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HIGH ORDER PERIODIC IMPULSIVE PROBLEMS

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Abstract. The theory of impulsive problem is experiencing a rapid development in the last few years. Mainly because they have been used to describe some phenomena, arising from different disciplines like physics or biology, subject to instantaneous change at some time instants called moments. Second order periodic impulsive problems were studied to some extent, however very few papers were dedicated to the study of third and higher order impulsive problems.

The high order impulsive problem considered is composed by the fully nonlinear equation

$$u^{(n)}(x) = f \quad x, u(x), u'(x), ..., u^{(n-1)}(x)$$

for a. e. $x \in I := [0,1] \setminus \{x_1, ..., x_m\}$ where $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is L^1-Carathéodory function, along with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(1), \quad i = 0, ..., n-1,$$

and the impulsive conditions

$$u^{(i)} \quad x_j^+ \quad = g_j^i(u(x_j)), \quad i = 0, ..., n-1,$$

where g_j^i, for $j = 1, ..., m$, are given real valued functions satisfying some adequate conditions, and $x_j \in (0,1)$, such that $0 = x_0 < x_1 < ... < x_m < x_{m+1} = 1$.

The arguments applied make use of the lower and upper solution method combined with an iterative technique, which is not necessarily monotone, together with classical results such as Lebesgue Dominated Convergence Theorem, Ascoli-Arzèla Theorem and fixed point theory.

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1. **Introduction.** Problems with impulses have been experiencing a rapid develpement in the last few years. Their high aplicability in such different disciplines like physics, biology or finance is, most likely, one of the main reasons for that. The problem covered in this paper is a generalization to $n-th$ order of a periodic problem, with some impulses. First and second order periodic impulsive problems were studied to some extent, ([2, 3, 5, 6, 7, 8]), however very few papers were dedicated to the study of third and higher order impulsive problems. One can refer for instance ([1, 4, 9]) and the references therein. To the best of our knowledge, no paper generalizes and extends the results to higher order.

We consider the high order impulsive problem composed by the fully nonlinear equation

$$u^{(n)}(x) = f\left(x, u(x), u'(x), ..., u^{(n-1)}(x)\right) \tag{1}$$

for a. e. $x \in [0,1] \setminus \{x_1, ..., x_m\}$ where $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is L^1-Carathéodory function, along with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(1), \quad i = 0, ..., n-1, \tag{2}$$

and the impulsive conditions

$$u^{(i)}\left(x_j^+\right) = g_j^i\left(u\left(x_j\right)\right), \quad i = 0, ..., n-1, \tag{3}$$

where g_j^i, for $j = 1, ..., m$,are given real valued functions satisfying some adequate conditions, and $x_j \in (0,1)$, such that $0 = x_0 < x_1 < ... < x_m < x_{m+1} = 1$.

The arguments applied in this paper make use of the lower and upper solution method combined with an iterative technique (suggested in [1]) which is not necessarily monotone, together with classical results such as Lebesgue Dominated Convergence Theorem, Ascoli-Arzèla Theorem and fixed point theory.

An example is presented to illustrate the existence and location part of the lower and upper solution method.

2. **Definitions and auxiliary results.** In this section some notations, definitions and auxiliary results, needed for the main existence result, are presented. For $m \in \mathbb{N}$, let $0 = x_0 < x_1 < ... < x_m < x_{m+1} = 1$, $D = \{x_1, ..., x_m\}$ and define $x_j^\pm := \lim_{x \to x_j^\pm} x$, for $j = 1, ..., m$.

Consider $PC^{(s)}(I)$, $s = 1, ..., n-1$, as the space of the real-valued functions u, such that $u^{(s)} \in PC(I)$, $u^{(s)}\left(x_k^+\right)$ and $u^{(s)}\left(x_k^-\right)$ exist with $u^{(s)}\left(x_k^-\right) = u^{(s)}(x_k)$, for $k = 1, 2, ..., m$. Therefore $u \in PC^{n-1}(I)$, it can be written as

$$u(x) = \begin{cases} u_0(x) & \text{if } x \in [0, x_1], \\ u_1(x) & \text{if } x \in (x_1, x_2], \\ \vdots & \\ u_m(x) & \text{if } x \in (x_m, 1], \end{cases}$$

where $u_m(x) \in C^{n-1}((x_i, x_{i+1}))$ for $i = 1, ..., m$.

Denote

$$PC_D^{n-1}(I) = \left\{u \in PC^{n-1}(I) : u^{(n-1)} \in AC(x_i, x_{i+1}),\ i = 0, 1, ..., m\right\}$$

and for each $u \in PC_D^{n-1}(I)$ we set the norm

$$\|u\|_D = \|u\| + \|u'\| + ... + \left\|u^{(n-1)}\right\|,$$

where

$$\|w\| = \sup_{x\in I} |w(x)|.$$

Throughout this paper the following hypothesis will be assumed:

(I1) $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function, that is, $f(x,\cdot,...,\cdot)$ is a continuous function for *a.e.* $x \in I$; $f(\cdot, y_0, ..., y_{n-1})$ is measurable for $(y_0,...,y_{n-1}) \in \mathbb{R}^n$; and for every $M > 0$ there is a real-valued function $\psi_M \in L^1([0,1])$ such that

$$|f(y_0,...,y_{n-1})| \le \psi_M(x), \text{ for a. e. } x \in [0,1]$$

and for every $(y_0,...,y_{n-1}) \in \mathbb{R}^n$ with $|y_i| \le M$, for $i = 0,...,n-1$.

(I2) the real valued functions g_j^i are nondecreasing, for $j = 1,...,m$ and $i = 0,...,n-1$.

Definition 2.1. A function $u \in PC_D^{n-1}(I)$ is a solution of (1)-(3) if it satisfies (1) almost everywhere in $I \setminus D$, the periodic conditions (2) and the impulse conditions (3).

Next Lemma is a key tool to obtain the main result .

Lemma 2.2. *Let $p : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a L^1-Carathéodory function such that*

$$(v-w)[p(x,v) - p(x,w)] \le 0, \quad \forall x \in [0,1], \quad \forall v,w \in \mathbb{R}. \tag{4}$$

Then for each $a_j^i \in \mathbb{R}$, for $j = 1,...,m,$ and $i = 0,...,n-1,$ the initial value problem composed by the equation

$$u^{(n)}(x) = p\left(x, u^{(n-1)}(x)\right) \text{ for a. e. } x \in (0,1) \tag{5}$$

and the boundary conditions

$$u^{(i)}\left(x_j^+\right) = a_j^i, \text{ for } i = 0,...,n-1, \tag{6}$$

has a unique solution $u \in PC_D^{n-1}(I)$.

Proof. The solution of (5)-(6) can be written as

$$u(x) := \sum_{i=0}^{n-1} a_j^i \frac{\left(x-x_j^+\right)^i}{i!} + \int_{x_j^+}^{x} \frac{(x-r)^{n-1}}{(n-1)!} u^{(n)}(r)\, dr. \tag{7}$$

As $p\left(x, u^{(n-1)}(x)\right)$ is bounded in $I \times \mathbb{R}$, we can define $N := \left\|p\left(x, u^{(n-1)}(x)\right)\right\|_1,$ where $||\cdot||_1$ is the usual norm in $L^1(I \times \mathbb{R})$, and the following estimates can be obtained for $x \in (x_j, x_{j+1})$

$$\begin{aligned} |u(x)| &\le \sum_{i=0}^{n-1} \frac{|a_j^i|}{i!} + N, \\ |u'(x)| &\le \sum_{i=1}^{n-1} \frac{|a_j^i|}{(i-1)!} + N, \\ &\vdots \\ \left|u^{(n-1)}(x)\right| &\le \left|a_j^{n-1}\right| + N \end{aligned}$$

Hence, as n is finite, for $\delta := \sum_{i=0}^{n-1} \frac{|a_j^i|}{i!} + ... + \sum_{i=1}^{n-1} \frac{|a_j^i|}{(i-1)!} + |a_j^{n-1}| + nN$, it is obtained that

$$\|u\|_D = \sum_{i=0}^{n-1} \left\| u^{(i)} \right\| = \|u\| + \|u'\| + ... + \left\| u^{(n-1)} \right\| \leq \delta. \tag{8}$$

Let $u \in PC_D^{n-1}(I)$ be such that $\|u\|_D \leq \delta$.

Define the operator $\mathcal{T} : PC_D^{n-1}(I) \to PC_D^{n-1}(I)$ given by

$$\mathcal{T}u := \sum_{i=0}^{n-1} a_j^i \frac{\left(x - x_j^+\right)^i}{i!} + \int_{x_j^+}^{x} \frac{(x-r)^{n-1}}{(n-1)!} u^{(n)}(r)\, dr \tag{9}$$

As $p\left(x, u^{(n-1)}(x)\right)$ is a L^1-Carathéodory function, then $\mathcal{T}$ is continuous and, by (8),

$$\|\mathcal{T}u_n\|_D = \|\mathcal{T}u_n\| + \left\|(\mathcal{T}u_n)'\right\| + ... + \left\|(\mathcal{T}u_n)^{(n-1)}\right\| \leq \delta.$$

Moreover the operator $\mathcal{T}$ is uniformly bounded and equicontinuous, therefore, by Ascoli-Arzela's theorem, $\mathcal{T}$ is a compact operator. As the set of solutions of the equation $u = \mathcal{T}u$ is bounded, then using Schauder fixed point theorem, $\mathcal{T}$ has a fixed point $u \in PC_D^{n-1}(I)$ which satisfies (7) and

$$u^{(i)}\left(x_j^+\right) = a_j^i, \text{ for } i = 0, ..., n-1.$$

which proves the existence of solution for problem (5)-(6).

To show uniqueness, we assume that the problem (5)-(6) has two solutions, u_1 and u_2, define $z(x) = u_1^{(n-1)}(x) - u_2^{(n-1)}(x)$ for $x \in]x_j, x_{j+1}]$.

By (4), we have for $x \in]x_j, x_{j+1}]$

$$z(x)\; z'(x) = \left[u_1^{(n-1)}(x) - u_2^{(n-1)}(x)\right]\left[p(x, u_1^{(n-1)}(x)) - p(x, u_2^{(n-1)}(x))\right] \leq 0.$$

On the other hand as $z\left(x_j^+\right) = 0$

$$\int_{x_j^+}^{x} z(t)\; z'(t) dt = \frac{(z(x))^2}{2} - \frac{\left(z(x_j^+)\right)^2}{2} \geq 0.$$

So $z(x) = 0$, for every $x \in]x_j, x_{j+1}]$, and, by integration and (6), $u_1^{(n-1)}(x) = u_2^{(n-1)}(x)$ for $x \in]x_j, x_{j+1}]$. □

Lower and upper functions will be given by the next definition:

Definition 2.3. A function $\alpha \in PC_D^{n-1}(I)$ is said to be a lower solution of the problem (1)-(3) if:

(i) $\alpha^{(n)}(x) \leq f\left(x, \alpha(x), ..., \alpha^{(n-1)}(x)\right)$, for a.e. $x \in (0,1)$,
(ii) $\alpha^{(i)}(0) \leq \alpha^{(i)}(1)$, $i = 0, ..., n-1$,
(iii) $\alpha^{(i)}\left(x_j^+\right) \leq g_j^i(\alpha(x_j))$, for $i = 0, ..., n-1$.

A function $\beta \in PC_D^{n-1}(I)$ is said to be a upper solution of the problem (1)-(3) if the reversed inequalities hold.

3. **Existence of solutions.** In this section the main existence and location result is presented.

Theorem 3.1. *Let α, β be, respectively, lower and upper solutions of (1)-(3) such that*

$$\alpha^{(n-1)}(x) \leq \beta^{(n-1)}(x) \text{ on } I \setminus D, \tag{10}$$

and

$$\alpha^{(i)}(0) \leq \beta^{(i)}(0), \ i = 0, ..., n-2. \tag{11}$$

Assume that conditions (I1) and (I2) hold and

$$f\left(x, \alpha, ..., \alpha^{(n-2)}, y_{n-1}\right) \leq f(x, y_0, .., y_{n-2}, y_{n-1}) \leq f\left(x, \beta, ..., \beta^{(n-2)}, y_{n-1}\right), \tag{12}$$

for fixed $(x, y_{n-1}) \in I \times \mathbb{R}$, $\alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x)$, for $i = 0, ..., n-2$.

Also, for $x \in [0,1]$, $\alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x)$, for $i = 0, ..., n-2$ and for $v, w \in \mathbb{R}$,

$$(v - w)[f(x, y_0, y_1, .., y_{n-2}, v) - f(x, y_0, y_1, .., y_{n-2}, w)] \leq 0.$$

Then the problem (1)-(3) has a solution $u(x) \in PC_D^{n-1}(I)$, such that

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \text{ for } i = 0, ..., n-1$$

for $x \in I \setminus D$.

Remark 1. As one can notice by (11) the inequalities $\alpha^{(i)}(x) \leq \beta^{(i)}(x)$ hold for $i = 0, ..., n-2$ and every $x \in I$.

Proof. Consider the following modified problem composed by the equation

$$\begin{aligned} u^{(n)}(x) &= f\left(x, \delta_0(x, u(x)), ..., \delta_{n-1}\left(x, u^{(n-1)}(x)\right)\right) \\ &\quad - u^{(n-1)}(x) + \delta_{n-1}\left(x, u^{(n-1)}(x)\right), \end{aligned} \tag{13}$$

for $x \in (0,1)$ and $x \neq x_j$ where the continuous functions $\delta_i : \mathbb{R}^2 \to \mathbb{R}$, for $i = 0, ..., n-1$, are given by

$$\delta_i(x, y_i) = \begin{cases} \beta^{(i)}(x) & , \quad y_i > \beta^{(i)}(x) \\ y_i & , \quad \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x) \\ \alpha^{(i)}(x) & , \quad y_i < \alpha^{(i)}(x), \end{cases} \tag{14}$$

with the boundary conditions (2) and the impulse assumptions (3).

To prove the existence of solution for the problem (13),(2),(3) we apply an iterative method, which is not necessarily monotone. Let $(u_l)_{l \in \mathbb{N}}$ be the sequence of function in $PC_D^{n-1}(I)$ defined as follows

$$u_0 = \alpha \tag{15}$$

and for $l = 1, 2, ...$

$$\begin{aligned} u_l^{(n)}(x) = f\left(x, \delta_0(x, u_{l-1}(x)), ..., \delta_{n-2}\left(x, u_{l-1}^{(n-2)}(x)\right), \delta_{n-1}\left(x, u_l^{(n-1)}(x)\right)\right) \\ - u_l^{(n-1)}(x) + \delta_{n-1}\left(x, u_l^{(n-1)}(x)\right), \end{aligned} \tag{16}$$

for a.e. $x \in (0,1)$ with the boundary conditions

$$u_l^{(i)}(0) = u_{l-1}^{(i)}(1), \quad i = 0, ..., n-1 \tag{17}$$

and the impulsive conditions, for $j = 1, ..., m$,

$$u_l^{(i)}\left(x_j^+\right) = g_j^i\left(u_{l-1}(x_j)\right), \quad i = 0, ..., n-1. \tag{18}$$

By Lemma 2.2 the sequence $(u_l)_{l\in\mathbb{N}}$ is well defined.

Step 1 - *Every solution of (16)-(18) verifies*

$$\alpha^{(i)}(x) \leq u_l^{(i)}(x) \leq \beta^{(i)}(x), \text{ for } i = 0, ..., n-1, \tag{19}$$

for all $l \in \mathbb{N}$ and every $x \in I$.

Let u be a solution of the problem (16)-(18). The proof of the inequalities (19) will be done using mathematical induction.

For $i = n-1$, consider the inequalities

$$\alpha^{(n-1)}(x) \leq u_l^{(n-1)}(x) \leq \beta^{(n-1)}(x).$$

For $l = 0$, by (15)

$$\alpha^{(n-1)}(x) = u_0^{(n-1)}(x) \leq \beta^{(n-1)}(x), \text{ for } x \in I \backslash D,$$

and by Remark 1

$$\alpha^{(i)}(x) = u_0^{(i)}(x) \leq \beta^{(i)}(x), \text{ for } i = 0, ..., n-2.$$

Suppose that for $k = 1, ..., n-1$, for $x \in I$,

$$\alpha^{(n-1)}(x) \leq u_k^{(n-1)}(x) \leq \beta^{(n-1)}(x). \tag{20}$$

For $x = 0$, by (17), (20) and Definition 2.3,

$$u_l^{(n-1)}(0) = u_{l-1}^{(n-1)}(1) \geq \alpha^{(n-1)}(1) \geq \alpha^{(n-1)}(0).$$

If $x = x_j^+$, $j = 1, ..., m$, from (18), (I2), (20) and Definition 2.3,

$$u_l^{(n-1)}\left(x_j^+\right) = g_j^{n-1}\left(u_{l-1}^{(n-1)}(x_j)\right) \geq g_j^{n-1}\left(\alpha^{(n-1)}(x_j)\right) \geq \alpha^{(n-1)}\left(x_j^+\right).$$

For $x \in \left]x_j, x_{j+1}\right]$, $j = 1, 2, ..., m$, suppose, by contradiction, that there exists $x^* \in \left]x_j, x_{j+1}\right]$ such that $\alpha^{(n-1)}(x^*) > u_l^{(n-1)}(x^*)$ and define

$$\min_{x\in\left]x_j, x_{j+1}\right]} u_l^{(n-1)}(x) - \alpha^{(n-1)}(x) := u_l^{(n-1)}(x^*) - \alpha^{(n-1)}(x^*) < 0.$$

As by (18), $u_l^{(n-1)}\left(x_j^+\right) \geq \alpha^{(n-1)}\left(x_j^+\right)$, then there is an interval $(\underline{x}, \overline{x}) \subset (x_j, x^*)$ such that

$$u_l^{(n-1)}(x) < \alpha^{(n-1)}(x) \text{ and } u_l^{(n)}(x) \leq \alpha^{(n)}(x), \ \forall x \in (\underline{x}, \overline{x}).$$

From (13) and (12) the following contradiction is obtained for $x \in (\underline{x}, \overline{x})$

$$\begin{aligned} 0 \ &\geq \ u_l^{(l)}(x) - \alpha^{(n)}(x) \\ &= \ f\left(x, \delta_0(x, u_{l-1}(x)), ..., \delta_{n-2}\left(x, u_{l-1}^{(n-2)}(x)\right), \alpha^{(n-1)}(x)\right) \\ &\quad -u^{(n-1)}(x) + \alpha^{(n-1)}(x) - \alpha^{(n)}(x) \\ &\geq \ f\left(x, \alpha(x), ..., \alpha^{(n-1)}(x)\right) - u^{(n-1)}(x) + \alpha^{(n-1)}(x) \\ &\quad -f\left(x, \alpha(x), ..., \alpha^{(n-1)}(x)\right) \geq \alpha^{(n-1)}(x) - u^{(n-1)}(x) > 0. \end{aligned}$$

Therefore $u_l^{(n-1)}(x) \geq \alpha^{(n-1)}(x)$, for all $l \in \mathbb{N}$ and every $x \in I$. In the same way it can be shown that $u_l^{(n-1)}(x) \leq \beta^{(n-1)}(x)$, $\forall x \in I$, $\forall l \in \mathbb{N}$, and so (19) is proved when $i = n-1$.

Consider now the inequality $\alpha^{(n-2)}(x) \leq u_l^{(n-2)}(x) \leq \beta^{(n-2)}(x)$, for all $l \in \mathbb{N}$ and every $x \in I$.

To justify (19) for $i=n-2$, notice that for $n=0$, the proof is obtained in a similar way as in above.

Assuming that for $l=1,...,n-1$ and every $x\in I$,

$$\alpha^{(n-2)}(x)\leq u_l^{(n-2)}(x)\leq \beta^{(n-2)}(x). \tag{21}$$

then for $x\in[0,x_1]$, by integration of the inequality $u_l^{(n-1)}(x)\geq \alpha^{(n-1)}(x)$ in $[0,x]$ we have

$$u_l^{(n-2)}(x)-u_l^{(n-2)}(0)\geq \alpha^{(n-2)}(x)-\alpha^{(n-2)}(0).$$

By (17) and (21),

$$\begin{aligned} u_l^{(n-2)}(x) &\geq \alpha^{(n-2)}(x)-\alpha^{(n-2)}(0)+u_{l-1}^{(n-2)}(1) \\ &\geq \alpha^{(n-2)}(x)-\alpha^{(n-2)}(0)+\alpha^{(n-2)}(1)\geq \alpha^{(n-2)}(x) \end{aligned}$$

hence $u_l^{(n-2)}(x)\geq \alpha^{(n-2)}(x)$, for all $x\in[0,x_1]$.

For $x\in\,]x_j,x_{j+1}]$, $j=1,2,...,m$, by integration of the inequality $u_l^{(n-1)}(x)\geq \alpha^{(n-1)}(x)$ in $x\in\,]x_j,x_{j+1}]$,

$$u_l^{(n-2)}(x)\geq \alpha^{(n-2)}(x)-\alpha^{(n-2)}\left(x_j^+\right)+u_l^{(n-2)}\left(x_j^+\right),$$

and by (18) and Definition 2.3

$$u_l^{(n-2)}(x)\geq \alpha^{(n-2)}(x)-\alpha^{(n-2)}\left(x_j^+\right)+g_j^{n-2}\left(u_{l-1}^{(n-2)}(x_j)\right)\geq \alpha^{(n-2)}(x).$$

obtaining that $u_l^{(n-2)}(x)\geq \alpha^{(n-2)}(x)$, for all $l\in\mathbb{N}$ and every $x\in I$. Using similar arguments it can be proved that $u_l^{(n-2)}(x)\leq \beta^{(n-2)}(x)$ and therefore

$$\alpha^{(n-2)}(x)\leq u_l^{(n-2)}(x)\leq \beta^{(n-2)}(x),\ \forall x\in I,\forall l\in\mathbb{N}. \tag{22}$$

The remaining inequalities in (19) can be proved as in above, by integration of (21) in $[0,x_1]$, applying the correspondent induction hypothesis as well as conditions (17), (18) and Definition 2.3.

Step 2 - The sequence $(u_l)_{l\in\mathbb{N}}$ is convergent to u solution of (16)-(18).

Let $C_i=\max\left\{\left\|\alpha^{(i)}\right\|,\left\|\beta^{(i)}\right\|\right\}$, for $i=0,...,n-1$, so there exists $M>0$, with $M:=\sum_{i=0}^{n-1}C_i$, and for all $l\in\mathbb{N}$,

$$\|u_l\|_D\leq M. \tag{23}$$

Let Ω be a compact subset of $\mathbb{R}^n$ given by

$$\Omega=\left\{(w_0,...,w_{n-1})\in\mathbb{R}^n:\|w_i\|\leq C_i,\ i=0,...,n-1\right\}.$$

As f is a L^1-Carathéodory function in Ω, then there exists a real-valued function $\psi_M(x)\in L^1(I)$, such that

$$|f(x,w_0,...,w_{n-1})|\leq \psi_M(x),\ \text{for every }\ (w_0,...,w_{n-1})\in\Omega. \tag{24}$$

By Step1 and (23), $\left(u_l,u_l',...,u_l^{(n-1)}\right)\in\Omega$, for all $l\in\mathbb{N}$. From (16) and (24) we obtain

$$\left|u_l^{(n)}(x)\right|\leq \psi_M(x)+2C_{n-1},\ \text{for a.e. }\ x\in I,$$

hence $u_l^{(n)}(x)\in L^1(I)$.

By integration in I we obtain that

$$u_l^{(n-1)}(x) = u_l^{(n-1)}(0) + \int_0^x u_l^{(n)}(s)\,ds + \sum_{0<x_j\leq x} g_j^{n-1}\left(u_{l-1}^{(n-1)}(x_j)\right),$$

therefore $u_l^{(n-1)} \in AC(x_j, x_{j+1})$ and $u_l \in PC_D^{n-1}(I)$. By Ascoli-Arzèla Theorem there exists a subsequence denoted by $(u_l)_{l\in\mathbb{N}}$, which converges to $u \in PC_D^{n-1}(I)$. Then $\left(u, u', ..., u^{(n-1)}\right) \in \Omega$.

Using the Lebesgue dominated convergence theorem, for $x \in (x_j, x_{j+1})$,

$$\int_{x_j}^x \left[\begin{array}{c} f\left(s, \delta_0(s, u_{l-1}(s)), ..., \delta_{n-2}\left(s, u_{l-1}^{(n-2)}(s)\right), \delta_{n-1}\left(s, u_l^{(n-1)}(s)\right)\right) \\ -u_l^{(n-1)}(x) + \delta_{n-1}\left(x, u_l^{(n-1)}(x)\right) \end{array} \right] ds$$

is convergent to

$$\int_{x_j}^x \left[\begin{array}{c} f\left(s, \delta_0(s, u(s)), ..., \delta_{n-2}\left(s, u^{(n-2)}(s)\right), \delta_{n-1}\left(s, u^{(n-1)}(s)\right)\right) \\ -u^{(n-1)}(s) + \delta_{n-1}\left(s, u^{(n-1)}(s)\right) \end{array} \right] ds$$

as $l \to \infty$.

Therefore as $l \to \infty$

$$u_l^{(n-1)}(x) = u_l^{(n-1)}(x_j) +$$
$$\int_{x_j}^x \left[\begin{array}{c} f\left(s, \delta_0(s, u_{l-1}(s)), ..., \delta_{n-2}\left(s, u_{l-1}^{(n-2)}(s)\right), \delta_{n-1}\left(s, u_l^{(n-1)}(s)\right)\right) \\ -u_l^{(n-1)}(x) + \delta_{n-1}\left(x, u_l^{(n-1)}(x)\right) \end{array} \right] ds$$

is convergent to

$$u^{(n-1)}(x) = u^{(n-1)}(x_j) +$$
$$\int_{x_j}^x \left[\begin{array}{c} f\left(s, \delta_0(s, u(s)), ..., \delta_{n-2}\left(s, u^{(n-2)}(s)\right), \delta_{n-1}\left(s, u^{(n-1)}(s)\right)\right) \\ -u^{(n-1)}(s) + \delta_{n-1}\left(s, u^{(n-1)}(s)\right) \end{array} \right] ds.$$

As the function f is L^1-Carathéodory function in (x_j, x_{j+1}), then $u^{(n-1)}(x) \in AC(x_j, x_{j+1})$. Therefore $u \in PC_D^{n-1}(I)$ and u is a solution of (16)-(18).

To prove that u is a solution of the initial problem (1)-(3) we note that taking the limit in (17) and (18), as $l \to \infty$, by the convergence of u_l then u verifies (2) and, by the continuity of the impulsive functions, u verifies (3). By (14), Step 1 and the convergence of u_l, u verifies (1).

Then problem (1)-(3) has a solution $u(x) \in PC_D^{n-1}(I)$, such that

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \text{for } i = 0, ..., n-1,$$

for $x \in I$. □

4. **Example.** Let us consider the fifth order nonlinear impulsive boundary value problem, composed by the equation

$$u^{(v)}(x) = u(x) + u'(x) - u''(x) + \left(u'''(x) + 1\right)^3 + k\left|u^{(iv)}(x)\right|^\theta \tag{25}$$

where $0 < \theta \leq 2$ and $k \leq -32$, for all $x \in [0,1] \setminus \left\{\frac{1}{2}\right\}$ along with the boundary conditions (2) and for $x = \frac{1}{2}$ the impulse conditions

$$\begin{aligned} u\left(\tfrac{1}{2}^{+}\right) &= \mu_1\left(u\left(\tfrac{1}{2}\right)\right) \\ u'\left(\tfrac{1}{2}^{+}\right) &= \mu_2\left(u'\left(\tfrac{1}{2}\right)\right)^{\frac{1}{2}} \\ u''\left(\tfrac{1}{2}^{+}\right) &= \mu_3\left(u''\left(\tfrac{1}{2}\right)\right)^{3} \\ u'''\left(\tfrac{1}{2}^{+}\right) &= \mu_4\left(u'''\left(\tfrac{1}{2}\right)\right)^{5} \\ u^{(iv)}\left(\tfrac{1}{2}^{+}\right) &= \mu_5\left(u^{(iv)}\left(\tfrac{1}{2}\right)\right)^{\frac{1}{3}} \end{aligned} \tag{26}$$

with $\mu_i \in \mathbb{R}^+$, $i = 1,2,3,4$.

Obviously this problem is a particular case of (1)-(3) with

$$f\left(x, y_0, y_1, y_2, y_3, y_4\right) = y_0 + y_1 - y_2 + \left(y_3 + 1\right)^3 + k\left|y_4\right|^{\theta},$$

for all $x \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, $m = 1$, $x_1 = \frac{1}{2}$ and the nondecreasing functions g_1^i, $i = 0,1,2,3,4$ given by $g_1^0(x) = \mu_1 x$, $g_1^1(x) = \mu_2 x^{\frac{1}{2}}$, $g_1^2(x) = \mu_3 x^3$, $g_1^3(x) = \mu_4 x^5$, $g_1^4(x) = \mu_5 x^{\frac{1}{3}}$.

One can verify that the functions $\alpha(x) = 0$ and

$$\beta(x) = \begin{cases} \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1 & , \ x \in \left[0, \frac{1}{2}\right] \\ \frac{x^4}{24} & , \ x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

are $PC_D^4(I)$ for $D = \left\{\frac{1}{2}\right\}$ and considering

$$\beta'(x) = \begin{cases} \frac{x^3}{6} + \frac{x^2}{2} + x + 1 & , \ x \in \left[0, \frac{1}{2}\right] \\ \frac{x^3}{6} & , \ x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

$$\beta''(x) = \begin{cases} \frac{x^2}{2} + x + 1 & , \ x \in \left[0, \frac{1}{2}\right] \\ \frac{x^2}{2} & , \ x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

and

$$\beta'''(x) = \begin{cases} x + 1 & , \ x \in \left[0, \frac{1}{2}\right] \\ x & , \ x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

they are lower and upper solutions, respectively, for the problem (25), (2), (26), with

$$0 \leq \mu_1 \leq \frac{1}{633}, \quad \mu_2 \leq \frac{\sqrt{237}}{948}, \quad \mu_3 \leq \frac{64}{2197}, \quad \mu_4 \leq \frac{1}{64}, \quad \mu_5 \leq 1.$$

As f verifies (12), therefore by Theorem 3.1 there is a solution $u(x) \in PC_D^4(I)$, such that, $\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x)$, for $i = 0,1,2,3,4$.

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