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Review

Fisheries management in random environments: Comparison of harvesting policies for the logistic model

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ABSTRACT

We describe the growth dynamics of a harvested fish population in a random environment using a stochastic differential equation logistic model, where the harvest term depends on a constant or a variable fishing effort. We consider revenues to be proportional to the yield and costs to be quadratic in terms of effort. We compare the optimal expected profit obtained with two types of harvesting policies, one based on variable effort, which is inapplicable, and the other based on a constant effort, which is applicable and sustainable. We answer two new questions: (a) What is the constant effort that optimizes the expected profit per unit time? (b) How do the two policies compare in terms of performance? We show that, in a realistic situation, there is only a slight reduction in profit when choosing the applicable constant effort policy instead of the inapplicable policy with variable effort.

1. Introduction

In a deterministic environment, the logistic growth model for a harvested population can be described, in terms of the *per capita* growth rate, by the ordinary differential equation (ODE)

$$\frac{1}{X(t)}\frac{dX(t)}{dt} = r\left(1-\frac{X(t)}{K}\right) - qE(t), \quad X(0)=x, \tag{1}$$

where $X(t)$ is the population size at time t, measured as biomass or as number of individuals, $r > 0$ is the intrinsic growth rate of the population, $K > 0$ is the carrying capacity of the environment, $q > 0$ is the catchability coefficient, $E(t) \geq 0$ is the fishing effort and $X(0) = x > 0$ represents the population size at time 0. The yield per unit time from harvesting is denoted by $H(t) = qE(t)X(t)$.

However, the environment is subject to significant random fluctuations that affect the population *per capita* natural growth rate. The effect of these fluctuations can be approximated by a white noise $\sigma\varepsilon(t)$, where $\varepsilon(t)$ is a standard white noise and $\sigma > 0$ measures the strength of environmental fluctuations. Therefore, the above ODE Eq. (1) must be updated to the stochastic differential equation (SDE)

$$\frac{1}{X(t)}\frac{dX(t)}{dt} = r\left(1-\frac{X(t)}{K}\right) + \sigma\varepsilon(t) - qE(t), \quad X(0)=x,$$

which can be written in the standard format

$$dX(t) = rX(t)\left(1-\frac{X(t)}{K}\right)dt - qE(t)X(t)dt + \sigma X(t)dW(t), \quad X(0)=x, \tag{2}$$

where $W(t) = \int_0^t \varepsilon(s)ds$ is a standard Wiener process. We will assume that $r > \sigma^2/2$, otherwise the population will rendered extinct, even in the absence of harvesting (see Braumann, 1985).

Stochastic differential equations have been studied as a way to explain many physical, biological, economic and social phenomena. A particular case is the application (starting with the pioneering work of Beddington and May (1977)) to the growth dynamics of a harvested population subject to a randomly varying environment, with the purpose of obtaining optimal harvesting policies. Such policies usually are intended to maximize the expected yield or profit over a finite or infinite time horizon T. Since population size depends on the fishing effort, it seems natural to consider $E(t)$ as a control and apply optimal control techniques to achieve either yield or profit optimization, discounted by a social rate.

The profit per unit time can be defined as the difference between sales revenue and fishing costs, i.e.,

$$P(t) := R(t) - C(t),$$

where $R(t)$ and $C(t)$ are respectively the revenue and cost per unit time. We consider the revenue per unit time to be proportional to the yield,

$$R(t) = pH(t),$$

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where $p > 0$ is the price per unit of yield. The cost of harvest per unit time is assumed to depend on effort and to have a quadratic form given by

$$C(t) = c(E(t))E(t), \quad \text{with} \quad c(E(t)) = c_1 + c_2E(t),$$

where $c(E(t))$ is the cost per unit effort and $c_1, c_2 > 0$ are constants. The quadratic cost structure incorporates the case where the fishermen need to use less efficient vessels and fishing technologies or pay higher overtime wages to implement an extraordinary high effort (see Clark, 1976, 1990). However, other more complicated profit structures can be used, as well as other population growth models (for instance, the Gompertz model instead of the logistic). The methodology would be similar to the one we use in this paper.

In the deterministic case, there is a quite comprehensive account of optimal harvesting policies regarding yield or profit optimization (Clark, 1976, 1990). Under general assumptions, unless we are close to the end of a finite time horizon T, the optimal policy is to harvest with maximum intensity (which can be limited to a maximum harvesting effort or be unlimited) when the population is above a critical threshold and stop harvesting (zero effort) when the population is below that threshold. Once the threshold is reached, one just needs to keep the harvesting rate constant at an appropriate value so that the population remains at the threshold size. However, when the population is below the threshold, the fishery should be closed until the threshold is reached, which may take a while.

Stochastic optimal control methods were also applied to derive optimal harvesting strategies in a randomly varying environment (e.g. Alvarez, 2000a,b; Alvarez and Shepp, 1998; Arnason et al., 2004; Hanson and Ryan, 1998; Lande et al., 1994, 1995; Lungu and Øksendal, 1996; Suri, 2008). The optimal policy is similar to the deterministic case, i.e., harvest with maximum intensity when the population is above a critical threshold (not necessarily the same as in the deterministic case) and stop harvesting when below the threshold. However, after the threshold size is attained, due to random fluctuations of the environment, population size will keep varying. In this case, fishing effort must be adjusted at every instant, so that the size of the population does not go above the equilibrium value. Such policies imply that the effort changes frequently and abruptly, according to the random fluctuations of the population. Sudden frequent transitions between quite variable effort levels are not compatible with the logistics of fisheries. Besides, the period of low or no harvesting poses social and economical undesirable implications. In addition to such shortcomings, these optimal policies require the knowledge of the population size at every instant, to define the appropriate level of effort. The estimation of the population size is a difficult, costly, time consuming and inaccurate task and, for these reasons, and the others pointed above, these policies should be considered unacceptable and inapplicable.

In Braumann, 1981, 1985, 2008, a constant fishing effort, $E(t) \equiv E$, was assumed, providing an alternative approach to optimal harvesting. For a large class of models (including the logistic), it was found that, taking a constant effort in Eq. (2), there is, under mild conditions, a stochastic sustainable behaviour. Namely, the probability distribution of the population size at time t will converge, as $t \to +\infty$, to an equilibrium probability distribution (the so-called stationary or steady-state distribution) having a probability density function (the so-called stationary density). For the logistic model, the stationary density function was found, and the effort E that optimizes the steady-state yield was determined. The issue of profit optimization, however, was not addressed.

This paper considers this issue of profit optimization for the sustainable constant effort harvesting policy. This policy, rather than switching between large and small or null fishing effort, keeps a constant effort and is therefore compatible with the logistics of fisheries. Furthermore, this alternative policy does not require knowledge of the population size. However, it will result in a reduction of the profit when we compare it with the inapplicable optimal policy. We will examine if such reduction is appreciable or negligible.

Section 2 presents the approach to solve the optimization variable effort problem through a dynamic programming method. In Section 3 we present the alternative sustainable approach based on constant effort. Section 4 shows an application with realistic biological and fishing parameters in which the two policies are compared using numerical and Monte Carlo methods. We end up, in Section 5, with the conclusions.

Computations were carried out with R (http://r-project.org) and the code is available as supplementary material.

2. Variable effort optimal policy

This section will summarize the variable effort optimal policy under a randomly varying environment. We will start the optimization at time $t = 0$. Let $X(0) = x$ be the corresponding population size. Furthermore, harvesting continues up to the time horizon $T < +\infty$ and we work with the profit present value, i.e., future profits are discounted by a rate $\delta > 0$ accounting for interest rate and cost of opportunity losses and for other social rates. For a time t in the horizon $[0, T]$, we define

$$J(y, t) := \mathbb{E}\left[\int_t^T e^{-\delta(\tau-t)}P(\tau)d\tau | X(t) = y\right], \tag{3}$$

which is, at time t, the expected discounted future profits when the population size at that time is y.

We want to optimize the expected accumulated discounted profit earned by the harvester in the interval $[0, T]$,

$$\begin{aligned} V := J(x, 0) &= \mathbb{E}_x[\int_0^T e^{-\delta\tau}P(\tau)d\tau] \\ &= \mathbb{E}_x[\int_0^T e^{-\delta\tau}(pqX(\tau) - c_1 - c_2E(\tau))E(\tau)d\tau], \end{aligned}$$

where we denote $\mathbb{E}[\cdots|X(0) = x]$ by $\mathbb{E}_x[\cdots]$.

Given that $E(t)$ is used as a control, the optimization is carried out with respect to $E(t)$. A very important issue emerges when dealing with fishing effort: should one consider any constraints on effort? In practice, the effort is always non-negative, hence we must consider $E(t) \geq 0$. On the other hand, the number of tools, gears, hours, vessels and manpower is finite and limited, so we will consider effort to be constrained as

$$0 \leq E(t) \leq E_{\max} < \infty\ . \tag{4}$$

The optimization problem can be solved by stochastic dynamic programming theory through Bellman's principle of optimality (see Bellman, 1957). In terms of optimization theory, our problem is to find the effort that maximizes V, subject to the growth dynamics given by Eq. (2) and to the constrains on effort given by Eq. (4). In addition, from Eq. (3) we get $J(X(T), T) = 0$, which is a boundary condition. Summing up, the stochastic optimal control problem is to determine

$$V^* := J^*(x, 0) = \max_{\substack{E(\tau) \\ 0\leq\tau\leq T}} \mathbb{E}_x\left[\int_0^T e^{-\delta\tau}(pqX(\tau) - c_1 - c_2E(\tau))E(\tau)d\tau\right], \tag{5}$$

s.t.

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)dt - qE(t)X(t)dt + \sigma X(t)dW(t), \quad X(0) = x,$$

$$0 \leq E(t) \leq E_{\max} < \infty\ ,$$

$$J^*(X(T), T) = 0.$$

The maximizer, i.e., the effort function $E(t)$ that leads to the maximum V^*, will be called the optimal variable effort and will be denoted by $E^*(t)$.

To solve Eq. (5), one can employ stochastic dynamic programming to derive the Hamilton-Jacobi-Bellman (HJB) equation (see Hanson, 2007)

$$-\frac{\partial J^*(X(t),t)}{\partial t} = (pqX(t) - c_1 - c_2E^*(t))E^*(t) - \delta J^*(X(t),t) + \frac{\partial J^*(X(t),t)}{\partial X(t)}\left(rX(t)\left(1-\frac{X(t)}{K}\right) - qE^*(t)X(t)\right) + \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}\sigma^2 X^2(t), \quad (6)$$

where the optimal variable effort is given by

$$E^*(t) = \begin{cases} 0 & \text{if} \quad E^*_{\text{free}}(t) < 0 \\ E^*_{\text{free}}(t), & \text{if} \quad 0 \le E^*_{\text{free}}(t) \le E_{\max} \\ E_{\max}, & \text{if} \quad E^*_{\text{free}}(t) > E_{\max}, \end{cases}$$

with

$$E^*_{\text{free}}(t) = \left(p - \frac{\partial J^*(X(t),t)}{\partial X(t)}\right)\frac{qX(t)}{2c_2} - \frac{c_1}{2c_2}$$

being the unconstrained effort (see Hanson and Ryan, 1998).

Eq. (6) does not have an explicit solution and needs to be solved numerically. This was done (see Appendix A) by discretizing Eq. (6) according to a Crank-Nicolson scheme (as in Thomas, 1995).

3. Constant effort optimal policy

To apply a constant effort policy, one considers the particular case of Eq. (2) where $E(t) \equiv E$:

$$dX(t) = rX(t)(1 - X(t)/K)dt - qEX(t)dt + \sigma X(t)dW(t), \quad X(0) = x. \quad (7)$$

The solution of Eq. (7) is (e.g. Øksendal, 2003) a homogeneous diffusion process with drift and diffusion coefficients given, respectively, by

$$a(X) = rX(1 - X/K) - qEX \quad \text{and} \quad b(X) = \sigma^2 X^2.$$

As before, to avoid extinction, we will assume that $r - qE > \sigma^2/2$ (see Braumann, 1985). The boundaries of the state space are $X = 0$ and $X = +\infty$ and, from Braumann (1985), one can see that they are both non-attractive and thus the solution $X(t)$ exists, is unique and will stay inside the interval $(0, +\infty)$ for all $t \ge 0$. It also shows that there exists a stationary distribution for the population size. In other words, there exists an equilibrium probability distribution, with probability density function

$$f(X) = \frac{m(X)}{\int_0^{+\infty} m(z)dz}, \quad 0 < X < +\infty,$$

where

$$m(X) := \frac{1}{b(X)}\exp\left(\int_{x^*}^{X}\frac{2a(z)}{b(z)}dz\right)$$

is the speed density ($x^* > 0$ is an arbitrary constant). The speed density is proportional to the time the population trajectories spend near the population size X (see Karlin and Taylor, 1981). For our model Eq. (7), f is given by (see Braumann, 1985)

$$f(X) = \frac{1}{\Gamma(\rho)}\alpha^\rho X^{\rho-1}e^{-\alpha X}, \quad 0 < X < +\infty,$$

where $\Gamma(\cdot)$ represents the Gamma function, $\rho = \frac{2(r-qE)}{\sigma^2} - 1$ and $\alpha = \frac{2r}{K\sigma^2}$.

Let us be clear that this is a stochastic equilibrium, not a deterministic one. In fact, the population size $X(t)$, due to the environmental fluctuations, does not stabilize into an equilibrium size as in the deterministic case. It is the probability distribution of $X(t)$ that stabilizes into an equilibrium distribution with probability density function given by $f(X)$, the stationary density. For small values of the noise intensity σ, the mode of the stationary distribution is close to the stable equilibrium size of the deterministic model ($\sigma = 0$).

The existence of the stationary density plays a central role when defining the sustainable optimal policy that we are going to study below, allowing us to take a steady-state approach. We denote by X_∞ the random variable with density f, i.e., the random variable exhibiting the steady-state probabilistic behaviour. Therefore, the expected value of X_∞ is

$$\mathbb{E}[X_\infty] = \frac{\rho}{\alpha} = K\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right).$$

This is a good approximation of the expected size of the population $\mathbb{E}[X_t]$ for large t.

The steady-state structure of the profit per unit time is given by

$$P_\infty = R_\infty - C = (pqX_\infty - c_1 - c_2E)E,$$

where $R_\infty = pH_\infty = pqEX_\infty$ is the sale price and $C = c(E)E = (c_1 + c_2E)E$ is the fishing cost. The steady-state optimization problem consists in maximizing the expected sustainable profit per unit time, i.e.,

$$\max_{E\ge 0}\mathbb{E}[P_\infty] = \max_{E\ge 0}\left\{pqEK\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right) - c_1E - c_2E^2\right\}.$$

This is a second degree polynomial and its maximum with respect to E is obtained by the root of the first derivative. The resulting optimal sustainable effort, E^{**}, and the optimal expected sustainable profit per unit time, $\mathbb{E}[P^{**}_\infty] := \max_{E\ge 0}\mathbb{E}[P_\infty]$, are given, respectively, by

$$E^{**} = \frac{r}{2q}\cdot\frac{1 - \frac{\sigma^2}{2r} - \frac{c_1}{pqK}}{1 + \frac{c_2 r}{pq^2K}} \quad (8)$$

and

$$\mathbb{E}[P^{**}_\infty] = \frac{rK}{4}\cdot\frac{p\left(1 - \frac{\sigma^2}{2r} - \frac{c_1}{pqK}\right)^2}{1 + \frac{c_2 r}{pq^2K}}. \quad (9)$$

Note that the first factor $r/(2q)$ in the expression of E^{**} is the effort that leads to the MSY (Maximum Sustainable Yield) $rK/4$ in the deterministic case ($\sigma = 0$). The second factor in the expression of E^{**} corrects that value to account for the profit structure and the intensity σ of environmental fluctuations.

From expressions (8) and (9) one can see that an increase of the environmental fluctuations (large σ) will reduce both the optimal sustainable effort and the optimal expected sustainable profit per unit time. Regarding the economic parameters, one can see from these equations that an increase of p leads to an increase of both effort and profit. In contrast, an increase of costs, i.e., an increase on c_1 or on c_2, reduces the effort and the profit, as one would expect.

To determine the optimal effort E^{**} using expression (8), we just need estimates of the population and fishing parameters (from some stock assessment, e.g. Prager (1994)) and of the economic parameters (from an economic database collection). The quality of the estimate of E^{**} depends on the quality of the parameters estimates. This is a difficult problem, not specific to our models, but of deterministic models as well and is beyond the scope of this paper.

4. Comparison of policies

The comparison between the optimal harvesting policy (variable effort) and the sustainable optimal policy (constant effort) cannot be done directly, since the first one yields the optimal expected accumulated discounted profit over a finite time horizon, V^*, and the latter yields the optimal expected profit per unit time, $\mathbb{E}[P^{**}_\infty]$, for a large time horizon $T \to +\infty$. Even so, both policies can be compared if one uses Monte Carlo simulations. Let

$$P^*(t) := (pqX(t) - c_1 - c_2E^*(t))E^*(t)$$
$$P^{**}(t) := (pqX(t) - c_1 - c_2E^{**})E^{**}$$

be the profit per unit time for the two policies. We can compute, for both policies, four comparable quantities of interest (* refers to the optimal policy and ** refers to the optimal sustainable policy):

1. **Expected accumulated discounted profit** in the interval [0, T]:

$$V^* := \mathbb{E}_x\left[\int_0^T e^{-\delta\tau}P^*(\tau)d\tau\right],\quad V^{**} := \mathbb{E}_x\left[\int_0^T e^{-\delta\tau}P^{**}(\tau)d\tau\right]. \tag{10}$$

Since we cannot obtain the expectations $\mathbb{E}[\cdot]$ analytically nor the integrals $\int_0^T e^{-\delta\tau}P^*(\tau)d\tau$ and $\int_0^T e^{-\delta\tau}P^{**}(\tau)d\tau$, we use numerical methods. We approximate the integrals by discretizing time. The expectations are approximated by the average of 1000 Monte Carlo simulated trajectories. In the case of V^*, the integral for a trajectory can also be estimated by the value corresponding to $T = 0$ of the numerical solution J^* of the HJB equation. We did not use that method (which gives numerical values almost indistinguishable from the method we use) since we want a full comparability with V^{**}, for which such method is not possible.

2. **Expected accumulated undiscounted profit** in the interval [0, T]:

$$V_u^* = \mathbb{E}_x\left[\int_0^T P^*(\tau)d\tau\right],\quad V_u^{**} = \mathbb{E}_x\left[\int_0^T P^{**}(\tau)d\tau\right]. \tag{11}$$

3. **Average expected profit per unit time (average weighted by the discount factors)**:

$$\overline{P^*} = \frac{V^*}{\int_0^T e^{-\delta\tau}d\tau},\quad \overline{P^{**}} = \frac{V^{**}}{\int_0^T e^{-\delta\tau}d\tau}. \tag{12}$$

4. **Average expected profit per unit time (unweighted average)**:

$$\overline{P_u^*} = \frac{V_u^*}{T},\quad \overline{P_u^{**}} = \frac{V_u^{**}}{T}. \tag{13}$$

Note, for the constant effort optimal policy, that we determine the constant effort E^{**} that maximizes $\mathbb{E}[P_\infty]$, thus obtaining the optimal expected profit per unit time at the steady-state $\mathbb{E}[P_\infty^{**}]$ given by Eq. (9). This quantity is, due to the ergodicity of $X(t)$, also the limit as $T \to +\infty$ of both the time-average expected profit $\overline{P_u^{**}} = \mathbb{E}_x[\int_0^T P^{**}(\tau)d\tau]/T$ and (with probability 1) of the observed time-average profit $\int_0^T P^{**}(\tau)d\tau/T$ actually experienced by harvesters.

4.1. Basic scenario

The determination of the expected profit values Eqs. (10)–(13) requires numerical computations. Thus, instead of arbitrary parameters values, we have decided to set up a basic scenario S_0 using realistic values. We found a quite complete set of parameter values (namely r, K, q, p, c_1 and c_2) for the Pacific halibut (*Hippoglossus hippoglossus*) in Clark (1990) and Hanson and Ryan (1998) and these are the ones chosen for S_0. The other parameters, for which we had no information (E_{max}, σ, x and δ), where chosen at reasonable values and the time horizon was set at $T = 50$ years. The complete set of parameter values is listed in Table 1.

Table 1
Parameter values used in the simulations of the basic scenario S_0. The Standardized Fishing Unit (SFU) measure is defined in Hanson and Ryan (1998).

Item	Description	Values	Units
r	Intrinsic growth rate	0.71	year^{-1}
K	Carrying capacity	$80.5 \cdot 10^6$	kg
q	Catchability coefficient	$3.30 \cdot 10^{-6}$	SFU^{-1} year^{-1}
E_{max}	Maximum fishing effort	$0.7r/q$	SFU
σ	Strength of environmental fluctuations	0.2	year$^{-1/2}$
x	Initial population size	$0.5K$	kg
δ	Discount factor	0.05	year^{-1}
p	Price per unit yield	1.59	\$kg^{-1}
c_1	Linear cost parameter	$96 \cdot 10^{-6}$	\$SFU^{-1} year^{-1}
c_2	Quadratic cost parameter	$0.10 \cdot 10^{-6}$	\$SFU^{-2} year^{-1}
T	Time horizon	50	years

Table 2
Numerical comparison between policies of the expected profits 1. to 4. (see expressions (10)–(13)) for the basic scenario S_0. The percent relative difference between the two policies is denoted by Δ. Besides the expected values, we also present the standard deviations (sd). Units are in million dollars for 1. and 2. and in million dollars per year for 3. and 4.

	Optimal policy	Optimal sustainable policy	Δ (%)
1.	$V^* \simeq 413.586$ (sd = 38.32)	$V^{**} \simeq 400.313$ (sd = 35.24)	−3.2
2.	$V_u^* \simeq 1129.130$ (sd = 88.63)	$V_u^{**} \simeq 1073.867$ (sd = 88.54)	−4.9
3.	$\overline{P_u^*} \simeq 22.529$ (sd = 2.09)	$\overline{P^{**}} \simeq 21.806$ (sd = 1.92)	−3.2
4.	$\overline{P_u^*} \simeq 22.583$ (sd = 1.77)	$\overline{P_u^{**}} \simeq 21.477$ (sd = 1.77)	−4.9

The resulting profit values Eqs. (10)–(13) for this basic scenario are shown in Table 2, where the first column refers to the optimal variable effort policy, the second column refers to the optimal constant effort policy, and the third column indicates the percent loss in the profit value when using the second policy instead of the first.

The first line of Table 2 compares the expected accumulated discounted profit Eq. (10) over the time horizon $T = 50$ years, V^* or V^{**} according to the policy used. One can see that the second policy implies a reduction in the expected profit of only 3.2% compared to the first policy. If one forgets depreciation and looks at the expected accumulated undiscounted profit Eq. (11), it shows a 4.9% expected profit reduction when comparing the second policy with the first. Obviously, the percent reductions are the same for the corresponding profits per year Eqs. (12) and (13), obtained by taking time averages of these quantities over the 50 year horizon. Therefore, the profit reductions that occur when considering a constant effort instead of a variable effort are quite small and, with a constant effort, the fishery manager does not need to worry about changes on the number of vessels, number of hooks or number of hours worked (just to name a few possibilities). This is extremely advantageous in terms of implementation and avoids out-of-model costs such as the purchase of new equipment to sustain increased effort periods or payment of unemployment benefits during effort reduction periods.

Besides the average profits, it is also interesting to look at their standard deviations, which measure the variability across the simulated trajectories. The variability of both policies is very similar (Table 2), with the optimal sustainable policy having a slightly lower variability.

Fig. 1 shows, for scenario S_0, what will happen when applying the optimal variable effort harvesting policy (left side) and the optimal constant effort sustainable policy (right side), in terms of the evolution, from time $t = 0$ to time $t = T = 50$ years, of the following quantities:

- Population size $X(t)$, on top;
- Optimal effort, in the middle: $E^*(t)$ (left) and E^{**} (right);
- Profit per unit time, at the bottom: $P^*(t)$ (left) and $P^{**}(t)$ (right).

The thin lines of Fig. 1 show one randomly chosen trajectory, corresponding to a possible particular environmental behaviour. It shows what the harvester would typically observe. The figure also presents the expected values of the variables, which are averages taken over all possible environmental behaviours (the one effectively seen and all the others that might have occurred); dashed lines show the exact values (only available for the constant effort policy) and solid lines show good approximations (based on averaging over a 1000 simulated trajectories). Looking at what the harvester typically experiences (thin lines in Fig. 1), one can see that the two policies behave quite differently. While for the constant effort policy, we apply the same effort E^{**} irrespective of the population size path and of the environmental conditions (middle right, where, of course, we cannot distinguish between the solid, the thin and the dashed lines), in the optimal policy the effort $E^*(t)$ changes quite frequently and abruptly (thin line on the middle left). We see that its values depend on time and on the fish population size (which is influenced by the random fluctuations of environmental

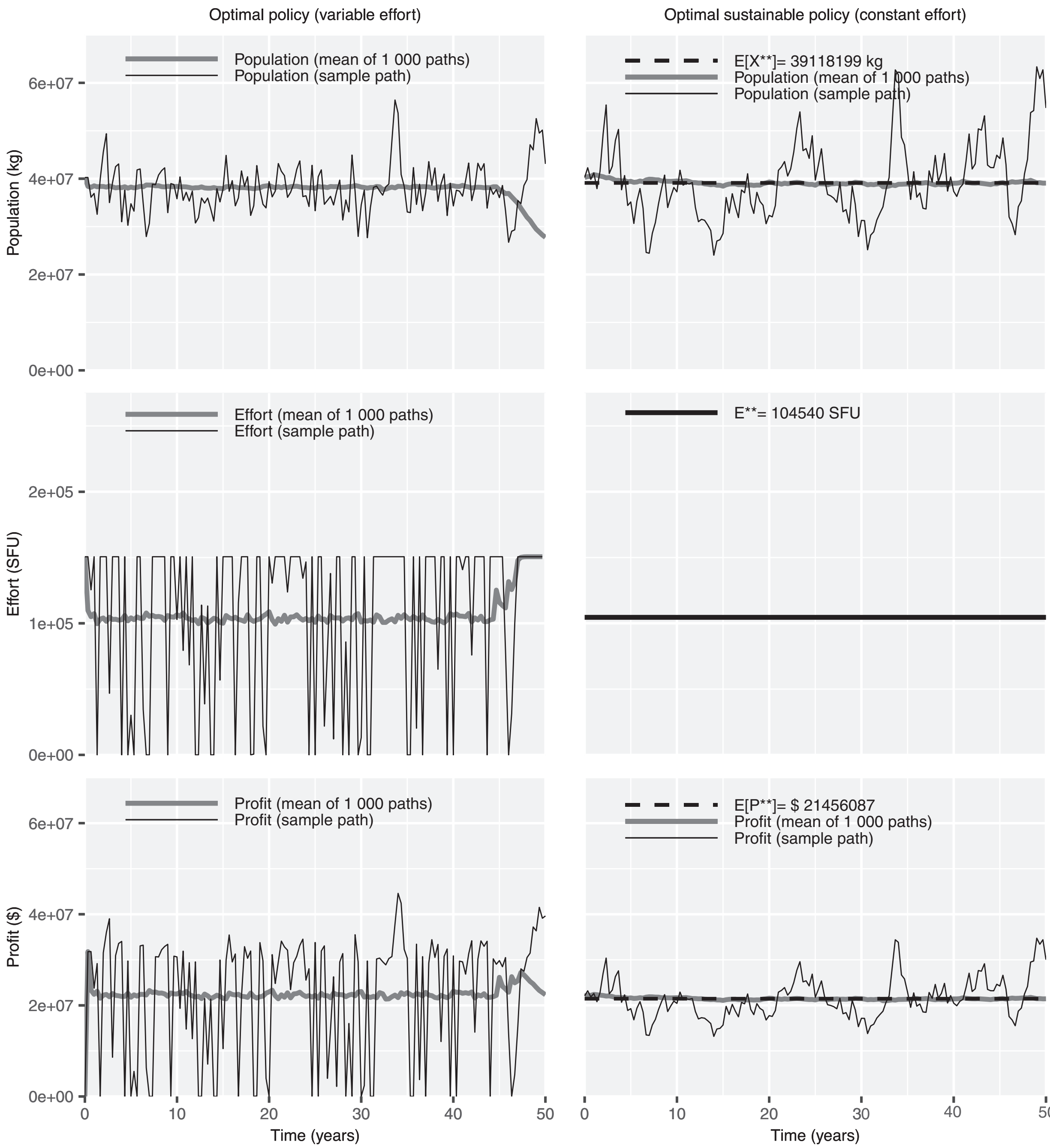

Fig. 1. Basic scenario S_0: mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side.

conditions), requiring constant evaluation of the fish stock. Furthermore, it exhibits periods of no or low harvesting, posing social burdens and possible extra costs of unemployment compensation (not considered in our cost structure), and periods of harvesting at the maximum effort E_{max}, which may also involve extra costs (e.g. investment in backup equipment or hiring of extra employees not trained in fishing).

Besides looking at the variation of the effort over time, it is also interesting to look at the time variability experienced by the harvester on the profit per unit time. For the basic scenario S_0, if we look at the thin lines at the bottom of Fig. 1 (corresponding to the environmental conditions randomly selected), we see that the optimal policy has frequent periods of zero profit (the periods of zero effort) and a much larger profit variability over time. A good measure of this variability for the chosen trajectory is the sample standard deviation of the profit per unit time values observed at the time instants of the simulations. Such standard deviation is 13.79 million dollars per year for the optimal policy and only 4.34 million dollars per year for the optimal sustainable policy, which provides the harvester with a much steadier profit. Similar results hold when we select other trajectories.

One can also see in Fig. 1 (left side) that the optimal variable effort policy exhibits a possibly dangerous effect near the time horizon, consisting in a considerable drop of the average population size (see solid line on top left), corresponding to an increase on the average effort (see solid line on middle left). This final effort increase is quite natural. Since “there is no tomorrow”, it is better profitwise to harvest as much as is profitable “now”, without worrying about stock preservation for near future fishing.

With the optimal sustainable policy, population size is driven to an

Table 3
Profit values for the different scenarios S_i ($i = 1, \ldots, 18$). The parameters are the same as those indicated in Table 1 for the base scenario S_0, with the exception of the parameter indicated in the second column, which has the value shown there (using the units of Table 1). The profit values are in million dollars for V^*, V^{**}, V_u^* and V_u^{**} and in million dollars per year for $\overline{P^*}$, $\overline{P^{**}}$, $\overline{P_u^*}$ and $\overline{P_u^{**}}$. The Δ columns indicate the percent relative difference between the two policies.

S_i	Changed parameter	V^*	V^{**}	$\overline{P^*}$	$\overline{P^{**}}$	Δ (%)	V_u^*	V_u^{**}	$\overline{P_u^*}$	$\overline{P_u^{**}}$	Δ (%)
S_0		413.586	400.313	22.529	21.806	−3.2	1129.130	1073.867	22.583	21.477	−4.9
S_1	$x = 0.25K$	378.677	362.817	20.627	19.763	−4.2	1092.856	1030.569	21.857	20.611	−5.7
S_2	$x = 0.75K$	440.168	422.449	23.977	23.011	−4.0	1157.066	1098.360	23.141	21.967	−5.1
S_3	$E_{max} = 0.5r/q$	400.383	400.313	21.809	21.806	0.0	1092.263	1073.867	21.845	21.477	−1.7
S_4	$E_{max} = 0.9r/q$	415.555	400.313	22.636	21.806	−3.7	1139.690	1073.867	22.794	21.477	−5.8
S_5	$\delta = 0.00$	1130.837	1081.011	22.617	21.620	−4.4	1130.837	1073.867	22.617	21.477	−5.0
S_6	$\delta = 0.10$	226.805	219.400	22.834	22.089	−3.3	1113.768	1073.867	22.275	21.477	−3.6
S_7	$r = 0.10$	62.853	46.925	3.424	2.556	−25.3	148.820	122.197	2.976	2.444	−17.9
S_8	$r = 0.40$	233.516	218.961	12.720	11.927	−6.2	628.749	582.937	12.575	11.659	−7.3
S_9	$\sigma = 0.10$	418.497	416.128	22.796	22.667	−0.6	1138.575	1120.752	22.772	22.415	−1.6
S_{10}	$\sigma = 0.40$	379.404	337.350	20.667	18.376	−11.1	1026.306	887.617	20.526	17.752	−13.5
S_{11}	$T = 10$	187.693	173.281	23.851	22.020	−7.7	238.555	218.376	23.855	21.838	−8.5
S_{12}	$T = 25$	326.509	311.682	22.881	21.842	−4.5	574.000	539.991	22.960	21.600	−5.9
S_{13}	$p = 1.19$	310.182	300.229	16.896	16.354	−3.2	846.828	805.385	16.937	16.108	−4.9
S_{14}	$p = 1.99$	516.990	500.397	28.161	27.257	−3.2	1411.431	1342.348	28.229	26.847	−4.9
S_{15}	$c_1 = 72 \cdot 10^{-6}$	413.586	400.313	22.529	21.806	−3.2	1129.130	1073.867	22.583	21.477	−4.9
S_{16}	$c_1 = 120 \cdot 10^{-6}$	413.586	400.313	22.529	21.806	−3.2	1129.130	1073.866	22.583	21.477	−4.9
S_{17}	$c_2 = 0.75 \cdot 10^{-7}$	413.593	400.319	22.529	21.806	−3.2	1129.149	1073.881	22.583	21.478	−4.9
S_{18}	$c_2 = 1.25 \cdot 10^{-7}$	413.579	400.307	22.528	21.805	−3.2	1129.111	1073.852	22.582	21.477	−4.9

equilibrium probability distribution with an average population size higher than the one of the variable effort policy. This expected size at equilibrium is the mean value of the stationary distribution referred in Section 3. With the constant effort policy, there is no decay of the expected population size near the end of the time horizon.

4.2. Alternative scenarios

We will now evaluate the influence of the parameters, by considering alternative values, usually one lower and one higher than those in Table 1 for S_0. For r and T, since S_0 values are high, we consider as alternatives two lower values. Since x and K can be scaled together, we choose only to change x. The same applies to r and q, so we did not study changes in q.

This leads to alternative scenarios S_1 to S_{18} (Table 3). For each scenario, we have computed the profit values as in Table 2 and a similar set of images as in Fig. 1. The profit values are shown in Table 3 and the figures are presented as supplementary material. The figures for scenarios S_{15}, S_{16}, S_{17} and S_{18} are not shown since they are almost undistinguishable from Fig. 1 of scenario S_0.

A general comment concerning Table 3 is that, for almost all the scenarios, the percent reduction of profit Δ incurred by using the optimal constant effort policy instead of the optimal variable effort policy is quite small, of the order of 0% to 6% for discounted profits and of 2% to 7% for undiscounted profits. The exceptions are the scenarios S_7, S_{10}, and, in a quite more attenuated way, S_{11}, which exhibit quite higher Δ values. S_7 corresponds to a substantially lower value of r than in the basic scenario. S_{10} corresponds to a quite higher value of the intensity of environmental fluctuations. S_{11} has a very short time horizon ($T = 10$ years), far away from the steady-state for which the constant effort policies were designed. Actually, designing non-sustainable policies, like the optimal variable effort policies considered here, for such short time horizons is ill-advised since these policies care not about future preservation of the stock.

To check the effect of changes in a given parameter, we can compare the results for the basic scenario S_0 with the results of the scenarios corresponding to alternative values of such parameter, using Table 3 and the figures (shown as supplementary material) associated to those scenarios.

Changing the initial population x (scenarios S_1 and S_2) affects, for the optimal variable effort policy, the expected values of population size, harvesting effort and profit per unit time. This happens only at the start of the projection period and is due to the longer time it takes for these expected values to approach their main trends (compare the left side of Fig. 1 with the left sides of Supplementary Figures 1 and 2). As for the constant effort policy, since the effort is designed assuming a steady-state, x has no effect on effort. However, like in the variable effort policy, it has an effect on the expected population size and on the expected profit per unit time at the beginning (before the approach to the mean trend), but, since the process is ergodic, has no effect in the long-term (as $T \to \infty$).

Constraining the maximum effort to 0.5 r/q (scenario S_3), which is very close to E^{**}, almost mimics the behaviour of the constant effort policy. In fact, in this scenario, the difference between V^* and V^{**} is very small. On Supplementary Figure 3 we can confirm the similarities between the two policies. Raising the maximum effort to 0.9 r/q (scenario S_4) gives, for the optimal variable effort policy, similar results to the ones in scenario S_0, although with slightly higher profit due to the fact that the restriction on effort is milder. Obviously, for the optimal constant effort policy, if $E_{max} \geq E^{**}$, the value of E_{max} is irrelevant.

Changing δ (scenarios S_5 and S_6) will, of course, not change the undiscounted profits for the optimal constant effort policy since this policy is designed to optimize the steady-state undiscounted profit per unit time. It has, however, a large effect on the accumulated discounted profit of both constant effort and variable effort policies, but has little effect on average profits per unit time (both discounted and undiscounted). It is also slight the effect, under the variable effort optimal policy, on the time evolution of the expected values of population size, optimal variable effort and profit per unit time (see left sides of Fig. 1 and Supplementary Figures 5 and 6).

Intrinsic growth rates lower than 0.7 (scenarios S_7 and S_8) imply lower biomass growth and, consequently, also lower profit values, since the optimal harvesting rates will be smaller for both policy types.

Comparing scenarios S_9 and S_{10} with S_0 shows that a higher intensity of environmental fluctuations reduces the expected profit for both types of policies, although the effect is quite mild. Contrary to the average, the influence on sample trajectories (which is what is experienced) of population size is quite profound. Although averages do not change much, fluctuations of the population size about its average will be more intense when σ is high and will almost fade away as σ approaches zero (deterministic environment). Obviously, sample paths of the profit per unit time will respond to changes in population size and, in the case of the variable effort policy, the same happens to the effort. For the sustainable policy, we had already seen on Section 3, from the steady-state expressions (8) and (9) of the optimal effort E^{**} and the optimal profit per unit time $\mathbb{E}[P_{\infty}^{**}]$, that these quantities decrease with a higher σ.

As already mentioned, when the terminal time T decreases (scenarios S_{11} and S_{12}), the differences between the two policies are more pronounced. In fact, the optimal sustainable policy 'needs more time' to get close to the stochastic steady-state. The accumulated profits are, in relation to the base scenario, much smaller since we are talking about shorter periods of time. However, the average profits per unit time are very close to the ones in the S_0 scenario.

A decrease of 25% (scenario S_{13}) or increase of 25% (scenario S_{14}) in the unit price p will have an effect of similar magnitude in profit. This is due to the fact that profit is dominated by the effect of price p, since the cost parameters c_1 and c_2 have, in this case, a low magnitude. For the same reason, variations in the cost parameters, c_1 and c_2, have very little influence on profit values.

5. Conclusions

Fish populations live in randomly varying environments and the effect of that variability on fish dynamics has to be taken into account when choosing optimal harvesting policies. For that reason, the use of a stochastic differential equation version of the classical logistic model with harvesting is appropriate.

The typical approach in the literature is, as for deterministic models, to use control theory (the harvesting effort being the control) to maximize the expected accumulated discounted profit over some time horizon T. We have used a profit structure where revenues per unit time are proportional to the yield and costs per unit time are quadratic functions of the effort.

In the stochastic case, the population fluctuations induced by the randomly varying environment lead to optimal policies with a highly variable effort (with frequent periods of no or low harvesting, or of harvesting at the maximum possible rate). This is not compatible with the logistics of fishing and causes social and economical problems (intermittent unemployment is just one of them). Besides, since population size has random fluctuations, knowledge of the population size at all times is required to determine the optimal effort and this is not feasible.

So, we consider as an alternative, sustainable constant effort fishing policies, which are extremely easy to implement and lead to a stochastic steady-state. We determine the constant effort that maximizes the expected profit per unit time at the steady-state. One might think that a constant effort policy would result in a substantial profit reduction compared with the optimal variable effort policy, but we have shown this is not the case.

In order to compare the two harvesting policies, we have considered four ways of evaluating the expected profit (discounted or not discounted and, for each case, the total accumulated value over a time horizon or the value per year).

We set up a basic scenario S_0 using a 50 year time horizon and parameter values based on the Pacific halibut (*Hippoglossus hippoglossus*) data. To compute expected profits with good accuracy, we simulated 1000 trajectories of the fish population size. We have also considered alternative scenarios corresponding to changes of the different parameter values used in scenario S_0, to see the influence of such parameters.

For the basic scenario, the constant effort policy implies a slight reduction in the average expected accumulated discounted profit of only 3.2% compared to the variable effort policy. In terms of the undiscouted profit, the reduction is 4.9%. The corresponding profits per year present, obviously, the same percent reductions. For the alternative scenarios, the percent reduction in the expected profits ranges from 0% to 7%, except in some extreme scenarios.

So, optimal constant effort policies will typically involve a slight reduction in profit compared to the optimal variable effort policies, but are quite easy to implement and do not have the shortcomings of the optimal variable effort policies.

Fishery managers/regulators do not have to worry about logistic problems of effort changes and equipment requirements and employment are kept at a constant level. In contrast, optimal variable effort policies have frequent strong changes in effort, including frequent closings of the fishery, posing logistic applicability problems, producing social burdens and out-of-model costs (such as unemployment compensations) and leading to a great instability in the profit earned by the harvester.

Furthermore, unlike optimal variable effort policies, in the optimal constant effort policies there is no need to keep adjusting the effort to the randomly varying population size, and so there is no need to determine the size of the population at all times. Constant effort policies also lead the probability distribution of population size to a sustainable equilibrium with an average population size higher than the final average size of the optimal harvesting policy.

These methodologies can be applied to similar comparison studies and other fishery models. This will be the subject of a further paper.

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Appendix A. Crank-Nicolson numerical solution of Eq. (6)

To discretize equation Eq. (6), we consider that:

- the optimization starts at time $t = 0$ and ends at time $t = T < +\infty$;
- the time interval is uniformly partitioned as

$0 = t_0 < t_1 < \ldots < t_n = T$,

with $t_{j+1} - t_j = \Delta t = T/n,\ j = 0, 1, \ldots, n - 1$;

- the state variable takes values within the interval $[0, 2K]$, which is uniformly partitioned as

$0 = X_0 < X_1 < \ldots < X_m = 2K$,

with $X_{i+1} - X_i = \Delta X = 2K/m,\ \ i = 0, 1, \ldots, m - 1$;

- since we have a boundary condition $J^*(X(T), T) = 0$, which is terminal instead of initial, the computation uses time moving backwards from T to 0;
- $J^*_{i,j} := J^*(X_i, t_j),\ \ E^*_{i,j} := E^*(X_i, t_j)$, with $0 \le i \le m$ and $0 \le j \le n$.

The following derivatives are discretized using a Crank-Nicolson scheme (as in Thomas, 1995 and Suri, 2008):

- For $1 \le i \le m - 1$ and $0 \le j \le n - 1$,

$$\frac{\partial J^*_{i,j}}{\partial t} = \frac{J^*_{i,j+1} - J^*_{i,j}}{\Delta t}.$$

$$\frac{\partial J^*_{i,j}}{\partial X} = \frac{1}{2}\left(\frac{J^*_{i+1,j+1} - J^*_{i-1,j+1}}{2\Delta X} + \frac{J^*_{i+1,j} - J^*_{i-1,j}}{2\Delta X}\right).$$

$$\frac{\partial^2 J^*_{i,j}}{\partial X^2} = \frac{1}{2}\left(\frac{(J^*_{i+1,j+1} - J^*_{i,j+1}) + (J^*_{i-1,j+1} - J^*_{i,j+1})}{\Delta X^2} + \frac{(J^*_{i+1,j} - J^*_{i,j}) + (J^*_{i-1,j} - J^*_{i,j})}{\Delta X^2}\right).$$

- For $i = m$ and $0 \le j \le n - 1$,

$$\frac{\partial J^*_{m,j}}{\partial X} = \frac{\partial J^*_{m-1,j}}{\partial X} + \Delta X \frac{\partial^2 J^*_{m-1,j}}{\partial X^2} = \frac{1}{2}\left(\frac{3J^*_{m,j} - 4J^*_{m-1,j} + J^*_{m-2,j}}{2\Delta X} + \frac{3J^*_{m,j+1} - 4J^*_{m-1,j+1} + J^*_{m-2,j+1}}{2\Delta X}\right).$$

$$\frac{\partial^2 J^*_{m,j}}{\partial X^2} = \frac{2J^*_{m,j+1} - 5J^*_{m-1,j+1} + 4J^*_{m-2,j+1} - J^*_{m-3,j+1}}{2\Delta X^2} + \frac{2J^*_{m,j} - 5J^*_{m-1,j} - 4J^*_{m-2,j} - J^*_{m-3,j}}{2\Delta X^2}.$$

Therefore, the discretized version of Eq. (6) is:

- For $1 \le i \le m - 1$ and $0 \le j \le n - 1$,

$$\frac{J^*_{i,j} - J^*_{i,j+1}}{\Delta t} = (pqX_i - c_1 - c_2E^*_{i,j+1})E^*_{i,j+1} - \delta\left(\frac{J^*_{i,j+1}}{2} + \frac{J^*_{i,j}}{2}\right) + \left(\frac{J^*_{i+1,j+1} - J^*_{i-1,j+1}}{4\Delta X} + \frac{J^*_{i+1,j} - J^*_{i-1,j}}{4\Delta X}\right) \times \left(rX_i\left(1 - \frac{X_i}{K}\right) - qE^*_{i,j+1}X_i\right) + \left(\frac{J^*_{i+1,j+1} - 2J^*_{i,j+1} + J^*_{i-1,j+1}}{4\Delta X^2} + \frac{J^*_{i+1,j} - 2J^*_{i,j} + J^*_{i-1,j}}{4\Delta X^2}\right)\sigma^2 X_i^2\ ;$$

- For $i = m$ and $0 \le j \le n - 1$,

$$\frac{J^*_{m,j} - J^*_{m,j+1}}{\Delta t} = (pqX_m - c_1 - c_2E^*_{m,j+1})E^*_{m,j+1} - \delta\left(\frac{J^*_{m,j+1}}{2} + \frac{J^*_{m,j}}{2}\right) + \left(\frac{3J^*_{m,j+1} - 4J^*_{m-1,j+1} + J^*_{m-2,j+1}}{4\Delta X} + \frac{3J^*_{m,j} - 4J^*_{m-1,j} + J^*_{m-2,j}}{4\Delta X}\right) \times \left(rX_m\left(1 - \frac{X_m}{K}\right) - qE^*_{m,j+1}X_m\right) + \left(\frac{2J^*_{m,j+1} - 5J^*_{m-1,j+1} + 4J^*_{m-2,j+1} - J^*_{m-3,j+1}}{4\Delta X^2} + \frac{2J^*_{m,j} - 5J^*_{m-1,j} + 4J^*_{m-2,j} - J^*_{m-3,j}}{4\Delta X^2}\right)\sigma^2 X_m^2.$$

We first compute the free optimal effort using the following discretization:

- For $1 \le i \le m - 1$ and $0 \le j \le n - 1$,

$$E^*_{free\ i,j+1} = \left(p - \frac{J^*_{i+1,j+1} - J^*_{i-1,j+1}}{2\Delta X}\right)\frac{qX_i}{2c_2} - \frac{c_1}{2c_2};$$

- For $i = m$ and $0 \le j \le n - 1$,

$$E^*_{free\ m,j+1} = \left(p - \frac{3J^*_{m,j+1} - 4J^*_{m-1,j+1} + J^*_{m-2,j+1}}{2\Delta X}\right)\frac{qX_m}{2c_2} - \frac{c_1}{2c_2}.$$

In each iteration, we then correct the free optimal effort to obtain the constrained optimal effort. Namely, for $1 \le i \le m$ and $0 \le j \le n$:

- If $E^*_{\text{free}_{i,j+1}} < 0$, then $E^*_{i,j+1} = 0$;
- If $0 \le E^*_{\text{free}_{i,j+1}} \le E_{\max}$, then $E^*_{i,j+1} = E^*_{\text{free}_{i,j+1}}$;
- If $E^*_{\text{free}_{i,j+1}} > E_{\max}$, then $E^*_{i,j+1} = E_{\max}$.

The discretized version of the HJB equation can be written as a system of M equations:

- For $1 \le i \le m - 1$ and $0 \le j \le n - 1$,

$$\left(\frac{(rX_i(1 - X_i/K) - qE^*_{i,j+1}X_i)\Delta t}{4\Delta X} - \frac{\sigma^2 X_i^2 \Delta t}{4\Delta X^2}\right)\cdot J^*_{i-1,j} + \left(1 + \frac{\delta\Delta t}{2} + \frac{\sigma^2 X_i^2 \Delta t}{2\Delta X^2}\right)\cdot J^*_{i,j} - \left(\frac{(rX_i(1 - X_i/K) - qE^*_{i,j+1}X_i)\Delta t}{4\Delta t} + \frac{\sigma^2 X_i^2 \Delta t}{4\Delta X^2}\right)\cdot J^*_{i+1,j}$$
$$= -\left(\frac{(rX_i(1 - X_i/K) - qE^*_{i,j+1}X_i)\Delta t}{4\Delta X} - \frac{\sigma^2 X_i^2 \Delta t}{4\Delta X^2}\right)\cdot J^*_{i-1,j+1} + \left(1 - \frac{\delta\Delta t}{2} - \frac{\sigma^2 X_i^2 \Delta t}{2\Delta X^2}\right)\cdot J^*_{i,j+1} + \left(\frac{(rX_i(1 - X_i/K) - qE^*_{i,j+1}X_i)\Delta t}{4\Delta X} + \frac{\sigma^2 X_i^2 \Delta t}{4\Delta X^2}\right)\cdot J^*_{i+1,j+1} + (pqX_i - c_1 - c_2E^*_{i,j+1})E^*_{i,j+1}\Delta t\ ;$$

- For $i = m$ and $0 \le j \le n - 1$,

$$\frac{\sigma^2 X_m^2 \Delta t}{4\Delta X^2}\cdot J^*_{m-3,j} - \left(\frac{(rX_m(1-X_m/K) - qE^*_{m,j+1}X_m)\Delta t}{4\Delta X} + \frac{\sigma^2 X_m^2 \Delta t}{4\Delta X^2}\right)\times J^*_{m-2,j} + \left(\frac{(rX_m(1-X_m/K) - qE^*_{m,j+1}X_m)\Delta t}{\Delta X} + \frac{5\sigma^2 X_m^2 \Delta t}{4\Delta X^2}\right)\times J^*_{m-1,j} + \left(\frac{2+\delta\Delta t}{2} - \frac{3(rX_m(1-X_m/K) - qE^*_{m,j+1}X_m)\Delta t}{4\Delta X} - \frac{\sigma^2 X_m^2 \Delta t}{2\Delta X^2}\right)\cdot J^*_{m,j}$$

$$= -\frac{\sigma^2 X_m^2 \Delta t}{4\Delta X^2}\cdot J^*_{m-3,j+1} + \left(\frac{(rX_m(1-X_m/K) - qE^*_{m,j+1}X_m)\Delta t}{4\Delta X} + \frac{\sigma^2 X_m^2 \Delta t}{4\Delta X^2}\right)\times J^*_{m-2,j+1} - \left(\frac{(rX_m(1-X_m/K) - qE^*_{m,j+1}X_m)\Delta t}{\Delta X} + \frac{5\sigma^2 X_m^2 \Delta t}{4\Delta X^2}\right)\times J^*_{m-1,j+1} + \left(\frac{2-\delta\Delta t}{2} + \frac{3(rX_m(1-X_m/K) - qE^*_{m,j+1}X_m)\Delta t}{4\Delta X} - \frac{\sigma^2 X_m^2 \Delta t}{2\Delta X^2}\right)\cdot J^*_{m,j+1} + (pqX_m - c_1 - c_2 E^*_{m,j+1})E^*_{m,j+1}\Delta t.$$

The system can be written, using appropriate matrices A, B and C, in the form

$$A\boldsymbol{J}^*_- = B\boldsymbol{J}^*_+ + C,$$

with

$$J^*_j = [J^*_{0,j} \quad J^*_{1,j} \quad \cdots \quad J^*_{m,j}]^T, \quad 0 \le j \le n,$$

where T is the transpose operator. The optimal solution when the system is at a given value x_0 at time t_0 is obtained by linear interpolation.

Appendix B. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.fishres.2017.07.016.

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