Riemannian problems with a fundamental differential system

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Abstract We introduce the reader to a fundamental exterior differential system of Riemannian geometry which arises naturally with every oriented Riemannian $n+1$-manifold M. Such system is related to the well-known metric almost contact structure on the unit tangent sphere bundle SM, so we endeavor to include the theory in the field of contact systems. Our EDS is already known in dimensions 2 and 3, where it was used by Ph. Griffiths in applications to mechanical problems and Lagrangian systems. It is also known in any dimension but just for flat Euclidean space. Having found the Lagrangian forms $\alpha_i \in \Omega^n$, $0 \le i \le n$, we are led to the associated functionals $\mathscr{F}_i(N) = \int_N \alpha_i$, on the set of hypersurfaces $N \subset M$, and to their Poincaré-Cartan forms. A particular functional relates to scalar curvature and thus we are confronted with an interesting new equation.

1 Geometric structures and the fundamental differential system

1.1 The manifold SM

Let M be any smooth oriented $n+1$-dimensional Riemannian manifold. Our study is centred on the geometry of the tangent bundle TM as an oriented Riemannian $2n+2$-manifold, endowed with the well-known Sasaki metric. Let $\pi : TM \longrightarrow M$ denote the canonical projection. The vector bundle $V := \ker \mathrm{d}\pi \simeq \pi^\star TM \longrightarrow TM$ agrees fibrewise with the tangent bundle to the fibres of TM. Moreover the tangent bundle of TM splits as $TTM = H \oplus V$, where H is a sub-vector bundle depending on ∇, the Levi-Civita connection. Clearly the *horizontal* sub-bundle H is also isomorphic to $\pi^* TM$ through the map $\mathrm{d}\pi$. We thus define an endomorphism

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$$B : TTM \longrightarrow TTM \tag{1}$$

transforming H in V and vanishing on the *vertical* sub-bundle V. This is used by many authors perhaps not giving it so much importance. Partly because one simply recurs to lifts of the same vector on M to either horizontal or vertical parts.

There also exists a connection independent vector field ξ over TM defined by $\xi_u = u$, or maybe more precisely $\xi_u = \pi^\star u,\ \forall u \in TM$, turning explicit the vertical lift. Henceforth, there exists a unique horizontal ∇-dependent vector field, formally, $B^{\mathrm{ad}}\xi \in H$, such that $B(B^{\mathrm{ad}}\xi) = \xi$. That field is the *geodesic spray* of the connection, cf. [17]. One can see easily that $\pi^\star\nabla_w\xi = w^v$, one reason being that $H = \ker(\pi^\star\nabla.\xi)$.

The manifold TM also inherits a linear connection, denoted ∇^*, which is just

$$\pi^*\nabla \oplus \pi^\star\nabla$$

preserving the canonical splitting $TTM = H \oplus V \simeq \pi^*TM \oplus \pi^\star TM$. We observe then that the connecting endomorphism B is parallel for such ∇^*. The torsion of ∇^* is given by $\pi^*T^\nabla(v,w) \oplus \mathscr{R}^\xi(v,w),\ \forall v,w \in TTM$, where the vertical part is $\mathscr{R}^\xi(v,w) = R^{\pi^\star\nabla}(v,w)\xi = \pi^\star R^\nabla(v,w)\xi$.

Now we come forward with the metric tensor of M. The Sasaki metric $\langle\ ,\ \rangle$ on TM is given naturally by the pull-back of the metric on M both to H and V. The parallel *mirror* morphism $B_| : H \to V$ is then metric-preserving. Now B^{ad} really denotes the adjoint endomorphism of B and the map $J = B - B^{\mathrm{ad}}$ is the Sasaki almost complex structure on TM.

Any frame in H extended with its mirror in V clearly determines an orientation on the manifold TM. We convention to adopt the order 'first H, then V', which is a relevant issue when $\dim M$ is odd.

Let us suppose ∇ is the Levi-Civita connection and consider the radius 1 tangent sphere bundle

$$SM = \{u \in TM :\ \|u\| = 1\}\ . \tag{2}$$

∇^* is a metric connection and so, differentiating $\langle\xi,\xi\rangle = 1$, we deduce $TSM = \xi^\perp$. Since the manifold TM is orientable, SM is also always orientable — the restriction of ξ being a unit *outward normal*. By the Gram-Schmidt process and the orthogonal group action, for any $u \in SM$ we may find a local horizontal orthonormal frame $e_0, e_1 \ldots, e_n$ on a neighbourhood of u in SM and such that $e_0 = B^{\mathrm{ad}}\xi$ or, equivalently, $e_0 = u \in H$.

With the dual horizontal coframing, clearly the identity $\pi^*\mathrm{vol}_M = e^0 \wedge e^1 \wedge \cdots \wedge e^n$ is satisfied. Adding the *mirror* subset $\{\xi^\flat, e^{n+1}, \ldots, e^{2n}\}$, with $e^{n+i} = e^i \circ B^{\mathrm{ad}},\ \forall i \geq 1$ (equivalently $e^{n+i}(e_j) = e^i(e_{j+n}) = 0,\ e^{n+i}(e_{j+n}) = e^i(e_j) = \delta^i_j,\ \forall i,j$), we find the volume form of TM:

$$\mathrm{Vol}_{TM} = e^0 \wedge e^1 \wedge \cdots \wedge e^n \wedge \xi^\flat \wedge e^{n+1} \wedge \cdots \wedge e^{(2n)} = (-1)^{n+1}\,\xi^\flat \wedge \mathrm{vol} \wedge \alpha\ . \tag{3}$$

We use $\mathrm{vol} = \pi^*\mathrm{vol}_M$; whereas α denotes the n-form on TM which is defined as the interior product of ξ with the vertical pull-back of the volume form of M. Hence, choosing appropriately $\pm\xi$ as unit normal direction, the canonical orientation of the Riemannian submanifold SM, given by $\pm\xi\lrcorner\mathrm{Vol}_{TM}$, agrees with $\mathrm{vol}\wedge\alpha=e^{01\cdots(2n)}$. A direct orthonormal frame as the one introduced previously is said to be *adapted*.

1.2 Further metric properties

The submanifold SM admits a metric linear connection $\nabla^\bigstar$. For any vector fields y,z on SM, the covariant derivative $\nabla^*_y z$ is well-defined and, admitting y,z perpendicular to ξ, we just have to add a correction term:

$$\nabla^\bigstar_y z \;=\; \nabla^*_y z-\langle\nabla^*_y z,\xi\rangle\xi \;=\; \nabla^*_y z+\langle y^v,z^v\rangle\xi \;. \tag{4}$$

Since $\langle\mathscr{R}^\xi(y,z),\xi\rangle=0$, then a torsion-free connection D is easy to find as $D_y z=\nabla^\bigstar_y z-\frac{1}{2}\mathscr{R}^\xi(y,z)$. This connection is most useful for some computations, but ceases to be metric. For the Levi-Civita connection we must add to D another term, A, given by:

$$\langle A_y z,w\rangle=\frac{1}{2}(\langle\mathscr{R}^\xi(y,w),z\rangle+\langle\mathscr{R}^\xi(z,w),y\rangle)\;. \tag{5}$$

Details on metric connections on SM are described in [4, 5].

We have found in [9] the conditions for natural maps to become isometries between tangent sphere bundles of different radius, including weighted Sasaki metric and conformal variation of the metric on the base manifold M when $\dim M\geq 3$. Notice the induced horizontal subspaces on SM are not fixed on the same conformal class on M. We do not explore here these results with the weights and radius, which are all aloud to be pullbacks of functions on M.

Just with the Sasaki metric we have a particular, new result which may catch the readers' attention for those theorems. Consider the constant norm $s>0$ sphere bundle $S_sM=sSM$ and let $M=M^\pm_R$ denote the space-form with metric g of constant sectional curvature $\pm 1/R^2$, where $R>0$.

Proposition 1. *Let g^S denote the Sasaki metric on the tangent bundle induced from the metric g on $M^\pm_R$. Then $(S_sM^\pm_R,g^S)$ is isometric to $(S_sM^\pm_1,(R^2g)^S)$.*

Proof. We use the map F defined in [9, section 2.6] and then apply twice corollary 2.2 from the same article, so the notation now is also from there:

$$(S_sM^\pm_R,g^S)\simeq(S_{\frac{s}{R}}M^\pm_1,g^{R^2,R^2})\simeq(S_1M^\pm_1,(R^2g)^{1,s^2})\simeq(S_sM^\pm_1,(R^2g)^S)\;.$$

We recall the notation, $g^{f_1,f_2}=f_1\pi^*g\oplus f_2\pi^\star g,\; g^S=g^{1,1}$. □

We have also computed in [4] the scalar curvature of the metrics above. For the weighted metric with f_1,f_2 constant, we have

$$\mathrm{Scal}_{(S_sM_R,g^{f_1,f_2})}=\pm\frac{n(n+1)}{f_1R^2}-\frac{f_2}{4f_1^2}\frac{s^2}{R^4}2n+\frac{(n-1)n}{f_2s^2}\;, \tag{6}$$

which is a positive (negative) constant for small (large) s, although we do not have an Einstein metric. The value of these results from [4, 9] has only recently been understood. Of course it is fun to verify the isometric invariance of our formulas.

1.3 The contact structure

We denote by θ the 1-form on SM defined by

$$\theta=(B^{\mathrm{ad}}\xi)^\flat=\langle\xi,B\,\cdot\rangle=e^0\ . \tag{7}$$

Y. Tashiro discovered in the 1960's that θ defines a metric contact structure, cf. [10]. In our adapted frame we find $\mathrm{d}\theta=e^{(1+n)1}+\cdots+e^{(2n)n}$. In other words, $\forall v,w\in TSM$, $\mathrm{d}\theta(v,w)=\langle v,Bw\rangle-\langle w,Bv\rangle$.

Now we present the set of natural n-forms $\alpha_0,\alpha_1,\ldots,\alpha_n$ existing always on SM. Together with θ they consist of the fundamental differential system we have announced. But we begin with the low dimension cases before a general definition.

In case $n=1$ we have a global coframing of SM with θ and two 1-forms $\alpha_0=e^2$ and $\alpha_1=e^1$, which are global forms. The following formulas were probably already known (to Cartan?), where k denotes the Gauss curvature of M:

$$\mathrm{d}\theta=\alpha_0\wedge\alpha_1\qquad \mathrm{d}\alpha_0=k\,\alpha_1\wedge\theta\qquad \mathrm{d}\alpha_1=\theta\wedge\alpha_0\ . \tag{8}$$

For the case $n=2$, $\alpha_0=e^{34},\ \alpha_1=e^{14}+e^{32},\ \alpha_2=e^{12}$, or the case $n=3$, $\alpha_0=e^{456},\ \alpha_1=e^{156}+e^{264}+e^{345},\ \alpha_2=e^{126}+e^{234}+e^{315},\ \alpha_3=e^{123}$, we do not have any special example or easier way of computing the exterior derivatives other than that which we use in [8] with the connections ∇^*,D above — except in case $n=3$ and flat metric coordinates, as shown in [3], because the 3-sphere is parallelizable and so we may explicit an adapted frame (just as with $n=1$).

Finally we define the $n+1$ natural n-forms on SM. First, for $0\leq i\leq n$, let

$$n_i=\frac{1}{i!(n-i)!}\ . \tag{9}$$

Continuing with the adapted frame introduced earlier, we then define:

$$\alpha_0\,=\,\alpha\,=\,\xi\lrcorner(\pi^\star\mathrm{vol}_M)=e^{(n+1)}\wedge\cdots\wedge e^{(2n)} \tag{10}$$

where $\pi^\star\mathrm{vol}_M$ is the vertical pull-back of the volume form of M. Now for each i we write, $\forall v_1,\ldots,v_n\in TSM$,

$$\alpha_i(v_1,\ldots,v_n)=n_i\sum_{\sigma\in S_n}\mathrm{sg}(\sigma)\,\alpha(Bv_{\sigma_1},\ldots,Bv_{\sigma_i},v_{\sigma_{i+1}},\ldots,v_{\sigma_n}). \tag{11}$$

We remark that $\alpha_n=e^{1\ldots n}$, which justifies the introduction of the weight n_i. For convenience we define $\alpha_{-1}=\alpha_{n+1}=0$.

Only a scarce number of references have used the exterior differential system of θ and the α_i, yet not rising them to the level of a field of study. It seems the n-forms have only been considered as an auxiliary tool in the solution of very few mechanical systems problems. First for 2 or 3 dimensional base space in Ph. Griffiths' book [13]. Then in [11, p.152] with emphasis on a 3-dimensional metric and an algebraic problem. The same being true regarding later articles in [14], as well as in [15].

Regarding the n-dimensional case, we suppose to be correct in saying it appears for the first time, though only for the Euclidean base space, in [12, p. 32]. To the best of our knowledge, the definition in full generality (11) is introduced first by the author in [8].

Our differential system is original for we do not have any other reference for the following formulas deduced in [8]. On a manifold with constant sectional curvature k we have

$$\begin{aligned} \mathrm{d}\alpha_0 &= \theta\wedge(\qquad\qquad -k\,\alpha_1) \\ \mathrm{d}\alpha_1 &= \theta\wedge(\qquad n\,\alpha_0-2k\,\alpha_2) \\ \mathrm{d}\alpha_2 &= \theta\wedge((n-1)\,\alpha_1-3k\,\alpha_3) \\ &\ \vdots \\ \mathrm{d}\alpha_{n-1} &= \theta\wedge(2\,\alpha_{n-2}-nk\,\alpha_n) \\ \mathrm{d}\alpha_n &= \theta\wedge\alpha_{n-1} \end{aligned} \tag{12}$$

or simply $\mathrm{d}\alpha_i=\theta\wedge((n-i+1)\,\alpha_{i-1}-k(i+1)\,\alpha_{i+1}),\ \forall i=0,\dots,n$. The particular case of formula (12) with sectional curvature $k=0$ is already known, as we referred.

1.4 Some structural relations

The proofs of the following are quite easy, cf. [8]. For any $0\leq i\leq n$ we have:

$$\begin{aligned} *(\mathrm{d}\theta)^i &= (-1)^{\frac{n(n+1)}{2}}\,\frac{i!}{(n-i)!}\,\theta\wedge(\mathrm{d}\theta)^{n-i} \\ *\alpha_i &= (-1)^i\,\theta\wedge\alpha_{n-i}\ . \end{aligned} \tag{13}$$

Also $\alpha_i\wedge\mathrm{d}\theta=0$ and $\alpha_i\wedge\alpha_j=0,\ \forall j\neq n-i$. Of course $*$ denotes the Hodge star-operator on SM, which satisfies $**=1$ on Λ^*_{SM}. In our notation,

$$R_{ijkl}=\langle\nabla_{e_i}\nabla_{e_j}e_k-\nabla_{e_j}\nabla_{e_i}e_k-\nabla_{[e_i,e_j]}e_k,e_l\rangle\ . \tag{14}$$

Theorem 1 (1st-order structure equations, [8]). *We have*

$$\mathrm{d}\alpha_i=(n-i+1)\,\theta\wedge\alpha_{i-1}+\mathscr{R}^\xi\,\alpha_i \tag{15}$$

where

$$\mathscr{R}^{\xi}\alpha_i = \sum_{0\leq j<q\leq n}\sum_{p=1}^{n} R_{jq0p}\, e^{jq}\wedge e_{p+n}\lrcorner\alpha_i\ . \tag{16}$$

This theorem is proved with the tools of connection theory introduced in the first section. We do not have any other method which could ease the computations.

Defining $r=\mathrm{Ric}\,(\xi,\xi)=\sum_{j=1}^{n}R_{j00j}$ as a smooth function on SM determined by the Ricci curvature of M, we have after computations ([8])

$$\mathrm{d}\alpha_n=\theta\wedge\alpha_{n-1} \qquad \mathrm{d}\alpha_{n-1}=2\theta\wedge\alpha_{n-2}-r\,\mathrm{vol}\ , \tag{17}$$

i.e. $\mathscr{R}^{\xi}\alpha_n=0$ and $\mathscr{R}^{\xi}\alpha_{n-1}=-r\theta\wedge\alpha_n$. Then clearly

$$\mathrm{d}(\mathscr{R}^{\xi}\alpha_i)=(n-i+1)\theta\wedge\mathscr{R}^{\xi}\alpha_{i-1} \qquad \mathrm{d}\theta\wedge\mathscr{R}^{\xi}\alpha_i=0\ . \tag{18}$$

Proposition 2. *The differential forms* $\theta,\ \alpha_0$ *and* α_1 *are always coclosed. Moreover, for all* $0\leq i\leq n$*,*

$$\mathrm{d}(i*\alpha_i+(-1)^i\mathscr{R}^{\xi}\alpha_{n-i+1})=0\ . \tag{19}$$

Proof. One just applies (13) and (18):

$$\mathrm{d}i*\alpha_i=i(-1)^{i+1}\theta\wedge\mathrm{d}\alpha_{n-i}=i(-1)^{i+1}\theta\wedge\mathscr{R}^{\xi}\alpha_{n-i}=(-1)^{i+1}\mathrm{d}(\mathscr{R}^{\xi}\alpha_{n-i+1})$$

Clearly, $\mathscr{R}^{\xi}\alpha_{n+1}=0$ and it is true $\mathrm{d}*\alpha_n=\mathrm{dvol}=0$. □

No further assumptions on M are required, so we believe there are good reasons to refer to the d-closed differential ideal $\mathscr{I}=\mathrm{span}\{\theta,\alpha_0,\ldots,\alpha_n\}$ as a *fundamental* object of any oriented Riemannian $n+1$-manifold.

It is quite interesting to consider the case of constant sectional curvature k in any dimension. The Riemann curvature tensor is $R_{ijpq}=k(\delta_{iq}\delta_{jp}-\delta_{ip}\delta_{jq})$, so one may prove that $\mathscr{R}^{\xi}\alpha_i=-k(i+1)\,\theta\wedge\alpha_{i+1}$, cf. (12).

1.5 Gwistor space and problems for calibrated geometries

The author's discovery of the exterior differential system $\mathscr{I}$ came after and with that of a natural G_2 structure on SM for M of dimension 4.

In [1, 2] it is proved that the total space of the radius 1 tangent sphere bundle $SM\longrightarrow M$ of any given oriented Riemannian 4-manifold M carries a natural G_2-structure. The space is now called G_2-*twistor* or *gwistor space*. Its fundamental structure 3-form is

$$\phi_1=\theta\wedge\mathrm{d}\theta+\alpha_2-\alpha_0\ . \tag{20}$$

Gwistor space is being studied as a subject of its own importance. It has had several developments in [1, 2, 3, 6, 7] in relation with G_2 geometry. We know that ϕ_1 is never closed and it is coclosed if and only if M is Einstein. The G_2 structure $\phi_2=\theta\wedge\mathrm{d}\theta+\alpha_3-\alpha_1$ is more restrictive. There is a *circle* of G_2 structures on SM within ϕ_1 and ϕ_2 compatible with the Sasaki metric.

An important open problem in linear algebra is to find the conditions for which a linear combination $\varphi=\sum_{i=0}^{n}b_i\alpha_i+c\theta^{\varepsilon}\wedge(\mathrm{d}\theta)^{[\frac{n}{2}]}$, with $b_i,c\in\mathrm{C}^{\infty}_{SM}$, $\varepsilon=0,1$, becomes a calibration. Recall a *calibration* is a closed p-form φ such that $\varphi_{|V}\leq\mathrm{vol}_V$ for every oriented tangent p-plane V, cf. [16]. One expects all b_i,c to be constant, yet we are unable to eliminate other possibility.

For even $n=\dim M-1$ we have an obvious φ of degree n. For $n=1$ the question may be solved easily recurring to (8). For $n=2$ and 3 we have a complete linear algebra classification of the calibrations in [16, Theorems 4.3.2 and 4.3.4]. In case $n=3$, we recover gwistor space.

The following result is quite interesting. Let $\rho=\xi\lrcorner\pi^{\star}\mathrm{Ric}$, the vertical lift of the Ricci tensor. With an adapted frame, we deduce

$$\rho=\sum_{a,b=1}^{n}R_{ab0a}e^{b+n}\ . \tag{21}$$

We have the following theorem giving a reduction of the degree of a differential equation.

Theorem 2 ([8]). *In any dimension we have* $\mathrm{d}*\alpha_2=\rho\wedge\mathrm{vol}$. *Henceforth, the metric on* M *is Einstein if and only if* $\delta\alpha_2=0$.

2 Geometric applications

2.1 Recalling Euler-Lagrange systems

We wish to study the Euler-Lagrange system (SM,θ,φ), where φ is a calibration, in applications to Riemannian geometry. We start further above recalling the theory of contact systems from [12]. In this section we assume (S,θ) is any given contact manifold, not necessarily metric, of dimension $2n+1$.

The *contact differential ideal* $\mathscr{I}$ is defined as the d-closed ideal generated by $\theta\in\Omega^1_S$. A generalisation of the famous Darboux Theorem assures that locally S is the 1-jet manifold $J^1(\mathbb{R}^n)$ of Euclidean flat space, with (Pfaff) coordinates (z,x^i,p_i) and contact form $\theta=\mathrm{d}z-\sum_{i=1}^{n}p_i\,\mathrm{d}x^i$. The submanifold N given by $z=0,p_i=0$ satisfies $\theta_{|N}=0$. That is also the case with any submanifold $\{(z(x),x^i,\partial_i z)\}$ where z is a C^1 function on the x^i.

An *integral* submanifold of S consists of a submanifold N together with an immersion $f:N\to S$ such that $f^*\theta=0$. Then of course $f^*\mathscr{I}=0$. A *Legendre submanifold* is a C^1-differentiable integral n-dimensional submanifold N. The Legendre submanifolds which appear as the graph of a function on N in the Pfaff coordinates are called *transverse*. Equivalently, N is transverse if and only if $f^*(\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n)\neq0$.

Any form $\Lambda \in \Omega^n_S$ is called a *Lagrangian*. An equivalence relation is immediately associated with equivalence class $\Lambda + \mathscr{I}^n + \mathrm{d}\Omega^{n-1}$, where Λ is a representative and $\mathscr{I}^n = \mathscr{I} \cap \Omega^n$.

An algebraic identity deduced in [12] carries over to the whole contact manifold as:

$$\mathscr{I}^k = \Omega^k, \qquad \forall\, k > n\ . \tag{22}$$

Hence there exist two forms α, β on S such that $\mathrm{d}\Lambda = \theta \wedge \alpha + \mathrm{d}\theta \wedge \beta = \theta \wedge (\alpha + \mathrm{d}\beta) + \mathrm{d}(\theta \wedge \beta)$. By [12, Theorem 1.1] there exists a unique global exact form Π such that $\Pi \wedge \theta = 0$ and $\Pi \equiv \mathrm{d}\Lambda$ in $\bar{H}^{n+1}(\Omega^*/\mathscr{I}, \mathrm{d})$. The *Poincaré-Cartan* form is $\Pi = \mathrm{d}(\Lambda - \theta \wedge \beta) = \theta \wedge (\alpha + \mathrm{d}\beta)$. The form $\Psi = \alpha + \mathrm{d}\beta$ turns out to be of great importance.

Now one wishes to find the critical points of a functional on the set of smooth, compact Legendre submanifolds $N \hookrightarrow S$, possibly with boundary, defined by:

$$\mathscr{F}_\Lambda(N) = \int_N f^*\Lambda\ . \tag{23}$$

Note that Λ clearly induces the same functional on its class for Legendre submanifolds without boundary.

Suppose we have a variation of Legendre submanifolds *with fixed boundary*, i.e. suppose there is a curve of smooth maps $f_t : N \to S$ which defines a Legendre submanifold N_t for each t and $\partial(N_t) = \partial(N_0)$. Differentiating $\mathscr{F}_\Lambda(N_t)$, cf. [12], leads to the conclusion that

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathscr{F}_\Lambda(N_t) = 0 \qquad \text{if and only if} \qquad f^*\Psi = 0\ . \tag{24}$$

A Legendre submanifold satisfying (24) is called a *stationary* Legendre submanifold. The exterior differential system algebraically generated by $\theta, \mathrm{d}\theta, \Psi$ is called the *Euler-Lagrange system* of (S, θ, Λ); its Poincaré-Cartan form Π is said to be non-degenerate if it has no other degree 1 factors besides the multiples of θ.

In order to determine conditions on the stationary submanifolds of $\mathscr{F}_\Lambda$ one proceeds as follows: find the Poincaré-Cartan n-form, transform it into the product $\theta \wedge \Psi$ and then do the analysis of $f^*\Psi = 0$, i.e. study the Euler-Lagrange equation.

2.2 On the unit tangent sphere bundle

Let us admit again an oriented $n+1$-dimensional Riemannian manifold M together with its unit tangent sphere bundle $SM \xrightarrow{\pi} M$. Now we let $f : N \to M$ be a compact oriented isometric immersed hypersurface.

Then we have also a smooth lift $\hat{f} : N \to SM$ of f, the unique unit normal $\nu \in T_{f(x)}M$ chosen according to the orientations of N and M. Note that $\hat{f}$ is also defined on ∂N. It is easy to see that we have the decomposition into horizontal plus vertical:

$$\mathrm{d}\hat{f}(w) = (\mathrm{d}f(w))^h + (f^*\nabla)_w f^*\nu\ , \quad \forall w \in T_xN\ . \tag{25}$$

Indeed, at each point $x \in N$ the vertical part is $\nabla^*_{\mathrm{d}\hat{f}(w)}\xi = (\hat{f}^*\nabla^*)_w\hat{f}^*\xi$, where ξ is the canonical vertical vector field on SM. Clearly, $(\hat{f}^*\xi)_x = \hat{f}(x) = \nu_{f(x)} = (f^*\nu)_x$ and $\hat{f}^*\pi^* = f^*$. By definition of $\hat{f}$ we clearly have that $\hat{f} : N \longrightarrow SM$ defines a Legendre submanifold of the natural contact structure, $\hat{f}^*\theta = 0$, and that it is a transverse submanifold.

A smooth Legendre submanifold Y is locally the lift $N \to Y \subset SM$ of an oriented smooth n-submanifold $N \hookrightarrow M$ if and only if ${e^{1\cdots n}}_{|Y} \neq 0$, i.e. precisely when Y is transverse. We are thus going to assume throughout such *open* condition on submanifolds, defined by the top differential form: $\alpha_{n|Y} \neq 0$.

Let us consider an adapted direct orthonormal coframe $e^0, e^1, \ldots, e^n, e^{n+1}, \ldots, e^{2n}$ locally defined on SM. Then it may not be tangent to $N \subset SM$. Yet we have also a direct orthonormal coframe $e^1, \ldots, e^n$ for N (we use the same letters for the pull-back). Now, from (25), for any $1 \leq j \leq n$ we have

$$\hat{f}^*e^j = e^j \qquad \text{and} \qquad \hat{f}^*e^{j+n} = -\sum_{k=1}^{n} A^j_k e^k \tag{26}$$

with A the second fundamental form of N. We recall, $A = -\nabla\nu : TN \to TN$ is a symmetric endomorphism; the associated tensor $H = \frac{1}{n}(\mathrm{Tr}A)\nu$ is the mean curvature vector field.

We now consider the n-forms α_i, which give in their own right interesting Lagrangian systems on the contact manifold SM. We wish to study the functionals $\mathscr{F}_i = \mathscr{F}_{\alpha_i}$ on the set of compact immersed hypersurfaces of M with fixed boundary.

Let $\sigma_i(A)$ denote the elementary symmetric polynomial of degree i in the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. Then we have that

$$\hat{f}^*\alpha_{n-i} = (-1)^i\sigma_i(A)\,\mathrm{vol}_N\ . \tag{27}$$

We do not know a more simple proof for the following result (only for Euclidean base space it is in [12] with the same method), than by using (24) on $\mathscr{F}_n$ and the Poincaré-Cartan form given by $\mathrm{d}\alpha_n = \theta \wedge \alpha_{n-1}$.

Theorem 3 (Classical theorem, [8]). *Let N be a compact isometrically immersed hypersurface in the Riemannian manifold M. Then, $\forall v \in \Gamma_0(N, f^*TM)$,*

$$\delta\mathrm{vol}(N)(v) = -\int_N n\langle v, H\rangle\,\mathrm{vol}_N\ . \tag{28}$$

In particular, N is minimal for the volume functional within all compact hypersurfaces with fixed boundary ∂N if and only if $H = 0$.

As used previously, one deduces $\hat{f}^*\alpha_{n-1} = -n\langle H, \nu\rangle\,\mathrm{vol}_N = -n\|H\|\,\mathrm{vol}_N$, hence the functional $\mathscr{F}_{n-1}$ corresponds with

$$\mathscr{F}_{n-1}(N) = -n\int_N \|H\|\,\mathrm{vol}_N\ , \tag{29}$$

i.e. the integral of the mean curvature on immersed submanifolds $N \subset M$.

Theorem 4 ([8]). *Suppose the Riemannian manifold M has dimension $n+1>2$. Then a compact isometric immersed hypersurface $f: N \to M$ with fixed boundary is stationary for the mean curvature functional $\mathscr{F}_{n-1}$ if and only if*

$$\mathrm{Scal}^N = \mathrm{Scal}^M - r_\nu \tag{30}$$

where $r_\nu = \mathrm{Ric}(\nu,\nu)$ is induced from the Ricci tensor of M and Scal *denotes scalar curvature functions.*

In particular, if M is an Einstein manifold, say where $\mathrm{Ric} = cg$ with c a constant, then N has stationary mean curvature volume if and only if N has constant scalar curvature $\mathrm{Scal}^N = nc$.

For an Einstein metric on the ambient manifold M, a formula in the last proof shows that $\mathscr{F}_{n-2}$ leads to an Euler-Lagrange equation essentially on the scalar curvature of N.

Theorem 5 ([8]). *Let M be a Riemannian manifold of dimension $n+1>2$ and constant sectional curvature k. Then a compact hypersurface N is a critical point of the scalar curvature functional $\int_N \mathrm{Scal}^N \mathrm{vol}_N$ with fixed boundary if and only if the eigenvalues $\lambda_1,\ldots,\lambda_n$ of A satisfy (assume $\lambda_3 = 0$ for $n=2$)*

$$6\sum_{j_1<j_2<j_3} \lambda_{j_1}\lambda_{j_2}\lambda_{j_3} + k(n-1)(n-2)(\lambda_1+\cdots+\lambda_n) = 0\ . \tag{31}$$

In other words, $6\sigma_3(A) + kn(n-1)(n-2)\|H\| = 0$.

The case $n=2$ is always satisfied and invariant of the ambient manifold - that is partly the theorem of Gauss-Bonnet.

Acknowledgements The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n PIEF-GA-2012-332209.

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