Existence, non-existence and multiplicity results for a third order eigenvalue three-point boundary value problem*

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Abstract

This paper provides sufficient conditions to guarantee the existence, non-existence and multiplicity of solutions for a third order eigenvalue fully differential equation, coupled with three point boundary value conditions.

Although the change of sign, it is obtained some bounds for the second derivative of the Green's function, which allow to define a different kind of cone that, as far as we know, has not been previously used in the literature.

The main arguments are based on fixed point index theory for bounded and unbounded sets. Some examples are presented to show that the different existence theorems proved are not comparable.

1 Introduction

In this work we study the existence of solution of the third order nonlinear differential equation

$$-u^{(3)}(t) = \lambda\, f(t,\, u(t),\, u'(t),\, u''(t)), \quad t \in [0,1], \tag{1}$$

with $\lambda > 0$ a parameter and $f\colon [0,1] \times \mathbb{R}^3 \to [0,\infty)$ a L^1-Carathéodory function, coupled with three point boundary value conditions

$$u(0) = u'(0) = 0, \quad u'(1) = \alpha\, u'(\eta), \tag{2}$$

*First and second authors was partially supported by Ministerio de Educación y Ciencia, Spain and FEDER, project MTM2013-43014-P. Third author was partially suported by National Founds through FCT-Fundação para a Ciência e a Tecnologia as part of project: SFRH/BSAB/114246/2016

where $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$ are given constants. Moreover, sufficient conditions for non-existence and multiplicity of solutions are given.

A precedent problem

$$u^{(3)}(t) + a(t)\, f(u(t)) = 0, \quad t \in [0,1],$$
$$u(0) = u'(0) = 0, \quad u'(1) = \alpha\, u'(\eta),$$

was considered in [5]. There, the authors constructed the Green's function related to the problem and established some of their properties. From them, they built a suitable cone and applied Guo-Krasnoselskii theorem to assure the existence of a positive solution of the problem.

Recently, in [9], the authors considered the following system

$$\begin{cases} -u^{(3)}(t) = f(t,\, v(t),\, v'(t)), & t \in [0,1], \\ -v^{(3)}(t) = f(t,\, u(t),\, u'(t)), & t \in [0,1], \\ u(0) = u'(0) = 0, \quad u'(1) = \alpha\, u'(\eta), \\ v(0) = v'(0) = 0, \quad v'(1) = \alpha\, v'(\eta). \end{cases}$$

There, they studied the properties of the first derivative of the Green's function related to the problem and used them to construct a cone K such that there exist $u,\, v \in K$ which constitute a positive solution of the system. To do this, they also use Guo-Krasnoselskii Theorem [4].

A similar nonlinear fourth-order boundary value problem is treated in [8], where the authors study the existence of nonzero and positive solutions by means of monotone iterative techniques and lower and upper solutions.

In this paper, we pretend to study a generalization of the previous equations by considering that the nonlinearity f depends on the solution and its first and second order derivatives. Because of this, we need to study the properties of the second derivative of the Green's function. However, as we will see along the paper, contrarily to what happened with the Green's function and its first derivative, in this case it is not possible to find a function Φ such that

$$\left| \frac{\partial^2 G}{\partial t^2}(t,s) \right| \leq \Phi(s), \quad \forall\, (t,s) \in [0,1] \times [0,1],$$

and

$$\frac{\partial^2 G}{\partial t^2}(t,s) \geq c\, \Phi(s) \quad \forall\, (t,s) \in [a,b] \times [0,1],$$

for some $[a,b] \subset [0,1]$ and $c \in (0,1)$.

This makes necessary to construct a different kind of cone that, as far as we know, has not been previously used in the literature. With this cone, we will give some conditions to assure the existence of a positive and increasing solution to problem (1)-(2), which will also be convex in a certain subset of its interval of definition.

The paper is organized in the following way: In Section 2 we compiled the known properties for the Green's function related to the problem and its first derivative, and we study its second

one. In Section 3 we define our cone and give some existence results by means of the fixed point index for unbounded sets (see [3]). In Section 4, we consider the fixed point index theory for bounded sets in order to obtain some results regarding existence and multiplicity of solutions. We follow the line of results given in [2, 6, 7]. In Section 5, we give some conditions under which there is not any solution for the considered problem. Finally, in Section 6, we give some examples which show that the existence results obtained in Sections 3 and 4 are not comparable.

2 Preliminary results

The Green's function related to the homogeneous problem

$$\begin{aligned} -u^{(3)}(t) &= 0, \quad t \in [0,1], \\ u(0) = u'(0) &= 0, \quad u'(1) = \alpha\, u'(\eta), \end{aligned}$$

is given by the following expression ([5])

$$G(t,s) = \frac{1}{2\,(1-\alpha\,\eta)} \begin{cases} (2\,t\,s - s^2)\,(1-\alpha\,\eta) + t^2\,s\,(\alpha-1), & s \le \min\{\eta,\, t\}, \\ t^2\,(1-\alpha\,\eta) + t^2\,s\,(\alpha-1), & t \le s \le \eta, \\ (2\,t\,s - s^2)\,(1-\alpha\,\eta) + t^2\,(\alpha\,\eta - s), & \eta \le s \le t, \\ t^2\,(1-s), & \max\{\eta,\, t\} \le s. \end{cases}$$

Next lemmas establish some properties of the Green's function and its first and second order derivatives.

Lemma 2.1. *([5, Lemma 2.2]) Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,*

$$0 \le G(t,s) \le g_0(s) = \frac{1+\alpha}{1-\alpha\,\eta}\, s\,(1-s) \quad \forall\,(t,s) \in [0,1] \times [0,1].$$

Lemma 2.2. *([5, Lemma 2.3]) Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,*

$$G(t,s) \ge \kappa_0\, g_0(s) \quad \forall\,(t,s) \in \left[\frac{\eta}{\alpha}, \eta\right] \times [0,1],$$

with $0 < \kappa_0 = \frac{\eta^2}{2\,\alpha^2\,(1+\alpha)}\, \min\{\alpha - 1, 1\} < 1$.

The first derivative of G is given by

$$\frac{\partial\, G}{\partial\, t}(t,s) = \frac{1}{(1-\alpha\,\eta)} \begin{cases} s\,(1-\alpha\,\eta) + t\,s\,(\alpha-1), & s \le \min\{\eta,\, t\}, \\ t\,(1-\alpha\,\eta) + t\,s\,(\alpha-1), & t \le s \le \eta, \\ s\,(1-\alpha\,\eta) + t\,(\alpha\,\eta - s), & \eta \le s \le t, \\ t\,(1-s), & \max\{\eta,\, t\} \le s, \end{cases}$$

and satisfies the following properties.

Lemma 2.3. *([9, Lemma 3]) Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,*

$$0 \le \frac{\partial G}{\partial t}(t,s) \le g_1(s) = \frac{1-s}{1-\alpha\,\eta} \quad \forall\,(t,s) \in [0,1]\times[0,1].$$

Lemma 2.4. *([9, Lemma 4]) Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,*

$$\frac{\partial G}{\partial t}(t,s) \ge \kappa_1\, g_1(s) \quad \forall\,(t,s) \in \left[\frac{\eta}{\alpha},\eta\right]\times[0,1],$$

with $0 < \kappa_1 = \eta < 1$.

The second derivative of G is given by

$$\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{1}{1-\alpha\,\eta}\begin{cases} s\,(\alpha-1), & s \le \min\{\eta,\, t\},\\ 1-\alpha\,\eta+s\,(\alpha-1), & t \le s \le \eta,\\ \alpha\,\eta - s, & \eta \le s \le t,\\ 1-s, & \max\{\eta,\, t\} \le s. \end{cases}$$

It is immediate to verify that it satisfies the following conditions:

Lemma 2.5. *Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,*

$$\frac{\partial^2 G}{\partial t^2}(t,s) \ge 0 \quad \forall\,(t,s) \in ([0,1]\times[0,1]) \setminus A,$$

where

$$A = \{(t,s) \in [0,1]\times[0,1];\, \alpha\,\eta < t < 1,\, \alpha\,\eta < s < t\}.$$

***Remark* 2.6.** Note that, in particular, $\frac{\partial^2 G}{\partial t^2}(t,s) \ge 0$ for all $(t,s) \in [0,\alpha\,\eta]\times[0,1]$.

Next two results will allow us to define a suitable cone in $\mathcal{C}^2[0,1]$.

Lemma 2.7. *Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,*

$$\frac{\partial^2 G}{\partial t^2}(t,s) \le g_2(s) \quad \forall\,(t,s) \in \left[\frac{\eta}{\alpha},1\right]\times[0,1]$$

and

$$-1 \le \frac{\partial^2 G}{\partial t^2}(t,s) \le \max\left\{g_2(s),\, \frac{1-\eta}{1-\alpha\,\eta}\right\} \quad \forall\,(t,s) \in [0,1]\times[0,1],$$

with

$$g_2(s) = \frac{\alpha\,(1-\alpha\,\eta)+\eta\,(\alpha-1)}{\eta\,(\alpha-1)}\,\frac{\partial^2 G}{\partial t^2}(\eta,s) = \frac{\alpha\,(1-\alpha\,\eta)+\eta\,(\alpha-1)}{\eta\,(\alpha-1)(1-\alpha\,\eta)}\begin{cases} s\,(\alpha-1), & 0 \le s \le \eta,\\ 1-s, & \eta \le s \le 1. \end{cases}$$

Proof. First, we will prove that $\frac{\partial^2 G}{\partial t^2}(t,s) \leq g_2(s)$ for all $(t,s) \in \left[\frac{\eta}{\alpha}, 1\right] \times [0,1]$.

For $s \leq \min\{\eta, t\}$ and $s \geq \max\{\eta, t\}$ we have that $\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{\partial^2 G}{\partial t^2}(\eta,s)$ and, since $\frac{\alpha\,(1-\alpha\,\eta)}{\eta\,(\alpha-1)} + 1 > 1$, it is obvious that $\frac{\partial^2 G}{\partial t^2}(t,s) \leq g_2(s)$.

For $t \leq s \leq \eta$, we have that

$$\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{1-\alpha\,\eta+s\,(\alpha-1)}{1-\alpha\,\eta} = \frac{(\alpha-1)\left(\frac{\eta\,\alpha\,(1-\alpha\,\eta)}{\alpha\,\eta\,(\alpha-1)}+s\right)}{1-\alpha\,\eta} \leq \frac{s\,(\alpha-1)\left(\frac{\alpha(1-\alpha\,\eta)}{\eta\,(\alpha-1)}+1\right)}{1-\alpha\,\eta} = g_2(s).$$

Finally, for $\eta \leq s \leq t$,

$$\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{\alpha\,\eta - s}{1-\alpha\,\eta} \leq \frac{1-s}{1-\alpha\,\eta} = \frac{\partial^2 G}{\partial t^2}(\eta,s) \leq g_2(s).$$

Now, we will prove that $\frac{\partial^2 G}{\partial t^2}(t,s) \geq -1$ for all $(t,s) \in [0,1] \times [0,1]$. It is immediate to verify that $\frac{\partial^2 G}{\partial t^2}(t,s) \geq 0$ for $s \leq \min\{\eta, t\}$, $t \leq s \leq \eta$ and $\max\{\eta, t\} \leq s$. On the other hand, we have that for $\eta \leq s \leq t$,

$$\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{\alpha\,\eta - s}{1-\alpha\,\eta} \geq \frac{\alpha\,\eta - 1}{1-\alpha\,\eta} = -1$$

and so the result holds.

Finally, we will prove that $\frac{\partial^2 G}{\partial t^2}(t,s) \leq \max\left\{g_2(s), \frac{1-\eta}{1-\alpha\,\eta}\right\}$ for all $(t,s) \in [0,1] \times [0,1]$. Obviously, it is enough to prove it for $(t,s) \in \left[0, \frac{\eta}{\alpha}\right] \times [0,1]$.

For $s \leq t$ and $s \geq \eta$, we have just seen that $\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{\partial^2 G}{\partial t^2}(\eta,s) \leq g_2(s)$.

On the other hand, for $t \leq s \leq \eta$, the following inequality holds

$$\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{1-\alpha\,\eta+s\,(\alpha-1)}{1-\alpha\,\eta} \leq \frac{1-\alpha\,\eta+\eta\,(\alpha-1)}{1-\alpha\,\eta} = \frac{1-\eta}{1-\alpha\,\eta}$$

and so the result is proved. □

***Remark* 2.8.** We note that for any $c > 0$ it would be possible to find a continuous function $g \neq g_2$ such that

$$\frac{\partial^2 G}{\partial t^2}(t,s) \leq g(s) \quad \forall\,(t,s) \in [c,1] \times [0,1].$$

Lemma 2.9. *Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then, for all $(t,s) \in \left[\frac{\eta}{\alpha}, \eta\right] \times [0,1]$,*

$$\frac{\partial^2 G}{\partial t^2}(t,s) \geq \frac{\partial^2 G}{\partial t^2}(\eta,s) \equiv \kappa_2\, g_2(s)$$

with $0 < \kappa_2 = \frac{\eta\,(\alpha-1)}{\alpha(1-\alpha\,\eta)+\eta\,(\alpha-1)}$.

Proof. For $s \leq t$ and $s \geq \eta$, we have that $\frac{\partial^2 G}{\partial t^2}(t,s) = \kappa_2\, g_2(s)$.

On the other hand, for $t \leq s \leq \eta$, it holds that

$$\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{1-\alpha\,\eta+s\,(\alpha-1)}{1-\alpha\,\eta} \geq \frac{s\,(\alpha-1)}{1-\alpha\,\eta} = \kappa_2\, g_2(s).$$

□

***Remark* 2.10.** We note that for any $[a,b] \subset (0,\alpha\,\eta)$ it would be possible to find a constant κ such that

$$\frac{\partial^2 G}{\partial t^2}(t,s) \geq \kappa\, g_2(s) \quad \forall\,(t,s) \in [a,b]\times[0,1].$$

However, for the sake of simplicity, we have chosen the interval $\left[\frac{\eta}{\alpha},\eta\right]$ to maintain the same interval than in Lemmas 2.2 and 2.4.

***Remark* 2.11.** We point out that, on the contrary to function G and $\frac{\partial G}{\partial t}$, it is not possible to find a continuous function $\tilde{g}_2(s)$ such that

$$\left|\frac{\partial^2 G}{\partial t^2}(t,s)\right| \leq \tilde{g}_2(s) \quad \forall\,(t,s) \in [0,1]\times[0,1]$$

and

$$\frac{\partial^2 G}{\partial t^2}(t,s) \geq \tilde{\kappa}_2\, \tilde{g}_2(s) \quad \forall\,(t,s) \in [a,b]\times[0,1],$$

with $[a,b] \subset [0,1]$ and $\tilde{\kappa}_2 \in (0,1)$.

This is due to the fact that for $s \geq \alpha\,\eta$,

$$\left|\frac{\partial^2 G}{\partial t^2}(t,s)\right| = \begin{cases} \frac{s-\alpha\,\eta}{1-\alpha\,\eta}, & s \leq t, \\ \frac{1-s}{1-\alpha\,\eta}, & t \leq s. \end{cases}$$

As a consequence, if there exists $\tilde{g}_2$ satisfying the previous conditions, it would necessarily verify that

$$\tilde{g}_2(s) \geq \max\left\{\frac{s-\alpha\,\eta}{1-\alpha\,\eta}, \frac{1-s}{1-\alpha\,\eta}\right\} = \begin{cases} \frac{1-s}{1-\alpha\,\eta}, & s \leq \frac{1+\alpha\,\eta}{2}, \\ \frac{s-\alpha\,\eta}{1-\alpha\,\eta}, & s \geq \frac{1+\alpha\,\eta}{2}, \end{cases}$$

for $s \geq \alpha\,\eta$ and so $\tilde{g}_2(1) \geq 1$.

On the other hand, we have that $\frac{\partial^2 G}{\partial t^2}(t,1) = 0$ so, if there exists $\tilde{g}_2$ in the previous conditions, it would happen that

$$0 = \frac{\partial^2 G}{\partial t^2}(t,1) \geq \tilde{\kappa}_2\, \tilde{g}_2(1) \geq \tilde{\kappa}_2 > 0,$$

which is a contradiction.

3 Main results

Let's consider $E = \mathcal{C}^2([0,1],\, \mathbb{R})$ equipped with the norm

$$\|u\| = \max\{\|u\|_\infty,\, \|u'\|_\infty,\, \|u''\|_\infty\},$$

where $\|v\|_\infty = \sup_{t\in[0,1]} |v(t)|$. It is very well known that $(E,\, \|\cdot\|)$ is a Banach space.

Taking into account the properties satisfied by the Green's function and its derivatives, we define

$$K = \big\{u \in \mathcal{C}^2([0,1],\mathbb{R}) :\, u(t) \geq 0,\, t \in [0,1],\, u'(t) \geq 0,\, t \in [0,1],\, u''(t) \geq 0,\, t \in [0, \alpha\,\eta],$$
$$\min_{t\in\left[\frac{\eta}{\alpha},\eta\right]} u(t) \geq \kappa_0\, \|u\|_\infty,\, \min_{t\in\left[\frac{\eta}{\alpha},\eta\right]} u'(t) \geq \kappa_1\, \|u'\|_\infty,\, \min_{t\in\left[\frac{\eta}{\alpha},\eta\right]} u''(t) \geq \kappa_2\, \|u''\|_{\left[\frac{\eta}{\alpha},\eta\right]}\big\},$$

where

$$\|u''\|_{\left[\frac{\eta}{\alpha},\eta\right]} :\, = \max_{t\in\left[\frac{\eta}{\alpha},\eta\right]} |u''(t)|$$

and κ_0, κ_1 and κ_2 are defined in previous section.

It is obvious that K is a cone in E.

It is very well known that the solutions of the problem (1)-(2) correspond with the fixed points of the integral operator

$$Tu(t) = \lambda \int_0^1 G(t,s)\, f(s, u(s), u'(s), u''(s))\, ds, \quad t \in [0,1]. \tag{3}$$

We make the following assumptions on the elements that take part in the previous expression:

(H1) λ is a positive parameter.

(H2) The nonlinearity $f\colon [0,1] \times \mathbb{R}^3 \to [0,\infty)$ satisfies L^1-Carathéodory conditions, that is,

- $f(\cdot, u, v, w)$ is measurable for each (u, v, w) fixed.
- $f(t, \cdot, \cdot, \cdot)$ is continuous for $a.\,e.\ t \in [0,1]$.
- For each $r > 0$ there exists $\phi_r \in L^1[0,1]$ such that

$$f(t,u,v,w) \leq \phi_r(t) \quad \forall\, (u,v,w) \in (-r,r) \times (-r,r) \times (-r,r),\ a.\,e.\ t \in [0,1].$$

Under these assumptions, coupled with additional properties on the function f, we will ensure the existence of solutions of the considered problem (1)-(2). Before doing that, we obtain some previous technical results.

Lemma 3.1. *$T\colon K \to K$ is a completely continuous operator.*

Proof. We divide the proof into several steps.

Step 1. T is well defined in K.

Let $u \in K$. We will prove that $Tu \in K$.

It is obvious that $Tu(t) \geq 0$ for all $t \in [0,T]$. Moreover, using Lemma 2.1, we have

$$Tu(t) = \lambda \int_0^1 G(t,s)\, f(s,u(s),u'(s),u''(s))\, ds \leq \lambda \int_0^1 g_0(s)\, f(s,u(s),u'(s),u''(s))\, ds,$$

and, taking the supremum for $t \in [0,1]$, we deduce that

$$\|Tu\|_\infty \leq \lambda \int_0^1 g_0(s)\, f(s,u(s),u'(s),u''(s))\, ds.$$

So, for $t \in \left[\frac{\eta}{\alpha},\eta\right]$, from Lemma 2.2, we have

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)\, f(s,u(s),u'(s),u''(s))\, ds \\ &\geq \lambda \int_0^1 \kappa_0\, g_0(s)\, f(s,u(s),u'(s),u''(s))\, ds \geq \kappa_0\, \|Tu\|_\infty \end{aligned}$$

and we deduce that

$$\min_{t\in\left[\frac{\eta}{\alpha},\eta\right]} Tu(t) \geq \kappa_0\, \|Tu\|_\infty.$$

Analogously, since $\frac{\partial G}{\partial t}(t,s) \geq 0$ on $[0,1]\times[0,1]$, it is immediate to verify that

$$(Tu)'(t) = \lambda \int_0^1 \frac{\partial\, G}{\partial\, t}(t,s)\, f(s,u(s),u'(s),u''(s))\, ds \geq 0.$$

Moreover, Lemma 2.3 implies

$$(Tu)'(t) \leq \lambda \int_0^1 g_1(s)\, f(s,u(s),u'(s),u''(s))\, ds,$$

and, taking the supremum for $t \in [0,1]$,

$$\|(Tu)'\|_\infty \leq \lambda \int_0^1 g_1(s)\, f(s,u(s),u'(s),u''(s))\, ds.$$

So, for $t \in \left[\frac{\eta}{\alpha},\eta\right]$, Lemma 2.4 gives us

$$(Tu)'(t) \geq \lambda \int_0^1 \kappa_1\, g_1(s)\, f(s,u(s),u'(s),u''(s))\, ds \geq \kappa_1\, \|(Tu)'\|_\infty$$

and we can affirm that

$$\min_{t\in\left[\frac{\eta}{\alpha},\eta\right]} (Tu)'(t) \geq \kappa_1\, \|(Tu)'\|_\infty.$$

Finally, from Lemma 2.5, we have that for $t \in [0, \alpha\,\eta]$,

$$(Tu)''(t) = \lambda \int_0^1 \frac{\partial^2 G}{\partial\, t^2}(t,s)\, f(s, u(s), u'(s), u''(s))\, ds \geq 0.$$

In addition, for $t \in \left[\frac{\eta}{\alpha}, \eta\right]$, Lemma 2.7 assures that

$$(Tu)''(t) \leq \lambda \int_0^1 g_2(s)\, f(s, u(s), u'(s), u''(s))\, ds,$$

and, taking the supremum for $t \in \left[\frac{\eta}{\alpha}, \eta\right]$,

$$\|(Tu)''\|_{\left[\frac{\eta}{\alpha},\eta\right]} \leq \lambda \int_0^1 g_2(s)\, f(s, u(s), u'(s), u''(s))\, ds.$$

So, for $t \in \left[\frac{\eta}{\alpha}, \eta\right]$, from Lemma 2.9 we know that

$$(Tu)''(t) \geq \lambda \int_0^1 \kappa_2\, g_2(s)\, f(s, u(s), u'(s), u''(s))\, ds \geq \kappa_2\, \|(Tu)''\|_{\left[\frac{\eta}{\alpha},\eta\right]}$$

and we deduce that

$$\min_{t \in \left[\frac{\eta}{\alpha},\eta\right]} (Tu)''(t) \geq \kappa_2\, \|(Tu)'\|_{\left[\frac{\eta}{\alpha},\eta\right]}.$$

Therefore, we can conclude that $Tu \in K$.

Step 2. T is a compact operator.

Let's consider

$$B = \{u \in E;\ \|u\| \leq r\}.$$

First, we will prove that $T(B)$ is uniformly bounded in $\mathcal{C}^2[0,1]$. We find the following bounds for $u \in B$:

$$\begin{aligned}\|Tu\|_\infty &= \sup_{t\in[0,1]} \left|\lambda \int_0^1 G(t,s)\, f(s, u(s), u'(s), u''(s))\, ds\right| \\ &\leq \lambda \int_0^1 g_0(s)\, f(s, u(s), u'(s), u''(s))\, ds \leq \lambda \int_0^1 g_0(s)\, \phi_r(s)\, ds := M_1.\end{aligned}$$

$$\begin{aligned}\|(Tu)'\|_\infty &= \sup_{t\in[0,1]} \left|\lambda \int_0^1 \frac{\partial G}{\partial\, t}(t,s)\, f(s, u(s), u'(s), u''(s))\, ds\right| \\ &\leq \lambda \int_0^1 g_1(s)\, f(s, u(s), u'(s), u''(s))\, ds \leq \lambda \int_0^1 g_1(s)\, \phi_r(s)\, ds := M_2.\end{aligned}$$

$$\begin{aligned}\|(Tu)''\|_\infty &= \sup_{t\in[0,1]}\left|\lambda\int_0^1 \frac{\partial^2 G}{\partial t^2}(t,s)\, f(s,u(s),u'(s),u''(s))\, ds\right| \\ &\le \sup_{t\in[0,1]} \lambda\int_0^1 \left|\frac{\partial^2 G}{\partial t^2}(t,s)\right| f(s,u(s),u'(s),u''(s))\, ds \\ &\le \lambda\int_0^1 \max\left\{\frac{1-\eta}{1-\alpha\,\eta},\, g_2(s)\right\} f(s,u(s),u'(s),u''(s))\, ds \\ &\le \lambda\int_0^1 \max\left\{\frac{1-\eta}{1-\alpha\,\eta},\, g_2(s)\right\} \phi_r(s)\, ds := M_3.\end{aligned}$$

So, it is deduced that

$$\|Tu\| \le \max\{M_1,\, M_2,\, M_3\} \quad \forall\, u \in B.$$

Now, we will prove that $T(B)$ is equicontinuous in $\mathcal{C}^2[0,1]$.

Let $t_1,\, t_2 \in [0,1]$. Without loss of generality, suppose that $t_1 < t_2$. Then,

$$\begin{aligned}|Tu(t_1) - Tu(t_2)| &\le \lambda\int_0^1 |G(t_1,s) - G(t_2,s)|\, f(s,u(s)u'(s),u''(s))\, ds \\ &\le \lambda\int_0^1 |G(t_1,s) - G(t_2,s)|\, \phi_r(s)\, ds\end{aligned}$$

and, since $G(\cdot,s)$ is continuous, we have that for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|t_2 - t_1| < \delta$ then $|Tu(t_1) - Tu(t_2)| < \varepsilon$ for all $u \in B$.

Analogously,

$$\begin{aligned}\left|(Tu)'(t_1) - (Tu)'(t_2)\right| &\le \lambda\int_0^1 \left|\frac{\partial G}{\partial t}(t_1,s) - \frac{\partial G}{\partial t}(t_2,s)\right| f(s,u(s)u'(s),u''(s))\, ds \\ &\le \lambda\int_0^1 \left|\frac{\partial G}{\partial t}(t_1,s) - \frac{\partial G}{\partial t}(t_2,s)\right| \phi_r(s)\, ds\end{aligned}$$

and, since $\frac{\partial G}{\partial t}(\cdot,s)$ is also continuous, we reason as in the previous case and conclude that $|(Tu)'(t_1) - (Tu)'(t_2)| < \varepsilon$ for all $u \in B$.

Finally, we have that

$$\begin{aligned}\left|(Tu)''(t_1)-(Tu)''(t_2)\right| &\leq \lambda\int_0^1\left|\frac{\partial^2 G}{\partial t^2}(t_1,s)-\frac{\partial^2 G}{\partial t^2}(t_2,s)\right| f(s,u(s)u'(s),u''(s))\,ds\\ &\leq \lambda\int_0^1\left|\frac{\partial^2 G}{\partial t^2}(t_1,s)-\frac{\partial^2 G}{\partial t^2}(t_2,s)\right| \phi_r(s)\,ds\\ &= \lambda\int_0^{t_1}\left|\frac{\partial^2 G}{\partial t^2}(t_1,s)-\frac{\partial^2 G}{\partial t^2}(t_2,s)\right| \phi_r(s)\,ds\\ &\quad+\lambda\int_{t_1}^{t_2}\left|\frac{\partial^2 G}{\partial t^2}(t_1,s)-\frac{\partial^2 G}{\partial t^2}(t_2,s)\right| \phi_r(s)\,ds\\ &\quad+\lambda\int_{t_2}^{1}\left|\frac{\partial^2 G}{\partial t^2}(t_1,s)-\frac{\partial^2 G}{\partial t^2}(t_2,s)\right| \phi_r(s)\,ds.\end{aligned}$$

In this case, we have that $\frac{\partial^2 G}{\partial t^2}(\cdot,s)$ is continuous in $[0,s)\cup(s,1]$ and has a jump discontinuity at $t=s$. Because of this, we can apply the same reasoning that in previous cases to assure that the first and last terms in the previous inequality tend to zero with independence of the function $u\in B$. On the other hand, we have that $\left|\frac{\partial^2 G}{\partial t^2}(t_1,s)-\frac{\partial^2 G}{\partial t^2}(t_2,s)\right|\phi_r(s)\in L^1[0,1]$ so it is obvious that

$$\lambda\int_{t_1}^{t_2}\left|\frac{\partial^2 G}{\partial t^2}(t_1,s)-\frac{\partial^2 G}{\partial t^2}(t_2,s)\right| \phi_r(s)\,ds \xrightarrow[t_1\to t_2]{} 0$$

with independence of the function $u\in B$.

Therefore we conclude that $T(B)$ is equicontinuous in $\mathcal{C}^2[0,1]$.

As a consequence, by Ascoli-Arzèla theorem, we can affirm that $T(B)$ is relatively compact in $\mathcal{C}^2[0,1]$ and so T is a completely continuous operator. □

We introduce now the following notation (see Remark 3.4 for details)

$$\Lambda_1=\int_0^1 g_0(s)\,ds,\quad \Lambda_2=\int_0^1 g_1(s)\,ds,\quad \Lambda_3=\int_0^1\max\left\{g_2(s),\,\frac{1-\eta}{1-\alpha\,\eta}\right\}ds,$$

$$\Lambda_4=\int_{\frac{\eta}{\alpha}}^{\eta}\kappa_0\,g_0(s)\,ds,\quad \Lambda_5=\int_{\frac{\eta}{\alpha}}^{\eta}\kappa_1\,g_1(s)\,ds$$

and we define

$$\bar{\Lambda}=3\,\max\{\Lambda_1,\,\Lambda_2,\,\Lambda_3\}\quad\text{and}\quad \underline{\Lambda}=\max\{\kappa_0\,\Lambda_4,\,\kappa_1\,\Lambda_5\}.$$

We also denote:

$$f_0=\lim_{|x|,|y|,|z|\to 0}\min_{t\in[0,1]}\frac{f(t,x,y,z)}{|x|+|y|+|z|},$$

$$f^{\infty}=\lim_{|x|,|y|,|z|\to \infty}\max_{t\in[0,1]}\frac{f(t,x,y,z)}{|x|+|y|+|z|}.$$

We will give now our first existence result.

Theorem 3.2. *Assume that hypothesis* $(H1)-(H2)$ *hold. If* $\bar{\Lambda}\, f^{\infty} < \underline{\Lambda}\, f_0$*, then for all*

$$\lambda \in \left(\frac{1}{\underline{\Lambda}\, f_0}, \frac{1}{\bar{\Lambda}\, f^{\infty}} \right)$$

problem (1)-(2) *has at least a positive solution that belongs to the cone* K*.*

Proof. Let $\lambda \in \left(\frac{1}{\underline{\Lambda}\, f_0}, \frac{1}{\bar{\Lambda}\, f^{\infty}} \right)$ and choose $\varepsilon \in (0, f_0)$ such that

$$\frac{1}{\underline{\Lambda}\,(f_0 - \varepsilon)} \leq \lambda \leq \frac{1}{\bar{\Lambda}\,(f^{\infty} + \varepsilon)}.$$

Taking into account the definition of f_0, we know that there exists $\delta_1 > 0$ such that when $\|u\| \leq \delta_1$,

$$f(t, u(t), u'(t), u''(t)) > \, (f_0 - \varepsilon)\,(|u(t)| + |u'(t)| + |u''(t)|), \quad \forall\, t \in [0,1].$$

Let

$$\Omega_{\delta_1} = \{u \in K;\, \|u\| < \delta_1\}$$

and choose $u \in \partial\, \Omega_{\delta_1}$. We will prove that $Tu \not\preceq u$. We have that

$$\begin{aligned}
Tu(t) &= \lambda \int_0^1 G(t,s)\, f(s, u(s), u'(s), u''(s))\, ds \geq \, \lambda \int_0^1 \kappa_0\, g_0(s)\, f(s, u(s), u'(s), u''(s))\, ds \\
&\geq \lambda \int_{\frac{\eta}{\alpha}}^{\eta} \kappa_0\, g_0(s)\, f(s, u(s), u'(s), u''(s))\, ds \\
&> \lambda \int_{\frac{\eta}{\alpha}}^{\eta} \kappa_0\, g_0(s)\, (f_0 - \varepsilon)\, \left(|u(s)| + |u'(s)| + |u''(s)| \right)\, ds \\
&\geq \lambda\, (f_0 - \varepsilon)\, \left(\kappa_0\, \|u\|_{\infty} + \kappa_1\, \|u'\|_{\infty} + \kappa_2\, \|u''\|_{\left[\frac{\eta}{\alpha}, \eta\right]} \right) \int_{\frac{\eta}{\alpha}}^{\eta} \kappa_0\, g_0(s)\, ds \\
&= \lambda\, (f_0 - \varepsilon)\, \left(\kappa_0\, \|u\|_{\infty} + \kappa_1\, \|u'\|_{\infty} + \kappa_2\, \|u''\|_{\left[\frac{\eta}{\alpha}, \eta\right]} \right)\, \Lambda_4 \geq \, \lambda\, (f_0 - \varepsilon)\, \Lambda_4\, \kappa_0\, u(t) \quad \forall\, t \in [0,1]
\end{aligned}$$

and

$$\begin{aligned}
(Tu)'(t) &= \lambda \int_0^1 \frac{\partial\, G}{\partial\, t}(t,s)\, f(s, u(s), u'(s), u''(s))\, ds \geq \, \lambda \int_0^1 \kappa_1\, g_1(s)\, f(s, u(s), u'(s), u''(s))\, ds \\
&\geq \lambda \int_{\frac{\eta}{\alpha}}^{\eta} \kappa_1\, g_1(s)\, f(s, u(s), u'(s), u''(s))\, ds \\
&> \lambda \int_{\frac{\eta}{\alpha}}^{\eta} \kappa_1\, g_1(s)\, (f_0 - \varepsilon)\, \left(|u(s)| + |u'(s)| + |u''(s)| \right)\, ds \\
&\geq \lambda\, (f_0 - \varepsilon)\, \left(\kappa_0\, \|u\|_{\infty} + \kappa_1\, \|u'\|_{\infty} + \kappa_2\, \|u''\|_{\left[\frac{\eta}{\alpha}, \eta\right]} \right) \int_{\frac{\eta}{\alpha}}^{\eta} \kappa_1\, g_1(s)\, ds \\
&= \lambda\, (f_0 - \varepsilon)\, \left(\kappa_0\, \|u\|_{\infty} + \kappa_1\, \|u'\|_{\infty} + \kappa_2\, \|u''\|_{\left[\frac{\eta}{\alpha}, \eta\right]} \right)\, \Lambda_5 \geq \, \lambda\, (f_0 - \varepsilon)\, \Lambda_5\, \kappa_1\, u'(t) \quad \forall\, t \in [0,1].
\end{aligned}$$

As a consequence we have that either $Tu(t) > u(t)\ \forall\, t \in [0,1]$ or $(Tu)'(t) > u'(t)\ \forall\, t \in [0,1]$ so it is proved that $Tu \not\preceq u$. We deduce (see [4, Theorem 2.3.3]) that

$$i_K(T,\, \Omega_{\delta_1}) = 0.$$

On the other hand, due to the definition of f^∞, we know that there exists $\tilde{\delta}_2 > 0$ such that when $\min\{|u(t)|,\, |u'(t)|,\, |u''(t)|\} \geq \tilde{\delta}_2$,

$$f(t, u(t), u'(t), u''(t)) \leq (f^\infty + \varepsilon)\, \left(|u(t)| + |u'(t)| + |u''(t)|\right) \leq 3\,(f^\infty + \varepsilon)\, \|u\|, \quad \forall\, t \in [0,1].$$

Let $\delta_2 > \{\delta_1,\, \tilde{\delta}_2\}$ and define

$$\Omega_{\delta_2} = \left\{u \in K;\ \min_{t\in[0,1]} |u(t)| < \delta_2\right\} \cup \left\{u \in K;\ \min_{t\in[0,1]} |u'(t)| < \delta_2\right\} \cup \left\{u \in K;\ \min_{t\in[0,1]} |u''(t)| < \delta_2\right\}.$$

We note that Ω_{δ_2} is an unbounded subset of the cone K. Because of this, the fixed point index of operator T with respect to Ω_{δ_2}, $i_K(T,\, \Omega_{\delta_2})$, is only defined in the case that the set of fixed points of operator T in Ω_{δ_2}, that is, $(I-T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$, is compact (see [3] for the details). We will see that $i_K(T,\, \Omega_{\delta_2})$ can be defined in this case.

First of all, since $(I-T)$ is a continuous operator, it is obvious that $(I-T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$ is closed.

Moreover, we can assume that $(I-T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$ is bounded. Indeed, on the contrary, we would have infinite fixed points of operator T on Ω_{δ_2} and it would be immediately deduced that problem (1)-(2) has an infinite number of positive solutions. Therefore we may assume that there exists a constant $M > 0$ such that $\|u\| < M$ for all $u \in (I-T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$.

Finally, we will see that $(I-T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$ is equicontinuous. This property follows from the fact that $(I-T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$ is bounded. The proof is totally analogous to Step 2 in the proof of Lemma 3.1.

Now, we will calculate $i_K(T,\, \Omega_{\delta_2})$. In particular, we will prove that $\|Tu\| \leq \|u\|$ for all $u \in \partial\, \Omega_{\delta_2}$. Let $u \in \partial\, \Omega_{\delta_2}$, that is, $u \in K$ such that

$$\min\left\{\min_{t\in[0,1]} |u(t)|,\ \min_{t\in[0,1]} |u'(t)|,\ \min_{t\in[0,1]} |u''(t)|\right\} = \delta_2.$$

Then,

$$\begin{aligned}
|Tu(t)| = Tu(t) =\ & \lambda \int_0^1 G(t,s)\, f(s, u(s), u'(s), u''(s))\, ds \\
& \leq \lambda \int_0^1 g_0(s)\, f(s, u(s), u'(s), u''(s))\, ds \leq\ 3\,\lambda \int_0^1 g_0(s)\, (f^\infty + \varepsilon)\, \|u\|\, ds \\
& = 3\,\lambda\, (f^\infty + \varepsilon)\, \|u\|\, \Lambda_1 \leq\ \lambda\, (f^\infty + \varepsilon)\, \|u\|\, \bar{\Lambda} \leq \|u\|
\end{aligned}$$

$$
\begin{aligned}
|(Tu)'(t)| = (Tu)'(t) = & \ \lambda \int_0^1 \frac{\partial\, G}{\partial\, t}(t,s)\, f(s,u(s),u'(s),u''(s))\, ds \\
& \leq \lambda \int_0^1 g_1(s)\, f(s,u(s),u'(s),u''(s))\, ds \leq \ 3\,\lambda \int_0^1 g_1(s)\, (f^\infty + \varepsilon)\, \|u\|\, ds \\
& = 3\,\lambda\, (f^\infty + \varepsilon)\, \|u\|\, \Lambda_2 \leq \ \lambda\, (f^\infty + \varepsilon)\, \|u\|\, \bar{\Lambda} \leq \|u\|
\end{aligned}
$$

and

$$
\begin{aligned}
|(Tu)''(t)| \leq & \lambda \int_0^1 \left| \frac{\partial^2\, G}{\partial\, t^2}(t,s) \right| \, f(s,u(s),u'(s),u''(s))\, ds \\
& \leq \lambda \int_0^1 \max\left\{ g_2(s),\ \frac{1-\eta}{1-\alpha\,\eta} \right\} \, f(s,u(s),u'(s),u''(s))\, ds \\
& \leq 3\,\lambda \int_0^1 \max\left\{ g_2(s),\ \frac{1-\eta}{1-\alpha\,\eta} \right\} \, (f^\infty + \varepsilon)\, \|u\|\, ds \\
& = 3\,\lambda\, (f^\infty + \varepsilon)\, \|u\|\, \Lambda_3 \leq \ \lambda\, (f^\infty + \varepsilon)\, \|u\|\, \bar{\Lambda} \leq \|u\|.
\end{aligned}
$$

We deduce that

$$\|Tu\| \leq \|u\|$$

and as a consequence ([3, Corollary 7.4]) we have that

$$i_K(T,\, \Omega_{\delta_2}) = 1.$$

Then, we conclude that T has a fixed point in $\bar{\Omega}_{\delta_2} \setminus \Omega_{\delta_1}$, that is, there exists at least a positive solution for problem (1)-(2). □

Consequently, we obtain the following

Corollary 3.3. *Assume that hypothesis* $(H1) - (H3)$ *hold. Then,*

(i) If $f_0 = \infty$ *and* $f^\infty = 0$, *then for all* $\lambda \in (0, \infty)$, *problem* (1)-(2) *has at least a positive solution.*

(ii) If $f_0 = \infty$ *and* $0 < f^\infty < \infty$, *then for all* $\lambda \in \left(0, \frac{1}{\Lambda\, f^\infty}\right)$, *problem* (1)-(2) *has at least a positive solution.*

(iii) If $0 < f_0 < \infty$ *and* $f^\infty = 0$, *then for all* $\lambda \in \left(\frac{1}{\Lambda\, f_0}, \infty\right)$, *problem* (1)-(2) *has at least a positive solution.*

***Remark* 3.4.** For the sake of completeness, we will give the exact expression of Λ_i, $i = 1, \dots, 5$:

$$\Lambda_1 = \frac{\alpha + 1}{6\,(1 - \alpha\,\eta)},$$

$$\Lambda_2 = \frac{1}{2\,(1 - \alpha\,\eta)},$$

$$\Lambda_3 = \frac{\alpha^2 - 2\,\alpha\left(\alpha^2 + 1\right)\eta + \left(\alpha^4 + 3\,\alpha^3 + \alpha + 1\right)\eta^2 - 2\,(\alpha\,(\alpha\,(\alpha\,(2\,\alpha - 3) + 5) - 3) + 1)\,\eta^3}{2\,(\alpha - 1)\,\eta\,(\alpha\,\eta - 1)\,(\alpha\,((\alpha - 1)\,\eta - 1) + \eta)} + \frac{\alpha^2(\alpha((\alpha - 2)\alpha + 3) - 1)\eta^4}{2\,(\alpha - 1)\,\eta\,(\alpha\,\eta - 1)\,(\alpha\,((\alpha - 1)\,\eta - 1) + \eta)},$$

$$\Lambda_4 = \frac{\eta^4\left(\alpha^3\,(2\,\eta - 3) + 3\,\alpha - 2\,\eta\right)}{12\,\alpha^5\,(\alpha\,\eta - 1)}\ \min\{\alpha - 1,\, 1\},$$

$$\Lambda_5 = \frac{(\alpha - 1)\,\eta^2\,(\alpha\,(\eta - 2) + \eta)}{2\,\alpha^2\,(\alpha\,\eta - 1)}.$$

4 Existence and multiplicity of solutions

In this section we will give some conditions to ensure the existence of multiple solutions of the boundary problem (1)-(2). To do that, we will use the fixed point index theory. Similar arguments have been applied in [2] to functional equations that only depends on the values of the solution u. First of all, we will compile some classical results regarding to this theory (see [1, 4] for more details).

Lemma 4.1. *Let D be an open bounded set with $D_K = D \cap K \neq \emptyset$ and $\bar{D}_K \neq K$. Assume that $F\colon \bar{D}_K \to K$ is a compact map such that $x \neq F\,x$ for $x \in \partial D_K$. Then the fixed point index $i_K(F, D_K)$ has the following properties:*

(1) If there exists $e \in K \setminus \{0\}$ such that $x \neq F\,x + \alpha\,e$ for all $x \in \partial D_K$ and all $\alpha > 0$, then $i_K(F, D_K) = 0$.

(2) If $\mu\,x \neq F\,x$ for all $x \in \partial D_K$ and for every $\mu \geq 1$, then $i_K(F, D_K) = 1$.

(3) Let D^1 be open in X with $\bar{D}^1 \subset D_K$. If $i_K(F, D_K) = 1$ and $i_K(F, D_K^1) = 0$, then F has a fixed point in $D_K \setminus \bar{D}_K^1$. The same result holds if $i_K(F, D_K) = 0$ and $i_K(F, D_K^1) = 1$.

We will consider the following sets:

$$K_\rho = \{u \in K;\, \|u\| < \rho\},$$

$$V_\rho = \left\{u \in K;\, \min_{t \in \left[\frac{\eta}{\alpha}, \eta\right]} u(t) < \rho,\, \min_{t \in \left[\frac{\eta}{\alpha}, \eta\right]} u'(t) < \rho,\, \|u''\|_\infty < \rho\right\}.$$

It is clear that

$$K_\rho \subset V_\rho \subset K_{\frac{\rho}{c}}$$

where $c = \min\{\kappa_0,\, \kappa_1,\, \kappa_2\}$.

In the two following lemmas we give some sufficient conditions to ensure that the index is either 1 or 0.

Lemma 4.2. *Let (see Remark 4.5)*

$$\frac{1}{m} = \max\left\{ \sup_{t\in[0,1]} \int_0^1 G(t,s)\,ds,\ \sup_{t\in[0,1]} \int_0^1 \frac{\partial G}{\partial t}(t,s)\,ds,\ \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial^2 G}{\partial t^2}(t,s)\right| ds \right\}$$

and

$$f^\rho = \sup\left\{ \frac{f(t,u,v,w)}{\rho};\ (t,u,v,w) \in [0,1]\times[0,\rho]\times[0,\rho]\times[-\rho,\rho] \right\}.$$

If there exists $\rho > 0$ such that

$$\lambda\,\frac{f^\rho}{m} < 1, \tag{I^1_ρ}$$

then $i_K(T, K_\rho) = 1$.

Proof. We will prove that $Tu \neq \mu\, u$ for all $u \in \partial K_\rho$ and for every $\mu \geq 1$.

Suppose, on the contrary, that there exist some $u \in \partial K_\rho$ and $\mu \geq 1$ such that

$$\mu\, u(t) = \lambda \int_0^1 G(t,s)\, f(s, u(s), u'(s), u''(s))\, ds.$$

Taking the supremum for $t \in [0,1]$, we obtain that

$$\begin{aligned}\mu\, \|u\|_\infty = \lambda \sup_{t\in[0,1]} \int_0^1 G(t,s)\, f(s,u(s),u'(s),u''(s))\, ds &\leq \lambda\,\rho\, f^\rho \sup_{t\in[0,1]} \int_0^1 G(t,s)\, ds \\ &\leq \lambda\,\rho\,\frac{f^\rho}{m} < \rho.\end{aligned}$$

On the other hand, we have that

$$\mu\, u'(t) = \lambda \int_0^1 \frac{\partial G}{\partial t}(t,s)\, f(s, u(s), u'(s), u''(s))\, ds$$

and so

$$\begin{aligned}\mu\, \|u'\|_\infty = \lambda \sup_{t\in[0,1]} \int_0^1 \frac{\partial G}{\partial t}(t,s)\, f(s,u(s),u'(s),u''(s))\, ds &\leq \lambda\,\rho\, f^\rho \sup_{t\in[0,1]} \int_0^1 \frac{\partial G}{\partial t}(t,s)\, ds \\ &\leq \lambda\,\rho\,\frac{f^\rho}{m} < \rho.\end{aligned}$$

Finally, it holds that

$$\mu\, u''(t) = \lambda \int_0^1 \frac{\partial^2 G}{\partial\, t^2}(t,s)\, f(s,u(s),u'(s),u''(s))\, ds$$

and so

$$\begin{aligned}\mu\, \|u''\|_\infty &\leq \lambda \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial^2 G}{\partial\, t^2}(t,s)\right| f(s,u(s),u'(s),u''(s))\, ds \leq \lambda\, \rho\, f^\rho \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial^2 G}{\partial\, t^2}(t,s)\right| ds \\ &\leq \lambda\, \rho\, \frac{f^\rho}{m} < \rho.\end{aligned}$$

As a consequence, it can be deduced that

$$\mu\, \rho = \mu\, \max\{\|u\|_\infty,\, \|u'\|_\infty,\, \|u''\|_\infty\} < \rho,$$

which is a contradiction with the assumption that $\mu \geq 1$. Therefore, the result is proved. □

Lemma 4.3. *Let (see Remark 4.5)*

$$\frac{1}{M} = \max\left\{ \inf_{t\in\left[\frac{\eta}{\alpha},\eta\right]} \int_{\frac{\eta}{\alpha}}^{\eta} G(t,s)\, ds,\ \inf_{t\in\left[\frac{\eta}{\alpha},\eta\right]} \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial\, G}{\partial\, t}(t,s)\, ds,\ \inf_{t\in\left[\frac{\eta}{\alpha},\eta\right]} \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial^2 G}{\partial\, t^2}(t,s)\, ds \right\}$$

and

$$f_\rho = \inf\left\{ \frac{f(t,u,v,w)}{\rho};\ (t,u,v,w) \in \left[\frac{\eta}{\alpha},\eta\right] \times \left[0,\frac{\rho}{\kappa_0}\right] \times \left[0,\frac{\rho}{\kappa_1}\right] \times [0,\rho] \right\}.$$

If there exists $\rho > 0$ such that

$$\lambda\, \frac{f_\rho}{M} > 1, \tag{I_ρ^0}$$

then $i_K(T, V_\rho) = 0$.

Proof. We will prove that there exists $e \in K \setminus \{0\}$ such that $u \neq Tu + \alpha\, e$ for all $u \in \partial V_\rho$ and all $\alpha > 0$.

Let us take $e(t) = 1$ and suppose that there exists some $u \in \partial V_\rho$ and $\alpha > 0$ such that $u = Tu + \alpha$. Then,

$$\begin{aligned}u(t) &= \lambda \int_0^1 G(t,s)\, f(s,u(s),u'(s),u''(s))\, ds + \alpha > \lambda \int_0^1 G(t,s)\, f(s,u(s),u'(s),u''(s))\, ds \\ &\geq \lambda \int_{\frac{\eta}{\alpha}}^{\eta} G(t,s)\, f(s,u(s),u'(s),u''(s))\, ds \geq \lambda\, \rho\, f_\rho \int_{\frac{\eta}{\alpha}}^{\eta} G(t,s)\, ds,\end{aligned}$$

$$\begin{aligned}u'(t) &= \lambda \int_0^1 \frac{\partial\, G}{\partial\, t}(t,s)\, f(s,u(s),u'(s),u''(s))\, ds \geq \lambda \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial\, G}{\partial\, t}(t,s)\, f(s,u(s),u'(s),u''(s))\, ds \\ &\geq \lambda\, \rho\, f_\rho \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial\, G}{\partial\, t}(t,s)\, ds\end{aligned}$$

and for $t \in \left[\frac{\eta}{\alpha}, \eta\right]$,

$$u''(t) = \lambda \int_0^1 \frac{\partial^2 G}{\partial t^2}(t,s)\, f(s, u(s), u'(s), u''(s))\, ds \geq \lambda \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial^2 G}{\partial t^2}(t,s)\, f(s, u(s), u'(s), u''(s))\, ds$$
$$\geq \lambda\, \rho\, f_\rho \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial^2 G}{\partial t^2}(t,s)\, ds.$$

Consequently, either $u(t) > \rho$, $u'(t) > \rho$ or $u''(t) > \rho$ for $t \in \left[\frac{\eta}{\alpha}, \eta\right]$, which is a contradiction. Therefore, we conclude the veracity of the result. □

From the previous lemmas, it is possible to formulate the following theorem, in which we give some conditions under which problem (3) has multiple solutions. In this case, we establish conditions to ensure the existence of one, two or three solutions. However, it must be pointed out that similar results can be formulated to ensure the existence of four or more solutions.

Theorem 4.4. *The integral equation* (3) *has at least one non trivial solution in K if one of the following conditions hold*

(C1) There exist ρ_1, $\rho_2 \in (0, \infty)$, $\frac{\rho_1}{c} < \rho_2$, such that $(I^0_{\rho_1})$ and $(I^1_{\rho_2})$ are verified.

(C2) There exist ρ_1, $\rho_2 \in (0, \infty)$, $\rho_1 < \rho_2$, such that $(I^1_{\rho_1})$ and $(I^0_{\rho_2})$ are verified.

The integral equation (3) *has at least two non trivial solutions in K if one of the following conditions hold*

(C3) There exist ρ_1, ρ_2, $\rho_3 \in (0, \infty)$, $\frac{\rho_1}{c} < \rho_2 < \rho_3$, such that $(I^0_{\rho_1})$, $(I^1_{\rho_2})$ and $(I^0_{\rho_3})$ are verified.

(C4) There exist ρ_1, ρ_2, $\rho_3 \in (0, \infty)$, with $\rho_1 < \rho_2$ and $\frac{\rho_2}{c} < \rho_3$, such that $(I^1_{\rho_1})$, $(I^0_{\rho_2})$ and $(I^1_{\rho_3})$ are verified.

The integral equation (3) *has at least three non trivial solutions in K if one of the following conditions hold*

(C5) There exist ρ_1, ρ_2, ρ_3, $\rho_4 \in (0, \infty)$, with $\frac{\rho_1}{c} < \rho_2 < \rho_3$ and $\frac{\rho_3}{c} < \rho_4$, such that $(I^0_{\rho_1})$, $(I^1_{\rho_2})$, $(I^0_{\rho_3})$ and $(I^1_{\rho_4})$ are verified.

(C6) There exist ρ_1, ρ_2, ρ_3, $\rho_4 \in (0, \infty)$, with $\rho_1 < \rho_2$ and $\frac{\rho_2}{c} < \rho_3 < \rho_4$, such that $(I^1_{\rho_1})$, $(I^0_{\rho_2})$, $(I^1_{\rho_3})$ and $(I^0_{\rho_4})$ are verified.

The proof of the previous result is an immediate consequence of the properties of the fixed point index.

***Remark* 4.5.** For the sake of completeness, we give the exact expression of the components which take part in the formulas of $\frac{1}{m}$ and $\frac{1}{M}$:

$$\int_0^1 G(t,s)\, ds = \frac{1}{12}\, t^2 \left(\frac{3\left(\alpha\, \eta^2 - 1\right)}{\alpha\, \eta - 1} - 2\, t \right)$$

and

$$\sup_{t\in[0,1]}\int_0^1 G(t,s)\,ds=\int_0^1 G\left(1,s\right)\,ds=\frac{\alpha\,\eta\,(2-3\,\eta)+1}{12\,(1-\alpha\,\eta)}.$$

Moreover,

$$\int_0^1 \frac{\partial\,G}{\partial\,t}(t,s)\,ds=\frac{t\,(\alpha\,\eta\,(\eta-t)+t-1)}{2\,(\alpha\,\eta-1)}$$

and

$$\sup_{t\in[0,1]}\int_0^1 \frac{\partial\,G}{\partial\,t}(t,s)\,ds=\int_0^1 \frac{\partial\,G}{\partial\,t}\left(1,s\right)\,ds=\frac{\alpha\,\eta\,(1-\eta)}{2\,(1-\alpha\,\eta)}.$$

Finally,

$$\int_0^1 \frac{\partial^2\,G}{\partial\,t^2}(t,s)\,ds=\begin{cases}\frac{\alpha\,\eta\,(\eta-2\,t)+2\,t-1}{2\,(\alpha\,\eta-1)}, & t\leq\alpha\,\eta\\ \frac{-2\,\alpha^2\,\eta^2+\alpha\,\eta\,(\eta+2\,t)-2\,(t-1)\,t-1}{2\,(\alpha\,\eta-1)}, & t>\alpha\,\eta\end{cases}$$

and

$$\sup_{t\in[0,1]}\int_0^1 \frac{\partial^2\,G}{\partial\,t^2}(t,s)\,ds=\int_0^1 \frac{\partial^2\,G}{\partial\,t^2}\left(0,s\right)\,ds=\frac{1-\alpha\,\eta^2}{2\,(1-\alpha\,\eta)}.$$

Now, it is easy to verify that

$$\frac{1}{m}=\max\left\{\frac{\alpha\,\eta\,(2-3\,\eta)+1}{12\,(1-\alpha\,\eta)},\,\frac{\alpha\,\eta\,(1-\eta)}{2\,(1-\alpha\,\eta)},\,\frac{1-\alpha\,\eta^2}{2\,(1-\alpha\,\eta)}\right\}=\frac{1-\alpha\,\eta^2}{2\,(1-\alpha\,\eta)}.$$

On the other hand, for $t\in\left[\frac{\eta}{\alpha},\eta\right]$,

$$\int_{\frac{\eta}{\alpha}}^{\eta} G(t,s)\,ds=\frac{1}{12}\left(\frac{2\,\eta^3}{\alpha^3}-\frac{6\,\eta^2\,t}{\alpha^2}+\frac{3\,\eta\,t^2\,(\alpha\,(\alpha\,(\alpha\,\eta+\eta-2)+\eta)-\eta)}{\alpha^2\,(\alpha\,\eta-1)}-2\,t^3\right)$$

and

$$\inf_{t\in\left[\frac{\eta}{\alpha},\eta\right]}\int_{\frac{\eta}{\alpha}}^{\eta} G(t,s)\,ds=\int_{\frac{\eta}{\alpha}}^{\eta} G\left(\frac{\eta}{\alpha},s\right)\,ds=\frac{(\alpha-1)\,\eta^3\,(\alpha\,(2-\alpha\,\eta)-\eta)}{4\,\alpha^4\,(1-\alpha\,\eta)}$$

In addition,

$$\int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial\,G}{\partial\,t}(t,s)\,ds=\frac{-\alpha^2\,\eta\,t\,(\alpha\,t+2)+\alpha^2\,t^2+\eta^2\,\left(\left(\alpha^3+\alpha^2+\alpha-1\right)\,t+1\right)-\alpha\,\eta^3}{2\,\alpha^2\,(\alpha\,\eta-1)}$$

and

$$\inf_{t\in\left[\frac{\eta}{\alpha},\eta\right]}\int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial\,G}{\partial\,t}(t,s)\,ds=\int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial\,G}{\partial\,t}\left(\frac{\eta}{\alpha},s\right)\,ds=\frac{(\alpha-1)\,\eta^2\,(\alpha\,(2-\alpha\,\eta)-\eta)}{2\,\alpha^3\,(1-\alpha\,\eta)}.$$

Finally,

$$\int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial^2\,G}{\partial\,t^2}(t,s)\,ds=\frac{\eta\,(\alpha\,(\alpha\,(\alpha\,\eta+\eta-2)+\eta)-\eta)}{2\,\alpha^2\,(\alpha\,\eta-1)}-t$$

and

$$\inf_{t\in\left[\frac{\eta}{\alpha},\eta\right]}\int_{\frac{\eta}{\alpha}}^{\eta}\frac{\partial^2 G}{\partial t^2}(t,s)\,ds=\int_{\frac{\eta}{\alpha}}^{\eta}\frac{\partial^2 G}{\partial t^2}(\eta,s)\;ds=\frac{(\alpha-1)^2(\alpha+1)\eta^2}{2\alpha^2(1-\alpha\eta)}.$$

Now we can calculate

$$\begin{aligned}\frac{1}{M}&=\max\left\{\frac{(\alpha-1)\eta^3(\alpha(2-\alpha\eta)-\eta)}{4\alpha^4(1-\alpha\eta)},\ \frac{(\alpha-1)\eta^2(\alpha(2-\alpha\eta)-\eta)}{2\alpha^3(1-\alpha\eta)},\ \frac{(\alpha-1)^2(\alpha+1)\eta^2}{2\alpha^2(1-\alpha\eta)}\right\}\\&=\frac{(\alpha-1)\eta^2(\alpha(2-\alpha\eta)-\eta)}{2\alpha^3(1-\alpha\eta)}.\end{aligned}$$

5 Results of non existence of solution

If the following theorem we give some conditions to ensure that the integral equation (3) has not non trivial solution in K.

Theorem 5.1. *If one of the following conditions hold*

(i) $f(t,x,y,z)<\tilde{m}\,\max\{x,\,y,\,|z|\}$ *for every* $t\in[0,1]$, $x,\,y\geq 0$ *and* $z\in\mathbb{R}$, *where*

$$\frac{1}{\tilde{m}}=\max\left\{\sup_{t\in[0,1]}\lambda\int_0^1 G(t,s)\,ds,\ \sup_{t\in[0,1]}\lambda\int_0^1\frac{\partial G}{\partial t}(t,s)\,ds,\ \sup_{t\in[0,1]}\lambda\int_0^1\left|\frac{\partial^2 G}{\partial t^2}(t,s)\right|\,ds\right\}$$

(ii) $f(t,x,y,z)>M\,x$ *for every* $t\in[a,b]\subset\left[\frac{\eta}{\alpha},\eta\right]$, *with* $a\neq b$, $x,\,y\geq 0$ *and* $z\in\mathbb{R}$, *where*

$$\frac{1}{M}=\inf_{t\in[a,b]}\lambda\int_a^b G(t,s)\,ds,$$

(iii) $f(t,x,y,z)>\tilde{M}\,y$ *for every* $t\in[a,b]\subset\left[\frac{\eta}{\alpha},\eta\right]$, *with* $a\neq b$, $x,\,y\geq 0$ *and* $z\in\mathbb{R}$, *where*

$$\frac{1}{\tilde{M}}=\inf_{t\in[a,b]}\lambda\int_a^b\frac{\partial G}{\partial t}(t,s)\,ds,$$

then the integral equation (3) *has not non trivial solution in* K.

Proof. We will prove just (i) and (ii) since item (iii) is totally analogous to (ii).

(i) Suppose, on the contrary, that there exists $u \in K$ such that $u = Tu$. Let $t_0 \in [0,1]$ be such that $\|u\|_\infty = u(t_0)$. Then,

$$\begin{aligned}\|u\|_\infty &= \lambda \int_0^1 G(t_0,s)\, f(s,u(s),u'(s),u''(s))\, ds \\ &< \lambda \int_0^1 G(t_0,s)\, \tilde{m}\, \max\{u(s),\, u'(s),\, |u''(s)|\}\, ds \\ &\leq \lambda\, \tilde{m}\, \|u\| \int_0^1 G(t_0,s)\, ds \leq \|u\|.\end{aligned}$$

Now, let $t_1 \in [0,1]$ such that $\|u'\|_\infty = u'(t_1)$. Then,

$$\begin{aligned}\|u'\|_\infty &= \lambda \int_0^1 \frac{\partial\, G}{\partial\, t}(t_1,s)\, f(s,u(s),u'(s),u''(s))\, ds \\ &< \lambda\, \|u\| \int_0^1 \frac{\partial\, G}{\partial\, t}(t_1,s)\, \tilde{m}\, \max\{u(s),\, u'(s),\, |u''(s)|\}\, ds \\ &\leq \lambda\, \tilde{m}\, \|u\| \int_0^1 \frac{\partial\, G}{\partial\, t}(t_1,s)\, ds \leq \|u\|.\end{aligned}$$

Finally, let $t_2 \in [0,1]$ such that $\|u''\|_\infty = |u''(t_2)|$. Then,

$$\begin{aligned}\|u''\|_\infty &= \left|\lambda \int_0^1 \frac{\partial^2\, G}{\partial\, t^2}(t_2,s)\, f(s,u(s),u'(s),u''(s))\, ds\right| \\ &\leq \lambda \int_0^1 \left|\frac{\partial^2\, G}{\partial\, t^2}(t_2,s)\right|\, f(s,u(s),u'(s),u''(s))\, ds \\ &< \lambda \int_0^1 \left|\frac{\partial^2\, G}{\partial\, t^2}(t_2,s)\right|\, \tilde{m}\, \max\{u(s),\, u'(s),\, |u''(s)|\}\, ds \\ &\leq \lambda\, \tilde{m}\, \|u\| \int_0^1 \left|\frac{\partial^2\, G}{\partial\, t^2}(t_2,s)\right|\, ds \leq \|u\|.\end{aligned}$$

Consequently, we reach to

$$\|u\| = \max\{\|u\|_\infty,\, \|u'\|_\infty\, \|u''\|_\infty\} < \|u\|,$$

which is a contradiction.

(ii) Suppose, on the contrary, that there exists $u \in K$ such that $u = Tu$. Let $t_0 \in [0,1]$ be such that $u(t_0) = \min_{t\in[a,b]} u(t)$. Then, for $t \in [a,b]$ we have that

$$\begin{aligned}u(t) &= \lambda \int_0^1 G(t,s)\, f(s,u(s),u'(s),u''(s))\, ds \geq \lambda \int_a^b G(t,s)\, f(s,u(s),u'(s),u''(s))\, ds \\ &> M\, \lambda \int_a^b G(t,s)\, u(s)\, ds.\end{aligned}$$

Therefore, we arrive at

$$u(t_0) = \min_{t\in[a,b]} u(t) > M \inf_{t\in[a,b]} \lambda \int_a^b G(t,s)\, u(s)\, ds \geq M\, u(t_0) \inf_{t\in[a,b]} \lambda \int_a^b G(t,s)\, ds = u(t_0),$$

which is a contradiction.

□

6 Examples

In this section we will consider several examples which show that the existence results proved in Theorems 3.2 and 4.4 are not comparable.

Example 6.1. Let's consider the problem with $f(t,x,y,z) = \frac{h(t)}{x^2+y^2+z^2}$, where $c_1 \geq h(t) \geq c_2 > 0$ for all $t \in [0,1]$, and η and α arbitrarily chosen, that is,

$$\begin{cases} -u^{(3)}(t) = \lambda \frac{h(t)}{(u(t))^2+(u'(t))^2+(u''(t))^2}, \quad t \in [0,1] \\ u(0) = u'(0) = 0, \quad u'(1) = \alpha\, u'(\eta). \end{cases}$$

In this case,

$$f_0 = \lim_{|x|,|y|,|z|\to 0} \min_{t\in[0,1]} \frac{f(t,x,y,z)}{|x|+|y|+|z|} = \lim_{|x|,|y|,|z|\to 0} \frac{\min_{t\in[0,1]} h(t)}{(x^2+y^2+z^2)\,(|x|+|y|+|z|)} = +\infty$$

and

$$f^\infty = \lim_{|x|,|y|,|z|\to \infty} \max_{t\in[0,1]} \frac{f(t,x,y,z)}{|x|+|y|+|z|} = \lim_{|x|,|y|,|z|\to \infty} \frac{\max_{t\in[0,1]} h(t)}{(x^2+y^2+z^2)\,(|x|+|y|+|z|)} = 0$$

so Theorem 3.2 assures that there exists at least a positive solution of the problem for all $\lambda > 0$.

On the other hand, let $\rho > 0$. Then,

$$f^\rho = \sup\left\{ \frac{h(t)}{\rho\,(x^2+y^2+z^2)};\ (t,x,y,z) \in [0,1]\times[0,\rho]\times[0,\rho]\times[-\rho,\rho] \right\} = \infty$$

so it is not possible to find a positive ρ such that $\lambda \frac{f^\rho}{m} < 1$ and, consequently, Theorem 4.4 can not be applied in this case.

Example 6.2. Let's consider the problem with $f(t,x,y,z) = h(t)\,(x^2+y^2+z^2+1)$, where $c_1 \geq h(t) \geq c_2 > 0$ for all $t \in [0,1]$, $\eta = \frac{1}{2}$ and $\alpha = \frac{3}{2}$, that is,

$$\begin{cases} -u^{(3)}(t) = \lambda\, h(t) \left((u(t))^2 + (u'(t))^2 + (u''(t))^2 + 1\right), \quad t \in [0,1] \\ u(0) = u'(0) = 0, \quad u'(1) = \frac{3}{2}\, u'(\frac{1}{2}). \end{cases}$$

In this case,

$$f_0 = \lim_{|x|,|y|,|z|\to 0} \min_{t\in[0,1]} \frac{f(t,x,y,z)}{|x|+|y|+|z|} = \lim_{|x|,|y|,|z|\to 0} \frac{(\min_{t\in[0,1]} h(t))\,(x^2+y^2+z^2+1)}{|x|+|y|+|z|} = \infty$$

and

$$f^\infty = \lim_{|x|,|y|,|z|\to \infty} \max_{t\in[0,1]} \frac{f(t,x,y,z)}{|x|+|y|+|z|} = \lim_{|x|,|y|,|z|\to \infty} \frac{(\max_{t\in[0,1]} h(t))\,(x^2+y^2+z^2+1)}{|x|+|y|+|z|} = \infty$$

so Theorem 3.2 can not be applied.

However, we will see that Theorem 4.4 lets us ensure the existence of at least one positive solution for certain values of λ.

Let $\rho_1,\ \rho_2 > 0$. Then,

$$f_{\rho_1} = \frac{1}{\rho_1} \inf_{t\in\left[\frac{1}{3},\frac{1}{2}\right]} h(t)$$

and

$$f^{\rho_2} = \frac{1+3\,\rho_2^2}{\rho_2} \sup_{t\in[0,1]} h(t).$$

Moreover, $\frac{1}{m} = \frac{5}{4}$ and $\frac{1}{M} = \frac{11}{108}$. As a consequence of $(C1)$ in Theorem 4.4, for any $\rho_1,\ \rho_2 > 0$ such that $\rho_1 < c\,\rho_2 = \frac{\rho_2}{90}$ and

$$\frac{108\,\rho_1}{11\,\inf_{t\in\left[\frac{1}{3},\frac{1}{2}\right]} h(t)} < \frac{4\,\rho_2}{5\,(1+3\,\rho_2^2)\,\sup_{t\in[0,1]} h(t)},$$

there exists at least a non trivial solution of problem (1)-(2) for all

$$\lambda \in \left(\frac{108\,\rho_1}{11 \inf\limits_{t\in\left[\frac{1}{3},\frac{1}{2}\right]} h(t)},\ \frac{4\,\rho_2}{5\,(1+3\,\rho_2^2) \sup\limits_{t\in[0,1]} h(t)} \right).$$

In particular, it can be deduced that there exists at least a non trivial solution of problem (1)-(2) for all

$$\lambda \in \left(0,\ \frac{2}{5\sqrt{3} \sup\limits_{t\in[0,1]} h(t)} \right).$$

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