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Agradecimentos

O trabalho apresentado só foi possível graças ao apoio e colaboração de muitas pessoas a quem devo os meus sinceros agradecimentos. Algumas delas merecem especial reconhecimento e gratidão. Em primeiro lugar, o Professor Imme P. van den Berg, orientador da presente dissertação, pela enorme disponibilidade demonstrada em todos os momentos, pela generosidade e prontidão com que partilhou comigo os seus conhecimentos assim como pelas críticas, correcções e sugestões relevantes feitas durante a orientação. Em segundo lugar, o Ulf Clotz, por conversas interessantes e esclarecedoras sobre os fundamentos de conjuntos externos e por me ajudar a melhor compreender a teoria HST. Em terceiro lugar, aos meus familiares, especialmente os meus pais e irmãs pela paciência e pelo amor que me transmitiram em todas as horas e aos meus sobrinhos Tiago e ao Eduardo que desde o seu nascimento sempre me fazem sorrir e voltar a sentir criança, mesmo nos momentos mais críticos. Finalmente à Inês pelos seus constantes incentivos e pelo amor e paciência que todos os dias demonstrou.

Resumo

"Axiomática para os Números Externos da Análise Não-standard e modelação de incertezas".

Nesta tese apresentamos uma axiomática para os números externos. Neutrices e números externos foram propostos como modelos de ordens de grandeza no contexto da Análise Não-standard. Mostramos que os números externos são um semigrupo comutativo regular para a adição e que os números externos que não são neutrices são um semigrupo comutativo regular para a multiplicação. A distributividade tem uma validade restringida mas que pode ser completamente caracterizada. Os números externos têm, em larga escala, propriedades semelhantes às dos números reais o que justifica a introdução de estruturas algébricas comuns, definidas por regras axiomáticas. As estruturas resultantes têm elementos neutros individualizados tanto para a adição como para a multiplicação e uma distributividade restringida. As estruturas apresentadas têm no entanto muitas propriedades em comum com estruturas clássicas tais como grupos, anéis e corpos. Mostramos que os axiomas apresentados têm um modelo nos números externos. Modelamos os paradoxos que surjem quando se consideram várias ordens de magnitude, chamados paradoxos Sorites, usando números externos.

Abstract

"Axiomatics for the External Numbers of Nonstandard analysis and modelation of uncertainties"

In this thesis we present an axiomatics for external numbers. Neutrices and external numbers were proposed as models of orders of magnitude within nonstandard analysis. We show that the external numbers form a commutative regular semigroup for addition and that the external numbers which are not neutrices form a commutative regular semigroup for multiplication. The validity of the distributive law is restricted, but it can be fully characterized. External numbers have to a large extent algebraic properties similar to those of real numbers. This justifies the introduction of common algebraic structures defined by axiomatic rules. The resulting structures have individualized neutral elements for both addition and multiplication and a restricted distributive law, but have to a large extent properties in common with classical structures such as groups, rings and fields. We show that the axioms presented have a model in the external numbers. We model the paradoxes which arise when several orders of magnitude are considered, called Sorites paradoxes, using external numbers.

Introduction

Ars negligendi longa, vita brevis.
(Van der Corput)

Neutrices and external numbers were introduced in [38] and [39], as mathematical models of orders of magnitude within nonstandard analysis. Within the external numbers we distinguish neutrices, a sort of generalized zeros. We use an axiomatic approach in which all infinite standard sets have nonstandard elements, so most neutrices are external sets. Being stable for some translations, additions and multiplications, external numbers are models of orders of magnitude or transitions with imprecise boundaries. In the nonstandard framework there are many neutrices, enabling to solve paradoxes which arise when several orders of magnitude are simultaneously considered. These paradoxes are called Sorites paradoxes. One can be stated in the following way: a single grain of wheat cannot be considered as a heap. Neither can two grains of wheat. One must admit the presence of a heap sooner or later, so where to draw the line? In fact, the heap and the grain of wheat are not of the same order of magnitude and we might say that the set of individual grains may be modeled by the external set of limited numbers (positive part of a neutrix) and the set of grains that form a heap may be modeled by the external set of the infinitely large numbers. It should also be possible to capture in this way some modalities, like the difference between a "good" approximation, allowing to obtain an adequately precise numerical result in some context, and a "bad", useless, one.

The stability of orders of magnitude under some repeated additions justifies to model them by (convex) groups of real numbers. Historically, the term neutrix was first used by Van der Corput [11], referring to groups of functions. Among others, his objective was to deal with imprecisions arising from neglecting terms of expansions. There are other approaches to this kind of problems, such as the $o\left(.\right)$ and $O\left(.\right)$ notation (Landau notation) [8], confidence intervals of statistics, interval arithmetic [25] [44] and fuzzy sets [73]. Hardy in his book on the Infinitärcalcül of Du Bois-Reymond [26] used the Landau notation to study "how fast" a function grows to infinity. This allows to distinguish different scales of functions. Hardy defends that

> The notions of the 'order of greatness' or 'order of smallness' of a function $f(n)$ of a positive integral variable n, when n is 'large,' or

> of a function $f(x)$ of a continuous variable x, when x is 'large' or 'small' or 'nearly equal to a,' are of the greatest importance even in the most elementary stages of mathematical analysis. [26]

These other approaches are not without fault as models of imprecisions, because they ultimately recourse to functions or precise intervals of numbers to model imprecise situations, and do not work with the actual error but only with an upper bound of the error. On the contrary, with external numbers it is possible to work directly with imprecisions and errors without recourse to upper bounds, for they have neither infimum nor supremum and satisfy the algebraic laws mentioned above. Moreover, the external numbers are totally ordered, even allowing for a sort of generalized Dedekind completeness property [3] [4] [39].

Chapter 1

On the foundations of external sets

> It was from them that I learned that hard work in stable surroundings could yield rewards, even if only in infinitesimally small increments.
>
> (Sidney Altman - Nobel Prize in Chemistry)

1.1 Introduction

In [55] [56] Robinson founded Nonstandard Analysis giving, for the first time, a correct treatment of infinitesimals. This was indeed a great achievement because infinitesimals had been present in mathematics in one way or the other since Archimedes's 'The Method of Mechanical Theorems', without having a proper and rigorous formulation. Nevertheless, Robinson's original treatment of infinitesimals seemed overcomplicated.

> It was developed [...] within a type-theoretical version of higher-order logic. [57]

Those tools made that first approach not very appealing to many mathematicians[1]. For this reason, Robinson and Zakon later gave a much simpler, purely set-theoretical, approach using model theory and superstructures [57] published in [42], further developed in [74]. Also in [42], Kreisel [40] raised the following questions[2]:

> 1. Is there a simple formal system (in the usual sense, that is, with a recursive, preferably finite, list of rules and axiom schemata)

[1] More recently a simplified version of this approach was given in [45]

[2] Kreisel [40] says that the first question posed itself as soon as Robinson published [55]. As for the second question he says that Robinson posed a related question, concerning non-standard arithmetic, at a meeting in London in 1965.

in which existing practice of nonstandard analysis can be codified? And if the answer is positive:

2. Is this formal system a conservative extension of current systems of analysis (in which existing practice of standard analysis has been codified)?

It turns out that the answer to both questions is positive and several set theories that axiomatize the nonstandard methods have been made since (see [34] for a book on axiomatic nonstandard theories and also [46] for a survey paper of nonstandard set theories). Some of these theories will be reviewed in this chapter. Nonstandard set theories can be classified into two groups: *internal theories*, which axiomatize standard and internal sets only and *external theories*, which axiomatize external sets as well. So, external theories tend to be more intricate and complicated. The main goal of this chapter is to give some foundational background for external sets.

We start by recalling the axioms and some basic set theoretical notions of ZFC, the usual framework for (standard) mathematics. Actually, almost all nonstandard theories are conservative extensions of ZFC. This means that every theorem of the nonstandard theory expressible in the language of ZFC is also a theorem of ZFC. However, it will be seen that in order to deal with external sets some axioms of ZFC need to be adapted. Nonstandard theories also typically allow some sort of transfer in order to connect the standard and the (internal) nonstandard worlds.

We recall the model theoretic construction of nonstandard analysis via superstructures by Robinson and Zakon [57] [74] and discuss how the external numbers may be developed in an appropriate superstructure. We then review the nonstandard internal set theories IST and BST. The latter has an extension to the external theory HST which, as we will see, allows to construct some neutrices and external numbers which are in general external sets.

1.2 ZFC

The most common axiomatic set theory and so the most common foundational background for mathematics is the theory Zermelo-Fraenkel-Choice which is usually abbreviated ZFC. For a thorough and modern treatment of ZFC we refer to [30]. In ZFC every object is a set. This theory has a very simple language, containing (apart from the logical symbols) only the binary membership predicate "$\in$". One should read $x \in y$ as 'x is a member of y' meaning that x is an element appearing in the set y. Classes in ZFC are informally defined as extensions of formulas in the following way. If $\varphi(x, p_1, ..., p_n)$ is a formula with parameters $p_1, ..., p_n$ then $\mathcal{C} = \{x : \varphi(x, p_1, ..., p_n)\}$ is a *class*. The members of a class $\mathcal{C}$ are the sets x that satisfy $\varphi(x, p_1, ..., p_n)$. We say that $\mathcal{C}$ is definable from $p_1, ..., p_n$. If $\varphi(x)$ has no parameters p_i then the class $\mathcal{C}$ is *definable*. The reason to consider classes is because classes are simpler to deal with than formulas. In the following we recall the axioms of ZFC with some comments.

Extensionality

The first axiom postulates the intuitive notion that sets with the same elements are equal. This axiom is also useful to prove that a given set is unique.

$$\forall X \forall Y \left(\forall x \left(x \in X \Leftrightarrow x \in Y \right) \Leftrightarrow X = Y \right).$$

Pair

Given two sets a and b there is a set (unique by extensionality) containing exactly a and b.

$$\forall a, b \exists A \forall x \left(x \in A \Leftrightarrow x = a \vee x = b \right).$$

One calls *ordered pair* to the (unique) set $(x, y) \equiv \{\{a\}, \{a, b\}\}$[3].

Separation (Scheme)

The elements x of a set X that verify the property $\varphi(x)$ form a new set Y (unique by extensionality). For each formula the following is an axiom.

$$\forall X \exists Y \forall x \left(x \in Y \Leftrightarrow (x \in X \wedge \varphi(x)) \right).$$

By separation a subclass of a set is also a set. It follows from separation that there is a set with no elements (unique by extensionality) called the *empty set* and denoted $\emptyset$. Moreover, if X and Y are sets the *intersection*

$$X \bigcap Y \equiv \{x \in X : x \in Y\}$$

is also a set.

Union

The next axiom states that for every set A there is a (unique) set $B \equiv \bigcup A$.

$$\forall A \exists B \forall x \left(x \in B \Leftrightarrow \exists X \in A \left(x \in X \right) \right).$$

One usually writes $X \bigcup Y$ instead of $\bigcup \{X, Y\}$.

Power Set

A set B is a *subset* of a set A if for every $x \in B$ one has that $x \in A$ and writes $B \subseteq A$. If $B \subseteq A$ and $A \neq B$ one says that B is a *proper subset* of A and writes simply $B \subset A$. For any set X there is a (unique) set $Y \equiv \mathcal{P}(X)$ that contains all subsets of X. This set is called the *power set* of X.

$$\forall X \exists Y \forall x \left(x \in Y \Leftrightarrow x \subseteq X \right).$$

[3] This definition was given by Kuratowski in 1921.

The power set axiom allows to define the following notions. One defines the (Cartesian) *product* of X and Y in the following way.

$$X \times Y \equiv \{(x,y) \in \mathcal{P}(\mathcal{P}(X \cup Y)) : x \in X \wedge y \in Y\}.$$

A *binary relation* R between two sets X and Y is a subset of the Cartesian product $X \times Y$. A binary relation f between two sets X and Y is a *function* if $(x,y) \in f$ and $(x,z) \in f$ implies $y = z$. A *partial order relation* "$\leq$" over a set X is a binary relation satisfying the following properties:

(O1) $\forall a \in X\,(a \leq a)$ (*reflexivity*).

(O2) $\forall a, b \in X\,(a \leq b \wedge b \leq a \Rightarrow a = b)$ (*antisymmetry*).

(O3) $\forall a, b, c \in X\,(a \leq b \wedge b \leq c \Rightarrow a \leq c)$ (*transitivity*).

A partial order relation is called a *total order relation* if it also satisfies

(O4) $\forall a, b \in X\,(a \leq b \vee b \leq a)$ (*totality*).

A *well-ordering* of a set is a total ordering of it according to which every non-empty subset has a least element.

Infinity

The next axiom states that there is an infinite set.

$$\exists X\,(\emptyset \in X \wedge \forall x\,(x \in X \Rightarrow x \cup \{x\} \in X)).$$

Replacement (Scheme)

The following axiom states that if a class $F = \{(x,y) : \varphi(x,y,p)\}$, where p is a parameter, is functional, then for every set X, $F(X)$ is a set.

$$\forall x \forall y \forall z\,(\varphi(x,y,p) \wedge \varphi(x,z,p) \Rightarrow y = z) \\ \Rightarrow \forall X \exists Y \forall y\,(y \in Y \Leftrightarrow (\exists x \in X)\,\varphi(x,y,p)).$$

A set A is called *transitive* if whenever $x \in A$, and $y \in x$, then $y \in A$. A set S which is strictly well-ordered with respect to $\in$ is an *ordinal* if and only if every element of S is also a subset of S. The class of all ordinals (which is not a set) is denoted Ord. An ordinal $\alpha = \beta + 1 = \beta \bigcup \{\beta\}$ is called a *successor ordinal*. If α is not a successor ordinal, then $\alpha = \sup\{\beta : \beta < \alpha\}$ is called a *limit ordinal*. In the presence of the following axiom it is possible to prove that $\alpha \in Ord$ if and only if α is a transitive set of transitive sets.

Regularity

$$\forall S\left(S\neq\emptyset\Rightarrow\left(\exists x\in S\right)\left(S\cap x=\emptyset\right)\right).$$

This means that every nonempty set has an $\in$-minimal element and, as a consequence, there can be no infinite sequences

$$...\in x_2\in x_1\in x_0.$$

Regularity is very useful in the construction of models but less relevant for the development of natural and real numbers, and in fact of all ordinary mathematics. Indeed, regularity's main feature is to give a "nice picture" of the universe of sets [46]. The universe of sets V is given by the *Von Neumann cumulative hierarchy* described below. The cumulative hierarchy is a collection of sets V_α indexed by Ord in the following way:

- $V_0=\emptyset$
- $V_{\alpha+1}=P\left(V_\alpha\right)$
- For any limit ordinal α, $V_\alpha\equiv\bigcup\limits_{\beta<\alpha}V_\alpha$
- $V\equiv\bigcup\limits_{\alpha\in Ord}V_\alpha$

The axiom of regularity implies that every set is obtained at some level of the cumulative hierarchy over the empty set, i.e. every set is in some V_α [30]. One defines the *rank* of x as the least α such that $x\in V_{\alpha+1}$.

Choice

Let S be a nonempty family of disjoint sets. A *choice function* for S is a function f such that $f\left(X\right)\in X$, for every $X\in S$. The following axiom states that every family of nonempty sets has a choice function.

$$\forall S\exists Y\forall X\in S\backslash\left\{\emptyset\right\}\left(\exists z\left(Y\cap X=\left\{z\right\}\right)\right).$$

1.3 Model theoretical nonstandard analysis

Model theory (see [9]) is a branch of mathematical logic that deals with models for axiomatic systems. A *model* for a theory can be seen as the assignment of meaning to the symbols of the language of the theory. So, for a given set of axioms a model is a mathematical object that satisfies the axioms. A *nonstandard model* is a model of a theory that is not isomorphic to the intended model (standard model). If the standard model is infinite and the language is first-order then by the Löwenheim–Skolem theorem [61] it has nonstandard models. Moreover, the nonstandard models can be chosen as elementary extensions or elementary substructures of the intended model. In particular, $\mathbb{R}$, the set of real numbers, has nonstandard models ${}^*\mathbb{R}$. This fact was deeply explored by Abraham Robinson [55] [56], leading to the foundation of nonstandard analysis. Nonstandard analysis was to a great extent inspired by Leibniz's ideas and intuitions towards the use of infinitesimal and infinitely large quantities. These "ghosts of departured quantities" [6] had been used informally throughout the early stages of the development of calculus. They were posteriorly abandoned when the $\epsilon - \delta$ based notions of analytic concepts such as limit, continuity, derivative and integral were introduced by the works of Cauchy, Bolzano and Weierstrass. In Robinson's words [56]:

> It is shown in this book that Leibniz's ideas can be fully vindicated and that they lead to a novel and fruitful approach to classical Analysis and to many other branches of mathematics. The key to our method is provided by the detailed analysis of the relation between mathematical languages and mathematical structures which lies at the bottom of contemporary model theory.

With nonstandard analysis, for the first time in history, a rigorous foundation for the use of infinitesimals was found.

Later, Robinson and Zakon [57] [74] (see also [21] for a more recent treatment of these constructions) gave a simpler approach to nonstandard analysis using set-theoretical objects called superstructures. A *superstructure* $V(S)$ over a set S is defined in the following way:

$$\begin{aligned} V_0(S) &= S \\ V_{n+1}(S) &= V_n(S) \cup P(V_n(S)) \\ V(S) &= \bigcup_{n\in\mathbb{N}} V_n(S). \end{aligned}$$

Superstructures of the empty set consist of sets of finite rank in the cumulative hierarchy and therefore do not satisfy the infinity axiom. Making $S = \mathbb{R}$ will suffice for virtually any construction necessary in analysis.

Bounded formulas are formulas where all quantifiers occur in the form

$$\forall x\, (x \in y \Rightarrow ...)$$

or

$$\exists x\, (x \in y \wedge ...).$$

A *nonstandard embedding* is a mapping $* : V(X) \to V(Y)$ from a superstructure $V(X)$ called the *standard model*, into another superstructure $V(Y)$, called *nonstandard model*, satisfying the following:

- ${}^*X = Y$
- *Transfer Principle*

  For every bounded formula $\varphi(x_1, ..., x_n)$ and elements $a_1, ..., a_n \in V(X)$, the property φ is true for $a_1, ..., a_n$ in the standard model if and only if it is true for ${}^*a_1, ..., {}^*a_n$ in the nonstandard model:

  $$\langle V(X), \in\rangle \models \varphi(a_1, ..., a_n) \Leftrightarrow \langle V(Y), \in\rangle \models \varphi({}^*a_1, ..., {}^*a_n).$$

- *Non-triviality*

  For every infinite set A in the standard model, the set $\{^*a : a \in A\}$ is a proper subset of $^*A$.

A set x is *internal* if and only if x is an element of $^*A$ for some element A of $V(\mathbb{R})$. Let X be a set with $A = \{A_i\}_{i\in I}$ a family of subsets of X. Then the collection A has the *finite intersection property*, if any finite subcollection $J \subset I$ has non-empty intersection. A model is *κ-saturated* if whenever $\{A_i\}_{i\in I}$ is a collection of internal sets with the finite intersection property and the cardinal of I is less than or equal to κ, $\bigcap\limits_{i\in I} A_i \neq \emptyset$.

As we will see in the next chapter neutrices are (external) convex subgroups of the real number system. In the model theoretic context, a nonstandard model of the theory of real numbers $^*\mathbb{R}$ generates external neutrices in an obvious way, since $^*\mathbb{R}$ contains such external convex subgroups. It is convenient to choose a superstructure large enough to be able to apply the usual operations of analysis on neutrices and external numbers. It is also convenient to suppose the superstructure to be κ-saturated, for some infinite cardinal κ. Neutrices defined over a (standard) set X of cardinality less than or equal to κ satisfy strong properties. Indeed, it is not difficult to adapt the proofs of axiomatic nonstandard analysis in [4] [39] to derive firstly that these neutrices are of the lowest complexity, i.e. are of the form $\cup_{x\in X}[-a_x, a_x]$ and $\cap_{x\in X}[-a_x, a_x]$, secondly that every neutrix N of this form is the multiple $N = \lambda I$ by a hyperreal number λ of an *idempotent* neutrix I (i.e., with $I.I = I$) and thirdly that every external lower halfline of $^*\mathbb{R}$ is bounded from above by an external number, which is cofinal with it, or just beyond [4] [39].

The superstructure approach allows one to use the methods of nonstandard analysis without paying much attention to the foundational details. However, it has some limitations if one wishes to use the nonstandard methods in its full generality. Superstructures can only model a fragment of ZFC. Take the following example [46], let $N \subseteq X$ be a copy of the natural numbers. Consider the function F with domain N such that $n \to \{...\{\emptyset\}...\}$ (n brackets). Then F is definable in $V(X)$ but range$(F) \notin V(X)$. Another objection to the superstructure approach's non-unicity is the fact that different problems will most likely need different superstructures and different embeddings. Also, the model theoretic concepts are not very appealing to most mathematicians. Finally one can argue that nonstandard analysis, in principle, is not concerned with superstructures. This makes way for axiomatic foundational frameworks for the nonstandard methods. Some of them will be reviewed in the following.

1.4 Theories for internal sets: IST and BST

We present two theories that concern internal sets only: IST [47] [48] [49] and BST [32] [33] [34]. The philosophical position of the internal approach to nonstandard analysis (IST and BST) is that

> ...we do not enlarge the world of mathematical objects in any way, we merely construct a richer language to discuss the same objects as before. [49]

Having a richer language enables one to say new things about the mathematical objects that were not possible to say using only the language of ZFC.

Both IST and BST are conservative extensions of ZFC and therefore, unlike the superstructure approach, all ZFC axioms remain valid. This is a very desirable fact because it means that also all ZFC theorems, definitions and constructions remain valid (as long as one considers only formulas in the ZFC language). Also, every statement in the language of ZFC that is provable in IST or BST can in principle be proved in ZFC (the conservative part). Working in IST is very much like working in ZFC and perhaps for this reason it was so far the only axiomatic nonstandard theory to be used in practice. See for example [50] [13] [3] [4] [14].

The axiomatics IST (Internal Set Theory) was presented in 1977 [47] and in a sense formulates within first-order language the behaviour of standard and internal sets of a (strong) nonstandard model. This is done by adding the unary standardness predicate "st" to the language of ZFC as well as adding to the axioms of ZFC three new axiom schemes involving the predicate "st": Idealization, Standardization and Transfer. One should read $st\,(x)$ as 'x is standard'. Nelson suggests to interpret "standard" informally as "fixed". Formulas which do not use the predicate st are called *internal* formulas (or $\in -$formulas) and formulas that use this new predicate are called *external* formulas (or $st- \in -$formulas). A formula φ is *standard* if only standard constants occur in φ. Before formulating the new axioms we introduce some useful abbreviations. We write $fin\,(x)$ meaning 'x is finite'. Let $\varphi\,(x)$ be a $st- \in -$formula:

$\forall^{st} x\varphi\,(x)$	abbreviates	$\forall x\,(stx \Rightarrow \varphi\,(x))$
$\exists^{st} x\varphi\,(x)$	abbreviates	$\exists x\,(st\,(x) \wedge \varphi\,(x))$
$\forall^{fin} x\varphi\,(x)$	abbreviates	$\forall x\,(fin\,(x) \Rightarrow \varphi\,(x))$
$\exists^{fin} x\varphi\,(x)$	abbreviates	$\exists x\,(fin\,(x) \wedge \varphi\,(x))$
$\forall^{stfin} x\varphi\,(x)$	abbreviates	$\forall x\,(st\,(x) \wedge fin\,(x) \Rightarrow \varphi\,(x))$
$\exists^{stfin} x\varphi\,(x)$	abbreviates	$\exists x\,(st\,(x) \wedge fin\,(x) \wedge \varphi\,(x))$

We are now able to formulate the new axioms of IST:

Idealization

$$\forall^{stfin} F \exists y \forall x \in F\ R\,(x, y) \Leftrightarrow \exists b \forall^{st} x\ R\,(x, b)\,, \tag{I}$$

for any internal relation R.

The idealization axiom states that saying that for any fixed finite set F there is a y such that $R\,(x, y)$ holds for all $x \in F$ is the same as saying that there is a b such that for all fixed x the relation $R\,(x, b)$ holds.

According to Nelson:

> The intuition behind (I) is that we can only fix a finite number of objects at a time. [49]

Standardization

$$\forall^{st} A \exists^{st} B \forall^{st} x \ \ (x \in B \Leftrightarrow x \in A \wedge \varphi(x)), \tag{S}$$

for every $st- \in -$ formula φ with arbitrary (internal) parameters. According to Nelson:

> The intuition behind (S) is that if we have a fixed set, then we can specify a fixed subset of it by giving a criterion for judging whether each fixed element is a member of it or not. [49]

Transfer

$$\forall^{st} y_1, ..., y_n \forall^{st} x \left(\varphi(x, y_1, ..., y_n) \Rightarrow \forall x \varphi(x, y_1, ..., y_n) \right), \tag{T}$$

for all internal standard $\varphi(x, y_1, ..., y_n)$. According to Nelson:

> the intuition behind (T) is that if something is true for a fixed, but arbitrary, x then it is true for all x. [49]

One interesting consequence of (I) is a theorem by Nelson [47] which states that there is a finite set F containing all standard sets. This implies that for any set A, there is a finite set F containing all standard elements of A, so all infinite sets have nonstandard elements. A useful consequence of (S) is the principle of *External Induction,* which states that for any (external or internal) formula φ, one has

$$\varphi(0) \wedge \left(\forall^{st} n \left(\varphi(n) \Rightarrow \varphi(n+1) \right) \right) \Rightarrow \left(\forall^{st} n \varphi(n) \right). \tag{EI}$$

In [48] Nelson showed that in IST is valid a reduction algorithm, that converts external formulas into internal equivalent formulas, for standard values of the parameters.

Some care is needed though, when working in IST, because external predicates are not allowed to define subsets. The violation of this rule is called *illegal set formation.* This is somewhat inconvenient because it implies that there are no external sets in IST. As argued in [33] [34], the axiomatics IST must be modified in order to admit an extension to external sets in a reasonable way. This is done by postulating a boundedness axiom that says that every set belongs to a standard set.

Boundedness

$$\forall x \exists^{st} y (x \in y),$$

and since this contradicts idealization the following (bounded) form is taken instead:

Bounded Idealization

$$\forall^{st} Y \left[\forall^{st fin} F \exists y \in Y (\forall x \in F \ R(x,y) \Leftrightarrow \exists b \in Y, \forall^{st} x \ R(x,b))\right],$$

for every $\in -$ formula R. This yields the subsystem BST, which corresponds to the bounded sets of IST. Notice that in usual analysis all sets are bounded. This means that BST is completely equivalent to IST in the known applications of IST. Nevertheless BST is somewhat more suitable for foundational purposes, as argued in [32] [34]. For example, in BST is valid a theorem [32] (valid in IST only for bounded formulas [48]) that allows to convert every formula φ in an equivalent formula φ' that is of the form $\exists^{st}\forall^{st}\psi$, where ψ is an $\in -$formula (formulas of this type are called $\sum_2^{st}$ formulas). This theorem permits to apply Nelson's *reduction algorithm* [48] without any restrictions.

1.5 Theories for external sets: HST

A "perfect" external set theory (a nonstandard set theory that includes external sets) should satisfy some requirements. It should be a conservative extension of classical mathematics (usually ZFC) so that all classical mathematical theorems and constructions remain valid. The theory should also allow to perform nonstandard constructions in its full generality and therefore include a strong version of saturation (called idealization in IST and bounded idealization in BST) and transfer principles. Finally it should allow to build, for any given set, the standard set of all its standard elements. This is called standardization. This means that ideally it should be something like an extension of IST allowing external sets and quantification over external formulas. However, as pointed out by Hrbáček [27] such a theory cannot exist. In fact, the axiom of regularity cannot be extended to the external universe. To see that let $\not\infty$ denote the external set of infinitely large real numbers. Observe that for all ω in the (nonempty) external set $\not\infty \cap \mathbb{N}$, one has $\not\infty \cap \mathbb{N} \cap \omega \neq \emptyset$. Additionally, if one wishes to formulate a nonstandard set theory with IST-style saturation[4], the replacement axiom in the external universe contradicts both power set and choice. Let n be a nonstandard natural number. By saturation there is a $1-1$ embedding into n, for all ordinals. So by power set and transfer the class Ord is a set (see Theorem 1.3.9 and Remark 1.3.10 in [34]). To be of *standard size* means to be an image of the set of all standard elements of a standard set[5].To see that choice fails, let x be well-ordered by a relation $\prec$. Consider the class of all standard ordinals ${}^{\sigma}Ord$, well-ordered by $\in$. We use the theorem that whenever two sets are well-ordered there is an order preserving embedding of one into the other (see Theorem 2.8 in [30]). Clearly ${}^{\sigma}Ord$ cannot be embedded into x, otherwise ${}^{\sigma}Ord$ would be a set. Then there is an embedding of x into ${}^{\sigma}Ord$. In fact, to an initial segment of ${}^{\sigma}Ord$. This means that x is of standard size. As

[4] In fact, standard size saturation is enough for the arguments that follow and lead to the "paradoxes".

[5] In HST, a set X is standard size if and only if X is well-ordered (see Section 1.3a in [34]).

a consequence, sets which are not of standard size cannot be well-ordered (see Theorem 1.3.1 in [34]).

These results are known as the *Hrbáček's paradoxes*. The first problem is not in fact a "real" problem because the regularity axiom is given so that every set is obtained at some level of the cumulative hierarchy over $\emptyset$ as mentioned in Section 1.2 and has no great impact on which theorems are true. This "nice picture" of the universe is contested by some mathematicians and a suitable anti-foundation axiom can be taken instead (see for example [1]).

In [27] Hrbáček considered already two possibilities to avoid this. The first one was to lose both power set and choice for external sets, leading to the system $\mathfrak{NS}_1$. The second one was to lose the replacement axiom for external sets, which lead to his theory $\mathfrak{NS}_2$[6]. A third possibility was developed by Kanovei and Reeken (see Part 3 of [32] and chapter 6 of [33]). The idea is to restrict saturation by a standard infinite cardinal κ in order to reinstate the power set axiom. This is a system of partially saturated external sets which modifies the system HST (described below), called HST_k. This may be a solution for many practical purposes but not a solution as a foundational system for the nonstandard methods.

The theory BST possesses an extension to HST [33] [34], which formulates within first-order language essential aspects of the behaviour of standard, internal and external sets within a nonstandard model, much as in Hrbáček's system $\mathfrak{NS}_1$. The system HST is conservative over ZFC [27] [32] and equiconsistent with both BST and ZFC (see Chapter 5 of [34]).

A set in HST is called *internal* if it is element of a standard set (see also the "Boundedness" axiom).

Below we use (definable) classes, they only should be interpreted as abbreviations of formulas with sets. Two important definable classes in HST are the class of standard sets

$$\mathbb{S} \equiv \{x : st\ x\}$$

and the class of internal sets

$$\mathbb{I} \equiv \{x : \exists y\,(st\ y \wedge x \in y)\}\,.$$

The definition of internal set in HST is different from the definition of internal set in IST. In fact, $(\mathbb{I}, \in, st)$ is a model of BST but not a model of IST (see Chapter 3 of [34]).

The axioms of HST are divided into three categories. The first deals with axioms which are valid for all sets, the second deals with axioms which are valid for standard or internal sets and the third deals with axioms which are valid for standard size sets. As aforementioned, to be of *standard size* means to be an image of the set of all standard elements of a standard set. Let X be a set. So a set is standard size if it is of the form

$$\{f\,(x) : x \in X \cap \mathbb{S}\}\,.$$

[6] Hrbacek proposed still a third theory called $\mathfrak{NS}_3$ that we do not consider here because it is not a conservative extension of ZFC.

As mentioned above in HST the power set axiom does not hold for external sets. However, if X is an internal set then

$$\mathcal{P}_{int}(X) = \mathcal{P}(X) \cap \mathbb{I}$$

exists. So power set holds for internal sets. Moreover if a set is standard size then $\mathcal{P}(X)$ is also standard size.

1.5.1 HST Axioms

Axioms for all sets

The axioms of this group are valid for all sets. These axioms are similar to the respective ones of ZFC with the difference that in HST they are presented in the full language. This implies in particular, by the axiom of separation, that the theory HST deals with external sets; for example if X is standard and infinite $\{x \in X : stx\}$ is an external set.

1. *Extensionality*
$$\forall X \forall Y (\forall x (x \in X \Leftrightarrow x \in Y) \Leftrightarrow X = Y).$$

2. *Pair*
$$\forall a, b \exists A \forall x (x \in A \Leftrightarrow x = a \vee x = b).$$

3. *Union*
$$\forall A \exists B \forall x (x \in B \Leftrightarrow \exists X \in A (x \in X)).$$

4. *Infinity*
$$\exists X (\emptyset \in X \wedge \forall x (x \in X \Rightarrow x \cup \{x\} \in X)).$$

5. *Separation*
$$\forall X \exists Y \forall x (x \in Y \Leftrightarrow (x \in X \wedge \varphi(x))).$$

6. *Collection*
$$\forall X \exists Y \forall x \in X (\exists y \varphi(x, y) \Rightarrow \exists y \in Y \varphi(x, y)).$$

The power set, regularity and choice axioms of ZFC are not valid in general. This is because, as mentioned above, each one of these axioms (if considered in the full language of HST) leads to a contradiction.

Axioms for standard and internal sets

In this group as well as in the next there are axioms which are not valid for all sets. The first axiom scheme states that all ZFC axioms, when restricted to standard parameters are valid in HST.

1. $$ZFC^{st}.$$

   This means, in particular, that the following are axioms of HST:

   (a) *Regularity*st

   $$\forall^{st} S \left(S \neq \emptyset \Rightarrow \left(\exists^{st} x \in S\right)(x \cap S \neq \emptyset)\right).$$

   (b) *Power Set*st

   $$\forall^{st} X \exists^{st} Y \forall^{st} x \, (x \in Y \iff x \subseteq X).$$

   (c) *Choice*st

   $$\forall^{st} S \exists^{st} Y \forall^{st} x \in S \backslash \{\emptyset\} \left(\exists^{st} z \, (Y \cap x = \{z\})\right).$$

   The fact that every axiom of ZFC restricted to standard sets is an axiom of HST means that the class $\mathbb{S}$ models ZFC.

2. *Transfer*

   $$\forall^{st} x_1 ... \forall^{st} x_n \left(\varphi^{st}(x_1, ...x_n) \Leftrightarrow \varphi^{int}(x_1, ...x_n)\right),$$

   where φ is an arbitrary closed $\in -$formula containing only standard parameters. This means that the universe $\mathbb{I}$ is an elementary extension of $\mathbb{S}$ in the ZFC language.

3. *Transitivity of* $\mathbb{I}$

   $$\forall^{int} x \forall y \, (y \in x \Rightarrow int \ y).$$

   The next axiom states that the class $\mathbb{I}$ is regular. This means that sets in HST are built over $\mathbb{I}$ in a way similar to the Von Neumann hierarchy of sets in ZFC over $\emptyset$.

4. *Regularity over* $\mathbb{I}$

   $$\forall X \neq \emptyset \exists x \in X \, (x \cap X \subset \mathbb{I}).$$

5. *Standardization*

   $$\forall X \exists^{st} Y \, (X \cap \mathbb{S} = Y \cap \mathbb{S}).$$

   This axiom implies that the only sets consisting entirely of standard sets are of the form $Y \cap \mathbb{S}$, where $Y \in \mathbb{S}$.

Axioms for sets of standard size

1. *Saturation*

   If $\mathcal{A} \subset \mathbb{I}$ is a standard size set then

$$((\forall X, Y \in \mathcal{A} \Rightarrow X \cap Y \in \mathcal{A}) \wedge (X \in \mathcal{A} \Rightarrow X \neq \emptyset)) \Rightarrow \cap\mathcal{A} \neq \emptyset.$$

2. *Standard Size Choice*

   Choice is available in the case where the domain of the choice function is of standard size.

   Let X be a set of standard size and F a function on X. Then

$$\forall x \in X\left((F(x) \neq \emptyset) \Rightarrow \exists f\left(f(x) \in F(x)\right)\right).$$

3. *Dependent Choice*

   Any nonempty partially ordered set without maximal elements includes a nonempty linearly ordered subset (sequence) where any element has its immediate successor[7].

It is possible to maintain the notation of IST in the way that the traditional symbols for the uniquely defined objects of a ZFC universe are attributed to the objects of the internal subuniverse.

In HST neutrices and external numbers are incorporated in the following way. Each neutrix, being a convex (bounded) subgroup of $\mathbb{R}$, is a genuine set within HST. Neutrices are defined by Π^{st} or Σ^{st} formulas, with reference to standard sets of all possible cardinals [4]. As shown in [33] this implies that we may not speak of the set of all neutrices. In fact, the neutrices form a definable class because quantification within HST may range over the whole universe. Then the algebraic operations on neutrices and external numbers are also definable classes, as well as the function which associates to each external number its neutrix.

[7] See [34], page 19 for an alternative formulation of this axiom.

Chapter 2

Neutrices and External Numbers

Sometimes it is useful to know how large your zero is.
(Author Unknown)

2.1 Introduction

In this chapter we study, from an algebraic point of view, a class of external sets called the external numbers. External numbers are, as a rule, bounded without having an infimum and supremum and invariant under at least some additions or translations, and therefore are models of orders of magnitude or transitions with imprecise boundaries. We show that the external numbers form a commutative regular semigroup for addition and that the external numbers which are not reduced to neutrices form a commutative regular semigroup for multiplication. Although the operations are not connected by a complete distributive law, we give necessary and sufficient conditions for distributivity to hold. Then we apply those results in order to obtain some binomial formulas. We also prove in an algebraic way some results which in [38] and [39] were proved by set-theoretic arguments. Most of the contents of this chapter were published in [15].

2.2 Preliminaries

We start by recalling some notions and results that will be useful in the remainder of this thesis. Unless otherwise mentioned we follow [38] and [39].

A real number x is *ilimited* (or infinitely large) if

$$\forall^{st} n \in \mathbb{N} \ \ |x| > n,$$

is *limited* if

$$\exists^{st} n \in \mathbb{N} \ \ |x| \leq n,$$

is *infinitesimal* (or infinitely small) if

$$\forall^{st} r \in \mathbb{R}^{+} \ \ |x| < r$$

and is *appreciable* if it is limited but not infinitesimal.

We recall that a *group* $(G, *)$ is is an algebraic structure, where $*$ is a binary operation satisfying the following axioms:

$$\forall a, b, c \in G \, ((a * b) * c = a * (b * c))$$

$$\exists^{1} e \forall \in G \, (a * e = e * a = a)$$

$$\forall a \in G \exists a' \in G \, (a * a' = a' * a = e)$$

This means that the operation $*$ is associative, has an identity element and an inverse for each element. If the operation $*$ also satisfies the following axiom

$$\forall a, b \in G \, (a * b = b * a)$$

we say that $(G, *)$ is a *commutative group*. A *subgroup* $(F, *)$ of a group $(G, *)$ is a group such that $F \subseteq G$. We say that $C \subseteq \mathbb{R}$ is *convex* if, for all x and y in C and all $t \in [0, 1]$, $(1 - t)\, x + ty \in C$.

Definition 2.2.1 *A* neutrix *is an additive convex subgroup of* $\mathbb{R}$*.*

Except for $\{0\}$ and $\mathbb{R}$ itself all neutrices are external sets (with internal elements). The most obvious (external) neutrices are

$$\pounds \equiv \bigcup_{st(n) \in \mathbb{N}}]-n, n[\,,$$

the external set of all *limited* numbers and

$$\oslash \equiv \bigcap_{st(n) \in \mathbb{N}} \left] -\frac{1}{n}, \frac{1}{n} \right[,$$

the external set of all *infinitesimal* numbers. It has been shown by Van den Berg [4] [5] that there are many neutrices not isomorphic by internal homomorphisms. Van den Berg also proved that all neutrices are external sets of the form $\bigcup_{x \in X} [-a_x, a_x]$ or $\bigcap_{x \in X} [-a_x, a_x]$, where X contains only standard elements (or is of standard size) and $a : X \to \mathbb{R}$. One important external set which is not a neutrix is

$$@ \equiv \bigcup_{st(n) \in \mathbb{N}} \left] \frac{1}{n}, n \right[,$$

the external set of all *appreciable* positive numbers.

We denote the external class of neutrices from now on by $\mathcal{N}$.

Addition and multiplication on $\mathcal{N}$ are defined by the Minkowski operations. So, if A, B are neutrices we define their sum by

$$A + B = \{a + b |\, (a, b) \in A \times B\}$$

and their product by

$$AB = \{ab |\, (a, b) \in A \times B\}\,.$$

Let A be a neutrix. Then, because neutrices are convex groups, $A+A = A$. We define the binary operation "$\leq$" on $\mathcal{N}$ by

$$A \leq B \Leftrightarrow A \subseteq B.$$

N.B. We use the symbol $\subseteq$ to indicate inclusion and $\subset$ to indicate strict inclusion. This means that neutrices are ordered by inclusion. An important consequence of this fact is that the sum of two neutrices is the larger of the two:

Proposition 2.2.2 *If A,$B \in \mathcal{N}$ then $A + B = \max(A, B)$.*

Proof. Suppose, without loss of generality, that B contains A. Then,

$$B \subseteq A + B \subseteq B + B = B.$$

Hence, $A + B = B$. ■

Neutrices are stable under multiplication by appreciable numbers, i.e. limited numbers which are not infinitesimal.

Proposition 2.2.3 *Let A be a neutrix. Then*

1. For all standard $n \in \mathbb{N}$, $nA = A$.

2. $@A = A$.

Proof. 1. The proof goes by external induction (EI). Clearly the statement is true if $n = 1$. Suppose that for some standard natural number n, $nA = A$. Then

$$(n + 1)\, A = nA + A = A + A = A.$$

2. Let a be an appreciable positive real number. Then there is a standard natural number such that $\frac{1}{n} \leq a \leq n$. Using Part 1 one has

$$A = \frac{nA}{n} = \frac{1}{n}A \subseteq aA \subseteq nA = A.$$

■

Definition 2.2.4 *An* external number α *is the algebraic sum of a real number a with a neutrix A.*

The external class of external numbers will be denoted by $\mathbb{E}$. If $\alpha = a + A$ and $\beta = b + B$ are two such external numbers, the Minkowski sum and product are given by

$$\begin{aligned} \alpha + \beta &= a + b + A + B \\ \alpha\beta &= ab + aB + bA + AB. \end{aligned}$$

Notice that by Proposition 2.2.2

$$\begin{aligned} \alpha + \beta &= a + b + \max(A, B) \\ \alpha\beta &= ab + \max(aB, bA, AB). \end{aligned}$$

Definition 2.2.5 *If $\alpha = a + A$ is an external number, then A is called the* neutrix part *of α and is denoted $N(\alpha)$. An external number which is not a neutrix is called* zeroless. *One defines $-\alpha = -a + A$ and $1/\alpha = 1/(a + A)$, if α is zeroless.*

Proposition 2.2.6 *Let $\alpha = a + A$ be an external number. Then*

$$\forall y \in \alpha \, (\alpha = y + A).$$

Proof. Let $y \in \alpha$. Then, there is $x \in A$ such that $y = a + x$ and hence, $y + A = a + x + A = a + A = \alpha$. ■

Definition 2.2.7 *Let A be a neutrix and α be an external number. We say that α is* appreciable with respect to *A if $\alpha A = A$, and that α is an* absorber *of A if $\alpha A \subset A$.*

Note that numbers which are appreciable with respect to $\oslash$ or $\pounds$ are simply appreciable.

Proposition 2.2.8 *Let $\alpha = a + A$ be a zeroless external number. Then $\frac{A}{a} = \frac{A}{\alpha} \subseteq \oslash$.*

Proof. If $x \in \frac{A}{a}$, then x cannot be appreciable, otherwise $a \in \frac{1}{x}A = A$, contradicting the fact that α is zeroless. Hence $\frac{A}{a} \subseteq \oslash$. Moreover $\frac{\alpha}{a} = 1 + \frac{A}{a}$. Hence $\frac{\alpha}{a} \subseteq 1 + \oslash$. So $\frac{a}{\alpha} \subseteq 1 + \oslash$. Therefore $\frac{A}{\alpha} = \frac{A}{a}\frac{a}{\alpha} = \frac{A}{a}$. ■

Hence, if $a \notin A$

$$\frac{1}{1 + \frac{A}{a}} = 1 + \frac{A}{a}$$

and

$$\left(1 + \frac{A}{a}\right)\left(1 + \frac{A}{a}\right) = \left(1 + \frac{A}{a}\right).$$

We state here some elementary properties of the multiplication. The first property is a direct consequence of the definition of multiplication.

Lemma 2.2.9 *Let $\alpha = a + A$ is zeroless. Then $\alpha B = aB + AB$ for all $B \in \mathcal{N}$.*

Lemma 2.2.10 *Let $\alpha = a + A$ and $\beta = b + B$ be zeroless external numbers. Then $\alpha\beta = ab + \max(aB, bA)$.*

Proof. Since $B \subseteq \oslash b$ by Proposition 2.2.8, $\max(BA, bA) = bA$. Hence $\alpha\beta = ab + \max(aB, bA, BA) = ab + \max(aB, bA)$. ■

Lemma 2.2.11 *Let $\alpha = a + A$ and $\beta = b + B$ be zeroless. Then $\alpha\beta = \alpha b + \alpha B$.*

Proof. Using Lemma 2.2.9 and 2.2.10, we derive

$$\begin{aligned} \alpha b + \alpha B &= (a + A)\, b + aB + AB \\ &= ab + bA + aB + AB \\ &= \alpha\beta. \end{aligned}$$

■

An order relation on $\mathbb{E}$ is given by the following.

Definition 2.2.12 *Given $\alpha, \beta \in \mathbb{E}$, we say that α is less than or equal to β and we write (with abuse of notation) $\alpha \leq \beta$, if and only if*

$$(\forall x \in \alpha)(\exists y \in \beta)(x \leq y). \tag{2.1}$$

We say that α is less than β and write $\alpha < \beta$, if $\alpha \leq \beta$ and $\alpha \cap \beta = \emptyset$.

Note that if $\alpha \cap \beta = \emptyset$, formula (2.1) is equivalent to $(\forall x \in \alpha)(\forall y \in \beta)(x < y)$. Two external numbers are always either disjoint or one contains the other.

Lemma 2.2.13 *Let α and β be two external numbers. Then*

$$\alpha \cap \beta = \emptyset \vee \alpha \subseteq \beta \vee \beta \subseteq \alpha.$$

Proof. Suppose that $\alpha \cap \beta \neq \emptyset$. Then, there is a real number x such that $x \in \alpha$ and $x \in \beta$. By Proposition 2.2.6 we may write $\alpha = x + A$ and $\beta = x + B$. Hence, if $\max(A, B) = A$, $\beta \subseteq \alpha$ and if $\max(A, B) = A$, $\alpha \subseteq \beta$. ■

Proposition 2.2.14 *Let A be a neutrix and let β and γ be external numbers such that $\beta \leq \gamma$. Then $A\beta \subseteq A\gamma$.*

Proof. Suppose that $\beta \leq \gamma$. Then there is $x \in \beta$ and $y \in \gamma$ such that $x \leq y$. Let $a \in A$. Then $|a|\, x \leq |a|\, z \in A\gamma$. Hence $A\beta \subseteq A\gamma$. ■

Theorem 2.2.15 *The relation $\leq$ is a total order relation compatible with addition and multiplication in the following way:*

1. $\forall\alpha\forall\beta\forall\gamma\, (\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma)$.
2. $\forall\alpha\forall\beta\, (N(\alpha) \leq \alpha \wedge N(\beta) \leq \beta \Rightarrow N(\alpha\beta) \leq \alpha\beta)$.
3. $\forall\alpha\forall\beta\forall\gamma\, (N(\alpha) \leq \alpha \leq \beta \wedge N(\gamma) \leq \gamma \Rightarrow \alpha\gamma \leq \beta\gamma)$.

Proof. Let α, β and γ be arbitrary external numbers.

We prove firstly that the relation $\leq$ is a total order relation. Let $x \in \alpha$. Because $x \leq x$ one has $\alpha \leq \alpha$, so the relation is reflexive. Suppose that $\alpha \leq \beta$ and $\beta \leq \gamma$. Then there is $y \in \beta$ and $z \in \gamma$ such that $x \leq y$ and $y \leq z$. Hence $x \leq z$ and the relation is transitive. Suppose now that $\alpha \leq \beta$ and $\beta \leq \alpha$. Then $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$. Hence by Lemma 2.2.13, one has $\alpha = \beta$ and the relation is antisymmetric. To prove the totality property suppose that $\alpha \nleq \beta$. Then there is $x \in \alpha$ such that $x > y$, for all $y \in \beta$. Hence $\beta \leq \alpha$. We conclude that the relation $\leq$ is a total order relation.

1. Suppose that $\alpha \leq \beta$. Let $a \in \alpha$. Then there is $b \in \beta$ such that $a \leq b$. Let $x \in \alpha + \gamma$. Then there is $c \in \gamma$ such that $x = a + c$. Hence $a + c \leq b + c \in \beta + \gamma$ and one concludes $\alpha + \gamma \leq \beta + \gamma$.

2. Suppose that $N(\alpha) \leq \alpha$ and $N(\beta) \leq \beta$. If $\alpha\beta$ is not zeroless then $\alpha\beta = N(\alpha\beta)$. If $\alpha\beta$ is zeroless by Lemma 2.2.10 one has $\alpha\beta = ab + \max(aB, bA)$. Let $x \in N(\alpha\beta)$. By Proposition 2.2.3.2 one has $|x| \in \max(aB, bA)$. If $\max(aB, bA) = aB$ there is $b \in B$ such that $|x| = ab$. Because $N(\alpha) \leq \alpha$, there are positive numbers $a' \in N(\alpha)$ and $a'' \in \alpha$ such that $a \leq a' \leq a''$. Also, there is $b' \in \beta$ such that $b \leq b'$, because $N(\beta) \leq \beta$. Then $x \leq |x| = ab \leq a^{''}b' \in \alpha\beta$ and $N(\alpha\beta) \leq \alpha\beta$. If $\max(aB, bA) = bA$ the proof is analogous.

3. Finally, suppose that $N(\alpha) \leq \alpha$, $\alpha \leq \beta$ and $N(\gamma) \leq \gamma$. Let $x \in \alpha\gamma$. Then, there is $a \in \alpha$ and $c \in \gamma$ such that $x = ac$. Moreover, a and c may be taken positive because $N(\alpha) \leq \alpha$ and $N(\gamma) \leq \gamma$. Because $\alpha \leq \beta$ there is $b \in \beta$ such that $a \leq b$. Then $ac \leq bc \in \beta\gamma$ and $\alpha\gamma \leq \beta\gamma$.

■

2.3 Algebraic properties for addition and multiplication

We start by showing that external numbers when equipped with addition and zeroless external numbers when equipped with multiplication are regular commutative semigroups. The neutral and unity elements appear in the form of external functions. We study these functions in some detail, also in order to obtain cancellation laws.

We give algebraic proofs of some properties of addition and multiplication, which were originally proved in [38] by set theoretical arguments.

2.3.1 External numbers and regular semigroups

We recall that a *semigroup* is a structure $(S, *)$ such that S is nonempty and "$*$" is a binary operation that satisfies the following axiom

$$\forall x, y, z \in S \, (x * (y * z) = (x * y) * z) \, .$$

A *regular* semigroup is a semigroup S such that every element is *regular*, that is, for every $a \in S$ there is $x \in S$ such that $axa = a$. We shall prove that $(\mathbb{E}, +)$ and $(\mathbb{E} \backslash \mathcal{N}, \cdot)$ are both commutative regular semigroups.

Theorem 2.3.1 *The structures* $(\mathbb{E}, +)$ *and* $(\mathbb{E} \backslash \mathcal{N}, \cdot)$ *are commutative regular semigroups.*

Proof. Let $\alpha = a + A$, $\beta = b + B$ and $\gamma = c + C$ be arbitrary external numbers. Firstly we prove that both operations are commutative. In fact, one has

$$\begin{aligned} \alpha + \beta &= (a + A) + (b + B) = a + b + \max(A, B) \\ &= b + a + \max(B, A) = (b + B) + (a + A) = \beta + \alpha \end{aligned}$$

and

$$\begin{aligned} \alpha\beta &= (a + A)(b + B) = ab + \max(aB, bA, AB) \\ &= ba + \max(bA, aB, AB) = (b + B)(a + A) = \beta\alpha. \end{aligned}$$

Secondly we prove that both operations are associative. Indeed,

$$\begin{aligned} (\alpha + \beta) + \gamma &= ((a + A) + (b + B)) + (c + C) \\ &= (a + b + \max(A, B)) + (c + C) \\ &= (a + b) + c + \max(\max(A, B), C) \\ &= a + (b + c) + \max(A, \max(B, C)) \\ &= (a + A) + (b + c + \max(B, C)) = \alpha + (\beta + \gamma) \end{aligned}$$

and

$$\begin{aligned} (\alpha\beta)\gamma &= ((a + A)(b + B))(c + C) \\ &= (ab + \max(aB, bA, AB))(c + C) \\ &= (ab)c + \max(abC, c\max(aB, bA, AB), \max(aB, bA, AB)C) \\ &= (ab)c + \max(abC, acB, bcA, cAB, aBC, bAC, ABC) \\ &= a(bc) + \max(a\max(bC, cB, CB), bcA, A\max(bC, cB, CB)) \\ &= (a + A)(bc + \max(bC, cB, CB)) \\ &= (a + A)((b + B)(c + C)) \\ &= \alpha(\beta\gamma). \end{aligned}$$

Finally, we prove the regularity properties. Let $\alpha = a + A$ be an arbitrary external number. Put $\delta = (-a + A)$ and $\zeta = \left(\frac{1}{a} + \frac{A}{a^2}\right)$. Then

$$\alpha + \delta + \alpha = (a + A) + (-a + A) + (a + A) = (a + A) = \alpha.$$

If α is zeroless, necessarily $a \neq 0$, so applying Lemma 2.2.10

$$\begin{aligned} \alpha\zeta\alpha &= (a + A)\left(\frac{1}{a} + \frac{A}{a^2}\right)(a + A) = \left(1 + \frac{aA}{a^2} + \frac{A}{a}\right)(a + A) \\ &= \left(1 + \frac{A}{a}\right)(a + A) = a + A + \frac{aA}{a} = a + A = \alpha. \end{aligned}$$

Hence $(\mathbb{E}, +)$ and $(\mathbb{E} \backslash \mathcal{N}, .)$ are commutative regular semigroups. ■

2.3.2 Properties of neutral and inverse elements

Uniqueness for identity and inverse elements holds neither for addition nor for multiplication, as seen by the following examples.

Example 2.3.2 *Let $\alpha = 1 + \oslash$ and $\varepsilon \simeq 0$.*

1. $\alpha + \oslash = 1 + \oslash + \oslash = 1 + \oslash = \alpha$ *and* $\alpha + \varepsilon\oslash = 1 + \oslash + \varepsilon\oslash = 1 + \oslash = \alpha$.
2. *Let* $\beta = 1 + \varepsilon\oslash$. *Then* $\alpha\alpha = (1+\oslash)(1+\oslash) = 1 + \oslash = \alpha$ *and* $\alpha\beta = (1+\oslash)(1+\varepsilon\oslash) = 1 + \oslash + \varepsilon\oslash = 1 + \oslash = \alpha$.

We introduce functions which generalize the concept of neutral element in order to have individualized neutral elements for addition and multiplication. For $\alpha \in \mathbb{E}$ these functions give us the maximal elements that leave α invariant; as such, they are unique neutral elements. We investigate the properties of such functions and will see that addition and multiplication have a common structure.

Proposition 2.3.3 *There is a unique function $e : \mathbb{E} \to \mathbb{E}$ such that (i) $\alpha + e(\alpha) = \alpha$ for all $\alpha \in \mathbb{E}$ and (ii) if $f : \mathbb{E} \to \mathbb{E}$ is such that $\alpha + f(\alpha) = \alpha$ for all $\alpha \in \mathbb{E}$, then $e(\alpha) + f(\alpha) = e(\alpha)$. In fact, $e(\alpha) = N(\alpha)$.*

Proof. We prove that $e : \mathbb{E} \to \mathbb{E}$ defined by $e(\alpha) = N(\alpha)$ is the required function. Let $\alpha = a + A$ be an arbitrary external number. Then

$$\alpha + e(\alpha) = \alpha + N(\alpha) = (a + A) + A = a + A = \alpha.$$

Hence, for all $f : \mathbb{E} \to \mathbb{E}$ such that $\alpha + f(\alpha) = \alpha$

$$e(\alpha) + f(\alpha) = A + f(\alpha) = \alpha - \alpha + f(\alpha) = -\alpha + \alpha = e(\alpha).$$

■

Proposition 2.3.4 *There is a unique function $u : \mathbb{E}\backslash\mathcal{N} \to \mathbb{E}$ such that (i) $\alpha u(\alpha) = \alpha$ for all $\alpha \in \mathbb{E}\backslash\mathcal{N}$ and (ii) if $v : \mathbb{E}\backslash\mathcal{N} \to \mathbb{E}$ verifies $\alpha v(\alpha) = \alpha$ for all $\alpha \in \mathbb{E}\backslash\mathcal{N}$, then $u(\alpha)v(\alpha) = u(\alpha)$. In fact, $u(\alpha) = 1 + \frac{N(\alpha)}{\alpha}$.*

Proof. We prove that $u : \mathbb{E}\backslash\mathcal{N} \to \mathbb{E}$ defined by $u(\alpha) = 1 + \frac{N(\alpha)}{\alpha}$ is the required function. Let $\alpha = a + A$ be an arbitrary zeroless external number. Then using Proposition 2.2.8 and Lemma 2.2.10,

$$\begin{aligned} \alpha u(\alpha) &= \alpha\left(1 + \frac{N(\alpha)}{\alpha}\right) = (a + A)\left(1 + \frac{A}{a}\right) \\ &= a + \max\left(a\frac{A}{a}, A\right) = a + A = \alpha. \end{aligned}$$

Let $v : \mathbb{E}\backslash\mathcal{N} \rightarrow \mathbb{E}$ be such that $\alpha v(\alpha) = \alpha$ for all $\alpha \in \mathbb{E}\backslash\mathcal{N}$. Then applying Lemma 2.2.11

$$\begin{aligned} u(\alpha)\, v(\alpha) &= \left(1 + \frac{N(\alpha)}{\alpha}\right) v(\alpha) = \left(1 + \frac{A}{a}\right) v(\alpha) \\ &= \frac{(a+A)\, v(\alpha)}{a} = \frac{a+A}{a} = 1 + \frac{A}{a} = u(\alpha). \end{aligned}$$

■

Corollary 2.3.5 *Let α be a zeroless external number. Then $u(\alpha) \neq e(\alpha)$.*

Proposition 2.3.6 *There is a unique function $s : \mathbb{E} \rightarrow \mathbb{E}$ such that $\alpha + s(\alpha) = e(\alpha)$ and $e(s(\alpha)) = e(\alpha)$ for all $\alpha \in \mathbb{E}$. In fact, $s(\alpha) = -\alpha$.*

Proof. Let $\alpha = a + A$ be an arbitrary external number. We prove that $s : \mathbb{E} \rightarrow \mathbb{E}$ defined by $s(\alpha) = -\alpha$ is the required function. one has

$$\alpha + s(\alpha) = (a + A) + (-a + A) = A = e(\alpha)$$

and

$$e(s(\alpha)) = e(-a + A) = A = e(\alpha).$$

Suppose that $t : \mathbb{E} \rightarrow \mathbb{E}$ is such that $\alpha + t(\alpha) = e(\alpha)$ and $e(t(\alpha)) = e(\alpha)$. Then $e(t(\alpha)) = e(\alpha) = e(s(\alpha))$ and

$$\begin{aligned} t(\alpha) &= t(\alpha) + e(t(\alpha)) = t(\alpha) + e(\alpha) = t(\alpha) + \alpha + s(\alpha) \\ &= e(\alpha) + s(\alpha) = e(s(\alpha)) + s(\alpha) = s(\alpha). \end{aligned}$$

■

Proposition 2.3.7 *There is a unique function $d : \mathbb{E}\backslash\mathcal{N} \rightarrow \mathbb{E}$ such that $\alpha d(\alpha) = u(\alpha)$ and $u(d(\alpha)) = u(\alpha)$, for all $\alpha \in \mathbb{E}\backslash\mathcal{N}$. In fact $d(\alpha) = \frac{1}{\alpha} = \frac{1}{a} + \frac{A}{a^2}$.*

Proof. Let $\alpha = a + A$ be an arbitrary external number. We prove that $d : \mathbb{E}\backslash\mathcal{N} \rightarrow \mathbb{E}$, defined by

$$d(\alpha) = \frac{1}{\alpha} = \frac{1}{a}\left(\frac{1}{1 + \frac{A}{a}}\right) = \frac{1}{a}\left(1 + \frac{A}{a}\right) = \frac{1}{a} + \frac{A}{a^2}$$

verifies the required conditions. one has

$$\alpha d(\alpha) = (a + A)\left(\frac{1}{a} + \frac{A}{a^2}\right) = 1 + \frac{A}{a} = u(\alpha)$$

and, by Proposition 2.3.4 and Proposition 2.2.8,

$$u(d(\alpha)) = u\left(\frac{1}{a} + \frac{A}{a^2}\right) = 1 + \frac{\frac{A}{a^2}}{\frac{1}{a}} = 1 + \frac{A}{a} = u(\alpha).$$

Uniqueness is shown in the same way as in the proof of Proposition 2.3.6. ■

It has been proved in Proposition 2.2.2 that always $e(\alpha + \beta) = e(\alpha)$ or $e(\alpha + \beta) = e(\beta)$. We prove an analogous property for multiplication.

Proposition 2.3.8 *Let α and β be zeroless external numbers. One has $u(\alpha\beta) = u(\alpha)$ or $u(\alpha\beta) = u(\beta)$.*

Proof. Let $\alpha = a + A$ and $\beta = b + B$ be zeroless. Then

$$\begin{aligned} u(\alpha\beta) &= u(ab + aB + bA) = 1 + \frac{aB + bA}{ab} \\ &= 1 + \max(\frac{B}{b}, \frac{A}{a}) = 1 + \max(\frac{N(\beta)}{\beta}, \frac{N(\alpha)}{\alpha}). \end{aligned}$$

Hence, $u(\alpha\beta) = u(\alpha)$ or $u(\alpha\beta) = u(\beta)$. ■

The fact that addition and multiplication have the above properties in common suggests the definition of an algebraic structure which we will call assembly. Assemblies and their properties will be studied in Section 3.3 of Chapter 3.

We finish by exploring the connection between the neutral and inverse functions of addition and multiplication.

Proposition 2.3.9 *The functions e and s have the following properties with respect to multiplication. For all $\alpha, \beta \in \mathbb{E}$*

1. $e(\alpha\beta) = \beta e(\alpha) + \alpha e(\beta)$.
2. $s(\alpha\beta) = s(\alpha)\beta = \alpha s(\beta)$.
3. $\alpha\beta = e(\alpha\beta) \Leftrightarrow \alpha = e(\alpha) \vee \beta = e(\beta)$.

Proof. Let $\alpha = a + A$ and $\beta = b + B$ be arbitrary external numbers. By Proposition 2.3.3 one has $e(\alpha) = N(\alpha) = A$ and $e(\beta) = N(\beta) = B$.

1. one has

$$\begin{aligned} e(\alpha\beta) &= e((a + A)(b + B)) \\ &= e(ab + aB + bA + AB) = aB + bA + AB \end{aligned}$$

and

$$\begin{aligned} \beta e(\alpha) + \alpha e(\beta) &= (b + B)A + (a + A)B \\ &= bA + AB + aB + AB = aB + bA + AB. \end{aligned}$$

Therefore $e(\alpha\beta) = \beta e(\alpha) + \alpha e(\beta)$.

2It holds that

$$\begin{aligned} s(\alpha\beta) &= s(ab + aB + bA + AB) \\ &= -ab + aB + bA + AB \\ &= (-a + A)(b + B) = s(\alpha)\beta. \end{aligned}$$

3. To prove the direct implication we assume that $\alpha\beta = e(\alpha\beta)$. Then

$$ab + aB + bA + AB = aB + bA + AB.$$

This implies that $ab \in aB$, $ab \in bA$ or $ab \in AB$. Suppose that $\alpha \neq e(\alpha)$ and $\beta \neq e(\beta)$. Then $a \notin A$ and $b \notin B$. Hence $ab \notin aB$, $ab \notin bA$ and $ab \notin AB$, a contradiction. We conclude that

$$\alpha\beta = e(\alpha\beta) \Rightarrow \alpha = e(\alpha) \vee \beta = e(\beta). \tag{2.2}$$

Assume now that $\alpha = e(\alpha)$ or $\beta = e(\beta)$. If $\alpha = e(\alpha)$, by Lemma 2.2.9

$$\alpha\beta = A\beta = Ab + AB = e(\alpha\beta).$$

The other case is analogous. Hence

$$\alpha = e(\alpha) \vee \beta = e(\beta) \Rightarrow \alpha\beta = e(\alpha\beta). \tag{2.3}$$

Combining (2.2) and (2.3), we obtain Part 3. ∎

Observe that the interpretation of neutrices as generalized zeros is further justified by Part 3 of the previous proposition, which states that in a sense zero divisors can only be neutrices.

This interpretation is enhanced by the next proposition, stating that neutral elements for addition are invariant for neutral elements for multiplication.

Proposition 2.3.10 *For all $\alpha \in \mathbb{E}$ and $\beta \in \mathbb{E} \backslash \mathcal{N}$ it holds that $e(\alpha)\,u(\beta) = e(\alpha)$.*

Proof. Let $\alpha = a + A$ be an external number and assume that $\beta = b + B$ is zeroless. Then by Proposition 2.2.8

$$e(\alpha)\,u(\beta) = A\left(1 + \frac{B}{b}\right) = A + \frac{AB}{b} = A = e(\alpha).$$

∎

The final proposition determines the neutral function for addition of the neutral function for multiplication.

Proposition 2.3.11 *For all $\alpha \in \mathbb{E} \backslash \mathcal{N}$ it holds that $e(u(\alpha)) = e(\alpha)\,d(\alpha)$.*

Proof. Assume that $\alpha = a + A$ is zeroless. Then by Proposition 2.3.7 and 2.2.8

$$e(\alpha)\,d(\alpha) = A\left(\frac{1}{a} + \frac{A}{a^2}\right) = \frac{A}{a} = e(u(\alpha)).$$

∎

2.4 Distributivity

External numbers are intervals of real numbers and therefore multiplication is *subdistributive* with respect to addition [44], i.e. for all external numbers α, β and γ

$$\alpha(\beta + \gamma) \subseteq \alpha\beta + \alpha\gamma. \tag{2.4}$$

However, proper distributivity does not always hold. Take for example $\alpha = \oslash, \beta = \omega + 1$ and $\gamma = -\omega$, where ω is an unlimited number. Then $\alpha(\beta+\gamma) = \oslash((\omega+1)-\omega) = \oslash$ and $\alpha\beta + \alpha\gamma = (\omega+1)\oslash - \omega\oslash = \omega\oslash$. Nevertheless the validity of the distributive law can be completely characterized. To this end we introduce the following notions.

Definition 2.4.1 *Let $\alpha \in \mathbb{E}$. If α is zeroless, we define the* relative uncertainty *of α to be the neutrix $\mathcal{R}(\alpha) = \frac{A}{a}$. If $\alpha \in \mathcal{N}$, we define $\mathcal{R}(\alpha) = \mathbb{R}$.*

Remark 2.4.2 *Let $\alpha = a + A$ be a zeroless external number. Then $\mathcal{R}(\alpha) \subseteq \oslash$ by Proposition 2.2.8. Moreover $\alpha = a(1 + \mathcal{R}(\alpha))$, because Lemma 2.2.11 implies that $a(1+\mathcal{R}(\alpha)) = a + a\mathcal{R}(\alpha) = a + A = \alpha$.*

Definition 2.4.3 *Let α and β be external numbers. We say that α is* (asymptotically) more precise *than β if $\mathcal{R}(\alpha) \subseteq \mathcal{R}(\beta)$.*

Observe that each zeroless external number is more precise than any neutrix.

Definition 2.4.4 *Let A be a neutrix and β and γ be external numbers. Then β and γ are called* opposite with respect to A *if $(\beta+\gamma)A \subset \max(|\beta|, |\gamma|)A$.*

Examples. Two real numbers b and c which are opposite, i.e. such that $b = -c$ are opposite with respect to all neutrices $N \supset \{0\}$. Appreciable real numbers β and γ are opposite with respect to $\oslash$ and $\pounds$ if and only if $\beta \simeq -\gamma$. Let ω be unlimited. Then $\omega + 1$ and $-\omega$ are opposite with respect to $\oslash$ without being nearly equal, because as we already saw

$$((\omega+1)-\omega)\oslash = \oslash \subset \omega\oslash = (\omega+1)\oslash .$$

If two numbers are opposite with respect to a given neutrix, none of them can be a neutrix. To see this observe first that $\beta + \gamma = \max(|\beta|, |\gamma|)$ if both are a neutrix or if one of them, say β, is a neutrix and $\gamma \subseteq \beta$. In the remaining case we may suppose that β is a neutrix and $\beta < |\gamma|$. Clearly $\gamma A \subseteq (\beta+\gamma)A$. Since $\beta/\gamma \subseteq \oslash$, it follows from Lemma 2.2.11 that $\beta + \gamma \subseteq \oslash\gamma + \gamma = (1+\oslash)\gamma$, so $(\beta+\gamma)A \subseteq (1+\oslash)\gamma A = \gamma A$. Hence $(\beta+\gamma)A = \gamma A = \max(|\beta|, |\gamma|)A$.

Numbers of the same sign are never opposite with respect to a given neutrix. In fact, two external numbers β and γ which are opposite with respect to a given neutrix A must satisfy $\beta/\gamma \subseteq -1 + \oslash$. Indeed, if $-1 + a \in \beta/\gamma$ with $a \not\simeq 0$,

$$(\beta+\gamma)A = \begin{cases} \beta A = \max(|\beta|, |\gamma|)A & |a| \simeq \infty \\ (-1+a)\gamma A = \gamma A = \beta A & -1+a \text{ appreciable} \\ \gamma A = \max(|\beta|, |\gamma|)A & -1+a \simeq 0. \end{cases}$$

The latter observation enables a characterization in terms of absorbers: If β and γ are opposite with respect to A, both $(\beta+\gamma)A \subset \beta A$ and $(\beta+\gamma)A \subset \gamma A$, hence $(\beta+\gamma)/\gamma$ and $(\beta+\gamma)/\beta$ are absorbers of A. Oppositeness is directly related to distributivity. Indeed, if β and γ are opposite with respect to A,

$$(\beta+\gamma)A \subset \max(|\beta|, |\gamma|)A = \max(\beta A, \gamma A) = \beta A + \gamma A,$$

and if β and γ are not opposite with respect to A,

$$(\beta + \gamma) A = \max(|\beta|, |\gamma|) A = \max(\beta A, \gamma A) = \beta A + \gamma A. \tag{2.5}$$

The next two lemmas are useful in dealing with oppositeness with respect to linear combinations of neutrices.

Lemma 2.4.5 *Let $\alpha = a + A$, $\beta = b + B$ and $\gamma = c + C$ be external numbers. Let M and N be neutrices.*

1. *If α and β are not opposite with respect to M nor N, then α and β are not opposite with respect to $M + N$.*
2. *If α and β are not opposite with respect to M, then α and β are not opposite with respect to γM.*

Proof. Suppose that α and β are not opposite with respect to M nor N. ■

1. As a consequence of Proposition 2.2.2 and formula (2.5) one has

$$\begin{aligned}(\alpha + \beta)(M + N) &= (\alpha + \beta) M + (\alpha + \beta) N \\ &= \alpha M + \beta M + \alpha N + \beta N \\ &= \alpha(M + N) + \beta(M + N).\end{aligned}$$

2. By formula (2.5)

$$\begin{aligned}(\alpha + \beta)\gamma M &= \gamma(\alpha + \beta) M \\ &= \gamma(\alpha M + \beta M) \\ &= \alpha\gamma M + \beta\gamma M.\end{aligned}$$

We are now able to state the criterion for distributivity:

Theorem 2.4.6 *Let α, β and γ be external numbers. Then $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ if and only if (i) α is more precise than β or γ, or (ii) β and γ are not opposite with respect to $N(\alpha)$.*

If one of the numbers is a neutrix, we may identify the following special cases:

Theorem 2.4.7 *Let α, β and γ be external numbers.*

1. *If $\alpha \in \mathcal{N}$ and neither $\beta \in \mathcal{N}$, nor $\gamma \in \mathcal{N}$, then $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ if and only if β and γ are not opposite with respect to α.*
2. *If $\beta \in \mathcal{N}$, or $\gamma \in \mathcal{N}$, then $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.*

A second important special case concerns external numbers of the same sign.

Theorem 2.4.8 *If α is an external number and β and γ are external numbers of the same sign, then $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.*

Let us illustrate the above results with some examples:

1. Let $\alpha = 1+\oslash$, $\beta = 1+\varepsilon$ with $\varepsilon \simeq 0$, and $\gamma = 1$. Then $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$ by Theorem 2.4.8. The equality follows also from Theorem 2.4.6 (ii). On one hand,

$$\begin{aligned} \alpha(\beta+\gamma) &= (1+\oslash)(1+\varepsilon+1) = (1+\oslash)(2+\varepsilon) \\ &= 2+\varepsilon+(2+\varepsilon)\oslash \quad \text{by Theorem 2.4.6 (ii)} \\ &= 2+\varepsilon+\oslash \quad \text{because } 2+\varepsilon \text{ is appreciable} \\ &= 2+\oslash. \end{aligned}$$

On the other hand,

$$\begin{aligned} \alpha\beta+\alpha\gamma &= (1+\oslash)(1+\varepsilon)+(1+\oslash)1 \\ &= 1+\oslash+(1+\oslash)\varepsilon+1+\oslash \quad \text{by Theorem 2.4.6 (ii)} \\ &= 1+\oslash+\varepsilon+\varepsilon\oslash+1+\oslash \quad \text{by Theorem 2.4.6 (ii)} \\ &= 2+\oslash. \end{aligned}$$

2. Let $\alpha = 1+\oslash$, $\beta = 1+\varepsilon$ with $\varepsilon \simeq 0$, and $\gamma = -1$. Then

$$\alpha(\beta+\gamma) = (1+\oslash)(1+\varepsilon-1) = (1+\oslash)\varepsilon = \varepsilon+\varepsilon\oslash$$

and

$$\begin{aligned} \alpha\beta+\alpha\gamma &= (1+\oslash)(1+\varepsilon)+(1+\oslash)(-1) \\ &= 1+\oslash+(1+\oslash)\varepsilon-1+\oslash \quad \text{by Theorem 2.4.6 (ii)} \\ &= \oslash+\varepsilon+\oslash\varepsilon \quad \text{by Theorem 2.4.6 (ii)} \\ &= \oslash. \end{aligned}$$

Because $\varepsilon\oslash \subset \oslash$, subdistributivity holds, but distributivity does not. This is in line with the fact that α is less precise than both β and γ and $1+\varepsilon$ and -1 are opposite with respect to $\oslash$. If we change β into $\beta' = 1+\oslash$, then α is as precise as β' and one verifies indeed that

$$\alpha\left(\beta'+\gamma\right) = \oslash = \alpha\beta'+\alpha\gamma.$$

If ε is appreciable, the numbers $1+\varepsilon$ and -1 are no longer opposite with respect to $\oslash$ and we see that

$$\alpha(\beta+\gamma) = \varepsilon+\varepsilon\oslash = \varepsilon+\oslash = \oslash+\varepsilon+\oslash\varepsilon = \alpha\beta+\alpha\gamma.$$

3. If $\alpha = \oslash$, $\beta = \omega+\pounds$, $\gamma = -\omega+\oslash$, where $\omega \simeq +\infty$, distributivity does not hold by Theorem 2.4.7.1, for β and γ are opposite with respect to $\oslash$. Indeed, one shows with the aid of Theorem 2.4.7.2 that

$$\begin{aligned} \alpha(\beta+\gamma) &= \oslash\pounds = \oslash \subset \oslash\omega \\ &= \oslash\omega+\oslash\pounds = \oslash(\omega+\pounds) = \alpha\max(|\beta|,|\gamma|). \end{aligned}$$

4. If $\alpha = \sqrt{\omega} + \pounds$, $\beta = \omega + \pounds$, $\gamma = \pounds$,where $\omega \simeq +\infty$, distributivity holds. Indeed, by Theorem 2.4.6 (i) and Theorem 2.4.7.2

$$\begin{aligned} \alpha(\beta+\gamma) &= (\sqrt{\omega}+\pounds)(\omega+\pounds) = (\sqrt{\omega}+\pounds)\omega + (\sqrt{\omega}+\pounds)\pounds \\ &= \omega\sqrt{\omega} + \pounds\omega + \sqrt{\omega}\pounds + \pounds\pounds = \omega\sqrt{\omega} + \pounds\omega. \end{aligned}$$

Note that we have calculated in fact $\alpha\beta$, and that $\alpha\gamma = (\sqrt{\omega} + \pounds)\pounds$ $= \sqrt{\omega}\pounds + \pounds\pounds = \sqrt{\omega}\pounds$ is contained in $N(\alpha\beta)$. Hence $\alpha\beta + \alpha\gamma = \alpha\beta = \alpha(\beta+\gamma)$.

Some of these calculations may be obtained directly by the definition of multiplication for external numbers. In fact this rule plays a substantial part in the proof of Theorem 2.4.6. We prove Theorem 2.4.6 considering two separate cases: the case where one of the numbers is a neutrix (Section 2.4.1) and the case where none of the external numbers is a neutrix (Section 2.4.2).

2.4.1 Distributivity with neutrices

In this section we prove Theorem 2.4.6 in the case where at least one of the external numbers α, β, γ is a neutrix. We have to consider two subcases, (i) β or γ is a neutrix and (ii) α is a neutrix.

Proof of Theorem 2.4.6 in the case where β or γ is a neutrix: one has $\mathcal{R}(\beta) = \mathbb{R}$ or $\mathcal{R}(\gamma) = \mathbb{R}$. Then the criterion

$$\mathcal{R}(\alpha) \leq \max(\mathcal{R}(\beta), \mathcal{R}(\gamma)) \vee (\beta+\gamma) A = A \max(|\beta|, |\gamma|)$$

is trivially satisfied. Conversely, suppose without loss of generality that $\gamma = C \in \mathcal{N}$. Because aB, aC, bA, AB, AC are neutrices, by Proposition 2.2.2

$$\begin{aligned} \alpha(\beta+\gamma) &= (a+A)(b+\max(B,C)) \\ &= ab + a\max(B,C) + bA + A\max(B,C) \\ &= ab + \max(aB, aC, bA, AB, AC) \end{aligned}$$

and

$$\begin{aligned} \alpha\beta + \alpha\gamma &= ab + \max(aB, bA, AB) + \max(aC, AC) \\ &= ab + \max(aB, aC, bA, AB, AC). \end{aligned}$$

Hence $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$.■

For the sake of clarity we mention in the form of a corollary the special cases which involve at least two neutrices.

Corollary 2.4.9 *Let α be an arbitrary external number and let B, $C \in \mathcal{N}$. Then $\alpha(B+C) = \alpha B + \alpha C$.*

Corollary 2.4.10 *If A, B, $C \in \mathcal{N}$, then $A(B+C) = AB + AC$.*

Corollary 2.4.11 *Let A, $C \in \mathcal{N}$ and let β be zeroless. Then $A(\beta + C) = A\beta + AC$.*

Proof of Theorem 2.4.6 in the case where α is a neutrix: Let $\alpha = A$ be an arbitrary neutrix. Without loss of generality we may assume that $|\beta| \geq |\gamma|$. Firstly suppose that $A(\beta + \gamma) = A\beta + A\gamma$. Then $A\beta + A\gamma = A\beta = A\max(|\beta|, |\gamma|)$. Hence β and γ are not opposite with respect to A.

Suppose now that A is more precise than β or γ, or that β and γ are not opposite with respect to A. In the first case β or γ has to be a neutrix because $\mathcal{R}(A) = \mathbb{R}$. This case is contained in Corollaries 2.4.10 and 2.4.11. In the second case $A(\beta + \gamma) = A\max(|\beta|, |\gamma|) = A\beta = A\beta + A\gamma$.■

2.4.2 Distributivity with zeroless external numbers

Let α, β and γ be zeroless external numbers. Unless stated otherwise, we always write $\alpha = a + A$, $\beta = b + B$ and $\gamma = c + C$, where a, b and c are real numbers and A, B and C are neutrices, with $A < |a|$, $B < |b|$ and $C < |c|$. Hence by Remark 2.4.2 $\alpha = a(1 + \mathcal{R}(\alpha))$, $\beta = b(1 + \mathcal{R}(\beta))$ and $\gamma = c(1 + \mathcal{R}(\gamma))$.

In order to prove the criterion for distributivity we suppose without loss of generality that $|\beta| \geq |\gamma|$. Then we may also suppose that $0 < \left|\frac{c}{b}\right| \leq 1$.

We prove the criterion for distributivity first in the case that $a = 1$, $b = 1$ and $0 < |c| \leq 1$; then $A \leq \oslash$ and $B \leq \oslash$ by Lemma 2.2.8. The general case will be obtained by rescaling.

To do so, we need to give direct proofs of distributivity in some relatively easy special cases.

Lemma 2.4.12 *One has $a(\beta + \gamma) = a\beta + a\gamma$.*

Proof. By Lemma 2.2.11 and Corollary 3.9.2 one has

$$\begin{aligned}
a(\beta + \gamma) &= a((b + c) + \max(B, C)) \\
&= a(b + c) + a\max(B, C) \\
&= ab + ac + aB + aC \\
&= a(b + B) + a(c + C) \\
&= a\beta + a\gamma.
\end{aligned}$$

■

Theorem 2.4.13 *Let α, β and γ be arbitrary external numbers such that $\alpha = a + A$. Then*

$$\alpha\beta + \alpha\gamma = \alpha(\beta + \gamma) + A\beta + A\gamma.$$

Proof. Using Lemma 2.2.11 and Lemma 2.4.12 one has

$$\begin{aligned}
\alpha(\beta + \gamma) + A\beta + A\gamma &= (a + A)(\beta + \gamma) + A\beta + A\gamma \\
&= a(\beta + \gamma) + A(\beta + \gamma) + A\beta + A\gamma \\
&= a\beta + a\gamma + A(\beta + \gamma) + A\beta + A\gamma.
\end{aligned}$$

Because $A\beta$ and $A\gamma$ are neutrices and Formula (2.4) one has

$$\alpha(\beta+\gamma)+A\beta+A\gamma = a\beta+a\gamma+A\beta+A\gamma.$$

Hence, by Lemma 2.2.11,

$$\alpha(\beta+\gamma)+A\beta+A\gamma = \alpha\beta+\alpha\gamma.$$

■

Proposition 2.4.14 *If α, β and γ are external numbers with β and γ not opposite with respect to $\pounds$, then $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$.*

Proof. We assume without loss of generality that α and β are positive. Because $A\pounds = A$, by Lemma 2.4.5.2 and by formula (2.5) it holds that $(\beta+\gamma)A = \beta A+\gamma A$. Notice that $\beta+\gamma$ is zeroless. If not, both $\pounds(\beta+\gamma) = \pounds(B+C) = B+C$ and $\pounds(\beta+\gamma) = \pounds\beta \geq b$, with $b > B$ and $b \geq |c| > C$, a contradiction. Then by Lemma 2.2.11 and Lemma 2.4.12

$$\begin{aligned}\alpha(\beta+\gamma) &= a(\beta+\gamma)+A(\beta+\gamma)\\ &= a\beta+a\gamma+A\beta+A\gamma\\ &= \alpha\beta+\alpha\gamma.\end{aligned}$$

■

Lemma 2.4.15 *Assume that $a = 1$, $b = 1$ and $0 < |c| \leq 1$. Then*

1. *If $\beta+\gamma \in \mathcal{N}$,*

$$\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma \Leftrightarrow A \leq \max(B,C) \Leftrightarrow \alpha\beta = \beta \vee \alpha\gamma = \gamma.$$

2. *If $\beta+\gamma \notin \mathcal{N}$,*

$$\begin{aligned}\alpha(\beta+\gamma) &= \alpha\beta+\alpha\gamma \Leftrightarrow A \leq \max(B,C) \vee (1+c)A = A\\ &\Leftrightarrow \alpha\beta = \beta \vee \alpha\gamma = \gamma \vee (\beta+\gamma)A = A.\end{aligned}$$

Proof. First observe that, by Lemma 2.2.10 and Lemma 2.2.9, one has

$$\alpha\beta = (1+A)(1+B) = 1+A+B \tag{2.6}$$

and

$$\alpha\gamma = (1+A)(c+C) = c+cA+C. \tag{2.7}$$

Hence

$$\alpha\beta+\alpha\gamma = 1+A+B+c+cA+C = 1+c+A+B+C = 1+c+\max(A,B,C). \tag{2.8}$$

1. We start with the first equivalence. By hypothesis and Lemma 2.2.11, one has

$$\alpha(\beta+\gamma) = (1+A)(B+C) = \max(B,C). \tag{2.9}$$

Hence, by formula (2.8) and formula (2.9), $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$ if and only if $A \leq \max(B, C)$.

To prove the second equivalence, suppose first that $\alpha\beta = \beta$ or $\alpha\gamma = \gamma$. In the first case, by formula (2.6)

$$1 + A + B = 1 + B,$$

which implies that $A \leq B$. In the second case by formula (2.7)

$$c + cA + C = c + C. \tag{2.10}$$

Because β and γ are zeroless, one has $B, C \subseteq \oslash$. Now $\beta + \gamma = 1 + c + B + C \subseteq 1 + c + \oslash \in \mathcal{N}$, which implies that $c \simeq -1$, hence $cA = A$. Then we derive from (2.10) that $c + A + C = c + C$, so $A \leq C$. We conclude that $A \leq \max(B, C)$. Conversely, suppose that $A \leq \max(B, C)$. Hence $A \leq B$ or $A \leq C$. Then clearly $1 + A + B = 1 + B$ or $c + A + C = c + C$ and $\alpha\beta = \beta$ or $\alpha\gamma = \gamma$ by formula (2.6) and formula (2.7). Hence

$$A \leq \max(B, C) \Leftrightarrow \alpha\beta = \beta \vee \alpha\gamma = \gamma.$$

2. One has $\beta + \gamma \neq \max(B, C)$. Hence

$$(\beta + \gamma) A = (1 + B + c + C) A = (1 + c) A. \tag{2.11}$$

We prove first that

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \Leftrightarrow A \leq \max(B, C) \vee (1 + c) A = A. \tag{2.12}$$

Lemma 2.2.10 yields

$$\begin{aligned} \alpha(\beta + \gamma) &= (1 + A)(1 + B + c + C) \\ &= 1 + c + B + C + (1 + c) A. \end{aligned}$$

If $A \leq \max(B, C)$, because $(1 + c) A \subseteq A$ both $\alpha\beta + \alpha\gamma = 1 + c + \max(B, C)$ and $\alpha(\beta + \gamma) = 1 + c + \max(B, C)$. If $(1 + c) A = A$, we conclude from (2.8) that $\alpha\beta + \alpha\gamma = \alpha(\beta + \gamma)$. If $\alpha\beta + \alpha\gamma = \alpha(\beta + \gamma)$, then, by formula (2.8) it holds that $A + B + C = B + C + (1 + c) A$, so $A \leq \max(B, C)$ or $A = (1 + c)A$. Hence formula (2.12) holds. Finally we prove that

$$A \leq \max(B, C) \vee (1 + c) A = A \Leftrightarrow \alpha\beta = \beta \vee \alpha\gamma = \gamma \vee (\beta + \gamma) A = A. \tag{2.13}$$

By (2.6) and (2.7) one has

$$\alpha\beta = \beta \Leftrightarrow A \leq B$$

and

$$\alpha\gamma = \gamma \Leftrightarrow C + cA = C.$$

Assume $A \leq \max(B, C)$ or $(1 + c) A = A$. If $A \leq B$, then $\alpha\beta = \beta$. If $A \leq C$, then $C + cA = C$, hence $\alpha\gamma = \gamma$. If $(1 + c) A = A$, then $(\beta + \gamma) A =$

$(1+c+B+C)A = (1+c)A + BA + CA = A$. Assume now that $\alpha\beta = \beta$ or $\alpha\gamma = \gamma$, or $(\beta+\gamma)A = A$. If $\alpha\beta = \beta$, then $A \leq B \leq \max(B, C)$. If $\alpha\gamma = \gamma$, then $cA \leq C$. If $cA = A$ one has that $A \leq C \leq \max(B, C)$. If $cA < A$ then $1 + c \simeq 1$, hence $(1+c)A = A$. If $(\beta+\gamma)A = A$, then $(1+c)A = A$, because $\beta + \gamma$ is zeroless. We conclude that formula (2.13) holds. ■

We are now able to characterize distributivity for zeroless external numbers:

Theorem 2.4.16

1. If $\beta + \gamma \in \mathcal{N}$,
$$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma \Leftrightarrow \mathcal{R}(\alpha) \leq \max(\mathcal{R}(\beta), \mathcal{R}(\gamma)).$$
2. If $\beta + \gamma \notin \mathcal{N}$,
$$\begin{aligned} \alpha(\beta+\gamma) &= \alpha\beta + \alpha\gamma &(2.14)\\ &\Leftrightarrow \mathcal{R}(\alpha) \leq \max(\mathcal{R}(\beta), \mathcal{R}(\gamma)) \vee (\beta+\gamma)A = A\max(|\beta|, |\gamma|). \end{aligned}$$

Proof. First, we put the products in a convenient form. Then by Lemma 2.2.9 and by Lemma 2.4.12 one has

$$\begin{aligned} \alpha(\beta+\gamma) &= a(1+\mathcal{R}(\alpha))(b(1+\mathcal{R}(\beta)) + c(1+\mathcal{R}(\gamma))) &(2.15)\\ &= ab(1+\mathcal{R}(\alpha))\left(1+\mathcal{R}(\beta) + \frac{c}{b} + \frac{c}{b}\mathcal{R}(\gamma)\right) \end{aligned}$$

and

$$\begin{aligned} \alpha\beta + \alpha\gamma &= a(1+\mathcal{R}(\alpha))b(1+\mathcal{R}(\beta)) + a(1+\mathcal{R}(\alpha))c(1+\mathcal{R}(\gamma)) &(2.16)\\ &= ab\left((1+\mathcal{R}(\alpha))(1+\mathcal{R}(\beta)) + (1+\mathcal{R}(\alpha))\left(\frac{c}{b} + \frac{c}{b}\mathcal{R}(\gamma)\right)\right). \end{aligned}$$

From (2.15) and (2.16) we conclude that distributivity is equivalent to

$$\begin{aligned} &(1+\mathcal{R}(\alpha))\left(1+\mathcal{R}(\beta) + \frac{c}{b} + \frac{c}{b}\mathcal{R}(\gamma)\right) &(2.17)\\ = \;&(1+\mathcal{R}(\alpha))(1+\mathcal{R}(\beta)) + (1+\mathcal{R}(\alpha))\left(\frac{c}{b} + \frac{c}{b}\mathcal{R}(\gamma)\right). \end{aligned}$$

Since by assumption $|\gamma| \leq |\beta|$ and $|c| \leq |b|$, we are able to apply Lemma 2.4.15. To prove Part 1, suppose that $\beta+\gamma \in \mathcal{N}$. Then $b+B+c+C = B+C \subseteq \oslash b + \oslash c = \oslash b$, so $1 + \frac{c}{b} + \oslash \subseteq \oslash$. Hence $\frac{c}{b} \simeq -1$. This implies that $\frac{c}{b}\mathcal{R}(\gamma) = \mathcal{R}(\gamma)$. Then

$$\mathcal{R}(\alpha) \leq \max\left(\mathcal{R}(\beta), \frac{c}{b}\mathcal{R}(\gamma)\right) \Leftrightarrow \mathcal{R}(\alpha) \leq \max(\mathcal{R}(\beta), \mathcal{R}(\gamma)).$$

Hence, by Lemma 2.4.15.1,

$$\begin{aligned} \alpha(\beta+\gamma) &= \alpha\beta + \alpha\gamma \Leftrightarrow \mathcal{R}(\alpha) \leq \max\left(\mathcal{R}(\beta), \frac{c}{b}\mathcal{R}(\gamma)\right)\\ &\Leftrightarrow \mathcal{R}(\alpha) \leq \max(\mathcal{R}(\beta), \mathcal{R}(\gamma)). \end{aligned}$$

To prove Part 2, suppose that $\beta+\gamma \notin \mathcal{N}$. Then, by Lemma 2.4.15.2 and formula (2.17)

$$\begin{aligned} \alpha\left(\beta+\gamma\right) &= \alpha\beta+\alpha\gamma \Leftrightarrow \left(1+\mathcal{R}\left(\alpha\right)\right)\left(1+\mathcal{R}\left(\beta\right)\right)=1+\mathcal{R}\left(\beta\right) \\ & \vee\left(1+\mathcal{R}\left(\alpha\right)\right)\left(\frac{c}{b}+\frac{c}{b}\mathcal{R}\left(\gamma\right)\right)=\frac{c}{b}+\frac{c}{b}\mathcal{R}\left(\gamma\right) \\ & \vee\left(1+\mathcal{R}\left(\beta\right)+\frac{c}{b}+\frac{c}{b}\mathcal{R}\left(\gamma\right)\right)A=A. \end{aligned} \tag{2.18}$$

First, we prove the direct implication of (2.14), using (2.18). With respect to (2.18) there are three cases to consider:
(i) $\left(1+\mathcal{R}\left(\alpha\right)\right)\left(1+\mathcal{R}\left(\beta\right)\right)=1+\mathcal{R}\left(\beta\right)$, (ii) $\left(1+\mathcal{R}\left(\alpha\right)\right)\left(\frac{c}{b}+\frac{c}{b}\mathcal{R}\left(\gamma\right)\right)=\frac{c}{b}+\frac{c}{b}\mathcal{R}\left(\gamma\right)$ and (iii) $\left(1+\mathcal{R}\left(\beta\right)+\frac{c}{b}+\frac{c}{b}\mathcal{R}\left(\gamma\right)\right)A=A$.

(i) By Lemma 2.2.10

$$1+\mathcal{R}\left(\alpha\right)+\mathcal{R}\left(\beta\right)=1+\mathcal{R}\left(\beta\right), \tag{2.19}$$

hence

$$\mathcal{R}\left(\alpha\right)\leq\mathcal{R}\left(\beta\right).$$

This implies that $\mathcal{R}\left(\alpha\right)\leq\max\left(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right)\right)$.

(ii) One has similarly to (2.19)

$$\frac{c}{b}\left(1+\mathcal{R}\left(\alpha\right)+\mathcal{R}\left(\gamma\right)\right)=\frac{c}{b}\left(1+\mathcal{R}\left(\gamma\right)\right),$$

implying that

$$\mathcal{R}\left(\alpha\right)\leq\mathcal{R}\left(\gamma\right).$$

Hence $\mathcal{R}\left(\alpha\right)\leq\max\left(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right)\right)$.

(iii) By Lemma 2.2.11

$$\left(b+B+c+C\right)A=bA,$$

and therefore

$$\left(\beta+\gamma\right)A=\beta A{=}A\max\left(\left|\beta\right|,\left|\gamma\right|\right).$$

Combining the three cases, we conclude that

$$\begin{aligned} \alpha\left(\beta+\gamma\right) &= \alpha\beta+\alpha\gamma \\ &\Rightarrow \mathcal{R}\left(\alpha\right)\leq\max(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right))\vee\left(\beta+\gamma\right)A{=}A\max\left(\left|\beta\right|,\left|\gamma\right|\right). \end{aligned} \tag{2.20}$$

To prove the reverse implication we need to consider two cases: (i) $\left(\beta+\gamma\right)A{=}A\max\left(\left|\beta\right|,\left|\gamma\right|\right)$ and (ii) $\mathcal{R}\left(\alpha\right)\leq\max(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right))$.

(i) One has $\left(\beta+\gamma\right)A{=}\left|\beta\right|A$. Then

$$\left(b+B+c+C\right)A=(b+B)A.$$

This implies that

$$\left(1+\mathcal{R}\left(\beta\right)+\frac{c}{b}+\frac{c}{b}\mathcal{R}\left(\gamma\right)\right)A=A.$$

Then by (2.18) we conclude that

$$(\beta+\gamma)\,A{=}A\max\left(|\beta|\,,|\gamma|\right)\Rightarrow\alpha\left(\beta+\gamma\right)=\alpha\beta+\alpha\gamma. \tag{2.21}$$

(ii) If $\mathcal{R}\left(\alpha\right)\leq\max(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right))$, then $\mathcal{R}\left(\alpha\right)\leq\mathcal{R}\left(\beta\right)$ or $\frac{c}{b}\mathcal{R}\left(\alpha\right)\leq\frac{c}{b}\mathcal{R}\left(\gamma\right)$ and by Lemma 2.2.10

$$\left(1+\mathcal{R}\left(\alpha\right)\right)\left(1+\mathcal{R}\left(\beta\right)\right)=1+\mathcal{R}\left(\beta\right)$$

or

$$\left(1+\mathcal{R}\left(\alpha\right)\right)\left(\frac{c}{b}+\frac{c}{b}\mathcal{R}\left(\gamma\right)\right)=\frac{c}{b}+\frac{c}{b}\mathcal{R}\left(\gamma\right).$$

Then by (2.18) we conclude that

$$\mathcal{R}\left(\alpha\right)\leq\max(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right))\Rightarrow\alpha\left(\beta+\gamma\right)=\alpha\beta+\alpha\gamma. \tag{2.22}$$

From (2.21) and (2.22) we obtain

$$\begin{aligned}\mathcal{R}\left(\alpha\right) &\leq \max(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right))\vee\left(\beta+\gamma\right)A{=}A\max\left(|\beta|\,,|\gamma|\right) \\ &\Rightarrow \alpha\left(\beta+\gamma\right)=\alpha\beta+\alpha\gamma.\end{aligned} \tag{2.23}$$

We conclude by combining (2.20) and (2.23). ■

To complete the proof of Theorem 2.4.6 for the case of zeroless numbers we need one more lemma.

Lemma 2.4.17 *If $\beta+\gamma\in\mathcal{N}$ and $\alpha\left(\beta+\gamma\right)\neq\alpha\beta+\alpha\gamma$, then $\left(\beta+\gamma\right)A\subset A\max$ $\left(|\beta|\,,|\gamma|\right)$.*

Proof. Suppose that $\alpha\left(\beta+\gamma\right)\neq\alpha\beta+\alpha\gamma$. Then, since $\beta+\gamma=B+C$, by Theorem 2.4.16 we obtain $\mathcal{R}\left(\alpha\right)>\max\left(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right)\right)$. We prove that

$$\max\left(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right)\right)\mathcal{R}\left(\alpha\right)\subset\mathcal{R}\left(\alpha\right).$$

Let $t\in\mathcal{R}\left(\alpha\right)$. Because $\mathcal{R}\left(\alpha\right)\subseteq\oslash$, one has $t\mathcal{R}\left(\alpha\right)\subseteq\mathcal{R}\left(\alpha\right)$. Suppose by contradiction that $t\mathcal{R}\left(\alpha\right)=\mathcal{R}\left(\alpha\right)$. Then $t\in t\mathcal{R}\left(\alpha\right)$. Then there exists $x\in\mathcal{R}(\alpha)$ such that $tx=t$. Hence $1\in\mathcal{R}\left(\alpha\right)$. This is in contradiction with the fact that $\mathcal{R}\left(\alpha\right)\subseteq\oslash$. Hence $t\mathcal{R}\left(\alpha\right)\subset\mathcal{R}\left(\alpha\right)$. Applying this inclusion to $t>\max\left(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right)\right)$ we obtain

$$\max\left(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right)\right)\mathcal{R}\left(\alpha\right)\subseteq t\mathcal{R}\left(\alpha\right)\subset\mathcal{R}\left(\alpha\right).$$

This implies that

$$\left(\frac{B}{b}+\frac{C}{c}\right)\frac{A}{a}\subset\frac{A}{a}.$$

Because $c\leq b$,

$$\left(\frac{B}{b}+\frac{C}{b}\right)A\subset A.$$

Hence, because $\beta+\gamma=B+C$,

$$(\beta+\gamma)\,A\subset A\beta.$$

We conclude that

$$(\beta+\gamma)\,A\subset A\max\left(|\beta|\,,|\gamma|\right).$$

■

Proof of Theorem 2.4.6 for zeroless external numbers: If follows immediately from Theorem 2.4.16 that

$$\begin{array}{rcl}\alpha\left(\beta+\gamma\right) & = & \alpha\beta+\alpha\gamma\\ & \Rightarrow & \mathcal{R}\left(\alpha\right)\leq\max\left(\mathcal{R}\left(\beta\right),\mathcal{R}\left(\gamma\right)\right)\vee\left(\beta+\gamma\right)A{=}A\max\left(|\beta|\,,|\gamma|\right).\end{array}$$

If $\beta+\gamma\notin\mathcal{N}$, the reverse implication follows from Theorem 2.4.16.2. If $\beta+\gamma\in\mathcal{N}$, the reverse implication follows from Theorem 2.4.16.1 and Lemma 2.4.17.■

2.4.3 Binomial formulas

As an application we study the effect of the distributive law on some binomial forms. Let $\alpha=a+A$, $\beta=b+B$ and $\gamma=c+C$ with a,b and c real numbers and A,B and C neutrices.

Firstly, because α is more precise than α, we always have

$$\alpha\left(\alpha+\beta\right)=\alpha^{2}+\alpha\beta. \tag{2.24}$$

Secondly, we investigate the validity of the equality

$$(\alpha-\beta)(\alpha+\beta)=\alpha^{2}-\beta^{2}. \tag{2.25}$$

If α and β are neutrices it is easy to verify the equality (2.25) directly. In the remaining case, we suppose without loss of generality that both α and β are non-negative. Then by Theorem 2.4.8

$$(\alpha-\beta)(\alpha+\beta)=(\alpha-\beta)\alpha+(\alpha-\beta)\beta$$

Hence by (2.24)

$$(\alpha-\beta)\alpha+(\alpha-\beta)\beta=\alpha^{2}-\alpha\beta+\alpha\beta-\beta^{2}=\alpha^{2}-\beta^{2}+N(\alpha\beta).$$

Hence always $(\alpha-\beta)(\alpha+\beta)\supseteq\alpha^{2}-\beta^{2}$. Observe that $N(\alpha^{2}-\beta^{2})=\alpha A+\beta B$ and that $N(\alpha\beta)=\alpha B+\beta A$. Hence (2.25) holds if $\alpha B+\beta A\subseteq\alpha A+\beta B$, say, if $B\leq A$ and $\beta\leq\alpha$.

Thirdly, we show that if α and β are neither opposite with respect to A nor to B,

$$\left(\alpha+\beta\right)^{2}=\alpha^{2}+2\alpha\beta+\beta^{2} \tag{2.26}$$

Indeed, by Lemma 2.4.5.1 the numbers α and β are not opposite with respect to $A+B=N(\alpha+\beta)$. Then by Theorem 2.4.6 and (2.24)

$$\begin{aligned}(\alpha+\beta)^2 &= (\alpha+\beta)(\alpha+\beta)=\alpha(\alpha+\beta)+\beta(\alpha+\beta)\\ &= \alpha^2+\alpha\beta+\beta\alpha+\beta^2=\alpha^2+2\alpha\beta+\beta^2.\end{aligned}$$

Finally we extend the equality (2.26) to a Binomial Theorem for external numbers. We need some properties of the relative uncertainty.

Lemma 2.4.18 *Let $\alpha=a+A$ and $\beta=b+B$ be external numbers. Then*

1. $\mathcal{R}(\alpha\beta)=\mathcal{R}(\alpha)+\mathcal{R}(\beta)$
2. If $k\in\mathbb{N}$ is standard, $\mathcal{R}(\alpha^k)=\mathcal{R}(\alpha)$.

If α and β are zeroless the lemma is an easy consequence of Proposition 2.3.9.1. Else the equalities are trivially satisfied.

Theorem 2.4.19 *Let $\alpha=a+A$ and $\beta=b+B$ be external numbers. If α and β are neither opposite with respect to A nor to B, for standard $n\in\mathbb{N}, n\geq 1$*

$$(\alpha+\beta)^n=\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k. \tag{2.27}$$

Proof. The proof is by external induction. If $n=1$, then (2.27) is clearly true. Suppose that (2.27) is true for standard n. Then

$$(\alpha+\beta)^{n+1}=(\alpha+\beta)(\alpha+\beta)^n=(\alpha+\beta)\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k.$$

The neutrix $C\equiv N\left(\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k\right)$ is a sum with a standard finite number of multiples of A and B. Hence Lemma 2.4.5.1 and Lemma 2.4.5.2 imply that α and β are not opposite with respect to C. By Theorem 2.4.6

$$(\alpha+\beta)\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k=\alpha\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k+\beta\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k.$$

It follows from Lemma 2.4.18 that $\mathcal{R}(\alpha)\leq\mathcal{R}\left(\binom{n}{k}\alpha^{n-k}\beta^k\right)$ for all k such that $0\leq k\leq n-1$, and $\mathcal{R}(\beta)\leq\mathcal{R}\left(\binom{n}{k}\alpha^{n-k}\beta^k\right)$ for all k such that $1\leq k\leq n$. Repeated application of Theorem 2.4.6 yields

$$\begin{aligned}&\alpha\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k+\beta\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k\\ =\;&\sum_{k=0}^{n}\binom{n}{k}\alpha^{n+1-k}\beta^k+\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^{k+1}.\end{aligned}$$

Because the relative uncertainty of natural numbers is zero, again by Theorem 2.4.6

$$\sum_{k=0}^{n}\binom{n}{k}\alpha^{n+1-k}\beta^{k}+\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^{k+1}=\sum_{k=0}^{n+1}\binom{n+1}{k}\alpha^{n+1-k}\beta^{k}.$$

■

Lemma 2.4.20 *Let a be a real number and A be a neutrix such that $|a|>A$. Then, for all standard n,*

$$a^nA+a^{n-1}A^2=a^nA.$$

Proof. We prove this result by external induction.

If $n=1$, we obtain $aA+a^0A^2=aA+A^2$. Because $|a|>A$ one has $aA\geq A^2$. Then, $aA+A^2=aA$ and the formula is valid for $n=1$.

Suppose now that $a^nA+a^{n-1}A^2=a^nA$ is valid for some standard n. Then, by Lemma 2.4.12,

$$\begin{aligned}a^{n+1}A+a^nA^2&=aa^nA+aa^{n-1}A^2\\&=a\left(a^nA+a^{n-1}A^2\right)\\&=a\left(a^nA\right)=a^{n+1}A.\end{aligned}$$

Hence $a^nA+a^{n-1}A^2=a^nA$, for all standard n, by external induction. ■

The following theorem gives a decomposition for the n-th power of a zeroless external number.

Theorem 2.4.21 *Let $\alpha=a+A$ be a zeroless external number. Then for all standard n*

$$\alpha^n=a^n+a^{n-1}A.$$

Proof. Again, our proof is by external induction.

If $n=1$ the result is obvious.

Suppose that, for some standard n, $\alpha^n=a^n+a^{n-1}A$. Then, using the definition of multiplication and Lemma 2.4.20,

$$\begin{aligned}\alpha^{n+1}&=\alpha^n\alpha=\left(a^n+a^{n-1}A\right)(a+A)\\&=a^na+a^nA+a^{n-1}aA+a^{n-1}A^2\\&=a^{n+1}+a^nA.\end{aligned}$$

Hence $\alpha^n=a^n+a^{n-1}A$, for all standard n, by external induction. ■

We finish by giving a decomposition formula for the product of n zeroless external numbers.

Theorem 2.4.22 *Let $\alpha_i=a_i+A_i$, $i=1,...,n$ be zeroless external numbers. Then for all standard n*

$$\prod_{i=1}^{n}(a_i+A_i)=\prod_{i=1}^{n}a_i+\sum_{j=1}^{n}\left[\left(\prod_{\substack{K=1\\K\neq j}}^{n}a_k\right)A_j\right].$$

Proof. If $n = 1$ the equality is obvious.

Suppose the equality is true for some standard n. Then, because $a_{n+1}+A_{n+1}$ is zeroless, using Lemma 2.2.10,

$$\begin{aligned}
\prod_{i=1}^{n+1}(a_i+A_i) &= \left(\prod_{i=1}^{n}(a_i+A_i)\right)(a_{n+1}+A_{n+1}) \\
&= \left(\prod_{i=1}^{n}a_i+\sum_{j=1}^{n}\left[\left(\prod_{\substack{K=1\\K\neq j}}^{n}a_k\right)A_j\right]\right)(a_{n+1}+A_{n+1}) \\
&= \prod_{i=1}^{n+1}a_i+\prod_{i=1}^{n}a_iA_{n+1}+a_{n+1}\sum_{j=1}^{n}\left[\left(\prod_{\substack{K=1\\K\neq j}}^{n}a_k\right)A_j\right] \\
&= \prod_{i=1}^{n+1}a_i+\sum_{j=1}^{n+1}\left[\left(\prod_{\substack{K=1\\K\neq j}}^{n+1}a_k\right)A_j\right].
\end{aligned}$$

Hence $\prod_{i=1}^{n}(a_i+A_i) = \prod_{i=1}^{n}a_i+\sum_{j=1}^{n}\left[\left(\prod_{\substack{K=1\\K\neq j}}^{n}a_k\right)A_j\right]$, for all standard n, by external induction. ∎

We finish with a result by Justino and Van den Berg [31] that gives a majoration of the neutrix part of the determinant of a matrix whose elements are external numbers. Let $\mathcal{A} = [\alpha_{ij}]$ be a $m \times n$ matrix. We say that $\mathcal{A}$ is *non-singular* if $\Delta = \det \mathcal{A}$ is zeroless. We denote $\overline{a} \equiv \max\limits_{\substack{1\leq i\leq m\\1\leq j\leq n}}|a_{ij}|$ and $\overline{A} \equiv \max\limits_{\substack{1\leq i\leq m\\1\leq j\leq n}}A_{ij}$.

Corollary 2.4.23 *Let $n, m \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be a non-singular $m \times n$ matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$, such that $\overline{a} = 1$, and $\Delta = \det \mathcal{A} = d + D$. Then*

$$D = N(\Delta) \subseteq \overline{A}.$$

Proof. Let S_n denote the set of all permutations of the set $\{1, 2, ..., n\}$ and $\sigma = (p_1, ..., p_n) \in S_n$. Let $\gamma_\sigma = (a_{1p_1} + A_{1p_1}) ... (a_{np_n} + A_{np_n})$. Because $\overline{a} = 1$ one has $|a_{kp_k}| \leqslant \overline{a} = 1$ and $A_{kp_k} \subseteq \overline{A} \subseteq \oslash$ for all $k \in \{1, ..., n\}$. So, by Theorem 2.4.21, one has $N(\gamma_\sigma) \subseteq N\left(\left(1+\overline{A}\right)^n\right) = \overline{A}$. Now,

$$\begin{aligned}
\Delta &= \begin{vmatrix} a_{11}+A_{11} & ... & a_{1n}+A_{1n} \\ \vdots & & \vdots \\ a_{n1}+A_{n1} & ... & a_{nn}+A_{nn} \end{vmatrix} = \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\gamma_\sigma \\
&= \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\left(a_{1p_1}...a_{np_n}+N(\gamma_\sigma)\right),
\end{aligned}$$

with $\operatorname{sgn}(\sigma) \in \{-1, 1\}$. Then

$$N(\Delta) = \sum_{\sigma\in S_n} N(\gamma_\sigma) \subseteq n!\overline{A} = \overline{A}.$$

∎

Chapter 3

Algebraic structures with individualized neutral elements

> Finally there are simple ideas of which no definition can be given; there are also axioms or postulates, or in a word primary principles, which cannot be proved and have no need of proof.
> (Gottfried Leibniz)

3.1 Introduction

In Chapter 2 (see also [15]) it was shown that the class of external numbers equipped with addition and the class of external numbers which are not neutrices equipped with multiplication form commutative regular semigroups. Unlike real numbers, external numbers have individualized neutral and inverse elements for both addition and multiplication. It was also shown that the distributive law is valid under some restrictions that can be completely characterized. Moreover, the external numbers are totally ordered, even allowing for a sort of generalized completeness property [3] [4] [39]. Hence external numbers have to a large extent algebraic properties similar to those of real numbers. This justifies the introduction of common algebraic structures defined by axiomatic rules.

For the purpose of clarity we start, in Section 3.2, with a full list of the axioms used. In Section 3.3 we define a structure called assembly which is a sort of group with neutral elements given in the form of functions. We derive some basic algebraic properties of assemblies and study subassemblies (substructures of assemblies which are also assemblies). In Section 3.4 we study some structure-preserving functions between assemblies, called assembly homomorphisms. We show that this extends the usual notion of group homomorphisms because the usual properties of homomorphisms remain valid. In Section 3.5 we study ordered assemblies. The notion of absolute value is extended to ordered assemblies. We show that the basic properties of the absolute value hold for ordered assemblies. In Section 3.7 we present axioms for multiplication and define a structure called association which is, roughly speaking, a sort of nondistributive ring with individualized neutral elements for both addition and multiplication. Further axioms are presented on mixing addition and multiplication. In Section 3.8 we study associations equipped with a total order relation. We show that ordered associations have, to some extent, properties similar to ordered rings. A very important difference between rings and associations is that the distributive law is lacking in the latter. In Section 3.9 we reduce that gap by giving an axiom which allows to have a restricted distributive law. Further axioms are presented, related to completeness. The resulting structure, called solid, has many properties in common with ordered fields. Indeed one can, roughly speaking, understand a solid as a semi-distributive ordered field with generalized neutral elements for both operations.

3.2 Axioms

In this section we give a list of the axioms used. The first group of axioms concerns only one operation called addition. These axioms are somewhat similar to the group axioms. In fact, it is our intention to generalize the group axioms so that each element has an individualized neutral element. In the second group we give axioms for a second operation called multiplication. The axioms of this group are similar to the axioms of the first group. In this way addition and multiplication have almost the same basic structure, much like what happens within rings and fields. The third group of axioms states, roughly speaking, that there is a total order relation compatible with the operations of addition and multiplication. A fourth group of axioms is given so that addition and multiplication can be connected. We give axioms that connect the individualized neutral and inverse elements for addition with multiplication and an axiom that gives a restricted distributive law. In the fifth group we add axioms related to completion, stating that there are (minimal) neutral elements for addition and multiplication (0 and 1) and that there is a maximal individualized neutral element (denoted M). We also give an axiom which allows to decompose each element in terms of the individualized neutral element. Finally we give an axiom that postulates the existence of nontrivial neutral elements, i.e. neutral elements besides 0 and M. The axioms are written in the first-order language $L = \{+, -, \leq\}$.

3.2.1 Axioms for addition

Axiom 3.2.1 (Associativity)

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z).$$

Axiom 3.2.2 (Commutativity)

$$\forall x\forall y(x+y=y+x).$$

Axiom 3.2.3 (Individualized neutral element)

$$\forall x\exists e\left(x+e=x\wedge\forall f\left(x+f=x\Rightarrow e+f=e\right)\right)^{1}.$$

Axiom 3.2.4 (Symmetric element)

$$\forall x\exists s\left(x+s=e\left(x\right)\wedge e\left(s\right)=e\left(x\right)\right).$$

Axiom 3.2.5 (Neutral element of sum)

$$\forall x\forall y\left(e\left(x+y\right)=e\left(x\right)\vee e\left(x+y\right)=e\left(y\right)\right).$$

3.2.2 Axioms for multiplication

Axiom 3.2.6 (Associativity)

$$\forall x\forall y\forall z(x\left(yz\right)=\left(xy\right)z).$$

Axiom 3.2.7 (Commutativity)

$$\forall x\forall y(xy=yx).$$

Axiom 3.2.8 (Individualized unity element)

$$\forall x\neq e\left(x\right)\exists u\left(xu=x\wedge\forall v\left(xv=x\Rightarrow uv=u\right)\right)^{1}.$$

Axiom 3.2.9 (Division element)

$$\forall x\neq e\left(x\right)\exists d\left(xd=u\left(x\right)\wedge u\left(d\right)=u\left(x\right)\right).$$

Axiom 3.2.10 (Unity element of product)

$$\forall x\neq e\left(x\right)\forall y\neq e\left(y\right)\left(u\left(xy\right)=u\left(x\right)\vee u\left(xy\right)=u\left(y\right)\right).$$

3.2.3 Order axioms

Axiom 3.2.11 (Reflexivity)

$$\forall x(x\leq x).$$

Axiom 3.2.12 (Antisymmetry)

$$\forall x\forall y(x\leq y\wedge y\leq x\Rightarrow x=y).$$

[1] The dependence of e and u on x is justified in Remark 3.2.28 because they appear to be functional, i.e. $e=e\left(x\right)$ and $u=u\left(x\right)$.

Axiom 3.2.13 (Transitivity)

$$\forall x \forall y \forall z (x \leq y \wedge y \leq z \Rightarrow x \leq z).$$

Axiom 3.2.14 (Totality)

$$\forall x \forall y (x \leq y \vee y \leq x).$$

Axiom 3.2.15 (Compatibility with addition)

$$\forall x \forall y \forall z \left(x \leq y \Rightarrow x + z \leq y + z \right).$$

Axiom 3.2.16 (Invariance)

$$\forall x \forall y \left(y + e(x) = e(x) \Rightarrow y \leq e(x) \right).$$

Axiom 3.2.17 (Compatibility with multiplication)

$$\forall x \forall y \forall z \left(\left(e\left(x\right) \leq x \wedge y \leq z \right) \Rightarrow xy \leq xz \right).$$

3.2.4 Mixed Axioms

Axiom 3.2.18 (Scale)

$$\forall x \forall y \exists z (e(x) y = e(z)).$$

Axiom 3.2.19 (Symmetric of product)

$$\forall x \forall y \left(s(xy) = s(x) y \right).$$

Axiom 3.2.20 (Neutral element of product)

$$\forall x \forall y \left(e(xy) = e(x) y + e(y) x \right).$$

Axiom 3.2.21 (Neutral element of unity)

$$\forall x \neq e(x) \left(e(u(x)) = e(x) d(x) \right).$$

Axiom 3.2.22 (Distributivity)

$$\forall x \forall y \forall z \left(xy + xz = x \left(y + z \right) + e \left(x \right) y + e \left(x \right) z \right).$$

3.2.5 Completion axioms

Axiom 3.2.23 (Minimal element addition)

$$\exists m \forall x \left(m + x = x \right).$$

Axiom 3.2.24 (Maximal element addition)

$$\exists M \forall x (e\left(x\right) + M = M).$$

Axiom 3.2.25 (Minimal element multiplication)

$$\exists u \forall x \left(ux = x \right).$$

Axiom 3.2.26 (Decomposition)

$$\forall x \exists a \left(x = a + e\left(x\right) \wedge e\left(a\right) = 0 \right).$$

Axiom 3.2.27 (Existence of Magnitudes)

$$\exists x \left(e\left(x\right) \neq m \wedge e\left(x\right) \neq M \right).$$

Remark 3.2.28 *The functional notation used in Axiom 3.2.4 and in Axiom 3.2.5 is justified by the fact that the element e of Axiom 3.2.3 is unique. Indeed, if* e' *satisfies Axiom 3.2.3, one has* $e' = e' + e = e + e\prime = e$. *Nontrivial existence is postulated in Axiom 3.2.27. Also s is unique and may be considered functional. Indeed, if* s' *satisfies Axiom 3.2.4 one has* $s' = s' + e(s') = s' + e(x) = s' + x + s = x + s' + s = e(x) + s = e(s) + s = s$. *The functional notation used in Axiom 3.2.9 and in Axiom 3.2.10 is justified in an analogous way. We may write* $-x$ *instead of* $s\left(x\right)$ *and* $x - y$ *instead of* $x + s\left(y\right)$. *Also, instead of* $d\left(x\right)$ *we may write* x^{-1}. *The elements m of Axiom 3.2.23 and u of Axiom 3.2.25 are also unique (see Proposition 3.9.6 below).*

Some of the axioms are field axioms or gentle generalizations of field axioms. Others are new: Axiom 3.2.3 and 3.2.8, the idempotency of Axioms 3.2.5, 3.2.10 and 3.2.16 and the scaling properties of Axioms 3.2.18, 3.2.20 and 3.2.21. Axiom 3.2.3 is given in order to capture the idea that every element has is own (additive) imprecision. Indeed we may interpret $e(x)$ as the 'imprecision' or 'error' of x. It can also be seen as a 'magnitude' or as a 'generalized zero'. Similarly we interpret $u\left(x\right)$, given by Axiom 3.2.8 as a generalized unity element and as a multiplicative or relative imprecision. These error functions allow one to keep track of the imprecisions involved throughout the several operations. With this interpretation in mind the remaining new axioms have a clearer meaning. Indeed, Axiom 3.2.5 states that the imprecision of a sum is equal to the maximum of the imprecision of its elements. In a similar way Axiom 3.2.10 states that the relative imprecision of a product is equal to the maximum of the relative imprecisions of its elements. Axiom 3.2.16 states that if a number leaves a magnitude invariant it is smaller than this magnitude. Axiom 3.2.18 postulates that a scaled magnitude, i.e. an magnitude multiplied by other element, remains a magnitude. Axiom 3.2.20 states that the imprecision of the product of two elements is the maximum of scaled imprecisions. Axiom 3.2.21 says that the magnitude of a unity is also a scaled magnitude. The decomposition axiom states that every number is the sum of a sharp element and a magnitude. The last axiom postulates the existence of nontrivial magnitudes.

3.3 Assemblies

Roughly speaking, an assembly may be seen as a group with individualized neutral and inverse elements. These elements appear in the form of functions, as mentioned in Remark 3.2.28. Throughout this section we investigate to what extent group properties may be generalized to assemblies. In fact, we show that many group properties are valid or may be adapted within assemblies. The fact that each element of an assembly has his own neutral element implies that some properties of subgroups are not valid for subassemblies (substructures of assemblies which are also assemblies). For instance, it is a basic fact from group theory that if $(A, +)$ and $(B, +)$ are two subgroups of a given group $(G, +)$ then $A \cap B \neq \emptyset$. As we will see that does not remain valid for assemblies. Nevertheless it is possible to adapt some properties of subgroups and a characterization of subassembly is given.

We emphasize that we aim to study algebraic properties of neutrices and external numbers which are external sets and even classes of nonstandard analysis, as mentioned in Section 1.5. So natural models of the introduced structures tend to be such external sets and classes.

Definition 3.3.1 *Given a nonempty class A and a binary operation $+$ on A, we say that $(A, +)$ is an* assembly *if A satisfies axioms 3.2.1-3.2.5.*

Examples of assemblies are the following.

Example 3.3.2

1. Commutative groups. In fact, commutative groups are assemblies on which the functions e and s are constant.

2. $(B, \cup)$ where B is the set of all ordinals less than a given ordinal. Observe that given two ordinals $\alpha, \beta \in B$ one has $\alpha \cup \beta = \alpha$ or $\alpha \cup \beta = \beta$. Then clearly commutativity and associativity hold. The remaining axioms trivially hold by making $e(\alpha) = s(\alpha) = \alpha$.

3. $(\mathbb{E}, +)$, where $\mathbb{E}$ is the external class of external numbers. This will be proved in Chapter 4.

The following are not assemblies.

Example 3.3.3

1. $(B, +)$ where B is the set of all ordinals less than a given ordinal and $+$ is the usual addition of ordinals is not an assembly because commutativity fails. In fact, $1 + \omega = \omega \neq \omega + 1$.

2. $(\mathbb{M}_{2\times 2}(\mathbb{E}), \cdot)$ where $\mathbb{M}_{2\times 2}(\mathbb{E})$ is the class of all 2-by-2 matrices with external numbers as coefficients equipped with the usual matrix multiplication[2]. Indeed, if N is a neutrix such that $N \neq \{0\}$ one has

$$\left(\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right)\begin{pmatrix}N & N\\ N & N\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}N & N\\ N & N\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

[2] In [31] matrices with external numbers as coefficients are considered and Cramer's Rule is applied to systems of linear equations with external numbers as coefficients, called *flexible systems*.

and

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\left(\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}N & N\\ N & N\end{pmatrix}\right) = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}N-N & N-N\\ -N+N & -N+N\end{pmatrix} =$$
$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}N & N\\ N & N\end{pmatrix} = \begin{pmatrix}N+N & N+N\\ N+N & N+N\end{pmatrix} = \begin{pmatrix}N & N\\ N & N\end{pmatrix}.$$

Hence multiplication is not associative.

We recall that the structure $(A,+)$ is *Von Neumann regular* if for all $a \in A$ there is $x \in A$ such that $a + x + a = a$. In this way one may think of x as a 'weak inverse' of a.

Proposition 3.3.4 *If $(A,+)$ is an assembly, then A is Von Neumann regular.*

Proof. Let $a \in A$. By Axiom 3.2.4, there is $s \in A$ such that $a+s = e(a)$. Then $a + s + a = e(a) + a = a$ by Axiom 3.2.3. ■

Within assemblies cancellation holds up to individual neutral elements.

Theorem 3.3.5 (Cancellation law) *Let $(A,+)$ be an assembly and let $x, y, z \in A$ be arbitrary. Then $x + y = x + z$ if and only if $e(x) + y = e(x) + z$.*

Proof. Suppose firstly that $x + y = x + z$. Then

$$e(x) + y = -x + x + y = -x + x + z = e(x) + z.$$

Suppose now that $e(x) + y = e(x) + z$. Then

$$x + y = x + e(x) + y = x + e(x) + z = x + z.$$

Hence $x + y = x + z \Leftrightarrow e(x) + y = e(x) + z$. ■

3.3.1 Neutral function

In the following we prove that the neutral function e is linear and idempotent with respect to addition and idempotent with respect to composition. We show also that an element can be equal to no more than one magnitude.

Proposition 3.3.6 *Let $(A,+)$ be an assembly. Then for all x and $y \in A$*

1. (Idempotency for addition) $e(x) + e(x) = e(x)$.
2. (Linearity of e) $e(x + y) = e(x) + e(y)$.

Proof. Let x and y be arbitrary elements of A.

1. By Axiom 3.2.3 one has $x + e(x) = x$. Then by the cancellation law, $e(x) + e(x) = e(x)$.

2. By commutativity, associativity and Axiom 3.2.3 one has

$$x + y + e(x) + e(y) = x + e(x) + y + e(y) = x + y.$$

Then by cancellation,

$$e(x+y)+e(x)+e(y)=e(x+y). \tag{3.1}$$

By Axiom 3.2.5 one has that $e(x+y)=e(x)$ or $e(x+y)=e(y)$. Suppose that $e(x+y)=e(x)$. Then by (3.1) and Part 1,

$$e(x+y)=e(x)+e(x)+e(y)=e(x)+e(y).$$

If $e(x+y)=e(y)$ the proof is analogous. Hence $e(x+y)=e(x)+e(y)$. ■

Proposition 3.3.7 (Idempotency for composition) *Let $(A,+)$ be an assembly. Then for all $x\in A$*

$$e(e(x))=e(x).$$

Proof. Let $x\in A$. By Axiom 3.2.4 and Proposition 3.3.6

$$e(e(x))=e(x-x)=e(x)+e(-x)=e(x)+e(x)=e(x).$$

■

We show that if an element is a magnitude then that element is equal to its own magnitude (imprecision).

Theorem 3.3.8 (Representation) *Let $(A,+)$ be an assembly and let x and $y\in A$. If $x=e(y)$ then $x=e(x)$.*

Proof. Suppose $y\in A$ is such that $x=e(y)$. Then, using Proposition 3.3.7,

$$e(x)=e(e(y))=e(y)=x.$$

■

Corollary 3.3.9 *Let $(A,+)$ be an assembly and let x and $y\in A$. If $x=e(y)$ then $e(x)=e(y)$.*

Corollary 3.3.10 *Let $(A,+)$ be an assembly and let x and $y\in A$. If $x\neq e(x)$, then $x\neq e(y)$.*

3.3.2 Inverse function for addition

The inverse function s is an injective mapping which is linear with respect to addition and has the symmetry property, meaning that the inverse of the inverse of a given element is the element itself.

Proposition 3.3.11 (Symmetry) *Let $(A,+)$ be an assembly. Then for all $x\in A$*

$$-(-x)=x.$$

Proof. Let $x \in A$. Observe that by Axiom 3.2.4

$$e(-(-x)) = e(-x) = e(x) = -x + x.$$

Hence

$$\begin{aligned} -(-x) &= -(-x) + e(-(-x)) = -(-x) - x + x \\ &= e(-x) + x = e(x) + x = x. \end{aligned}$$

■

Proposition 3.3.12 (Linearity) *Let $(A, +)$ be an assembly and let x and $y \in A$. Then*

$$-(x + y) = -x - y.$$

Proof. By Proposition 3.3.6.2 and Axiom 3.2.4

$$\begin{aligned} -(x + y) + x + y &= e(x + y) = e(x) + e(y) \\ &= -x + x - y + y = -x - y + x + y. \end{aligned}$$

Then by cancellation

$$-(x + y) + e(x + y) = -x - y + e(x + y).$$

Again using Axiom 3.2.4 one obtains

$$\begin{aligned} -(x + y) + e(-(x + y)) &= -x - y + e(-x) + e(-y) \\ &= -x + e(-x) - y + e(-y), \end{aligned}$$

and consequently

$$-(x + y) = -x - y.$$

■

Proposition 3.3.13 (Injectivity) *Let $(A, +)$ be an assembly and $x, y \in A$. Then*

1. *$x = y$ if and only if $-x = -y$.*
2. *$-x = y$ if and only if $x = -y$.*

Proof. 1. Because the inverse function is functional, one only needs to prove the direct implication. To prove it, suppose that $-x = -y$. Then

$$x = -(-x) = -(-y) = y.$$

2. Suppose that $-x = y$. By Proposition 3.3.11 $-x = -(-y)$. Hence $x = -y$, by Part 1. The other implication is analogous. ■

3.3.3 Magnitude and inverse

The composition of the inverse function with the neutral function is equal to the neutral function.

Proposition 3.3.14 *Let $(A,+)$ be an assembly and let $x \in A$. Then*

$$e(-x) = -e(x) = e(x).$$

Proof. By Axiom 3.2.4 one only has to show that $-e(x) = e(x)$. Using Proposition 3.3.11 and Proposition 3.3.12 one derives

$$e(x) = -x + x = -x - (-x) = -(x - x) = -e(x).$$

■

Proposition 3.3.15 *Let $(A,+)$ be an assembly and let $x \in A$. Then*

$$e(x) - x = -x.$$

Proof. Using Proposition 3.3.14, Proposition 3.3.12 and Axiom 3.2.3, one has

$$e(x) - x = -e(x) - x = -(x + e(x)) = -x$$

■

The following proposition states, roughly speaking, that the neutral function is not affected by the inverse function.

Proposition 3.3.16 *Let $(A,+)$ be an assembly and let x and $y \in A$.*

1. $-x = e(x)$ if and only if $x = e(x)$.
2. If $-x = -y$ or $-x = y$ then $e(x) = e(y)$.

Proof. 1. Assume firstly that $-x = e(x)$. Then, $e(x) = -x + x = e(x) + x = x$.

Assume secondly that $x = e(x)$. Then $e(x) = x - x = e(x) - x = -x$ by Proposition 3.3.15.

2. If $-x = -y$, then $e(x) = e(-x) = e(-y) = e(y)$.

If $-x = y$, then

$$e(x) = e(-x) = e(y).$$

■

Let $(A,+)$ be an assembly and $x, y, z \in A$. We consider linear additive equations of the type $x + y = e(x)$ and $x + y = z$ where x is variable and y and z are constant. The next result shows how to solve such equations, assuming $e(x) = e(y)$.

Proposition 3.3.17 *Let $(A,+)$ be an assembly and $x, y \in A$. Suppose that $e(x) = e(y)$. Then*

1. $x - y = e(x)$ if and only if $x = y$.
2. $x + y = e(x)$ if and only if $x = -y$.
3. If $x + y = z$ then $x = z - y$.

Proof. 1. If $x - y = e(x)$ then, from the uniqueness of the inverse function, $-x = -y$. Hence $x = y$ by Proposition 3.3.13.1. If $x = y$ then $x - y = x - x = e(x)$.

2. If $x + y = e(x)$ then the result follows from the first case because $x + y = x - (-y)$. If $x = -y$ then $x + y = -y + y = e(y) = e(x)$.

3. If $x + y = z$ then $x + e(y) = z - y$. Hence $x = x + e(x) = x + e(y) = z - y$. ■

3.3.4 Subassemblies

In the following we show that substructures of assemblies which are also assemblies (subassemblies) have properties similar to the properties of subgroups. An important difference though is that subassemblies do not need to contain 0, allowing both $(\mathbb{E}\backslash\mathbb{R}, +)$ and $(\mathbb{R}, +)$ to be subassemblies of $(\mathbb{E}, +)$.

Definition 3.3.18 *Given an assembly $(A, +)$ we say that $(B, +)$ is a* subassembly *of $(A, +)$ if $(B, +)$ is an assembly and $B \subseteq A$.*

Within group theory it is possible to characterize the subgroups of a given group $(G, +)$ as any nonempty subset of G which is closed under addition and under inversions. The following theorem gives a similar characterization of subassemblies of a given assembly.

Theorem 3.3.19 *A structure $(B, +)$ is a subassembly of an assembly $(A, +)$ if and only if $\emptyset \subset B \subseteq A$ and for all $x, y \in B$, $x - y \in B$.*

Proof. Suppose firstly that $(B, +)$ is a subassembly of an assembly $(A, +)$. Let $x, y \in B$. Then because $(B, +)$ is an assembly B is nonempty and for all $x, y \in B$, $x - y \in B$. Suppose secondly that $\emptyset \subset B \subseteq A$ and for all $x, y \in B$, $x - y \in B$. Then the binary operation $+$ is associative and commutative in B. With $x = y \in B$, one has $x - x = e(x) \in B$ and $e(x) - x = e(-x) - x = -x \in B$. Hence $(B, +)$ is an assembly. ■

To prove that a structure is a subassembly of a given assembly becomes quite simpler using the previous theorem. We illustrate this with some examples.

Example 3.3.20 *The following are subassemblies of $(\mathbb{E}, +)$.*

1. $(\mathbb{R}, +)$, because $\mathbb{R} \subset \mathbb{E}$ and $(\mathbb{R}, +)$ is a group.
2. $B = \{x + A | x \in \mathbb{R}\}$, where A is a given neutrix. We have $A \in B$ and $B \subseteq \mathbb{E}$. If $\alpha = a + A$, $\beta = b + A \in B$ then $\alpha - \beta = (a + A) - (b + A) = (a - b) + A \in B$.

3. $(\mathcal{N}, +)$, where $\mathcal{N}$ is the class of all neutrices. The class of all neutrices is nonempty because $0 \in \mathcal{N}$ and the difference of two neutrices is equal to the larger of the two.

4. $(\mathbb{E} \backslash \mathbb{R}, +)$. Clearly $\oslash \in \mathbb{E} \backslash \mathbb{R}$, hence $\mathbb{E} \backslash \mathbb{R}$ is nonempty. Let $x = a + A$, $y = b + B \in \mathbb{E} \backslash \mathbb{R}$. Then $x - y = (a - b) + \max (A, B) \in \mathbb{E} \backslash \mathbb{R}$.

5. $(A_\rho, +)$, where $\rho \in \mathbb{R}$ and $A_\rho = \left\{ x \in \mathbb{E} : x \subseteq \bigcup\limits_{st\ n} [-\rho^n, \rho^n] \right\}$. Clearly $\emptyset \neq A_\rho \subseteq \mathbb{E}$. Let $x, y \in A_\rho$. Then there are standard m, n such that $x \subseteq [-\rho^m, \rho^m]$ and $y \subseteq [-\rho^n, \rho^n]$. Let $p = \max \{m, n\}$. Then $|x - y| \leq 2 \max \{x, y\} \leq 2\rho^p \leq \rho^{p+1}$.

The set of all magnitudes of a given assembly forms a subassembly.

Proposition 3.3.21 *Let $(A, +)$ be an assembly. If $N_A \equiv \{x \in A : x = e(x)\}$ is nonempty then $(N_A, +)$ is a subassembly of $(A, +)$.*

Proof. Assuming that N_A is nonempty let $x = e(x), y = e(y) \in N_A$. Because A is an assembly and $N_A \subseteq A$ one has $x - y \in A$. Then, by the linearity of e,

$$\begin{aligned} x - y &= e(x) - e(y) \\ &= e(x) + e(-y) = e(x - y) \in N_A. \end{aligned}$$

Hence $(N_A, +)$ is a subassembly of $(A, +)$ by Theorem 3.3.19. ∎

Let $(B, +), (C, +)$ be subassemblies of an assembly $(A, +)$. The fact that both $(\mathbb{E} \backslash \mathbb{R}, +)$ and $(\mathbb{R}, +)$ are subassemblies of $(\mathbb{E}, +)$ shows that, unlike what happens with groups, it is possible that $B \cap C = \emptyset$. Moreover, $B \cup C$ may be a subassembly of A and both $B \nsubseteq C$ and $C \nsubseteq B$. However, the following holds.

Proposition 3.3.22 *Let $(B, +), (C, +)$ be subassemblies of an assembly $(A, +)$. Then $(B + C, +)$ is a subassembly of $(A, +)$ and if $B \cap C$ is nonempty $(B \cap C, +)$ is also a subassembly of $(A, +)$.*

Proof. Suppose that $B \cap C$ is nonempty. Let $x, y \in B \cap C$. Then, because B and C are assemblies, $x - y \in B$ and $x - y \in C$ and then $x - y \in B \cap C$. Hence $(B \cap C, +)$ is a subassembly of $(A, +)$, by Theorem 3.3.19.

Suppose now that $x, y \in B + C$. Then there are $u, v \in B$ and $r, t \in C$, such that $x = u + r$ and $y = v + t$. Because B and C are assemblies, $u - v \in B$ and $r - t \in C$ and then

$$\begin{aligned} x - y &= (u + r) + (-(v + t)) \\ &= u - v + r - t \in B + C. \end{aligned}$$

Hence by Theorem 3.3.19 $(B + C, +)$ is a subassembly of $(A, +)$. ∎

By the previous proposition $A = \{x + N | x \in \mathbb{Z}, N \in \mathcal{N}\}$, $B = \{x + \oslash | x \in \mathbb{Q}\}$ and $C = \{x \in \mathbb{Z} : x \text{ is limited}\}$ are assemblies because $A = \mathbb{Z} + \mathcal{N}$, $B = \mathbb{Q} + \oslash$ and $C = \mathbb{Z} \cap \pounds$.

3.4 Homomorphisms

A very important notion in group theory (as well as in other algebraic structures) is the notion of homomorphism, i.e. a function that respects the algebraic structure. The kernel of a homomorphism is, roughly speaking, a measure of the degree to which the homomorphism fails to be injective. These notions can readily be extended to assemblies. We start by giving some examples of assembly homomorphisms. Then we derive some basic properties of homomorphisms and prove that the homomorphic image of an assembly and the kernel of an assembly homomorphism are both assemblies.

Definition 3.4.1 *Let $(A,+)$ and $(B,\cdot)$ be assemblies. A map $\varphi : A \longrightarrow B$ is an* assembly homomorphism *(or simply an homomorphism) if for all $x, y \in A$, $\varphi(x+y) = \varphi(x) \cdot \varphi(y)$.*

The following are assembly homomorphisms:

1. All group homomorphisms, because every group is an assembly.

2. Let A be a neutrix. Then $f : (\mathbb{E},+) \longrightarrow (\mathbb{E},+)$ such that $f(x) = x + A$ is an homomorphism. In fact, if $x, y \in \mathbb{E}$,
$$f(x+y) = (x+y)+A = x+y+A+A = (x+A)+(y+A) = f(x)+f(y).$$

3. Let $(A,+)$ be an assembly. Then $f : (A,+) \longrightarrow (A,+)$ such that $f(x) = e(x)$ is an homomorphism because if $x, y \in A$
$$f(x+y) = e(x+y) = e(x) + e(y) = f(x) + f(y).$$

4. The function $f : (\mathbb{E},+) \longrightarrow (\mathbb{E},+)$ such that $f(x) = \omega x$ for some $\omega \approx +\infty$ is an homomorphism. Let $x, y \in \mathbb{E}$. Then, using Theorem 2.4.6,
$$f(x+y) = \omega(x+y) = \omega x + \omega y = f(x) + f(y).$$

5. The function $f : (\mathcal{N},+) \longrightarrow (\mathcal{N},+)$, $f(x) = \oslash x$, where $\oslash$ is the external set of infinitesimal numbers. Let $x, y \in \mathcal{N}$. Using Theorem 2.4.6,
$$f(x+y) = \oslash(x+y) = \oslash x + \oslash y = f(x) + f(y).$$

6. [38] The function $f : (\mathcal{N},+) \longrightarrow (\mathbb{E}\backslash\{0\},\cdot)$ such that $f(x) = \exp_S(x) \equiv [-e^x, e^x]$. Let $A, B \in \mathcal{N}$. Then
$$\begin{aligned}\exp_S(A+B) &= [\left(-e^A\right)e^B, \left(e^A\right)e^B] = [-e^A, e^A]e^B \\ &= [-e^A, e^A][-e^B, e^B] = \exp_S(A)\exp_S(B)\end{aligned}$$

Obvious examples of functions which are not homomorphisms are nonlinear functions. Consider for instance the function $f : (\mathbb{E}, +) \longrightarrow (\mathbb{E}, +)$ such that $f(x) = x^2$. In fact, if $x = -1 + \oslash$ and $y = 1 + \oslash$ then

$$f(x + y) = f(\oslash) = \oslash^2$$

and

$$f(x) + f(y) = (1 + \oslash)^2 + (-1 + \oslash)^2 = (1 + \oslash) + (1 + \oslash) = 2 + \oslash.$$

However there are also functions which may appear to be linear but are really not. As such one may not extend example 5 to $\mathbb{E}$:

$$\oslash(1 - 1) = 0,$$

while

$$\oslash 1 - \oslash 1 = \oslash.$$

We intend to prove that the homomorphic image of a generalized zero is a generalized zero and that the homomorphic image of the inverse of a given element is the inverse of the homomorphic image of that same element. These properties generalize similar properties for group homomorphisms. In order to do that we need the following lemma.

Lemma 3.4.2 *Let $\varphi : (A, +) \longrightarrow (B, \cdot)$ be an assembly homomorphism. Then*

1. $\varphi(e(x)) = e(\varphi(e(x)))$
2. $\varphi(e(x)) \cdot \varphi(x)^{-1} = \varphi(x)^{-1}$
3. $\varphi(-x) = \varphi(-x) \cdot \varphi(e(x))$
4. $\varphi(e(x)) \cdot \varphi(x)^{-1} = e(\varphi(x)) \cdot \varphi(-x)$

Proof. 1. One has

$$\varphi(e(x)) = \varphi(e(x) + e(x)) = \varphi(e(x)) \cdot \varphi(e(x)).$$

Then by cancellation

$$e(\varphi(e(x))) = e(\varphi(e(x))) \cdot \varphi(e(x)) = \varphi(e(x)).$$

2. Using Part 1 it holds that

$$\begin{aligned}
\varphi(e(x)) \cdot \varphi(x)^{-1} &= e(\varphi(e(x))) \cdot \varphi(x)^{-1} \\
&= e(\varphi(e(x)))^{-1} \cdot \varphi(x)^{-1} = e(\varphi(e(x))^{-1} \cdot \varphi(x)) \\
&= (\varphi(e(x)) \cdot \varphi(x))^{-1} = \varphi(e(x) + x)^{-1} \\
&= \varphi(x)^{-1}.
\end{aligned}$$

3. Because φ is an assembly homomorphism one has

$$\varphi(-x) = \varphi(-x + e(x)) = \varphi(-x) \cdot \varphi(e(x)).$$

4. Observe that

$$\varphi(e(x)) = \varphi(x - x) = \varphi(x) \cdot \varphi(-x). \tag{3.2}$$

Then

$$\varphi(e(x)) \cdot \varphi(x)^{-1} = \varphi(x) \cdot \varphi(-x) \cdot \varphi(x)^{-1} = e(\varphi(x)) \cdot \varphi(-x).$$

■

We are now able to show that assembly homomorphisms have the expected properties.

Proposition 3.4.3 *Let $\varphi : (A, +) \longrightarrow (B, \cdot)$ be an assembly homomorphism and let $x \in A$. Then*

1. $\varphi(-x) = \varphi(x)^{-1}$.

2. $\varphi(e(x)) = e(\varphi(x))$.

Proof. 1. Because φ is a homomorphism

$$\begin{aligned} \varphi(-x) &= \varphi(x - x - x) = \varphi(x) \cdot \varphi(-x) \cdot \varphi(-x) \\ &= e(\varphi(x)) \cdot \varphi(x) \cdot \varphi(-x) \cdot \varphi(-x). \end{aligned}$$

Then by Lemma 3.4.2.3

$$\varphi(-x) = e(\varphi(x)) \cdot \varphi(e(x)) \cdot \varphi(-x) = e(\varphi(x)) \cdot \varphi(-x).$$

Hence by Lemma 3.4.2.4

$$\varphi(-x) = \varphi(e(x)) \cdot \varphi(x)^{-1} = \varphi(x)^{-1}.$$

2. By formula (3.2) and Part 1 one has

$$\varphi(e(x)) = \varphi(x) \cdot \varphi(-x) = \varphi(x) \cdot \varphi(x)^{-1} = e(\varphi(x)).$$

■

The homomorphic image of an assembly is also an assembly.

Theorem 3.4.4 *Let $\varphi : (A, +) \longrightarrow (B, \cdot)$ be an assembly homomorphism. Then $\varphi(A)$ is a subassembly of B.*

Proof. Clearly $\varphi(A) \subseteq B$. Because A and B are assemblies they are nonempty and then $\varphi(A)$ is also nonempty. Let $u, v \in \varphi(A)$. Then there are $a, b \in A$ such that $u = \varphi(a)$ and $v = \varphi(b)$. Then, by Proposition 3.4.3.1,

$$u \cdot (v)^{-1} = \varphi(a) \cdot \varphi(b)^{-1} = \varphi(a) \cdot \varphi(-b) = \varphi(a - b) \in \varphi(A).$$

Hence $\varphi(A)$ is a subassembly of B by Theorem 3.3.19. ■

Definition 3.4.5 *Let $\varphi : (A,+) \longrightarrow (B,\cdot)$ be an assembly homomorphism. We call* kernel *of φ the set*

$$Ker_\varphi \equiv \{x \in A : \varphi(x) = e(\varphi(x))\}.$$

The kernel of an assembly homomorphism is also an assembly.

Proposition 3.4.6 *Let $\varphi : (A,+) \longrightarrow (B,\cdot)$ be an assembly homomorphism. Then Ker_φ is a subassembly of A. Moreover, if φ is injective then*

$$Ker_\varphi = \{x \in A : x = e(x)\}.$$

Proof. It is clear that $Ker_\varphi \subseteq A$. By Proposition 3.4.3.2 $\varphi(e(x)) = e(\varphi(x))$ and then $Ker_\varphi \neq \emptyset$. Let $x, y \in Ker_\varphi$. Then $\varphi(x) = e(\varphi(x))$ and $\varphi(y) = e(\varphi(y))$. Using Proposition 3.4.3 one derives

$$\begin{aligned} \varphi(x-y) &= \varphi(x)\cdot\varphi(-y) = \varphi(x)\cdot\varphi(y)^{-1} \\ &= e(\varphi(x))\cdot e(\varphi(y))^{-1} = e(\varphi(x))\cdot e(-\varphi(y)) \\ &= e(\varphi(x)\cdot(-\varphi(y))) = e(\varphi(x-y)). \end{aligned}$$

Then $x - y \in Ker_\varphi$. Hence Ker_φ is a subassembly of A, by Theorem 3.3.19.

Suppose that φ is injective. If $x \in Ker_\varphi$, one has $\varphi(x) = e(\varphi(x)) = \varphi(e(x))$. Then $x = e(x)$. If $x = e(x)$, then $\varphi(x) = \varphi(e(x)) = e(\varphi(x))$. Hence $x \in Ker_\varphi$. ∎

The kernel of an assembly homomorphism is reduced to neutral elements if and only if a sort of relaxed injectivity holds for the homomorphism.

Theorem 3.4.7 *Let $\varphi : (A,+) \longrightarrow (B,\cdot)$ be an assembly homomorphism and let $x, y \in A$. Then the following are equivalent*

1. $\varphi(x) = \varphi(y) \Rightarrow x + e(y) = y + e(x)$
2. $\varphi(x) = \varphi(y) \Rightarrow x - y = e(x+y)$.
3. $Ker_\varphi = \{x \in A : x = e(x)\}$.

Proof. We prove that (1)⇒(2)⇒(3)⇒(1).

(1)⇒(2) Assume that $\varphi(x) = \varphi(y)$. Then $x + e(y) = y + e(x)$ and $x - y = x + e(y) - y = y + e(x) - y = e(x) + e(y) = e(x+y)$.

(2)⇒(3) Observe firstly that if $x \in Ker_\varphi$ then $\varphi(x) = e(\varphi(x)) = \varphi(e(x))$. Then $x - e(x) = e(x+x)$. Hence $x = e(x)$. Then $Ker_\varphi \subseteq \{x \in A : x = e(x)\}$. If $y \in \{x \in A : x = e(x)\}$, by Proposition 3.4.3.2 $\varphi(y) = \varphi(e(y)) = e(\varphi(y))$. Hence $y \in Ker_\varphi$ and $\{x \in A : x = e(x)\} \subseteq Ker_\varphi$. Then $Ker_\varphi = \{x \in A : x = e(x)\}$.

(3)⇒(1) Let $x \in A$. Then

$$\varphi(x) = e(\varphi(x)) \Leftrightarrow x = e(x). \tag{3.3}$$

If $\varphi(x) = \varphi(y)$, by Proposition 3.4.3

$$\begin{aligned}\varphi(x-y) &= \varphi(x)\cdot\varphi(-y) = \varphi(y)\cdot(-\varphi(y)) = e(\varphi(y)) = e(\varphi(y))\cdot e(\varphi(y))\\ &= e(\varphi(x))\cdot e(\varphi(y)) = e(\varphi(x))\cdot e(-\varphi(y))\\ &= e(\varphi(x))\cdot e(\varphi(-y)) = e(\varphi(x)\cdot\varphi(-y)) = e(\varphi(x-y))\end{aligned}$$

Then, by formula (3.3), $x - y = e(x-y) = e(x) + e(-y) = e(x) + e(y)$. Hence

$$x + e(y) = y + e(x).$$

■

3.5 Ordered Assemblies

An ordered assembly is an assembly together with a total ordering of its elements that is compatible with addition. So, an ordered assembly is an assembly equipped with an order relation "$\leq$" satisfying axioms 3.2.11-3.2.15.

Let $(A, +, \leq)$ be an ordered assembly and let $x \in A$. Continuing our interpretation of $e(x)$ as a generalized zero we say that x is *positive* if $e(x) \leq x$ and that x is *negative* if $x \leq e(x)$. We prove that if x is positive then $-x$ is negative and that if x is negative then $-x$ is positive. Then we give an algebraic characterization of the order relation for the function e. We also generalize this result in Theorem 3.5.10. Finally we consider the notion of absolute value and prove that some classical properties such as the triangular inequality remain valid for assemblies and that other properties may be generalized.

Remark 3.5.1 *Let $(A, +, \leq)$ be an ordered assembly and let $x, y \in A$. Observe that if $x \leq y$,*

$$e(x) \leq y - x \tag{3.4}$$

and

$$-y + e(x) \leq -x + e(y). \tag{3.5}$$

Remark 3.5.2 *Let $(A, +, \leq)$ be an ordered assembly. It follows immediately from the axioms that for all $x, y \in A$ one has*

1. x is positive if and only if $-x$ is negative.

2. x is negative if and only if $-x$ is positive.

3. If $x \leq y$, then $x + e(y) \leq y$.

4. If $x \leq y$, then $x \leq y + e(x)$.

Let $(A, +, \leq)$ be an ordered assembly. As stated in the previous remark, $x \in A$ is positive if and only if $-x$ is negative and vice-versa. A similar result is valid for strict inequalities.

Proposition 3.5.3 *Let $(A, +, \leq)$ be an ordered assembly. Then for all $x, y \in A$*

1. $e(x) < x$ if and only if $-x < e(x)$.
2. $x < e(x)$ if and only if $e(x) < -x$.

Proof. Part 1 follows from Proposition 3.3.16.1 and Remark 3.5.2.1.
Part 2 follows from Proposition 3.3.16.1 and Remark 3.5.2.2. ■
We show that x is positive if and only if it is larger than or equal to $-x$.

Proposition 3.5.4 *Let $(A, +, \leq)$ be an ordered assembly and let $x \in A$. Then $-x \leq x$ if and only if $e(x) \leq x$.*

Proof. Assume firstly that $-x \leq x$. by Axiom 3.2.14, one has $e(x) \leq x$ or $x \leq e(x)$. In the first case there is nothing to show. In the second case, by Remark 3.5.2.2 one has $e(x) \leq -x$ then by transitivity $e(x) \leq x$. Assume secondly that $e(x) \leq x$. Then by Remark 3.5.2.1 $-x \leq e(x)$ and by transitivity $-x \leq x$. ■

From now on we will assume Axiom 3.2.16. We recall that this axiom says, according to our intended interpretation, that if an element leaves a magnitude invariant for addition, i.e. if the sum of the element with a magnitude is equal to the magnitude, then the element is smaller than this magnitude. In the following theorem we use this axiom in order to give an algebraic characterization of the order relation for the function e in terms of addition.

Theorem 3.5.5 *Let $(A, +, \leq)$ be an ordered assembly. Then, for all $x, y \in A$, $e(x) + e(y) = e(x)$ if and only if $e(y) \leq e(x)$.*

Proof. By Axiom 3.2.16 we only need to prove the necessary part. Suppose that $e(y) \leq e(x)$. By compatibility with addition

$$e(y) = e(y) + e(y) \leq e(x) + e(y).$$

Now $e(x) + e(y) = e(x)$ or $e(x) + e(y) = e(y)$ by Proposition 3.3.6.2. If $e(x) + e(y) = e(x)$, there is nothing to prove. If $e(x) + e(y) = e(y)$, by Axiom 3.2.16, $e(x) \leq e(y)$. But then, by antisymmetry, $e(x) = e(y)$. Hence

$$e(x) + e(y) = e(x) + e(x) = e(x).$$

■

Corollary 3.5.6 *Let $(A, +, \leq)$ be an ordered assembly and let $x, y \in A$. Then*

$$e(x + y) \leq e(x) \Rightarrow e(y) \leq e(x).$$

Proof. Suppose that $e(x + y) \leq e(x)$. Then by Theorem 3.5.5

$$e(x) + e(x + y) = e(x).$$

Hence $e(x) + e(y) = e(x)$ and $e(y) \leq e(x)$ by Theorem 3.5.5. ■

Corollary 3.5.7 *Let $(A, +, \leq)$ be an ordered assembly and let $x, y \in A$. Then*

$$x + y = x \Rightarrow e(y) \leq e(x).$$

Proof. Suppose that $x + y = x$. Then by the cancellation law

$$e(x) + y = e(x). \tag{3.6}$$

Then by Axiom 3.2.16,

$$y \leq e\,(x)\,. \tag{3.7}$$

Using Proposition 3.5.8.1 and formula (3.6), one has

$$e\,(x + y) = e\,(x) + e\,(y) = y + e\,(x) + e\,(y) = y + e\,(x) \leq e\,(x)\,.$$

Hence by Corollary 3.5.6

$$e(y) \leq e(x).$$

■

We prove a generalization of Axiom 3.2.15. As a consequence one has that the sum of two positive elements is also positive. Also, any element which is larger than or equal to a positive element is also positive.

Proposition 3.5.8 *Let $(A, +, \leq)$ be an ordered assembly and let $x, y \in A$.*

1. If $x \leq y$ and $z \leq w$, then $x + z \leq y + w$.

2. If x and y are both positive then $x + y$ is also positive.

3. If $e\,(y) \leq y \leq x$ then $e\,(x) \leq x$.

Proof. To prove Part 1 suppose that $x \leq y$ and $z \leq w$. By compatibility with addition one has

$$x + z \leq y + z$$

and

$$y + z \leq y + w.$$

Then by transitivity

$$x + z \leq y + w.$$

Part 2 follows from Part 1.

As for Part 3, suppose that $e\,(y) \leq y \leq x$. If $e\,(y) \leq e\,(x)$ then $e\,(x) + e\,(y) = e\,(x)$ by Theorem 3.5.5. By compatibility with addition

$$e\,(x) = e\,(x) + e\,(y) \leq x + e\,(x) = x.$$

If $e\,(x) \leq e\,(y)$, one has $e\,(x) \leq x$, by Axiom 3.2.13. ■

Proposition 3.5.9 *Let $(A, +, \leq)$ be an ordered assembly and let $x, y \in A$. Then $x \leq x + e\,(y)$.*

Proof. If $e(y) \leq e(x)$. Then by Theorem 3.5.5

$$x + e(y) = x + e(x) + e(y) = x + e(x) = x.$$

If $e(x) \leq e(y)$ then by compatibility with addition

$$x = x + e(x) \leq x + e(y).$$

■

A positive number leaves a magnitude invariant if and only if it is smaller than this magnitude. This means that if y is positive we can replace the implication in Axiom 3.2.16 by an equivalence. In this way one has, for positive elements, a complete connection between the invariance of magnitudes for addition and the order relation.

Theorem 3.5.10 *Let $(A, +, \leq)$ be an ordered assembly and let $x, y \in A$. If y is positive then*

$$y \leq e(x) \Leftrightarrow e(x) + y = e(x).$$

Proof. Suppose that y is positive. By Axiom 3.2.16 we only need to prove the sufficiency. Suppose that $y \leq e(x)$. Then by transitivity $e(y) \leq e(x)$ and by Theorem 3.5.5

$$e(x) + e(y) = e(x). \tag{3.8}$$

Because y is positive, using the compatibility with addition and formula (3.8),

$$e(x) = e(y) + e(x) \leq e(x) + y.$$

And, because $y \leq e(x)$,

$$e(x) + y \leq e(x) + e(x) = e(x).$$

Hence

$$e(x) = e(x) + y.$$

■

3.6 Absolute Value

In ordered assemblies it is possible to define a notion of absolute value. Let $(A, +, \leq)$ be an ordered assembly and $x \in A$. We show that classical properties such as the fact that the absolute value of x is equal to the absolute value of $-x$ and the triangular inequality remain valid for ordered assemblies. Other properties need to be adapted. For example, the absolute value of x is greater than or equal to $e(x)$ and equality is verified if and only if $x = e(x)$. Keeping in mind the interpretation of $e(x)$ as a generalized zero one may see those properties as generalizations of the usual properties of the absolute value.

Definition 3.6.1 *Let $(A, +, \leq)$ be an ordered assembly and $x \in A$. The* absolute value of x *is defined as*

$$|x| \equiv \begin{cases} x, & \text{if } e(x) \leq x \\ -x, & \text{if } x < e(x) \end{cases}$$

Remark 3.6.2 *Let $(A, +, \leq)$ be an ordered assembly and $x \in A$. It is clear from the previous definition that the absolute value is idempotent, i.e. $||x|| = |x|$.*

Proposition 3.6.3 *Let $(A, +, \leq)$ be an ordered assembly and let $x, y \in A$. Then*

$$e(x) + y = e(x) \Rightarrow |y| \leq e(x).$$

Proof. If $y \geq e(y)$ the result follows by Axiom 3.2.16 because $|y| = y$.

If $y < e(y)$, suppose that $e(x) + y = e(x)$. Then $e(x) - y = -(-e(x) + y) = -(e(x) + y) = -e(x) = e(x)$. Then by Axiom 3.2.16 $|y| = -y \leq e(x)$. ■

3.6.1 Absolute value and the magnitudes

We start by showing that the absolute value is always positive.

Proposition 3.6.4 *Let $(A, +, \leq)$ be an ordered assembly and $x \in A$. Then*

$$e(x) \leq |x|.$$

Proof. If $e(x) \leq x$ then

$$e(x) \leq x = |x|.$$

If $x < e(x)$, then by Remark 3.5.2.2 $e(x) \leq -x$. Hence $e(x) \leq |x|$. ■

By Proposition 3.3.14 and the definition of absolute value we may conclude that the absolute value of the imprecision of a given element and the imprecision of its absolute value are both equal to its imprecision.

Proposition 3.6.5 *Let $(A, +, \leq)$ be an ordered assembly and $x \in A$. Then*

$$e(|x|) = e(x) = |e(x)|.$$

The next result states the positive-definiteness of the absolute value, i.e. the absolute value of an element is a generalized zero if and only if the element is a generalized zero.

Proposition 3.6.6 *Let $(A, +, \leq)$ be an ordered assembly and let $x \in A$. Then $|x| = e(x)$ if and only if $x = e(x)$.*

Proof. Assume firstly that $|x| = e(x)$.

If $e(x) \leq x$,

$$e(x) = |x| = x.$$

If $x < e(x)$,

$$e(x) = |x| = -x.$$

Hence $x = e(x)$, by Proposition 3.3.16.1.

Assume secondly that $x = e(x)$. Then by Proposition 3.6.5

$$|x| = |e(x)| = e(x).$$

■

Proposition 3.6.7 (Symmetry) *Let $(A, +, \leq)$ be an ordered assembly and $x, y \in A$. Then*

$$|x| = |-x|.$$

Proof. If $e(x) \leq -x$ then $x \leq e(x)$ by Proposition 3.5.2 and $|-x| = -x = |x|$. If $-x < e(x)$ then $e(x) < x$ by Proposition 3.5.2.2. Hence $|-x| = -(-x) = x = |x|$. ■

Let R be an ordered ring and let $x, y \in R$. Then $|x - y| = 0$ if and only if $x = y$. This property is called identity of indiscernibles. We prove an adapted version of this property within ordered assemblies.

Proposition 3.6.8 *Let $(A, +, \leq)$ be an ordered assembly and $x, y, z \in A$. Then $|x - y| = e(z)$ if and only if $x + e(y) = y + e(x)$. Moreover $e(z) = e(x) + e(y)$.*

Proof. By Theorem 3.3.8 one has $e(z) = e(|x - y|)$. By Proposition 3.6.5 it holds that

$$e(|x - y|) = e(x - y) = e(x) + e(y).$$

Hence $e(z) = e(x) + e(y)$.

Suppose firstly that $|x - y| = e(z)$. Then it follows from Proposition 3.6.6 that

$$x - y = e(z) = e(x) + e(y).$$

Hence $x + e(y) = y + e(x)$.

Suppose secondly that $x + e(y) = y + e(x)$. Then, using Proposition 3.6.5,

$$\begin{aligned} |x - y| &= |x - y + e(y)| = |e(x) + e(y)| \\ &= e(x) + e(y) = e(z). \end{aligned}$$

■

3.6.2 Basic properties of the absolute value

Any element is bounded above by its absolute value and bounded below by the inverse of its absolute value.

Proposition 3.6.9 *Let $(A, +, \leq)$ be an ordered assembly and $x, y \in A$. Then*

$$-|x| \leq x \leq |x|.$$

Proof. If $e(x) \leq x$, then, $-x \leq e(x)$. Then $-x \leq e(x) \leq x$ and by reflexivity and transitivity $-x \leq x$. Moreover, because $e(x) \leq x$, $|x| = x$ and $-|x| = -x$. Then
$$-|x| = -x \leq x = |x|.$$

If $x < e(x)$, then $|x| = -x$ and $-|x| = -(-x)$. By Remark 3.5.2.2, $e(x) \leq -x$ Then
$$-|x| = -(-x) = x < e(x) \leq -x = |x|,$$
and by transitivity one concludes the result. ■

Proposition 3.6.10 *Let $(A, +, \leq)$ be an ordered assembly and $x, y \in A$. If y is positive then $|x| = y$ if and only if $x = y$ or $x = -y$.*

Proof. Let y be positive.

Suppose firstly that $|x| = y$. If $e(x) \leq x$ one has
$$x = |x| = y.$$
If $x < e(x)$, then
$$-x = |x| = y.$$
Hence $x = -y$, by Proposition 3.3.13.2.

Suppose secondly that $x = y$ or $x = -y$. In the first case one has
$$|x| = |y| = y,$$
because y is positive. In the second case, again because y is positive,
$$|x| = |-y| = |y| = y.$$
■

Proposition 3.6.11 *Let $(A, +, \leq)$ be an ordered assembly and $x, y \in A$. Then*

1. *$|x| \leq y$ if and only if $x \leq y$ and $-x \leq y$.*
2. *$y \leq |x|$ if and only if $y \leq x$ or $y \leq -x$.*

Proof. 1. Suppose firstly that $|x| \leq y$.

If $e(x) \leq x$ then $-x \leq e(x)$. Hence
$$-x \leq e(x) \leq x = |x| \leq y.$$
By transitivity $x \leq y$ and $-x \leq y$.

If $x < e(x)$, then $e(x) < -x$. Then
$$x < e(x) < -x = |x| \leq y$$
Again by transitivity $x \leq y$ and $-x \leq y$.

Suppose secondly that $x \leq y$ and $-x \leq y$. If $e(x) \leq x$, then

$$|x| = x \leq y.$$

If $x < e(x)$, then

$$|x| = -x \leq y.$$

2. Suppose firstly that $y \leq |x|$. If $e(x) \leq x$, then

$$y \leq |x| = x.$$

If $x < e(x)$, then

$$y \leq |x| = -x.$$

Hence $y \leq |x| \Rightarrow y \leq x$ or $y \leq -x$.

Suppose secondly that $y \leq x$ or $y \leq -x$. Assume firstly that $y \leq x$. If $e(x) \leq x$ then

$$y \leq x = |x|.$$

If $x < e(x)$ and $y \leq x$, then

$$y \leq x < e(x) \leq -x = |x|.$$

Assume now that $y \leq -x$. If $e(x) \leq x$ one has

$$y \leq -x \leq e(x) \leq x = |x|.$$

If $x < e(x)$

$$y \leq -x = |x|.$$

Hence $y \leq |x|$. ■

3.6.3 Triangular inequalities

Proposition 3.6.12 (Triangular inequality) *Let $(A, +, \leq)$ be an ordered assembly and $x, y \in A$. Then $|x + y| \leq |x| + |y|$.*

Proof. Using Proposition 3.6.9 and Proposition 3.5.8.1 one has

$$x + y \leq |x| + |y|. \tag{3.9}$$

In order to prove that also $-(x + y) \leq |x| + |y|$ we observe that also

$$-|x| - |y| \leq x + y. \tag{3.10}$$

By Proposition 3.5.8.1,

$$-|x| - |y| + |x| + |y| \leq x + y + |x| + |y|.$$

By Axiom 3.2.4

$$e(|x|) + e(|y|) \leq x + y + |x| + |y|,$$

and by Proposition 3.6.5

$$e(x) + e(y) \leq x + y + |x| + |y| \,.$$

Adding $-x-y$ to both sides of the inequality and using Proposition 3.5.8.1 and the compatibility with addition one has

$$-x - y \leq e(x) + e(y) + |x| + |y| \,.$$

Then by Proposition 3.6.5,

$$-(x+y) = -x - y \leq |x| + |y| \,. \tag{3.11}$$

From inequalities (3.9) and (3.11), using Proposition 3.6.11.1, one concludes

$$|x+y| \leq |x| + |y| \,.$$

■

Corollary 3.6.13 *Let $(A, +, \leq)$ be an ordered assembly and $x, y, z \in A$. Then*

$$|x-y| \leq |x-z| + |z-y| \,.$$

Proof. Using Proposition 3.6.12 one has

$$|x-y| \leq |x-z+z-y| \leq |x-z| + |z-y| \,.$$

■

Proposition 3.6.14 *Let $(A, +, \leq)$ be an ordered assembly and $x, y \in A$. Then*

$$|x + e\,(y)| = |x| + e\,(y) \,.$$

Proof. Assume firstly that $e\,(x) \geq e\,(y)$ then by Theorem 3.5.5 $e\,(x) + e\,(y) = e\,(x)$, hence also $e\,(|x|) + e\,(y) = e\,(|x|)$, so

$$\begin{aligned} |x + e\,(y)| &= |x| = |x| + e\,(|x|) \\ &= |x| + e\,(|x|) + e\,(y) = |x| + e\,(y) \,. \end{aligned}$$

Assume now that $e\,(x) \leq e\,(y)$. We study two cases: $e\,(x) \leq x$ and $x < e\,(x)$. In the first case one has $e\,(x + e\,(y)) = e\,(x) + e\,(y) \leq x + e\,(y)$ by Proposition 3.5.8.1. Hence

$$|x + e\,(y)| = x + e\,(y) = |x| + e\,(y) \,.$$

In the second case one has $e\,(x) < -x$. Then $e\,(x)+e\,(y) \leq -x+e\,(y)$ by Proposition 3.5.8.1. Then $e\,(-x + e\,(y)) = e\,(x) + e\,(y) \leq -x + e\,(y) = -\,(x + e\,(y))$. Hence

$$|x + e\,(y)| = -\,(x + e\,(y)) = -x + e\,(y) = |x| + e\,(y) \,.$$

■

In a an ordered ring R the inequalities $|x| - |y| \leq |x+y|$ and $||x| - |y|| \leq |x-y|$ hold for all $x, y \in R$. We show that these inequalities also hold within ordered assemblies. But first we need the following lemma.

Lemma 3.6.15 *Let $(A,+,\leq)$ be an ordered assembly and $x,y\in A$. Then*

$$|x-y|+e(y)=|x-y|+e(x)=|x-y|. \tag{3.12}$$

Proof. One has

$$e(|x-y|)=e(x-y)=e(x)+e(y).$$

Hence

$$|x-y|=|x-y|+e(x)+e(y),$$

which implies (3.12). ■

Proposition 3.6.16 *Let $(A,+,\leq)$ be an ordered assembly and $x,y\in A$. Then*

$$|x|-|y|\leq|x+y|.$$

Proof. Using Proposition 3.6.7 and Proposition 3.6.12 one has

$$|x|\leq|x+e(y)|=|x+y-y|\leq|x+y|+|-y|=|x+y|+|y|.$$

Then, by compatibility with addition,

$$|x|-|y|\leq|x+y|+e(|y|).$$

Hence

$$|x|-|y|\leq|x+y|,$$

by Lemma 3.6.15. ■

Proposition 3.6.17 *Let $(A,+,\leq)$ be an ordered assembly and $x,y\in A$. Then*

$$||x|-|y||\leq|x-y|.$$

Proof. By Proposition 3.6.12

$$|x|\leq|x+e(y)|=|x+y-y|\leq|x-y|+|y|$$

and

$$|y|\leq|y+e(x)|=|y+x-x|\leq|y-x|+|x|.$$

Then, using symmetry,

$$|x|-|y|\leq|x-y|+e(|y|)=|x-y|+e(y)$$

and

$$|y|-|x|\leq|y-x|+e(|x|)=|x-y|+e(x).$$

Hence

$$|x|-|y|\leq|x-y|,$$

and

$$-(|x|-|y|)=|y|-|x|\leq|x-y|$$

by Lemma 3.6.15. By Definition 3.6.1 and symmetry one concludes $||x|-|y||\leq|x-y|$. ■

Proposition 3.6.18 *Let $(A,+,\leq)$ be an ordered assembly and $x,y\in A$. Then*

$$e(x)+y=e(x)\Leftrightarrow |y|\leq e(x).$$

Proof. By Proposition 3.6.3 one only needs to show the converse implication.

If y is positive then $|y|=y$ and the result follows from Theorem 3.5.10.

If y is negative then $-y$ is positive by Proposition 3.5.3.2. Suppose firstly that $|y|\leq e(x)$. Then

$$e(x)-y=e(x),$$

by Theorem 3.5.10. Then

$$e(x)+y=-(-e(x)-y)=-(e(x)-y)=-e(x)=e(x).$$

■

3.7 Associations

In this section we introduce a second operation " $\cdot$ " called multiplication and consider structures which are assemblies for both addition and multiplication. This yields a structure called association which is meant to be a sort of nondistributive ring with individualized neutral elements for both operations. We investigate to what extent the properties of rings remain valid within associations. Axioms 3.2.18 and 3.2.20 allow us to gain the necessary tools to understand how magnitudes behave with respect to multiplication. Indeed, with these additional axioms we are able to show that a generalized unity multiplied by a magnitude (generalized zero) is equal to a magnitude. This is related to the well-known result, valid in rings, saying that $1.0=0$. We also introduce Axiom 3.2.21 in order to be able to calculate the magnitude of a unity $e(u(x))$. We show how to compute the magnitude of an inverse $e\left(x^{-1}\right)$. Furthermore we show (see Theorem 3.7.19 below) that the magnitude of a unity, the magnitude of an inverse and the product of a magnitude by a unity are strongly connected. Let R be a ring and let $x,y\in R$. It is well-known that if $xy=0$ implies that $x=0$ or $y=0$ for all $x,y\in R$ one says that the ring has no *zero divisors* and call the ring an *integral domain*. We show that an association is a sort of (nondistributive) integral domain because associations have no generalized zero divisors.

3.7.1 Multiplicative assemblies

Definition 3.7.1 *A structure $(A,+,e,s,\cdot,u,d)$ is called an association if A satisfies axioms 3.2.1-3.2.10.*

The previous definition means that if $(A,+,e,s,\cdot,u,d)$ is an association both $(A,+,e,s)$ and $(A^{*},\cdot,u,d)$ are assemblies, where $A^{*}=\{x\in A:x\neq e(x)\}$, and multiplication is both commutative and associative in A. If no confusion is possible by abuse of language we say that A is an association.

Definition 3.7.2 *Let A be an association and $x \in A$. We say that x is* zeroless *if $x \neq e(x)$.*

Because $(A^*, ., u, d)$ is an assembly, the multiplicative cancellation law holds taking the following form.

Proposition 3.7.3 (Multiplicative cancellation law) *Let A be an association. Let $x, y, z \in A^*$ be arbitrary. Then*

$$xy = xz \Leftrightarrow u(x)y = u(x)z.$$

Results which hold for additive assemblies should also hold for multiplicative assemblies. A list of those results is given in Proposition 3.7.5. However, to fully justify those results one needs to ensure that $x \neq e(x)$ whenever $u(x)$ or $d(x)$ are considered. For example in Part 4 of Proposition 3.7.5 one needs to verify that $e(u(x)) \neq u(x)$. In order to prove all such verifications we assume from now on Axiom 3.2.18. This axiom states, roughly speaking, that one can treat scaled magnitudes just as magnitudes. The verifications mentioned above are made in the following lemma.

Lemma 3.7.4 *Let A be an association and let $x, y \in A^*$*

1. $u(x) \neq e(u(x))$.
2. $x^{-1} \neq e(x^{-1})$.
3. $-u(x) \neq e(-u(x))$.
4. $u(x) \neq e(y)$, for all $y \in A$.

Proof. 1. Suppose that $u(x) = e(u(x))$. Then

$$x = xu(x) = xe(u(x)).$$

By Axiom 3.2.18, there is z such that $x = e(z)$ and by Theorem 3.3.8, $x = e(x)$. Hence, if $x \neq e(x)$, then $u(x) \neq e(u(x))$.

2. Suppose that $x^{-1} = e(x^{-1})$. Then

$$u(x) = xx^{-1} = xe(x^{-1}).$$

By Axiom 3.2.18 there is z such that $u(x) = e(z)$. By Theorem 3.3.8 $u(x) = e(u(x))$, in contradiction with Part 1.

3. By Part 1,

$$u(x) \neq e(u(x)).$$

By Proposition 3.3.16.1 and Axiom 3.2.4

$$-u(x) \neq e(u(x)) = e(-u(x)).$$

4. Suppose towards a contradiction that $u(x) = e(y)$. Then $x = e(u(x))$, by Theorem 3.3.8, in contradiction with Part 1. ■

By Lemma 3.7.4, because $(A^*, ., u, d)$ is an assembly, the following holds.

Proposition 3.7.5 *Let A be an association. For all $x, y \in A^*$*

1. $u(x)\,u(x) = u(x)$.
2. $u(xy) = u(x)\,u(y)$.
3. $u(u(x)) = u(x)$.
4. $u(x) = (u(x))^{-1}$
5. $\left(x^{-1}\right)^{-1} = x$.
6. $(xy)^{-1} = x^{-1}y^{-1}$.
7. $u\left(x^{-1}\right) = (u(x))^{-1} = u(x)$.
8. $x \neq u(x) \Rightarrow x \neq u(y)$.
9. $u(x)\,x^{-1} = x^{-1}$.
10. $x \neq e(x) \Rightarrow x^{-1}u(x) = x^{-1}$.

3.7.2 Symmetric of the product

Let R be a ring and $x, y \in R$. A well-known ring property says that $-(xy) = (-x)y$. From now on we assume Axiom 3.2.19 which links multiplication with the inverse for addition in the same manner. We show other properties involving multiplication and the inverse for addition which are also valid in rings.

Proposition 3.7.6 *Let A be an association and let $x, y \in A$. Then*

1. $(-x)y = (-y)x$.
2. $(-x)(-y) = xy$.

Proof. 1. By Axiom 3.2.19 one has

$$(-x)y = -(xy) = -(yx) = (-y)x.$$

2. By Axiom 3.2.19 and Proposition 3.3.11

$$(-x)(-y) = -(x(-y)) = -((-y)x) = -(-(xy)) = xy.$$

■

Next proposition says that the product of a magnitude by a given element is equal to the product of the magnitude by the inverse for addition of that element. So, in a way, when one multiplies a given element by a magnitude the sign of that element can be neglected.

Proposition 3.7.7 *Let A be an association and let $x, y \in A$. Then*

$$e(y)(-x) = e(y)x.$$

Proof. By Proposition 3.7.6.1 and Proposition 3.3.14

$$e(y)(-x) = -(e(y))x = e(y)x.$$

■

The following proposition shows that the usual relations between the inverse for addition, the inverse for multiplication and the unity are preserved in associations.

Proposition 3.7.8 *Let A be an association and let $x \in A$ be zeroless. Then*

1. $(-x)\,u(x) = -x.$
2. $(-x)\,x^{-1} = -u(x).$
3. $u(-x) = u(x)$
4. $-u(x) = (-u(x))^{-1}$

Proof. 1. Because $x \neq e(x)$, by Axiom 3.2.19,

$$(-x)u(x) = -(xu(x)) = -x.$$

2. Because $x \neq e(x)$, by Axiom 3.2.19

$$(-x)x^{-1} = -(xx^{-1}) = -u(x).$$

3. Using Proposition 3.7.5 and Proposition 3.7.6.2

$$\begin{aligned} u\,(-x) &= u\,(-x)\,u\,(-x) \\ &= u\,((-x)\,(-x)) = u\,(xx) \\ &= u\,(x)\,u\,(x) = u\,(x)\,. \end{aligned}$$

4. By Lemma 3.7.4.3 one has $-u(x) \neq e(-u(x))$, then using Part 3, Proposition 3.7.5 and Proposition 3.7.6.2

$$\begin{aligned} (-u(x))(-u(x))^{-1} &= u(-u(x)) = u(u(x)) \\ &= u(x) = u(x)u(x) \\ &= (-u(x))(-u(x)) \end{aligned}$$

Hence by cancellation

$$u\,(-u\,(x))\,(-u(x))^{-1} = u(-u(x))(-u(x))$$

and one concludes that

$$(-u(x))^{-1} = -u(x).$$

■

The inverse for addition of the inverse for multiplication is equal to the inverse for multiplication of the inverse for addition.

Proposition 3.7.9 *Let A be an association and let $x \in A$ be zeroless. Then*

$$-\left(x^{-1}\right) = (-x)^{-1}.$$

Proof. By Proposition 3.7.8.4

$$\begin{aligned} -(x^{-1}) &= -(x^{-1}u(x)) = x^{-1}(-u(x)) \\ &= x^{-1}(-u(x))^{-1} = (x(-u(x)))^{-1} \\ &= (-(xu(x)))^{-1} = (-x)^{-1}. \end{aligned}$$

■

3.7.3 Neutral element of product

We assume now Axiom 3.2.20. This axiom states, according to the intended interpretation, that the imprecision of the product is equal to the maximum of the scaled imprecisions. This enables us to connect magnitudes with multiplication.

If an element is not a magnitude then its square (element multiplied by itself) is also not a magnitude.

Proposition 3.7.10 *Let A be an association and let $x \in A^*$, then*

$$x^2 \neq e\left(x^2\right).$$

Proof. Suppose that $x \neq e(x)$ and $x^2 = e\left(x^2\right)$. Then, by Axiom 3.2.20

$$x^2 = e\left(x^2\right) = xe(x) + e(x)x = xe(x).$$

Hence by cancellation and Axiom 3.2.18, for some $z \in A^*$

$$x = e(x)u(x) = e(z).$$

Then $x = e(x)$ by Theorem 3.3.8 in contradiction with our initial assumption. Hence $x^2 \neq e\left(x^2\right)$. ■

Proposition 3.7.11 *Let A be an association and let $x, y \in A$. If $x = e(x)$ then $e(x)y = e(xy)$.*

Proof. Suppose $x = e(x)$. Then

$$e(xy) = e(e(x)y) = e(x)y.$$

■

We intend to prove that the imprecision of an element multiplied by its unity is equal to the imprecision. This generalizes in a way the classical result $1.0 = 0$ which is fundamental in rings. In order to do so we need the following lemmas.

Lemma 3.7.12 *Let A be an association and let $x \in A^*$, then*

$$e(u(x))u(x) = e(u(x)).$$

Proof. Using Axiom 3.2.20 and Proposition 3.7.5.1 one derives

$$\begin{aligned} e(u(x))u(x) &= e(u(x))u(x) + u(x)e(u(x)) \\ &= e(u(x)u(x)) \\ &= e\left(u(x)\right). \end{aligned}$$

■

Lemma 3.7.13 *Let A be an association. If $x \in A^*$ then $e(x) = e(x)u(x) + xe(u(x))$.*

Proof. Suppose that $x \neq e(x)$. Then, by Axiom 3.2.8 and Axiom 3.2.20

$$e\left(x\right) = e\left(xu\left(x\right)\right) = e\left(x\right)u\left(x\right) + xe\left(u\left(x\right)\right).$$

■

3.7.4 Neutral element of unity

From now on we assume Axiom 3.2.21. This axiom allows to calculate the imprecision of the unity of an element in terms of the imprecision and the inverse for multiplication of that element. This connects additive and multiplicative functions in a strong way. Indeed, we are now able to show that the generalized unity of an element multiplied by the generalized zero of that element is equal to the generalized zero.

Theorem 3.7.14 *Let A be an association. Let $x \in A^*$, then*

$$e(x) = e(x)u(x).$$

Proof. Suppose that $x \neq e(x)$. Then by Lemma 3.7.13 one has

$$e(x) = e(x)u(x) + xe(u(x)).$$

Then by Axiom 3.2.21 one has

$$e(x) = e(x)u(x) + xe(x)x^{-1} = e(x)u(x) + e(x)u(x).$$

Hence

$$e(x) = e(x)u(x).$$

■

Corollary 3.7.15 *Let A be an association. Let $x \in A^*$, then*

$$xe\left(u\left(x\right)\right) = e\left(x\right).$$

Proof. By Axiom 3.2.21 and Theorem 3.7.14

$$xe(u(x)) = xe(x)x^{-1} = e(x)u(x) = e\left(x\right).$$

■

In the following we explore the connection between magnitudes and division.

Lemma 3.7.16 *Let A be an association and let $x \in A^*$. Then*

$$e(x)x^{-1} = e(x)x^{-1} + xe(x^{-1}).$$

Proof. Using Axiom 3.2.20 one has

$$e(u(x)) = e(xx^{-1}) = e(x)x^{-1} + xe(x^{-1}).$$

Hence $e(x)x^{-1} = e(x)x^{-1} + xe(x^{-1})$, by Axiom 3.2.21. ■

Proposition 3.7.17 *Let A be an association and let $x \in A^*$. Then*

$$e(u(x))x^{-1} = e(x^{-1})u(x).$$

Proof. Observe that by Lemma 3.7.4.2 $x \neq e(x)$ implies $x^{-1} \neq e(x^{-1})$. Putting $y = x^{-1}$ and using Proposition 3.7.5.5 one has

$$e(x^{-1})x = e(x^{-1})x + x^{-1}e(x). \tag{3.13}$$

By formula (3.13) and Axiom 3.2.20 one has

$$\begin{aligned} e(x^{-1})u(x) &= e(x^{-1})xx^{-1} = \left(e(x^{-1})x + x^{-1}e(x)\right)x^{-1} \\ &= e(xx^{-1})x^{-1} = e(u(x))x^{-1}. \end{aligned} \tag{3.14}$$

■

In the next proposition we show a relation between the magnitude of the inverse for multiplication of an element and the magnitude of the element.

Proposition 3.7.18 *Let A be an association and let $x \in A^*$. Then*

$$e(x^{-1}) = e(x)x^{-1}x^{-1}.$$

Proof. Suppose that $x \neq e(x)$. Then using Axiom 3.2.20

$$e(x^{-1}) = e(x^{-1}u(x)) = e(x^{-1})u(x) + x^{-1}e(u(x)). \tag{3.15}$$

Combining (3.15) and (3.14), one has

$$e(x^{-1}) = e(u(x))x^{-1} + e(u(x))x^{-1} = e(u(x))x^{-1}.$$

Hence $e(x^{-1}) = e(x)x^{-1}x^{-1}$, by Axiom 3.2.21. ■

We show that Theorem 3.7.14, Axiom 3.2.21 and Proposition 3.7.18 are equivalent.

Theorem 3.7.19 *Let A be an association. Suppose $x \in A^*$. Then the following are equivalent*

1. $e(u(x)) = e(x)x^{-1}$.
2. $e(x)u(x) = xe(u(x)) = e(x)$.

3. $e(x^{-1}) = e(x)x^{-1}x^{-1}$.

Proof. By Proposition 3.7.18 one has (1)⇒(3).

(2)⇒(1). If $e(x)u(x) = xe(u(x))$, then

$$e(x)u(x)x^{-1} = xe(u(x))x^{-1}.$$

Hence by Proposition 3.7.5.9

$$e(x)x^{-1} = e(u(x))u(x)$$

and by Lemma 3.7.12,

$$e(x)x^{-1} = e(u(x)).$$

(3)⇒(2). If $e(x^{-1}) = e(x)x^{-1}x^{-1}$, by Axiom 3.2.20 one has

$$\begin{aligned} xe(u(x)) &= xe(xx^{-1}) = x\left(xe(x^{-1}) + e(x)x^{-1}\right) \\ &= x\left(xe(x)x^{-1}x^{-1} + e(x)x^{-1}\right) \\ &= x\left(e(x)u(x)x^{-1} + e(x)x^{-1}\right) \\ &= x\left(e(x)x^{-1} + e(x)x^{-1}\right). \end{aligned}$$

Hence $e(x)u(x) = xe(u(x)) = e(x)$, by Corollary 3.7.15. ■

The product of an element by the magnitude of its inverse for multiplication is equal to the product of the magnitude by the inverse for multiplication.

Proposition 3.7.20 *Let A be an association and let $x \in A^*$, then*

$$e(x^{-1})x = e(x)x^{-1} = e(u(x)).$$

Proof. Suppose that $x \neq e(x)$. Then by Axiom 3.2.21 and Theorem 3.7.19 one has

$$e(x^{-1})x = e(x)x^{-1}x^{-1}x = e(x)x^{-1}u(x) = e(x)x^{-1} = e(u(x)).$$

■

In some rings (integral domains) the product of two nonzero elements is always nonzero. Associations are a sort of (nondistributive) integral domains in the sense that they have no generalized zero divisors.

Theorem 3.7.21 *Let A be an association and let $x, y \in A$. Then $xy = e(xy)$ if and only if $x = e(x)$ or $y = e(y)$.*

Proof. Suppose firstly that $x = e(x)$ or $y = e(y)$.

If $x = e(x)$, then $xy = e(x)y$ and by Axiom 3.2.18 there is z such that $xy = e(z)$. By Theorem 3.3.8 we conclude that $xy = e(xy)$.

If $y = e(y)$ the proof is analogous. Hence if $x = e(x)$ or $y = e(y)$, then $xy = e(xy)$.

Suppose secondly, towards a contradiction, that $x \neq e(x)$ and $y \neq e(y)$ and $xy = e(xy)$. We may assume, without loss of generality that $u(x)u(y) = u(x)$. Then

$$\begin{aligned} u(x) &= u(x)u(y) = xx^{-1}yy^{-1} \\ &= xyx^{-1}y^{-1} = e(xy)x^{-1}y^{-1}. \end{aligned}$$

By Axiom 3.2.18 there is z such that

$$e(z) = e(xy)x^{-1}y^{-1}.$$

Then by Proposition 3.3.7

$$e(xy)x^{-1}y^{-1} = e\left(e(xy)x^{-1}y^{-1}\right) = e(u(x))$$

and one concludes that $u(x) = e(u(x))$ in contradiction with Proposition 3.7.4.1. Hence if $xy = e(xy)$, then $x = e(x)$ or $y = e(y)$. ■

3.8 Ordered associations

In this section we consider associations equipped with a total order relation. We observe that an association is totally ordered and compatible with the operations if there is an order relation "$\leq$" satisfying Axioms 3.2.11-3.2.17. The compatibility with multiplication is ensured by Axiom 3.2.17.

Results in this section are in analogy with results of ordered rings. We prove that if x and y are both positive then their product is also positive. This implies that all squares (i.e. the multiplication of an element by itself) are positive. Like what happens in ordered rings, the order relation is always preserved under addition and is preserved under multiplication (in general) only when multiplying by positive elements, even by multiplication by magnitudes.

3.8.1 Preservation properties

Let R be an ordered ring and let $x, y, z, w \in R$ such that $y, z \geq 0$. It is well-known that if $x \leq y$ and $z \leq w$ then $xz \leq yw$. In ordered associations this property remains valid.

Proposition 3.8.1 *Let A be an ordered association and let $x, y, z \in A$ such that y and z are both positive. If $x \leq y$ and $z \leq w$ then $xz \leq yw$.*

Proof. Suppose that $x \leq y$ and $z \leq w$. Then by compatibility with multiplication $xz \leq yz$ and $yz \leq yw$. Hence $xz \leq yw$ by transitivity. ■

The order relation is preserved under scaling.

Proposition 3.8.2 *Let A be an ordered association and let $x, y, z \in A$. If $e(y) \leq e(z)$ then $xe(y) \leq xe(z)$.*

Proof. If $e(x) \leq x$, by Axiom 3.2.17 $xe(y) \leq xe(z)$.

If $x < e(x)$, by Proposition 3.5.3.2 $e(-x) = e(x) < -x$. Hence $-xe(y) \leq -xe(z)$ by compatibility with multiplication. By Proposition 3.7.7 $xe(y) \leq xe(z)$. We conclude that for all x, $xe(y) \leq xe(z)$. ■

Lemma 3.8.3 *Let A be an ordered association and let $x, y \in A$. Then*

$$e(x)e(y) \leq xe(y).$$

Proof. If $e(x) \leq x$, the result follows by compatibility with multiplication because $e(y) \leq e(y)$.

If $x < e(x)$, by Proposition 3.5.3.2, $e(x) < -x$. Then, by Proposition 3.7.7 and Axiom 3.2.17

$$e(x)e(y) \leq -xe(y) = xe(y).$$

■

The magnitude of the product of two elements is larger than the product of the magnitudes of its elements.

Proposition 3.8.4 *Let A be an ordered association and let $x, y \in A$. Then*

$$e(x)e(y) \leq e(xy).$$

Proof. By Lemma 3.8.3, $e(x)e(y) \leq xe(y)$ and $e(x)e(y) \leq ye(x)$. Hence by Proposition 3.5.2.1,

$$e(x)e(y) = e(x)e(y) + e(x)e(y) \leq xe(y) + ye(x) = e(xy).$$

■

In ordered rings the product of two positive elements is also positive. This implies that all squares (i.e. an element multiplied by itself) are positive (larger than or equal to zero). We show that remains true within ordered associations and conclude that all squares are positive (larger than or equal to its magnitude).

Theorem 3.8.5 *Let A be an ordered association and let $x, y \in A$. If x and y are both positive then $e(x)e(y) \leq e(xy) \leq xy$.*

Proof. Suppose that x and y are both positive. By Axiom 3.2.17, $xe(y) \leq xy$ and $ye(x) \leq xy$. Then by Proposition 3.5.2.1 and Axiom 3.2.20,

$$e(xy) = xe(y) + ye(x) \leq xy + xy.$$

By adding $-(xy)$ to both sides of the equation one has

$$-xy \leq xy.$$

Then by Proposition 3.5.4,

$$e(xy) \leq xy.$$

Hence

$$e(x)e(y) \leq xy,$$

by Proposition 3.8.4 and transitivity. ■

Corollary 3.8.6 *Let A be an ordered association and let $x \in A$. Then $e(x^2) \leq x^2$. Moreover, equality holds if and only if $x = e(x)$.*

Proof. We show firstly that $e(x)^2 \leq x^2$. If x is positive, by Theorem 3.8.5, $e(x)^2 \leq x^2$. If x is negative, by Proposition 3.5.2.2

$$e(-x) = e(x) \leq -x.$$

Then

$$e(x)e(x) = e(-x)e(-x) \leq -x(-x) = x^2,$$

by Theorem 3.8.5 and Proposition 3.7.6.2. Hence $e(x)^2 \leq x^2$, by Axiom 3.2.14. By compatibility with addition and Proposition 3.8.4

$$e(x)^2 + e(x^2) \leq x^2 + e(x)^2 = x^2.$$

Hence $e(x^2) \leq x^2$, by Theorem 3.5.5.

If $x = e(x)$ then

$$x^2 = x^2 + e\left(x^2\right) = e\left(x^2\right) + e(x)^2 = e\left(x^2\right),$$

by Proposition 3.8.4.

If $x \neq e(x)$, by Proposition 3.7.10 one has $x^2 \neq e\left(x^2\right)$. Hence equality holds if and only if $x = e(x)$. ■

3.8.2 Order and division

Let R be an ordered ring and let $x \in R$. It is well-known that if $x > 0$ then $x^{-1} > 0$. An adapted version of this result is valid within ordered associations. Indeed, we show that the inverse for multiplication of a positive element is also positive.

Proposition 3.8.7 *Let A be an association and let $x \in A$. If $e(x) < x$ then $e\left(x^{-1}\right) < x^{-1}$.*

Proof. Suppose that $e(x) < x$. By compatibility with multiplication and Theorem 3.8.5

$$xe\left(x^{-1}x^{-1}\right) \leq xx^{-1}x^{-1} = u(x)\,x^{-1} = x^{-1}.$$

Then by Axiom 3.2.20

$$\begin{aligned} u(x)\,e\left(x^{-1}\right) &= xe\left(x^{-1}\right)x^{-1} \\ &= x\left(e\left(x^{-1}\right)x^{-1} + x^{-1}e\left(x^{-1}\right)\right) \\ &= xe\left(x^{-1}x^{-1}\right) \leq x^{-1}. \end{aligned}$$

Hence $e\left(x^{-1}\right) \leq x^{-1}$, by Theorem 3.7.14 because $u(x) = u\left(x^{-1}\right)$. Because $x \neq e(x)$, by Lemma 3.7.4.2 one has $x^{-1} \neq e\left(x^{-1}\right)$. Hence $e\left(x^{-1}\right) < x^{-1}$. ■

The unity function is always (strictly) positive.

Proposition 3.8.8 *Let A be an ordered association and let $x \in A^*$. Then*

$$e\left(u\left(x\right)\right) < u\left(x\right).$$

Proof. By Lemma 3.7.4.1 one has $u\left(x\right) \neq e\left(u\left(x\right)\right)$. To show that $e\left(u\left(x\right)\right) \leq u\left(x\right)$ we consider the cases: $e\left(x\right) \leq x$ and $x < e\left(x\right)$. On the first case, by compatibility with multiplication and Axiom 3.2.21

$$e\left(u\left(x\right)\right) = e\left(x\right)x^{-1} \leq xx^{-1} = u\left(x\right).$$

On the second case $e\left(x\right) \leq -x$ and $e\left(x^{-1}\right) < x^{-1}$ by Proposition 3.5.3.2 and Proposition 3.8.7. Then, using compatibility with multiplication, Proposition 3.7.7 and Proposition 3.7.20

$$e\left(u\left(x\right)\right) = xe\left(x^{-1}\right) = -xe\left(x^{-1}\right) \leq \left(-x\right)\left(-x^{-1}\right) = xx^{-1} = u\left(x\right).$$

■

Corollary 3.8.9 *Let A be an ordered association and let $x \in A^*$. Then*

1. If $u\left(x\right) \leq x$ then $e\left(x\right) \leq x$.
2. $e\left(x\right) < u\left(x\right)$.

Proof. 1. Put $y = u\left(x\right)$ in Proposition 3.5.8.3, because $u\left(x\right)$ is positive by Proposition 3.8.8.

2. Put $x = u\left(x\right)$ in Part 1. Then $e\left(x\right) \leq u\left(x\right)$. Hence $e\left(x\right) < u\left(x\right)$, by Lemma 3.7.4.4. ■

It is well-known that in ordered rings when multiplying by a negative element inequalities are not preserved. Indeed, let R be an ordered ring and let $x, y, z \in R$. If $x < 0$ and $y \leq z$ then $xy \geq xz$. In ordered associations this property is somewhat modified in order to take into account the magnitudes of the products of the elements.

Proposition 3.8.10 *Let A be an ordered association and let $x, y, z \in A$. Suppose that $y \leq z$. If $x \leq e\left(x\right)$ then $xz + e(xy) \leq xy + e(xz)$. Moreover, if $x = e\left(x\right)$ then $xy \leq xz$.*

Proof. Suppose that $y \leq z$. If $x \leq e\left(x\right)$, then $e(-x) = e\left(x\right) \leq -x$. By compatibility with multiplication

$$\left(-x\right)y \leq \left(-x\right)z.$$

Then, by Axiom 3.2.19,

$$-\left(xy\right) \leq -\left(xz\right).$$

and by (3.5) one concludes that

$$xz + e(xy) \leq xy + e(xz).$$

If $x = e\left(x\right)$, by compatibility with multiplication $e\left(x\right)y \leq e\left(x\right)z$. Hence $xy \leq xz$. ■

Let R be an ordered ring and let $x \in R$. One has $x \geq 1$ if and only if $x^{-1} \leq 1$. In ordered associations an adapted form of this property remains valid.

Proposition 3.8.11 *Let A be an ordered association and let $x \in A^*$. Then*

1. *If $u(x) \leq x$ then $x^{-1} \leq u(x)$.*

2. *If $e(x) < x$ and $x \leq u(x)$ then $u(x) \leq x^{-1}$.*

Proof. 1. Suppose that $u(x) \leq x$. Then by Corollary 3.8.9.1 one has $e(x) \leq x$ and then $e(x^{-1}) \leq x^{-1}$, by Proposition 3.8.7. Hence

$$x^{-1} = x^{-1}u(x) \leq xx^{-1} = u(x),$$

by Axiom 3.2.17.

2. Suppose that $e(x) < x$ and $x \leq u(x)$. Then $e(x^{-1}) \leq x^{-1}$, by Proposition 3.8.7. Hence

$$u(x) = xx^{-1} \leq u(x)x^{-1} = x^{-1},$$

by Axiom 3.2.17. ■

Proposition 3.8.12 *Let A be an ordered association and let $x \in A^*$. If $e\left(x^{-1}\right) \leq x^{-1} \leq x$ then $e\left(x^{-1}\right) \leq e(x)$.*

Proof. Suppose $e\left(x^{-1}\right) \leq x^{-1} \leq x$. Using Axiom 3.2.17

$$x^{-1}x^{-1} \leq xx^{-1} = u(x).$$

By Proposition 3.7.18, Axiom 3.2.17 and Theorem 3.7.14 one has

$$e\left(x^{-1}\right) = x^{-1}x^{-1}e(x) \leq u(x)e(x) = e(x).$$

■

3.8.3 Absolute value of product

We show that the product of a magnitude by an element is equal to the product of the magnitude by the absolute value of the element. We also prove that the absolute value has the usual properties for multiplication and division.

Proposition 3.8.13 *Let A be an ordered association and let $x, y \in A$. Then $e(x)|y| = e(x)y$.*

Proof. If $e(y) \leq y$ then $|y| = y$. Hence $e(x)|y| = e(x)y$.

If $y < e(y)$ then $|y| = -y$. By Proposition 3.7.6.1 and Proposition 3.3.14

$$e(x)|y| = e(x)(-y) = -(e(x))y = e(x)y.$$

■

Because the order relation is compatible with multiplication, the absolute value is linear for multiplication.

Proposition 3.8.14 *Let A be an ordered association and let $x, y \in A$. Then*

$$|xy| = |x|\,|y|\,.$$

Proof. If x and y are both positive by compatibility with multiplication $e(xy) \leq xy$. Then

$$|xy| = xy = |x|\,|y|\,.$$

If x and y are both negative then $e(x) \leq -x$ and $e(y) \leq -y$. Then by compatibility with multiplication, $e(xy) \leq (-x)\,(-y) = xy$. Hence

$$|xy| = xy = (-x)\,(-y) = |-x|\,|-y|\,.$$

If x is positive and y is negative, then by Proposition 3.6.7, Axiom 3.2.19 and the definition of absolute value,

$$\begin{aligned} |xy| &= |-(xy)| = |x(-y)| \\ &= x(-y) = |x|\,|-y| = |x|\,|y|\,. \end{aligned}$$

The case where y is positive and x is negative is analogous to the previous case. ■

The absolute value of the quotient of two elements is the quotient of the absolute values of each element.

Proposition 3.8.15 *Let A be an ordered association and let $x, y \in A$. If $y \neq e(y)$ then*

$$\left|xy^{-1}\right| = |x|\,|y|^{-1}\,.$$

Proof. By Proposition 3.8.14 we only need to show that $\left|y^{-1}\right| = |y|^{-1}$.

If $e\,(y) < y$ then $e\left(y^{-1}\right) < y^{-1}$ by Proposition 3.8.7. Hence

$$\left|y^{-1}\right| = y^{-1} = |y|^{-1}\,.$$

If $y < e\,(y)$ then $y^{-1} < e\left(y^{-1}\right)$ by Proposition 3.8.7 and Proposition 3.7.9. Hence

$$\left|y^{-1}\right| = -\left(y^{-1}\right) = (-y)^{-1} = |y|^{-1}\,.$$

■

3.9 Distributivity

In this section we connect addition with multiplication via an adapted distributivity axiom. We recall that usual distributivity holds if given any elements x, y, z one has $x\,(y+z) = xy + xz$. Roughly speaking the adapted distributivity axiom states that there is a price to pay when distributivity is used. The price is that a certain imprecision is added. However this imprecision can be easily calculated and depends only on y, z and the imprecision of x.

Nevertheless, in some elementary cases distributivity can be deduced using only the axioms for ordered associations. We start by giving some examples of these elementary properties.

We then assume the completion axioms. By assuming these axioms we are able to show that the elements m and u given by Axioms 3.2.23 and 3.2.8 are unique and have properties similar to those of 0 and 1 in rings. Then we consider precise elements. Precise elements are such that their magnitude is minimal. We show that the set of precise elements of an ordered association on which the completion axioms are valid is closed under addition, multiplication and inversion. We show that the unity of a precise element is also precise.

We define a structure called solid which is roughly speaking an ordered field with individualized neutral and unity elements. This means that results in this section are in analogy with results valid in fields. So, a solid is a structure satisfying Axioms 3.2.1-3.2.22. By assuming the distributivity axiom we are able to prove that subdistributivity always holds and that the set of precise elements of a solid is a field. This means in particular that distributivity always holds for precise elements. We then give a complete characterization of the distributive law (3.18). We also show that in the case where y and z are both positive or both negative and in the case where $y = e\left(y\right)$ or $z = e\left(z\right)$ distributivity holds.

We finish by showing that the distributivity axiom and the characterization of distributivity (3.18) are indeed equivalent.

3.9.1 Elementary cases

As mentioned above we start by giving some elementary distributive properties without recurring to the distributivity axiom.

Proposition 3.9.1 *Let A be an association and let $x, y \in A$. Then*

1. $e(x)y = e(x)y + e(x)e(y)$.
2. $e(x)(y + e(y)) = e(x)y + e(x)e(y)$.
3. *If $x \in A^*$, then* $x(u(x) + e(x)x^{-1}) = xu(x) + xe(x)x^{-1}$.

Proof. 1. By Axiom 3.2.3

$$e(x)y = e(x)y + e\left(e\left(x\right)y\right).$$

Then by Axiom 3.2.20 and Proposition 3.3.7

$$\begin{aligned} e(x)y &= e\left(x\right)y + e\left(e\left(x\right)\right)y + e\left(x\right)e\left(y\right) \\ &= e\left(x\right)y + e\left(x\right)y + e\left(x\right)e\left(y\right) \end{aligned}$$

and by Proposition 3.3.6.1

$$e(x)y = e\left(x\right)y + e\left(x\right)e\left(y\right).$$

2. By Part 1 one has

$$e(x)(y+e(y)) = e(x)y = e(x)y + e(x)e(y).$$

3. Suppose that $x \neq e(x)$. On one hand, using Axiom 3.2.21,

$$\begin{aligned} x(u(x)+e(x)x^{-1}) &= x(u(x)+e(u(x))) \\ &= xu(x) \\ &= x. \end{aligned}$$

On the other hand, by Proposition 3.7.14,

$$\begin{aligned} xu(x)+xe(x)x^{-1} &= x+e(x)u(x) \\ &= x+e(x) \\ &= x. \end{aligned}$$

Hence

$$x(u(x)+e(x)x^{-1}) = xu(x)+xe(x)x^{-1}.$$

■

Distributivity holds in the case where y and z are both magnitudes.

Proposition 3.9.2 *Let A be an ordered association and let $x, y, z \in A$. Then*

$$x\,(e(y)+e(z)) = xe(y)+xe(z).$$

Proof. We may suppose without loss of generality that $e(z) \leq e(y)$. Then by Theorem 3.5.5

$$e(y)+e(z) = e(y). \tag{3.16}$$

We prove firstly that $e(y)x + e(z)x = e(y)x$.

By Proposition 3.8.2 one has $e(z)x \leq e(y)x$. Then by Axiom 3.2.18 and Theorem 3.5.5

$$e(y)x+e(z)x = e(y)x. \tag{3.17}$$

Hence by (3.16) and (3.17)

$$x\,(e(y)+e(z)) = xe(y) = xe(y)+xe(z).$$

■

An interesting special case of the previous proposition is that distributivity holds if all the elements are magnitudes.

Corollary 3.9.3 *Let A be an ordered association and let $x, y, z \in A$. Then*

$$e(x)\,(e(y)+e(z)) = e(x)e(y)+e(x)e(z).$$

Proposition 3.9.4 *Let A be an ordered association and let $x, y, z \in A$. If $e(z) \leq e(y)$ then $x(y+e(z)) = xy+xe(z)$.*

Proof. Suppose that $e(z) \leq e(y)$. Then by Theorem 3.5.5

$$\begin{aligned} x(y+e(z)) &= x(y+e(y)+e(z)) \\ &= x(y+e(y)) = xy. \end{aligned}$$

Now $xe\,(z) \leq xe\,(y)$ by Proposition 3.8.2 and $xe\,(y)+xe\,(z) = xe\,(y)$ by Theorem 3.5.5. Then using Axiom 3.2.20

$$\begin{aligned} xy + xe(z) &= xy + e(xy) + xe(z) \\ &= xy + e(x)y + xe(y) + xe(z) \\ &= xy + e(x)y + xe(y) \\ &= xy + e(xy) = xy. \end{aligned}$$

Hence $x(y+e(z)) = xy + xe(z)$. ■

Corollary 3.9.5 *Let A be an ordered association and let $x, y, z \in A$. Then*

$$xy = x(y+e(y)) = xy + xe(y).$$

3.9.2 Completion Axioms

In this section we assume that A is an ordered association on which the completion axioms are also valid. We prove unicity of the elements m and u given by Axioms 3.2.23 and 3.2.8 and call them zero and one respectively. Then we prove that zero has the expected properties concerning the order relation and multiplication.

Proposition 3.9.6 *Let $x \in A$. Then there is exactly one m such that $x+m = x$ and exactly one u such that $xu = x$.*

Proof. Suppose that there are m and m' such that, for all x, $x+m = x$ and $x+m' = x$. Then $m+m' = m$ and $m'+m = m'$. Hence $m = m'$.

Suppose now that there are u and u' such that, for all x, $xu = x$ and $xu' = x$. Then $u.u' = u$ and $u'u = u'$. Hence $u = u'$. ■

Definition 3.9.7 *We call* zero *the unique element m such that, for all x, $x + m = x$ and it will be denoted by* 0 *and* one *the unique element u such that, for all x, $xu = x$ and it will be denoted by* 1.

In the following proposition we show some elementary properties of the elements 0 and 1.

Proposition 3.9.8 *Let $x \in A$. Then*

1. $0 < 1$.
2. $0 \leq e\,(x)$.

3. $e(0) = 0$.
4. $u(1) = 1$.
5. If $0 \leq x$ and $0 \leq y$, then $0 \leq x + y$.
6. If $x \leq 0$ and $y \leq 0$, then $x + y \leq 0$.
7. $-0 = 0$.
8. If $y \leq z$ and $0 \leq x$, then $yx \leq zx$.
9. If $0 \leq x$ then $e(x) \leq x$.

Proof. 1. This is an immediate consequence of Corollary 3.8.9.2.

2. By Axiom 3.2.23,

$$e(x) + 0 = e(x).$$

By Axiom 3.2.16 one has $0 \leq e(x)$.

3. By Axiom 3.2.23 and Proposition 3.3.6.2

$$e(x) = e(x + 0) = e(x) + e(0).$$

Then

$$x + e(0) = x + e(x) + e(0) = x + e(x) = x.$$

Hence

$$e(0) = 0,$$

by Proposition 3.9.6.

4. By Axiom 3.2.25 and Proposition 3.7.5.2

$$u(x) = u(x.1) = u(x)\, u(1).$$

Then

$$xu(1) = xu(x)\, u(1) = xu(x) = x.$$

Hence

$$u(1) = 1,$$

by Proposition 3.9.6.

5. Suppose that $0 \leq x$ and $0 \leq y$. Then by Proposition 3.5.8.1

$$0 = 0 + 0 \leq x + y.$$

6. Suppose that $x \leq 0$ and $y \leq 0$. Then by Proposition 3.5.8.1

$$x + y \leq 0 + 0 = 0.$$

7. Using Part 3 and Proposition 3.3.14

$$-0 = -e(0) = e(0) = 0.$$

8. Directly from Axiom 3.2.17 and Part 3.

9. Suppose that $0 \leq x$. Then $e(x) = 0 + e(x) \leq x + e(x) = x$ by compatibility with addition. ■

We prove that zero is the absorbing element for multiplication.

Proposition 3.9.9 *Let $x \in A$. Then*

$$x0 = 0.$$

Proof. By Proposition 3.9.8.3 and Proposition 3.9.4, for all x

$$x(1+0) = x1 + x0$$

Hence

$$x = x + x0$$

for all x. Then, $x0 = 0$, for all x, by Proposition 3.9.6. ∎

3.9.3 Precise elements

In this section we suppose that A is an ordered association such that the completion axioms are also valid.

Definition 3.9.10 *If $x \in A$ is such that $e(x) = 0$ we call x* precise.

By Proposition 3.9.8.3, the element 0 is precise. We show that the element 1 is also precise.

Proposition 3.9.11 *In A one has*

$$e(1) = 0.$$

Proof. Let $x \in A$. One has

$$x(1 + e(1)) = x1 = x.$$

By Proposition 3.9.4

$$x(1 + e(1)) = x1 + xe(1) = x + xe(1).$$

Hence $x = x + xe(1)$. Then by Proposition 3.9.6 one has $xe(1) = 0$. Putting $x = 1$, one obtains $1e(1) = 0$. Hence $e(1) = 0$, by Axiom 3.2.25. ∎

Precise numbers are closed under addition, multiplication and inversion. The unity of a precise element is also precise.

Proposition 3.9.12 *Let $a, b \in A$ be precise. Then $a + b$, ab are precise. If $a \neq 0$ then $u(a)$ and a^{-1} are also precise.*

Proof. Since a and b are precise, $e(a) = e(b) = 0$. Then,

$$\begin{aligned} e(a+b) &= e(a) + e(b) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Hence $a+b$ is precise. By Axiom 3.2.20 and Proposition 3.9.9

$$\begin{aligned} e(ab) &= e(a)b + ae(b) \\ &= 0b + a0 \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Hence ab is precise. Suppose that $a \neq 0$. Then by Axiom 3.2.21 and Proposition 3.9.9

$$e(u(a)) = e(a)a^{-1} = 0a^{-1} = 0.$$

Hence $u(a)$ is precise. By Proposition 3.7.18 and Proposition 3.9.9

$$e\left(a^{-1}\right) = e(a)a^{-1}a^{-1} = 0a^{-1}a^{-1} = 0.$$

Hence a^{-1} is precise. ■

Proposition 3.9.13 *If $a \in A^*$ is precise, then $u(a)$ is precise.*

Proof. By Axiom 3.2.21, Lemma 3.7.4.2 and Proposition 3.9.9

$$e(u(a)) = e(a)a^{-1} = 0a^{-1} = 0.$$

Then $u(a)$ is precise. ■

3.9.4 Solids

In this section we will assume all the axioms presented in Section 3.2. This yields a structure called solid. A solid is then a sort of semi-distributive ordered field with individualized neutral elements for both addition and multiplication. For this matter the results on this section will be related to field properties. In solids it is possible to completely characterize the distributivity property in a way similar to Theorem 2.4.6. So, the main result in this section is a criterion for distributivity given in Theorem 3.9.19. We show that distributivity holds when multiplying by a precise element and that the set of precise elements of a solid is a field. We prove the subdistributivity of multiplication over addition. We also prove that any unity multiplied by any magnitude is equal to the magnitude. We finish by showing that the criterion for distributivity given by formula (3.18) of Theorem 3.9.19 is equivalent to Axiom 3.2.22.

Definition 3.9.14 *A structure $(A, +, e, s, \cdot, u, d)$ is called a* solid *if A satisfies axioms 3.2.1-3.2.27.*

Definition 3.9.15 *A zeroless precise element of a solid is called* non-zero.

Distributivity always holds when multiplying by a precise number.

Theorem 3.9.16 *Let A be a solid. Let $x, y, a \in A$ such that a is precise. Then*

$$a(x+y) = ax + ay.$$

Proof. By Proposition 3.9.9

$$e(a)\,x + e(a)\,y = 0 + 0 = 0.$$

Then, by Axiom 3.2.22,

$$ax + ay = a(x+y) + e(a)\,x + e(a)\,y = a(x+y).$$

■

Theorem 3.9.17 *The set of precise elements of a solid A is a field.*

Proof. The theorem follows from Theorem 3.9.16 and Proposition 3.9.12. ■

Theorem 3.9.18 (Subdistributivity) *Let A be a solid and let $x, y, z \in A$. Then*

$$x(y+z) \leq xy + xz.$$

Proof. Using Proposition 3.5.9 one has

$$x(y+z) \leq x(y+z) + e(x)\,y + e(x)\,z = xy + xz.$$

■

The main goal of this section is to show the following theorem.

Theorem 3.9.19 (Distributivity criterion) *Let A be a solid and let $x, y, z \in A$ be zeroless. Then*

$$xy + xz = x(y+z) \Leftrightarrow e(x)(y+z) = e(x)\,y + e(x)\,z \vee R(x) \leq R(y) + R(z). \tag{3.18}$$

Definition 3.9.20 *The* relative uncertainty of x, *noted $R(x)$ is defined as follows: if $x \neq e(x)$, then $R(x) = e(u(x))$. If $x = e(x)$, then $R(x) = M$, where M is given by Axiom 3.2.24.*

Lemma 3.9.21 *Let A be a solid and let $x, y \in A$ such that $x \neq e(x)$ and $y \neq e(y)$. If $R(x) \leq R(y)$ then $e(x)\,y \leq e(y)\,x$.*

Proof. Suppose that $R(x) \leq R(y)$. Then

$$e(x)\,x^{-1} = e(u(x)) \leq e(u(y)) = e(y)\,y^{-1}.$$

By Proposition 3.8.2

$$e(x)\,u(x)\,y \leq e(y)\,y^{-1}xy = e(y)\,u(y)\,x.$$

Then $e(x)\,y \leq e(y)\,x$, by Theorem 3.7.14. ■

Lemma 3.9.22 *Let A be a solid and let $x, y, z \in A$. If $e(x)(y+z) \neq e(x)\,y + e(x)\,z$, then $e(x)\,y = e(x)\,z$.*

Proof. Suppose $e(x)(y+z) \neq e(x)y + e(x)z$. Then $e(x)(y+z) < e(x)y + e(x)z$, by Theorem 3.9.18. We may assume, without loss of generality that $|y| \leq |z|$. Then by Proposition 3.8.13 and compatibility with multiplication

$$e(x)y = e(x)|y| \leq e(x)|z| = e(x)z.$$

Hence

$$e(x)y \leq e(x)z, \tag{3.19}$$

so $e(x)y + e(x)z = e(x)z$, by Theorem 3.5.5 and we conclude that

$$e(x)(y+z) < e(x)z. \tag{3.20}$$

In order to show that $e(x)z \leq e(x)y$ we prove firstly that $y < e(y)$. Suppose towards a contradiction that $e(y) \leq y$. Then by compatibility with addition and multiplication

$$e(x)z \leq e(x)(z + e(y)) \leq e(x)(z+y).$$

In contradiction with (3.20). Hence $|y| = -y$.

Secondly we show that $|y| > e(z)$. Suppose towards a contradiction that $|y| \leq e(z)$. Then

$$|y| + e(z) = -y + e(z) \leq e(z) + e(z) = e(z).$$

On the other hand

$$|y| + e(z) = -y + e(z) \geq e(y) + e(z) \geq e(z),$$

because $y < e(y)$. Hence $|y| + e(z) = e(z)$. But then

$$\begin{aligned} e(x)(y+z) &= e(x)(-y-z) = e(x)(|y| + e(z) - z) \\ &= e(x)(e(z) - z) = e(x)(-z) = e(x)z, \end{aligned}$$

in contradiction with (3.20). Hence $|y| > e(z)$.

Thirdly we show that $y + y < -z$. Suppose towards a contradiction that $-z \leq y + y$. Using Theorem 3.9.18 and (3.20) one has

$$\begin{aligned} e(x)z &= e(x)(z+z-z) \leq e(x)(z+z+y+y) \\ &\leq e(x)(y+z) + e(x)(y+z) = e(x)(y+z) < e(x)z, \end{aligned}$$

which is a contradiction. Hence $y + y < -z$. Then by (3.5)

$$z + e(y) \leq -y - y + e(z).$$

Then using Proposition 3.5.9 and Theorem 3.9.18

$$\begin{aligned} e(x)z &\leq e(x)(z + e(y)) \\ &\leq e(x)(-y - y + e(z)) \\ &\leq e(x)(-y-y-y) \leq e(x)y, \end{aligned}$$

because $y + y < -z$ and $-y > e(z)$. Hence

$$e(x)\,z \le e(x)\,y. \tag{3.21}$$

Combining (3.19) and (3.21) one derives that $e(x)\,z = e(x)\,y$. ■

The following lemma is the converse implication of Theorem 3.9.19.

Lemma 3.9.23 *Let A be a solid and let $x, y, z \in A$ be zeroless. If $e(x)(y+z) = e(x)\,y + e(x)\,z \vee R(x) \le R(y) + R(z)$, then $xy + xz = x(y+z)$.*

Proof. We start by showing that if $e(x)(y+z) = e(x)\,y + e(x)\,z$, distributivity holds. Using Axiom 3.2.20, one has

$$\begin{aligned}
xy + xz &= x(y+z) + e(x)\,y + e(x)\,z \\
&= x(y+z) + e(x(y+z)) + e(x)(y+z) \\
&= x(y+z) + xe(y+z) + e(x)(y+z) + e(x)(y+z) \\
&= x(y+z) + xe(y+z) + e(x)(y+z) \\
&= x(y+z) + e(x(y+z)) \\
&= x(y+z).
\end{aligned}$$

Suppose now that $R(x) \le R(y) + R(z)$. We may suppose that $e(x)(y+z) \ne e(x)\,y + e(x)\,z$. Then $e(x)\,y = e(x)\,z$ by Lemma 3.9.22 and $e(x)\,y \le e(y)\,x$ by Lemma 3.9.21. Hence

$$\begin{aligned}
xy + xz &= x(y+z) + e(x)\,y + e(x)\,z \\
&= x(y+z) + e(x)(y+z) + xe(y) + xe(z) + e(x)\,y \\
&= x(y+z) + e(x)(y+z) + xe(y) + xe(z) \\
&= x(y+z) + e(x(y+z)) \\
&= x(y+z).
\end{aligned}$$

■

Remark 3.9.24 *By Axiom 3.2.26 there is a precise element a such that $x = a + e(x)$ and there is a precise element b such that $u(x) - 1 = b + e(u(x) - 1) = b + e(u(x))$. So one can write $u(x) = 1 + b + e(u(x))$.*

Proposition 3.9.25 *Let A be a solid and let $a \in A$ be a non-zero element. Then*

$$u(a) = 1.$$

Proof. Suppose that a is a precise element such that $a \ne 0$. According to Remark 3.9.24 one may write $u(a) = 1 + b + e(u(a))$. By Proposition 3.9.13 one has $e(u(a)) = 0$. Then $u(a) = 1 + b$. So

$$a = au(a) = a(1+b).$$

Then by Theorem 3.9.16

$$a = a + ab.$$

By cancellation

$$0 = e(a) = e(a) + ab = ab.$$

Then $b = 0$, by Theorem 3.7.21. Hence

$$u(a) = 1.$$

■

By Axiom 3.2.26 we can decompose any element as the sum of its magnitude with a precise element. We show that if an element is zeroless then the absolute value of the precise part of the decomposition must be larger than the magnitude. We also show that, for zeroless elements, dividing the magnitude by the precise part is the same as dividing the magnitude by the element.

Proposition 3.9.26 *Let A be a solid. Let $x = a + e(x) \in A$ be zeroless and such that $e(a) = 0$. Then*

1. $e(x) < |a|$.
2. $e(x)\, a^{-1} = e(x)\, x^{-1} = e(u(x))$.

Proof. 1. Suppose firstly that $e(x) < x$. Suppose towards a contradiction that $a \leq e(x)$. Then by compatibility with addition

$$x = a + e(x) \leq e(x) + e(x) = e(x),$$

which is a contradiction. Then $0 < a$, by Proposition 3.5.8.3. Hence

$$e(x) < a = |a|.$$

Suppose secondly that $x < e(x)$. Suppose towards a contradiction that $e(x) \leq a$. Then by compatibility with addition

$$e(x) = e(x) + e(x) \leq a + e(x) = x,$$

which is a contradiction. Then $0 < -a$, by Proposition 3.5.8.3. Hence

$$e(x) = -e(x) < -a = |a|.$$

2. By Part 1 and Proposition 3.9.8.2, one has $0 \leq e(x) < |a|$, so $a \neq 0$. Then $a \leq a + e(x) = x$, by Proposition 3.5.9. Then by compatibility with multiplication, Proposition 3.9.25 and Theorem 3.7.14

$$e(u(x)) = e(x)\, x^{-1} \leq e(x)\, u(x)\, a^{-1} = e(x)\, a^{-1}. \tag{3.22}$$

Using Theorem 3.7.14, Theorem 3.9.16 and Proposition 3.9.25 one derives

$$\begin{aligned} e(x)a^{-1} &= e(x)u(x)a^{-1} = e(x)x^{-1}xa^{-1} \\ &= e(x)x^{-1}(a+e(x))a^{-1} \\ &= e(x)x^{-1}\left(1+e(x)a^{-1}\right). \end{aligned}$$

Then by Part 1 and Proposition 3.9.25

$$e(x)a^{-1} \leq e(x)x^{-1}\left(1+aa^{-1}\right) = e(x)x^{-1}(1+1).$$

Hence

$$e(x)a^{-1} \leq e(x)x^{-1} + x^{-1}e(x) = x^{-1}e(x) = e(u(x)), \tag{3.23}$$

by Theorem 3.9.18. Combining (3.22) and (3.23) one concludes the result. ■

One shows that the element b of Remark 3.9.24 has to be "small" in the sense that the absolute value of b is less than or equal to the magnitude of the unity of x.

Lemma 3.9.27 *Let A be a solid. Let $x = a + e(x) \in A$ be zeroless. Suppose $u(x) = 1 + b + e(u(x))$. Then $|b| \leq e(u(x))$.*

Proof. Suppose that $u(x) = 1 + b + e(u(x))$. Because $R(e(x)) = M = R(e(u(x)))$, using Lemma 3.9.23 one derives

$$\begin{aligned} a + e(x) &= x = xu(x) = (a+e(x))(1+b+e(u(x))) \\ &= a(1+b) + e(x)(1+b) + (a+e(x))e(u(x)) \end{aligned}$$

Then by Theorem 3.9.16 and Corollary 3.7.15

$$\begin{aligned} a + e(x) &= a + ab + e(x)(1+b) + xe(u(x)) \\ &= a + ab + e(x)(1+b) + e(x). \end{aligned}$$

Then

$$e(x) = ab + e(x)(1+b) + e(x).$$

We show that $e(x)(1+b) \leq e(x)$. Suppose towards a contradiction that $e(x) < e(x)(1+b)$. Then

$$\begin{aligned} e(e(x)) &= e(x) = e(ab) + e(e(x)) + e(e(x)(1+b)) \\ &= e(x) + e(x)(1+b) = e(x)(1+b), \end{aligned}$$

in contradiction with our initial supposition. Then $e(x)(1+b) \leq e(x)$. Hence

$$e(x) = ab + e(x).$$

Then $|ab| \leq e(x)$, by Proposition 3.6.18. Hence

$$|b| \leq e(x)a^{-1} = e(u(x)),$$

by Proposition 3.9.26.2. ■

A unity can be written as the sum of one with the magnitude of the unity. This gives a sort of expansion of the unity in terms of the minimal unity and the imprecision of the unity.

Proposition 3.9.28 (Expansion) *Let A be a solid. Let $x \in A$ be zeroless. Then*

$$u(x) = 1 + e(u(x)).$$

Proof. By Lemma 3.9.27 one may suppose $u(x) = 1 + b + e(u(x))$, with $|b| \leq e(u(x))$. Then

$$b + e(u(x)) \leq |b| + e(u(x)) \leq e(u(x)) + e(u(x)) = e(u(x)).$$

Moreover,

$$\begin{aligned} e(u(x)) &= -(e(u(x)) + e(u(x))) \\ &\leq -(|b| + e(u(x))) \\ &= -|b| - e(u(x)) \\ &= -|b| + e(u(x)) \leq b + e(u(x)). \end{aligned}$$

Hence

$$b + e(u(x)) = e(u(x))$$

and

$$u(x) = 1 + e(u(x)).$$

■

Lemma 3.9.29 *Let A be a solid and let $x \in A^*$. Then $1 > e(u(x))$.*

Proof. Suppose towards a contradiction that $1 \leq e(u(x))$. By Corollary 3.8.9.2 the element 1 is positive. Then using Proposition 3.9.28 and Theorem 3.5.10

$$u(x) = 1 + e(u(x)) = e(u(x)),$$

in contradiction with Lemma 3.7.4.4. Hence $1 > e(u(x))$. ■

A magnitude multiplied by a unity is equal to this magnitude.

Theorem 3.9.30 *Let A be a solid. Let $x \in A$ and $y \in A^*$. Then*

$$e(x)\, u(y) = e(x).$$

Proof. By Proposition 3.9.28

$$e(x)\, u(y) = e(x)\,(1 + e(u(y))).$$

One has $R(e(x)) = M = R(1) + R(e(u(x)))$. Then

$$e(x)\, u(y) = e(x)\, 1 + e(x)\, e(u(y)) = e(x) + e(x)\, e(u(y)).$$

By Lemma 3.9.29 one has $1 > e(u(y))$. Then by compatibility with multiplication $e(x) \geq e(x)\, e(u(y))$. Hence

$$e(x)\, u(y) = e(x),$$

by Theorem 3.5.5. ■

Corollary 3.9.31 *Let A be a solid. Let $x, y \in A$. Then $R(xy) = R(x) + R(y)$.*

Proof. If $x = e(x)$ or $y = e(y)$ then $R(xy) = M = R(x) + R(y)$.
If $x \neq e(x)$ and $y \neq e(y)$ then by Axiom 3.2.20 and Theorem 3.9.30

$$\begin{aligned} R(xy) &= e(u(xy)) = e(u(x)u(y)) \\ &= e(u(x))u(y) + u(x)e(u(y)) \\ &= e(u(x)) + e(u(y)) \\ &= R(x) + R(y). \end{aligned}$$

■

Lemma 3.9.32 *Let A be a solid and let $x, y \in A$ such that $x \neq e(x)$ and $y \neq e(y)$. If $e(x)y \leq e(y)x$ then $R(x) \leq R(y)$.*

Proof. Suppose that $e(x)y \leq e(y)x$. Then by compatibility with multiplication

$$e(x)u(y)x^{-1} \leq e(y)y^{-1}u(x).$$

By Theorem 3.9.30

$$e(x)x^{-1} \leq e(y)y^{-1}.$$

Hence

$$R(x) = e(u(x)) \leq e(u(y)) = R(y),$$

by Axiom 3.2.21. ■

We are now able to prove Theorem 3.9.19**.**

Proof of Theorem 3.9.19: By Lemma 3.9.23 we only need to show the direct implication.

Proof. Suppose that $xy + xz = x(y+z)$. Then by Axiom 3.2.22

$$x(y+z) = x(y+z) + e(x)y + e(x)z.$$

By cancellation

$$e(x(y+z)) = e(x(y+z)) + e(x)y + e(x)z.$$

Then

$$\begin{aligned} e(x)y + e(x)z &\leq e(x(y+z)) \\ &= e(x)(y+z) + xe(y+z) \\ &= e(x)(y+z) + xe(y) + xe(z). \end{aligned}$$

Hence

$$e(x)y + e(x)z \leq e(x)(y+z) + xe(y) + xe(z). \tag{3.24}$$

We must consider three cases: (i) $e(x)y + e(x)z \leq e(x)(y+z)$, and if (i) does not hold, (ii) $e(x)y + e(x)z \leq xe(y)$ and (iii) $e(x)y + e(x)z \leq xe(z)$.

(i) One has

$$e(x)\,y + e(x)\,z = e(x)\,(y+z)\,,$$

by (3.24) and Theorem 3.9.18.

(ii) One has $e(x)\,y = e(x)\,z$, by Lemma 3.9.22. Then

$$e(x)\,y \leq xe(y)\,.$$

Hence $R(x) \leq R(y) \leq R(y) + R(z)$, by Lemma 3.9.32.

(iii) One has $e(x)\,y = e(x)\,z$, by Lemma 3.9.22. Then

$$e(x)\,z \leq xe(z)\,.$$

Hence $R(x) \leq R(z) \leq R(y) + R(z)$, again by Lemma 3.9.32. ■

It is an immediate consequence of Theorem 3.9.19 that if $R(x) \leq R(y) + R(z)$, then distributivity holds. We show that if either y or z is a magnitude then distributivity holds. This generalizes Proposition 3.9.4 because there are no conditions on the magnitudes.

Theorem 3.9.33 *Let A be a solid and let $x, y,\ z \in A$. If $y = e(y)$ or $z = e(z)$, then distributivity holds.*

Proof. If $y = e(y)$ or $z = e(z)$, then $R(x) \leq R(y) + R(z)$ and hence distributivity holds, by Theorem 3.9.19. ■

The following result gives a characterization of distributivity in the case where $x = e(x)$.

Proposition 3.9.34 *Let A be a solid and let $x, y, z \in A$. If $x = e(x)$, $y \neq e(y)$ and $z \neq e(z)$ then $x(y+z) = xy + xz$ if and only if $e(x)\,(y+z) = e(x)\,y$ or $e(x)\,(y+z) = e(x)\,z$.*

Proof. Let A be a solid and let x, y, z be such that $x = e(x)$, $y \neq e(y)$ and $z \neq e(z)$.

Suppose firstly that $x(y+z) = xy + xz$. Then

$$e(x)\,(y+z) = e(x)\,y + e(x)\,z,$$

because $x = e(x)$. Then by Axiom 3.2.5, $e(x)\,(y+z) = e(x)\,y$ or $e(x)\,(y+z) = e(x)\,z$.

Suppose secondly that $e(x)\,(y+z) = e(x)\,y$ or $e(x)\,(y+z) = e(x)\,z$. We may assume without loss of generality that $e(x)\,y \geq e(x)\,z$. Then by Theorem 3.5.5

$$e(x)\,y + e(x)\,z = e(x)\,y.$$

If $e(x)\,(y+z) = e(x)\,y$ then there is nothing to prove. Suppose that

$$e(x)\,(y+z) = e(x)\,z. \tag{3.25}$$

If $z \leq e(y)$, by Theorem 3.5.10,

$$e(y) + z = e(y)\,.$$

Hence

$$\begin{aligned} e(x)(y+z) &= e(x)(y+e(y)+z) \\ &= e(x)(y+e(y)) \\ &= e(x)y \\ &= e(x)y+e(x)z. \end{aligned}$$

If $e(y) \le z$, by compatibility with addition and multiplication,

$$e(x)y+e(x)z = e(x)y \le e(x)(y+z). \tag{3.26}$$

On the other hand, using Proposition 3.5.2.1 and (3.25), one has

$$e(x)(y+z) = e(x)z = e(x)z+e(x)z \le e(x)y+e(x)z. \tag{3.27}$$

Then, from (3.26) and (3.27), by antisymmetry, we conclude that

$$e(x)(y+z) = e(x)y+e(x)z.$$

■

We prove that distributivity always holds if y and z are both positive or both negative.

Theorem 3.9.35 *Let A be a solid and let $x, y, z \in A$. If y and z are both positive then $x(y+z) = xy+xz$.*

Proof. Suppose that y and z are both positive. By Theorem 3.9.18 one has $e(x)(y+z) \le e(x)y+e(x)z$. We show that also $e(x)y+e(x)z \le e(x)(y+z)$. We may suppose without loss of generality that $e(y) \le y \le z$. Then $e(x)y \le e(x)z$, by Proposition 3.8.1. Then

$$\begin{aligned} e(x)y+e(x)z &\le e(x)z+e(x)z = e(x)z \\ &\le e(x)(z+e(y)) \le e(x)(y+z). \end{aligned}$$

Hence $e(x)(y+z) = e(x)y+e(x)z$, and the result follows by Theorem 3.9.19. ■

Corollary 3.9.36 *Let A be a solid and let $x, y, z \in A$. If y and z are both negative, then $x(y+z) = xy+xz$*

Proof. Suppose that y and z are both negative. Then, by Proposition 3.5.2.2, $e(y) \le -y$ and $e(z) \le -z$. Then by Proposition 3.7.7

$$\begin{aligned} e(x)(y+z) &= e(x)(-(y+z)) = e(x)(-y-z) \\ &= e(x)(-y-z). \end{aligned}$$

Then by Theorem 3.9.35 and Proposition 3.7.7,

$$\begin{aligned} e(x)(y+z) &= e(x)(-y)+e(x)(-z) \\ &= e(x)y+e(x)z. \end{aligned}$$

Hence distributivity holds, by Axiom 3.2.22. ■

Corollary 3.9.37 *Let A be a solid and let $x, y \in A$. Then $x(y+y) = xy + xy$.*

Proof. If y is positive by Theorem 3.9.35

$$x(y+y) = xy + xy.$$

If y is negative, then by Corollary 3.9.36, also

$$x(y+y) = xy + xy.$$

■

3.9.5 Equivalent form of the distributivity axiom

We show that the distributivity condition on Axiom 3.2.22 and (3.18) are equivalent. In order to do so we show the following lemma, without using the distributivity axiom. In the proof of the following lemma we use a corollary of Theorem 3.9.35, so it can be shown supposing (3.18) and without recurring to this axiom.

Lemma 3.9.38 *Let $x, y, z \in A$. Suppose (3.18). Then*

$$e(x)(y+z) \leq e(x)y + e(x)z.$$

Proof. Suppose without loss of generality that $y \leq z$. Then by compatibility with addition $y + z \leq z + z$. Hence

$$e(x)y \leq e(x)z$$

and

$$e(x)(y+z) \leq e(x)(z+z),$$

by Axiom 3.2.16. Then using Corollary 3.9.37

$$\begin{aligned} e(x)(y+z) &\leq e(x)(y+z) + e(x)y \\ &\leq e(x)(z+z) + e(x)y \\ &= e(x)(z) + e(x)(z) + e(x)y = e(x)(z) + e(x)y. \end{aligned}$$

■

Theorem 3.9.39 *Let $x, y, z \in A$. Then (3.18) if and only if $xy + xz = x(y+z) + e(x)y + e(x)z$.*

Proof. By Theorem 3.9.19 we only need to prove the necessary part.

Suppose (3.18). By Lemma 3.9.38

$$e(x)(y+z) \leq e(x)y + e(x)z.$$

Then by Theorem 3.5.5,

$$e(x)(y+z) + e(x)y + e(x)z = e(x)y + e(x)z. \tag{3.28}$$

By Axiom 3.2.26 there is a precise number a such that $x = a + e(x)$. Then

$$x(y+z) + e(x)y + e(x)z = (a + e(x))(y+z) + e(x)y + e(x)z.$$

Because $R(y+z) \leq M = R(e(x))$, using (3.18) one has

$$x(y+z) + e(x)y + e(x)z = a(y+z) + e(x)(y+z) + e(x)y + e(x)z.$$

By (3.28)

$$x(y+z) + e(x)y + e(x)z = a(y+z) + e(x)y + e(x)z$$

Then

$$e(a)(y+z) = 0(y+z) = 0 = 0y + 0z = e(a)y + e(a)z,$$

by Proposition 3.9.9. Hence

$$x(y+z) + e(x)y + e(x)z = ay + az + e(x)y + e(x)z,$$

by (3.18). Because $R(y) \leq M = R(e(x))$, using (3.18) one has

$$x(y+z) + e(x)y + e(x)z = (a + e(x))y + (a + e(x))z = xy + xz.$$

■

Chapter 4

Models for the axioms

> The happiness of life is made up of minute fractions - the little, soon-forgotten charities of a kiss or smile, a kind look, a heart-felt compliment, and the countless infinitesimals of pleasurable and genial feeling.
>
> (Samuel Coleridge)

We show that the axioms presented in Section 3.2 have a model in the class of external numbers.

Let F be an ordered field and let $x \in F$. Then $e(x) = 0$. This means that ordered fields trivially satisfy some of the axioms. In fact, we show that ordered fields satisfy all the axioms with exception of Axiom 3.2.24 and the last one. Axiom 3.2.24 is not satisfied because $F \notin F$.

In the class of external numbers there are individualized neutral elements for both addition and multiplication. This means that the functions e and u don't have to be constant. We show that external numbers, provided with addition and multiplication as defined in Section 2.2, are a (nontrivial) model for the axioms because external numbers satisfy all the axioms, including Axiom 3.2.27.

4.1 Ordered fields and the axioms

Theorem 4.1.1 *Every ordered field $(F, +, \cdot, \geq)$ satisfies Axioms 3.2.1-3.2.26, with the exception of Axiom 3.2.24.*

Proof. Let $(F, +, \cdot, \geq)$ be an ordered field and let $x, y, z \in F$.

We start by proving that the axioms of the first two groups are satisfied. Clearly $(F, +)$ is an assembly with $e(x) = 0$ and $(F \setminus \{0\}, \cdot)$ is an assembly with $u(x) = 1$. Furthermore, because F is a field, multiplication is both commutative and associative in F.

Regarding the order axioms, by the definition of ordered field, we only need to prove Axioms 3.2.16 and 3.2.17. To prove that Axiom 3.2.16 holds suppose that $y + e(x) = e(x)$. Then $y + 0 = 0$, because $e(x) = 0$. Hence $e(x) = 0 \geq y$.

To prove that the order relation is compatible with multiplication suppose that $x \geq e(x) \wedge y \geq z$. This means that $x \geq 0$ and $y \geq z$. Then, because F is an ordered field, $xy \geq xz$.

We prove now that the mixed axioms hold. Observe that, because $e(x) = 0$, the axiom of Scale and Axiom 3.2.20 simply state that $0 = 0$ and Axiom 3.2.21 states that if $x \neq 0$ then $0 = 0$. So we only need to prove the axiom of distributivity. In a field distributivity holds, so

$$x(y+z) + e(x)y + e(x)z = x(y+z) + 0y + 0z = x(y+z) = xy + xz.$$

Hence the distributivity axiom holds.

Axiom 3.2.23 holds clearly by making $m = 0$, and Axiom 3.2.8 holds by making $u = 1$. To prove that Axiom 3.2.26 holds put $a = x$. Then, because $e(x) = 0$, $x = a = a + 0 = a + e(x)$. ■

Definition 4.1.2 *We say that a structure* $(A, +, \cdot)$ *is a* distributive association *if* $(A, +, \cdot)$ *is an association and if* $x(y+z) = xy + xz$*, for all* $x, y, z \in A$.

Corollary 4.1.3 *Every field is a distributive association.*

Corollary 4.1.4 *Every ordered field is an ordered distributive association.*

4.2 External numbers and the axioms

In a sense, ordered fields are a trivial model for (most of) the axioms. We prove that a nontrivial model for the axioms, i.e. a model such that the functions e and u are not identically zero and one respectively, exists in the external numbers (Chapter 2). To do so we start by proving that $(\mathbb{E}, +)$ and $(\mathbb{E} \backslash \mathcal{N}, \cdot)$ are both assemblies. We recall that by Theorem 2.3.1 both $(\mathbb{E}, +)$ and $(\mathbb{E} \backslash \mathcal{N}, \cdot)$ are commutative regular semigroups.

Theorem 4.2.1 *The commutative regular semigroup* $(\mathbb{E}, +)$ *is an assembly.*

Proof. Let $\alpha = a + A$ and $\gamma = c + C \in \mathbb{E}$. Proposition 2.3.3 states that $e(\alpha) = A$ and Proposition 2.3.6 states that $s(\alpha) = -\alpha$. So we only need to verify Axiom 3.2.5. By Proposition 2.2.2

$$e(\alpha + \gamma) = A + C = \max(A, C).$$

Then $e(\alpha + \gamma) = e(\alpha)$ or $e(\alpha + \gamma) = e(\gamma)$. Hence $(\mathbb{E}, +)$ is an assembly. ■

Theorem 4.2.2 *The commutative regular semigroup* $(\mathbb{E} \backslash \mathcal{N}, \cdot)$ *is an assembly.*

Proof. Let $\beta = b + B$ and $\delta = d + D \in \mathbb{E} \backslash \mathcal{N}$. Proposition 2.3.4 states that $u(\beta) = 1 + \frac{B}{b} = 1 + R(\beta)$ and Proposition 2.3.7 states that $d(\beta) = \frac{1}{b} + \frac{B}{b^2}$. So we only need to verify Axiom 3.2.10. By Lemma 2.2.10

$$u(\beta\delta) = u(bd + bD + dB) = 1 + \frac{bD + dB}{bd} = 1 + \frac{D}{d} + \frac{B}{b}.$$

If $\max(bD + dB) = bD$, then $u(\beta\delta) = 1 + \frac{D}{d} = u(\delta)$. If $\max(bD + dB) = dB$, then $u(\beta\delta) = 1 + \frac{B}{b} = u(\beta)$. Hence $u(\beta\delta) = u(\delta)$ or $u(\beta\delta) = u(\beta)$ and we conclude that $(\mathbb{E} \backslash \mathcal{N}, \cdot)$ is an assembly. ■

Because $(\mathbb{E}, +)$ and $(\mathbb{E} \backslash \mathcal{N}, \cdot)$ are assemblies the following results hold (see also [38] [39]).

Corollary 4.2.3 *Let $\alpha = a + A, \beta = b + B$ and $\gamma = c + C$ be external numbers. Then*

1. $\alpha + \beta = \alpha + \gamma \Leftrightarrow A + \beta = A + \gamma$.
2. *The function N is idempotent for sum and for composition.*
3. *The function N is a homomorphism for addition.*
4. *The composition of the inverse function with itself is the identity map.*
5. *The inverse function is a homomorphism for addition.*
6. $-A = A$.
7. *If $\alpha \neq A$, then $\alpha \neq B$, for all $b \in N$.*

Corollary 4.2.4 *Let α, β, γ be zeroless external numbers. Then*

1. $\alpha\beta = \alpha\gamma \Leftrightarrow (1 + R(\alpha))\beta = (1 + R(\alpha))\gamma$.
2. *The function $1 + R(\alpha)$ is idempotent for multiplication and for composition.*
3. *The function $1 + R(\alpha)$ is an homomorphism for multiplication.*
4. *The composition of the inverse function with itself is the identity map.*
5. *The inverse function is a homomorphism for multiplication.*
6. $(1 + R(\alpha))^{-1} = 1 + R(\alpha)$.
7. *If $\alpha \neq (1 + R(\alpha))$, then $\alpha \neq (1 + R(\beta))$.*

Corollary 4.2.5 *The structure $(\mathbb{E}, +, \cdot)$ is an association.*

Proof. Directly from Theorem 4.2.1 and Theorem 4.2.2. ■

External numbers equipped with addition, multiplication and the order relation presented in Definition 2.2.12 satisfy the order axioms.

Theorem 4.2.6 *The structure* $(\mathbb{E}, +, \cdot, \leq)$ *is an ordered association.*

Proof. By Theorem 2.2.15 and Corollary 4.2.5 we only need to show that Axiom 3.2.16 and Axiom 3.2.17 are satisfied. Let $\alpha = a + A$, $\beta = b + B$ and $\gamma = c + C \in \mathbb{E}$.

To prove that Axiom 3.2.16 is satisfied suppose that $\alpha + N(\beta) = N(\beta)$, i.e., $a + A + B = B$. Then $a + A \subseteq B$. Hence $\alpha \leq N(\beta)$.

To prove that Axiom 3.2.17 is satisfied suppose that $N(\alpha) \leq \alpha$ and $\beta \leq \gamma$. If $\alpha = A$, then $A\beta \subseteq A\gamma$ by Proposition 2.2.14. If $\alpha > A$ and $x \in \alpha$ then $x > 0$. Let $y \in \beta$. Then there is $z \in \gamma$ such that $y \leq z$, because $\beta \leq \gamma$. Then $xy \leq xz$. Hence $\alpha\beta \leq \alpha\gamma$. ■

As shown in Theorem 4.1.1 real numbers, as well as ordered fields in general, satisfy the completion axioms with the exception of Axiom 3.2.24 and Axiom 3.2.27. The class of external numbers satisfies all the completion axioms, noting that $\mathbb{R}$ itself is a neutrix.

Theorem 4.2.7 *The structure* $(\mathbb{E}, +, \cdot, \leq)$ *is a solid.*

Proof. By Theorem 4.2.6, Theorem 2.4.13, Proposition 2.3.9 and Proposition 2.3.11 we only need to verify that the Scale axiom and the completion axioms are satisfied. Let $\alpha = a + A$,$\beta = b + B$ and $\gamma = c + C \in \mathbb{E}$.

Put $\delta = bA + AB$. One has

$$N(\alpha)\beta = A(b + B) = bA + AB = N(\delta).$$

Hence the Scale axiom is satisfied.

Axiom 3.2.26 is trivially satisfied. By making $m = 0$, $M = \mathbb{R}$ and $u = 1$, one has $0 + x = x$, $A + \mathbb{R} = \mathbb{R}$ and $1x = x$. Clearly $\pounds \neq 0$ and $\pounds \neq \mathbb{R}$, so Axiom 3.2.27 also holds. ■

We conclude that all axioms of Section 3.2 are satisfied within the structure of external numbers. So the following holds.

Theorem 4.2.8 *The structure* $(\mathbb{E}, +, \cdot, \leq)$ *is a model for axioms 3.2.1-3.2.27.*

Chapter 5

Sorites

> The knowledge came upon me, not quickly, but little by little and grain by grain.
> (Charles Dickens in David Copperfield)

5.1 To be or not to be a heap

In this chapter we consider the paradoxes which arise when several orders of magnitude are considered. This is part of the larger phenomenon known as vagueness.

Vague predicates share (at least) the following common features:

1. Admit borderline cases
2. Lack sharp boundaries
3. Are susceptible to sorites paradoxes

Borderline cases are the ones where it is not clear whether or not the predicate applies, independently of how much we know about it. For instance, most basketball players are clearly tall and most jockeys are clearly not tall. But in many cases is rather unclear if the person in question is tall, even if one knows its height with great precision. Furthermore, there is no clear distinction between the set of all tall people and the set of people that are not tall. These sets lack sharp boundaries. This leads to a collection of paradoxes called *Sorites* paradoxes. One can be stated in the following way: a single grain of wheat cannot be considered as a heap. Neither can two grains of wheat. One must admit the presence of a heap sooner or later, so where to draw the line? This was first discovered by the Greek philosopher Eubulides of Miletus. In fact, the name Sorites derives from the Greek word for heap. However, one can reconstruct the paradox by replacing the term 'heap' by other vague concepts such as 'tall', 'beautiful', 'bald', 'heavy', 'cold', 'rich',...

The argument consists of a predicate S (the soritical predicate) and a subject expression a_n in the series with regards to which S is soritical. The terms of the series are supposed to be ordered. According to Barnes [2] a predicate S must satisfy three constraints in order to be considered soritical:

1. Appear to be valid for a_1, the first item in the series;

2. Appear to be false for a_i, the last item in the series;

3. Each adjacent pair in the series, a_n and a_{n+1} must be sufficiently similar as to appear indiscriminable in respect to S.

This means that the predicate S needs to be sufficiently vague in order to allow small changes. Small changes do not determine the difference between a set of individual grains and a heap, between a bald man and a hairy one, between a rich person and a poor one. However, and in spite of the vagueness involved, it also needs to have a certain area on which S is clearly true and an area on which S is clearly false.

> The difference of one grain would seem to be too small to make any difference to the application of the predicate; it is a difference so negligible as to make no apparent difference to the truth-values of the respective antecedents and consequents.
>
> Yet the conclusion seems false. [29]

In fact, we claim that a heap and a set of individual grains of wheat are not of the same order of magnitude. A set of individual grains may be modeled by a standard subset of the external set of limited numbers (positive part of a neutrix) and the set of grains that form a heap may be modeled by the external set of the infinitely large numbers.

It should also be possible to capture with external sets some modalities, like the difference between a "good" approximation, allowing to obtain an adequately precise numerical result in some context, and a "bad", useless, one. The stability of orders of magnitude under some repeated additions justifies to model them by (convex) groups of real numbers.

5.2 Paradoxical forms

The Sorites paradox can be stated in various forms. This implies that one cannot hope to solve the paradox by pointing out a fault particular to any one of the forms. One should instead try and reveal a common fault to all possible forms that the paradox can take. We consider the (standard) mathematical induction and conditional forms and then present a nonstandard point of view on those forms. Let S represent the predicate 'is not a heap' and let a_n represent the sentence 'n grains of wheat'.

5.2.1 Induction

Mathematical induction is generally used (within standard mathematics) to prove that a mathematical statement involving a natural number n holds for all possible values of n. This is done in two steps. On the first step (*basis*) one proves that there is a first element for which the statement holds. On the second step (*inductive step*) one shows that if the statement holds for some n then it also holds for $n+1$. Then the statement is valid for all n.

The Sorites paradox can now be represented in the following way:

$$\begin{cases} (Sa_1 \wedge \forall n\, (Sa_n \rightarrow Sa_{n+1})) \rightarrow \forall n Sa_n \\ \exists \omega\, (\neg Sa_\omega) \end{cases}$$

So, if one admits that:

1. A single grain of wheat is not a heap.
2. If a collection of n grains of wheat is not a heap then a collection of $n+1$ grains of wheat is also not a heap.

One concludes (by induction) that the heap will never appear. Since at some point the heap is obviously there one might come to the conclusion that there is something wrong with induction or, at least, with applying induction to this case.

5.2.2 Conditional form

The conditional form of the Sorites paradox is the most common form throughout the literature. It can be formalized in the following way:

If

$$\begin{cases} Sa_1 \\ Sa_1 \rightarrow Sa_2 \\ Sa_2 \rightarrow Sa_3 \\ \ldots \\ Sa_i \rightarrow Sa_{i+1} \\ \exists j\, (\neg Sa_j) \end{cases}$$

Assuming Sa_1, $Sa_1 \rightarrow Sa_2$, $Sa_2 \rightarrow Sa_3$, ..., $Sa_i \rightarrow Sa_{i+1}$, by *modus ponens,* the conclusion is Sa_i, where i can be arbitrarily large. This is a fairly simple reasoning where the premises are: a single grain of wheat does not make a heap; if one grain of wheat does not make a heap then two grains of wheat do not form a heap either; if two grains of wheat do not make a heap then three grains of wheat do not form a heap either... if i grains of wheat do not make a heap then $i+1$ grains of wheat do not form a heap either. The conclusion is that a set of an arbitrarily large number of grains i does not make a heap. However if one observes that there is a set of j grains that form a heap it generates a paradox.

5.2.3 A nonstandard point of view on paradoxical forms

We propose to replace the standard forms presented above by the following forms which involve reasoning with nonstandard methods.

If one replaces mathematical induction by external induction (EI) in IST, the reasoning becomes:

$$\begin{cases} (Sa_1 \wedge \forall^{st} n\,(Sa_n \rightarrow Sa_{n+1})) \rightarrow \forall^{st} n Sa_n \\ \exists \omega\,(\neg Sa_\omega) \end{cases}$$

So, if one admits that:

1. A single grain of wheat is not a heap.
2. If n is a standard number and if a set of n grains of wheat is not a heap then a set of $n+1$ grains of wheat is also not a heap.

One concludes that in the presence of a standard number of grains of wheat one does not have a heap. The heap arises when one has a nonstandard number $\omega \simeq +\infty$ of grains of wheat.

The conditional form, using nonstandard analysis, becomes the following.

Let i be a standard natural number. If

$$\begin{cases} Sa_1 \\ Sa_1 \rightarrow Sa_2 \\ Sa_2 \rightarrow Sa_3 \\ \ldots \\ Sa_i \rightarrow Sa_{i+1} \end{cases}$$

Then, by *modus ponens,* the conclusion is Sa_i, for i arbitrarily large but (naive) standard. In nonstandard analysis this is modeled by allowing *modus ponens* but only a standard (naive) number of times. We call "naive" the natural numbers which can be obtained from zero by the successive addition of one. This corresponds to Reeb's famous slogan:

> Les entiers naïfs ne remplissent pas ℕ. [14]

We are in fact claiming that the formalization of the predicate 'is not a heap' should be an external predicate, where not being a heap means to possess a standard number of grains.

5.3 Responses

There are several attempts to solve the Sorites paradox. These responses are divided into the following four types ([35], pages 19-20). A first type of response would be to deny the validity of the argument, refusing to grant that the conclusion follows from the premises. Alternatively one can question the strict truth of the inductive premise (or of one of the conditionals). A third possibility is to accept the validity of the argument and the truth of its inductive premise (or of all the conditional premises) but contest the truth of the conclusion. Finally one can grant that there good reasons to consider both the argument form as valid, and accept the premises and deny the conclusion hence proving that the predicate is incoherent.

In this section we review some of the responses to the paradox. For a wider account on this matter see for example [35] [62] [59] and [69]. We emphasize that the theories presented below correspond to a wide variety of related points of view. This means that there are many versions of the theories presented. So, when reviewing a theory we tend to give only the general lines, common to the various versions of that theory.

5.3.1 Ideal Languages

Natural languages such as English or Portuguese distinguish between intension and extension of terms. The *intension* is the internal content of a term or concept while the *extension* is the range of applicability of a term by naming the particular objects that the term denotes. The two predicates 'is a creature with a heart' and 'is a creature with a kidney' (see [53]) have the same extension because the set of creatures with hearts and the set of creatures with kidneys are the same. However, having a heart and having a kidney are very different things, so one concludes that

> terms can name the same thing but differ in meaning. [53]

Hence, the distinction between intension and extension leads necessarily to vagueness, ambiguity and indeterminacy of meaning for words and phrases. This is in part the reason why natural languages are so powerful and allow the special beauty only achieved by poetry. But it also means that those who want clarity and precision of language will be unsatisfied with natural languages. According to Quine

> The sorites paradox is one imperative reason for precision in science, along with more familiar reasons. [54]

An ideal language would left out all such factors in order to eliminate any vagueness.

The defenders of this response (see Frege [19], Russell [58] and Wittgenstein [71]) consider vagueness as an eliminable feature of natural language. This would mean that sorites arguments are not valid since they contain vague expressions.

As stated by Russell,

> The fact is that all words are attributable without doubt over a certain area, but become questionable within a penumbra, outside which they are again certainly not attributable. Someone might seek to obtain precision in the use of words by saying that no word is to be applied in the penumbra, but unfortunately the penumbra

> is itself not accurately definable, and all the vagueness which apply the primary use of words apply also when we try to fix a limit to their indubitable applicability. [58]

So, this response implies that it is the philosopher's job to discover a logically ideal language. However, this doesn't seem possible using classical logic:

> All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life, but only to an imagined celestial existence. [58]

Russell also believed that

> Vagueness, clearly, is a matter of degree, depending upon the extent of the possible differences between different systems represented by the same representation. Accuracy, on the contrary, is an ideal limit. [58]

Criticism

As follows from the above, ideal languages as a response to the sorites paradox seem to have unsatisfying features. According to Keefe,

> denying the validity of the sorites argument seems to require giving up absolutely fundamental rules of inference. [35]

So, if one chooses to go in this direction fundamental rules such as *modus ponens* or mathematical induction are to be put in question. In fact, most philosophers nowadays believe that vagueness is an important part of natural language and cannot be separated from it.

5.3.2 The Epistemic Theory

The Epistemic Theory is based on the idea that the precise boundaries to knowledge itself cannot be known. Vagueness is seen as a particular type of ignorance.

The fact that this theory is built in the classical logic framework implies that there are precise bounds for the extensions of vague predicates even if we do not know where they are located. For instance, the defenders of the epistemic theory claim that there is in fact a last grain of wheat in the series before the heap turns up, even if one is not (nor ever will be) able to identify it definitively. In fact, Williamson [69] has shown that if there is a precise boundary for penumbral cases we cannot not know where it is. So, soritical predicates are indeterminate in extension but not semantically.

This position, counter-intuitive as it may seem, has been notably defended by Williamson [69] [70] and Sorensen [63] [64].

Criticism

The first and major objection to this theory is its counter-intuitive nature. The meaning of a word is (usually) determined by its use. According to Wittgenstein

> For a large class of cases - though not for all - in which we employ the word 'meaning' it can be defined thus: the meaning of a word is its use in the language [71]

and

> if we had to name anything which is the life of the sign, we should have to say that it is its use. [72]

For instance, the word 'piano' means an actual piano because we use that word to mean an actual piano (even if one does not know how to play). Now, one does not usually use the word 'heap' as if a single grain of wheat could make a difference. Neither, more generally, does one use any vague term as if it were not tolerant to small changes. One does not use vague terms as if they had precise borders. In this sense Smith [62] claims that the epistemicist is forced to deny a link between meaning and use.

Another point that deserves criticism is that nothing is said about how predicates get the precise extensions that they do. It is claimed that there is in fact a last grain of wheat in the series before the heap turns up. So there should be attempts to find which one is it [35]. We agree that ignorance is no excuse for the lack of attempts to find the precise boundaries of vague concepts. There should be at least some reasons to believe about where these boundaries are.

5.3.3 Supervaluationism

According to Fine, vagueness is a semantic notion not to be confused with ambiguity nor undecidability:

> Let us say, in a preliminary way, what vagueness is. I take it to be a semantic notion. Very roughly, vagueness is deficiency of meaning. As such, it is to be distinguished from generality, undecidability, and ambiguity. These latter are, if you like, lack of content, possible knowledge, and univocal meaning, respectively. [16]

Supervaluationism proposes to solve the problem of vagueness by modifying classical semantics, using Van Fraassen's supervaluations. According to Van Fraassen:

> A *supervaluation over a model* is a function that assigns T (F) exactly to those statements assigned T (F) by all the classical valuations over that model. [18]

And he concludes that

> Supervaluations have truth-value gaps. [18]

In classical logic the connectives have truth values in a functional way. We recall that a connective of statements is *truth-functional* if and only if the truth value of any compound statement obtained by applying that connective is a function of the individual truth values of the constituent statements that form the compound. The classical logic connectives are all truth-functional (truth tables). Supervaluationists abandon the concept of truth-functionality.

Fine applies the distinction between extension and intension [53] to vagueness:

> Extensional vagueness is deficiency of extension, intensional vagueness deficiency of intension. Moreover, if intension is the possibility of extension, then intensional vagueness is the possibility of extensional vagueness. [16]

According to this theory, a vague predicate does not need to have a unique, sharply bounded, truth function. Vague predicates have things to which they definitely apply (positive extension), things to which they definitely do not (negative extension) and a penumbra (penumbral connections). The penumbra involves cases which seem to be neither true nor false[1] (borderline cases). These penumbral connections are instances of truth-value gaps. Truth-value gaps are related with extensional vagueness. However,

> Despite the connection, extensional vagueness should not be defined in terms of truth-value gaps. This is because gaps can have other sources, such as failure of reference or presupposition. [16]

Supervaluationists claim, roughly speaking, that a vague sentence is true if and only if it is true for all ways of making it completely precise [16], called *precisifications*. There are then many interpretations or precisifications. Each one of these precisifications has no penumbra because it behaves according to classical bivalence. The assignment of truth value for all such precisifications is a supervaluation.

A sentence which is true in all precisifications is called supertrue and a sentence which is false in all precisifications is called superfalse. A sentence which is true for some precisifications and false on others is neither true nor false[2]. This means in particular that tautologies from classical logic are supertrue.

According to Keefe,

> truth is supertruth, [35]

[1] Fine warns about the general confusion of under- and over-determinacy.

> A vague sentence can be made more precise; and this operation should preserve truth-value. But a vague sentence can be made to be either true or false, and therefore the original sentence can be neither. [16]

[2] In fact, not all supervaluationists accept this last sentence.

meaning that a sentence is true if and only if it is true on all admissible precisifications. A precisification is acceptable only if the extensions of the concepts do not overlap. The truth of a compound sentence is determined by its truth on every precisification.

For a wider account on supervaluationism see for example [18] [16] [35].

Criticism

Fodor and Lepore are particularly critic of the supervaluationist approach to vagueness:

> [...] there is something fundamentally wrong with using supervaluation techniques either for preserving classical logic or for providing a semantics for linguistic expressions ordinarily thought to produce truth-value gaps. [17]

However the fault they point out is not of a logical nature. Indeed they say:

> Right from the start , however, we want to emphasize that the objections we are raising are *philosophical* rather than *logical*. We have no argument with supervaluations considered as a piece of formal mathematics. [17]

Fodor and Lepore point out as the main flaws of supervaluationism the violation of intuitive semantic principles concerning disjunctions and existential quantification, the abandonment of classical rules of inference and the violation of core principles concerning the concept of truth.

Let S represent the predicate 'is not a heap'. Since all tautologies are supertrue,

$$\neg\left(\forall n\left(Sa_n \rightarrow Sa_{n+1}\right)\right)$$

is equivalent to

$$\exists n\left(Sa_n \wedge \neg Sa_{n+1}\right),$$

which, semantically speaking, seems to postulate the existence of a sharp boundary and looks for that matter like a step back towards the epistemic theory. Also $(S \vee \neg S)$ is supertrue. So, for all precisifications one of the statements is true. However, the statement S is borderline and therefore neither true nor false.

Keefe [35] argues that it is possible to surpass these difficulties at the price of adding a new operator to the language: the 'definitely' operator D. This operator is however not closed under certain operations such as contraposition and conditional introduction. So alternatives to the classical closure principles are proposed. However, this implies that the logic used is no longer classical.

Another argument against supervaluationism is that little information is given on what makes a precisification acceptable other than saying that precisifications must respect penumbral connections and therefore admissibility is a vague matter. Also, supervaluationism states that precisifications behave in a classical way and have no penumbra. However, each precisification may divide the positive and negative extensions in different places.

5.3.4 Many-valued logics

Many-valued logics is a general term that refers to logics which have more than two truth-values. In these logics the principle of truth-functionality is accepted and so a sentence remains unaffected when one of its components is replaced by another with the same truth value. Many-valued logics became accepted as an independent part of logic with the works of Łukasiewicz and Post in the 1920's. Since then many many-valued logics emerged (e.g. [41] [20] [24] [51]) and it is not possible nor desirable to describe them all in these pages. However we shall discuss an application of Kleene's three-valued logic and applications of fuzzy logics because these are the most relevant in what concerns the phenomenon of vagueness. For a more complete reference concerning many-valued logics see for example [23].

Kleene's three-valued logic

Perhaps one of the simplest and best-known examples of a many-valued logic is Kleene's three-valued logic [37]. Kleene thought of the third truth value as undefined or underdetermined[3]. So one has three truth-values: 1 (true), 0 (false) and $\frac{1}{2}$ (undefined or unknown). One has truth-tables for which the connectives are *regular*, i.e. in terms of ordering, undefined is placed below both true and false. This means that the behavior of the third truth value should be compatible with any increase in information. Kleene proposed the following truth-tables for the so-called *strong* connectives:

p	$\neg p$
1	0
0	1
$\frac{1}{2}$	$\frac{1}{2}$

p	q	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$p \longleftrightarrow q$
1	1	1	1	1	1
1	0	1	0	0	0
1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	0	1	0
0	0	0	0	1	1
0	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$
$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$
$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

These tables are uniquely determined as the strongest possible regular extensions of the classical two-valued tables. Quantifiers can be defined in the following way: $\exists x : P(x)$ is true if $P(x)$ is true for some value of x and it is false if $P(x)$ is false for all values and indefinite otherwise; $\forall x P(x)$ is true if $P(x)$ is true for all values of x and false if $P(x)$ is false for some value and indefinite otherwise. Tye [66] applies Kleene's three-valued logic to the sorites

[3] Priest [52] gave an alternative three-valued logic conceiving the third truth-value as overdetermined, interpreting the symbol $\frac{1}{2}$ as being both true and false.

paradox. However, the objections made to the bipartite division can also be used to refute a tripartite division. In fact, Tye [66] claims that

> [...] vagueness cannot be reconciled with any precise dividing lines.

because

> there is no determinate fact of the matter about where truth-value changes occur.

That is to say that there is no way to assign precise truth-values to vague terms. So, as a solution, Tye proposes to use a vague metalanguage. He claims that there are sets which are genuinely vague items. For instance the set of tall men has borderline members (men which are neither clearly members nor clearly non-members of the set).

> There is no determinate fact of the matter about there are objects that are neither members, borderline members, nor non-members. [66]

Kleene's three-valued logic has the undesirable feature of having no tautologies, because the two-valued tautologies can take the value $\frac{1}{2}$ in the three-valued case. As an example consider the law of excluded middle $p \vee \neg p$. In Kleene's three-valued logic the truth table is the following:

p	$\neg p$	$p \vee \neg p$
1	0	1
0	1	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Tye tries to avoid this flaw by saying that a statement is a *quasi-tautology* if it has no false substitution instances. So two-valued tautologies become three-valued quasi-tautologies.

Kleene's three-valued logic is still a precise formalization and having no tautologies seems a price too high to pay in order to be able to deal in the above sense with vagueness. Also, according to Keefe: [35]

> [...] the appeal to quasi-tautologies adds nothing: if earning this title is enough for his [Tye's] purposes, then the fact that $p \vee \neg p$ also earns it should be of concern. Moreover, what matters for validity does not relate to quasitautologies [sic], and assertion depends on sentences being true not being either true or indefinite, so the role for the notion seems to be merely one of appeasement.

Fuzzy logics

Fuzzy logics propose a graded notion of inference. Truth-values range in degree between 0 and 1 in order to capture different degrees of truth. In this way, the value 0 is attributed to sentences which are completely false and the value 1 to sentences which are completely true. The remaining sentences are truer than the false sentences, but not as true as the true ones so they have intermediate logical values according to "how true" they are. According to Bogenberger

> In fuzzy logic, the truth of any statement becomes a matter of degree. [7]

Fuzzy logic is related to Zadeh's work on fuzzy sets [73]. A *fuzzy set* A on X is characterized by a membership function $f_A(x)$ with values in the interval $[0, 1]$. So, a fuzzy set A is a class of objects that allow a continuum of grades of membership. The membership degree is then the degree to which the sentence 'x is a member of A' is true. So, one can interpret the membership degrees of fuzzy sets as truth degrees of the membership predicate in a suitable many-valued logic.

Theories of vagueness which recourse to fuzzy logics are advocated most notably by Machina [43] and Smith [62].

According to these theories the notion of heap is a vague one and it may hold true of given objects only to some (truth) degree. The premises should be considered partially true to a degree which is quite near to the maximal degree 1. This inference has to involve truth degrees for the premises and has to provide a truth degree for the conclusion in a way that in each step the truth degree becomes smaller. The sentence 'n grains of sand do not make a heap' tends toward being false for an increasing number of grains.

Criticism

The problem of saying whether the sentence 'a set of n grains makes a heap' is true or not is essentially the same as to say that that sentence is true with a certain (precise) fixed degree. This false precision is perhaps the main objection to the application of many-valued logics to the sorites paradox. According to Keefe

> [T]he degree theorist's assignments impose precision in a form that is just as unacceptable as a classical true/false assignment. In so far as a degree theory avoids determinacy over whether a is F, the objection here is that it does so by enforcing determinacy over the degree to which a is F. All predications of "is red" will receive a unique, exact value, but it seems inappropriate to associate our vague predicate "red" with any particular exact function from objects to degrees of truth. For a start, what could determine which is the correct function, settling that my coat is red to degree 0.322 rather than 0.321? [35]

Also, Urquhart states that

> One immediate objection which presents itself to [fuzzy logic's] line of approach is the extremely artificial nature of the attaching of precise numerical values to sentences like '73 is a large number' or 'Picasso's Guernica is beautiful'. In fact, it seems plausible to say that the nature of vague predicates precludes attaching precise numerical values just as much as it precludes attaching precise classical truth values. [67]

Smith [62] tries to solve this problem, suggesting several possible solutions and concluding that the best answer is to mix fuzzy logic with a theory called plurivaluationism (not to be confused with supervaluationism[4]) called *fuzzy plurivaluationism*. So, Smith accepts the semantic realism implied by the Epistemic view, but denies that vague predicates have to refer to a single bivalent model.

5.3.5 Contextualism

Contextualism[5] defends that interpretations change over time or according to context. Such shifts of contexts may occur instantaneously. For instance, at the beginning of a conversation the context is empty. Then, as the conversation goes along, these notions are sharpened in such a way that borderline cases (undecided so far) get assigned to either the extension or the anti-extension of the vague predicates in question. In fact, borderline sentences can express something true in one context and something false in another, so they are context-sensitive. In this way one can disagree about the truth-values of the propositions expressed by borderline sentences, even in situations where all the relevant information is available. This view is most prominently elaborated by Shapiro [59] [60] and DeRose [12].

Besides contex-sensitivity Shapiro defines as central the concepts of *judgment dependence*, *open texture*, and the *principle of tolerance*. Judgment dependence means that both the extensions and anti-extensions for the borderline cases are solely determined by the decisions of competent speakers. These decisions are put in (and can be removed from) the *conversational record*. Open texture means that for a vague predicate S there exists an object a such that a competent speaker can decide whether Sa holds or not without her competency being compromised. The principle of tolerance is defined as follows. Suppose that

[4] Supervaluationism involves only one intended (non-classical) model relevant to questions concerning meaning and truth, while plurivaluationism allows that there may be multiple (classical) models.

[5] Contextualism is often seen as an argument against philosophical skepticism. Skepticism claims that we don't actually know what we think we know.

> But, according to contextualists, the skeptic, in presenting her argument, manipulates the semantic standards for knowledge, thereby creating a context in which she can *truthfully* say that we know nothing or very little. [12]

two objects a, b differ only marginally in the relevant respect on which a vague predicate S is tolerant. Then if one competently judges Sa to hold, then Sb also holds.

Criticism

One reason for skepticism about contextualism is that the problems with vague expressions seem to remain whether context-sensitivity is taken into account or not. By taking context into account one can reduce vagueness but not eliminate completely. Indeed, sets with vague boundaries are invariant to some translations. Take for instance the word 'ugly'. Even if a particular context is given (and even if one knows a great deal about another one's ugliness) there is still no reason to suppose that there is a sharp boundary between what 'ugly' applies to and what it does not.

Smith [62] argues that contextualism should not be seen as a theory of vagueness in its own right. He claims that this theory is compatible with all other mentioned theories.

5.4 External numbers as a model

In this section we present an approach to the Sorites paradox which takes advantage of the notions and concepts of nonstandard analysis. Indeed, we believe that neutrices (Chapter 2) are adequate to model the type of vagueness involved in the Sorites paradox. We want to emphasize that our response solves only a specific type of vagueness (of the type Sorites) and therefore is not intended as a theory for vagueness in general. Also, we are not claiming that other theories are without value. For instance, the fuzzy logic approach has been quite successful in solving vagueness related to traffic and transportation processes (see for example [65] [7], also [75] for other examples of applications of fuzzy set theory). According to Teodorović [65]

> [...] a wide range of traffic and transportation engineering parameters are characterized by uncertainty, subjectivity, imprecision and ambiguity. Human operators, dispatchers, drivers and passengers use this subjective knowledge or linguistic information on a daily basis when making decisions.

Also,

> The results obtained show that fuzzy set theory and fuzzy logic present a promising mathematical approach to model complex traffic and transportation processes [...]

However the fuzzy logic approach is also not without fault as model of imprecision, because it ultimately recourses to precise intervals to model imprecise situations. Moreover, it does not work with the actual error but only with an upper bound of the error. On the contrary, with external numbers it is possible to work directly with imprecisions and errors without recourse to upper bounds, for they have neither infimum nor supremum and satisfy strong algebraic laws as seen in Chapter 2.

Soritical arguments share with external numbers the fact of being tolerant to small changes but not tolerant to large changes in relevant aspects. In fact, with external numbers it is even possible to define rigorously what we mean with terms such as 'small changes' or 'large changes'. The fact that large changes come as the result of the accumulation of small changes comes as no surprise because it is a very well known fact from nonstandard analysis that an infinitely large sum of infinitesimals may become appreciable or even infinitely large.

A simple shift from the classical forms presented in Section 5.2.1 and in Section 5.2.2 to the forms using nonstandard concepts presented in Section 5.2.3 does not solve the problem. A million grains of wheat should form a heap and yet that is clearly a standard number of grains. However, both these forms suggest that the set of individual grains may be modeled by the external set of limited numbers (positive part of a neutrix) and the set of grains that form a heap may be modeled by the external set of the infinitely large numbers. Indeed "precise" objects possess sharp bounds and can be modeled by standard sets. "Vague" objects have no clear bounds and should be for this matter modeled by nonstandard sets which are given by external properties.

As mentioned above epistemicists believe in the existence of sharp bounds for vague concepts, claiming that ignorance is somehow inevitable. We disagree completely with that point of view. Indeed, the tolerance of vague terms such as 'heap' to small changes indicates that such terms do not have a sharp, definite bound. By using neutrices to model such terms it is possible to avoid the paradox and explain the tolerance to small changes.

According to Keefe [35], degree theories fail to provide an acceptable account of vagueness and are forced to make an implausible commitment to a unique numerical assignment for each sentence. Smith [62] argues that an adequate account of vagueness must involve degrees of truth and that the main objections to this theory may be overcome. His fuzzy plurivaluationism theory seems overcomplicated for our approach on the Sorites paradox. We believe that the problem with the fuzzy logic approach is the fact that precise numbers are used to model imprecise predicates. This problem is overcome if one uses the external set of limited numbers because this external set is tolerant to appreciable (but not infinitely large) imprecisions.

A final remark concerns the strength of nonstandard axioms, which may introduce undesirable consequences of external modelling. As such, within IST, the proposed solution of the Sorites paradox

$$\left\{ \begin{array}{c} st\,(0) \\ \forall n\,(st\,(n) \rightarrow st\,(n+1)) \\ \exists \omega\,(\neg st\,(\omega)) \end{array} \right. \tag{5.1}$$

implies, by the group property of the standard numbers, invariance by doubling, i.e.

$$\forall n\,(st\,(n) \to st\,(2n))\,.$$

One easily imagines a soritical context where this is inappropriate. However,

$$\exists n\,(st\,(n) \wedge \neg st\,(2n)) \tag{5.2}$$

is consistent with (5.1). In such a context {(5.1), (5.2)} might be an acceptable axiom system indeed, though of course some calculation properties will be lost.

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