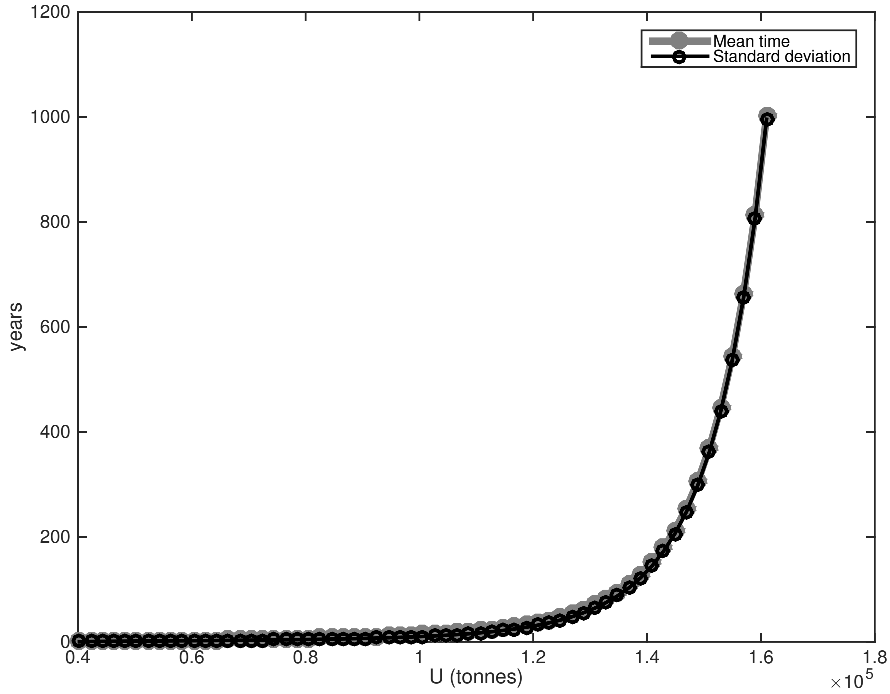


Figure 7.1: Logistic model without Allee effects: mean and standard deviation in natural scale (on top) and logarithmic scale (on bottom) of the first passage time by several values of L, when the initial population size is $x = 4.03 \times 10^4$ tonnes.



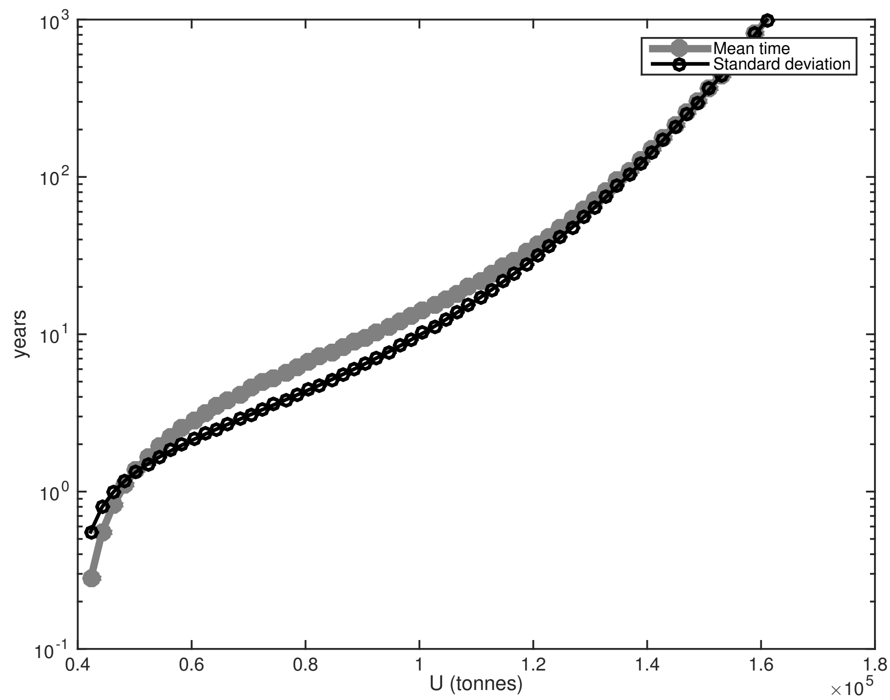


Figure 7.2: Logistic model without Allee effects: mean and standard deviation in natural scale (on top) and logarithmic scale (on bottom) of the first passage time by several values of U, when the initial population size is $x = 4.03 \times 10^4$ tonnes.

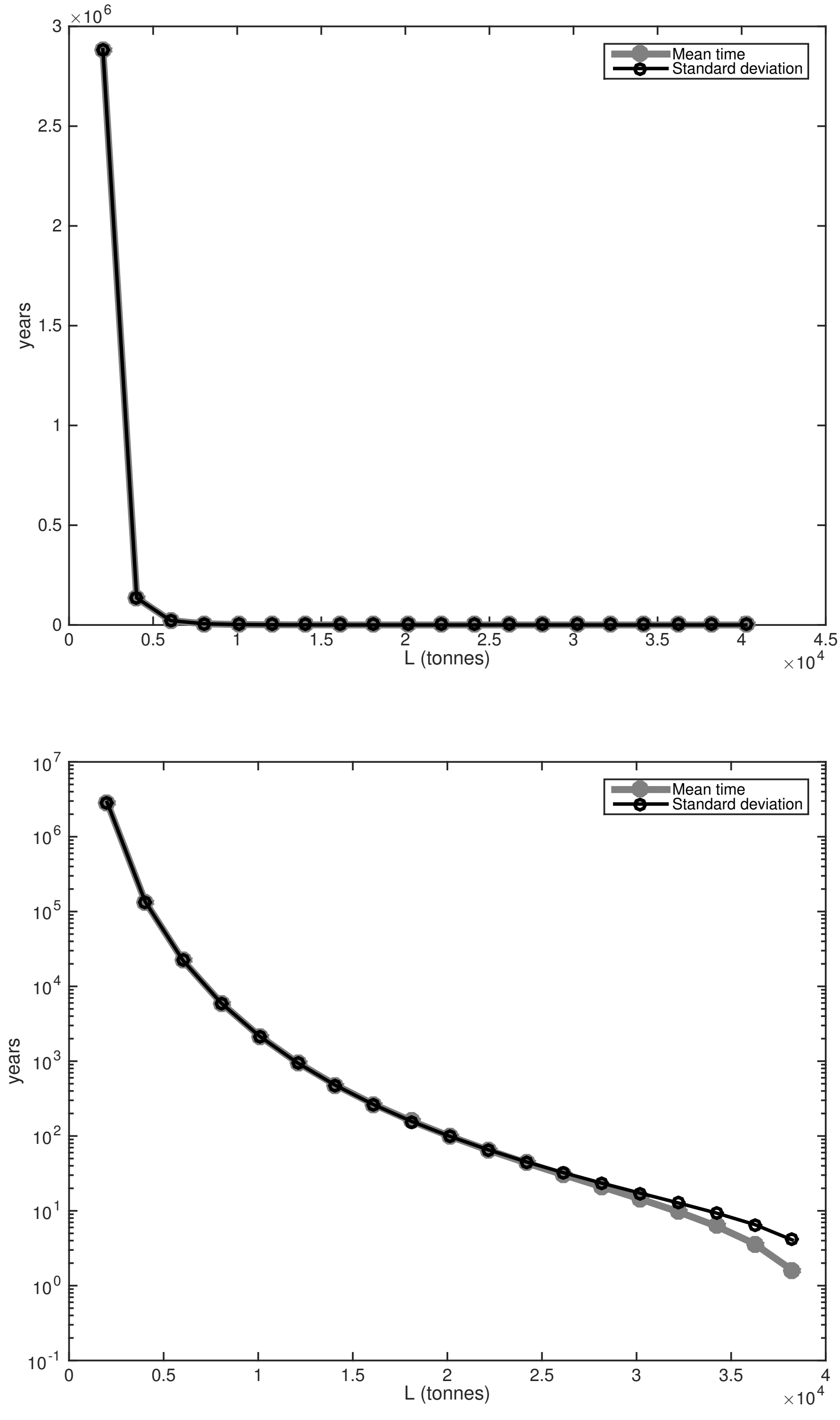


Figure 7.3: Logistic-like model with weak Allee effects: mean and standard deviation in natural scale (on top) and logarithmic scale (on bottom) of the first passage time by several values of L, when the initial population size is $x = 4.03 \times 10^4$ tonnes.

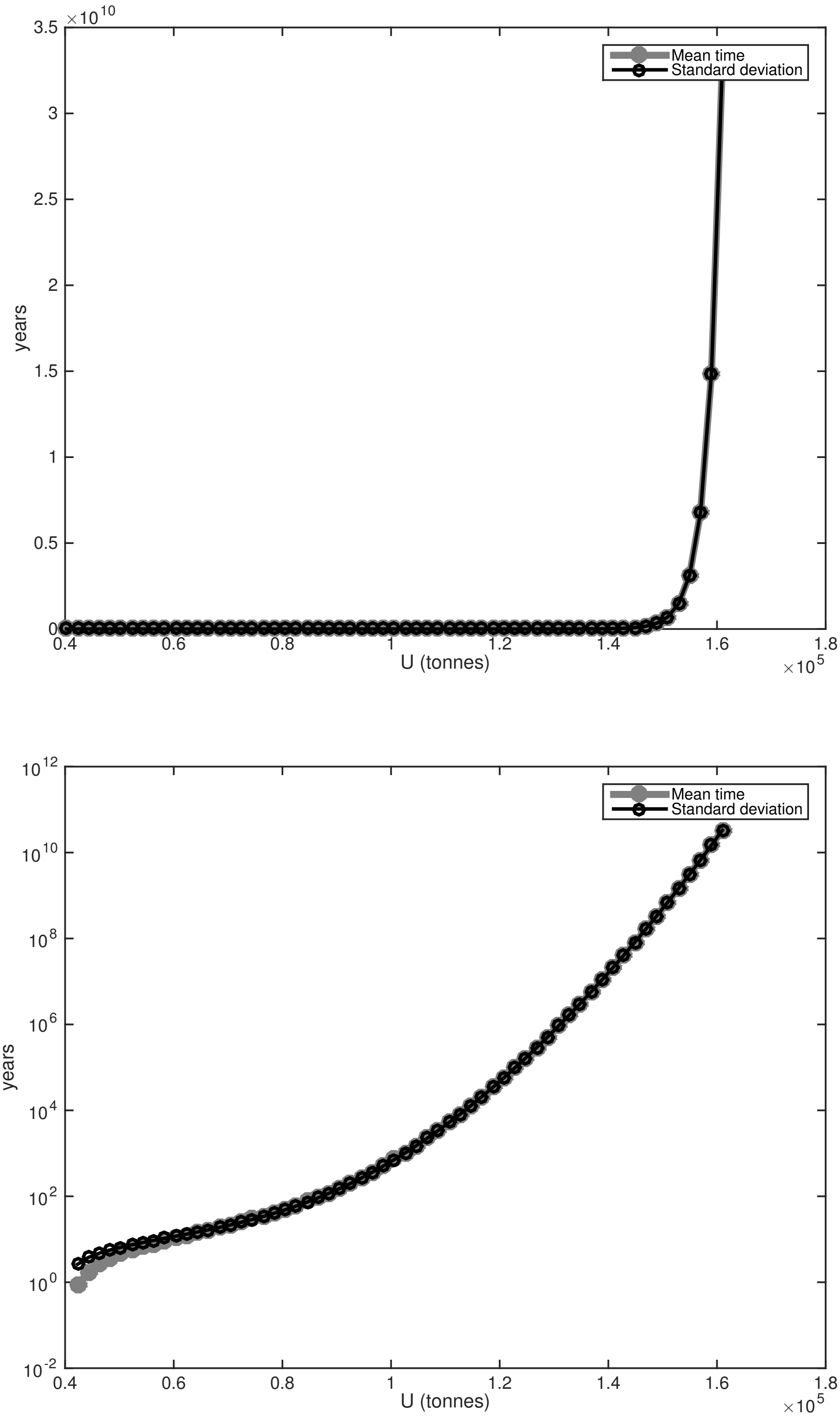


Figure 7.4: Logistic-like model with weak Allee effects: mean and standard deviation in natural scale (on top) and logarithmic scale (on bottom) of the first passage time by several values of U, when the initial population size is $x = 4.03 \times 10^4$ tonnes.

7.2 Estimation of the density probability functions of T_L and T_U

The Laplace transforms of the first passage times T_L and T_U, when the initial population size is x, are given respectively by

$$\mathbb{E}_x[\exp(-\lambda T_L)] \qquad \text{and} \qquad \mathbb{E}_x[\exp(-\lambda T_U)]. \tag{7.7}$$

In Giet et al. (2015), for the stochastic logistic model without harvesting,

$$dX(t) = r_1 X(t)\left(1 - \frac{X(t)}{K_1}\right)dt + \sigma X(t)dW(t), \quad X(0) = x, \tag{7.8}$$

one can find the following expressions for (7.7):

$$\mathbb{E}_x[\exp(-\lambda T_L)] = \left(\frac{x}{L}\right)^{\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u} \cdot \frac{\Psi\left(\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u, 1+2\sqrt{\frac{2\lambda}{\sigma^2}+u^2}, vx\right)}{\Psi\left(\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u, 1+2\sqrt{\frac{2\lambda}{\sigma^2}+u^2}, vL\right)} \tag{7.9}$$

$$\mathbb{E}_x[\exp(-\lambda T_U)] = \left(\frac{x}{U}\right)^{\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u} \cdot \frac{\Phi\left(\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u, 1+2\sqrt{\frac{2\lambda}{\sigma^2}+u^2}, vx\right)}{\Phi\left(\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u, 1+2\sqrt{\frac{2\lambda}{\sigma^2}+u^2}, vU\right)}, \tag{7.10}$$

where $u = \frac{1}{2}\left(1 - \frac{2r_1}{\sigma^2}\right)$, $v = \frac{2r_1}{K_1\sigma^2}$, and Ψ and Φ are the hypergeometric confluent functions. Note that Ψ and Φ are also denoted by U and M, respectively (see Abramowitz and Stegun, 1964, Chapter 13).

The stochastic logistic model with harvesting based on constant effort, can be written as (see Section 4.2)

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)dt - qEX(t)dt + \sigma X(t)dW(t), \quad X(0) = x,$$

or as,

$$dX(t) = r_1 X(t)\left(1 - \frac{X(t)}{K_1}\right)dt + \sigma X(t)dW(t), \quad X(0) = x,$$

if one considers $r_1 = (r - qE)$ and $K_1 = K\left(1 - \frac{qE}{r}\right)$, which gives (7.8). So, the Laplace transforms

(7.9) and (7.10) are valid for the model with harvesting based on constant effort if one uses $r_1 = (r - qE)$ and $K_1 = K\left(1 - \frac{qE}{r}\right)$.

The inversion of the Laplace transforms (7.9) and (7.10) returns the probability density functions of T_L and T_U. Due to the non-linearity of (7.9) and (7.10), it is not possible to obtain explicitly those densities. However, in Valsa and Brancik (1998), one can find an algorithm to implement numerically the Laplace transform inversion. After several runs, we have concluded that the algorithm is quite unstable and the necessary tuning is not straightforward. However, it allows us to obtain very satisfactory results.

We have run the algorithm using the data from the basic scenario S_0 (see Section 5.2.1), with $E = E^{**}$ and considering $L = 0.90x$ and $U = 1.10x$. The algorithm returns the p.d.f. of T_L for $L = 0.90x$ and T_U for $U = 1.10x$, which graphics are depicted on Figures 7.5 and 7.6, respectively. In both graphics the area under the lines is approximately 1 (more specifically 0.99), which indicates that they are a very good approximation of the p.d.f.. We also have computed the mean value of T_L for $L = 0.90x$ and T_U for $U = 1.10x$ from the graphics and from expressions (7.1) and (7.3). The differences are less than 0.1%, which reinforces the idea that the algorithm is credible and, therefore, the approximations of the p.d.f. are very accurate.

The graphics of the estimated densities are, for visibility reasons, in log-log scale. In fact, in the graphics in natural scale the depicted lines are too close to the axes.

The fact that the mean and the standard deviation of the first passage times are very close to each other suggests that the distribution of first passage times might be approximately exponential. The p.d.f. obtained on Figures 7.5 and 7.6 are also not too different from the exponential distribution case.

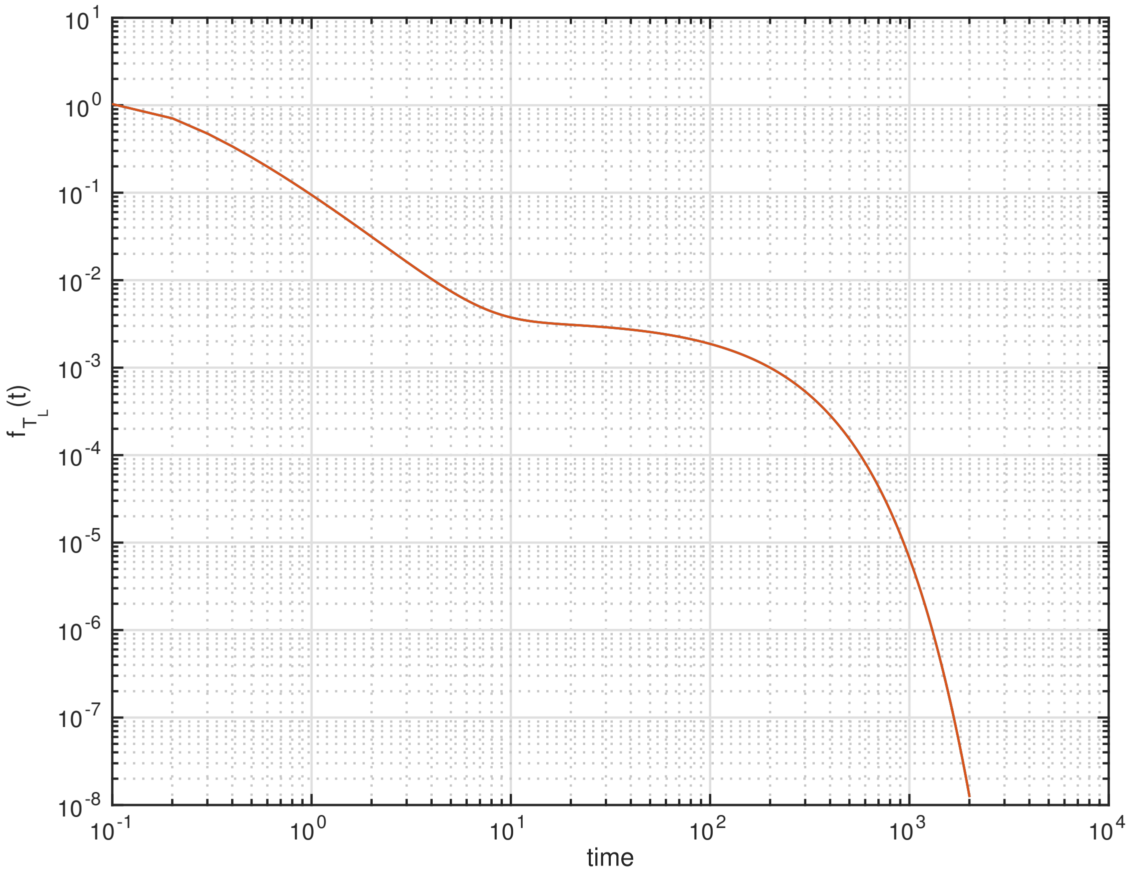


Figure 7.5: Estimation of the density probability function of T_L with $L = 0.90x$, when the initial population size is $x = 4.03 \times 10^4$ tonnes.

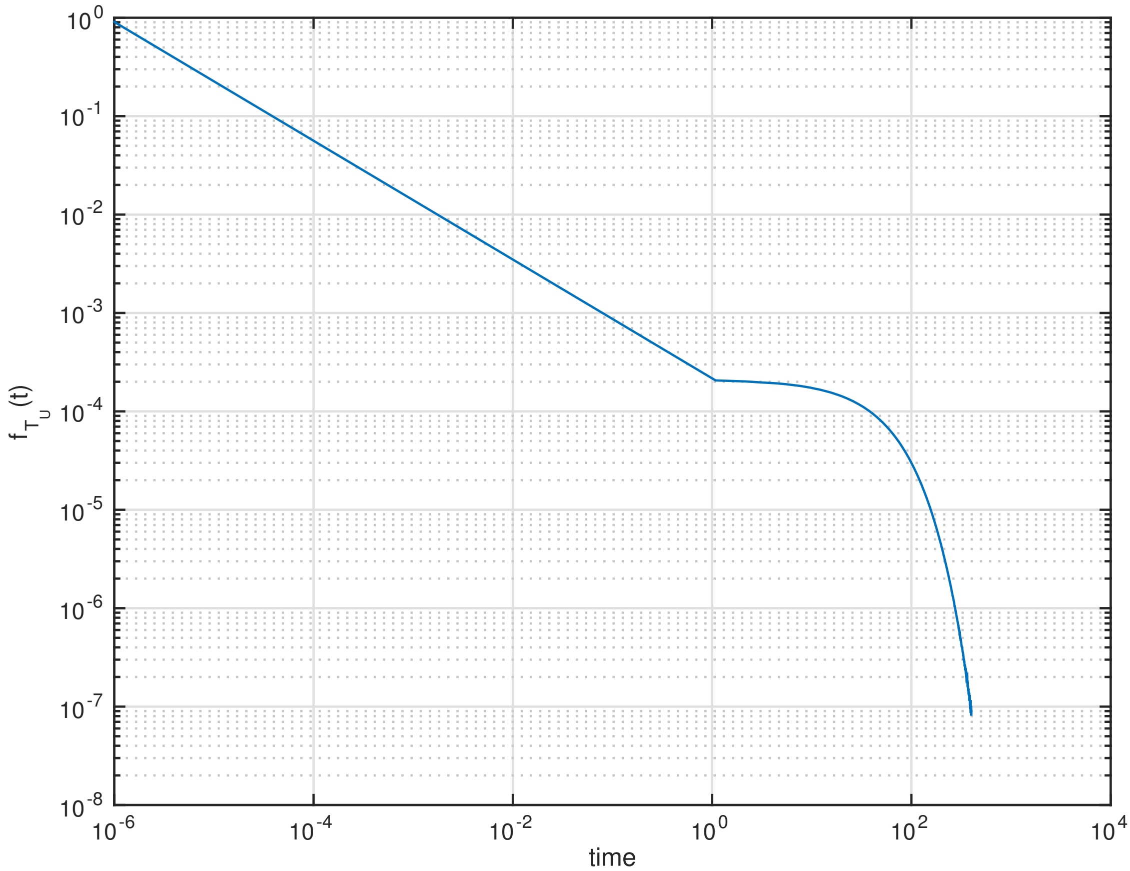


Figure 7.6: Estimation of the density probability function of T_U for $U = 1.10x$, when the initial population size is $x = 4.03 \times 10^4$ tonnes.

7.3 Conclusions

In this chapter we have presented, for the logistic model and for the logistic-like model with weak Allee effects, the expressions for the mean and standard deviation of the first passage times by lower and by upper thresholds. For several lower and upper threshold values and for the data and parameters from scenarios S_0 and S_{A4} previously studied, we have computed the mean and standard deviation time of the first passage by lower and by upper thresholds. For both models, as illustrated by Figures 7.1 to 7.4, the results are qualitatively similar but, in the case of the logistic model, the population needs more time to reach lower thresholds and less time to reach upper thresholds. We have also seen that the mean and the standard deviation have the same order of magnitude, which suggests distributions of first passage times not far from being exponential.

For the logistic model, we have set up 9 scenarios with parameters variations, namely the lower and upper thresholds, the initial population and the effort. For these scenarios, the mean and standard deviation of the first passage time by lower and by upper thresholds were computed. The conclusions were very similar to the ones based on the Figures. The general idea is that there exists a decrease of the mean and the standard deviation values of the first passage time by the threshold when the threshold values approaches the initial population value.

For the logistic model without harvesting, we have found in the bibliography the expressions of the Laplace transform of the first passage time by the lower and by the upper thresholds. With some mild adaptations, we deduced the expressions for the logistic model with harvesting. The inversion of the Laplace transform gives the probability density functions of the first passage time by the lower and by the upper thresholds. The expressions cannot be obtained explicitly, so we resort to numerical methods to compute them. Using Matlab we were able to plot an approximation of the probability densities functions for first passage time by the lower and by the upper thresholds. The resulting graphics support our idea that the first passage times have distributions not far from being exponential.

8

Conclusions

Fish populations live in randomly varying environments and the effect of that variability on fish dynamics has to be taken into account when choosing optimal harvesting policies. For that reason, the use of a stochastic differential equation model with harvesting is appropriate.

The typical approach in the literature is, as for deterministic models, to use control theory (the harvesting effort being the control) to maximize the expected accumulated discounted profit over some time horizon T. We have used a profit structure where revenues per unit time are quadratic functions of the yield and costs per unit time are quadratic functions of the effort.

In the stochastic case, the population fluctuations induced by the randomly varying environment lead to optimal policies with a highly variable effort (with frequent periods of no or low harvesting, or of harvesting at the maximum possible rate). This is not compatible with the logistics of fishing and causes social and economical problems (intermittent unemployment is just one of them). Besides, due to the random fluctuations that affect the population size, knowledge of its size at all times is required to determine the optimal effort. This is not feasible because the estimation of the population size is a difficult, costly, time consuming and inaccurate task.

So, we consider as an alternative, sustainable constant effort fishing policies, which are extremely easy to implement and lead to a stochastic steady-state. We determine the constant effort that maximizes the expected profit per unit time at the steady-state in the general case and for the specific cases of the logistic and the Gompertz models. One might think that a constant effort policy would result in a substantial profit reduction compared with the optimal variable effort policy, but we have shown this is not the case.

To compare the two harvesting policies, we have considered four ways of evaluating the expected profit: the expected accumulated discounted profit in a finite time interval, the expected accumulated undiscounted profit in a finite time interval, the average expected profit per unit time weighted by the discount factors and the average expected profit per unit time unweighted.

To obtain the profit values we have performed 1000 Monte Carlo simulations using a Crank-Nicolson discretization scheme in time and space of the HJB equation and an Euler scheme for the population paths. To compute the simulations we have applied the logistic and the Gompertz models to realistic data of fished populations and, for each model, we set up a basic scenario with the original data using a 50 year time horizon. To study the influence of the parameters on the policies performance, we have considered 20 alternative scenarios by considering changes on the parameter values, usually one lower and one higher than the original value.

For the logistic model, the optimal sustainable policy produces a slightly smaller profit in comparison with the optimal policy. For the Gompertz model, the profit differences between the two optimal policies are even smaller than in the logistic case. For both models we have indeed seen that the optimal policy have frequent strong changes in effort, including frequent closings of the fishery, posing logistic applicability problems, producing social burdens and out-of-model costs (such as unemployment compensations) and leading to a much greater instability in the profit earned by the harvester as compared with the constant effort policy. Furthermore, unlike the optimal variable effort policy, in the optimal constant effort policy there is no need to keep adjusting the effort to the randomly varying population size, and so there is no need to determine the size of the population at all times. The optimal policy also creates a possibly dangerous effect near the time horizon, implying a considerable drop on the population size. On the contrary, the optimal sustainable policy does not have these shortcomings, is very easy to implement and drives the population to a stochastic equilibrium. With a few exceptions, the alternative scenarios share the same behaviour as the basic scenario.

Since the optimal policy is not applicable, we have presented, for the logistic and the Gompertz models, a sub-optimal policy, named stepwise policy, based on variable effort but with periods of constant effort. This policy is not optimal, but has the advantage of being applicable, since the changes on effort are less frequent and compatible with the fishing activity. Furthermore, although we still need to keep estimating the fish stock size, we do not need to do it so often. Replacing the optimal variable effort policy by these stepwise policy has the advantage of applicability but, at best, considerably reduces the already small profit advantage the optimal effort policy have over the optimal sustainable policy based on constant effort. In some cases, the optimal sustainable policy even outperforms these stepwise policies in terms of profit.

In summary, optimal constant effort policies will typically involve a slight reduction in profit compared to the inapplicable optimal variable effort policies, and may even outperform the stepwise effort policies. They are quite easy to implement and do not have the serious shortcomings of the variable effort policies. Fishery managers/regulators do not have to worry about logistic problems of frequent changes in effort and equipment requirements, employment is kept at a constant level and, there is no need to frequently estimate the size of the population. Constant effort policies also lead the probability distribution of population size to a sustainable equilibrium with an average population size higher than the final average size of the optimal harvesting policy. Even the slight reduction in average profits (when compared to the optimal policies) are probably apparent, since the above mentioned out-of-model costs of the optimal policies (not considered in the model) are likely to outweight that reduction.

We have also compared the performance of both policies when the population is under the influence of weak Allee effects and the population natural growth follows a logistic-type model. For the constant effort model we proved that, if the effort is not too high, the state space boundaries are non-attractive, that there is a stationary density for the population size and we have found the stationary density expression. Both optimal policies were applied to the data used in the basic scenario of the logistic case without Allee effects. To see the influence of the weak Alee effects when comparing both policies, we have simulated 5 scenarios with variations on the Allee parameter. We have seen that, as the Allee parameter becomes smaller, the Allee effects have less influence on both policies and, therefore, the policies tend to behave as in the scenario without Allee effects. When the Allee parameter increases approaching zero, the Allee effects become more pronounced and imply huge differences in terms of profit values when comparing both harvesting policies; the profit becomes, for both types of policies, substantially lower than in the model without Allee effects.

For the logistic model and the logistic-like model with weak Allee effects, we present the expressions for the first and second moments of the first passage times by a lower and by an upper thresholds. For several lower and upper threshold values and for the data and parameters from scenarios S_0 and S_{A4} previously studied, we have computed the mean and standard deviation time of the first passage by the lower and by the upper thresholds. For both models, the closer U or L are to the initial population size x, the smaller are the first passage times. We have also seen that the mean and the standard deviation have the same order of magnitude.

When comparing the logistic model with the logistic-like model with Allee effects, one sees that Allee effects result in a much faster approach to lower thresholds (which can endanger the population) and much slower approach to higher thresholds (which affects population recovery). For the logistic model, we have also considered 8 alternative scenarios with parameters variations, namely the lower and upper thresholds, the initial population and the effort, in order to look at the effect of such variations. Still for the logistic model without harvesting, we have found in the bibliography the expressions of the Laplace transform of the first passage time by the lower and by the upper thresholds. With some mild adaptations, we deduced the expressions for the logistic model with harvesting. Using Matlab we were able to numerically invert the Laplace transform and plot an approximation of the probability densities functions for first passage time by lower and by upper thresholds, which suggest that the first passage time distributions are not far from being exponential.

We sincerely hope that the results in this thesis can contribute to better decisions in designing more efficient and safer harvesting policies, in face of the uncertainty in environmental conditions. If so, mathematical methods would again prove their usefulness.

As for future work, we think that it would be interesting to consider other population natural growth dynamics (like generalized logistic models, Gompertz-like models with Allee effects, for example), and more sophisticated sustainable harvesting policies. Some of these may be amenable to analytical methods, others should have to rely heavily on simulations. We would also like to reinforce the validation of our conclusions with other real population data. Furthermore, we have used parameter estimates obtained by researchers in the field based on deterministic models, and it would be nice to extend the estimation methods to the stochastic case, including the estimation of σ. We would expect that, if the fishing effort could be kept constant, the catches would be proportional to population size and this would allow reasonably good estimation methods of the population parameters.

A

SDE solutions

A.1 Solution of the logistic stochastic differential equation

To solve the SDE

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)dt - qEX(t)dt + \sigma X(t)dW(t), \quad X(0) = x, \tag{A.1}$$

we begin by converting it into an equivalent Stratonovich equation (see, for instance, Øksendal, 1998)

$$(S) \quad dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)dt - qEX(t)dt - \frac{\sigma^2}{2}X(t)dt + \sigma X(t)dW(t), \quad X(0) = x.$$

Applying the change of variable $Z(t) = K/X(t)$ to the latter equation, using Stratonovich rules of calculus (identical to ordinary rules) and rearranging terms yields

$$dZ(t) = -\left(r - qE - \frac{\sigma^2}{2}\right) Z(t)dt + rdt - \sigma Z(t)dW(t), \quad Z(0) = K/x.$$

Multiplying by the integrating factor $\exp\left\{\left(r - qE - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}$ and taking into account the rules of differentiation gives

$$\frac{d}{dt}\left(Z(t)\exp\left\{\left(r - qE - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}\right) = r\exp\left\{\left(r - qE - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}.$$

Integrating on the interval $[0, t]$ and after some trivial operations, results in

$$\begin{aligned} Z(t) &= Z(0)\exp\left\{-\left(r - qE - \frac{\sigma^2}{2}\right)t - \sigma W(t)\right\} \\ &+ r\exp\left\{-\left(r - qE - \frac{\sigma^2}{2}\right)t - \sigma W(t)\right\}\int_0^t \exp\left\{\left(r - qE - \frac{\sigma^2}{2}\right)s + \sigma W(s)\right\}ds. \end{aligned}$$

Finally by transforming to $X(t)$ gives

$$X(t) = \frac{K\exp\left\{\left(r - qE - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}}{\frac{K}{x} + r\int_0^t \exp\left\{\left(r - qE - \frac{\sigma^2}{2}\right)s + \sigma W(s)\right\}ds},$$

which is the solution of equation (A.1).

A.2 Solution of the Gompertz stochastic differential equation

To solve the SDE

$$dX(t) = rX(t)\ln\frac{K}{X(t)}dt - qEX(t)dt + \sigma X(t)dW(t), \quad X(0) = x, \tag{A.2}$$

we apply the change of variable $Z(t) = e^{rt}\ln X(t)$ and Ito´s Theorem 2.2.3. The resultant SDE is

$$dZ(t) = \left(r\ln K - qE - \frac{\sigma^2}{2}\right)e^{rt}dt + \sigma e^{rt}dW(t).$$

Integrating on the interval $[0, t]$ and after some trivial operations, results in

$$Z(t) = \ln x + \frac{1}{r}\left(r \ln K - qE - \frac{\sigma^2}{2}\right)\left(e^{rt} - 1\right) + \sigma \int_0^t e^{rs} dW(s).$$

Finally by transforming to $X(t)$ gives

$$X(t) = \exp\left\{e^{-rt} \ln x\right\} \exp\left\{\frac{1}{r}\left(r \ln K - qE - \frac{\sigma^2}{2}\right)\left(1 - e^{-rt}\right)\right\} \exp\left\{\sigma e^{-rt} \int_0^t e^{rs} dW(s)\right\}.$$

which is the solution of equation (A.2).

B

R code for Chapters 5 and 6

We present the code used to run the basic scenario S_0 of the logistic model. For other scenarios and models, the user just needs to make appropriate changes.

```
set.seed(123456789)
setwd("~/Desktop/Rcode")
g<-basename(getwd())
library(matrixcalc)
library(signal)
library(gridExtra)
% parmeters for scenario S0
% (for others, make appropriate changes)
T <- 50
r <- 0.71
```

```
K <- 80.5*10^6
q <- 3.30*10^(-6)
Emin <- 0
Emax <- 0.7*(r/q)
xmin <- 0
xmax <- 2*K
n <- 150
m <- 75
(deltat <- T/n)
(deltax <- (xmax - xmin)/m)
x <- seq(xmin, xmax, by = deltax)
t <- seq(0, T, by = deltat)
p <- 1.59
cone <- 96*10^(-6)
ctwo <- 0.10*10^(-6)
d <- 0.05
sigma <- 0.2
xone <- K/2
path <- 1000

J <- matrix(0, nrow = m + 1, ncol = n + 1)
E <- matrix(0, nrow = m + 1, ncol = n + 1)
A <- matrix(0, nrow = m, ncol = m)
B <- matrix(0, nrow = m, ncol = m)

% j for instants, i for states

for (j in 2:(n + 1)) {
for (i in 2:m) {
E[i, j - 1] <- (p - (J[i + 1, j - 1] - J[i - 1, j - 1])/(2 * deltax)) *
(q * x[i]/(2 * ctwo)) - cone/(2 * ctwo)
ifelse(E[i, j - 1] < Emin, E[i, j - 1] <- Emin, E[i, j - 1])
ifelse(E[i, j - 1] > Emax, E[i, j - 1] <- Emax, E[i, j - 1])
}

E[m + 1, j - 1] <- (p - (3 * J[m + 1, j - 1] - 4 * J[m, j - 1] + J[m -
1, j - 1])/(2 * deltax)) * (q * x[m + 1]/(2 * ctwo)) - cone/(2 *
ctwo)
ifelse(E[m + 1, j - 1] < Emin, E[m + 1, j - 1] <- Emin, E[m + 1, j -
1])
ifelse(E[m + 1, j - 1] > Emax, E[m + 1, j - 1] <- Emax, E[m + 1, j -
```

```
1])
% Solving HJB equation
source("func.aux1.R")
source("func.aux2.R")
source("func.aux3.R")
c1 <- rep(0, m + 1)
c2 <- rep(0, m + 1)
c3 <- rep(0, m + 1)
for (i in 2:(m + 1)) {
c1[i] <- constant1(r, K, q, x[i], E[i, j - 1])
c2[i] <- constant2(deltat, deltax, sigma, x[i])
c3[i] <- constant3(deltat, p, q, cone, ctwo, x[i], E[i, j - 1])
}
A[1, 1] <- 1 + d * deltat/2 + 0.5 * c2[2]
A[1, 2] <- -deltat * c1[2]/(4 * deltax) - 0.25 * c2[2]
A[m, m - 3] <- 0.25 * c2[m + 1]
A[m, m - 2] <- -deltat * c1[m + 1]/(4 * deltax) - 0.25 * c2[m + 1]
A[m, m - 1] <- deltat * c1[m + 1]/deltax + 1.25 * c2[m + 1]
A[m, m] <- 1 + d * deltat/2 - 3 * deltat * c1[m + 1]/(4 * deltax) -
0.5 * c2[m + 1]
firstcolumn <- 1
for (row in 2:(m - 1)) {
A[row, firstcolumn] <- deltat * c1[row + 1]/(4 * deltax) - 0.25 *
c2[row + 1]
A[row, firstcolumn + 1] <- 1 + d * deltat/2 + 0.5 * c2[row + 1]
A[row, firstcolumn + 2] <- -deltat * c1[row + 1]/(4 * deltax) -
0.25 * c2[row + 1]
firstcolumn <- firstcolumn + 1
}
B[1, 1] <- 1 - d * deltat/2 - 0.5 * c2[2]
B[1, 2] <- deltat * c1[2]/(4 * deltax) + 0.25 * c2[2]
B[m, m - 3] <- -0.25 * c2[m + 1]
B[m, m - 2] <- deltat * c1[m + 1]/(4 * deltax) + 0.25 * c2[m + 1]
B[m, m - 1] <- -deltat * c1[m + 1]/deltax - 0.5 * c2[m + 1]
B[m, m] <- 1 - d * deltat/2 + 3 * deltat * c1[m + 1]/(4 * deltax) +
0.25 * c2[m + 1]
firstcolumn <- 1
for (row in 2:(m - 1)) {
B[row, firstcolumn] <- -deltat * c1[row + 1]/(4 * deltax) + 0.25 *
c2[row + 1]
B[row, firstcolumn + 1] <- 1 - d * deltat/2 - 0.5 * c2[row + 1]
```

```
B[row, firstcolumn + 2] <- deltat * c1[row + 1]/(4 * deltax) +
0.25 * c2[row + 1]
firstcolumn <- firstcolumn + 1
}
C <- B %*% J[2:(m + 1), (j - 1)]
D <- c3[2:(m + 1)]
F <- C + D
lu.dec <- lu.decomposition(A)
L <- lu.dec$L
U <- lu.dec$U
temp.mat <- solve(U) %*% solve(L) %*% F
for (count in 2:(m + 1)) {
J[count, j] <- temp.mat[count - 1]
}
}
for (i in 2:m) {
E[i, n + 1] <- (p - (J[i + 1, n + 1] - J[i - 1, n + 1])/(2 * deltax)) *
(q * x[i]/(2 * ctwo)) - cone/(2 * ctwo)
ifelse(E[i, n + 1] < Emin, E[i, n + 1] <- 0, E[i, n + 1])
ifelse(E[i, n + 1] > Emax, E[i, n + 1] <- Emax, E[i, n + 1])
}
E[m + 1, n + 1] <- (p - (3 * J[m + 1, n + 1] - 4 * J[m, n + 1] + J[m -
1, n + 1])/(2 * deltax)) * (q * x[m + 1]/(2 * ctwo)) - cone/(2 * ctwo)
ifelse(E[m + 1, n + 1] < Emin, E[m + 1, n + 1] <- 0, E[m + 1, n + 1])
ifelse(E[m + 1, n + 1] > Emax, E[m + 1, n + 1] <- Emax, E[m + 1, n + 1])

write.table(round(J, digits = 0), "J.csv", sep = ";", dec = ",", row.names = FALSE)
write.table(round(E, digits = 0), "E.csv", sep = ";", dec = ",", row.names = FALSE)

% Interpolation

population <- matrix(xone, nrow = path, ncol = n + 1)
Estar <- matrix(0, nrow = path, ncol = n + 1)
Jstar <- matrix(0, nrow = path, ncol = n + 1)
% Wiener process
mat.estoc <- matrix(rnorm(path * (n + 1)), nrow = path, ncol = n + 1)
dW <- (sqrt(deltat)) * mat.estoc
W <- t(apply(dW, 2, sum))
for (i in 1:path) {
for (b in 1:(n)) {
if (population[i, b] < xmax & population[i, b] > xmin) {
```

```
Estar[i, b] <- interp1(x, E[, (n + 1) - (b - 1)], population[i,
b], method = "linear", extrap = FALSE)
Jstar[i, b] <- interp1(x, J[, (n + 1) - (b - 1)], population[i,
b], method = "linear", extrap = FALSE)
} else {
Estar[i, b] <- interp1(x, E[, (n + 1) - (b - 1)], population[i,
b], method = "linear", extrap = TRUE)
Jstar[i, b] <- interp1(x, J[, (n + 1) - (b - 1)], population[i,
b], method = "linear", extrap = TRUE)
}
if (Estar[i, b] < 0) {
Estar[i, b] <- 0
}
if (Estar[i, b] > Emax) {
Estar[i, b] <- Emax
}
% Euler scheme
population[i, b + 1] <- population[i, b] + (r * population[i, b] *
(1 - (population[i, b])/K) - q * Estar[i, b] * population[i,
b]) * deltat + sigma * population[i, b] * dW[i, b]
}
}

Estarmean <- apply(Estar, 2, mean)
Jstarmean <- apply(Jstar, 2, mean)
populationmean <- apply(population, 2, mean)
dWmean <- apply(dW, 2, mean)

%Sustainable policy
Estarstar <- (p * q * K * (r - 0.5 * sigma^2) - cone * r)/(2 * p * (q^2) *
K + 2 * r * ctwo)
Estarstar
EXstarstar <- K * (1 - (p * (q^2) * K * (r - 0.5 * sigma^2) - cone * r *
q)/(2 * r * p * (q^2) * K + 2 * (r^2) * ctwo) - (sigma^2)/(2 * r))
EXstarstar
Pistarstar <- ((r - 0.5 * sigma^2) * p * q * K - cone * r)^2/(4 * r * (p *
(q^2) * K + r * ctwo))
Pistarstar

write.table(round(populationmean, digits = 0), "Estar.csv", sep = ";",
dec = ",", row.names = FALSE)
```

```
write.table(round(Estar, digits = 0), "Estar.csv", sep = ";", dec = ",",
row.names = FALSE)
write.table(round(Jstar, digits = 0), "Jstar.csv", sep = ";", dec = ",",
row.names = FALSE)
write.table(round(Estarmean, digits = 0), "Estarmean.csv", sep = ";", dec = ",",
row.names = FALSE)
write.table(round(Jstarmean, digits = 0), "Jstarmean.csv", sep = ";", dec = ",",
row.names = FALSE)

##### Comparisons #####

#Auxiliar variable
expdt <- deltat * matrix(rep((exp(-d * t) + exp(-d * (t + deltat)))/2, path), nrow =
    path, ncol = n + 1, byrow = T)

###########
###  1  ###
###########
(PistarT <- mean(Jstar[, 1]))

############
###  1A  ###
############
expdtLEstar <- expdt * (p * q * population - cone - ctwo * Estar) * Estar
intexpdtLEstar <- apply(expdtLEstar[,1:n], 1, sum)
(PistarT2 <- mean(intexpdtLEstar))

###########
###  2  ###
###########
LEstar <- (p * q * population - cone - ctwo * Estar) * Estar
LEstardt <- LEstar * deltat
intLEstardt <- apply(LEstardt[,1:n], 1, sum)
(VstarT <- mean(intLEstardt))

###########
###  3  ###
###########
ifelse(d==0,int0T <- T,int0T <- (1 - exp(-d * T))/d)
(PistarT2/int0T)
```

```
###########
###  4  ###
###########
(VstarT/T)

###########
###  5  ###
###########
simX <- matrix(0, nrow = path, ncol = n + 1)
simX[, 1] <- rep(xone, path)
for (i in 1:path) {
for (j in 1:(n)) {
simX[i, j + 1] <- simX[i, j] + (r * simX[i, j] * (1 - (simX[i,
j])/K) - q * Estarstar * simX[i, j]) * deltat + sigma * simX[i,
j] * dW[i, j]
}
}
expdtLEstarstar <- expdt * (p * q * simX - cone - ctwo * Estarstar) * Estarstar
intexpdtLEstarstar <- apply(expdtLEstarstar[,1:n], 1, sum)
(PistarstarT <- mean(intexpdtLEstarstar))

###########
###  6  ###
###########
LEstarstar <- (p * q * simX - cone - ctwo * Estarstar) * Estarstar
LEstarstardt <- LEstarstar * deltat
intLEstarstardt <- apply(LEstarstardt[,1:n], 1, sum)
(VstarstarT <- mean(intLEstarstardt))

###########
###  7  ###
###########
ifelse(d==0,int0T <- T,int0T <- (1 - exp(-d * T))/d)
(PistarstarT/int0T)

###########
###  8  ###
###########
(VstarstarT/T)

####################################
```

```
###  Print values to a pdf file  ###
####################################
pdf(paste("Comparisons",g,'.pdf',sep=''), height=5, width=5)
V.old <- c(PistarT,0,0,0,0)
V.star <- c(PistarT2,sd(intexpdtLEstar),PistarstarT,sd(intexpdtLEstarstar),(
    PistarstarT-PistarT2)/PistarT2)
V.star.u <- c(VstarT,sd(intLEstardt),VstarstarT,sd(intLEstarstardt),(VstarstarT-VstarT
    )/VstarT)
P.star <- c(PistarT2/int0T,sd(intexpdtLEstar)/int0T,PistarstarT/int0T,sd(
    intexpdtLEstarstar)/int0T,((PistarstarT/int0T)-(PistarT2/int0T))/(PistarT2/int0T))
P.star.u <- c(VstarT/T,sd(intLEstardt)/T,VstarstarT/T,sd(intLEstarstardt)/T,((
    VstarstarT/T)-(VstarT/T))/(VstarT/T))
data.mat <- matrix(c(V.old,V.star,V.star.u,P.star,P.star.u),5,5,byrow=TRUE)
data.mat[,1:4] <- round(10^(-6)*data.mat[,1:4],3)
data.mat[,5] <- round(100*data.mat[,5],1)
colnames(data.mat) <- c('Opt.','sd','Opt. Sust.','sd','%')
rownames(data.mat) <- c('V.old','V.star','V.star.u','P.star','P.star.u')
data.mat
grid.table(data.mat)
dev.off()
##### End comparisons ######

##### Graphics #####
matLopt <- matrix(0, nrow = path, ncol = n + 1)
matLopt[, 1] <- rep(0, path)
for (i in 1:path) {
for (j in 1:n) {
matLopt[i, j + 1] <- (p * q * population[i, j] - cone - ctwo *
Estar[i, j]) * Estar[i, j]
}
}
Lopt <- apply(matLopt, 2, mean)
Lsust <- (p * q * simX - cone - ctwo * Estarstar) * Estarstar
Lsust <- apply(Lsust, 2, mean)
matLEstarstar <- (p * q * simX - cone - ctwo * Estarstar) * Estarstar
(num1<-floor(runif(1)*path)+1)
library(ggplot2)
source("func.aux4.R")
df11<-data.frame(Time=t,Population=populationmean)
df12<-data.frame(Time=t,Population=population[num1,])
plot1<-ggplot(df11, aes(Time,Population))+geom_line(size=1,aes(color='Population (mean
```

```
    of 1 000 paths)'))+
scale_y_continuous(limits = c(0,7e+07), expand = c(0, 0)) +
scale_x_continuous(limits = c(0,T), expand = c(0, 0)) +
geom_line(data=df12,size=0.25,aes(color='Population (sample path)'))+labs(y='
    Population (kg)',color='Legend')+
theme(legend.justification=c(0,1.10), legend.position=c(0,1.10),legend.background =
    element_rect(fill=alpha('white', 0)))+
theme(legend.key.size =  unit(0.15, "in"))+
theme(legend.text = element_text(size=7))+
theme(legend.title=element_blank())+
ggtitle("Optimal policy (variable effort)")+
theme(plot.title = element_text(size = 7,hjust=0.5))+
theme(axis.text=element_text(size=7),
axis.title=element_text(size=7))+
theme(axis.title.x=element_blank(),
axis.text.x=element_blank(),
axis.ticks.x=element_blank())+
theme(plot.margin= unit(c(.5, .5, 0, .5), "lines"))+
theme(legend.key = element_blank())+
theme(panel.background = element_rect(fill = "grey96",
colour = "grey96",
size = 0.5, linetype = "solid"))+
guides(colour = guide_legend(keywidth = 3.5, keyheight = .5,override.aes = list(
    linetype=c('solid','solid'),size=c(1,0.25))))+
scale_color_manual(values=c("grey50", "black"))
df21<-data.frame(Time=t[1:n],Effort=Estarmean[1:n])
df22<-data.frame(Time=t[1:n],Effort=Estar[num1, 1:n])
plot2<-ggplot(df21, aes(Time,Effort))+geom_line(size=1,aes(color='Effort (mean of 1
    000 paths)'))+
scale_y_continuous(limits = c(0,2.75e+05), expand = c(0, 0)) +
scale_x_continuous(limits = c(0,T), expand = c(0, 0)) +
geom_line(data=df22,size=0.25,aes(color='Effort (sample path)'))+labs(y='Effort (SFU)'
    ,x='Time (years)',color='Legend')+
theme(legend.justification=c(0,1.10), legend.position=c(0,1.10),legend.background =
    element_rect(fill=alpha('white', 0)))+
theme(legend.key.size =  unit(0.15, "in"))+
theme(legend.text = element_text(size=7))+
theme(legend.title=element_blank())+
theme(axis.text=element_text(size=7),
axis.title=element_text(size=7))+
theme(axis.title.x=element_blank(),
```

```
axis.text.x=element_blank(),
axis.ticks.x=element_blank())+
theme(plot.margin= unit(c(.5, .5, 0, .5), "lines"))+
theme(legend.key = element_blank())+
theme(panel.background = element_rect(fill = "grey96",
colour = "grey96",
size = 0.5, linetype = "solid"))+
guides(colour = guide_legend(keywidth = 3.5, keyheight = .5,override.aes = list(
    linetype=c('solid','solid'),size=c(1,0.25))))+
scale_color_manual(values=c("grey50", "black"))
df31<-data.frame(Time=t,Population=apply(simX, 2, mean))
df32<-data.frame(Time=t,Population=simX[num1,])
df33<-data.frame(Time=t,Population=rep(EXstarstar, n + 1))
plot3<-ggplot(df31, aes(Time,Population))+geom_line(size=1,aes(color='Population (mean
     of 1 000 paths)'))+
scale_y_continuous(limits = c(0,7e+07), expand = c(0, 0)) +
scale_x_continuous(limits = c(0,T), expand = c(0, 0)) +
geom_line(data=df32,size=0.25,aes(color='Population (sample path)'))+labs(y='
    Population (kg)',color='Legend')+
geom_line(linetype = "dashed",data=df33,size=.75,aes(color=paste("E[X**]=", round(
    EXstarstar/1000, 0),'tonnes')))+
scale_color_manual(values=c("black", "grey50", "black"))+
theme(legend.justification=c(0,1.10), legend.position=c(0,1.10),legend.background =
    element_rect(fill=alpha('white', 0)))+
theme(legend.text = element_text(size=7))+
theme(legend.title=element_blank())+
ggtitle("Optimal sustainable policy (constant effort)")+
theme(plot.title = element_text(size = 7,hjust=0.5))+
theme(axis.text=element_text(size=7),
axis.title=element_text(size=7))+
theme(axis.title.y=element_blank(),
axis.text.y=element_blank(),
axis.ticks.y=element_blank())+
theme(axis.title.x=element_blank(),
axis.text.x=element_blank(),
axis.ticks.x=element_blank())+
theme(plot.margin= unit(c(.5, 2.8, 0, .5), "lines"))+
theme(legend.key = element_blank())+
theme(
panel.background = element_rect(fill = "grey96",
colour = "grey96",
```

```
size = 0.5, linetype = "solid"))+
guides(colour = guide_legend(keywidth = 3.5, keyheight = .5,override.aes = list(
    linetype=c('dashed','solid','solid'),size=c(.75,1,0.25))))
df41<-data.frame(Time=t,Effort=rep(Estarstar, n + 1))
plot4<-ggplot(df41, aes(Time,Effort))+geom_line(size=1,aes(color=c(paste("E**=", round
    (Estarstar, 0),'SFU'))))+
scale_y_continuous(limits = c(0,2.75e+05), expand = c(0, 0)) +
scale_x_continuous(limits = c(0,T), expand = c(0, 0)) +
labs(y='Effort (SFU)', x='Time (years)',color='Legend')+
theme(legend.justification=c(0,1.10), legend.position=c(0,1.10), legend.background =
    element_rect(fill=alpha('white', 0)))+
theme(legend.key.size =  unit(0.15, "in"))+
theme(legend.text = element_text(size=7))+
theme(legend.title=element_blank())+
theme(axis.text=element_text(size=7),
axis.title=element_text(size=7))+
theme(axis.title.x=element_blank(),
axis.text.x=element_blank(),
axis.ticks.x=element_blank())+
theme(axis.title.y=element_blank(),
axis.text.y=element_blank(),
axis.ticks.y=element_blank())+
guides(colour = guide_legend(keywidth = 3.5, keyheight = .5,override.aes = list(
    linetype=c('solid'),size=1)))+
theme(plot.margin= unit(c(.5, 2.8, 0, .5), "lines"))+
theme(legend.key = element_blank())+
theme(panel.background = element_rect(fill = "grey96",
colour = "grey96",
size = 0.5, linetype = "solid"))+
scale_color_manual(values=c("black"))
df51<-data.frame(Time=t[2:n+1],Profit=Lopt[2:n+1])
df52<-data.frame(Time=t[2:n+1],Profit=matLopt[num1,2:n+1])
plot5<-ggplot(df51, aes(Time,Profit))+geom_line(size=1,aes(color='Profit (mean of 1
    000 paths)'))+
scale_y_continuous(limits = c(0,7e+07), expand = c(0, 0)) +
scale_x_continuous(limits = c(0,T), expand = c(0, 0)) +
geom_line(data=df52,size=0.25,aes(color='Profit (sample path)'))+labs(y='Profit ($)',x
    ='Time (years)',color='Legend')+
theme(legend.justification=c(0,1.10), legend.position=c(0,1.10),legend.background =
    element_rect(fill=alpha('white', 0)))+
theme(legend.key.size =  unit(0.15, "in"))+
```

```
theme(legend.text = element_text(size=7))+
theme(legend.title=element_blank())+
theme(axis.text=element_text(size=7),
axis.title=element_text(size=7))+
theme(plot.margin= unit(c(.5, .5, 0, .5), "lines"))+
theme(legend.key = element_blank())+
theme(panel.background = element_rect(fill = "grey96",
colour = "grey96",
size = 0.5, linetype = "solid"))+
guides(colour = guide_legend(keywidth = 3.5, keyheight = .5,override.aes = list(
    linetype=c('solid','solid'),size=c(1,0.25))))+
scale_color_manual(values=c("grey50", "black"))
df61<-data.frame(Time=t,Profit=Lsust)
df62<-data.frame(Time=t,Profit=matLEstarstar[num1,])
df63<-data.frame(Time=t,Profit=rep(Pistarstar, n + 1))
plot6<-ggplot(df61, aes(Time,Profit))+geom_line(size=1,aes(color='Profit (mean of 1
    000 paths)'))+
scale_y_continuous(limits = c(0,7e+07), expand = c(0, 0)) +
scale_x_continuous(limits = c(0,T), expand = c(0, 0)) +
geom_line(data=df62,size=0.25,aes(color='Profit (sample path)'))+labs(y='Profit ($)',
    color='Legend', x='Time (years)')+
geom_line(linetype = "dashed",data=df63,size=.75,aes(color=paste("E[Pi**]=",round(
    Pistarstar/10^6, 3), 'Million dollars')))+
scale_color_manual(values=c("black", "grey50", "black"))+
theme(legend.justification=c(0,1.10), legend.position=c(0,1.10),legend.background =
    element_rect(fill=alpha('white', 0)))+
theme(legend.text = element_text(size=7))+
theme(legend.title=element_blank())+
theme(plot.title = element_text(size = 7))+
theme(axis.text=element_text(size=7),
axis.title=element_text(size=7))+
theme(axis.title.y=element_blank(),
axis.text.y=element_blank(),
axis.ticks.y=element_blank())+
theme(plot.margin= unit(c(.5, 2.8, 0, .5), "lines"))+
theme(legend.key = element_blank())+
theme(panel.background = element_rect(fill = "grey96",
colour = "grey96",
size = 0.5, linetype = "solid"))+
guides(colour = guide_legend(keywidth = 3.5, keyheight = .5,override.aes = list(
    linetype=c('dashed','solid','solid'),size=c(.75,1,.25))))
```

```
g1 <- ggplotGrob(plot1)
g2 <- ggplotGrob(plot2)
g3 <- ggplotGrob(plot3)
g4 <- ggplotGrob(plot4)
g5 <- ggplotGrob(plot5)
g6 <- ggplotGrob(plot6)
ggsave(paste('britesbraumann',g,'.pdf',sep=''),grid.arrange(g1, g3, g2, g4, g5, g6,
    ncol=2),width=7,height=7)
save.image(paste(g,'.RData',sep=''))
###End


% Aux. function 1
constant1<-function(r,K,q,X,E){
return(r*X*(1-X/K)-q*E*X)
}
% Aux. function 2
constant2<-function(deltat,deltax,sigma,X){
return(deltat*(sigma^2)*(X^2)/(deltax^2))
}
% Aux. function 3
constant3<-function(deltat,p,q,c1,c2,X,E){
return(deltat*E*(p*q*X-cone-ctwo*E))
}
% Aux. function 4
multiplot <- function(..., plotlist=NULL, file, cols=1, layout=NULL) {
library(grid)
% Make a list from the ... arguments and plotlist
plots <- c(list(...), plotlist)
numPlots = length(plots)
% If layout is NULL, then use 'cols' to determine layout
if (is.null(layout)) {
% Make the panel
% ncol: Number of columns of plots
% nrow: Number of rows needed, calculated from % of cols
layout <- matrix(seq(1, cols * ceiling(numPlots/cols)),
ncol = cols, nrow = ceiling(numPlots/cols))
}
if (numPlots==1) {
print(plots[[1]])
} else {
```

```
% Set up the page
grid.newpage()
pushViewport(viewport(layout = grid.layout(nrow(layout), ncol(layout))))
% Make each plot, in the correct location
for (i in 1:numPlots) {
% Get the i,j matrix positions of the regions that contain this subplot
matchidx <- as.data.frame(which(layout == i, arr.ind = TRUE))
print(plots[[i]], vp = viewport(layout.pos.row = matchidx$row,
layout.pos.col = matchidx$col))}}}
```

C

Matlab code for Chapter 6

We present the code used to compute the mean and standard deviation of the first passage time by a lower threshold for the logistic-like model. For other thresholds and models, the user just needs to make appropriate changes.

```
r = 0.71; K = 1; A = -0.75.*K; sigma = 0.2; x = K./2; q = 3.30.*10.^(-6); E = 60546;
a = 2.*(r.*A+q.*E.*(K-A))./((K-A).*sigma.^2); b = r./(K.*(K-A).*sigma.^2); d = K+A;
u = 0.05:0.05:1;

syms y z w
s = AT(y) (y.^a).*exp(b.*(y-d).^2);
m = AT(y) (1./sigma.^2).*(y.^(-a-2)).*exp(-b.*(y-d).^2);

%Mean
```

```
M = AT(u)2.*integral(AT(y)s(y).*integral(AT(z)m(z),y,Inf),u,x,'ArrayValued',true)

%Standard Deviation
f1 = AT(z,y) integral(AT(t)((tan(t)).^(-a-2)).*((sec(t)).^2).*exp(b.*((y-d).^2)-b.*((
    tan(t)-d).^2)),atan(z),pi/2,'ArrayValued',true,'RelTol',1e-1)

f2 = AT(z) integral(AT(t)(tan(t).^(-a-2)).*((sec(t)).^2).*exp(b.*((z-d).^2)-b.*((tan(t
    )-d).^2)),atan(z),pi/2,'ArrayValued',true,'RelTol',1e-1)

f3 = AT(y) (y.^a).*integral(AT(z)(z.^a).*f2(z).*f1(z,y),y,Inf,'ArrayValued',true,'
    RelTol',1e-1)

sd1 = AT(u) sqrt(8./(sigma.^4).*integral(AT(y)f3(y),u,x,'ArrayValued',true,'RelTol',1e
    -1))

meanTu = arrayfun(M,u)
sdTu = arrayfun(sd1,u)

X = (u.*0.5.*80.5.*10.^6)./1000
figure
box on
set(semilogy(X,(meanTu)),'DisplayName','Mean time','LineWidth',4,'Color',[0.5 0.5
    0.5]), hold on
set(semilogy(X,(sdTu)),'DisplayName','Standard deviation','LineWidth',2,'Color',[0 0
    0]);
xlabel('L (tonnes)')
ylabel('years')
legend('show')

figure
box on
hold on
set(plot(X,(meanTu)),'DisplayName','Mean time','LineWidth',4,'Color',[0.5 0.5 0.5])
set(plot(X,(sdTu)),'DisplayName','Standard deviation','LineWidth',2,'Color',[0 0 0]);
xlabel('L (tonnes)')
ylabel('years')
legend('show')
```

D

Matlab code for Chapter 7

We present the code used to estimate the p.d.f. of T_L and T_U.

```
tmin = 0.001; tmax = 2000.1; npts = 20001;
[t,ft]=INVLAP('((x/u)^(q+sqrt((2*s/sigma^2)+q^2)))*kummerU(q+sqrt((2*s/sigma^2)+q^2)
    ,1+2*sqrt((2*s/sigma^2)+q^2),ro*x)/kummerU(q+sqrt((2*s/sigma^2)+q^2),1+2*sqrt((2*s/
    sigma^2)+q^2),ro*u)',tmin,tmax,npts);
ft(isnan(ft)) = 0;

figure(1)
set(1,'color','white')
plot(t,ft,'-'), grid on, hold on
xlabel('t'), ylabel('f(t)')
axis([0 tmax 0 1.1*max(ft)])
saveas(gcf,'Fig1.jpg')
```

```
figure(2)
set(2,'color','white')
loglog(t,ft,'-'), grid on, hold on
xlabel('time'), ylabel('f_{T_L}(t)')
%axis([tmin tmax 0 1.1.*max(ft)])
saveas(gcf,'Fig2.jpg')

%integral
s = sum(((tmax-tmin)/(npts-1)).*ft)
% 0.992

%mean
med = sum(((tmax-tmin)/npts).*ft.*t)
% 89.44

%%%%% Upper threshold
format long
tmin = 10^(-6);
tmax = 400;
npts = 370;

[t,ft]=INVLAP(funcN(s),tmin,tmax,npts,22,20,19);

figure(1)
set(1,'color','white')
plot(t,ft,'-'), grid on, hold on
xlabel('t'), ylabel('log(f(t))')
axis([0 tmax min(ft) max(ft)])
saveas(gcf,'Fig1.jpg')

figure(2)
set(2,'color','white')
loglog(t,ft,'-'), grid on, hold on
xlabel('time'), ylabel('f_{T_U}(t)')
%axis([0 log10(tmax)  1.1*min(log10(ft)) 1.1*max(log10(ft))])
saveas(gcf,'Fig2.jpg')

%integral
s = s1*(tmax-tmin)/(npts-1)
% 0.994
```

```
%mean
s2 = sum(ft.*t);
med = s2*(tmax-tmin)/(npts-1)
% 0.552

%%%% Invlap
% INVLAP - Numerical Inversion of Laplace Transforms
function [radt,ft]=INVLAP(Fs,tini,tend,nnt,a,ns,nd);
FF=strrep(strrep(strrep(Fs,'*','.*'),'/','./'),'^','.^');
r1 = 0.71; K1 = 80.5.*10.^6; E = 104540; q = 3.30.*10.^(-6);
r = r1-q.*E; sigma = 0.2; K = K1.*(1-q.*E./r1); x = K./2; u = 0.9.*x;
q = 0.5-r./sigma.^2; ro = (2.*r)./(K.*sigma.^2);
if nargin==4
a=6; ns=20; nd=19;  end;    % implicit parameters
radt=linspace(tini,tend,nnt); % time vector
if tini==0  radt=radt(2:1:nnt);  end;  % t=0 is not allowed
tic                                    % measure the CPU time
for n=1:ns+1+nd               % prepare necessary coefficients
alfa(n)=a+(n-1)*pi*j;
beta(n)=-exp(a)*(-1)^n;
end;
n=1:nd;
bdif=fliplr(cumsum(gamma(nd+1)./gamma(nd+2-n)./gamma(n)))./2^nd;
beta(ns+2:ns+1+nd)=beta(ns+2:ns+1+nd).*bdif;
beta(1)=beta(1)/2;
for kt=1:nnt                  % cycle for time t
tt=radt(kt);
s=alfa/tt;                 % complex frequency s
bt=beta/tt;
btF=bt.*eval(FF);          % functional value F(s)
ft(kt)=sum(real(btF));     % original f(tt)
end;
toc
```

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