Existence, localization and multiplicity results for nonlinear and functional

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Existence, localization and multiplicity results for nonlinear and functional high order boundary value problems

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To Sofia and Pedro

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Abstract

In this thesis several problems are addressed. The problems considered vary from second order problems up to high order problems where generalizations to n^{th} order are studied. Such problems range from problems without functional dependence up to problems where the functional dependence is featured both in the equation and on the boundary conditions.

Functional boundary conditions include most of the classical conditions as multipoint cases, conditions with delay and/or advances, nonlocal or integral, with maximum or minimum arguments,... Existence, nonexistence, multiplicity and localization results are then discussed in accordance with these conditions.

The method used is the lower and upper solutions combined with different techniques (degree theory, Nagumo condition, iterative technique, Green's function) to obtain such results.

Several applications are studied such as the periodic oscillations of the axis of a satellite and conjugate boundary value problems, to emphasize the applicability of the method used.

Keywords

Lower and upper solutions, degree theory, bilateral and one-sided Nagumo conditions, extremal solutions, high order functional boundary value prob-

lems, periodic problems, Green function, ϕ- Laplacian equation, impulsive problems.

Resumo

Resultados de existência, localização e multiplicidade para problemas não lineares e funcionais de ordem superior com valores na fronteira

Nesta tese, intitulada em português, "Resultados de existência, localização e multiplicidade para problemas não lineares e funcionais de ordem superior com valores na fronteira ", diferentes problemas são abordados. Estes problemas variam desde problemas de segunda ordem até problemas de ordem superior, onde generalizações de ordem n são feitas e onde os problemas apresentados variam desde o caso em que não existe dependência funcional até aos em que esta dependência funcional está presente tanto na equação como nas condições de fronteira.

Sobre estas condições, que incluem a maioria das condições clássicas, resultados de existência, não existência, multiplicidade e localização de solução são discutidos de acordo com estas condições.

O método utilizado é o método da sub e sobre-solução combinado com diferentes técnicas.

Várias aplicações são estudadas, nomeadamente as oscilações periódicas do eixo de um satélite e problemas conjugados, de forma a dar ênfase à aplicabilidade do método utilizado.

Palavras chave

Sub e sobre-solução, teoria do grau, condição de Nagumo bilateral e unilateral, soluções extremais, problemas funcionais de ordem superior, problemas periódicos, função de Green, equação ϕ- Laplaciano, problemas impulsivos.

Notations

$\|y\|_p -$ norm of y in L^p and given by

$$||y||_p = \begin{cases} \left(\int_0^1 |y(t)|^p dt\right)^{1/p}, & 1 \leq p < \infty, \\ \sup\{|y(t)| : t \in I\}, & p = \infty. \end{cases}$$

$C^k(I, \mathbb{R})$ or $C^k(I) -$ space of real valued functions with continuous $k -$ derivative in I, $k \in \mathbb{N}_0$, where $C^0(I) = C(I)$

$$||y|| = \max_{t \in I} |y(t)|$$

$W^{m,p}(I) := \left\{ f \in L^1_{loc}(I) : \|f\|_{W^{m,p}(I)} < \infty \right\}$, when $D^\alpha_w f$ exist for $|\alpha| \leq m$,
with $\|f\|_{W^{m,p}(I)} := \left(\sum_{|\alpha| \leq m} \|D^\alpha_w f\|^p_{L^p(I)} \right)^{1/p}$ for $1 \leq p < \infty$ and
for $p = \infty$, $\|f\|_{W^{m,\infty}(I)} := \max_{|\alpha| \leq m} \|D^\alpha_w f\|_{L^\infty(I)}$

$d(f, \Omega, p) -$ Leray-Schauder topological degree of f relative to Ω at $p \in \Omega$

$d_L(f, \Omega, p) -$ Coincidence degree relatively to L in Ω at $p \in \Omega$
(for details see [70])

$PC(I)$ $-$ set of functions $u : I \to \mathbb{R}$ continuous on $I \backslash D$ where
$u\left(x_k^+\right) = \lim\limits_{x \to x_k^+} u\left(x_k\right)$ and $u\left(x_k^-\right) = \lim\limits_{x \to x_k^-} u\left(x_k\right)$ exist for $k = 1, 2, ..., m$

$\|y\|_{PC} -$ norm of x in $PC\left(I\right)$ given by $\|y\|_{PC} = \sup\limits_{t \in I} \left|y\left(t\right)\right|$

$\|p\|_1 := \int_I \left|p\left(t\right)\right| dt$

$AC\left(I\right)$ - set of absolutely continuous functions $u : I \to \mathbb{R}$

$\mathbb{R}_0^+ = [0, +\infty)$

Introduction

This monograph is dedicated to higher order boundary value problems. The first part studies sufficient conditions to obtain existence and multiplicity results for nonlinear boundary value problems and the second part considers functional boundary value problems.

The main results of each chapter are original and they were presented in international events, published in international reviewed journals or submitted for publication.

The first chapter is dedicated to the study of higher order periodic problems. These problems have been studied by several authors, with different techniques and tools according the several types of goals and contexts. The method applied makes use of the lower and upper solutions technique and it was chosen by the following features: it provides an unification for the higher order problems, as up to now, these problems were studied, in the existent literature, in separate for the odd and the even case; it allows more general nonlinearities and full differential equations, generalizing the range of possible applications to real life phenomena, as beam theory, epidemiology, human scoliosis amongst others; it emphasizes some qualitative properties of the solution, such as sign, variation, type, ..., reason why the high order results obtained are illustrated with examples, in order to stress some of these particularities.

The same method is applied to Ambrosetti-Prodi type equations with Lidstone boundary conditions, in Chapter 2. The research made allows to show the role of lower and upper definition in the main results, namely in the type of assumptions to consider on the nonlinearity. These Lidstone problems are well known in the literature due to their applicability in beam theory. The ability to combine information about not only the existence and location results but also with nonexistence results are key points to ensure the usefulness of this method in applications.

Ambrosetti-Prodi type equations are responsible for leading to a new research trail still open: what are the sufficient conditions to obtain multiplicity results for Lidstone boundary conditions?

Multiplicity results analyzed in Chapter 3, via lower and upper solution method, are obtained for some two point separated boundary value problems. This study was motivated by a phenomena that occured in the London Millenium bridge in its openning day, for which it is given a physical meanning to the parameter s and positive force $p(x)$. From the research developped in this matter we stress two points about the multiplicity part of the so called Ambrosetti-Prodi equations:

- the generalization for higher order problems requires an assumption to define that the "speed growth" on some variables are greater than in other ones. This can be done by perturbation (see condition (3.3.11)) or by adequate assymptotic behaviour as in [36].

- the "*a priori*" upper bound on the second derivatives of every solution (see condition (3.3.9)) is, in our point of view, the weakness of these type of problems and moreover it constraints this method's application to real problems. How to replace such bound assumptions will require further research.

In Chapter 4, fourth order periodic problems with two types of impulsive effects are studied. The key tool in both cases is an iterative technique, not necessarily monotone, combined with lower and upper solutions. Remark that, due to the discontinuities caused by the instantaneous changes at some moments (impulses) the auxiliary functions used to define lower and upper solutions in Chapter 1, are not needed now, allowing different definitions, even for periodic cases.

The study of functional boundary value problems comprises the whole second part.

In the fifth chapter we start by combining the second order Ambrosetti-Prodi type equations with some functional boundary conditions. These conditions are extremely general and they include most of the classical cases as multipoint, conditions with delay and/or advances, nonlocal or integral, with maximum or minimum arguments... As this field of study is still in its early days, apart from the results presented, several thoughts and open problems remain, that still crave attention for future research.

Chapter 6 introduces the study of higher order problems with functional boundary conditions. This study was motivated by the attempt to formulate a result that would combine the works of several authors and also by the high potential in applications that this type of conditions can unleash. As such, an application to a theoretical result (conjugate boundary value problems) is presented. The problem considered can in fact be covered by the more general functional boundary conditions. As to that it is tackled as an application of the results presented in this chapter. In fact for this case, lower and upper solution method allow not only a sharper estimate then the ones existent in the literature, but it also unveils results for some unexplored parameters.

In Chapter 7 a more general equation is considered: a generalization of the $\phi-$ Laplacian equation. Combining this equation with the type of general boundary data as functional boundary conditions was the guideline for this chapter. The generalization of these results to higher order led to interesting conclusions, as for instance, the need to define different lower and upper solutions, depending on n odd or n even.

Chapter 8 introduces new functional boundary value problems. These problems are characterized by the fact that both the equation and the boundary conditions have functional dependence. As such, integro-differential equations, eventually with delay and/or advances, nonlocal maximum or minimum arguments can now be considered. Generalizing these results to higher order is one of the key aspects of this chapter. The other one is emphasizing the applicability and adaptability of such problems and methods. For that, one considers two applications: a theoretical one - the Lidstone problem - and an application to satellites. In this last case a problem that models the periodic oscillations of the axis of a satellite in the plane of the elliptic orbit around its centre of mass, is approached using lower and upper solutions. By this technique the solutions obtained for certain value of the parameters are different from the ones existent in the literature.

Through the combination of a different technique that included Green functions and a sharper version of Bolzano Theorem one was able to obtain extremal solutions to a fourth order problem with functional boundary conditions, in Chapter 9. By extremal solutions, we mean the existence of a maximal solution and a minimal solution. As in the previous chapters, the high potential of this tool not only to provide information and to be used in applications, but also to deal with so general boundary conditions. In fact, to the best of our knowledge is the first time where sufficient conditions for the existence of extremal solutions are given, to fourth order problems with functional dependence in every boundary function.

As final remark we mention that with this thesis we expect to illustrate that, through lower and upper solution method, one can study higher order problems as general as in the functional case and, moreover that, through this method, not only information about the existence of solution is obtained, but also location information on its derivatives, which can in fact be of extreme use in some applications and add value to this method.

Part I

Nonlinear boundary value problems: Existence and multiplicity results

Chapter 1

High order periodic problems

1.1 Introduction

High order periodic boundary value problems have been studied by several authors in the last decades. This type of problems arises several questions when approached and its usage has become widely spread due to its applicability, namely in beam theory, epidemiology and human scoliosis, amongst others.

In this chapter we consider the higher order periodic boundary value problem composed by the fully differential equation

$$u^{(n)}(x) = f\left(x, u(x), u'(x), ..., u^{(n-1)}(x)\right) \tag{1.1.1}$$

for $n \geq 3$, $x \in I := [a, b]$, and $f : I \times \mathbb{R}^n \to \mathbb{R}$ a continuous function and the periodic boundary conditions

$$u^{(i)}(a) = u^{(i)}(b), \ i = 0, 1, ..., n-1. \tag{1.1.2}$$

For first and second order these periodic problems are already well documented in the literature. In the study of periodic problems several different types of arguments and techniques were considered:

In [46, 47, 92] variational methods and the mountain pass theorem are used to derive homoclinic and periodic solutions for second and sixth-order ordinary differential equations;

Mawhin's continuation theorem and inequality techniques are used in [85] to approach third order periodic problems and in [4, 28], authors use the same method to approach a fourth order problem;

Monotone methods, such as the lower and upper solutions method, were considered in [93] to obtain extremal solutions to the second order periodic problem. These same methods were also considered in [82], to obtain existence results for the third order problem;

Fourth order periodic problems, approached via the lower and upper solution method, can be observed in [1, 97]. In this line, equation (1.1.1) generalizes these results to a higher order problem. Also remark that not only (1.1.1) is a fully differential equation but also f is a non-linear function, which generalizes [34, 59, 79] where a linear or quasi-linear n^{th} order periodic problem is discussed.

A nonlinear fully differential equation of higher order as in (1.1.1) was studied in [56], for f a bounded and periodic function verifying different assumptions for n even or odd. Moreover, in [61], the nonlinear part f of (1.1.1) must verify the following assumptions:

(A$_1$) There are continuous functions $e(x)$ and $g_i(x,y)$, $i=0,...,n-1$, such that

$$|f(x,y_0,...,y_{n-1})| \leq e(x) + \sum_{i=0}^{n-1} g_i(x,y_i)$$

with

$$\lim_{|y|\to\infty} \sup_{x\in[0,1]} \frac{|g_i(x,y)|}{|y|} = r_i \geq 0, \ \ i=0,1,...,n-1.$$

($\mathbf{A}_2$) There is a constant M such that, for $x \in [0,1]$,

$$f\left(x, y_0, 0, ..., 0\right) > 0, \text{ for } y_0 > M,$$

and

$$f\left(x, y_0, 0, ..., 0\right) < 0, \text{ for } y_0 < -M.$$

($\mathbf{A}_3$) There are real numbers $L \geq 0$, $\alpha > 0$ and $a_i \geq 0$, $i = 1, ..., n-1$, such that

$$\left|f\left(x, y_0, ..., y_{n-1}\right)\right| \geq \alpha \left|y_0\right| - \sum_{i=1}^{n-1} a_i \left|y_i\right| - L,$$

for every $x \in [0,1]$ and $(y_0, ..., y_{n-1}) \in \mathbb{R}^n$.

The arguments used in this chapter allow more general nonlinearities, namely, f does not need a sublinear growth in $y_0, ..., y_{n-1}$ (as in (A_1)) nor to change sign (as in (A_2)). In fact, condition (1.3.2) in the main result (see Theorem 1.3.1) refers an, eventually, opposite monotony to (A_2) and improve the existent results in the literature for periodic higher order boundary value problems.

In short, the technique used is based on lower and upper solutions not necessarily ordered, in the topological degree theory, like it was suggested in [29, 43], and has the following key points:

- A Nagumo-type condition on the nonlinearity, useful to obtain an *a priori* estimation for the $(n-1)^{th}$, derivative and to define an open and bounded set where the topological degree is well defined.

- A new kind of definition for lower and upper solutions, required to deal with the absence of a definite order for lower and upper functions and their derivatives up to the $(n-3)^{th}$ order. We remark that with such functions it is only required boundary data for the derivatives of order

$n-2$ and $n-1$. Therefore the set of admissible functions for lower and upper solutions is more general.

- An adequate auxiliary and perturbed problem, where the truncations and the homotopy are extended to some mixed boundary conditions, allowing a invertible linear operator and the evaluation of the Leray-Schauder degree.

This chapter contains an example where both existence and location of solution are shown as well as some emphasis is put on the fact that the lower and upper solutions are not well ordered.

Even though some new results are presented in this chapter there are still some open problems to be addressed in future investigations, for instance:

- relaxing the continuity condition on the f function
- introducing some functional dependence on the f function
- relaxing or removing condition (1.3.2) in Theorem 1.3.1

1.2 Definitions and *a priori* bounds

In this section it is introduced a Nagumo-type growth condition, initially presented in [80], and now useful to obtain an *a priori* estimate for the $(n-1)^{th}$ derivative.

Definition 1.2.1 *A continuous function $f : I \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy a Nagumo type condition in*

$$E = \left\{ \begin{array}{c} (x, y_0, ..., y_{n-1}) \in I \times \mathbb{R}^n : \gamma_i(x) \leq y_i \leq \Gamma_i(x), \\ i = 0, ..., n-2 \end{array} \right\}, \tag{1.2.1}$$

with $\gamma_i(x)$ *and* $\Gamma_i(x)$ *continuous functions such that,*

$$\gamma_i(x) \leq \Gamma_i(x), \text{ for } i = 0, 1, ..., n-2 \text{ and every } x \in I, \tag{1.2.2}$$

if there exists a real continuous function $h_E : [0, +\infty[\to]0, +\infty[$ *such that*

$$|f(x, y_0, ..., y_{n-1})| \leq h_E(|y_{n-1}|), \text{ for every } (x, y_0, ..., y_{n-1}) \in E, \tag{1.2.3}$$

with

$$\int_0^{+\infty} \frac{s}{h_E(s)} ds = +\infty. \tag{1.2.4}$$

The a priori bound is given by next lemma:

Lemma 1.2.2 *Consider* γ_i, $\Gamma_i \in C(I, \mathbb{R})$, *for* $i = 0, ..., n-2$, *such that (1.2.2) holds and* E *is defined by (1.2.1). Assume there is* $h_E \in C(\mathbb{R}_0^+, \mathbb{R}^+)$, *such that*

$$\int_\eta^{+\infty} \frac{s}{h_E(s)} ds > \max_{x \in I} \Gamma_{n-2}(x) - \min_{x \in I} \gamma_{n-2}(x), \tag{1.2.5}$$

where $\eta \geq 0$ *is given by*

$$\eta := \max\left\{ \frac{\Gamma_{n-2}(b) - \gamma_{n-2}(a)}{b-a}, \frac{\Gamma_{n-2}(a) - \gamma_{n-2}(b)}{b-a} \right\}.$$

Then there exists $R > 0$ *(depending on* $\gamma_{n-2}, \Gamma_{n-2}$ *and* h_E*) such that for every continuous function* $f : E \to \mathbb{R}$ *verifying (1.2.3) and for every* $u(x)$ *solution of (1.1.1) such that*

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x), \ i = 0, 1, ..., n-2 \tag{1.2.6}$$

for every $x \in I$, *we have*

$$\left\| u^{(n-1)} \right\|_\infty < R.$$

Proof. Let u be a solution of (1.1.1) and such that (1.2.6) is verified.

If for every solution u of (1.1.1), verifying (1.2.6), there is $\eta \geq 0$ such that

$$\left| u^{(n-1)}(x) \right| \leq \eta,$$

for every $x \in [a,b]$, then considering $\eta := R$ will conclude the proof.

Assume by contradiction that $\left|u^{(n-1)}(x)\right| > \eta$, for every $x \in [a,b]$. If $u^{(n-1)}(x) > \eta$, for every $x \in [a,b]$, then the following contradiction is obtained

$$\begin{aligned}
\Gamma_{n-2}(b) - \gamma_{n-2}(a) &\geq u^{(n-2)}(b) - u^{(n-2)}(a) \\
&= \int_a^b u^{(n-1)}(\tau)\, d\tau \\
&> \int_a^b \eta d\tau = \eta(b-a) \\
&\geq \Gamma_{n-2}(b) - \gamma_{n-2}(a).
\end{aligned}$$

The case $u^{(n-1)}(x) < \eta$, for every $x \in [a,b]$, leads to a similar contradiction. So there is $x \in [a,b]$ such that

$$\left|u^{(n-1)}(x)\right| \leq \eta.$$

Suppose that there is $x_1 \in [a,b]$ such that $u^{(n-1)}(x_1) > \eta$ and consider $J := [x_0, x_1]$ or $[x_1, x_0]$ such that

$$u^{(n-1)}(x_0) = \eta \text{ and } u^{(n-1)}(x) > \eta,$$

for every $x \in J \setminus \{x_0\}$. Since $\eta \geq 0$, then $u^{(n-1)}(x) \geq 0$, for every $x \in J$.

For $J = [x_0, x_1]$ (if $J = [x_1, x_0]$ the arguments are similar), applying a convenient change of variable, by (1.1.1), (1.2.5) and (1.2.3), we have

$$\begin{aligned}
\int_{u^{(n-1)}(x_0)}^{u^{(n-1)}(x_1)} \frac{s}{h_E(s)} ds &= \int_{x_0}^{x_1} \frac{u^{(n-1)}(x)}{h_E\left(u^{(n-1)}(x)\right)} u^{(n)}(x)\, dx \\
&= \int_{x_0}^{x_1} \frac{f\left(x, u(x), u'(x), ..., u^{(n-1)}(x)\right)}{h_E\left(u^{(n-1)}(x)\right)} u^{(n-1)}(x)\, dx \\
&\leq \int_{x_0}^{x_1} u^{(n-1)}(x)\, dx = u^{(n-2)}(x_1) - u^{(n-2)}(x_0) \\
&\leq \max_{x\in[a,b]} \Gamma_{n-2}(x) - \min_{x\in[a,b]} \gamma_{n-2}(x) < \int_\eta^R \frac{s}{h_E(s)} ds.
\end{aligned}$$

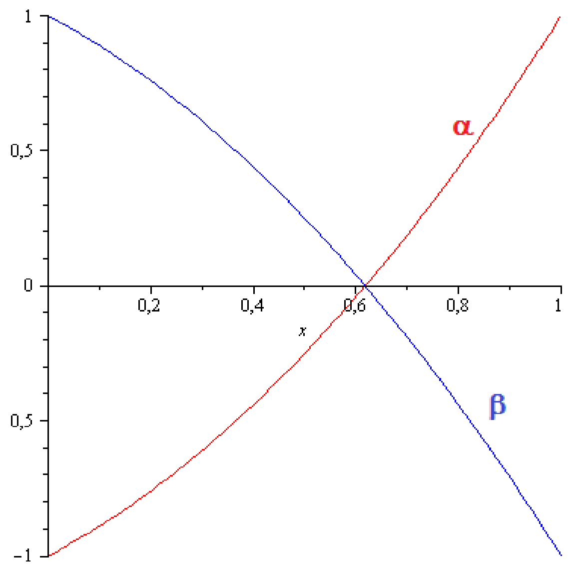

Figure 1.2.1: Example of non-ordered upper and lower solutions

Then $u^{(n-1)}(x_1) \leq R$. Since x_1 was taken arbitrarily in $[a, b]$ as long as $u^{(n-1)}(x) > \eta$, we can conclude that, for every $x \in [a, b]$, such that $u^{(n-1)}(x) > \eta$,

$$u^{(n-1)}(x) \leq R.$$

In a similar way it can be proved that $u^{(n-1)}(x) \geq -R$, for every $x \in [a, b]$, such that $u^{(n-1)}(x) < -\eta$. Therefore,

$$\left|u^{(n-1)}(x)\right| \leq \eta, \ \forall x \in [a, b].$$

■

In this Chapter it is used non-ordered lower and upper solutions, β and α, respectively. That is there exist $x_1, x_2 \in [a, b]$ such that $\alpha(x_1) > \beta(x_1)$ and $\alpha(x_2) < \beta(x_2)$. Therefore the set of admissible functions that can be considered as lower and upper solutions for the problem (1.1.1)-(1.1.2) is generalized. To recover some order needed to define the branches where the solution and its derivatives are localized, one must consider adequate auxiliary functions, as it can be seen in the next definition.

Definition 1.2.3 *For $n \geq 3$, the function $\alpha \in C^n(I)$ is a lower solution of problem (1.1.1)-(1.1.2) if:*

(i) $\alpha^{(n)}(x) \geq f\left(x, \alpha_0(x), \alpha_1(x), ..., \alpha_{n-3}(x), \alpha^{(n-2)}(x), \alpha^{(n-1)}(x)\right)$

with

$$\alpha_i(x) := \alpha^{(i)}(x) - \sum_{j=i}^{n-3} \left\|\alpha^{(j)}\right\|_\infty (x-a)^{j-i}, \ i = 0, ..., n-3. \tag{1.2.7}$$

(ii) $\alpha^{(n-1)}(a) \geq \alpha^{(n-1)}(b)$, $\alpha^{(n-2)}(a) = \alpha^{(n-2)}(b)$.

The function $\beta \in C^n(I)$ is an upper solution of problem (1.1.1)-(1.1.2) if:

(iii) $\beta^{(n)}(x) \leq f\left(x, \beta_0(x), \beta_1(x), ..., \beta_{n-3}(x), \beta^{(n-2)}(x), \beta^{(n-1)}(x)\right)$

where

$$\beta_i(x) := \beta^{(i)}(x) + \sum_{j=i}^{n-3} \left\|\beta^{(j)}\right\|_\infty (x-a)^{j-i}, \ i = 0, ..., n-3. \tag{1.2.8}$$

(iv) $\beta^{(n-1)}(a) \leq \beta^{(n-1)}(b)$, $\beta^{(n-2)}(a) = \beta^{(n-2)}(b)$.

Remark that the functions α, β are not necessarily ordered, but the auxiliary functions α_i and β_i are well ordered for $i = 0, ..., n-3$.

Moreover, there is no need of data on the values of the lower solution α or the upper solution β and their derivatives until order $(n-3)$ in the boundary. In fact, this is a key point to have more general sets of admissible functions as lower or upper solutions of problem (1.1.1)-(1.1.2).

1.3 Existence of periodic solutions

The main theorem provides an existence and location result for problem (1.1.1)-(1.1.2) in presence of lower and upper solutions, not necessarily ordered.

Theorem 1.3.1 *Assume that $\alpha, \beta \in C^n(I)$ are lower and upper solutions of (1.1.1)-(1.1.2) such that*

$$\alpha^{(n-2)}(x) \leq \beta^{(n-2)}(x), \ \forall x \in I. \tag{1.3.1}$$

Let $f : I \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function verifying a Nagumo-type condition in

$$E_* = \left\{ \begin{array}{c} (x, y_0, ..., y_{n-1}) \in I \times \mathbb{R}^n : \alpha_i \leq y_i \leq \beta_i, \ i = 0, 1, ..., n-3, \\ \alpha^{(n-2)} \leq y_{n-2} \leq \beta^{(n-2)} \end{array} \right\}$$

and

$$\begin{aligned} f(x, \alpha_0, ..., \alpha_{n-3}, y_{n-2}, y_{n-1}) &\geq f(x, y_0, ..., y_{n-3}, y_{n-2}, y_{n-1}) \\ &\geq f\left(x, \beta_0, ..., \beta_{n-3}, y_{n-2}, y_{n-1}\right) \end{aligned} \tag{1.3.2}$$

for fixed $(x, y_{n-2}, y_{n-1}) \in I \times \mathbb{R}^2$ and $\alpha_i \leq y_i \leq \beta_i$, $i = 0, 1, ..., n-3$.

Then problem (1.1.1)-(1.1.2) has at least a periodic solution $C^n(I)$ such that

$$\alpha_i(x) \leq u^{(i)}(x) \leq \beta_i(x), i = 0, 1, ..., n-3,$$

and

$$\alpha^{(n-2)}(x) \leq u^{(n-2)}(x) \leq \beta^{(n-2)}(x),$$

for $x \in I$.

Proof. Consider the homotopic and truncated auxiliary equation

$$\begin{aligned} u^{(n)}(x) &= \lambda f\left(\begin{array}{c} x, \delta_0(x, u(x)), ..., \delta_{n-2}\left(x, u^{(n-2)}(x)\right), \\ u^{(n-1)}(x) \end{array}\right) \\ &\quad + u^{(n-2)}(x) - \lambda \delta_{n-2}\left(x, u^{(n-2)}(x)\right) \end{aligned} \tag{1.3.3}$$

where the continuous functions $\delta_i, \delta_{n-2} : \mathbb{R}^2 \to \mathbb{R}$, $i = 0, ..., n-3$, are given by

$$\delta_i(x, y_i) = \begin{cases} \beta_i(x) & , \quad y_i > \beta_i(x) \\ y_i & , \quad \alpha_i(x) \le y_i \le \beta_i(x) \\ \alpha_i(x) & , \quad y_i < \alpha_i(x) \end{cases}$$

with α_i and β_i defined in (1.2.7) and (1.2.8), respectively,

$$\delta_{n-2}(x, y_{n-2}) = \begin{cases} \beta^{(n-2)}(x) & , \quad y_{n-2} > \beta^{(n-2)}(x) \\ y_{n-2} & , \quad \alpha^{(n-2)}(x) \le y_{n-2} \le \beta^{(n-2)}(x) \\ \alpha^{(n-2)}(x) & , \quad y_{n-2} < \alpha^{(n-2)}(x) \end{cases}$$

coupled with the boundary conditions

$$\begin{aligned} u^{(k)}(a) &= \lambda\ \eta_k\left(u^{(k)}(b)\right), \quad k = 0, ..., n-3 \\ u^{(n-2)}(a) &= u^{(n-2)}(b) \\ u^{(n-1)}(a) &= u^{(n-1)}(b) \end{aligned} \tag{1.3.4}$$

where the functions $\eta_k : \mathbb{R} \to \mathbb{R}$, $k = 0, ..., n-3$, are defined by

$$\eta_k\left(u^{(k)}(b)\right) = \begin{cases} \beta_k(a) & , \quad u^{(k)}(b) > \beta_k(a) \\ u^{(k)}(b) & , \quad \alpha_k(a) \le u^{(k)}(b) \le \beta_k(a) \\ \alpha_k(a) & , \quad u^{(k)}(b) < \alpha_k(a). \end{cases} \tag{1.3.5}$$

Take $r_{n-2} > 0$ such that, for every $x \in I$

$$-r_{n-2} < \alpha^{(n-2)}(x) \le \beta^{(n-2)}(x) < r_{n-2}, \tag{1.3.6}$$

$$f\left(x, \alpha_0(x), ..., \alpha_{n-3}(x), \alpha^{(n-2)}(x), 0\right) - \alpha^{(n-2)}(x) - r_{n-2} < 0, \tag{1.3.7}$$

$$f\left(x,\beta_0(x),...,\beta_{n-3}(x),\beta^{(n-2)}(x),0\right)-\beta^{(n-2)}(x)+r_{n-2}>0. \tag{1.3.8}$$

Step 1: *Every solution of the problem (1.3.3)-(1.3.4) satisfies in I*

$$\left|u^{(i)}(x)\right|<r_i,\ i=0,...,n-2,$$

independently of $\lambda\in[0,1]$*, with* r_{n-2} *given as above and for* $k=0,...,n-3$,

$$r_k=\xi_k+r_{n-2}(b-a)^{n-2-k}, \tag{1.3.9}$$

where

$$\xi_k:=\max\left\{\sum_{j=k}^{n-3}\beta_j(a)(b-a)^{j-k},\ -\sum_{j=k}^{n-3}\alpha_j(a)(b-a)^{j-k}\right\}. \tag{1.3.10}$$

Let u be a solution of (1.3.3)-(1.3.4).

Assume, by contradiction, that there exists $x\in I$ such that $\left|u^{(n-2)}(x)\right|\geq r_{n-2}$. Consider the case $u^{(n-2)}(x)\geq r_{n-2}$ and define

$$\max_{x\in I}u^{(n-2)}(x):=u^{(n-2)}(x_0)\qquad(\geq r_{n-2}>0).$$

If $x_0\in\,]a,b[$, then $u^{(n-1)}(x_0)=0$ and $u^{(n)}(x_0)\leq 0$. By (1.3.2),(1.3.6) and (1.3.8), for $\lambda\in[0,1]$ the following contradiction holds

$$\begin{aligned}
0&\geq u^{(n)}(x_0)\\
&=\lambda f\left(x_0,\delta_0(x_0,u(x_0)),...,\delta_{n-2}\left(x_0,u^{(n-2)}(x_0)\right),u^{(n-1)}(x_0)\right)\\
&+u^{(n-2)}(x_0)-\lambda\delta_{n-2}\left(x_0,u^{(n-2)}(x_0)\right)\\
&\geq\lambda\left[f\left(x_0,\beta_0(x_0),...,\beta^{(n-2)}(x_0),0\right)-\beta^{(n-2)}(x_0)+r_{n-2}\right]+u^{(n-2)}(x_0)-r_{n-2}\\
&>0
\end{aligned}$$

If $x_0=a$ then

$$\max_{x\in I}u^{(n-2)}(x):=u^{(n-2)}(a).$$

By (1.3.4),

$$0 \geq u^{(n-1)}(a) = u^{(n-1)}(b) \geq 0,$$

therefore $u^{(n-1)}(a) = 0$ and $u^{(n)}(a) \leq 0$. Applying the same technique as above, replacing x_0 by a, a similar contradiction is achieved.

The case $x_0 = b$ is analogous and so $u^{(n-2)}(x) < r_{n-2}$, for every $x \in I$. As the inequality $u^{(n-2)}(x) > -r_{n-2}$, for every $x \in I$, can be proved by the same arguments, then

$$\left| u^{(n-2)}(x) \right| < r_{n-2}, \ \forall x \in I.$$

By integration in $[a, x]$, (1.3.4) and (1.3.5),

$$\begin{aligned} u^{(n-3)}(x) &< u^{(n-3)}(a) + r_{n-2}(x-a) \\ &= \lambda \eta_{n-3}\left(u^{(n-3)}(b)\right) + r_{n-2}(x-a) \\ &\leq \lambda \beta_{n-3}(a) + r_{n-2}(b-a) \\ &\leq \beta^{(n-3)}(a) + r_{n-2}(b-a). \end{aligned}$$

and

$$\begin{aligned} u^{(n-3)}(x) &> u^{(n-3)}(a) - r_{n-2}(x-a) \\ &\geq \lambda \alpha_{n-3}(a) - r_{n-2}(b-a) \\ &\geq \alpha^{(n-3)}(a) - r_{n-2}(b-a). \end{aligned}$$

Applying similar arguments it can be proved that, for $k = 0, ..., n-3$,

$$\begin{aligned} &\sum_{j=k}^{n-3} \alpha_j(a)(b-a)^{j-k} - r_{n-2}(b-a)^{n-2-k} \\ \leq\ & u^{(i)}(x) \leq \sum_{j=k}^{n-3} \beta_j(a)(b-a)^{j-k} + r_{n-2}(b-a)^{n-2-k}. \end{aligned}$$

Defining

$$\xi_k := \max\left\{ \sum_{j=k}^{n-3} \beta_j(a)(b-a)^{j-k}, \ -\sum_{j=k}^{n-3} \alpha_j(a)(b-a)^{j-k} \right\}$$

and

$$r_k = \xi_k + r_{n-2}\,(b-a)^{n-2-k}\,,$$

than

$$\left|u^{(i)}(x)\right| < r_i,\ \ i = 0,...,n-2.$$

with r_k given by (1.3.9), for $k = 0,...,n-3$.

Step 2: *There exists $R > 0$ such that every solution u of problem (1.3.3)-(1.3.4) satisfies*

$$\left|u^{(n-1)}(x)\right| < R, \quad \forall x \in I,$$

independently of $\lambda \in [0,1]$.

For r_i, $i = 0,...,n-2$, given in the previous step, consider the set

$$E_1 = \{(x, y_0, ..., y_{n-1}) \in I \times \mathbb{R}^n : -r_i \leq y_i \leq r_i,\ \ i = 0,1,...,n-2\}$$

and the function $F_\lambda : E_1 \to \mathbb{R}$ given by

$$\begin{aligned} F_\lambda\,(x, y_0, ..., y_{n-1}) &= \lambda f\,(x, \delta_0\,(x, y_0)\,, ..., \delta_{n-2}\,(x, y_{n-2})\,, y_{n-1}) \\ &\quad + y_{n-2} - \lambda \delta_{n-2}\,(x, y_{n-2})\,. \end{aligned} \tag{1.3.11}$$

Considering $\alpha_i := -r_i$ and $\beta_i := r_i$, then f satisfies a Nagumo-type condition in E_1, consider the function $h_{E_1} \in C\left(\mathbb{R}_0^+, [k, +\infty[\right)$ for some $k > 0$, such that (1.2.3) and (1.2.4) hold by E replaced by E_1. Thus, for $(x, y_0, ..., y_{n-1}) \in E_1$, we have, by (1.3.3) and (1.3.6),

$$F_\lambda\,(x, y_0, ..., y_{n-1}) \leq h_{E_1}\,(|y_{n-1}|) + 2\ r_{n-2}.$$

For this function F_λ define $h_{E_1}\,(w) := h_{E_*}\,(|w|) + 2\ r_{n-2}$, therefore

$$\int_0^{+\infty} \frac{s}{h_{E_1}\,(s)} ds = \int_0^{+\infty} \frac{s}{h_{E_*}\,(|s|) + 2\ r_{n-2}} ds \geq \frac{1}{1 + \frac{2\ r_{n-2}}{k}} \int_0^{+\infty} \frac{s}{h_{E_*}\,(|s|)} ds,$$

and so $h_{E_1}(w)$ verifies (1.2.4), that is, F_λ satisfies the Nagumo condition in E_1 with $h_{E*}(w)$ replaced by $h_{E_1}(w)$, independently of λ.

Defining

$$\gamma_i(x) := -r_i, \quad \Gamma_i(x) := r_i, \quad i = 0, ..., n-2$$

the assumptions of Lemma 1.2.2 are satisfied with E replaced with E_1. So there exists $R > 0$, depending only on r_i, $i = 0, ..., n-2$, and φ, such that

$$\left|u^{(n-1)}(x)\right| < R, \ \forall x \in I.$$

Step 3: *For $\lambda = 1$ the problem (1.3.3)-(1.3.4) has a solution $u_1(x)$.*

Consider the operators

$$\mathcal{L} : C^n(I) \times \mathbb{R}^n \subset C^{n-1}(I) \to C(I) \times \mathbb{R}^n$$

and, for $\lambda \in [0,1]$,

$$\mathcal{N}_\lambda : C^{n-1}(I) \to C(I) \times \mathbb{R}^n$$

where

$$\mathcal{L}u = \left(u^{(n)} - u^{(n-2)}, u(a), ..., u^{(n-1)}(a)\right)$$

and

$$\mathcal{N}_\lambda u = \begin{pmatrix} \lambda f\left(x, \delta_0(x, u(x)), ..., \delta_{n-2}\left(x, u^{(n-2)}(x)\right), u^{(n-1)}(x)\right) \\ +u^{(n-2)}(x) - \lambda\delta_{n-2}\left(x, u^{(n-2)}(x)\right), \\ \lambda\eta_0(u(b)), ..., \lambda\eta_{n-3}\left(u^{(n-3)}(b)\right), u^{(n-2)}(b), u^{(n-1)}(b) \end{pmatrix}.$$

As $\mathcal{L}$ has a compact inverse it can be considered the completely continuous operator

$$\mathcal{T}_\lambda : \left(C^{n-1}(I), \mathbb{R}\right) \to \left(C^{n-1}(I), \mathbb{R}\right)$$

defined by

$$\mathcal{T}_\lambda(u) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u).$$

For R given by Step 2, consider the set

$$\Omega = \left\{ y \in C^{n-1}(I) : \left\| y^{(i)} \right\|_\infty < r_i,\ i = 0, ..., n-2,\ \left\| y^{(n-1)} \right\|_\infty < R \right\}.$$

By Steps 1 and 2, for every u solution of (1.3.3)-(1.3.4), $u \notin \partial\Omega$ and so the degree $d(\mathcal{I} - \mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0,1]$. By the invariance under homotopy

$$\pm 1 = d(\mathcal{I} - \mathcal{T}_0, \Omega, 0) = d(\mathcal{I} - \mathcal{T}_1, \Omega, 0).$$

Thus the equation $\mathcal{T}_1(x) = x$, equivalent to the problem given by the equation

$$\begin{aligned} u^{(n)}(x) &= f\left(x, \delta_0(x, u(x)), ..., \delta_{n-2}\left(x, u^{(n-2)}(x)\right), u^{(n-1)}(x)\right) + u^{(n-2)}(x) \\ &\quad -\delta_{n-2}\left(x, u^{(n-2)}(x)\right), \end{aligned}$$

coupled with the boundary conditions

$$\begin{aligned} u^{(k)}(a) &= \eta_k\left(u^{(k)}(b)\right), \quad k = 0, 1..., n-3, \\ u^{(n-2)}(a) &= u^{(n-2)}(b) \\ u^{(n-1)}(a) &= u^{(n-1)}(b), \end{aligned}$$

has at least a solution $u_1(x)$ in Ω.

Step 4: *$u_1(x)$ is a solution of (1.1.1)-(1.1.2).*

This solution $u_1(x)$ is a solution of (1.1.1)-(1.1.2) if it verifies

$$\begin{aligned} \alpha^{(n-2)}(x) \le u_1^{(n-2)}(x) &\le \beta^{(n-2)}(x), \qquad (1.3.12) \\ \alpha_i(x) \le u_1^{(i)}(x) \le \beta_i(x),\ i &= 0, 1..., n-3, \forall x \in I. \end{aligned}$$

Suppose, by contradiction, that there is $x \in I$ such that

$$\alpha^{(n-2)}(x) > u_1^{(n-2)}(x)$$

and define

$$\min_{x\in I}\left[u_1^{(n-2)}(x)-\alpha^{(n-2)}(x)\right]:=u_1^{(n-2)}(x_1)-\alpha^{(n-2)}(x_1)<0.$$

If $x_1\in\,]a,b[$, then $u_1^{(n-1)}(x_1)-\alpha^{(n-1)}(x_1)=0$ and $u_1^{(n)}(x_1)-\alpha^{(n)}(x_1)\geq 0$. Therefore, by (1.3.2) and Definition 1.2.3, we obtain the following contradiction

$$\begin{aligned}
0 &\leq u_1^{(n)}(x_1)-\alpha^{(n)}(x_1) \qquad (1.3.13)\\
&\leq f\left(x_1,\delta_0(x_1,u_1(x_1)),...,\delta_{n-3}\left(x_1,u_1^{(n-3)}(x_1)\right),\alpha^{(n-2)}(x_1),\alpha^{(n-1)}(x_1)\right)\\
&\quad +u^{(n-2)}(x_1)-\alpha^{(n-2)}(x_1)\\
&\quad -f\left(x_1,\alpha_0(x_1),...,\alpha_{n-3}(x_1),\alpha^{(n-2)}(x_1),\alpha^{(n-1)}(x_1)\right)\\
&\leq u^{(n-2)}(x_1)-\alpha^{(n-2)}(x_1)<0.
\end{aligned}$$

If $x_1=a$ then

$$\min_{x\in I}\left[u_1^{(n-2)}(x)-\alpha^{(n-2)}(x)\right]:=u_1^{(n-2)}(a)-\alpha^{(n-2)}(a)<0.$$

By Definition 1.2.3

$$0\leq u_1^{(n-1)}(a)-\alpha^{(n-1)}(a)\leq u_1^{(n-1)}(b)-\alpha^{(n-1)}(b)\leq 0$$

and, therefore,

$$u_1^{(n-1)}(a)=\alpha^{(n-1)}(a),\quad u_1^{(n)}(a)\geq\alpha^{(n)}(a).$$

The case where $x_1=b$ the proof is identical and so

$$\alpha^{(n-2)}(x)\leq u_1^{(n-2)}(x),\quad \forall x\in I.$$

Applying the same arguments, one can verify that $u_1^{(n-2)}(x)\leq\beta^{(n-2)}(x)$, for every $x\in I$, and (1.3.12) holds.

Integrating (1.3.12) in $[a, x]$, by (1.3.5) and (1.2.7)

$$\begin{aligned} u_1^{(n-3)}(x) &\geq u_1^{(n-3)}(a) + \alpha^{(n-3)}(x) - \alpha^{(n-3)}(a) \\ &\geq \alpha_{n-3}(a) + \alpha^{(n-3)}(x) - \alpha^{(n-3)}(a) \\ &= \alpha^{(n-3)}(x) \geq \alpha^{(n-3)}(x) - \left\|\alpha^{(n-3)}\right\|_\infty = \alpha_{n-3}(x). \end{aligned}$$

Analogously, by (1.3.5) and (1.2.8),

$$\begin{aligned} u_1^{(n-3)}(x) &\leq u_1^{(n-3)}(a) + \beta^{(n-3)}(x) - \beta^{(n-3)}(a) \\ &\leq \beta_{n-3}(a) + \beta^{(n-3)}(x) - \beta^{(n-3)}(a) \\ &= \beta^{(n-3)}(x) \leq \beta^{(n-3)}(x) + \left\|\beta^{(n-3)}\right\|_\infty = \beta_{n-3}(x), \end{aligned}$$

and, therefore,

$$\alpha_{n-3}(x) \leq u_1^{(n-3)}(x) \leq \beta_{n-3}(x), \quad \forall x \in I.$$

By integration and using the same technique it can be proved that

$$\alpha_i(x) \leq u_1^{(i)}(x) \leq \beta_i(x),$$

for $i = 0, 1, ..., n-3$ and $x \in I$. ■

1.4 Examples

In the literature n^{th} order periodic boundary value problems with fully differential equations are often considered only for n even or n odd, like it can be seen in [56]. So, we introduce two examples, including the odd and even cases.

Example 1.4.1 *Consider the fifth order fully differential equation*

$$\begin{aligned} u^{(v)}(x) &= -\arctan(u(x)) - \frac{(u'(x))^3}{7} - \frac{(u''(x))^5}{8} + \frac{(u'''(x))^6}{12} \\ &\quad + \left(u^{(iv)}(x) + 12\right)^{\frac{2}{3}} - 500, \end{aligned} \tag{1.4.1}$$

for $x \in [0,1]$ *, with the boundary conditions*

$$u^{(i)}(0) = u^{(i)}(1), \ i = 0,1,2,3,4. \tag{1.4.2}$$

The functions $\alpha, \beta : [0,1] \to \mathbb{R}$ *given by*

$$\begin{aligned} \alpha(x) &= -\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{6} + \frac{5}{2}x^2 + x + 1, \\ \beta(x) &= \frac{x^5}{5} - \frac{x^4}{2} + 6x^3 + 12x - 1 \end{aligned}$$

are non-ordered lower and upper solutions, respectively, of problem (1.4.1)-(1.4.2) verifying (1.3.1) for $n = 5$*, with the following auxiliary functions*

$$\begin{aligned} \alpha_0(x) &= -\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{6} - \frac{11}{2}x^2 - \frac{13}{2}x - \frac{119}{30}, \\ \alpha_1(x) &= -x^4 + 2x^3 + \frac{x^2}{2} - 3x - \frac{13}{2}, \\ \alpha_2(x) &= -4x^3 + 6x^2 + x - 3, \end{aligned}$$

and

$$\begin{aligned} \beta_0(x) &= \frac{x^5}{5} - \frac{x^4}{2} + 6x^3 + 34x^2 + 41x + \frac{157}{10}, \\ \beta_1(x) &= x^4 - 2x^3 + 18x^2 + 34x + 41, \\ \beta_2(x) &= 4x^3 - 6x^2 + 36x + 34. \end{aligned}$$

Figures 1.4.1 and 1.4.2 illustrate lower and upper solutions and the auxiliary functions.

The function

$$f(x, y_0, y_1, y_2, y_3, y_4) = -\arctan y_0 - \frac{(y_1)^3}{7} - \frac{(y_2)^5}{8} + \frac{(y_3)^6}{12} + (y_4 + 12)^{\frac{2}{3}} - 500$$

is continuous, verifies conditions (1.2.3) and (1.2.4) in

$$E = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3, y_4) \in I \times \mathbb{R}^5 : \alpha_i \le y_i \le \beta_i, \ i = 0,1,2, \\ \alpha''' \le y_3 \le \beta''' \end{array} \right\},$$

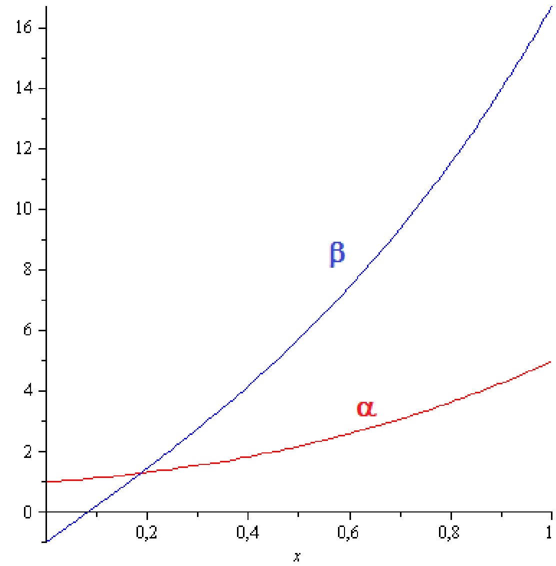

Figure 1.4.1: Functions $\alpha(x)$ and $\beta(x)$ are non-ordered.

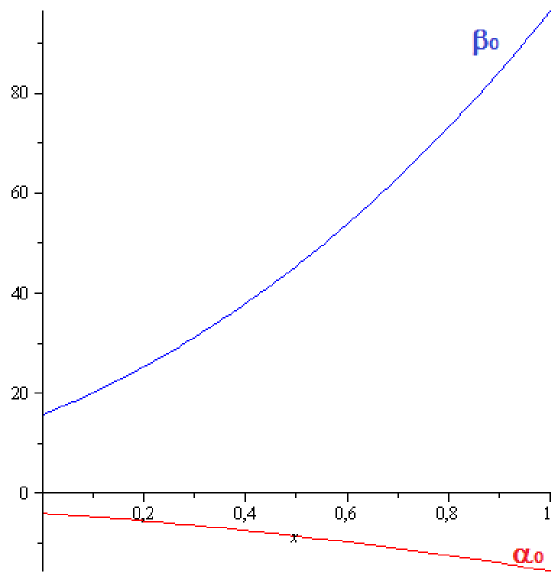

Figure 1.4.2: Functions $\alpha_0(x)$ and $\beta_0(x)$ are well ordered.

with

$$h_E(|y_4|) = 4.6 \times 10^7 + \frac{\pi}{2} + (y_4 + 12)^{\frac{2}{3}}$$

and it satisfies (1.3.2).

By Theorem 1.3.1 there is a non trivial periodic solution $u(x)$ *of problem (1.4.1)-(1.4.2), such that*

$$\begin{aligned} \alpha_i(x) &\leq u^{(i)}(x) \leq \beta_i(x), \quad \textit{for } i = 0, 1, 2, \\ -12x^2 + 12x + 1 &\leq u'''(x) \leq 12x^2 - 12x + 36, \quad \textit{for } x \in [0, 1]. \end{aligned}$$

Remark that this solution is a non trivial periodic one because a constant function cannot be a solution of (1.4.1). Moreover, despite α *and* β *are non-ordered, as shown in Figure 1.4.1, the solution* $u(x)$ *for the problem (1.1.1)-(1.1.2) exists within the area delimited by the well ordered functions,* α_0 *and* β_0.

Example 1.4.2 *For* $x \in [0, 1]$ *consider the sixth order differential equation*

$$\begin{aligned} u^{(vi)}(x) &= -(u(x))^3 - \arctan(u'(x)) - (u''(x))^3 \\ &\quad - \exp(u'''(x)) + 50 \left(u^{(iv)}(x)\right)^5 + \left|u^{(v)}(x) + 1\right|^\theta + 2, \end{aligned} \tag{1.4.3}$$

with $0 < \theta \leq 2$*, along with the boundary conditions*

$$u^{(i)}(0) = u^{(i)}(1), \; i = 0, 1, ..., 5. \tag{1.4.4}$$

The functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ *given by*

$$\begin{aligned} \alpha(x) &= -\tfrac{x^4}{4} + 1 \\ \beta(x) &= \tfrac{x^4}{4} - 1 \end{aligned}$$

are lower and upper solutions, respectively, of problem (1.4.3)-(1.4.4) verify-

ing (1.3.1) for $n = 6$ with the auxiliary functions given by Definition 1.2.3

$$\begin{aligned} \alpha_0(x) &= -\tfrac{x^4}{4} - 6x^3 - 3x^2 - x, \\ \alpha_1(x) &= -x^3 - 6x^2 - 3x - 1, \\ \alpha_2(x) &= -3x^2 - 6x - 3, \\ \alpha_3(x) &= -6x - 6 \end{aligned}$$

and

$$\begin{aligned} \beta_0(x) &= \tfrac{x^4}{4} + 6x^3 + 3x^2 + x, \\ \beta_1(x) &= x^3 + 6x^2 + 3x + 1, \\ \beta_2(x) &= 3x^2 + 6x + 3, \\ \beta_3(x) &= 6x + 6. \end{aligned}$$

The function

$$\begin{aligned} f(x, y_0, y_1, y_2, y_3, y_4, y_5) \; = \; & -y_0^3 - \arctan(y_1) - (y_2)^3 - \exp(y_3) + 50 \;\; (y_4)^5 \\ & + |y_5 + 1|^\theta + 2 \end{aligned}$$

is continuous, verifies conditions (1.2.3) and (1.2.4) in

$$E = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3, y_4, y_5) \in [a,b] \times \mathbb{R}^5 : \alpha_i \le y_i \le \beta_i, \;\; i = 0,1,2,3, \\ \alpha^{(iv)} \le y_4 \le \beta^{(iv)} \end{array} \right\},$$

with

$$h_E(|y_5|) = 41 + \frac{\pi}{2} + e^2 + |y_5 + 1|^\theta$$

and satisfies (1.3.2).

By Theorem 1.3.1 there is a solution $u(x)$ of problem (1.4.3)-(1.4.4), such that

$$\begin{gathered} -\tfrac{x^4}{4} - 6x^3 - 3x^2 - x \le u(x) \le \tfrac{x^4}{4} + 6x^3 + 3x^2 + x \\ -x^3 - 6x^2 - 3x - 1 \le u'(x) \le x^3 + 6x^2 + 3x + 1 \\ -3x^2 - 6x - 3 \le u''(x) \le 3x^2 + 6x + 3 \\ -6x - 6 \le u'''(x) \le 6x + 6 \\ -6 \le u^{(iv)}(x) \le 6 \end{gathered}$$

As seen in the previous example, the functions α_i, β_i *for* $i = 0, 1, 2, 3$ *are well ordered despite the functions* $\alpha(x)$ *and* $\beta(x)$ *are not ordered, for* $x \in [0, 1]$.

Moreover this solution is non trivial because the unique constant function solution of (1.4.3), $u = \sqrt[3]{3}$, *is not in the set* $[\alpha_0, \beta_0]$.

Chapter 2

New trends on Lidstone problems

2.1 Introduction

Fourth order differential equations are often said as beam equations due to their relevance in beam theory, namely in the study of the bending of an elastic beam. In this chapter it is considered the nonlinear fully equation

$$u^{(iv)}(x) + f(x, u(x), u'(x), u''(x), u'''(x)) = sp(x) \tag{2.1.1}$$

for $x \in [0,1]$, where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ and $p : [0,1] \to \mathbb{R}^+$ are continuous functions and s a real parameter. Problems involving these types of equations are known as Ambrosetti-Prodi problems, as they were introduced in [2]. In fact they provide the discussion of existence, nonexistence and multiplicity results on the parameter s. More precisely, sufficient conditions, for the existence of s_0 and s_1, are obtained, such that:

- if $s < s_0$, the problem has no solution.
- if $s = s_0$, the problem has a solution.

- if $s \in]s_0, s_1]$, the problem has at least two solutions.

As boundary conditions it is considered

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{2.1.2}$$

known as Lidstone boundary conditions. They appear in several physic and engineering situations such as simply supported beams, ([48, 49]), and suspension bridges, ([24, 57]). Different boundary conditions, meaning different types of support at the endpoints, are considered in the literature. As example one can refer [48, 69, 78, 90].

Related problems have been studied by many authors, either from a variational approach, ([45, 50]), or with topological techniques, ([5, 10, 66, 88]), or both [21]. Recently, some papers applied the lower and upper solutions method to more general boundary conditions such as nonlinear, ([14, 25, 33, 78]), and functional cases, ([15, 19, 27]), some of them including the Lidstone case.

The bilateral Nagumo condition, used in some of the above papers, plays an important role to control the growth of the third derivative. In this work it is applied a more general Nagumo-type assumption: an unilateral condition. From this point of view, the results existing in the literature for problem (2.1.1)-(2.1.2), ([76, 77]), are improved, because the nonlinearity can be unbounded from above or from below, following arguments suggested by [42, 43].

It is pointed out that, for Lidstone problems, where there is no information about the third derivative on the boundary, the replacement of the bilateral condition by an unilateral one is not trivial. It requires a new *a priori* lemma and a new auxiliary problem in the proof of the main result.

In this Chapter a reflexive section about the role of lower and upper solution in the main results is included. In this section it is discussed how conditions in the lower and upper solutions definition influence the main Theorem and vice-versa. This "power shift" between the Definition and Theorem makes it possible to present some new results. Perpetuating this line of thought it will be then possible to extend some results to a functional version of (2.1.1)-(2.1.2). This functional version, and subsequent results, will later be presented in Chapter 8 as an application.

As in the previous Chapter there are still several open problems in respect of (2.1.1)-(2.1.2), namely the relaxation of the continuity condition on f, on condition (2.4.3) in Theorem 2.4.2 and generalization of these results to a n^{th} order problem. Moreover the discussion for the multiplicity part on Ambrosetti-Prodi is still open for the Lidstone problem.

2.2 Definitions and auxiliary results

In this section some auxiliar results and definitions, essential to the proof of the main result, are presented.

In Chapter 1 a generalization of a Nagumo-type condition was presented. Now it is considered an one-sided Nagumo-type condition, meaning that the function f is only limited either from above (illustrated by condition (2.2.1)) or from below (see 2.2.2). Therefore two different Lemmas can be obtained, depending on the condition assumed on the nonlinearity f.

The one-sided Nagumo-type condition to be used and the consequent *a priori* estimation are precise as it follows:

Definition 2.2.1 *Given a subset* $E \subset [0,1] \times \mathbb{R}^4$*, a continuos function* $f : E \to \mathbb{R}$ *is said to satisfy the one-sided Nagumo-type condition in* E *if there* *exists a real continuous function* $h_E : \mathbb{R}_0^+ \to [k, +\infty[$*, for some* $k > 0$*, such that*

$$f(x, y_0, y_1, y_2, y_3) \leq h_E(|y_3|), \ \forall (x, y_0, y_1, y_2, y_3) \in E \tag{2.2.1}$$

or

$$f(x, y_0, y_1, y_2, y_3) \geq -h_E(|y_3|), \ \forall (x, y_0, y_1, y_2, y_3) \in E, \tag{2.2.2}$$

with

$$\int_0^{+\infty} \frac{t}{h_E(t)} dt = +\infty. \tag{2.2.3}$$

Lemma 2.2.2 *Let* $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *be a continuous function, verifying Nagumo-type conditions (2.2.1) and (2.2.3) in*

$$E = \left\{ (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \gamma_i(x) \leq y_i \leq \Gamma_i(x), \ i = 0, 1, 2 \right\} \tag{2.2.4}$$

where $\gamma_i(x)$ *and* $\Gamma_i(x)$ *are continuous functions such that, for* $i = 0, 1, 2,$ $\gamma_i(x) \leq \Gamma_i(x)$*, for every* $x \in [0,1]$*.*

Then for every $\rho > 0$ *there is* $R > 0$ *such that every solution* $u(x)$ *of equation (2.1.1) verifying*

$$u'''(0) \geq -\rho \ , \ u'''(1) \leq \rho \tag{2.2.5}$$

and

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x), \ \forall x \in [0,1], \tag{2.2.6}$$

for $i = 0, 1, 2,$ *satisfies*

$$\|u'''\|_\infty < R.$$

Proof. Consider u a solution of the equation (2.1.1) that satisfies (2.2.5) and (2.2.6) and define the non-negative real number

$$r := \max \{\Gamma_2(1) - \gamma_2(0), \Gamma_2(0) - \gamma_2(1)\}.$$

Suppose $\rho > 0$ large enough such that for every u solution of (2.1.1) we have $|u'''(x)| \leq \rho$, for every $x \in [0,1]$, and $\rho \geq r$. If $\rho = R$ then the proof is complete.

Consider now that there is u solution of (2.1.1) and $x_0 \in [0,1]$ such that $|u'''(x_0)| > \rho$. If $|u'''(x)| > \rho$, for every $x \in [0,1]$ then, for $u'''(x) > \rho$, it is obtained the following contradiction

$$\begin{aligned}
\Gamma_2(1) - \gamma_2(0) \quad &\geq \quad u''(1) - u''(0) = \int_0^1 u'''(\tau)\, d\tau \\
&> \quad \int_0^1 \rho\, d\tau \geq \int_0^1 r\, d\tau \geq \Gamma_2(1) - \gamma_2(0).
\end{aligned}$$

The case $u'''(x) \geq -\rho$, for every $x \in [0,1]$, follows similar arguments. So there is $x \in [0,1]$ such that $|u'''(x)| \leq \rho$.

As the integrals

$$\int_0^{+\infty} \frac{t}{h_E(t)} dt \text{ and } \int_0^{+\infty} \frac{\tau}{h_E(\tau) + |s|\,\|p\|} d\tau$$

are of the same type, as long as s belongs to a bounded set, by (2.2.3), take $R_1 > \rho$ such that

$$\int_\rho^{R_1} \frac{\tau}{h_E(\tau) + |s|\,\|p\|}\, d\tau > \max_{x \in [0,1]} \Gamma_2(x) - \min_{x \in [0,1]} \gamma_2(x). \tag{2.2.7}$$

Consider $x_1 \in [0,1[$ such that $u'''(x_1) < -\rho$ or $x_1 \in]0,1]$ such that $u'''(x_1) > \rho$. In the first case let $\hat{x}_1$ be such that $0 \leq \hat{x}_1 < x_1$ and, for every $x \in [\hat{x}_1, x_1[$,

$$u'''(\hat{x}_1) = -\rho \text{ and } u'''(x) < -\rho. \tag{2.2.8}$$

By an adequate change of variable and (2.2.7),we obtain

$$\begin{aligned}
\int_{-u'''(\widehat{x}_1)}^{-u'''(x_1)} \frac{\tau}{h_E(\tau)+|s|\,\|p\|}d\tau &= \int_{\widehat{x}_1}^{x_1} \frac{-u'''(x)}{h_E(-u'''(x))+|s|\,\|p\|}.\left(-u^{(iv)}(x)\right)dx \\
&= \int_{\widehat{x}_1}^{x_1} \frac{f\left(x,u,u',u'',u'''\right)-sp\left(x\right)}{h_E(-u'''(x))+|s|\,\|p\|}\;\left(-u'''(x)\right)dx \\
&\leq \int_{\widehat{x}_1}^{x_1} -u'''(x)dx = u''(\widehat{x}_1)-u''(x_1) \\
&\leq \max_{x\in[0,1]}\Gamma_2(x)-\min_{x\in[0,1]}\gamma_2(x) \\
&< \int_{\rho}^{R_1} \frac{\tau}{h_E(\tau)+|s|\,\|p\|}d\tau
\end{aligned}$$

and therefore that $u'''(x_1) > -R_1$. By the arbitrariness of x_1, for every $x \in [0,1[$ such that $u'''\left(x\right) < -\rho$ the inequality $u'''\left(x\right) > -R_1$ holds. In a similar way it can be proved that $u'''(x_1) < R_1$ and so $|u'''(x)| \leq R_1$, for every $x \in \left[0,1\right]$.

Consider now $\rho < r$ and take $R_2 > r$ such that

$$\int_{r}^{R_2} \frac{\tau}{h_E(\tau)+|s|\,\|p\|}d\tau > \max_{x\in[0,1]}\Gamma_2(x)-\min_{x\in[0,1]}\gamma_2(x). \tag{2.2.9}$$

By (2.2.5), there is $x \in [0,1]$ such that $|u'''(x)| \leq r$. If $|u'''(x)| \leq r$, holds for every $x \in [0,1]$ then the proof is concluded. Otherwise, it can be taken $x_2 \in [0,1[$ such that $u'''\left(x_2\right) < -r$ or $x_2 \in]0,1]$ such that $u'''(x_2) > r$. In the first case consider $0 \leq \hat{x}_2 \leq x_2$ with

$$u'''(\hat{x}_2) = -r \text{ and } u'''(x) < -r,\ \forall x \in [\hat{x}_2, x_2[$$

Applying a similar method as in (2.2.8) it is obtained

$$\int_{-u'''(\hat{x}_2)}^{-u'''(x_2)} \frac{\tau}{h_E(\tau)+|s|\,\|p\|}d\tau < \int_{r}^{R_2} \frac{\tau}{h_E(\tau)+|s|\,\|p\|}d\tau$$

and so $u'''(x_2) > -R_2$. Arguing as above it can be shown that when $u'''(x_2) > r$ the inequality $u'''(x_2) < R_2$ still holds. Therefore $|u'''(x)| \leq R_2$, for every $x \in \left[0,1\right]$.

Taking $R = \max\{R_1, R_2\}$ then $|u'''(x)| \leq R$, for every $x \in [0,1]$. ■

If the function f verifies (2.2.2) the following Lemma is obtained:

Lemma 2.2.3 *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function, verifying Nagumo-type conditions (2.2.2) and (2.2.3) in E given by (2.2.4).*

Then for every $\rho > 0$ there is $R > 0$ such that every solution $u(x)$ of equation (2.1.1) verifying

$$u'''(0) \leq \rho \ , \ u'''(1) \geq -\rho. \tag{2.2.10}$$

and

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x), \ \forall x \in [0,1], \tag{2.2.11}$$

for $i = 0, 1, 2,$ satisfies

$$\|u'''\| < R.$$

Proof. The proof follows similar arguments as in the proof of Lemma 2.2.2 with the adequate changes. ■

Remark 2.2.4 *Observe that R depends only on the functions h_E, γ_2 and Γ_2 and not on the boundary conditions. Moreover if s belongs to a bounded set, then R can be considered the same, independently of s.*

The functions used as lower and upper solutions are defined as a pair:

Definition 2.2.5 *The functions $\alpha, \beta \in C^4(]0,1[) \cap C^2([0,1])$ verifying*

$$\alpha^{(i)}(x) \leq \beta^{(i)}(x), \ i = 0, 1, 2, \ \forall x \in [0,1], \tag{2.2.12}$$

define a pair of lower and upper solutions of problem (2.1.1)-(2.1.2) if the following conditions are satisfied:

(i) $\alpha^{(iv)}(x) + f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)) \geq sp(x)$,

$\beta^{(iv)}(x) + f(x, \beta(x), \beta'(x), \beta''(x), \beta'''(x)) \leq sp(x)$;

(ii) $\alpha(0) \leq 0, \quad \alpha''(0) \leq 0, \quad \alpha''(1) \leq 0,$

$\beta(0) \geq 0, \quad \beta''(0) \geq 0, \quad \beta''(1) \geq 0;$

(iii) $\alpha'(0) - \beta'(0) \leq \min\{\beta(0) - \beta(1),\ \alpha(1) - \alpha(0)\}$.

As it was shown in [77], condition (iii) can not be removed for this type of definition. However if the minimum in (iii) is non-positive then assumption (2.2.12) can be replaced by $\alpha''(x) \leq \beta''(x)$, for every $x \in [0,1]$, as the other inequalities are obtained from integration.

2.3 Existence and location result

For values of the parameter s such that there are lower and upper solutions of (2.1.1)-(2.1.2) it can be obtained the following existence and location result, where the nonlinear part can be unbounded from above or from below.

Theorem 2.3.1 *Suppose that there is a pair of lower and upper solutions of the problem (2.1.1)-(2.1.2), $\alpha(x)$ and $\beta(x)$, respectively. Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying the one-sided Nagumo conditions (2.2.1) and (2.2.3) in*

$$E_* = \left\{(x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x),\ i = 0, 1, 2\right\}$$

and

$$f(x, \alpha, \alpha', y_2, y_3) \leq f(x, y_0, y_1, y_2, y_3) \leq f(x, \beta, \beta', y_2, y_3), \qquad (2.3.1)$$

for $\alpha(x) \leq y_0 \leq \beta(x)$, $\alpha'(x) \leq y_1 \leq \beta'(x)$ *and for fixed* $(x, y_2, y_3) \in [0,1] \times \mathbb{R}^2$*. Then the problem (2.1.1)-(2.1.2) has at least a solution* $u(x) \in C^4([0,1])$*, satisfying*

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \text{ for } i = 0,1,2, \ \forall x \in [0,1].$$

Proof. Consider the continuous truncations δ_i given by

$$\delta_i(x, y_i) = \begin{cases} \alpha^{(i)}(x) & \text{if} \quad y_i < \alpha^{(i)}(x) \\ y_i & \text{if} \quad \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x) \\ \beta^{(i)}(x) & \text{if} \quad y_i > \beta^{(i)}(x) \end{cases}, \tag{2.3.2}$$

for $i = 0, 1, 2$.

For $\lambda \in [0,1]$, consider the homotopic equation

$$\begin{aligned} u^{(iv)}(x) &= \lambda \left[sp(x) - f\left(x, \delta_0(x,u), \delta_1(x,u'), \delta_2(x,u''), u'''\right)\right] \\ &\quad + u''(x) - \lambda \delta_2(x, u''), \end{aligned} \tag{2.3.3}$$

and the boundary conditions

$$\begin{aligned} u(0) = u(1) &= 0, \\ (1-\lambda) u'''(0) &= \lambda |u''(0)|, \\ (1-\lambda) u'''(1) &= -\lambda |u''(1)|. \end{aligned} \tag{2.3.4}$$

Let $r_2 > 0$ large enough, such that, for every $x \in [0,1]$,

$$-r_2 < \alpha''(x) \leq \beta''(x) < r_2, \tag{2.3.5}$$

$$sp(x) - f(x, \beta(x), \beta'(x), \beta''(x), 0) + r_2 - \beta''(x) > 0, \tag{2.3.6}$$

$$sp(x) - f(x, \alpha(x), \alpha'(x), \alpha''(x), 0) - r_2 - \alpha''(x) < 0. \tag{2.3.7}$$

The proof follows similar steps to the proof of the main result in the previous Chapter (Theorem 1.3.1), therefore only the key points of the arguments are presented:

- *Every solution $u(x)$ of the problem (2.3.3)-(2.3.4) verifies*

$$|u''(x)| < r_2, \quad |u'(x)| < r_1, \quad |u(x)| < r_{1,} \; \forall x \in [0,1],$$

with $r_1 := r_2 + u'(0)$ *independently of* $\lambda \in [0,1]$.

As for interior points the technique is identical, assuming, by contradiction, that

$$\max_{x\in[0,1]} u''(x) := u''(0) \geq r_2 > 0.$$

Then for $\lambda \in]0,1]$, it is obtained

$$0 \geq (1-\lambda)\, u'''(0) = \lambda u''(0) \geq \lambda r_2 > 0.$$

For $\lambda = 0$, $u'''(0) = 0$. Therefore $u^{(iv)}(0) \leq 0$ and the case is identically to the interior points.

If

$$\max_{x\in[0,1]} u''(x) := u''(1) \geq r_2,$$

for $\lambda \in]0,1]$, the contradiction is similar

$$0 \leq (1-\lambda)\, u'''(1) = -\lambda\, |u''(1)| \leq -\lambda r_2 < 0.$$

The case $\lambda = 0$, implies $u'''(1) = 0$ and $u^{(iv)}(1) \geq 0$ and the contradiction is obtained by the same technique as in the interior points.

The case $u''(x) \leq -r_2$ is analogous and so

$$|u''(x)| < r_2, \; \forall x \in [0,1], \forall \lambda \in [0,1].$$

Integrating in $[0,x]$, $u'(x) - u'(0) = \int_0^x u''(s)ds < r_2$, and

$$|u'(x)| < r_2 + u'(0), \; \forall x \in [0,1], \forall \lambda \in [0,1].$$

By integration, $u(x) - u(0) = \int_0^x u'(s)ds \leq \int_0^x r_1 ds \leq r_1$.

With the same arguments it can be proved that $u(x) > -r_1$ and

$$|u(x)| < r_1, \forall x \in [0,1].$$

- *There is* $R>0$ *such that, every solution* $u(x)$ *of the problem (2.3.3)-(2.3.4) verifies*

$$|u'''(x)|<R,\ \forall x\in[0,1],$$

independently of $\lambda\in[0,1]$.

- *Problem (2.3.3)-(2.3.4) has at least a solution* $u_1(x)$ *for* $\lambda=1$.

The existence of at least a solution $u_1(x)$ for problem (2.3.3)-(2.3.4) is obtained with the operators $\mathcal{L}:C^4([0,1])\subset C^3([0,1])\to C([0,1])\times\mathbb{R}^4$ given by

$$\mathcal{L}u=\left(u^{(iv)}-u'',u(0),u(1),u'''(0),u'''(1)\right),$$

$\mathcal{N}_\lambda:C^3([0,1])\to C([0,1])\times\mathbb{R}^4$ by

$$\mathcal{N}_\lambda=\left(\begin{array}{c}\lambda\left[sp(x)-f\left(x,\delta_0(x,u),\delta_1(x,u'),\delta_2(x,u''),u'''(x)\right)\right]-\lambda\delta_2(x,u''),\\ 0,0,\lambda\left[u'''(0)+|u''(0)|\right],\lambda\left[u'''(1)-|u''(1)|\right]\end{array}\right)$$

and $\mathcal{T}_\lambda:(C^4([0,1]),\mathbb{R})\to(C^4([0,1]),\mathbb{R})$ by

$$\mathcal{T}_\lambda(u)=\mathcal{L}^{-1}\mathcal{N}_\lambda(u).$$

The function $u_1(x)$ will be a solution of the initial problem (2.1.1)-(2.1.2) if it verifies $\alpha^{(i)}(x)\le u_1^{(i)}(x)\le\beta^{(i)}(x)$, $i=0,1,2,\ \forall x\in[0,1]$.

Suppose, by contradiction, that there is $x\in[0,1]$ such that $\alpha''(x)>u_1''(x)$ and define

$$\min_{x\in[0,1]}\left[u_1''(x)-\alpha''(x)\right]:=u_1''(x_1)-\alpha''(x_1)<0.$$

If $x_1\in\,]0,1[$, then $u_1'''(x_1)=\alpha'''(x_1)$ and $u^{(iv)}(x_1)\ge\alpha^{(iv)}(x_1)$.

By Definition 2.2.5 and (2.3.1) it is obtained the contradiction:

$$\begin{aligned}
\alpha^{(iv)}(x_1) &\leq u_1^{(iv)}(x_1) \\
&= sp(x_1) - f\left(x_1, \delta_0(x_1, u), \delta_1(x_1, u'), \alpha''(x_1), \alpha'''(x_1)\right) \\
&\quad + u''(x_1) - \alpha''(x_1) \\
&< sp(x_1) - f\left(x_1, \alpha(x_1), \alpha'(x_1), \alpha''(x_1), \alpha'''(x_1)\right) \leq \alpha^{(iv)}(x_1).
\end{aligned}$$

If $x_1 = 0$ or $x_1 = 1$ the contradiction is trivial, by Definition 2.2.5 (ii).

Therefore $\alpha''(x) \leq u_1''(x)$, for every $x \in [0,1]$. In a similar way it can be proved that $u_1''(x) \leq \beta''(x)$, and so

$$\alpha''(x) \leq u_1''(x) \leq \beta''(x), \quad \text{for every } x \in [0,1]. \tag{2.3.8}$$

As, by (2.1.2),

$$\begin{aligned}
0 &= \int_0^1 u_1'(x)\, dx = \int_0^1 \left(u_1'(0) + \int_0^x u_1''(s)\, ds\right) dx \\
&= u_1'(0) + \int_0^1 \int_0^x u_1''(s)\, ds\; dx
\end{aligned}$$

then

$$u_1'(0) = -\int_0^1 \int_0^x u_1''(s)\, ds dx. \tag{2.3.9}$$

By this technique

$$\int_0^1 \int_0^x \alpha''(s)\, ds\; dx = \alpha(1) - \alpha(0) - \alpha'(0)$$

and, by Definition 2.2.5 (iii), (2.3.8) and (2.3.9)

$$\begin{aligned}
-\beta'(0) &\leq \alpha(1) - \alpha(0) - \alpha'(0) = \int_0^1 \int_0^x \alpha''(s)\, ds\; dx \\
&\leq \int_0^1 \int_0^x u_1''(s)\, ds\; dx = -u_1'(0).
\end{aligned}$$

Therefore $u_1'(0) \leq \beta'(0)$ and, by integration of (2.3.5), one obtains

$$u_1'(x) - u_1'(0) = \int_0^x u_1''(s)\, ds \leq \int_0^x \beta''(s)\, ds = \beta'(x) - \beta'(0)$$

and

$$u_1'(x) \leq \beta'(x) - \beta'(0) + u_1'(0) \leq \beta'(x), \forall x \in [0,1].$$

The relation $\alpha'(x) \leq u_1'(x)$, for every $x \in [0,1]$, can be proved by similar arguments. Then $\alpha'(x) \leq u_1'(x) \leq \beta'(x)$, for every $x \in [0,1]$. By Definition 2.2.5 (ii)

$$\begin{aligned}\alpha(x) &\leq \int_0^x \alpha'(s)\,ds \leq \int_0^x u_1'(s)\,ds = u_1(x) \\ &\leq \int_0^x \beta'(s)\,ds = \beta(x) - \beta(0) \leq \beta(x).\end{aligned}$$

Therefore $u_1(x)$ is a solution for problem (2.1.1)-(2.1.2). ■

Remark 2.3.2 *Theorem 2.3.1 still holds if condition (2.2.1) is replaced by (2.2.2) and conditions (2.3.4) are replaced by*

$$\begin{aligned}u(0) = u(1) &= 0, \\ (1-\lambda)\,u'''(0) &= -\lambda\,|u''(0)|, \\ (1-\lambda)\,u'''(1) &= \lambda\,|u''(1)|.\end{aligned}$$

2.4 Generalized lower and upper solutions

When looking at the definition of lower and upper solution one can wonder about its impact and importance in the existence and location results presented in these previous chapters and throughout all the thesis.

It is immediate that they provide a very graphical information about some qualitative properties of the solution, but one can ask how deep is their influence in the final results, for instance, in Definition 2.2.5 is it possible to relax condition (2.2.12) and condition iii)? How does this change affects the final result?

With this thought in mind we consider the following definitions for lower and upper solutions:

Definition 2.4.1 *Functions* $\alpha, \beta \in C^4\left(]0,1[\right) \cap C^2\left([0,1]\right)$ *are a pair of lower and upper solutions of (2.1.1)-(2.1.2) if the following conditions are satisfied:*

(i) $\alpha^{(iv)}(x) + f\left(x, \alpha_0(x), \alpha_1(x), \alpha''(x), \alpha'''(x)\right) \geq sp(x),$

where

$$\begin{array}{c} \alpha_0(x) = \int_0^x \alpha_1(s)\, ds, \\ \alpha_1(x) = \alpha'(x) - \alpha'(0) - \int_0^1 \int_0^x |\beta''(s)|\, ds dx; \end{array} \tag{2.4.1}$$

(ii) $\alpha''(0) \leq 0, \;\; \alpha''(1) \leq 0;$

(iii) $\beta^{(iv)}(x) + f\left(x, \beta_0(x), \beta_1(x), \beta''(x), \beta'''(x)\right) \leq sp(x),$

where

$$\begin{array}{c} \beta_0(x) = \int_0^x \beta_1(s)\, ds, \\ \beta_1(x) = \beta'(x) - \beta'(0) + \int_0^1 \int_0^x |\alpha''(s)|\, ds dx; \end{array} \tag{2.4.2}$$

(iv) $\beta''(0) \geq 0, \;\; \beta''(1) \geq 0.$

Now, the main existence and location result becomes:

Theorem 2.4.2 *Suppose that there is a pair of lower and upper solutions of the problem (2.1.1)-(2.1.2),* $\alpha(x)$ *and* $\beta(x)$*, respectively verifying*

$$\alpha''(x) \leq \beta''(x), \;\; \forall x \in [0,1]. \tag{2.4.3}$$

Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *be a continuous function satisfying the one-sided Nagumo conditions (2.2.1), or (2.2.2), and (2.2.3) in*

$$E_* = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \alpha_0(x) \leq y_0 \leq \beta_0(x), \\ \alpha_1(x) \leq y_1 \leq \beta_1(x), \;\; \alpha''(x) \leq y_2 \leq \beta''(x) \end{array} \right\}$$

and

$$f\left(x, \alpha_0, \alpha_1, y_2, y_3\right) \leq f\left(x, y_0, y_1, y_2, y_3\right) \leq f\left(x, \beta_0, \beta_1, y_2, y_3\right), \tag{2.4.4}$$

for $\alpha_0(x) \le y_0 \le \beta_0(x)$, $\alpha_1(x) \le y_1 \le \beta_1(x)$ *and for fixed* $(x, y_2, y_3) \in [0,1] \times \mathbb{R}^2$. *Then the problem (2.1.1)-(2.1.2) has at least a solution* $u(x) \in C^4([0,1])$, *satisfying*

$$\alpha_i(x) \le u^{(i)}(x) \le \beta_i(x), \text{ for } i = 0, 1, \ \forall x \in [0,1],$$

and

$$\alpha''(x) \le u''(x) \le \beta''(x), \ \forall x \in [0,1].$$

Proof. The arguments are similar to the proof of Theorem 2.3.1. So we only prove that the solution $u_1(x)$ of the modified problem will be a solution of the initial problem (2.1.1)-(2.1.2). For that it is sufficient to show that

$$\alpha''(x) \le u''(x) \le \beta''(x) \tag{2.4.5}$$

and

$$\alpha_i(x) \le u^{(i)}(x) \le \beta_i(x), \text{ for } i = 0, 1, \tag{2.4.6}$$

for every $x \in [0,1]$.

The inequalities (2.4.5) and (2.4.6) can be proved as in Theorem 2.3.1.

By integration,

$$u'(x) - u'(0) = \int_0^x u''(s)\,ds \le \int_0^x \beta''(s)\,ds = \beta'(x) - \beta'(0)$$

and it is obtained

$$u'(x) \le \beta'(x) - \beta'(0) + u'(0). \tag{2.4.7}$$

Furthermore by (2.1.2)

$$0 = \int_0^1 u'(x)\,dx \le u'(0) + \int_0^1 \int_0^x \beta''(s)\,dsdx.$$

Hence it is obtained

$$u'(0) \ge -\int_0^1 \int_0^x \beta''(s)\,dsdx$$

and in a similar way $u'(0) \leq -\int_0^1 \int_0^x \alpha''(s)\, dsdx$. Applying this in (2.4.7)

$$\begin{aligned} u'(x) &\leq \beta'(x) - \beta'(0) - \int_0^1 \int_0^x \alpha''(s)\, dsdx \\ &\leq \beta'(x) - \beta'(0) + \int_0^1 \int_0^x |\alpha''(s)|\, dsdx = \beta_1(x). \end{aligned}$$

Using the same arguments it is proved that

$$\alpha_1(x) \leq u'(x) \leq \beta_1(x), \ \forall x \in [0,1].$$

Integrating the previous inequality one obtains

$$\alpha_0(x) = \int_0^x \alpha_1(s) \leq u(x) \leq \int_0^x \beta_1(s)\, ds = \beta_0(x), \ \forall x \in [0,1].$$

■

As one can notice the inclusion of the auxiliary functions α_0, β_0 and α_1, β_1 allows not only the usage of non-ordered lower and upper solutions, increasing the range of admissible lower and upper solutions for the problem (2.1.1)-(2.1.2), but also to overcome the order relation between the first derivatives, where there is no information.

Next example illustrates a set of lower and upper solutions that were not covered by Definition 2.2.5 and Theorem 2.3.1 but are now included in Definition 2.4.1 and Theorem 2.4.2. In this example lower and upper solutions are not ordered and condition (iii) from Definition 2.2.5 is eliminated, case that was not possible by Definition 2.2.5 and Theorem 2.3.1.

Example 2.4.3 *For $x \in [0,1]$ consider the differential equation*

$$u^{(iv)}(x) + e^{u(x)} + \arctan(u'(x)) - (u''(x))^3 - |u'''(x)|^k = sp(x), \quad (2.4.8)$$

with $p : [0,1] \to \mathbb{R}^+$ a continuous function and $k \in [0,2]$, along with the boundary conditions (2.1.2).

The functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ given by

$$\alpha(x) = -x^2 + \tfrac{1}{2}$$
$$\beta(x) = x^2 - \tfrac{1}{2}$$

are lower and upper solutions, respectively, of problem (2.4.8),(2.1.2) verifying (2.4.3) with the auxiliary functions given by Definition 2.4.1

$$\alpha_0(x) = -x^2 - x,$$
$$\alpha_1(x) = -2x - 1,$$

and

$$\beta_0(x) = x^2 + x,$$
$$\beta_1(x) = 2x + 1,$$

for

$$\frac{e^2 + \arctan(3) - 8}{\max\limits_{x\in[0,1]} p(x)} \le s \le \frac{e^{-2} - \arctan(3) + 8}{\max\limits_{x\in[0,1]} p(x)}$$

The function

$$f(x, y_0, y_1, y_2, y_3) = e^{y_0} + \arctan(y_1) - (y_2)^3 - |y_3|^k \tag{2.4.9}$$

is continuous, verifies conditions (2.2.1)and (2.2.3) in

$$E = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3) \in [a,b] \times \mathbb{R}^5 : \alpha_i \le y_i \le \beta_i,\ i = 0,1 \\ \alpha'' \le y_2 \le \beta'' \end{array} \right\}.$$

and satisfies (2.4.4).

By Theorem 2.4.2 there is a solution $u(x)$ of problem (2.4.8),(2.1.2), such that

$$-x^2 - x \le u(x) \le x^2 + x$$
$$-2x - 1 \le u'(x) \le 2x + 1$$
$$-2 \le u''(x) \le 2$$

Notice that the nonlinearity f given by (2.4.9) does not verify the two-sided Nagumo type conditions and, therefore, [77] can not be applied to

(2.4.8)-(2.1.2). In fact, suppose by contradiction that there are a set E and a positive function φ such that $|f(x, y_0, y_1, y_2, y_3)| \leq \varphi(|y_3|)$ in E and

$$\int_0^{+\infty} \frac{s}{\varphi(s)} = +\infty.$$

Consider, in particular, that

$$f(x, y_0, y_1, y_2, y_3) \leq \varphi(|y_3|), \quad \forall (x, y_0, y_1, y_2, y_3) \in E$$

and $(0, 0, 0, y_3) \in E$. So, for $x \in [0, 1]$, $y_0 = 0$, $y_1 = 0$, $y_2 = 0$ and $y_3 \in \mathbb{R}^+$,

$$f(x, 0, 0, 0, y_3) = 1 + |y_3|^k \leq \varphi(|y_3|).$$

As $\int_0^{+\infty} \frac{s}{1+s^k} ds$, is finite, then the following contradiction is obtained:

$$+\infty > \int_0^{+\infty} \frac{s}{1+s^k} ds \geq \int_0^{+\infty} \frac{s}{\varphi(s)} ds = +\infty.$$

Chapter 3

Multiplicity of solutions

3.1 Introduction

In this Chapter two different sets of boundary conditions are considered. Firstly it is considered the problem composed by

$$u^{(iv)}(x) + f(x, u(x), u'(x), u''(x), u'''(x)) = sp(x), \tag{3.1.1}$$

and

$$u(1) = u'(1) = u''(0) = u'''(1) = 0,$$

with a bilateral Nagumo type condition and secondly with

$$u(a) = A, \ u'(a) = B, \ u'''(a) = C, \ u'''(b) = D,$$

with $A, B, C, D \in \mathbb{R}$ assuming an one-sided Nagumo type condition. In both cases existence, nonexistence and multiplicity results will be presented.

The arguments used were suggested by several papers namely [30], applied to second order periodic problems, [73, 89], to third order separated boundary value problems, [21] for incomplete fourth order equations with two-point boundary conditions. In short, the method makes use of Nagumo-type growth conditions, lower and upper solutions technique for higher order boundary value problems, suggestete, for example in ([33, 44, 78]), and degree theory, [70].

Last section contains an application of the beam theory to the London Millennium footbridge. During the opening day some unexpected lateral movements occurred as pedestrians crossed the bridge. This lateral movement was then found to be related with the lateral loads and the number of pedestrians [23]. Lower and upper solutions method used to obtain Ambrosetti-Prodi results is particularly well adapted for these applications. In fact it provides not only lower and upper bounds for the beam displacement under transverse and axial loads, but also it gives information on the range for the values of s where it can be obtained the existence, nonexistence or the multiplicity of solutions. Designating s the number of pedestrians, this number can then be easily bounded, making this method a sharp tool for some applications where bounds on the solution or its derivatives are important.

Several Ambrosetti-Prodi type problems remain open, for example, in the case of multiplicity of solutions, the Lidstone problems. Some experiments trying to overcome these issues were made with new types of lower and upper solutions and more general assumptions on f and can be seen in Section 8.5, dedicated to the Lidstone case. In our point of view, obtaining multiple solutions by this method has two weak points:

- to know *a priori* some bounds for the second derivative of all solutions (assumed in condition (3.3.9))

- the assumption of a "speed growth" condition on f, such as (3.3.11) or an equivalent assymptotic condition used in [36], that is, consider that the perturbation of some variables are stronger than other ones.

How to overcome them is an open issue.

3.2 Existence and nonexistence results

In this section we consider the nonlinear fully equation (3.1.1) with the boundary conditions

$$u(1) = u'(1) = u''(0) = u'''(1) = 0, \tag{3.2.1}$$

which can be seen as a clamped beam at the right endpoint.

A Nagumo-type growth condition is assumed on the nonlinear part of the differential equation. The Nagumo-type condition used in this section is a particular case of the one presented in Chapter 1, for $n = 4$. As in before, this will be an essential tool to prove an *a priori* bound for the third derivative of the corresponding solutions.

Definition 3.2.1 *Given a subset $E \subset [0,1] \times \mathbb{R}^4$, a continuous function $g : E \to \mathbb{R}$ is said to satisfy the Nagumo-type condition in E if there exists a real continuous function $h_E : \mathbb{R}_0^+ \to [a, +\infty[$, for some $a > 0$, such that*

$$|g(x, y_0, y_1, y_2, y_3)| \leq h_E(|y_3|), \ \forall (x, y_0, y_1, y_2, y_3) \in E \tag{3.2.2}$$

with

$$\int_0^{+\infty} \frac{t}{h_E(t)} dt = +\infty. \tag{3.2.3}$$

Lemma 3.2.2 *Let $g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function, verifying Nagumo-type conditions (3.2.2) and (3.2.3) in*

$$E = \left\{(x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \gamma_i(x) \leq y_i \leq \Gamma_i(x), \ i = 0, 1, 2\right\}$$

where $\gamma_i(x)$ and $\Gamma_i(x)$ are continuous functions such that, for $i = 0, 1, 2$,

$$\gamma_i(x) \leq \Gamma_i(x), \ \forall x \in [0,1].$$

Then there exists $r > 0$, such that every solution $u(x)$ of equation (3.1.1) verifying

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x), \ \forall x \in [0,1],$$

for $i = 0, 1, 2$, satisfies $\|u'''\| < r$.

Proof. The arguments for this proof are a particular case of Lemma 1.2.2, for $n = 4$ considering

$$g(x, y_0, y_1, y_2, y_3) = s\ p(x) - f(x, y_0, y_1, y_2, y_3) \tag{3.2.4}$$

and $\overline{h}_E(|y_3|) := |s|\ \|p\| + h_E(|y_3|)$, as the integrals

$$\int_0^{+\infty} \frac{t}{h_E(t)} dt \ \text{ and } \int_0^{+\infty} \frac{t}{h_E(t) + |s|\ \|p\|} dt$$

are of the same kind. ■

In the forward the useful functions to define such set E will be the adequate lower and upper solutions.

Definition 3.2.3 *A function $\alpha \in C^4([0,1])$ is a lower solution of problem (3.1.1),(3.2.1) if it verifies*

$$\alpha^{(iv)}(x) + f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)) \geq sp(x) \tag{3.2.5}$$

and

$$\begin{aligned} \alpha(1) &\leq 0, \\ \alpha'(1) &\geq 0, \\ \alpha''(0) &\leq 0, \\ \alpha'''(1) &\leq 0. \end{aligned} \tag{3.2.6}$$

A function $\beta \in C^4([0,1])$ is an upper solution of problem (3.1.1),(3.2.1) if the reversed inequalities hold.

Next Theorem is not a trivial consideration of Theorem 1.3.1 for $n = 4$. In fact lower and upper solutions are not well ordered for every derivative of the corresponding solution. This is a "natural" fact as different types of support at the endpoints will cause different interactions on the complete beam structure.

For values of the parameter s such that there are lower and upper solutions of problem (3.1.1),(3.2.1) it can be obtained the following existence and location result.

Theorem 3.2.4 *Suppose that there are lower and upper solutions of (3.1.1), (3.2.1), $\alpha(x)$ and $\beta(x)$, respectively, such that*

$$\alpha''(x) \leq \beta''(x), \forall x \in [0,1]. \tag{3.2.7}$$

Let $f : [0,1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ be a continuous function verifying Nagumo-type conditions (3.2.2) and (3.2.3) in

$$E_* = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \\ \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), \ i = 0, 2, \ \beta'(x) \leq y_1 \leq \alpha'(x) \end{array} \right\}.$$

satisfying

$$\begin{aligned} f(x, \alpha(x), \alpha'(x), y_2, y_3) &\leq f(x, y_0, y_1, y_2, y_3) \\ &\leq f(x, \beta(x), \beta'(x), y_2, y_3), \end{aligned} \tag{3.2.8}$$

for fixed $(x, y_2, y_3) \in [0,1] \times \mathbb{R}^2$ and $\alpha(x) \leq y_0 \leq \beta(x), \ \beta'(x) \leq y_1 \leq \alpha'(x)$.

Then problem (3.1.1),(3.2.1) has at least a solution $u(x) \in C^4([0,1])$ and there is $N > 0$ such that

$$\begin{aligned} \alpha^{(i)}(x) &\leq u^{(i)}(x) \leq \beta^{(i)}(x), \ i = 0, 2, \\ \beta'(x) &\leq u'(x) \leq \alpha'(x), \\ |u'''(x)| &< N, \ \forall x \in [0,1]. \end{aligned}$$

Remark 3.2.5 *If there are $\alpha(x)$ and $\beta(x)$ lower and upper solutions of the problem (3.1.1),(3.2.1) for some values of s, then s belongs to a bounded set, as*

$$\begin{aligned}\alpha^{(iv)}(x) + f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)) &\leq sp(x) \\ &\leq \beta^{(iv)}(x) + f(x, \beta(x), \beta'(x), \beta''(x), \beta'''(x)),\end{aligned}$$

for every $x \in [0,1]$.

Proof. By integration of (3.2.7) in $[x,1]$ and (3.2.6) it is obtained that

$$\beta'(x) \leq \alpha'(x), \forall x \in [0,1].$$

Therefore we cand define the truncations given by (2.3.2) for $i = 0, 2$ and

$$\bar{\delta}_1(x, y_1) = \begin{cases} \alpha'(x) & \text{if} \quad y_1 > \alpha'(x) \\ y_1 & \text{if} \quad \beta'(x) \leq y_1 \leq \alpha'(x) \\ \beta'(x) & \text{if} \quad y_1 < \beta'(x) \end{cases} . \tag{3.2.9}$$

Consider for $\lambda \in [0,1]$ the auxiliary problem composed by (2.3.3) with δ_1 replaced by $\bar{\delta}_1$,with the boundary conditions (3.2.1).

Taking $r_2 > 0$ verifying (2.3.5)-(2.3.7) the proof is analogous to Theorem 2.3.1. As such, only some of the modifications are remarked:

- Every solution $u(x)$ of the problem (2.3.3),(3.2.1) verifies

$$\left|u^{(i)}(x)\right| < r_2, \ i = 0, 1, 2, \ \forall x \in [0,1].$$

- The operators $L : C^4([0,1]) \subset C^3([0,1]) \to C([0,1]) \times \mathbb{R}^4$ and $N_\lambda : C^3([0,1]) \to C([0,1]) \times \mathbb{R}^4$ are defined as

$$\mathcal{L}u = \left(u^{(iv)} - u'', \ u(1), \ u'(1), u''(0), \ u'''(1)\right)$$

$$\mathcal{N}_\lambda = \begin{pmatrix} \lambda \left[sp(x) - f\left(x, \delta_0(x,u), \bar{\delta}_1(x,u'), \delta_2(x,u''), u'''(x)\right)\right] - \lambda\delta_2(x,u''), \\ 0, 0, 0, 0 \end{pmatrix}$$

- The location of u' in the set $[\beta', \alpha']$ with α' and β' in reversed order is obtained by integration of

$$\alpha''(x) \leq u''(x) \leq \beta''(x),$$

in $[x, 1]$.

■

The existence and nonexistence of solutions for problem (3.1.1),(3.2.1) will be discussed for some values of the parameter s.

Theorem 3.2.6 *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function verifying Nagumo-type conditions, (3.2.2) and (3.2.3). If*

(H_1) *$f(x, y_0, y_1, y_2, y_3)$ is nondecreasing on y_0 and nonincreasing on y_1 and y_2;*

(H_2) *there exist $s_1 \in \mathbb{R}$ and $r > 0$ such that, for $x \in [0,1]$, $y_0 \leq -r$ and $y_1 \geq r$,*

$$\frac{f(x,0,0,0,0)}{p(x)} < s_1 < \frac{f(x, y_0, y_1, -r, 0)}{p(x)}; \tag{3.2.10}$$

then there is $s_0 < s_1$ (with the possibility of $s_0 = -\infty$) such that:

1) for $s < s_0$, (3.1.1),(3.2.1) has no solution;

2) for $s_0 < s \leq s_1$, (3.1.1),(3.2.1) has, at least, a solution.

Proof. Define

$$s^* = \max_{x \in [0,1]} \frac{f(x,0,0,0,0)}{p(x)}.$$

By (3.2.10), there is $x^* \in [0,1]$ such that

$$\frac{f(x,0,0,0,0)}{p(x)} \leq s^* = \frac{f(x^*,0,0,0,0)}{p(x^*)} < s_1, \ \forall x \in [0,1].$$

For r given by (3.2.10), $\beta(x) \equiv 0$ is an upper solution of (3.1.1),(3.2.1) for $s = s^*$ and, as by (H_1) and (3.2.10),

$$\begin{aligned} 0 &> s_1 p(x) - f(x, -r, r, -r, 0) \\ &\geq sp(x) - f\left(x, -r\left(\frac{x^2}{2} - x + \frac{1}{2}\right), -rx + r, -r, 0\right) \end{aligned} \tag{3.2.11}$$

therefore $\alpha(x) = -r(\frac{x^2}{2} - x + \frac{1}{2})$ is a lower solution of (3.1.1),(3.2.1) for every $s \leq s_1$. So by Theorem 3.2.4 there exists a solution for problem (3.1.1),(3.2.1) for $s = s^*$.

Suppose that problem (3.1.1),(3.2.1) has a solution $u_\sigma(x)$ for $s = \sigma \leq s_1$. So $u_\sigma(x)$ is an upper solution of (3.1.1),(3.2.1) for $\sigma \leq s \leq s_1$.

Let $R > 0$ sufficiently large such that, for r given by (3.2.10),

$$r \leq R, \quad \max_{x\in[0,1]} u'_\sigma(x) \leq R \text{ and } u''_\sigma(1) \geq -R. \tag{3.2.12}$$

As in (3.2.11), $\overline{\alpha}(x) = -R(\frac{x^2}{2} - x + \frac{1}{2})$ is a lower solution of (3.1.1),(3.2.1), for s such that $s \leq s_1$. In order to apply Theorem 3.2.4 it must be proved that $\overline{\alpha}''(x) \leq u''_\sigma(x)$, in $[0,1]$. Suppose, by contradiction, that there exists $x \in]0,1[$, such that $\overline{\alpha}''(x) > u''_\sigma(x)$ and define

$$\min_{x\in[0,1]} u''_\sigma(x) := u''_\sigma(x_0) \ (< -R).$$

Therefore $u'''_\sigma(x_0) = 0, \quad u^{(iv)}_\sigma(x_0) \geq 0$ and, by (H_1)

$$\begin{aligned} 0 &\leq u^{(iv)}_\sigma(x_0) = \sigma p(x_0) - f(x_0, u_\sigma(x_0), u'_\sigma(x_0), u''_\sigma(x_0), 0) \\ &\leq \sigma p(x_0) - f(x_0, u_\sigma(x_0), R, -R, 0). \end{aligned}$$

By integration on $[x, 1]$ and (3.2.12),

$$u_\sigma(x) = -\int_x^1 u'_\sigma(s)\, ds \geq -\int_x^1 R\, ds = (x-1)R \geq -R$$

and the following contradiction is obtained, by (H_1) and (3.2.10),

$$\begin{aligned} 0 &\leq \sigma p(x_0) - f(x_0, u_\sigma(x_0), R, -R, 0) \\ &\leq \sigma p(x_0) - f(x_0, -R, R, -R, 0) < 0. \end{aligned}$$

Therefore, by Theorem 3.2.4, there is a solution to problem (3.1.1),(3.2.1) for $\sigma \leq s \leq s_1$.

Consider the set

$$S = \{s \in \mathbb{R} : \text{ (3.1.1),(3.2.1) has a solution}\}.$$

As, $s^* \in S$ then S is a non-empty set. Let $s_0 = \inf S$. Therefore, for $s < s_0$, problem (3.1.1),(3.2.1) has no solution. By the definition of s_0 and s^*, $s_0 \leq s^* < s_1$, (3.1.1),(3.2.1) has a solution for $s \in]s_0, s_1]$.

It is pointed out that if $s_0 = -\infty$ then, problem (3.1.1),(3.2.1) has a solution for every $s \leq s_1$. ■

Another version of Theorem 3.2.6 can be formulated, with similar proof:

Theorem 3.2.7 *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function verifying Nagumo-type conditions, (3.2.2) and (3.2.3). If*

$(\bar{H}_1)$ *$f(x, y_0, y_1, y_2, y_3)$ is nonincreasing on y_0 and nondecreasing on y_1 and y_2;*

$(\bar{H}_2)$ *there exist $s_1 \in \mathbb{R}$ and $r > 0$ such that, for $x \in [0,1]$, $y_0 \geq -r$ and $y_1 \leq r$,*

$$\frac{f(x,0,0,0,0)}{p(x)} > s_1 > \frac{f(x, y_0, y_1, -r, 0)}{p(x)}; \tag{3.2.13}$$

then there is $s_0 > s_1$ (with the possibility of $s_0 = +\infty$) such that:

1) for $s > s_0$, (3.1.1),(3.2.1) has no solution;

2) for $s_0 > s \geq s_1$, (3.1.1),(3.2.1) has, at least, a solution.

3.3 Multiple solutions to fully differential equations

To obtain multiplicity results using the lower and upper solution method, some additional tools are required. A stronger definition for lower and upper solutions is introduced with strict functions, as well as an extra assumption on f, a "speed growth" condition.

Definition 3.3.1 *A function $\alpha \in C^4(]0,1[) \cap C^3([0,1])$ is a strict lower solution of problem (3.1.1),(3.2.1) if*

$$\alpha^{(iv)}(x) + f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)) > sp(x) \tag{3.3.1}$$

and

$$\begin{aligned} \alpha(1) &\leq 0, \\ \alpha'(1) &\geq 0, \\ \alpha''(0) &< 0, \\ \alpha'''(1) &\leq 0. \end{aligned} \tag{3.3.2}$$

A function $\beta \in C^4(]0,1[) \cap C^3([0,1])$ is a strict upper solution of problem (3.1.1),(3.2.1) if the reversed inequalities hold.

Define the set

$$X = \left\{u \in C^3([0,1]) : u(1) = u'(1) = u''(0) = u'''(1) = 0\right\}$$

and the linear operator

$$L : dom\ L \to C([0,1]) \text{ with } dom\ L = C^4([0,1]) \cap X$$

given by $Lu = u^{(iv)}$.

For $s \in \mathbb{R}$ consider the nonlinear operator

$$N_s : C^3([0,1]) \cap X \to C([0,1])$$

given by

$$N_s u = f\left(x, u\left(x\right), u'\left(x\right), u''\left(x\right), u'''\left(x\right)\right) - sp\left(x\right). \tag{3.3.3}$$

For $\Omega \subset X$ an open and bounded set, the operator $L + N_s$ is L-compact in $\overline{\Omega}$. Remark that in *dom* L, problem (3.1.1),(3.2.1) is equivalent to equation

$$Lu + N_s u = 0.$$

Next lemma is useful to evaluate the topological degree of $L + N_s$ in Ω relatively to L at $p \in \Omega$, noted by $d_L\left(L + N_s, \Omega, p\right)$.

Lemma 3.3.2 *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying Nagumo-type conditions and (3.2.8). If there are strict lower and upper solutions of (3.1.1),(3.2.1), $\alpha(x)$ and $\beta(x)$, respectively, then there exists $\rho_3 > 0$ such that for*

$$\Omega = \left\{ \begin{array}{c} u \in dom\ L : \alpha^{(i)}\left(x\right) < u^{(i)}\left(x\right) < \beta^{(i)}\left(x\right),\ i = 0, 2, \\ \beta'(x) < u'(x) < \alpha'(x),\ ||u'''|| < \rho_3 \end{array} \right\}$$

we have $d_L\left(L + N_s, \Omega, 0\right) = \pm 1$.

Proof. Consider the continuous functions given by (2.3.2) and (3.2.9), along with the auxiliary problem

$$\left\{ \begin{array}{c} u^{(iv)}\left(x\right) + F\left(x, u\left(x\right), u'\left(x\right), u''\left(x\right), u'''\left(x\right)\right) = sp\left(x\right) \\ u\left(1\right) = u'\left(1\right) = u''\left(0\right) = u'''(1) = 0 \end{array} \right. \tag{3.3.4}$$

with $F : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ a continuous function given by

$$F\left(x, y_0, y_1, y_2, y_3\right) = f\left(x, \delta_0\left(x, y_0\right), \overline{\delta}_1\left(x, y_1\right), \delta_2\left(x, y_2\right), y_3\right) - y_2 + \delta_2\left(x, y_2\right).$$

Define the operator $F_s : C^3\left([0,1]\right) \cap X \to C\left([0,1]\right)$ given by $F_s u = F\left(x, u, u', u'', u'''\right) - sp\left(x\right)$.

In $dom\ L$ problem (3.3.4) is equivalent to the equation $Lu+F_s u=0$. For $\lambda \in [0,1]$ and $u \in dom\ L$ consider the homotopy

$$H_\lambda u := Lu - (1-\lambda)\, u'' + \lambda F_s u. \tag{3.3.5}$$

Take $\rho_2 > 0$ large enough such that, for $x \in [0,1]$,

$$-\rho_2 \leq \alpha''(x) < \beta''(x) \leq \rho_2 \tag{3.3.6}$$

$$sp(x) - f(x, \alpha(x), \alpha'(x), \alpha''(x), 0) - \rho_2 - \alpha''(x) < 0$$

and

$$sp(x) - f(x, \beta(x), \beta'(x), \beta''(x), 0) + \rho_2 - \beta''(x) > 0$$

Applying the technique suggested in the proof of Theorem 3.2.4, with obvious modifications, there is $\rho_3 > 0$ such that every solution u of $H_\lambda u = 0$ verifies

$$\|u''\| < \rho_2 \text{ and } \|u'''\| < \rho_3 \tag{3.3.7}$$

independently from $\lambda \in [0,1]$. Consider

$$\Omega_1 = \{y \in dom\ L : \|y''\| < \rho_2,\ \|y'''\| < \rho_3\}.$$

The inclusion $\Omega_1 \supset \Omega$ is obtained from (3.3.6). By (3.3.7), every solution u of $H_\lambda u = 0$ is in Ω_1 for every $\lambda \in [0,1]$.

Since $u \notin \partial\Omega_1$ then $d_L(H_\lambda, \Omega_1, 0)$ is well defined for every $\lambda \in [0,1]$. On the other hand, the linear part of equation $H_0 u = 0$, i.e., $Lu - u'' = 0$ has only the trivial solution and, by degree theory, $d_L(H_0, \Omega_1, 0) = \pm 1$. By the invariance under homotopy

$$\pm 1 = d_L(H_0, \Omega_1, 0) = d_L(H_1, \Omega_1, 0) = d_L(L + F_s, \Omega_1, 0). \tag{3.3.8}$$

Then, there is u_1 solution of $H_1 u = 0$, i.e., solution of $Lu + F_s u = 0$.

Applying the arguments referred in the proof of Theorem 3.2.4 it can be proved that

$$\alpha^{(i)}(x) < u_1^{(i)}(x) < \beta^{(i)}(x), i = 0, 2, \ \beta'(x) < u_1'(x) < \alpha'(x) \ \forall x \in [0,1],$$

and $\|u_1'''\| < \rho_3$. Therefore $u_1(x) \in \Omega$.

The degree $d_L(L + F_s, \Omega, 0)$ is well defined and by (3.3.8) and the excision property of the degree,

$$d_L(L + F_s, \Omega_1, 0) = d_L(L + F_s, \Omega, 0) = d_L(L + N_s, \Omega, 0) = \pm 1,$$

since equations $Lu + F_s u = 0$ and $Lu + N_s u = 0$ are equivalent in *dom* L. ■

To obtain the multiplicity result for problem (3.1.1),(3.2.1) it must be assumed some bound from below and a "speed growth" on f. Such assumption on f models the fact that the variation of some variables, will have different influences on the global monotony of f, that is, the perturbation of some variables are stronger than other ones.

Theorem 3.3.3 *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function verifying the assumptions of Theorem 3.2.6. Suppose that there are $M > -r$, with r given by (3.2.10), such that every solution u of (3.1.1),(3.2.1), with $s \leq s_1$, satisfies*

$$u''(x) < M, \ \forall x \in [0,1], \tag{3.3.9}$$

and $m \in \mathbb{R}$ such that

$$f(x, y_0, y_1, y_2, y_3) \geq m \ p(x), \tag{3.3.10}$$

for every $(x, y_0, y_1, y_2, y_3) \in [0,1] \times [-M_1, M_1]^2 \times [-r, M] \times \mathbb{R}$, where $M_1 := \max\{r, |M|\}$.

Then s_0 given by Theorem 3.2.6 is finite and:

1) *for* $s < s_0$*, (3.1.1),(3.2.1) has no solution;*

2) *for* $s = s_0$*, (3.1.1),(3.2.1) has at least a solution.*

Moreover, if there is $\theta > 0$*, such that*

$$f(x, y_0 + \theta\eta_0, y_1 - \theta\eta_1, y_2 + \theta, y_3) \leq f(x, y_0, y_1, y_2, y_3), \tag{3.3.11}$$

for every $(x, y_0, y_1, y_2, y_3) \in [0,1] \times [-M_1, M_1]^3 \times \mathbb{R}$ *and* $0 \leq \eta_i \leq 1,\ i = 0, 1,$ *then*

3) *for* $s \in \,]s_0, s_1]$*, (3.1.1),(3.2.1) has at least two solutions.*

Proof. *Step 1 - Every solution* $u(x)$ *of problem (3.1.1),(3.2.1) for* $s \in \,]s_0, s_1]$ *satisfies*

$$-r < u''(x) < M \text{ and } \left|u^{(i)}(x)\right| < M_1,\ i = 0, 1,\ \forall x \in [0,1].$$

For the first case, by (3.3.9), it will be enough to prove that $-r < u''(x)$, for every $x \in [0,1]$.

Suppose, by contradiction, that there are u solution of (3.1.1),(3.2.1), for some $s \in \,]s_0, s_1]$, and $x_0 \in]0,1]$ such that

$$\min_{x \in [0,1]} u''(x) := u''(x_0) \ \ (\leq -r).$$

Therefore $u'''(x_0) = 0$ and $u^{(iv)}(x_0) \geq 0$. By (H_1),

$$0 \leq u^{(iv)}(x_0) \leq s_1 p(x_0) - f(x_0, u(x_0), u'(x_0), -r, 0). \tag{3.3.12}$$

If $u(x_0) \leq -r$ and $u'(x_0) \geq r$, by (3.2.10) and (3.3.12), we get the contradiction

$$0 \leq s_1 p(x_0) - f(x_0, u(x_0), u'(x_0), -r, 0) < 0.$$

If $u(x_0) \geq -r$ and $u'(x_0) \geq r$ (the other cases are analogous) a similar contradiction is obtained by (H_1) and (3.2.10):

$$\begin{aligned} 0 &\leq s_1 p(x_0) - f(x_0, u(x_0), u'(x_0), -r, 0) \\ &\leq s_1 p(x_0) - f(x_0, -r, u'(x_0), -r, 0) < 0. \end{aligned}$$

So, every solution u of (3.1.1),(3.2.1) with $s \in]s_0, s_1]$, verifies $-r < u''(x) < M,\ \forall x \in [0,1]$. Integrating on $[x,1]$ we have

$$-r \le -r(1-x) < \int_x^1 u''(s)\,ds = -u'(x) < M(1-x) \le |M|.$$

Therefore $|u'(x)| < M_1$ and similarly $|u(x)| < M_1$.

<u>*Step 2*</u> - *The number s_0 is finite.*

Suppose that $s_0 = -\infty$. By Theorem 3.2.6, problem (3.1.1),(3.2.1) has a solution for every s such that $s \le s_1$. Let $u(x)$ be a solution of (3.1.1),(3.2.1), for $s \le s_1$. Then by (3.3.10),

$$u^{(iv)}(x) \le sp(x) - mp(x) = (s-m)\,p(x).$$

Define

$$p_1 := \min_{x\in[0,1]} p(x) > 0$$

and consider s small enough such that

$$m - s > 0 \ \text{ and } \ \frac{(m-s)\,p_1}{16} > M.$$

By boundary conditions (3.2.1),

$$\begin{aligned} u'''(x) &= -\int_x^1 u^{(iv)}(\xi)\,d\xi \ge \int_x^1 (m-s)\,p(\xi)\,d\xi \\ &\ge \int_x^1 (m-s)\,p_1\,d\xi = (m-s)(1-x)\,p_1 > 0. \end{aligned}$$

Defining $I = \left[0, \frac{1}{4}\right]$, then $|1-x| \ge \frac{1}{4}$, for $x \in I$,

$$u'''(x) \ge \frac{(m-s)\,p_1}{4},\ \forall x \in I,$$

and this contradiction with (3.3.9) is attained

$$u''\left(\frac{1}{4}\right) = \int_0^{\frac{1}{4}} u'''(x)\,dx \ge \int_0^{\frac{1}{4}} \frac{(m-s)\,p_1}{4}dx = \frac{1}{16}(m-s)\,p_1 > M.$$

Therefore s_0 is finite.

<u>*Step 3*</u> - *For $s \in]s_0, s_1]$, there is a second solution for (3.1.1),(3.2.1).*

By Step 2, there is $s_{-1} < s_0$ such that problem (3.1.1),(3.2.1), with $s = s_{-1}$, has no solution. By Lemma 3.3.2, there exists $\rho_3^* > 0$ such that $\|u'''\| < \rho_3^*$, for every solution u of (3.1.1),(3.2.1), with $s \in]s_{-1}, s_1]$. Defining the set

$$\Omega_2 = \{y \in domL : \|y''\| < M_1,\ \|y'''\| < \rho_3^*\} \tag{3.3.13}$$

we have

$$d_L\left(L + N_{s_{-1}}, \Omega_2, 0\right) = 0. \tag{3.3.14}$$

By Step 1, every solution u of (3.1.1),(3.2.1), with $s \in]s_{-1}, s_1]$, satisfies $u \notin \partial\Omega_2$ and for the homotopy on the parameter s,

$$H(\lambda) = (1-\lambda)\, s_{-1} + \lambda s_{1,}$$

the degree $d_L\left(L + N_{H(\lambda)}, \Omega_2, 0\right)$ is well defined for every $\lambda \in [0,1]$ and $s \in]s_{-1}, s_1]$. As the degree is invariant under homotopy and by (3.3.14) it results

$$0 = d_L\left(L + N_{s_{-1}}, \Omega_2, 0\right) = d_L\left(L + N_s, \Omega_2, 0\right), \tag{3.3.15}$$

for $s \in]s_{-1}, s_1]$.

For some $\sigma \in]s_0, s_1] \subset]s_{-1}, s_1]$, by Theorem 3.2.6, there is $u_\sigma(x)$, solution of (3.1.1),(3.2.1) with $s = \sigma$. Let $\varepsilon > 0$ small enough such that

$$|u''_\sigma(x) + \varepsilon| < M_1,\ \forall x \in [0,1]. \tag{3.3.16}$$

Following the arguments used in the proof of Theorem 3.2.6, for r given by (3.2.10),

$$\alpha(x) = -r\left(\frac{x^2}{2} - x + \frac{1}{2}\right)$$

is a strict lower solution of (3.1.1),(3.2.1) for $s \le s_1$ and, applying (3.3.11) with

$$\theta = \varepsilon,\quad \eta_0 = \frac{(x-1)^2}{2}$$

and $\eta_1 = 1 - x$, the function

$$\widetilde{u}(x) = u_\sigma(x) + \varepsilon \frac{(x-1)^2}{2}$$

is a strict upper solution of (3.1.1),(3.2.1) for $\sigma < s \leq s_1$. As, by (3.3.16), $\alpha''(x) = -r \leq \widetilde{u}''(x)$ then, by Lemma 3.3.2, there is $\overline{\rho}_3 > 0$, independent of s, such that for

$$\Omega_\varepsilon = \left\{ \begin{array}{c} y \in dom\ L : \alpha^{(i)}(x) < y^{(i)}(x) < \tilde{u}^{(i)}(x),\ i = 0, 2, \\ \beta''(x) \leq y' \leq \alpha'(x),\ \|y'''\| < \overline{\rho}_3 \end{array} \right\},$$

the degree for $L + N_s$ in Ω_ε verifies

$$d_L(L + N_s, \Omega_\varepsilon, 0) = \pm 1, \quad \text{for } s \in [\sigma, s_1]. \tag{3.3.17}$$

Take, in (3.3.13), ρ_3^* large enough such that $\Omega_\varepsilon \subset \Omega_2$. Therefore, by (3.3.15), (3.3.17) and the additivity of the degree, we obtain

$$d_L\left(L + N_s, \Omega_2 - \overline{\Omega}_\varepsilon, 0\right) = \mp 1, \quad \text{for } s \in \left]\sigma, s_1\right]. \tag{3.3.18}$$

Then problem (3.1.1),(3.2.1) has, at least, two solutions u_1 and u_2 such that $u_1 \in \Omega_\varepsilon$ and $u_2 \in \Omega_2 - \overline{\Omega}_\varepsilon$, for $s \in \left]\sigma, s_1\right]$ and, as σ is arbitrary, for $s \in \left]s_0, s_1\right]$.

<u>Step 4</u> - *For $s = s_0$, problem (3.1.1),(3.2.1) has one solution.*

Consider the sequence (s_m), with $s_m \in \left]s_0, s_1\right]$ and $\lim s_m = s_0$. By Theorem 3.2.6, for every s_m, problem (3.1.1),(3.2.1), with $s = s_m$, has a solution u_m. By Step 1, $\left\|u_m^{(i)}\right\| < M_1$, for $i = 0, 1, 2$, independently of m. So there is L, large enough, such that $\|u_m'''\| < L$, that is, the sequence $\left(u_m^{(iv)}\right)$, $m \in \mathbb{N}$, is bounded in $C([0,1])$. By the Arzéla-Ascoli Theorem, there is a subsequence, (u_m), convergent in $C^3([0,1])$ to a solution $u_0(x)$ of (3.1.1),(3.2.1), with $s = s_0$. ■

3.4 Existence and nonexistence results with unbounded nonlinearities

Let us consider the problem given by the equation (3.1.1) in $[a,b]$, where $f : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ and $p : [a,b] \to \mathbb{R}^+$ are continuous functions and the boundary conditions

$$u(a) = A,\ u'(a) = B,\ u'''(a) = C,\ u'''(b) = D, \tag{3.4.1}$$

with $A, B, C, D \in \mathbb{R}$.

The results presented in this section improve the existing results in the literature, as far as we know, by using a more general Nagumo-type condition, which allows the nonlinear part to be unbounded. This type of results, can be applied to beam equations, where the nonlinear part has some asymmetric growth.

In this section it is defined the one-sided Nagumo-type growth condition assumed on the nonlinear part of the differential equation which will be an important tool to obtain the *a priori* bound for the third derivative of the corresponding solutions, even with unbounded functions. The one-sided Nagumo type condition used in this section is given by Definition 2.2.1.

Lower and upper solutions will be defined as follows:

Definition 3.4.1 *Consider $A, B, C, D \in \mathbb{R}$. The function $\alpha \in C^4([a,b])$ is a lower solution of the problem (3.1.1),(3.4.1) if*

$$\alpha^{(iv)}(x) \geq sp(x) - f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)),$$

and

$$\alpha(a) \leq A, \quad \alpha'(a) \leq B, \quad \alpha'''(a) \geq C, \quad \alpha'''(b) \leq D. \tag{3.4.2}$$

The function $\beta \in C^4([a,b])$ *is an upper solution of the problem (3.1.1),(3.4.1) if the reversed inequalities hold.*

The following theorem provides a general existence and location result similar to Theorem 2.3.1.

Theorem 3.4.2 *Suppose that there are lower and upper solutions of the problem (3.1.1),(3.4.1)* $\alpha(x)$ *and* $\beta(x)$, *such that,*

$$\alpha''(x) \leq \beta''(x), \text{ for every } x \in [a,b].$$

Let $f : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ *be a continuous function satisfying the one-sided Nagumo conditions (2.2.1) (or (2.2.2)) and (2.2.3) in*

$$E_* = \left\{(x, y_0, y_1, y_2, y_3) \in [a,b] \times \mathbb{R}^4 : \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), i = 0,1,2\right\}.$$

If f *verifies*

$$f(x, \alpha, \alpha', y_2, y_3) \leq f(x, y_0, y_1, y_2, y_3) \leq f(x, \beta, \beta', y_2, y_3), \tag{3.4.3}$$

for $\alpha(x) \leq y_0 \leq \beta(x)$ *and* $\alpha'(x) \leq y_1 \leq \beta'(x)$, *in* $[a,b]$ *and for fixed* $(x, y_2, y_3) \in [a,b] \times \mathbb{R}^2$, *then problem (3.1.1),(3.4.1) has at least a solution* $u(x) \in C^4([a,b])$, *satisfying*

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \text{ for } i = 0,1,2 \text{ and every } x \in [a,b].$$

Proof. The proof is analogous to Theorem 2.3.1, with the following particularities:

- The auxiliar problem is composed by equation (2.3.3) and the boundary conditions,

$$\begin{aligned} u(a) &= \lambda A \\ u'(a) &= \lambda B \\ u'''(a) &= \lambda \left[C + u''(a) - \delta_2(a, u''(a))\right] \\ u'''(b) &= \lambda \left[D - u''(b) + \delta_2(b, u''(b))\right], \end{aligned} \tag{3.4.4}$$

  with $\lambda \in [0,1]$.

- The number $r_2 > 0$ will be considered large enough, such that, for every $x \in [a,b]$, it satisfies conditions (2.3.5)-(2.3.7) and

$$\begin{array}{ll} C - \beta''(a) > -r_2; & C - \alpha''(a) < r_2 \\ D + \alpha''(b) > -r_2; & D + \beta''(b) < r_2 \end{array} \tag{3.4.5}$$

- Every solution $u(x)$ of the problem (2.3.3),(3.4.4) satisfies

$$\left| u^{(i)}(x) \right| < r_i, \ \forall x \in [a,b], \ i = 0,1,2,$$

with $r_1 := r_2 (b-a) + |B|$ *and* $r_0 := r_2 (b-a)^2 + |B| (b-a) + |A|$, *independently of* $\lambda \in [0,1]$.

- If, by contradiction,

$$\max_{x \in [0,1]} u''(x) := u''(a) \geq r_2 > 0,$$

and $u'''(a^+) = u'''(a) \leq 0$, then for $\lambda \in]0,1]$, by (3.4.4) and (3.4.5) the following contradiction is obtained

$$\begin{array}{rcl} 0 & \geq & u'''(a) = \\ & = & \lambda [C + u''(a) - \beta''(a)] \geq \lambda [C + r_2 - \beta''(a)] > 0 \end{array}$$

If $\lambda = 0$, by (3.4.4), $u^{(iv)}(a) \leq 0$, and the contradiction is

$$0 \geq u^{(iv)}(a) = u''(a) \geq r_2 > 0.$$

The arguments for $x_0 = b$, are similar and therefore $u''(x) < r_2$, $\forall x \in [a,b]$, $\forall \lambda \in [0,1]$.

The case $u''(x) \leq -r_1$ is analogous and so

$$|u''(x)| < r_2, \ \forall x \in [a,b], \forall \lambda \in [0,1].$$

- By integration in $[a,x]$,

$$u'(x)-\lambda B=\int_a^x u''(s)ds<r_2(x-a)<r_2(b-a),$$

and so

$$|u'(x)|<r_2(b-a)+|B|\,,\ \forall x\in[a,b]\,,\forall\lambda\in[0,1]\,.$$

With similar procedure

$$\begin{aligned} u(x)-\lambda A &= \int_a^x u'(s)ds<\int_a^x (r_2(b-a)+|B|)\ ds \\ &= r_2(b-a)^2+|B|\,(x-a)<r_2(b-a)^2+|B|\,(b-a). \end{aligned}$$

Therefore

$$|u(x)|<r_2(b-a)^2+|B|\,(b-a)+|A|$$

- Lemma 2.2.3 can be applied in $[a,b]$ defining $\rho:=2r_2$, then by (3.4.4) and (3.4.5)

$$u'''\,(a)\leq\lambda\,[C+u''(a)-\alpha''\,(a)]\leq 2r_2:=\rho$$

and

$$u'''\,(b)\geq\lambda\,[D-u''(b)+\alpha''\,(b)]\geq[-u''(b)-r_2]>-2r_2:=-\rho.$$

To apply Lemma 2.2.2 in $[a,b]$ the technique is similar.

- The operators

$$\mathcal{L}:C^4\left([a,b]\right)\to C\left([a,b]\right)\times\mathbb{R}^4$$

given by

$$\mathcal{L}u=\left(u^{(iv)}-u'',u\left(a\right),u'\left(a\right),u'''\left(a\right),u'''\left(b\right)\right)$$

and, $\mathcal{N}_\lambda : C^3([a,b]) \to C([a,b]) \times \mathbb{R}^4$, by

$$\mathcal{N}_\lambda = \begin{pmatrix} \lambda\left[sp(x) - f\left(x, \delta_0(x,u), \delta_1(x,u'), \delta_2(x,u''), u'''\right)\right] \\ -\lambda\delta_2(x,u''), \lambda A, \lambda B, \\ \lambda\left[C + u''(a) - \delta_2(a, u''(a))\right], \lambda\left[D - u''(b) + \delta_2(b, u''(b))\right] \end{pmatrix}.$$

- To prove that a solution $u_1(x)$ of the problem (2.3.3),(3.4.4) for $\lambda = 1$, is also a solution of the initial problem, at the boundary points, assume by contradiction that

$$\max_{x\in[0,1]}\left[\alpha''(x) - u_1''(x)\right] := \alpha''(a) - u_1''(a) > 0.$$

Therefore $u_1'''(a) - \alpha'''(a) = u_1'''(a^+) - \alpha'''(a^+) \le 0$ and the following contradiction is obtained with (3.4.2)

$$\alpha'''(a) = C + u''(a) - \alpha''(a) < C.$$

For the other endpoint the arguments are similar.

■

For clearness, the dependence of the solution on s will be discussed in $[0,1]$ and for the particular case $A = B = C = D = 0$. Therefore the boundary conditions become

$$u(0) = 0,\ u'(0) = 0,\ u'''(0) = 0,\ u'''(1) = 0, \tag{3.4.6}$$

and the corresponding definitions of lower and upper solutions will verify these restrictions.

Next Theorem follows the same method as in Theorem 3.2.6.

Theorem 3.4.3 *Let $f : [0,1]\times\mathbb{R}^4 \to \mathbb{R}$ be a continuous function that verifies the one-sided Nagumo conditions (2.2.1) (or (2.2.2)) and (2.2.3). If:*

(H_1^*) $f(x, y_0, y_1, y_2, y_3)$ *is nondecreasing on* y_0 *and* y_1 *and nonincreasing on* y_2,

(H_2^*) *there are* $s_1 \in \mathbb{R}$ *and* $r > 0$ *such that for every* $x \in [0,1]$ *and* $y_0, y_1 \leq -r$,

$$\frac{f(x,0,0,0,0)}{p(x)} < s_1 < \frac{f(x,y_0,y_1,-r,0)}{p(x)}, \tag{3.4.7}$$

then there is $s_0 < s_1$ *(eventually* $s_0 = -\infty$*) such that:*

1) for $s < s_0$, *(3.1.1),(3.4.6) has no solution.*

2) for $s_0 < s \leq s_1$, *(3.1.1),(3.4.6) has at least one solution.*

Proof. As the proof follows the technique referred in Theorem 3.2.6 it is pointed out only the adequate modifications of this case:

- $\alpha(x) = -r\frac{x^2}{2}$ and $\beta(x) \equiv 0$ are lower and upper solutions, respectively, for the problem (3.1.1),(3.4.6), for $s = s^*$. So, by Theorem 3.2.4 there is a solution of (3.1.1),(3.4.6) for $s = s^*$.

- Assuming that problem (3.1.1),(3.4.6) has for $s = \sigma \leq s_1$, a solution $u_\sigma(x)$, this function $u_\sigma(x)$ is an upper solution of (3.1.1),(3.4.6), for $\sigma \leq s \leq s_1$.

- For $\eta > 0$ large enough such that

$$\eta \geq r, \ u''_\sigma(0) \geq -\eta, \ \ u''_\sigma(1) \geq -\eta, \ \min_{x\in[0,1]} u'_\sigma(x) \geq -\eta\ , \tag{3.4.8}$$

the function

$$\bar{\alpha}(x) = -\eta\frac{x^2}{2}$$

is a lower solution of (3.1.1),(3.4.6) for $s \leq s_1$.

As it can be proved that,

$$-\eta = \bar{\alpha}''(x) \leq u''_\sigma(x), \ \forall x \in \left]0,1\right[,$$

then by Theorem 3.2.4, there is a solution of (3.1.1),(3.4.6) for $\sigma \leq s \leq s_1$.

■

As in the previous section, a dual version of last theorem can be enunciated, with similar proof:

Theorem 3.4.4 *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function that verifies the one-sided Nagumo conditions ((2.2.1) or (2.2.2)) and (2.2.3).*

($\bar{H}_1^*$) $f(x, y_0, y_1, y_2, y_3)$ is nonincreasing on y_0 and $y_1$and nondecreasing on y_2;

($\bar{H}_2^*$) there are$s_1 \in \mathbb{R}$ and $r > 0$ such that, for every $x \in [0,1]$ and $y_0, y_1 \geq r$,

$$\frac{f(x,0,0,0,0)}{p(x)} > s_1 > \frac{f(x, y_0, y_1, -r, 0)}{p(x)},$$

then there is $s_0 > s_1$ (with the possibility that $s_0 = +\infty$) such that:

1) for $s > s_0$, (3.1.1),(3.4.6) has no solution.

2) for $s_0 > s \geq s_1$, (3.1.1),(3.4.6) has at least one solution.

3.5 Multiplicity results for unbounded f

As in before, to prove the existence of at least a second solution it is necessary to introduce strict lower and upper solutions

Definition 3.5.1 *The function $\alpha(x) \in C^4([0,1])$ is a strict lower solution of the problem (3.1.1),(3.4.6) if the following conditions are fulfilled:*

$$\alpha^{(iv)}(x) + f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)) > sp(x)$$

and

$$\alpha(0) < 0,\ \alpha'(0) < 0,\ \alpha'''(0) \geq 0,\ \alpha'''(1) \leq 0$$

The function $\beta(x) \in C^4([0,1])$ is called a strict upper solution of problem (3.1.1),(3.4.6) if the reversed inequalities hold

Let us consider the set

$$Y = \left\{ y \in C^3([0,1]) : y(0) = y'(0) = y'''(0) = y'''(1) = 0 \right\}$$

and the operator $L : dom\ L \to C([0,1])$ in which $dom\ L = C^4([0,1]) \cap Y$ given by $Lu = u^{(iv)}$. For $s \in \mathbb{R}$ consider $N_s : C^3([0,1]) \cap Y \to C([0,1])$ defined by (3.3.3).

In this case the tool to evaluate the degree will be given by the next lemma.

Lemma 3.5.2 *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function verifying Nagumo conditions, ((2.2.1) or (2.2.2)) and (2.2.3), and the condition (H_1^*). Let us suppose that there are strict lower and upper solutions of the problem (3.1.1),(3.4.6), $\alpha(x)$ and $\beta(x)$ respectively, such that $\alpha''(x) < \beta''(x)$, $\forall x \in [0,1]$. Thus there is $\bar{\rho} > 0$ such that for*

$$\Omega_* = \left\{ u \in dom\ L : \alpha^{(i)}(x) < u^{(i)}(x) < \beta^{(i)}(x),\ i = 0,1,2,\ \left\| u'''(x) \right\|_\infty < \rho \right\}$$

the degree $L+N_s$, relatively to L is well defined and given by $d_L(L+N_s, \Omega_*, 0) = \pm 1$.

Proof. The proof is analogous to Lemma 3.3.2, with obvious changes to prove the key points:

- There is an open bounded set $\Omega_1 \supset \Omega$ such that $d_L(L+F_s, \Omega_1, 0)$ is well defined and $d_L(L+F_s, \Omega_1, 0) = \pm 1$.

- If u is a solution of $H_1 u = 0$ then $u \in \Omega_*$.

■

Similar to Theorem 3.3.3, for the multiplicity result it will be needed a "speed growth condition".

Theorem 3.5.3 *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying the assumptions of Theorem 3.4.3. Suppose that there is $M > -r$, with r given by (3.4.7), such that for every u solution of the problem (3.1.1),(3.4.6), with $s \leq s_1$, verifies $u''(x) < M$, $\forall x \in [0,1]$, and there is $m \in \mathbb{R}$ such that*

$$f(x, y_0, y_1, y_2, y_3) \geq mp(x), \tag{3.5.1}$$

for $(x, y_0, y_1, y_2, y_3) \in [0,1] \times [-r - |M|, r + |M|]^2 \times [-r, |M|] \times \mathbb{R}$.

Then s_0 given by Theorem 3.4.3 is finite and:

1) *for $s < s_0$, (3.1.1),(3.4.6) has no solution;*

2) *for $s = s_0$, (3.1.1),(3.4.6) has, at least, one solution.*

Moreover, let $M_1 := \max\{r, |M|\}$ and suppose that there is $\theta > 0$ such that for $(x, y_0, y_1, y_2, y_3) \in [0,1] \times [-M_1, M_1]^3 \times \mathbb{R}$ and $0 \leq \eta_0, \eta_1 \leq 1$,

$$f(x, y_0 + \eta_0 \theta, y_1 + \eta_1 \theta, y_2 + \theta, y_3) \leq f(x, y_0, y_1, y_2, y_3). \tag{3.5.2}$$

Then

3) *for $s \in \left]s_0, s_1\right]$, (3.1.1),(3.4.6) has, at least, two solutions.*

Proof. An analogous method to the proof of Theorem 3.3.3 can be applied. So it is referred only the main changes:

- Every solution $u(x)$ of problem (3.1.1),(3.4.6), for $s \in \left]s_0, s_1\right]$, satisfies

$$-r < u''(x) < M, \ -r < u^{(i)}(x) < |M|, \ \forall x \in [0,1], \ i = 0, 1.$$

- s_0 is finite results from the contradiction between

$$u^{(iv)}(x) \leq (s-m)\,p(x), \ \forall x \in [0,1],$$

for s small enough and, by (3.4.6), there is $c \in \,]0,1[$, such that $u^{(iv)}(c) = 0$.

- To prove the existence of a second solution of (3.1.1),(3.4.6), for $s \in \,]s_0, s_1]$, assuming that there is a solution $u_\sigma(x)$ for $\sigma \in \,]s_0, s_1] \subset [s_{-1}, s_1]$, consider $\varepsilon > 0$, small enough, such that

$$\begin{aligned} \left|u''_\sigma(x) + \varepsilon\right| &< M_1, \\ \varepsilon^2\left(\frac{x^2}{2} + x + 1\right) &< 1, \\ \varepsilon^2(x+1), \ \forall x &\in [0,1]. \end{aligned}$$

Therefore the auxiliary function

$$\tilde{u}(x) := u_\sigma(x) + \varepsilon\frac{x^2}{2} + \varepsilon x + \varepsilon$$

is a strict upper solution of problem (3.1.1),(3.4.6) for $\sigma < s \leq s_1$.

In fact, applying (3.5.2) with $\theta := \frac{1}{\varepsilon}$, $\eta_0 := \varepsilon^2\left(\frac{x^2}{2} + x + 1\right)$ and $\eta_1 := \varepsilon^2(x+1)$, it is obtained

$$\begin{aligned} & \tilde{u}^{(iv)}(x) \\ = \ & u_\sigma^{(iv)}(x) = \sigma p(x) - f\left(x, u_\sigma(x), u'_\sigma(x), u''_\sigma(x), u'''_\sigma(x)\right) \\ < \ & sp(x) - f\left(x, u_\sigma(x), u'_\sigma(x), u''_\sigma(x), \tilde{u}'''(x)\right) \\ \leq \ & sp(x) \\ & -f\left(x, u_\sigma(x) + \varepsilon\left(\frac{x^2}{2} + x + 1\right), u'_\sigma(x) + \varepsilon(x+1), u''_\sigma(x) + \varepsilon, \tilde{u}'''(x)\right) \\ = \ & sp(x) - f\left(x, \tilde{u}(x), \tilde{u}'(x), \tilde{u}''(x), \tilde{u}'''(x)\right) \end{aligned}$$

and the boundary conditions

$$\begin{gathered}\tilde{u}(0) = u_\sigma(0) + \varepsilon > 0 \\ \tilde{u}'(0) = u'_\sigma(0) + \varepsilon > 0 \ . \\ \tilde{u}'''(0) = \tilde{u}'''(1) = 0\end{gathered}$$

The function

$$\alpha(x) = -\frac{r}{5}\left(\frac{x^2}{2} + 2x + 3\right)$$

is a strict lower solution of the problem (3.1.1),(3.4.6) for $s \leq s_1$.

As, $-r < u''_\sigma(x)$, for every $x \in [0,1]$, then,

$$\alpha''(x) < \tilde{u}''(x), \ \forall x \in [0,1]. \tag{3.5.3}$$

By integrating in $[0,x]$ and (3.5.1) it is easily obtained

$$\begin{gathered}\alpha(x) < \tilde{u}_\sigma(x) \\ \alpha'(x) < \tilde{u}'_\sigma(x), \ \forall x \in [0,1].\end{gathered}$$

The arguments follow in analogous way applying Lemma 3.5.2, to be sure that there is $\bar{\rho}_3 > 0$, independently of s, such that for

$$\Omega_\varepsilon = \left\{ x \in dom\ L : \begin{array}{c} -\frac{r}{5}\left(\frac{x^2}{2} + 2x + 3\right) < y(x) < u_\sigma(x) + \varepsilon\frac{x^2}{2} + \varepsilon x + \varepsilon, \\ -\frac{r}{5}(x+1) < y'(x) < u_\sigma(x) + \varepsilon x + \varepsilon, \\ -\frac{r}{5} < y''(x) < u_\sigma(x) + \varepsilon, \\ \|y'''\|_\infty < \bar{\rho}_3 \end{array} \right\}.$$

■

3.6 The London Millennium Bridge application

There is a huge potential for this multiplicity tool in terms of applications. The London Millennium bridge phenomenon is a clear example.

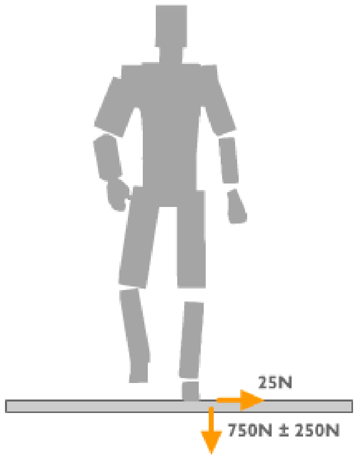

Figure 3.6.1: Walking, in addition to our weight, we create a repeating pattern of forces as our mass rises and falls. This creates a vertical fluctuating force of around 250 N which repeats with each step.

In the opening day this bridge revealed lateral movements, while pedestrians were crossing it. These lateral movements were related with the lateral load caused by the number os pedestrians crossing it.

To discover the causes of the lateral displacements, the number of pedestrians walking on the bridge was increased in groups. From a certain number, the excitation due to the lateral movement was greater then the damping, causing the lateral oscillations in the bridge.

If, in equation (3.1.1), the parameter s represents the number of pedestrians walking on the bridge at a given moment, then the lower and upper solution method applied to Ambrosetti-Prodi problems can be a very sharp tool to give some bounds on s.

In fact, if s_0 represents the number of pedestrians corresponding to the damping factor inherent to the bridge, then from Theorem 3.3.3 we obtain

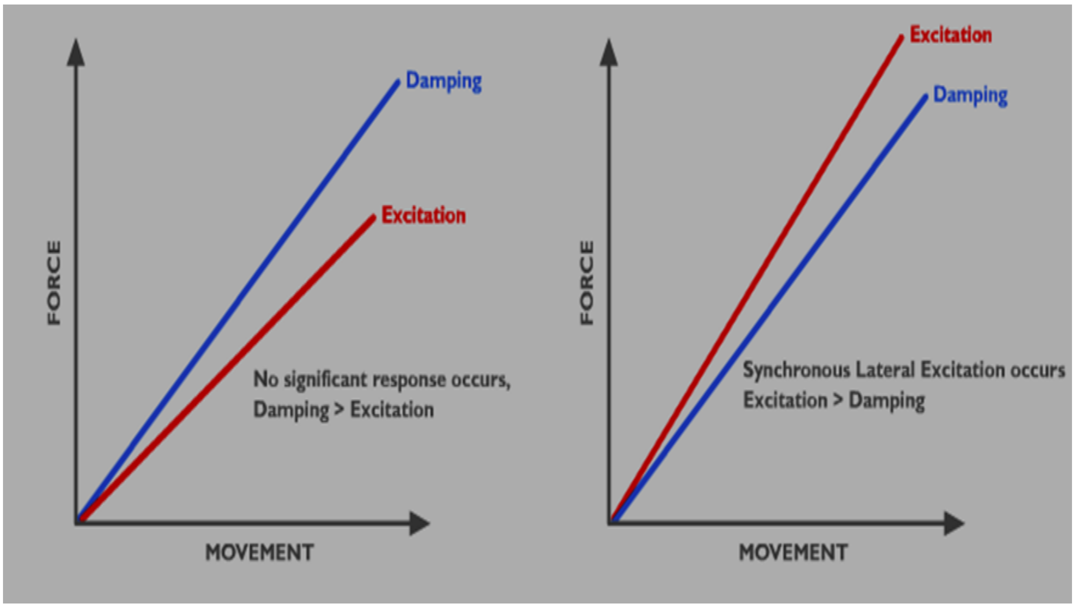

Figure 3.6.2: When the number of people increases, the excitation force is greater than the damping of the structure then Synchronous Lateral Excitation occurs and the sideways movements increase.

that:

- $s_0 = 166$;
- for $s \leq s_0$ there is no displacement because the excitation would be less than the damping;
- for $s \in]s_0, s_1]$ the bridge presents lateral oscillations as the load force is greater than the damping capacity.

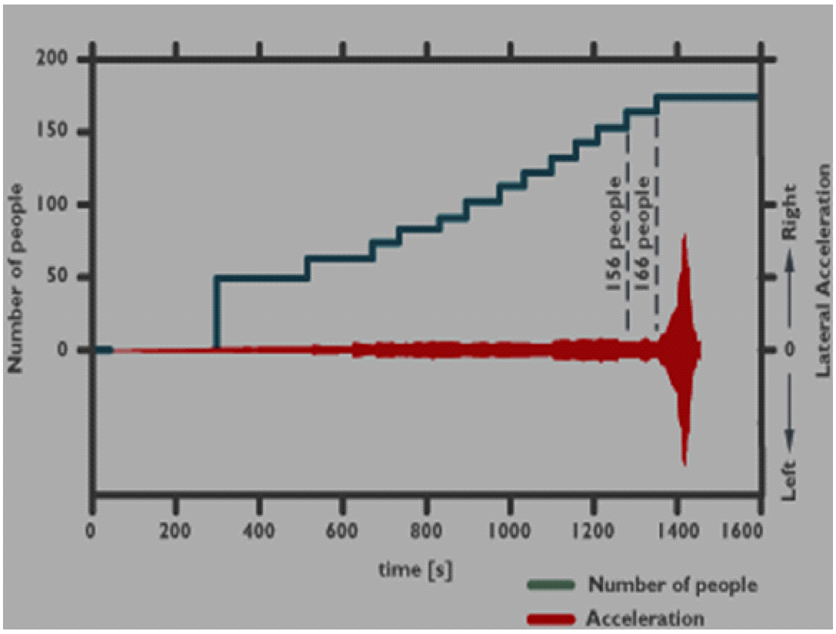

Figure 3.6.3: Relation between the number of pedestrians in the bridge at a given time and the lateral oscillations.

Chapter 4

Periodic impulsive fourth order problems

4.1 Introduction

The theory of impulsive problems is experiencing a rapid development in the last few years. Mainly because they have been used to describe some phenomena, arising from different disciplines like physics or biology, subject to instantaneous change at some time instants called moments. Second order periodic impulsive problems were studied to some extent, ([11, 64, 86]), however very few papers were dedicated to the study of third and higher order impulsive problems. One can refer for instance [63, 95] and the references therein.

Two types of fourth order impulsive problems will be considered in this chapter. Both are composed by the fully nonlinear equation

$$u^{(iv)}(x) = f\left(x, u(x), u'(x), u''(x), u'''(x)\right) \tag{4.1.1}$$

for a. e. $x \in I := [0,1] \setminus \{x_1, ..., x_m\}$ where $f : [0,1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ is L^1-

Carathéodory function, along with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(1), \; i = 0, 1, 2, 3. \tag{4.1.2}$$

The impulsive conditions are of two types. First problem contain the impulse assumptions

$$\begin{aligned} u\left(x_j^+\right) &= g_j\left(u\left(x_j\right)\right), \\ u'\left(x_j^+\right) &= h_j\left(u'\left(x_j\right)\right), \\ u''\left(x_j^+\right) &= k_j\left(u''\left(x_j\right)\right), \\ u'''\left(x_j^+\right) &= l_j\left(u'''\left(x_j\right)\right), \end{aligned} \tag{4.1.3}$$

and the second problem the mixed impulsive conditions

$$\begin{aligned} u\left(x_j^+\right) &= g_j\left(u\left(x_j\right)\right), \\ u'\left(x_j^+\right) &= h_j\left(u'\left(x_j\right)\right), \\ u''\left(x_j^+\right) &= k_j\left(u''\left(x_j\right)\right), \\ u''\left(x_j^+\right) &= u''\left(x_{j+1}\right), \end{aligned} \tag{4.1.4}$$

where g_j, h_j, k_j and l_j, for $j = 1, ..., m$, are given real valued functions satisfying some adequate conditions, and $x_j \in (0,1)$, such that $0 = x_0 < x_1 < ... < x_m < x_{m+1} = 1$.

The arguments applied in this chapter make use of the lower and upper solution method with an iterative technique (suggested in [9]) which is not necessarily monotone, together with classical results such as Lebesgue Dominated Convergence Theorem, Ascoli-Arzela Theorem and fixed point theory.

The sufficient conditions for the existence of solution for both problems are slightly different. In the problem (4.1.1)-(4.1.3) it is assumed that the third derivatives of the lower and upper solutions are well ordered. This is an "unusual" assumption in fourth order problems and it is "too strong". This condition can be weakened for problem (4.1.1), (4.1.2), (4.1.4), where it is assumed that the second derivative of the lower and upper solution are well ordered, or in reverse order. In addition to this more usual condition, two more features are required:

- a Nagumo-type condition to control the third derivative
- a "Dirichlet type" boundary condition in each subinterval defined by the impulsive moments.

For each problem it is presented an example to illustrate the existence and location parts of lower and upper solution method.

In both problems it is remarked that the definition of lower and upper solutions have different differential inequalities, as well as a different type of variation on the nonlinearity. As such it is reasonable to raise the question, for which more research is still required:

Is there a relation between the type of the differential inequality used in lower and upper solution and the "monotone" type assumption on the nonlinearity?

4.2 Definitions and auxiliary results

In this section some notations, definitions and auxiliary results, needed for the main existence result, are presented. For $m \in \mathbb{N}$, let $0 = x_0 < x_1 < ... < x_m < x_{m+1} = 1$ and $D = \{x_1, ..., x_m\}$ and define $x_j^{\pm} := \lim\limits_{x \to x_j^{\pm}} x$, for $j = 1, ..., m$.

Consider $PC^{(l)}(I)$, $l = 1, 2, 3$ as the space of the real-valued functions u, such that $u^{(l)} \in PC(I)$, $u^{(l)}\left(x_k^+\right)$ and $u^{(l)}\left(x_k^-\right)$ exist with $u^{(l)}\left(x_k^-\right) =$

$u^{(l)}(x_k)$, for $k = 1, 2, ..., m$. Therefore $u \in PC^3(I)$, it can be written as

$$u(x) = \begin{cases} u_0(x) & \text{if } x \in [0, x_1], \\ u_1(x) & \text{if } x \in (x_1, x_2], \\ \vdots & \\ u_m(x) & \text{if } x \in (x_m, 1], \end{cases}$$

where $u_m(x) \in C^3((x_i, x_{i+1}))$ for $i = 1, ..., m$.

Denote

$$PC^3_D(I) = \left\{u \in PC^3(I) : u''' \in AC(x_i, x_{i+1}), \ i = 0, 1, ..., m\right\}$$

and for each $u \in PC^3_D(I)$ we set the norm

$$\|u\|_D = \|u\| + \|u'\| + \|u''\| + \|u'''\|$$

.

Throughout this Chapter the following hypothesis will be assumed :

(I1) $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a L^1-Carathéodory function, that is, $f(x, \cdot, \cdot, \cdot, \cdot)$ is a continuous function for *a.e.* $x \in I$; $f(\cdot, y_0, y_1, y_2, y_3)$ is measurable for $(y_0, y_1, y_2, y_3) \in \mathbb{R}^4$; and for every $M > 0$ there is a real-valued function $\psi_M \in L^1([0,1])$ such that

$$|f(x, y_0, y_1, y_2, y_3)| \leq \psi_M(x), \text{ for a. e. } x \in [0,1]$$

and for every $(y_0, y_1, y_2, y_3) \in \mathbb{R}^4$ with $|y_i| \leq M$, for $i = 0, 1, 2, 3$.

(I2) the real valued functions g_j, h_j, k_j and l_j are nondecreasing, for $j = 1, ..., m$.

Definition 4.2.1 *A function $u \in PC^3_D(I)$ is a solution of (4.1.1)-(4.1.3) if it satisfies (4.1.1) almost everywhere in $I \setminus D$, the periodic conditions (4.1.2) and the impulse conditions (4.1.3).*

Next Lemma is a key tool to obtain the main result .

Lemma 4.2.2 *Let* $p : [0,1] \to \mathbb{R}$ *be a* L^1-*Carathéodory function. Then for each* $a_j,\ b_j,\ c_j,\ d_j \in \mathbb{R},\ j = 1,2...,m,$ *the initial value problem composed by the equation*

$$u^{(iv)}(x) = p(x) \ \textit{for a. e. } x \in (0,1) \tag{4.2.1}$$

and the boundary conditions

$$u\left(x_j^+\right) = a_j, \quad u'\left(x_j^+\right) = b_j, \quad u''\left(x_j^+\right) = c_j, \quad u'''\left(x_j^+\right) = d_j, \tag{4.2.2}$$

has a unique solution $u \in PC_D^3(I)$ *such that* $\|u\|_D \leq \delta$, *for* $\delta = |a_j| + 2|b_j| + \frac{5}{2}|c_j| + 4N.$

Proof. Define the operators $\mathcal{T} : PC_D^3(I) \to PC_D^3(I)$ given by

$$\begin{aligned} \mathcal{T}u \quad : \quad &= a_j + b_j\left(x - x_j^+\right) + c_j\frac{\left(x - x_j^+\right)^2}{2} \\ &+ d_j\frac{\left(x - x_j^+\right)^3}{3!} + \int_{x_j^+}^{x} \frac{(x-r)^3}{3!} u^{(iv)}(r)\, dr. \end{aligned} \tag{4.2.3}$$

As $p(x)$ is an L^1-Carathéodory function, then the operator $\mathcal{T}$ is continuous. As $p(x)$ is bounded in I, we can define $N = \|p(x)\|_1$. Therefore the following estimates can be obtained for $x \in (x_j, x_{j+1})$

$$|u(x)| \leq |a_j| + |b_j| + \frac{|c_j|}{2} + \frac{|d_j|}{3!} + N,$$

$$|u'(x)| \leq |b_j| + |c_j| + \frac{|d_j|}{2} + N,$$

$$|u''(x)| \leq |c_j| + |d_j| + N,$$

$$|u'''(x)| \leq |d_j| + N.$$

Hence, for $\delta := |a_j| + 2|b_j| + \frac{5}{2}|c_j| + \frac{8}{3}|d_j| + 4N$, it is obtained that

$$\|u\|_D = \|u\| + \|u'\| + \|u''\| + \|u'''\| \leq \delta. \tag{4.2.4}$$

Let $u \in PC_D^3(I)$ such that $\|u\|_D \leq \delta$, then by (4.2.4),

$$\begin{aligned}
\|\mathcal{T}u_n\|_D &= \|\mathcal{T}u_n\| + \left\|(\mathcal{T}u_n)'\right\| + \left\|(\mathcal{T}u_n)''\right\| + \left\|(\mathcal{T}u_n)'''\right\| \leq \\
&\leq |a_j| + |b_j| + \frac{|c_j|}{2} + \frac{|d_j|}{3!} + N + \\
&\quad |b_j| + |c_j| + \frac{|d_j|}{2} + N + |c_j| + |d_j| + N + |d_j| + N \\
&\leq \delta
\end{aligned}$$

As the operator $\mathcal{T}$ is uniformly bounded and equicontinuous by Ascoli-Arzela Theorem $\mathcal{T}$ is a compact operator. Moreover the set of solutions of the equation, $u = \mathcal{T}u$, is bounded. By Schauder fixed point Theorem this implies that $\mathcal{T}$ has a fixed point $u \in PC_D^3(I)$ given by

$$u(x) = \left[\begin{array}{c} a_j + b_j\left(x - x_j^+\right) + c_j\frac{\left(x-x_j^+\right)^2}{2} + d_j\frac{\left(x-x_j^+\right)^3}{3!} \\ +\int\limits_{x_j^+}^{x} \frac{(x-r)^3}{3!}p(r)\,dr \end{array}\right].$$

As

$$u'(x) = \left[\begin{array}{c} b_j + c_j\left(x - x_j^+\right) + d_j\frac{\left(x-x_j^+\right)^2}{2} \\ +\int\limits_{x_j^+}^{x} \frac{(x-r)^2}{2!}p(r)\,dr \end{array}\right],$$

$$u''(x) = \left[c_j + d_j\left(x - x_j^+\right) + \int\limits_{x_j^+}^{x} (x-r)\,p(r)\,dr\right],$$

$$u'''(x) = \left[d_j + \int\limits_{x_j^+}^{x} p(r)\,dr\right],$$

then this fixed point satisfies $u\left(x_j^+\right) = a_j$, $u'\left(x_j^+\right) = b_j$, $u''\left(x_j^+\right) = c_j$ and $u'''\left(x_j^+\right) = d_j$.

Assuming that the problem (4.1.1)-(4.1.3), has two solutions, u_1 and u_2, the uniqueness is easily obtained by the integration of (4.2.1) in (x_j, x_{j+1}).

∎

Lower and upper functions will be given by the next definition:

Definition 4.2.3 *A function $\alpha \in PC_D^3(I)$ is said to be a lower solution of the problem (4.1.1)-(4.1.3) if:*

(i) $\alpha^{(iv)}(x) \leq f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x))$, *for a.e.* $x \in (0,1)$.

(ii) $\alpha(0) \leq \alpha(1), \ \ \alpha'(0) \leq \alpha'(1), \ \ \alpha''(0) \leq \alpha''(1), \ \ \alpha'''(0) \leq \alpha'''(1),$

(iii) $\alpha\left(x_j^+\right) \leq g_j(\alpha(x_j)), \ \ \alpha'\left(x_j^+\right) \leq h_j(\alpha'(x_j)), \ \ \alpha''\left(x_j^+\right) \leq k_j(\alpha''(x_j)),$
$\alpha'''\left(x_j^+\right) \leq l_j(\alpha'''(x_j)),$

A function $\beta \in PC_D^3(I)$ is said to be a upper solution of the problem (4.1.1)-(4.1.3) if the reversed inequalities hold.

4.3 Existence of solutions

In this section the main existence and location result is presented.

Theorem 4.3.1 *Let α, β be, respectively, lower and upper solutions of (4.1.1)-(4.1.3) such that*

$$\alpha'''(x) \leq \beta'''(x) \ \ \text{on } I \setminus D, \tag{4.3.1}$$

and

$$\alpha^{(i)}(0) \leq \beta^{(i)}(0), \ \ i = 0, 1, 2. \tag{4.3.2}$$

Assume that conditions (I1) and (I2) hold and

$$f(x, \alpha, \alpha', \alpha'', y_3) \leq f(x, y_0, y_1, y_2, y_3) \leq f(x, \beta, \beta', \beta'', y_3), \tag{4.3.3}$$

for fixed $(x, y_3) \in I \times \mathbb{R}$ and $\alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x)$, for $i = 0, 1, 2$.

Then the problem (4.1.1)-(4.1.3) has a solution $u(x) \in PC_D^3(I)$, *such that*

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \text{ for } i = 0,1,2,3$$

for $x \in I \backslash D$.

Remark 4.3.2 *As one can notice by (4.3.2) the inequalities* $\alpha^{(i)}(x) \leq \beta^{(i)}(x)$ *hold for* $i = 0,1,2$ *and every* $x \in I$.

Proof. Consider the following modified problem composed by the equation

$$\begin{aligned} u^{(iv)}(x) = f\left(x, \delta_0(x, u(x)), \delta_1(x, u'(x)), \delta_2(x, u''(x)), \delta_3(x, u'''(x))\right) \\ -u'''(x) + \delta_3(x, u'''(x)), \end{aligned} \tag{4.3.4}$$

for $x \in (0,1)$ and $x \neq x_j$ where the continuous functions $\delta_i : \mathbb{R}^2 \to \mathbb{R}$, for $i = 0,1,2,3$, are given by

$$\delta_i(x, y_i) = \begin{cases} \beta^{(i)}(x) &, \quad y_i > \beta^{(i)}(x) \\ y_i &, \quad \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x) \\ \alpha^{(i)}(x) &, \quad y_i < \alpha^{(i)}(x), \end{cases} \tag{4.3.5}$$

with the boundary conditions (4.1.2) and the impulsive assumptions (4.1.3).

To prove the existence of solution for the problem (4.3.4),(4.1.2),(4.1.3) it is used an iterative, not monotone, technique. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of function in $PC_D^3(I)$ defined as follows

$$u_0 = \alpha \tag{4.3.6}$$

and for $n = 1,2,...$

$$\begin{aligned} u_n^{(iv)}(x) = f\left(x, \delta_0(x, u_{n-1}(x)), \delta_1\left(x, u'_{n-1}(x)\right), \delta_2\left(x, u''_{n-1}(x)\right), \delta_3(x, u'''_n(x))\right) \\ +u'''_n(x) - \delta_3(x, u'''_n(x)), \end{aligned} \tag{4.3.7}$$

for a.e. $x \in (0,1)$ with the boundary conditions

$$\begin{aligned} u_n(0) &= u_{n-1}(1), \quad u_n'(0) = u_{n-1}'(1), \\ u_n''(0) &= u_{n-1}''(1), \quad u_n'''(0) = u_{n-1}'''(1), \end{aligned} \tag{4.3.8}$$

and the impulsive conditions, for $j = 1, ..., m$,

$$\begin{array}{c} u_n\left(x_j^+\right) = g_j\left(u_{n-1}(x_j)\right), \quad u_n'\left(x_j^+\right) = h_j\left(u_{n-1}'(x_j)\right) \\ u_n''\left(x_j^+\right) = k_j\left(u_{n-1}''(x_j)\right), \quad u_n'''\left(x_j^+\right) = l_j\left(u_{n-1}'''(x_j)\right) \end{array} \tag{4.3.9}$$

By Lemma 4.2.2 the sequence $(u_n)_{n\in\mathbb{N}}$ is well defined.

Step 1 - *Every solution of (4.3.7)-(4.3.9) verifies*

$$\alpha^{(i)}(x) \leq u_n^{(i)}(x) \leq \beta^{(i)}(x), \text{ for } i = 0,1,2,3, \tag{4.3.10}$$

for all $n \in \mathbb{N}$ and every $x \in I$.

Let u be a solution of the problem (4.3.7)-(4.3.9). The proof of the inequalities (4.3.10) will be done by mathematical induction.

For $n = 0$, by (4.3.6)

$$\alpha'''(x) = u_0'''(x) \leq \beta'''(x), \text{ for } x \in I\backslash D,$$

and by Remark 4.3.2

$$\alpha^{(i)}(x) = u_0^{(i)}(x) \leq \beta^{(i)}(x), \text{ for } i = 0,1,2.$$

Suppose that for $k = 1, ..., n-1$, for $x \in I$

$$\alpha'''(x) \leq u_k'''(x) \leq \beta'''(x). \tag{4.3.11}$$

For $x = 0$, by (4.3.8), (4.3.11) and Definition 4.2.3,

$$u_n'''(0) = u_{n-1}'''(1) \geq \alpha'''(1) \geq \alpha'''(0).$$

If $x = x_j^+$, $j = 1, ..., m$, from (4.3.9), (I2), (4.3.11) and Definition 4.2.3,

$$u_n''' \left(x_j^+\right) = l_j \left(u_{n-1}''' \left(x_j\right)\right) \geq l_j \left(\alpha''' \left(x_j\right)\right) \geq \alpha''' \left(x_j^+\right).$$

For $x \in \left]x_j, x_{j+1}\right]$, $j = 1, 2, ..., m$, suppose, by contradiction, that there exists $x^* \in \left]x_j, x_{j+1}\right]$ such that $\alpha''' (x^*) > u_n''' (x^*)$ and define

$$\min_{x \in \left]x_j, x_{j+1}\right]} u_n''' (x) - \alpha''' (x) := u_n''' (x^*) - \alpha''' (x^*) < 0.$$

As by (4.3.9), $u_n''' \left(x_j^+\right) \geq \alpha''' \left(x_j^+\right)$, then there is an interval $(\underline{x}, \overline{x}) \subset (x_j, x^*)$ such that

$$u_n''' (x) < \alpha''' (x) \text{ and } u_n^{(iv)} (x) \leq \alpha^{(iv)} (x), \ \forall x \in (\underline{x}, \overline{x}).$$

From (4.3.4) and (4.3.3) the following contradiction is obtained for $x \in (\underline{x}, \overline{x})$

$$\begin{aligned}
0 &\geq u_n^{(iv)} (x) - \alpha^{(iv)} (x) \\
&= f \left(x, \delta_0 (x, u_{n-1} (x)), \delta_1 \left(x, u_{n-1}' (x)\right), \delta_2 \left(x, u_{n-1}'' (x)\right), \alpha''' (x)\right) \\
&\quad - u''' (x) + \alpha''' (x) - \alpha^{(iv)} (x) \\
&\geq f (x, \alpha (x), \alpha' (x), \alpha'' (x), \alpha''' (x)) - u''' (x) + \alpha''' (x) \\
&\quad - f (x, \alpha (x), \alpha' (x), \alpha'' (x), \alpha''' (x)) \\
&\geq \alpha''' (x) - u''' (x) > 0.
\end{aligned}$$

Therefore $u_n''' (x) \geq \alpha''' (x)$, for all $n \in \mathbb{N}$ and every $x \in I$. In the same way it can be shown that $u_n''' (x) \leq \beta''' (x)$, $\forall x \in I$, $\forall n \in \mathbb{N}$, and so (4.3.10) is proved for $i = 3$.

Consider now the inequality $\alpha'' (x) \leq u_n'' (x) \leq \beta'' (x)$, for all $n \in \mathbb{N}$ and every $x \in I$.

To justify (4.3.10) for $i = 2$, notice that for $n = 0$, the proof is obtained in a similar way as in above.

Assuming that for $k = 1, ..., n-1$ and every $x \in I$,

$$\alpha''(x) \leq u_k''(x) \leq \beta''(x). \tag{4.3.12}$$

then for $x \in [0, x_1]$, by integration of the inequality $u_n'''(x) \geq \alpha'''(x)$ in $[0, x]$ we have

$$u_n''(x) - u_n''(0) \geq \alpha''(x) - \alpha''(0).$$

By (4.3.8) and (4.3.12),

$$\begin{aligned} u_n''(x) &\geq \alpha''(x) - \alpha''(0) + u_{n-1}''(1) \\ &\geq \alpha''(x) - \alpha''(0) + \alpha''(1) \geq \alpha''(x) \end{aligned}$$

hence $u_n''(x) \geq \alpha''(x)$, for all $x \in [0, x_1]$.

For $x \in]x_j, x_{j+1}]$, $j = 1, 2, ..., m$, by integration of the inequality $u_n'''(x) \geq \alpha'''(x)$ in $x \in]x_j, x_{j+1}]$,

$$u_n''(x) \geq \alpha''(x) - \alpha''\left(x_j^+\right) + u_n''\left(x_j^+\right),$$

and by (4.3.9) and Definition 4.2.3

$$u_n''(x) \geq \alpha''(x) - \alpha''\left(x_j^+\right) + k_j\left(u_{n-1}''(x_j)\right) \geq \alpha''(x).$$

obtaining that $u_n''(x) \geq \alpha''(x)$, for all $n \in \mathbb{N}$ and every $x \in I$. Using similar arguments it can be proved that $u_n''(x) \leq \beta''(x)$ and therefore

$$\alpha''(x) \leq u_n''(x) \leq \beta''(x), \ \forall x \in I, \forall n \in \mathbb{N}. \tag{4.3.13}$$

For the inequality $\alpha'(x) \leq u_n'(x) \leq \beta'(x)$,for all $n \in \mathbb{N}$ and every $x \in I$.

For $n = 0$, the proof is obtained analogously to the previous cases.

Suppose that for $k = 1, ..., n-1$ and every $x \in I$ we have

$$\alpha'(x) \leq u_k'(x) \leq \beta'(x). \tag{4.3.14}$$

For $x \in [0, x_1]$, integrating (4.3.13) we have

$$u_n'(x) - u_n'(0) \geq \alpha'(x) - \alpha'(0).$$

By (4.3.8) and (4.3.14) it is obtained that

$$\begin{aligned} u_n'(x) &\geq \alpha'(x) - \alpha'(0) + u_{n-1}'(1) \\ &\geq \alpha'(x) - \alpha'(0) + \alpha'(1) \geq \alpha'(x), \end{aligned}$$

hence $u_n'(x) \geq \alpha'(x)$, for all $x \in [0, x_1]$.

For $x \in]x_j, x_{j+1}]$, $j = 1, 2, ..., m$. Integrating the same inequality in $]x_j, x_{j+1}]$, then

$$u_n'(x) \geq \alpha'(x) - \alpha'\left(x_j^+\right) + u_n'\left(x_j^+\right).$$

By (4.3.9), (I2) and Definition 4.2.3

$$\begin{aligned} u_n'(x) &\geq \alpha'(x) - \alpha'\left(x_j^+\right) + u_n'\left(x_j^+\right) \\ &\geq \alpha'(x) - \alpha'\left(x_j^+\right) + h\left(u_{n-1}'\left(x_j^+\right)\right) \geq \alpha'(x), \end{aligned}$$

obtaining that $u_n'(x) \geq \alpha'(x)$, for all $n \in \mathbb{N}$ and every $x \in I$. Using similar arguments it can be proved that $u_n'(x) \leq \beta'(x)$.

Moreover, through the same process, the inequality $\alpha(x) \leq u_n(x) \leq \beta(x)$, is obtained for all $n \in \mathbb{N}$ and every $x \in I$.

<u>*Step 2*</u> - *The sequence $(u_n)_{n\in\mathbb{N}}$ is convergent to u solution of (4.3.7)-(4.3.9).*

Let $C_i = \max\left\{\left\|\alpha^{(i)}\right\|, \left\|\beta^{(i)}\right\|\right\}$, for $i = 0, 1, 2, 3$, so there exists $M > 0$, with $M := \sum_{i=0}^{3} C_i$, and for all $n \in \mathbb{N}$,

$$\|u_n\|_D \leq M. \tag{4.3.15}$$

Let Ω be a compact subset of $\mathbb{R}^4$ given by

$$\Omega = \left\{(w_0, w_1, w_2, w_3) \in \mathbb{R}^4 : \|w_i\| \leq C_i, \ i = 0, 1, 2, 3\right\}.$$

As f is a L^1-Carathéodory function in Ω, then there exists a real-valued function $\psi_M(x) \in L^1(I)$, such that

$$|f(x, w_0, w_1, w_2, w_3)| \leq \psi_M(x), \text{ for every } (w_0, w_1, w_2, w_3) \in \Omega. \quad (4.3.16)$$

By Step1 and (4.3.15), $(u_n, u_n', u_n'', u_n''') \in \Omega$, for all $n \in \mathbb{N}$. From (4.3.7) and (4.3.16) we obtain

$$\left|u_n^{(iv)}(x)\right| \leq \psi_M(x) + 2C_3, \text{ for a.e. } x \in I,$$

hence $u_n^{(iv)}(x) \in L^1(I)$.

By integration in I we obtain that

$$u_n'''(x) = u_n'''(0) + \int_0^x u_n^{(iv)}(s)\, ds + \sum_{0<x_j\leq x} l_j\left(u_{n-1}'''(x_j)\right),$$

therefore $u_n''' \in AC(x_j, x_{j+1})$ and $u_n \in PC_D^3(I)$. By Ascoli-Arzela Theorem there exists a subsequence denoted by $(u_n)_{n\in\mathbb{N}}$, which converges to $u \in PC_D^3(I)$. Then $(u, u', u'', u''') \in \Omega$.

Using the Lebesgue dominated convergence theorem, for $x \in (x_j, x_{j+1})$,

$$\int_{x_j}^x \left[\begin{array}{c} f\left(s, \delta_0(s, u_{n-1}(s)), \delta_1\left(s, u_{n-1}'(s)\right), \delta_2\left(s, u_{n-1}''(s)\right), \delta_3(s, u_n'''(s))\right) \\ -u_n'''(s) + \delta_3(x, u_n'''(s)) \end{array}\right] ds$$

is convergent to

$$\int_{x_j}^x \left[\begin{array}{c} f(s, \delta_0(s, u(s)), \delta_1(s, u'(s)), \delta_2(s, u''(s)), \delta_3(s, u'''(s))) \\ -u'''(s) + \delta_3(x, u'''(s)) \end{array}\right] ds$$

as $n \to \infty$.

Therefore as $n \to \infty$

$$u_n'''(x) = u_n'''(x_j) +$$

$$\int_{x_j}^x \left[\begin{array}{c} f\left(s, \delta_0(s, u_{n-1}(s)), \delta_1\left(s, u_{n-1}'(s)\right), \delta_2\left(s, u_{n-1}''(s)\right), \delta_3(s, u_n'''(s))\right) \\ -u_n'''(s) + \delta_3(x, u_n'''(s)) \end{array}\right] ds$$

is convergent to

$$u'''(x) = u'''(x_j) + \int_{x_j}^{x} \left[\begin{array}{c} f(s, \delta_0(s, u(s)), \delta_1(s, u'(s)), \delta_2(s, u''(s)), \delta_3(s, u'''(s))) \\ -u'''(s) + \delta_3(x, u'''(s)) \end{array} \right] ds.$$

As the function f is L^1-Carathéodory function in (x_j, x_{j+1}), then $u'''(x) \in AC(x_j, x_{j+1})$. Therefore $u \in PC_D^3(I)$ and u is a solution of (4.3.7)-(4.3.9).

To prove that u is a solution of the initial problem (4.1.1)-(4.1.3) we note that taking the limit in (4.3.8) and (4.3.9), as $n \to \infty$, by the convergence of u_n then u verifies (4.1.2) and, by the continuity of the impulsive functions, u verifies (4.1.3). By (4.3.5), Step 1 and the convergence of u_n, u verifies (4.1.1).

Then problem (4.1.1)-(4.1.3) has a solution $u(x) \in PC_D^3(I)$, such that

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \text{for } i = 0, 1, 2, 3,$$

for $x \in I$. ■

Example 4.3.3 *Let us consider the following nonlinear impulsive boundary value problem, composed by the equation*

$$u^{(iv)}(x) = (u(x))^3 + \arctan(u'(x) + 1) + 0.01(u''(x))^5 + k|u'''(x)|^{\theta}, \tag{4.3.17}$$

where $0 < \theta \leq 2$ *and* $k \leq -677$*, for all* $x \in [0,1] \setminus \left\{\frac{1}{2}\right\}$ *along with the boundary conditions (4.1.2) and for* $x = \frac{1}{2}$ *the impulse conditions*

$$\begin{aligned} u\left(\tfrac{1}{2}^{+}\right) &= \mu_1\left(u\left(\tfrac{1}{2}\right)\right)^3 \\ u'\left(\tfrac{1}{2}^{+}\right) &= \mu_2\left(u'\left(\tfrac{1}{2}\right)\right) \\ u''\left(\tfrac{1}{2}^{+}\right) &= \mu_3\sqrt[3]{\left(u''\left(\tfrac{1}{2}\right)\right)} \\ u'''\left(\tfrac{1}{2}^{+}\right) &= \mu_4\left(u'''\left(\tfrac{1}{2}\right)\right)^5, \end{aligned} \tag{4.3.18}$$

with $\mu_i \in \mathbb{R}^+$, $i = 1, 2, 3, 4$.

Obviously this problem is a particular case of (4.1.1)-(4.1.3) with

$$f(x, y_0, y_1, y_2, y_3) = (y_0)^3 + \arctan(y_1 + 1) + 0.01 (y_2)^5 + k |y_3|^\theta,$$

for all $x \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, $m = 1$, $x_1 = \frac{1}{2}$ *and the nondecreasing functions* g_j, h_j, k_j *and* l_j *given by* $g(x) = \mu_1 x^3$, $h(x) = \mu_2 x$, $k(x) = \mu_3 \sqrt[3]{x}$, $l(x) = \mu_4 x^5$.

One can verify that the functions $\alpha(x) = 0$ *and*

$$\beta(x) = \begin{cases} x^3 + 3x^2 + 4x + \frac{3}{2} & , \ x \in \left[0, \frac{1}{2}\right] \\ x^3 & , \ x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

are $PC_D^3(I)$ *for* $D = \left\{\frac{1}{2}\right\}$ *and considering*

$$\beta'(x) = \begin{cases} 3x^2 + 6x + 4 & , \ x \in \left[0, \frac{1}{2}\right] \\ 3x^2 & , \ x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

and

$$\beta''(x) = \begin{cases} 6x + 6 & , \ x \in \left[0, \frac{1}{2}\right] \\ 6x & , \ x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Moreover, they are lower and upper solutions, respectivelly, for the problem (4.3.17), (4.1.2), (4.3.18), with

$$\mu_1 \leq \frac{64}{42875}, \quad \mu_2 \leq \frac{3}{31}, \ \mu_3 \leq \sqrt[3]{3}, \quad \mu_4 \leq \frac{1}{6^4}.$$

As f *verifies (4.3.3), therefore by Theorem 4.3.1 there is a solution* $u(x) \in PC_D^3(I)$, *such that*

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \ \text{for } i = 0, 1, 2, 3.$$

Remark that this solution can not be a trivial periodic solution, as the only constant solution of (4.3.17) is $u(x) = -\sqrt[3]{\frac{\pi}{4}}$ *which is not in region displayed in Figure 4.3.1.*

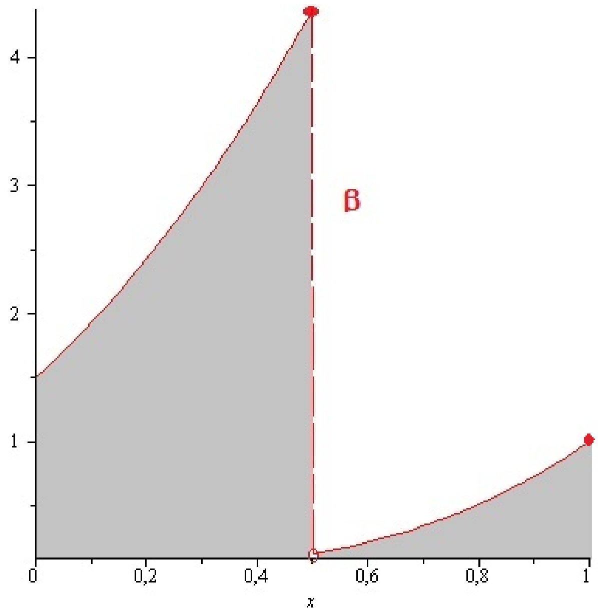

Figure 4.3.1: Region where solution u lies

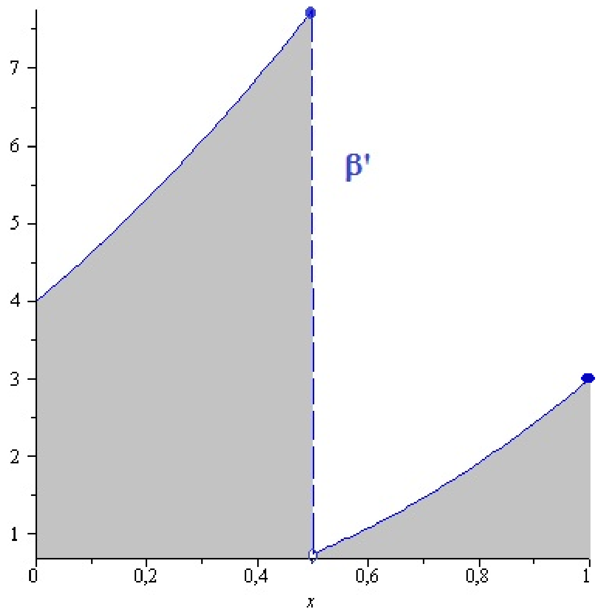

Figure 4.3.2: Region where solution u' lies

4.4 Mixed impulsive boundary conditions

In this section we deal with the impulsive problem composed by the fourth order fully nonlinear equation (4.1.1) with the periodic boundary conditions (4.1.2) and the mixed impulsive conditions of "Dirichlet type" (4.1.4).

The arguments of the proof require the following lemma:

Lemma 4.4.1 *[93, Lemma 2] For $z, w \in C(I)$ such that $z(x) \leq w(x)$, for every $x \in I$, define*

$$q(x,u) = max\{z\,(x)\,, min\{u, w\,(x)\}\}. \tag{4.4.1}$$

Then, for each $u \in C^1(I)$ the next two properties hold:

(a) $\dfrac{d}{dx}[q(x,u(x))]$ *exists for a.e. $x \in I$.*

(b) If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$ then

$$\frac{d}{dx}\left[q(x,u_m(x))\right] \to \frac{d}{dx}\left[q(x,u(x))\right] \text{ for a.e. } x \in I.$$

Next lemma will provide uniqueness of solution:

Lemma 4.4.2 *Let $p : [0,1] \to \mathbb{R}$ be a L^1-Carathéodory function verifying a Nagumo type condition in*

$$E = \left\{(x, y_0, y_1, y_2, y_3) \in I \times \mathbb{R}^4 : \gamma_i\,(x) \leq y_i \leq \Gamma_i\,(x)\,,\ i = 0,1,2\right\},$$

for some continuous functions γ_i, Γ_i, $i = 0,1,2$ such that $\gamma_i\,(x) \leq \Gamma_i\,(x)$, $\forall x \in [0,1]$. Then for each a_j, b_j, $c_j \in \mathbb{R}$, $j = 1,2...,m$, the boundary value problem composed by the equation

$$u^{(iv)}\,(x) = p\,(x) \text{ for a. e. } x \in (0,1) \tag{4.4.2}$$

$$u\left(x_j^+\right) = a_j, \quad u'\left(x_j^+\right) = b_j, \quad u''\left(x_j^+\right) = c_j, \quad u''\left(x_{j+1}\right) = u''\left(x_j^+\right), \tag{4.4.3}$$

has a unique solution $u \in PC_D^3\,(I)$.

Proof. From condition $u''\left(x_{j+1}\right)=u''\left(x_j^+\right)$ there is $\xi\in\left(x_j,x_{j+1}\right)$ such that $u'''\left(\xi\right)=0$. Define the operators $\mathcal{V}:PC_D^3\left(I\right)\to PC_D^3\left(I\right)$ given by

$$\mathcal{V}u:=a_j+b_j\left(x-x_j^+\right)+c_j\frac{\left(x-x_j^+\right)^2}{2}+\int_{x_j^+}^{x}\frac{\left(x-r\right)^2}{2}u'''\left(r\right)dr \qquad (4.4.4)$$

As $p\left(x\right)$ is a L^1-Carathéodory function, then the operator $\mathcal{V}$ is continuous. Since $p\left(x\right)$ is bounded in I, we can define $N=\max\left\{\left|p\left(x\right)\right|:x\in I\right\}$. Therefore the following estimates can be obtained

$$\begin{gathered}\gamma_0:=-\left|a_j\right|-\left|b_j\right|-\tfrac{\left|c_j\right|}{2}-N\leq\left|u\left(x\right)\right|\leq\left|a_j\right|+\left|b_j\right|+\tfrac{\left|c_j\right|}{2}+N:=\Gamma_0\\ \gamma_1:=-\left|b_j\right|-\left|c_j\right|-N\leq\left|u'\left(x\right)\right|\leq\left|b_j\right|+\left|c_j\right|+N:=\Gamma_1\\ \gamma_2:=-\left|c_j\right|-N\leq\left|u''\left(x\right)\right|\leq\left|c_j\right|+N:=\Gamma_2\\ -N\leq\left|u'''\left(x\right)\right|\leq N.\end{gathered}$$

Then by Lemma 1.2.2 there $R>0$, such that $\left|u'''\left(x\right)\right|<R$, for $x\in\left[0,1\right]$. Then defining $\delta_*:=\left|a_j\right|+2\left|b_j\right|+\frac{5}{2}\left|c_j\right|+4N$ it is obtained that

$$\left\|u\right\|_D=\left\|u\right\|+\left\|u'\right\|+\left\|u''\right\|+\left\|u'''\right\|\leq\delta_*. \qquad (4.4.5)$$

Let $u\in PC_D^3\left(I\right)$ such that $\left\|u\right\|_D\leq\delta$, then by (4.4.5),

$$\begin{aligned}&\left\|\mathcal{V}u_n\right\|_D=\left\|\mathcal{V}u_n\right\|+\left\|\left(\mathcal{V}u_n\right)'\right\|+\left\|\left(\mathcal{V}u_n\right)''\right\|+\left\|\left(\mathcal{V}u_n\right)'''\right\|\\ \leq\;&\left|a_j\right|+\left|b_j\right|+\frac{\left|c_j\right|}{2}+\left|b_j\right|+\left|c_j\right|+\left|c_j\right|+4N\leq\delta_*.\end{aligned}$$

As the operator $\mathcal{V}$ is uniformly bounded and equicontinuous by Ascoli-Arzela's theorem $\mathcal{V}$ is a compact operator. Moreover the set of solutions of the equation $u=\mathcal{V}u$, is bounded. By Schauder fixed point Theorem this implies that $\mathcal{V}$ has a fixed point $u\in PC_D^3\left(I\right)$ given by

$$u\left(x\right)=\left[a_j+b_j\left(x-x_j^+\right)+c_j\frac{\left(x-x_j^+\right)^2}{2}+\int_{x_j^+}^{x}\frac{\left(x-r\right)^2}{2}u'''\left(r\right)dr\right].$$

Suppose that the problem (4.4.2)-(4.4.3), has two solutions, u_1 and u_2. For $x \in (x_j, x_{j+1})$

$$u_1^{(iv)}(x) = p(x) \text{ and } u_2^{(iv)}(x) = p(x),$$

then

$$u_1^{(iv)}(x) = u_2^{(iv)}(x).$$

By integration for $x \in (x_j, x_{j+1})$,

$$\int_{x_j^+}^{x_{j+1}} \int_{x_j^+}^{x} u_1^{(iv)}(s)\, ds dx = \int_{x_j^+}^{x_{j+1}} \int_{x_j^+}^{x} u_2^{(iv)}(s)\, ds dx,$$

$$\int_{x_j^+}^{x_{j+1}} \left[u_1'''(x) - u_1'''\left(x_j^+\right)\right] dx = \int_{x_j^+}^{x_{j+1}} \left[u_2'''(x) - u_2'''\left(x_j^+\right)\right] dx,$$

and

$$\begin{aligned} & u_1''(x_{j+1}) - u_1''\left(x_j^+\right) - u_1'''\left(x_j^+\right)\left(x_{j+1} - x_j^+\right) \\ = \; & u_2''(x_{j+1}) - u_2''\left(x_j^+\right) - u_2'''\left(x_j^+\right)\left(x_{j+1} - x_j^+\right). \end{aligned}$$

As

$$u_1'''(x) - u_1'''\left(x_j^+\right) = \int_{x_j^+}^{x} u_1^{(iv)}(s)\, ds = \int_{x_j^+}^{x} u_2^{(iv)}(s)\, ds = u_2'''(x) - u_1'''\left(x_j^+\right), \tag{4.4.6}$$

then, $u_1'''(x) = u_2'''(x)$. Again by integration in (x_j, x_{j+1}) and (4.4.3) it is obtained that

$$u_1''(x) = u_2''(x),\; u_1'(x) = u_2'(x),\; u_1(x) = u_2(x).$$

for every $x \in I$. ■

Definition 4.4.3 *A function $\alpha \in PC_D^3(I)$ is said to be a lower solution of the problem (4.1.1), (4.1.2), (4.1.4) if:*

(i) $\alpha^{(iv)}(x) \geq f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x))$, *for a.e.* $x \in (0,1)$.

(ii) $\alpha(0) \leq \alpha(1), \quad \alpha'(0) \leq \alpha'(1), \quad \alpha''(0) \leq \alpha''(1),$

(iii) $\alpha\left(x_j^+\right) \leq g_j(\alpha(x_j)), \quad \alpha'\left(x_j^+\right) \leq h_j(\alpha'(x_j)), \quad \alpha''\left(x_j^+\right) \leq k_j(\alpha''(x_j)),$
$\alpha''\left(x_j^+\right) \leq \alpha''(x_{j+1})$.

A function $\beta \in PC_D^3(I)$ *is an upper solution of the problem (4.1.1), (4.1.2), (4.1.4) if the reversed inequalities hold.*

In this section the main existence and location result is presented.

Theorem 4.4.4 *Let* α, β *be, respectively, lower and upper solutions of (4.1.1)-(4.1.4) such that*

$$\alpha''(x) \leq \beta''(x) \ \text{ on } I \backslash D \tag{4.4.7}$$

and

$$\alpha^{(i)}(0) \leq \beta^{(i)}(0), \ i = 0, 1. \tag{4.4.8}$$

If f *verifies a Nagumo-type condition in*

$$E = \left\{(x, y_0, y_1, y_2, y_3) \in I \times \mathbb{R}^4 : \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), \ i = 0, 1\right\},$$

and conditions (I1) and (I2) hold,

$$f(x, \alpha, \alpha', y_2, y_3) \geq f(x, y_0, y_1, y_2, y_3) \geq f(x, \beta, \beta', y_2, y_3), \tag{4.4.9}$$

for fixed $(x, y_2, y_3) \in I \times \mathbb{R}^2$ *and* $\alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x)$, *for* $i = 0, 1, 2$, *then the problem 4.1.1), (4.1.2), (4.1.4) has a solution* $u(x) \in PC_D^3(I)$, *such that*

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \ \text{for } i = 0, 1, 2, \ \text{for } x \in I \backslash D.$$

Proof. Consider the following modified problem composed by the equation

$$u^{(iv)}(x) = f\left(x, \delta_0(x, u(x)), \delta_1(x, u'(x)), \delta_2(x, u''(x)), q\left(\tfrac{d}{dx}(\delta_2(x, u''(x)))\right)\right) \\ +u''(x) - \delta_2(x, u''(x)), \tag{4.4.10}$$

for $x \in (0,1)$ and $x \neq x_j$ where the continuous functions $\delta_i : \mathbb{R}^2 \to \mathbb{R}$, for $i = 0, 1$, are given by (4.3.5) and q by (4.4.1), the boundary conditions (4.1.2) and the impulsive conditions (4.1.4).

To prove the existence of solution for the problem (4.4.10),(4.1.2),(4.1.4) it is applied a non monotone iterative technique similar to the previous section. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions in $PC^3_D(I)$ defined by, $u_0 = \alpha$ and for $n = 1, 2, ...$

$$u_n^{(iv)}(x) = f\begin{pmatrix} x, \delta_0(x, u_{n-1}(x)), \delta_1\left(x, u'_{n-1}(x)\right), \delta_2\left(x, u''_{n-1}(x)\right), \\ q\left(\frac{d}{dx}(\delta_2(x, u''_n(x)))\right) \end{pmatrix} \\ +u''_n(x) - \delta_2(x, u''_n(x)), \tag{4.4.11}$$

for a.e. $x \in (0,1)$,

$$\begin{aligned} u_n(0) &= u_{n-1}(1), \quad u'_n(0) = u'_{n-1}(1), \\ u''_n(0) &= u''_{n-1}(1), \quad u'''_n(0) = u'''_{n-1}(1), \end{aligned} \tag{4.4.12}$$

and the impulsive conditions,

$$\begin{aligned} &u_n\left(x_j^+\right) = g_j(u_{n-1}(x_j)), \quad u'_n\left(x_j^+\right) = h_j\left(u'_{n-1}(x_j)\right) \\ &u''_n\left(x_j^+\right) = k_j\left(u''_{n-1}(x_j)\right), \quad u''_n\left(x_j^+\right) = u''_{n-1}(x_{j+1}), \end{aligned} \tag{4.4.13}$$

for $j = 1, ..., m$.

By Lemma 4.4.2 the sequence $(u_n)_{n\in\mathbb{N}}$ is well defined.

Step 1 - For every solution $u_n(x)$ of (4.4.11), (4.4.12), (4.4.13) we have for all $n \in \mathbb{N}$

$$\alpha^{(i)}(x) \leq u_n^{(i)}(x) \leq \beta^{(i)}(x), \text{ for } i = 0, 1, 2,$$

for every $x \in I \backslash D$.

Let $u_n(x)$ be a sequence of solutions of (4.4.11), (4.4.12), (4.4.13). By mathematical induction, for $n = 0$, by (4.4.8) the inequalities

$$\alpha^{(i)}(x) \leq u_0(x) \leq \beta^{(i)}(x), \text{ for } i = 0, 1, 2,$$

hold for $x \in I \backslash D$.

Suppose that for $k = 1, ..., n-1$ and every $x \in I \backslash D$ we have

$$\alpha''(x) \leq u_k''(x) \leq \beta''(x). \tag{4.4.14}$$

For $x = 0$, from (4.4.12), (4.4.14) and Definition 4.4.3 it is obtained that

$$u_n''(0) = u_{n-1}''(1) \geq \alpha''(1) \geq \alpha''(0).$$

For $x = x_j^+$, with $j = 1, ..., m$, from (4.4.13), (I2), (4.4.14) and Definition 4.4.3 we obtain

$$u_n''\left(x_j^+\right) = k_j\left(u_{n-1}''(x_j)\right) \geq k_j\left(\alpha''(x_j)\right) \geq \alpha''\left(x_j^+\right).$$

For $x \in]x_j, x_{j+1}[$, $j = 1, 2, ..., m$, suppose that there exists $x \in]x_j, x_{j+1}[$ such that $\alpha''(x) > u_n''(x)$ and define

$$\min_{x \in]x_j, x_{j+1}[} u_n''(x) - \alpha''(x) := u_n''(x^*) - \alpha''(x^*) < 0.$$

As $u_n''\left(x_j^+\right) \geq \alpha''\left(x_j^+\right)$ then there is an interval $(\underline{x}, \overline{x}) \subset (x_j, x_{j+1})$ such that $x^* \in (\underline{x}, \overline{x})$ and

$$u_n''(x) < \alpha''(x), \quad \forall x \in (\underline{x}, \overline{x}).$$

Then $u_n'''(x^*) - \alpha'''(x^*) = 0$ and, from (4.4.9), for $x \in (\underline{x}, \overline{x})$

$$\begin{aligned} 0 &\leq u_n^{(iv)}(x) - \alpha^{(iv)}(x) \\ &= f\left(x, \delta_0(x, u_{n-1}(x)), \delta_1\left(x, u_{n-1}'(x)\right), \alpha''(x), \alpha'''(x)\right) \\ &\quad + u''(x) - \alpha''(x) - \alpha^{(iv)}(x) \\ &\leq f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)) + u'''(x) - \alpha'''(x) \\ &\quad - f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)) \\ &\leq \alpha''(x) - u''(x) \leq 0. \end{aligned}$$

Therefore $(u_n''' - \alpha''')(x)$ is decreasing in $(\underline{x}, \overline{x})$ and $(u_n''' - \alpha''')(x) < 0$ in $(\underline{x}, \overline{x})$, which is a contradiction with the definition of x^*.

For $x = x_{j+1}$, with $j = 1, ..., m$, from (4.4.13), (4.4.14) and Definition 4.4.3 we obtain

$$u_n''(x_{j+1}) = u_{n-1}''\left(x_j^+\right) \geq \alpha''\left(x_j^+\right) \geq \alpha''(x_{j+1}).$$

Therefore $u_n''(x) \geq \alpha''(x)$, for all $n \in \mathbb{N}$ and every $x \in I \backslash D$. In the same way it can be shown that $u_n''(x) \leq \beta''(x)$ and so

$$\alpha''(x) \leq u_n''(x) \leq \beta''(x), \ \forall x \in I, \ \forall n \in \mathbb{N}. \tag{4.4.15}$$

To prove the inequalities $\alpha'(x) \leq u_n'(x) \leq \beta'(x)$, for all $n \in \mathbb{N}$ and every $x \in I$, suppose that for $k = 1, ..., n-1$ and every $x \in I$ we have

$$\alpha'(x) \leq u_k'(x) \leq \beta'(x). \tag{4.4.16}$$

Integrating the first inequality of (4.4.15) for $x \in [0, x_1]$

$$u_n'(x) - u_n'(0) \geq \alpha'(x) - \alpha'(0).$$

By (4.4.12) and (4.4.16) it is obtained that

$$\begin{aligned} u_n'(x) &\geq \alpha'(x) - \alpha'(0) + u_{n-1}'(1) \\ &\geq \alpha'(x) - \alpha'(0) + \alpha'(1) \geq \alpha'(x). \end{aligned}$$

For $x \in \left]x_j, x_{j+1}\right]$, $j = 1, 2, ..., m$, again by integration (4.4.13) and Definition 4.4.3

$$\begin{aligned} u_n'(x) &\geq \alpha'(x) - \alpha'\left(x_j^*\right) + u_n'\left(x_j^*\right) \\ &= \alpha'(x) - \alpha'\left(x_j^*\right) + h_j\left(u_{n-1}'(x_j)\right) \\ &\geq \alpha'(x), \end{aligned}$$

obtaining that $u_n'(x) \geq \alpha'(x)$, for all $n \in \mathbb{N}$ and every $x \in I$. Using similar arguments it can be proved that $u_n'(x) \leq \beta'(x)$ and so

$$\alpha'(x) \leq u_k'(x) \leq \beta'(x), \ \forall x \in I, \ \forall n \in \mathbb{N}. \tag{4.4.17}$$

For last inequalities $\alpha(x) \leq u_n(x) \leq \beta(x)$, for all $n \in \mathbb{N}$ and every $x \in I$, for $n = 0$, the proof is obtained analogously to the previous cases.

Assume that for $k = 1, ..., n-1$ and every $x \in I$ we have

$$\alpha(x) \leq u_k(x) \leq \beta(x). \tag{4.4.18}$$

By integration, Definition 4.4.3, (4.4.13) and (4.4.18), for $x \in [0, x_1]$, it is obtained that

$$\begin{aligned} u_n(x) &\geq \alpha(x) - \alpha(0) + u_{n-1}(1) \\ &\geq \alpha(x) - \alpha(0) + \alpha(1) \geq \alpha(x), \end{aligned}$$

hence $u_n(x) \geq \alpha(x)$, for all $x \in [0, x_1]$ and every $n \in \mathbb{N}$.

For $x \in \left]x_j, x_{j+1}\right]$, $j = 1, 2, ..., m$, integrating the same inequality, by (4.4.13), (I2) and Definition 4.4.3

$$u_n(x) \geq \alpha(x) - \alpha\left(x_j^+\right) + u_n\left(x_j^+\right) \geq$$
$$\alpha(x) - \alpha\left(x_j^+\right) + g\left(u_{n-1}\left(x_j^+\right)\right) \ \geq \alpha(x),$$

holds, for all $n \in \mathbb{N}$ and every $x \in I$. Using similar arguments it can be proved that $u_n(x) \leq \beta(x)$, $\forall n \in \mathbb{N}$ and every $x \in I$.

Step 2 - The sequence $(u_n)_{n\in\mathbb{N}}$ is convergent to u solution of (4.1.1), (4.1.2), (4.1.4).

As f verifies a Nagumo type condition in

$$\bar{E} = \left\{(x, y_0, y_1, y_2, y_3) \in I \times \mathbb{R}^4 : \alpha^{(i)}(x) \le y_i \le \beta^{(i)}(x),\ i = 0, 1, 2\right\},$$

then by Lemma 1.2.2, with $n = 4$ and applied to $\bar{E}$ we may define $C_i = \max\left\{\left\|\alpha^{(i)}\right\|, \left\|\beta^{(i)}\right\|\right\}$, for $i = 0, 1, 2$ and $C_3 := R$, with R given by Nagumo condition. Then we can conclude that exists $M > 0$, such that $M := \sum_{i=0}^{3} C_i$ and for all $n \in \mathbb{N}$

$$\|u_n\|_D \le M. \tag{4.4.19}$$

Let Ω be a compact subset of $\mathbb{R}^4$ given by

$$\Omega = \left\{(w_0, w_1, w_2, w_3) \in \mathbb{R}^4 : \|w_i\| \le C_i,\ i = 0, 1, 2, 3\right\},$$

as f is a L^1-Carathéodory function in Ω, then there exists a real-valued function $\psi_M(x) \in L^1(I)$, such that

$$|f(x, w_0, w_1, w_2, w_3)| \le \psi_M(x),\ \text{for every}\ (w_0, w_1, w_2, w_3) \in \Omega_I. \tag{4.4.20}$$

By (4.4.19), $(u_n, u_n', u_n'', u_n''') \in \Omega_I$, for all $n \in \mathbb{N}$. From (4.4.11) and (4.4.20) we obtain

$$\left|u_n^{(iv)}(x)\right| \le \psi_M(x),\ \text{for a.e. } x \in I,$$

and so $u_n^{(iv)}(x) \in L^1(I)$.

By integration in I,

$$u_n'''(x) = u_n'''(0) + \int_0^x u_n^{(iv)}(s)\,ds + \sum_{0<x_j\le x} l_j\left(u_{n-1}'''(x_j)\right),$$

therefore $u_n''' \in AC(x_j, x_{j+1})$ and $u_n \in PC_D^3(I)$. By Ascoli-Arzela Theorem there exists a subsequence denoted also by $(u_n)_{n\in\mathbb{N}}$, which converges to $u \in PC_D^3(I)$. Then $(u, u', u'', u''') \in \Omega$.

Using the Lebesgue dominated convergence theorem we have that, for all $x \in (x_j, x_{j+1})$,

$$\int_{x_j}^{x} \left[\begin{array}{c} f\left(\begin{array}{c} s, \delta_0 \left(s, u_{n-1}(s)\right), \delta_1 \left(s, u'_{n-1}(s)\right), \delta_2 \left(s, u''_{n-1}(s)\right), \\ q\left(\frac{d}{dx}\left(\delta_2\left(x, u''_n(x)\right)\right)\right) \end{array} \right) \\ +u''_n(s) - \delta_2\left(x, u''_n(s)\right) \end{array} \right] ds$$

is convergent to

$$\int_{x_j}^{x} \left[\begin{array}{c} f\left(s, \delta_0\left(s, u(s)\right), \delta_1\left(s, u'(s)\right), \delta_2\left(s, u''(s)\right), q\left(\frac{d}{dx}\left(\delta_2\left(x, u''(x)\right)\right)\right)\right) \\ +u''(s) - \delta_2\left(x, u''(s)\right) \end{array} \right] ds$$

as $n \to \infty$.

Therefore as $n \to \infty$

$$\begin{aligned} u'''_n(x) &= u'''_n(x_j) + \\ & \int_{x_j}^{x} \left[\begin{array}{c} f\left(\begin{array}{c} s, \delta_0 \left(s, u_{n-1}(s)\right), \delta_1 \left(s, u'_{n-1}(s)\right), \delta_2 \left(s, u''_{n-1}(s)\right), \\ q\left(\frac{d}{dx}\left(\delta_2\left(x, u''_n(x)\right)\right)\right) \end{array} \right) \\ +u''_n(s) - \delta_2\left(x, u''_n(s)\right) \end{array} \right] ds \end{aligned}$$

is convergent to

$$\begin{aligned} u'''(x) &= u'''(x_j) + \\ & \int_{x_j}^{x} \left[\begin{array}{c} f\left(\begin{array}{c} s, \delta_0 \left(s, u(s)\right), \delta_1 \left(s, u'(s)\right), \delta_2 \left(s, u''(s)\right), \\ q\left(\frac{d}{dx}\left(\delta_2\left(x, u''(x)\right)\right)\right) \end{array} \right) \\ +u''(s) - \delta_2\left(x, u''(s)\right) \end{array} \right] ds. \end{aligned}$$

As the function f is L^1-Carathéodory function in (x_j, x_{j+1}), then $u^{(iv)}(x) \in AC(x_j, x_{j+1})$ and u is a solution of *(4.4.11)-(4.4.13)*.

The proof that u is a solution of of the initial problem (4.1.1), (4.1.2), (4.1.4) can be done as in Theorem 4.3.1. ■

Example 4.4.5 *Let us consider the following nonlinear impulsive boundary value problem, composed by the equation*

$$u^{(iv)}(x) = -\arctan(u(x)) - (u'(x))^3 + (u''(x))^5 - k\left|u'''(x)+1\right|^{\frac{1}{3}}, \quad (4.4.21)$$

for all $x \in [0,1] \setminus \left\{\frac{1}{2}\right\}$ *and* $0 < k \leq -\frac{\pi}{2} + 5$ *along with the boundary conditions (4.1.2) and the impulsive conditions*

$$\begin{aligned} u\left(\tfrac{1}{2}^+\right) &= \mu_1\left(u\left(\tfrac{1}{2}\right)\right)^{\frac{1}{3}} \\ u'\left(\tfrac{1}{2}^+\right) &= \mu_2\left(u'\left(\tfrac{1}{2}\right)\right)^{\frac{1}{5}} \\ u''\left(\tfrac{1}{2}^+\right) &= \mu_3\left(u''\left(\tfrac{1}{2}\right)\right)^{3} \\ u''\left(\tfrac{1}{2}^+\right) &= u''(1), \end{aligned} \quad (4.4.22)$$

with $\mu_i \in \mathbb{R}^+$, $i = 1,2,3$.

The above problem is a particular case of (4.1.1), (4.1.2), (4.1.4), defining

$$f(x, y_0, y_1, y_2, y_3) = -\arctan(y_0) - (y_1)^3 + (y_2)^5 - k\left|y_3+1\right|^{\frac{1}{3}},$$

$m = 1$, $x_1 = \frac{1}{2}$ *and the nondecreasing functions* g_j, h_j *and* k_j *are given by, for all* $x \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, $g(x) = \mu_1\sqrt[3]{x}$, $h(x) = \mu_2\sqrt[5]{x}$ *and* $k(x) = \mu_3 x^3$.

It can be checked that functions $\alpha(x) = 0$ *and*

$$\beta(x) = \begin{cases} x^2+2x+1 & , \ x \in \left[0, \frac{1}{2}\right] \\ x^2 & , \ x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

are $PC_D^3(I)$ *for* $D = \left\{\frac{1}{2}\right\}$, *obtaining*

$$\beta'(x) = \begin{cases} 2x+2 & , \ x \in \left[0, \frac{1}{2}\right] \\ 2x & , \ x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

and lower and upper solutions, respectivelly, for the problem (4.4.21), (4.1.2), (4.4.22), for

$$\mu_1 \leq \frac{1}{2\sqrt[3]{18}}, \quad \mu_2 \leq \frac{1}{\sqrt[5]{3}}, \quad \mu_3 \leq \frac{1}{4}.$$

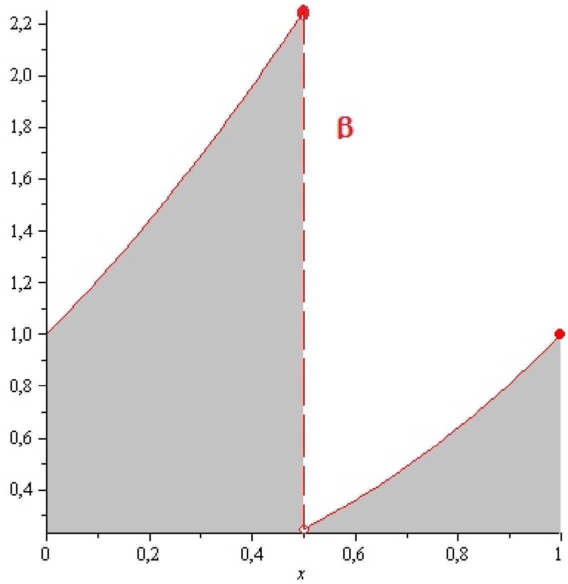

Figure 4.4.1: Region for the localization of solution u

As f verifies (4.4.9), therefore by Theorem 4.4.4 there is a solution $u(x) \in PC_D^3(I)$, such that

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \text{ for } i = 0, 1, 2,$$

for $x \in I$. Moreover, from the location part for $u(x)$, this solution is non negative (see Figure 4.4.1) and from (4.4.21) $u(x)$ is a non trivial solution as k is positive.

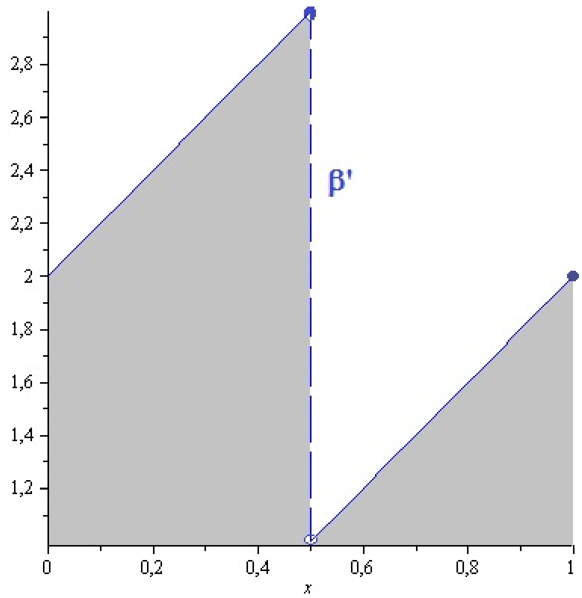

Figure 4.4.2: Location for u'

Part II

Functional Boundary Value Problems

Chapter 5

Existence and nonexistence results for problems with functional boundary conditions

5.1 Introduction

In this chapter it is considered the problem composed by the second order Ambrosetti-Prodi equation

$$u''(x) + f(x, u(x), u'(x)) = sp(x) \tag{5.1.1}$$

with $x \in [a, b]$, where $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ a Carathéodory function, $p : [a, b] \to \mathbb{R}^+$ a continuous functions and s a real parameter, with the functional boundary conditions given by

$$\begin{aligned} L_0(u, u(a), u'(a)) &= 0, \\ L_1(u, u(b), u'(b)) &= 0, \end{aligned} \tag{5.1.2}$$

where $L_0, L_1 : C([a, b]) \times \mathbb{R}^2 \to \mathbb{R}$ satisfy some adequate conditions.

For this problem it will be done a partial discussion of the solution on the parameter s, i. e., only the existence and nonexistence will be studied.

The relation between s and the multiplicity of solutions is, still, an open problem.

The technique used combines the methods suggested in Chapter 3 with the arguments applied in functional boundary value problems, as it can be seen, for instance, in ([15, 16, 19]). Functional boundary conditions as (5.1.2) are extremely general and they include most of the classical conditions as multipoint cases, conditions with delay and/or advances, nonlocal or integral, with maximum or minimum arguments,...

These type of Ambrosetti-Prodi results were never obtained for functional boundary value problems. In fact the functional dependence in the boundary conditions make the relation between the parameter s, the existence and multiplicity of solutions very delicate.The aim of this section is to initiate this study. Other than the results presented there are yet many open issues:

- Is it possible to obtain the standard multiplicity discussion with this kind of functional boundary data?

- If yes, what are the sufficient conditions for it on the nonlinearity? And on the boundary functions?

- Can these conditions be generalized for higher order functional problems? Under which terms?

5.2 General existence and location result

Throughout this Chapter the following hypotheses will be assumed:

(J1) $L_0 : C([a,b]) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function nondecreasing in the first and third variable.

(J2) $L_1 : C([a,b]) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function nondecreasing in the first and nonincreasing in the third variable.

A Nagumo-type growth condition, as presented in the previous Chapters, is an important tool to obtain the main result. Therefore in this section it is used a particular case of Definition 1.2.1, for $n = 2$, together with a similar version of Lemma 3.2.2 applied to

$$g\left(x,y,z\right) = s\ p(x) - f\left(x,y,z\right)$$

The lower and upper solution used are given by the definition:

Definition 5.2.1 *The function $\alpha \in C^2\left([a,b]\right)$ is a lower solution of the problem (5.1.1)-(5.1.2) if it verifies:*

$$\alpha''\left(x\right) \geq sp(x) - f\left(x, \alpha\left(x\right), \alpha'\left(x\right)\right), \tag{5.2.1}$$

and

$$\begin{aligned} L_0\left(\alpha, \alpha\left(a\right), \alpha'\left(a\right)\right) &\geq 0, \\ L_1\left(\alpha, \alpha\left(b\right), \alpha'\left(b\right)\right) &\geq 0. \end{aligned} \tag{5.2.2}$$

The function $\beta \in C^2\left([a,b]\right)$ is an upper solution of the problem (5.1.1)-(5.1.2) for the reversed inequalities.

The existence result is the following:

Theorem 5.2.2 *Let $f : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ be a Carathéodory function. Suppose that there are lower and upper solutions of the problem (5.1.1)-(5.1.2), $\alpha\left(x\right)$ and $\beta\left(x\right)$, respectively, such that,*

$$\alpha\left(x\right) \leq \beta\left(x\right), \forall x \in [a,b],$$

f satisfies a Nagumo condition in

$$E_* = \left\{ (x, y_0, y_1) \in [a,b] \times \mathbb{R}^2 : \alpha(x) \leq y_0 \leq \beta(x) \right\}.$$

If conditions (J1) and (J2) hold then the problem (5.1.1)-(5.1.2) has at least a solution $u(x) \in C^2([a,b])$, satisfying

$$\alpha(x) \leq u(x) \leq \beta(x), \ \forall \ x \in [a,b].$$

Remark 5.2.3 *If there are $\alpha(x)$ and $\beta(x)$ lower and upper solutions of the problem (5.1.1)-(5.1.2) for some values of s, then s belongs to a bounded set, as*

$$\alpha''(x) + f(x, \alpha(x), \alpha'(x)) \leq sp(x) \leq \beta''(x) + f(x, \beta(x), \beta'(x)),$$

for every $x \in [a,b]$.

Proof. Define the continuous functions

$$\delta(x,y) = \max\{\alpha(x), \min\{y, \beta(x)\}\} \tag{5.2.3}$$

and, for some $K > 0$,

$$q(v(x)) = \max\left\{-K, \min\left\{\frac{d}{dx}(v(x)), K\right\}\right\}, \text{ for } a.e.\ x \in \mathbb{R}.$$

Consider the modified problem composed by the equation

$$u''(x) = sp(x) - f\left(x, \delta(x, u(x)), q\left(\frac{d}{dx}(\delta(x, u(x)))\right)\right) \tag{5.2.4}$$

and the Dirichlet boundary conditions,

$$\begin{aligned} u(a) &= \delta(a, u(a) + L_0(\delta(\cdot, u), \delta(a, u(a)), u'(a))), \\ u(b) &= \delta(b, u(b) + L_1(\delta(\cdot, u), \delta(b, u(b)), u'(b))), \end{aligned} \tag{5.2.5}$$

The proof follows standard arguments of lower and upper solutions method, which were developed with detail in Part I. As such, only the points related with the functional boundary conditions are presented:

- Every solution u of problem (5.2.4)-(5.2.5), satisfies

$$\alpha(x) \leq u(x) \leq \beta(x) \text{ and } |u'(x)| < K,$$

for every $x \in [a,b]$, with $K > 0$ given by Nagumo condition.

For u solution of the modified problem (5.2.4)-(5.2.5), assume, by contradiction, that there exists $x \in [a,b]$ such that $\alpha(x) > u(x)$.

Defining

$$\min_{x \in I} (u - \alpha)(x) := (u - \alpha)(x_0) < 0, \tag{5.2.6}$$

as, by (5.2.5), $u(a) \geq \alpha(a)$ and $u(b) \geq \alpha(b)$, then $x_0 \in (a,b)$. So, there is $(x_1, x_2) \subset (a,b)$ such that $x_0 \in (x_1, x_2)$,

$$u(x) < \alpha(x), \forall x \in (x_1, x_2), \quad (u - \alpha)(x_1) = (u - \alpha)(x_2) = 0. \tag{5.2.7}$$

Therefore, for all $x \in (x_1, x_2)$ it is satisfied that $\delta(x,u) = \alpha(x)$ and $\frac{d}{dx}(\delta(x,u)) = \alpha'(x)$. Therefore we deduce that

$$\begin{aligned} u''(x) &= sp(x) - f\left(x, \delta(x, u(x)), q\left(\frac{d}{dx}(\delta(x, u(x)))\right)\right) \\ &= sp(x) - f(x, \alpha(x), \alpha'(x)) \\ &\leq \alpha''(x) \qquad \text{for a. e. } x \in (x_1, x_2). \end{aligned}$$

So $(u-\alpha)'(x)$ is nonincreasing on the interval (x_1, x_2).

As $(u-\alpha)'(x_0) = 0$, then $(u - \alpha)$ is nonincreasing on (x_0, x_2), which contradicts (5.2.6) and (5.2.7).

The inequality $u(x) \leq \beta(x)$, in $[a,b]$, can be proved in same way and, so,

$$\alpha(x) \leq u(x) \leq \beta(x), \ \forall\, x \in [a,b]. \tag{5.2.8}$$

- Problem (5.2.4)-(5.2.5) has at least one solution.

For $\lambda \in [0,1]$ let us consider the homotopic problem given by

$$u''(x) = \lambda \left[sp(x) - f\left(x, \delta(x, u(x)), q\left(\frac{d}{dx}(\delta(x, u(x))) \right) \right) \right] \tag{5.2.9}$$

and the boundary conditions

$$\begin{aligned} u(a) = &\ \lambda \delta \begin{pmatrix} a, u(a) + \\ L_0(\delta(\cdot, u), \delta(a, u(a)), u'(a)) \end{pmatrix} &:= \lambda L_{A,} \\ u(b) = &\ \lambda \delta \begin{pmatrix} b, u(b) + \\ L_1(\delta(\cdot, u), \delta(b, u(b)), u'(b)) \end{pmatrix} &:= \lambda L_B. \end{aligned} \tag{5.2.10}$$

Defining the operators $\mathcal{L} : C([a,b]) \to C([a,b]) \times \mathbb{R}^2$ by

$$\mathcal{L}u = (u'', u(a), u(b))$$

and $\mathcal{N}_\lambda : C([a,b]) \to C([a,b]) \times \mathbb{R}^2$ by

$$\mathcal{N}_\lambda u = \left(\lambda \left[sp(x) - f\left(x, \delta(x, u(x)), q\left(\frac{d}{dx}(\delta(x, u(x))) \right) \right) \right], \lambda L_A, \lambda L_B, \right)$$

It can be proved by degree theory (as in Part I) that (5.2.4)-(5.2.5) has a solution $u_1(x)$, for $\lambda = 1$.

- This function $u_1(x)$ is a solution of (5.1.1) – (5.1.2).

As $u_1(x)$ fulfills equation (5.1.1), it will be enough to prove that:

$$\begin{aligned} \alpha(a) \leq &\ u_1(a) + L_0(\delta(\cdot, u_1), \delta(a, u_1(a)), u_1'(a)) &\leq \beta(a), \\ \alpha(b) \leq &\ u_1(b) + L_1(\delta(\cdot, u_1), \delta(b, u_1(b)), u_1'(b)) &\leq \beta(b). \end{aligned}$$

So, assume that

$$u_1(a) + +L_0(\delta(\cdot, u_1), \delta(a, u_1(a)), u_1'(a)) > \beta(a). \tag{5.2.11}$$

Then, by (5.2.5), $u(a) = \beta(a)$. By (5.2.2) and previous steps, it is obtained the following contradiction with (5.2.11):

$$u_1(a) + +L_0(\delta(\cdot, u_1), \delta(a, u_1(a)), u_1'(a)) \leq \beta(a).$$

Applying similar arguments it can be proved that

$$\alpha(a) \leq u_1(a) + +L_0(\delta(\cdot, u_1), \delta(a, u_1(a)), u_1'(a)).$$

and

$$\alpha(b) \leq \quad u_1(b) + L_1(\delta(\cdot, u_1), \delta(b, u_1(b)), u_1'(b)) \quad \leq \beta(b).$$

■

5.3 Existence and nonexistence results

The dependence of solution on s will be discussed in $[0,1]$, only for clearness of arguments and without loss of generality. In the corresponding definitions of lower and upper solutions the corresponding modifications must be considered. Some extra hypotheses on the continuous functions L_0, L_1 are required to obtain the existence and nonexistence result:

Theorem 5.3.1 *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function that verifies the assumptions on Theorem 5.2.2. Moreover if :*

(i) $f(x, y, z)$ *is nonincreasing on* y;

(ii) there is $s_1 \in \mathbb{R}$ *and* $r > 0$ *such that*

$$\frac{f(x,0,0)}{p(x)} < s_1 < \frac{f(x,-r,0)}{p(x)}, \tag{5.3.1}$$

for every $x \in [0,1]$;

(iii) L_0 *and* L_1 *verify (J1), (J2) and*

(J3) $L_i(x, x, 0) \geq 0$, *for every* $x \leq -r$ *and* $L_i(0,0,0) \leq 0$, *for* $i = 0, 1$,

then there is $s_0 < s_1$ *(with the possibility that* $s_0 = -\infty$*) such that:*

1) for $s < s_0$, (5.1.1)-(5.1.2) has no solution.

2) for $s_0 < s \leq s_1$, (5.1.1)-(5.1.2) has at least one solution.

Proof. The technique is similar to the one used, for example, in Theorem 3.2.6.

We underline only some features:

- For
$$s^* = \max_{x\in[0,1]} \frac{f(x,0,0)}{p(x)}, \tag{5.3.2}$$
$\beta(x) \equiv 0$ and $\alpha(x) = -r$, with r given by (5.3.1), are, respectively, lower and upper solutions of (5.1.1)-(5.1.2) with $s = s^*$. Therefore this problem has a solution for $s = s^*$.

- Assuming that (5.1.1)-(5.1.2) has a solution $u_\sigma(x)$ for $s = \sigma \leq s_1$, consider $R > 0$ sufficiently large such that,
$$r \leq R, \quad \max_{x\in[0,1]} u_\sigma(x) \geq -R. \tag{5.3.3}$$
Then $u_\sigma(x)$ and $\overline{\alpha}(x) = -R$ are, respectively, lower and upper solutions of (5.1.1)-(5.1.2), for $s \in \left]s_0, s_1\right]$.

■

5.4 Examples

In this section we will consider two examples that illustrate conditions (J1), (J2) and (J3) and how do they relate with Theorem 5.2.2 and Theorem 5.3.1.

Example 5.4.1 *Let us consider, for* $x \in [0,1]$, *the problem given by the equation*

$$u''(x) + \arctan\left(u(x)^2\right) + \left(u'(x)\right)^{\frac{2}{3}} = sp(x), \tag{5.4.1}$$

with $p : [0,1] \to \mathbb{R}^+$ *a continuous function, along with the functional boundary conditions*

$$\begin{array}{c} \max\limits_{x\in[0,1]} u(x) + k_1 u(0) = 0 \\ \max\limits_{x\in[0,1]} \int_0^x u(s)\, ds + k_2 u(1) = 0. \end{array} \tag{5.4.2}$$

The functions

$$\alpha(x) = -x - 1$$

and

$$\beta(x) = x + 1$$

are, respectively, lower and upper solutions to the problem (5.4.1)-(5.4.2), for $k_1 \leq -2$ *and* $k_2 \leq \frac{-3}{4}$, *and for*

$$\frac{1 + \arctan 1}{\max\limits_{x\in[0,1]} p(x)} \leq s \leq \frac{1 + \arctan 1}{\min\limits_{x\in[0,1]} p(x)}.$$

This problem is a particular case of (5.1.1)-(5.1.2), defining

$$\begin{aligned} f(x,y,z) &= \arctan(y^2) + (z)^{\frac{2}{3}} \\ L_0(y_0,y_1,y_2) &= \max_{x\in[0,1]} y_0(x) + k_1 y_1 \\ L_1(y_0,y_1,y_2) &= \max_{x\in[0,1]} \int_0^x y_0(s)\, ds + k_2 y_1. \end{aligned}$$

As function f *verifies a Nagumo condition in*

$$E = \left\{(x,y_0,y_1) \in [0,1] \times \mathbb{R}^2 : -x - 1 \leq y_0 \leq x + 1\right\}, \tag{5.4.3}$$

therefore by Theorem 5.2.2 there is at least a solution $u(x)$ *of the problem (5.4.1)-(5.4.2), satisfying*

$$-x - 1 \leq u(x) \leq x + 1, \ \forall x \in [0,1].$$

We remark that in the previous example there is no information about non existence of solutions for problem (5.4.1)-(5.4.2). In fact Theorem 5.3.1 is not applicable as function f is not increasing in $u(x)$ and the boundary condition L_1, given in (5.4.2) does not verify (J3) for $k_2 \leq \frac{-3}{4}$. A new example, with a suitable function f and boundary conditions is presented, verifying assumptions of Theorem 5.3.1.

Example 5.4.2 *Let us consider, for* $x \in [0,1]$, *the problem given by the equation*

$$u''(x) - u(x)^3 + (u'(x)+1)^{\frac{2}{3}} = sp(x), \tag{5.4.4}$$

where $p : [0,1] \to \mathbb{R}^+$ *is a continuous function, along with the functional boundary conditions*

$$\begin{aligned} u'(0) - u(0)^3 &= 0 \\ \eta u(1) - u'(1) &= 0, \end{aligned} \tag{5.4.5}$$

with $\eta \leq 0$.

The functions

$$\alpha(x) = -x - 1$$

and

$$\beta(x) = x + 1$$

are, respectively, lower and upper solutions to the problem (5.4.4)-(5.4.5), for

$$\frac{-1+2^{\frac{2}{3}}}{\max\limits_{x\in[0,1]} p(x)} \leq s \leq \frac{8}{\min\limits_{x\in[0,1]} p(x)}.$$

Considering

$$\begin{gathered} f(x,y,z) = -y^3 + (z+1)^{\frac{2}{3}}, \\ L_0(y_0,y_1,y_2) = -(y_1)^3 + y_2, \\ L_1(y_0,y_1,y_2) = \eta y_1 - y_2, \end{gathered}$$

it can be seen easily that problem (5.4.4)-(5.4.5) is a particular case of (5.1.1)-(5.1.2) and f verifies Nagumo conditions in E given by (5.4.3).

For $r > 0$ and s_1 such that

$$\frac{1}{\max\limits_{x\in[0,1]} p\left(x\right)} < s_1 < \frac{r^3+1}{\min\limits_{x\in[0,1]} p\left(x\right)},$$

boundary conditions (5.4.5) satisfy condition (J3), therefore by Theorem 5.3.1 there is $s_0 < s_1$ where:

- *for $s < s_0$, (5.4.4)-(5.4.5) has no solution;*

- *for $s_0 < s \leq s_1$, (5.4.4)-(5.4.5) has at least one solution.*

Chapter 6

High order problems with functional boundary conditions

6.1 Introduction

In this chapter we consider initially the problem composed of the fully nonlinear fourth order equation

$$u^{(iv)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)) \tag{6.1.1}$$

with $x \in [0,1]$, where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function, coupled with the functional boundary conditions

$$\begin{aligned} L_0(u, u', u'', u(0)) &= 0, \\ L_1(u, u', u'', u'(0)) &= 0, \\ L_2(u, u', u'', u''(0), u'''(0)) &= 0, \\ L_3(u, u', u'', u''(1), u'''(1)) &= 0, \end{aligned} \tag{6.1.2}$$

where $L_0, L_1 : C([0,1])^3 \times \mathbb{R} \to \mathbb{R}$ and $L_2, L_3 : C(I)^3 \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying some adequate monotonicity assumptions.

This problem is generalized later on to a n^{th} order fully nonlinear equation

$$u^{(n)}(x) = f\left(x, u(x), u'(x), ..., u^{(n-1)}(x)\right) \tag{6.1.3}$$

with $x \in [0,1]$, where $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function along with the functional boundary conditions

$$\begin{aligned} &L_i(u, u', ..., u^{(n-2)}, u^{(n-3)}(0)) = 0, \text{ for } i = 0, ..., n-3 \\ &L_{n-2}(u, u', ..., u^{(n-2)}, u^{(n-2)}(0), u^{(n-1)}(0)) = 0, \\ &L_{n-1}(u, u', ..., u^{(n-2)}, u^{(n-2)}(1), u^{(n-1)}(1)) = 0, \end{aligned} \tag{6.1.4}$$

where $L_i : C([0,1])^{n-1} \times \mathbb{R} \to \mathbb{R}$ and $L_{n-2}, L_{n-1} : C([0,1])^{n-1} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying some monotone conditions to be defined.

Functional boundary conditions are very general in nature. They not only generalize most of the classical boundary conditions as they also cover the separated and multipoint cases, with delay and/or advances, with maximum or minimum arguments, nonlocal or integral conditions,...

The fourth order problems were studied by several authors with different boundary conditions and different methods, see for example [33, 35, 67, 77, 78]. The method used here was suggested by [15, 16, 19]. In [15] the boundary conditions considered are

$$\begin{aligned} u(b) = A;\ u'(b) = B,\ \ A, B \in \mathbb{R} \\ L_1(u, u', u'', u''(a), u'''(a)) = 0 \\ L_2(u, u', u'', u''(b), u'''(b)) = 0 \end{aligned} \quad ,$$

for $x \in [a,b]$ and L_1, L_2 some continuous functions. In [19] the boundary conditions used are

$$\begin{aligned} B_1(u, u', u'', u(a)) = 0 \\ B_2(u, u', u'', u'(a)) = 0 \\ B_3(u, u', u'', u''(a), u''(b), u'''(a)) = 0 \\ L_2(u''(a), u''(b)) = 0, \end{aligned}$$

where B_i, $i = 1, 2, 3$ and L_2 are suitable functions. As it can be seen easily, boundary conditions (6.1.2) generalize the above results. In spite such a general formulation, problem (6.1.1)-(6.1.2) can be studied by similar techniques and analogous methods as the separated boundary value problems studied in the first part.

In short, the key points of the arguments are: *a priori* estimates on the third derivative provided by a Nagumo-type condition, [77, 80]; an auxiliary and truncated problem, where the corresponding linear and homogeneous problem has only the trivial solution; an open and bounded set where the Leray-Schauder degree is well defined, [70].

Lower and upper solutions technique allows us to obtain not only the existence but also to locate the solution and its $(n-2)$ derivatives. In the final section two examples are presented. An example will illustrate how these features can be applied. In fact, defining lower and upper solutions well ordered, with a nonnegative lower function, implies that the solution is nonnegative. Moreover, if the second derivatives of lower and upper solutions have the same sign, the solution is not trivial and it can not be a straight line.

The location part provided by lower and upper solutions method can also be useful to some "theoretical" problems. In this sense, last section contains an application of problem (6.1.3)-(6.1.4) to the $(n-1, 1)$ conjugate boundary value problem. The key idea was suggested in [98], where some estimates for the solution of the problem

$$u^{(n)}(x) + g(x) f(u(x)) = 0$$

$$u^{(i)}(0) = u(1) = 0, \; i = 0, ..., n-2,$$

were obtained.

As it can be seen, the localization of the solution by lower and upper solutions allow more precise estimates than the ones existing in the literature.

6.2 Existence and location result in fourth order case

Consider a Nagumo-type growth condition on the nonlinear part of the differential equation (6.1.1). As in the previous chapters this will be useful to prove an *a priori* bound for the third derivative of the corresponding solutions. (See Lemma 1.2.2 for $n = 4$).

Throughout this section the boundary functions verify the following assumptions:

(M_1) $L_0, L_1 : C([0,1])^3 \times \mathbb{R} \to \mathbb{R}$ are nondecreasing in all variables except the fourth one.

(M_2) $L_2 : C([0,1])^3 \times \mathbb{R}^2 \to \mathbb{R}$ is nondecreasing in all variables, except the fourth one.

(M_3) $L_3 : C([0,1])^3 \times \mathbb{R}^2 \to \mathbb{R}$ is nondecreasing in the first, second and third variables and nonincreasing in the fifth one.

Lower and upper functions are given by next definition:

Definition 6.2.1 *A function $\alpha \in C^4([0,1])$ is a lower solution of problem (6.1.1)-(6.1.2) if:*

$$\alpha^{(iv)}(x) \geq f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)), \tag{6.2.1}$$

and

$$\begin{aligned} L_0\left(\alpha,\alpha',\alpha'',\alpha\left(0\right)\right) &\geq 0, \\ L_1\left(\alpha,\alpha',\alpha'',\alpha'\left(0\right)\right) &\geq 0, \\ L_2\left(\alpha,\alpha',\alpha'',\alpha''\left(0\right),\alpha'''\left(0\right)\right) &\geq 0, \\ L_3\left(\alpha,\alpha',\alpha'',\alpha''\left(1\right),\alpha'''\left(1\right)\right) &\geq 0. \end{aligned} \tag{6.2.2}$$

The function $\beta \in C^4\left([0,1]\right)$ is an upper solution of the problem (6.1.1)-(6.1.2) if the reversed inequalities hold.

The main theorem can be said to be an existence and location result as it provides the existence of a solution but also some strips where the solution and its first and second derivatives are located.

Theorem 6.2.2 *Let $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ be a continuous function. Suppose that there are lower and upper solutions of the problem (6.1.1)-(6.1.2), $\alpha\left(x\right)$ and $\beta\left(x\right)$, respectively, such that,*

$$\alpha\left(0\right)\leq\beta\left(0\right),\quad \alpha'\left(0\right)\leq\beta'\left(0\right),\quad \alpha''\left(x\right)\leq\beta''\left(x\right),\quad \forall x\in[0,1], \tag{6.2.3}$$

f satisfies Nagumo conditions in

$$E_* = \left\{(x,y_0,y_1,y_2,y_3)\in[0,1]\times\mathbb{R}^4 : \alpha^{(i)}\left(x\right)\leq y_i\leq\beta^{(i)}\left(x\right),\ i=0,1,2\right\}$$

and

$$f\left(x,\alpha,\alpha',y_2,y_3\right)\geq f\left(x,y_0,y_1,y_2,y_3\right)\geq f\left(x,\beta,\beta',y_2,y_3\right), \tag{6.2.4}$$

for $\alpha\left(x\right)\leq y_0\leq\beta\left(x\right)$, $\alpha'\left(x\right)\leq y_1\leq\beta'\left(x\right)$, in $[0,1]$, and fixed $(x,y_2,y_3)\in[0,1]\times\mathbb{R}^2$.

If conditions $(M_1)-(M_3)$ hold, then problem (6.1.1)-(6.1.2) has at least one solution $u\left(x\right)\in C^4\left([0,1]\right)$, such that

$$\alpha^{(i)}\left(x\right)\leq u^{(i)}\left(x\right)\leq\beta^{(i)}\left(x\right),\quad \forall x\in[0,1],\ \textit{for } i=0,1,2.$$

Proof. Let us consider the usual continuous truncations δ_i given by (2.3.2) and for $\lambda \in [0,1]$, a similar homotopic and perturbed equation to (2.3.3), with the boundary conditions

$$\begin{aligned} u(0) &= \lambda\delta_0\left(0, u(0) + L_0\left(u, u', u'', u(0)\right)\right), \\ u'(0) &= \lambda\delta_1\left(0, u'(0) + L_1\left(u, u', u'', u'(0)\right)\right), \\ u''(0) &= \lambda\delta_2\left(0, u''(0) + L_2\left(u, u', u'', u''(0), u'''(0)\right)\right), \\ u''(1) &= \lambda\delta_2\left(1, u''(1) + L_3\left(u, u', u'', u''(1), u'''(1)\right)\right). \end{aligned} \tag{6.2.5}$$

For $r_2 > 0$ large enough, such that, for every $x \in [0,1]$,

$$\begin{aligned} -r_2 < \alpha''(x) \leq \beta''(x) < r_2, \\ f\left(x, \beta(x), \beta'(x), \beta''(x), 0\right) + r_2 - \beta''(x) > 0, \\ f\left(x, \alpha(x), \alpha'(x), \alpha''(x), 0\right) - r_2 - \alpha''(x) < 0, \end{aligned} \tag{6.2.6}$$

the proof carries on with the standard steps of lower and upper solutions method. Therefore we present only the steps related to the boundary conditions:

- E*very solution* $u(x)$ *of the problem (2.3.3)-(6.2.5) we have*

  $$|u''(x)| < r_2 \quad |u'(x)| < r_1 \quad |u(x)| < r_0, \forall x \in [0,1],$$

  with $r_1 := r_2 + \max\{|\alpha'(0)|, |\beta'(0)|\}$ *and* $r_0 := r_1 + \max\{|\alpha(0)|, |\beta(0)|\}$, *independently of* $\lambda \in [0,1]$.

If, by contradiction,

$$\min_{x\in[0,1]} u''(x) := u''(0) \leq -r_2 < 0,$$

then by (6.2.5) and (2.3.2), the following contradiction is obtained, for $\lambda \in]0,1]$

$$\begin{aligned} -r_2 &\geq u''(0) = \lambda\delta_2\left(0, u''(0) + L_2(u, u', u'', u''(0), u'''(0))\right) \\ &\geq \lambda\alpha''(0) > -r_2. \end{aligned}$$

The arguments for $x_0 = 1$, are similar.

Proved that

$$|u''(x)| < r_2, \ \forall x \in [0,1], \forall \lambda \in [0,1],$$

then, integrating in $[0,x]$,

$$\begin{aligned} u'(x) &= \int_0^x u''(s)ds + u'(0) \\ &= \int_0^x u''(s)ds + \lambda\delta_1(0, u'(0) + L_1(u, u', u'', u'(0))). \end{aligned}$$

Therefore,

$$\begin{aligned} |u'(x)| &\leq \int_0^x |u''(s)|\,ds + |\lambda\delta_1(0, u'(0) + L_1(u, u', u'', u'(0)))| \\ &< r_2 + \max\{|\alpha'(0)|, |\beta'(0)|\} := r_1 \end{aligned}$$

Similarly, it can be proved that

$$|u(x)| < r_1 + \max\{|\alpha(0)|, |\beta(0)|\}, \forall x \in [0,1].$$

- The operators used to prove that problem (2.3.3)-(6.2.5) has at least a solution $u_1(x)$ for $\lambda = 1$ are $\mathcal{L} : C^4([0,1]) \to C([0,1]) \times \mathbb{R}^4$ given by

$$\mathcal{L}u = \left(u^{(iv)} - u'', u(0), u'(0), u''(0), u''(1)\right),$$

and $\mathcal{N}_\lambda : C^3([0,1]) \to C([0,1]) \times \mathbb{R}^4$, given by

$$\mathcal{N}_\lambda = \begin{pmatrix} \lambda f(x, \delta_0(x, u(x)), \delta_1(x, u'(x)), \delta_2(x, u''(x)), u'''(x)) \\ -\lambda\delta_2(x, u''(x)), A_{0\lambda}, \ A_{1\lambda}, \ A_{2\lambda}, \ A_{3\lambda} \end{pmatrix},$$

where

$$\begin{aligned}
A_{0\lambda} &:= \lambda\delta_0\left(0, u\left(0\right) + L_0\left(u, u', u'', u\left(0\right)\right)\right), \\
A_{1\lambda} &:= \lambda\delta_1\left(0, u'\left(0\right) + L_1\left(u, u', u'', u'\left(0\right)\right)\right), \\
A_{2\lambda} &:= \lambda\delta_2\left(0, u''\left(0\right) + L_2\left(u, u', u'', u''\left(0\right), u'''\left(0\right)\right)\right), \\
A_{3\lambda} &:= \lambda\delta_2\left(1, u''\left(1\right) + L_3\left(u, u', u'', u''\left(1\right), u'''\left(1\right)\right)\right).
\end{aligned}$$

- This function $u_1(x)$ will be a solution of the original problem (6.1.1)-(6.1.2) if

$$\alpha^{(i)}(x) \le u_1^{(i)}(x) \le \beta^{(i)}(x), \; i = 0,1,2, \; \forall x \in [0,1], \tag{6.2.7}$$

  and

$$\begin{gathered}
\alpha(0) \le u_1(0) + L_0\left(u_1, u_1', u_1'', u_1(0)\right) \le \beta(0) \\
\alpha'(0) \le u_1'(0) + L_1\left(u_1, u_1', u_1'', u_1'(0)\right) \le \beta'(0) \\
\alpha''(0) \le u_1''(0) + L_2\left(u_1, u_1', u_1'', u_1''(0), u_1'''(0)\right) \le \beta''(0) \\
\alpha''(1) \le u_1''(1) + L_3\left(u_1, u_1', u_1'', u_1''(1), u_1'''(1)\right) \le \beta''(1)
\end{gathered}$$

  hold.

The inequalities (6.2.7) can be proved as in Theorem 1.3.1 and applying (6.2.5).

As the boundary conditions, assume that

$$u_1(0) + L_0\left(u_1, u_1', u_1'', u_1(0)\right) < \alpha(0). \tag{6.2.8}$$

By (2.3.2) and (6.2.5)

$$u_1(0) = \delta_0(0, u_1(0) + L_0(u_1, u_1', u_1'', u_1(0))) = \alpha(0)$$

and, by (6.2.3), $u_1'(0) \ge \alpha'(0)$ and $u_1''(0) \ge \alpha''(0)$. Therefore, by (H_1) and (6.2.2) this contradiction with (6.2.8) is achieved:

$$\begin{aligned}
u_1(0) + L_0\left(u_1, u_1', u_1'', u_1(0)\right) &= \alpha(0) + L_0\left(u_1, u_1', u_1'', \alpha(0)\right) \\
&\ge \alpha(0) + L_0\left(\alpha, \alpha', \alpha'', \alpha(0)\right) \ge \alpha(0).
\end{aligned}$$

Analogously it is shown that $u_1(0) + L_0(u_1, u_1', u_1'', u_1(0)) \leq \beta(0)$.

Remaining inequalities can be proved by a similar technique. ■

Example 6.2.3 *Consider the fourth order multipoint problem*

$$\begin{cases} u^{(iv)}(x) = -0.1(u(x))^3 - 0.1\left|u''(x) - 2\right| e^{0.01u'(x)} + 20\sqrt[3]{|u'''(x)|} \\ \\ \sum\limits_{n=1}^{+\infty} a_n^0 u(x_n) + \sum\limits_{n=1}^{+\infty} b_n^0 u'(x_n) + \sum\limits_{n=1}^{+\infty} c_n^0 u''(x_n) - ku(0) = 0 \\ \sum\limits_{n=1}^{+\infty} a_n^1 u(\widehat{x}_n) + \sum\limits_{n=1}^{+\infty} b_n^1 u'(\widehat{x}_n) + \sum\limits_{n=1}^{+\infty} c_n^1 u''(\widehat{x}_n) - \eta u'(0) = 0 \\ u''(0) + 2u'''(0) = 0 \\ u''(1) = 2, \end{cases} \tag{6.2.9}$$

where $\sum_{n=1}^{+\infty} a_n^i, \sum_{n=1}^{+\infty} b_n^i, \sum_{n=1}^{+\infty} c_n^i$, *for* $i = 0, 1$, *are positive convergent series to* a^i, b^i *and* c^i, *respectively,* $x_n, \widehat{x}_n \in [0,1]$, $k \geq 7a^0 + 8b^0 + 8c^0$ *and* $\eta \geq \frac{1}{3}(7a^1 + 8b^1 + 8c^1)$.

The functions $\alpha, \beta \in [0,1] \to \mathbb{R}$ *given by*

$$\alpha(x) = x^2 \text{ and } \beta(x) = -x^3 + 4x^2 + 3x + 1$$

are, respectively, lower and upper solutions of (6.2.9) with

$$\begin{aligned} f(x, y_0, y_1, y_2, y_3) &= -0.1(y_0)^3 - 0.1|y_2 - 2|e^{0.01y_1} + 20\sqrt[3]{|y_3|} \\ L_0(z_1, z_2, z_3, z_4) &= \sum_{n=1}^{+\infty} a_n^0 z_1(x_n) + \sum_{n=1}^{+\infty} b_n^0 z_2(x_n) + \sum_{n=1}^{+\infty} c_n^0 z_3(x_n) - kz_4 \\ L_1(z_1, z_2, z_3, z_4) &= \sum_{n=1}^{+\infty} a_n^1 z_1(\widehat{x}_n) + \sum_{n=1}^{+\infty} b_n^1 z_2(\widehat{x}_n) + \sum_{n=1}^{+\infty} c_n^1 z_3(\widehat{x}_n) - \eta z_4 \\ L_2(z_1, z_2, z_3, z_4, z_5) &= z_4 + 2z_5 \\ L_3(z_1, z_2, z_3, z_4, z_5) &= z_4 - 2. \end{aligned}$$

As the continuous function f verifies the Nagumo condition for

$$h_{E_*}(y_3) = 34.3 + 0.6\ e^{0.08} + 20\sqrt[3]{|y_3|}$$

in

$$E_* = \left\{ (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \begin{array}{c} x^2 \leq y_0 \leq -x^3 + 4x^2 + 3x + 1 \\ 2x \leq y_1 \leq -3x^2 + 8x + 3 \\ 2 \leq y_2 \leq -6x + 8 \end{array} \right\}$$

then, by Theorem 6.2.2, there is a solution $u(x)$ of problem (6.2.9) such that, for every $x \in [0,1]$,

$$\begin{aligned} x^2 &\leq u(x) \leq -x^3 + 4x^2 + 3x + 1, & (6.2.10) \\ 2x &\leq u'(x) \leq -3x^2 + 8x + 3 \\ 2 &\leq u''(x) \leq -6x + 8. & (6.2.11) \end{aligned}$$

Remark that this solution u is nonnegative, by (6.2.10) and illustrated by Figure 6.2.1. Moreover, by (6.2.11), u can not be a straight line.

6.3 Higher order problem

In this section the previous results are generalized to the n^{th} order problem (6.1.3)-(6.1.4), where $L_i : C([0,1])^{n-1} \times \mathbb{R} \to \mathbb{R}$, for $i = 0, ..., n-3$ and $L_{n-2}, L_{n-1} : C([0,1])^{n-1} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying the monotonicity assumptions:

(M$'_1$) $L_i : C([0,1])^{n-1} \times \mathbb{R} \to \mathbb{R}$, for $i = 0, ..., n-3$, are nondecreasing in all variables except the last one.

(M$'_2$) $L_{n-2} : C([0,1])^{n-1} \times \mathbb{R}^2 \to \mathbb{R}$ is nondecreasing in all variables, except the last one.

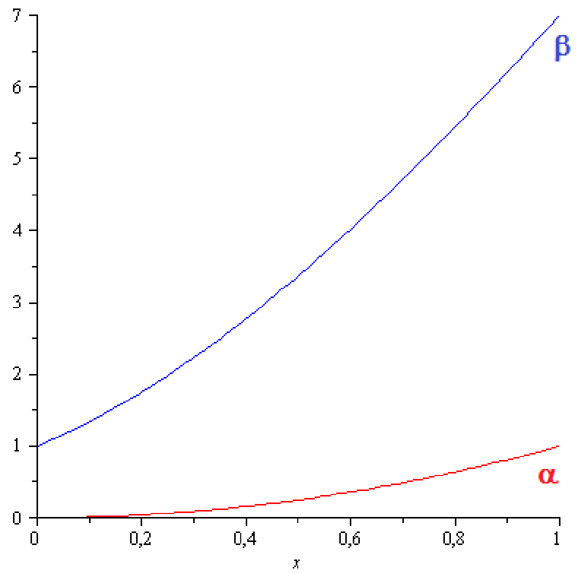

Figure 6.2.1: Both upper (α) and lower solution (β) are nonnegative

(M_3') $L_{n-1} : C([0,1])^{n-1} \times \mathbb{R}^2 \to \mathbb{R}$ is nondecreasing from the first up to the $(n-1)$ variable and nonincreasing in the last one.

Definitions of lower and upper solutions follow the same type:

Definition 6.3.1 *A function $\alpha \in C^n([0,1])$ is a lower solution of problem (6.1.3)-(6.1.4) if:*

$$\alpha^{(n)}(x) \geq f\left(x, \alpha(x), \alpha'(x), ..., \alpha^{(n-1)}(x)\right), \tag{6.3.1}$$

and

$$\begin{aligned} &L_i\left(\alpha, \alpha', ..., \alpha^{(n-2)}, \alpha^{(i)}(0)\right) \geq 0, \;\; i = 0, ..., n-3 \\ &L_{n-2}\left(\alpha, \alpha', ..., \alpha^{(n-2)}, \alpha^{(n-2)}(0), \alpha^{(n-1)}(0)\right) \geq 0, \\ &L_{n-1}\left(\alpha, \alpha', ..., \alpha^{(n-2)}, \alpha^{(n-2)}(1), \alpha^{(n-1)}(1)\right) \geq 0. \end{aligned} \tag{6.3.2}$$

The function $\beta \in C^n([0,1])$ is an upper solution of the problem (6.1.3)-(6.1.4) if the reversed inequalities hold.

The main theorem generalizes the existence and location result obtained in the previous section, to a n^{th} order problem.

Theorem 6.3.2 *Let $f:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ be a continuous function. Suppose that there are lower and upper solutions of the problem (6.1.3)-(6.1.4), $\alpha(x)$ and $\beta(x)$, respectively, such that,*

$$\begin{array}{c}\alpha^{(i)}(0)\leq\beta^{(i)}(0)\,,\ i=0,...,n-3\\ \alpha^{(n-2)}(x)\leq\beta^{(n-2)}(x)\,,\quad\forall x\in[0,1]\,,\end{array}\tag{6.3.3}$$

f satisfies Nagumo conditions

$$E_*=\left\{\begin{array}{c}(x,y_0,...,y_{n-1})\in[0,1]\times\mathbb{R}^n:\alpha^{(i)}(x)\leq y_i\leq\beta^{(i)}(x)\,,\\ i=0,...,n-2\end{array}\right\}$$

and

$$\begin{aligned}f\left(x,\alpha,...,\alpha^{(n-3)},y_{n-2},y_{n-1}\right)&\geq f\left(x,y_0,...,y_{n-3},y_{n-2},y_{n-1}\right)\\ &\geq f\left(x,\beta,...,\beta^{(n-3)},y_{n-2},y_{n-1}\right),\end{aligned}\tag{6.3.4}$$

for $\alpha^{(i)}(x)\leq y_i\leq\beta^{(i)}(x)$, $i=0,...,n-3$, in $[0,1]$, and fixed $(x,y_{n-2},y_{n-1})\in[0,1]\times\mathbb{R}^2$.

If conditions $(M_1')-(M_3')$ hold, then problem (6.1.3)-(6.1.4) has at least one solution $u(x)\in C^n([0,1])$, such that

$$\alpha^{(i)}(x)\leq u^{(i)}(x)\leq\beta^{(i)}(x)\,,\quad\forall x\in[0,1]\,,\ \textit{for } i=0,...,n-2.$$

Proof. The same type of arguments as in the fourth order case can be applied.

For the readers convenience we point out only some specific steps.

Considering now the truncations $\delta_i,\ i=0,...,n-2$

$$\delta_i(x,y_i)=\left\{\begin{array}{ccc}\alpha^{(i)}(x) & \text{if} & y^{(i)}<\alpha^{(i)}(x)\\ y^{(i)} & \text{if} & \alpha^{(i)}(x)\leq y^{(i)}\leq\beta^{(i)}(x)\\ \beta^{(i)}(x) & \text{if} & y^{(i)}>\beta^{(i)}(x)\,,\end{array}\right.\tag{6.3.5}$$

for $\lambda \in [0,1]$, the n^{th} order homotopic equation

$$u^{(n)}(x) = \tag{6.3.6}$$
$$\lambda\left(f\left(x,\delta_0(x,u(x)),...,\delta_{n-2}\left(x,u^{(n-2)}(x)\right),u^{(n-1)}(x)\right)\right)$$
$$+u^{(n-2)}(x) - \lambda\delta_{n-2}\left(x,u^{(n-2)}(x)\right),$$

with the boundary conditions

$$\begin{aligned}
u^{(i)}(0) &= \lambda\delta_i\left(0,u^{(i)}(0)+L_i\left(u,...,u^{(n-2)},u^{(i)}(0)\right)\right),\ i=0,...,n-3,\\
u^{(n-2)}(0) &= \lambda\delta_{n-2}\left(\begin{array}{c} 0,u^{(n-2)}(0)+ \\ L_{n-2}\left(u,...,u^{(n-2)},u^{(n-2)}(0),u^{(n-1)}(0)\right)\end{array}\right),\\
u^{(n-2)}(1) &= \lambda\delta_{n-2}\left(\begin{array}{c} 1,u^{(n-2)}(1)+ \\ L_{n-1}\left(u,...,u^{(n-2)},u^{(n-2)}(1),u^{(n-1)}(1)\right)\end{array}\right),
\end{aligned} \tag{6.3.7}$$

and for $r_{n-2} > 0$ large enough, such that, for every $x \in [0,1]$,

$$\begin{aligned}
-r_{n-2} < \alpha^{(n-2)}(x) \le \beta^{(n-2)}(x) < r_{n-2},\\
f\left(x,\beta(x),...,\beta^{(n-2)}(x),0\right) + r_{n-2} - \beta^{(n-2)}(x) > 0,\\
f\left(x,\alpha(x),...,\alpha^{(n-2)}(x),0\right) - r_{n-2} - \alpha^{(n-2)}(x) < 0.
\end{aligned} \tag{6.3.8}$$

- For every solution $u(x)$ of the problem (6.3.6)-(6.3.7) we have

  $$\left|u^{(i)}(x)\right| < r_i,\ \forall x \in [0,1],\ i=0,...,n-2,$$

  where r_{n-2} is given as above and

  $$r_j := r_{n-2} + \sum_{k=j}^{n-3} \max\left\{\left|\alpha^{(k)}(0)\right|,\left|\beta^{(k)}(0)\right|\right\},$$

  $j=0,...,n-3$ independently of $\lambda \in [0,1]$.

- The non null Leray-Schauder degree is evaluate in the open set

  $$\Omega = \left\{y \in C^{n-1}([0,1]) : \left\|y^{(i)}\right\| < r_i,\ i=0,...,n-2,\ \left\|y^{(n-1)}\right\| < R\right\},$$

where $R > 0$ is obtained from the Nagumo condition, for the operator

$$\mathcal{T}_\lambda : (C^n([0,1]), \mathbb{R}) \to (C^n([0,1]), \mathbb{R})$$

defined by

$$\mathcal{T}_\lambda(u) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u),$$

with

$$\mathcal{L} : C^n([0,1]) \to C([0,1]) \times \mathbb{R}^n.$$

given by

$$\mathcal{L}u = \left(u^{(n)} - u^{(n-2)}, u(0), ..., u^{(n-2)}(0), u^{(n-2)}(1)\right),$$

and $\mathcal{N}_\lambda : C^n([0,1]) \to C([0,1]) \times \mathbb{R}^n$, $\lambda \in [0,1]$ by

$$\mathcal{N}_\lambda = \left(\begin{array}{c} \lambda f\left(x, \delta_0(x, u(x)), ..., \delta_{n-2}\left(x, u^{(n-2)}(x)\right), u^{(n-1)}(x)\right) \\ -\lambda\delta_{n-2}\left(x, u^{(n-2)}(x)\right), A_{0,\lambda}, \ ..., \ A_{(n-1),\lambda} \end{array}\right),$$

where

$$\begin{aligned} A_{i,\lambda} :=& \lambda\delta_i\left(0, u^{(i)}(0) + L_i\left(u, ..., u^{(n-2)}, u^{(i)}(0)\right)\right), \ i = 0, ..., n-3 \\ A_{(n-2),\lambda} :=& \lambda\delta_{n-2}\left(\begin{array}{c} 0, u^{(n-2)}(0) + \\ L_{n-2}\left(u, ..., u^{(n-2)}, u^{(n-2)}(0), u^{(n-1)}(0)\right) \end{array}\right), \\ A_{(n-1),\lambda} :=& \lambda\delta_{n-2}\left(\begin{array}{c} 1, u^{(n-2)}(1) + \\ L_{n-1}\left(u, ..., u^{(n-2)}, u^{(n-2)}(1), u^{(n-1)}(1)\right) \end{array}\right). \end{aligned}$$

Therefore the auxiliary problem (6.3.6)-(6.3.7) has at least one solution $u_1(x)$ for $\lambda = 1$.

- This solution $u_1(x)$ is also a solution of the original problem (6.1.3)-(6.1.4) because it can be proved, arguing as in Theorem 6.2.2, that

$$\alpha^{(i)}(x) \leq u_1^{(i)}(x) \leq \beta^{(i)}(x), \ i = 0, ..., n-2, \ \forall x \in [0,1],$$

and, for $i = 0, ..., n-3$,

$$\alpha^{(i)}(0) \leq u_1^{(i)}(0) + L_i\left(u_1, ..., u_1^{(n-2)}, u_1^{(i)}(0)\right) \leq \beta^{(i)}(0),$$

$$\alpha^{(n-2)}(0) \leq u_1^{(n-2)}(0) + L_{n-2}\left(u_1, ..., u_1^{(n-2)}, u_1^{(n-2)}(0), u_1^{(n-1)}(0)\right)$$
$$\leq \beta^{(n-2)}(0)$$
$$\alpha^{(n-2)}(1) \leq u_1^{(n-2)}(1) + L_{n-1}\left(u_1, ..., u_1^{(n-2)}, u_1^{(n-2)}(1), u_1^{(n-1)}(1)\right)$$
$$\leq \beta^{(n-2)}(1).$$

■

6.4 Conjugate boundary value problems

In this section it is considered a $(n-1, 1)$ conjugate boundary value problem. These higher order problems are so called due to the way that the information in the boundary conditions is provided. In this case meaning that information about the solution and up to the $(n-1)$ derivatives is provided at the startpoint and, at the endpoint, only the information about the solution is given. So, consider the $(n-1, 1)$ conjugate boundary value problem, for $n \geq 2$,

$$\begin{aligned} u^{(n)}(x) + g(x) f(u(x)) &= 0 \\ u^{(i)}(0) = u(1) &= 0, \ i = 0, ..., n-2, \end{aligned} \tag{6.4.1}$$

where $x \in [0,1]$, $f : [0, \infty) \to [0, \infty)$ and $g : [0,1] \to [0, \infty)$ are continuous functions.

These boundary value problems have been studied by many authors, either from a theoretical point of view either with several kinds of real applications. For instance in the second order case they describe several phenomena such as nonlinear diffusion generated by nonlinear sources, thermal ignition of gases and concentration in chemical or biological problems where only positive solutions are meaningful.

The $(n-1,1)$ conjugate boundary value problem was first introduced, as far aas we know, in [26], with the problem

$$u^{(n)}(x)+a(x)f(u(x))=0$$
$$u^{(i)}(0)=u(1)=0,\ i=0,...,n-2,$$

where $x\in(0,1)$, $f:[0,\infty)\to[0,\infty)$, $a:[0,1]\to[0,\infty)$ are continuous functions, a does not vanish identically on any subinterval and f is either sublinear or superlinear. Existence results are obtained using cones and Krasnosel'skii fixed point theorems of cone compressions and cone expansions.

In [96], the nonlocal problem, for $n\geq 2$

$$u^{(n)}(x)+g(x)f(u(x))=0$$
$$u^{(i)}(0)=0,\ i=0,...,n-2,\quad u(1)=\alpha[u],$$

where $x\in(0,1)$, $f:[0,\infty)\to[0,\infty)$, $a:[0,1]\to[0,\infty)$ are continuous functions and $\alpha[u]=\int_0^1 u(s)\,dA(s)$, is studied by fixed point index.

The key idea developed in all the above papers is to find lower and upper estimates for the solutions. Moreover for problem (6.4.1) some sufficient conditions for the existence and nonexistence of solution are obtained.

Lower and upper solutions can be of extremely useful in this quest as they can in fact provide the lower and upper bounds needed for these results. Throughout this section one will expose an alternative way of obtaining the bounds mentioned in [98], where some lower and upper bounds to the solution are developed to obtain the main result.

For clearness we refer the functions used to obtain the estimations:

Consider $w_1,w_2:[0,1]\to[0,\infty)$ given by

$$w_1(t)=\begin{cases}(n-1)^{n-1}(n-2)^{2-n}\left(t^{n-2}-t^{n-1}\right), & \text{if}\quad t\geq p,\\ \\ t^{n-1} & \text{if}\quad t\leq p,\end{cases}$$

and

$$w_2(t) = \begin{cases} (n-1)^{n-1}(n-2)^{2-n}\left(t^{n-2} - t^{n-1}\right), & \text{if } \ t \le q, \\ \\ 1 & \text{if } \ t \ge q, \end{cases}$$

for the constants

$$p = \frac{(n-1)^{n-1}}{(n-1)^{n-1} + (n-2)^{n-2}}, \quad q = \frac{n-2}{n-1},$$

The main result is given by the theorem:

Theorem 6.4.1 *[98, Theorem 2.10] If $u \in C^n([0,1])$ satisfies the boundary conditions from (6.4.1), $u^{(n)}(t) \le 0$ and $u(t) > 0$ for $0 < t < 1,$ then*

$$w_2(t)\|u\| \ge u(t) \ge w_1(t)\|u\|, \quad 0 \le t \le 1, \tag{6.4.2}$$

and

$$u(t) \le u(p)\frac{w_2(t)}{w_1(p)}, \quad 0 \le t \le 1. \tag{6.4.3}$$

In particular, if $u \in C^n([0,1])$ is a positive solution of (6.4.1), then $u(t)$ satisfies (6.4.2) and (6.4.3).

The proof of this Theorem applies analytical and numerical methods. However lower and upper solutions are another tool very useful for these cases and it allows to obtain, eventually, sharper estimates.

To illustrate the role of lower and upper solutions in this field let us consider Example 3.6 from [98].

The $(3,1)$ conjugate boundary value problem is given by

$$u^{(iv)}(x) + \mu g(x)u(x) = 0, \ x \in (0,1)$$

with the boundary conditions

$$u^{(i)}(0) = u(1) = 0, \ i = 0,1,2. \tag{6.4.4}$$

In particular, for $g(x) \equiv 1$ and $\mu > 0$ the equation becomes

$$u^{(iv)}(x) = -\mu u(x), \ x \in (0,1). \tag{6.4.5}$$

Then by Theorem 6.4.1, for

$$w_1(x) = \begin{cases} \frac{27}{4}\left(x^2 - x^3\right), & \text{if } \ x \geq \frac{27}{31}, \\ \\ x^3 & \text{if } \ x \leq \frac{27}{31}, \end{cases}$$

and

$$w_2(x) = \begin{cases} \frac{27}{4}\left(x^2 - x^3\right), & \text{if } \ x \leq \frac{2}{3}, \\ \\ 1 & \text{if } \ x \geq \frac{2}{3}, \end{cases}$$

there is a solution $u(x)$, such that

$$1 \geq w_2(x) \geq u(x) \geq w_1(x) \geq 0,$$

for $x \in [0,1]$. This area is shown in Figure 6.4.1, meaning that the solution $u(x)$, lies inside the grey area.

It is easy to see that (6.4.5),(6.4.4) is a particular case of problem (6.1.3)-(6.1.4), with $n = 4$ and

$$\begin{aligned} f(x, y_0, y_1, y_2, y_3) &= -\mu y_0, \\ L_0(z_1, z_2, z_3, z_4) &= z_4, \\ L_1(z_1, z_2, z_3, z_4) &= z_4, \\ L_2(z_1, z_2, z_3, z_4, z_5) &= z_4, \\ L_3(z_1, z_2, z_3, z_4, z_5) &= z_1(1). \end{aligned}$$

Functions $\alpha, \beta \in [0,1] \to \mathbb{R}$ given by

$$\alpha(x) = 0 \text{ and } \beta(x) = -\frac{x^4}{8} + \frac{x^3}{4}$$

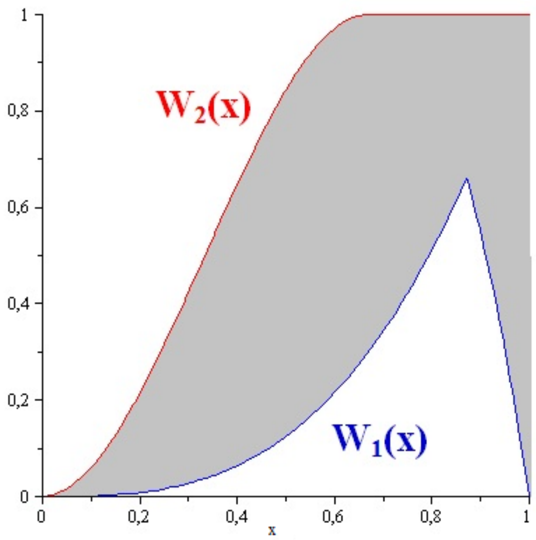

Figure 6.4.1: Functions $w_1(x)$ and $w_2(x)$ bound the solution

are, respectively, lower and upper solutions for (6.4.5),(6.4.4), for $\mu \leq 24$, according Definition 6.3.1.

As the continuous function f verifies Nagumo condition with

$$h_{E_*}(y_3) = \frac{\mu}{4} \leq 6,$$

in

$$E = \left\{ (x, y_0) \in [0,1] \times \mathbb{R}^4 : 0 \leq y_0 \leq -\frac{x^4}{8} + \frac{x^3}{4} \right\},$$

then, by Theorem 6.2.2, there is a solution $u(x)$ for problem (6.4.5),(6.4.4) such that

$$0 \leq u(x) \leq -\frac{x^4}{8} + \frac{x^3}{4}, \ \forall x \in [0,1]. \tag{6.4.6}$$

Moreover this solution, by (6.4.6) is a non negative solution for the problem (6.4.5),(6.4.4).

Comparing both Figure 6.4.1 and Figure 6.4.2 one can conclude that for $\mu \in (0, 24]$ the solution given Theorem 6.4.1 and (6.4.6) is not the same

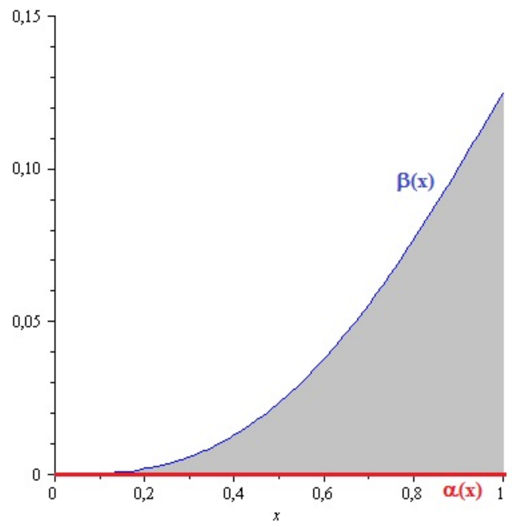

Figure 6.4.2: The non negative solution for problem (6.4.5),(6.4.4) is delimited by α and β

solution. However, in some sense, the estimation given by (6.4.6) is sharper than the bounds given by Theorem 6.4.1.

Furthermore lower and upper solutions method provide information for a different set of μ than the one shown in [98]: here the results are obtained for $\mu > 0$, and in our example the solution is generalized for $\mu \leq 24$.

Chapter 7

Generalized $\phi-$Laplacian equation with functional boundary conditions

7.1 Introduction

This chapter is devoted to the study of a generalized n^{th} order ϕ - Laplacian type differential equation

$$-\left(\phi\left(u^{(n-1)}(x)\right)\right)' = f(x, u(x), ..., u^{(n-1)}(x)), \tag{7.1.1}$$

for $x \in I := [0,1]$, where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, $n \geq 2$, and $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function, with the boundary conditions

$$\begin{aligned} &g_i\left(u, u', ..., u^{(n-2)}, u^{(i)}(1)\right) = 0, \quad i = 0, ..., n-3, \\ &g_{n-2}\left(u, u', ..., u^{(n-2)}, u^{(n-2)}(0), u^{(n-1)}(0)\right) = 0, \\ &g_{n-1}\left(u, u', ..., u^{(n-2)}, u^{(n-2)}(1), u^{(n-1)}(1)\right) = 0, \end{aligned} \tag{7.1.2}$$

where $g_i : (C(I))^{n-1} \times \mathbb{R} \to \mathbb{R}$, $i = 0, ..., n-3$, $g_{n-2}, g_{n-1} : (C(I))^{n-1} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions verifying adequate monotone assumptions.

Boundary conditions (7.1.2) cover many of the classical boundary conditions, such as various two-point and multipoint, integral with delay and/or advances boundary conditions, nonlocal, with maximum and/or minimum arguments,... One can refer ([39, 43, 61, 73]) for higher order separated problems, ([36, 37, 38, 54, 55, 60, 83, 100]), for the multipoint cases, and ([15, 16, 19, 74]), for higher order functional problems.

In these papers a variety of techniques and tools is used, with the lower and upper solution method. The same method was used in [40] to study the problem composed by (7.1.1) for $n = 2$ and the boundary conditions

$$\begin{gathered} g\left(u\left(0\right), u'\left(0\right), u'\left(1\right)\right) = 0 \\ u\left(1\right) = h\left(u\left(0\right)\right), \end{gathered}$$

and obtain sufficient conditions for the existence of solution.

The main existence and location result, here presented is based on [41] and it seems interesting to us, not only by the improvement on the existing related literature but also by some of its consequences and conclusions:

- for $n \geq 3$ the order between lower and upper solutions, and their derivatives until order $(n-3)$, is not relevant. In fact, these orders depend whether n is odd or even and on the relation between the $(n-2)$ derivatives of lower and upper solutions, as it can be seen in Remark 7.3.2;

- the behavior of the nonlinearity f, given by (7.3.4), depends on several factors: from the parity of n, from the relation between the $(n-2)$ derivatives of lower and upper solution and subsequent orders;

- the assumptions on the monotone behavior of the functions on the boundary data, depend on the parity of n (see assumptions (N_1) and (N_2)).

The above items were just "guessed" from the existent results in higher order boundary value problems of different orders, where lower and upper solutions are applied in the well ordered or reversed order cases. However, as far as we know, they were proved in [41] for the first time.

The arguments here applied follow the standard lower and upper solutions technique, together with a Nagumo-type condition, to control the growth of $u^{(n-1)}$, and a fixed-point result. Remark, also, that, due to a truncation tool, it is not considered the usual assumption on ϕ, that is, $\phi(\mathbb{R}) = \mathbb{R}$.

7.2 Preliminary results and definitions

This section will provide some definitions and results to be used forward.

Let $L^p(I)$, $1 \leq p \leq \infty$, be the usual Lebesgue spaces of functions with the standard norms.

The Nagumo-type condition for this case needs an adjustment on the integral assumption:

Definition 7.2.1 *Given a subset $E \subset I \times \mathbb{R}^n$, a function $f : I \times \mathbb{R}^n \to \mathbb{R}$ verifies a Nagumo-type condition in the set*

$$E := \{(x, y_0, \ldots, y_{n-1}) \in I \times \mathbb{R}^n : m_j(x) \leq y_j \leq M_j(x),\ j = 0, \ldots, n-2\},$$

with $m_j, M_j \in C(I, \mathbb{R})$ such that

$$m_j(x) \leq M_j(x),\ \forall x \in I,\ j = 0, \ldots, n-2,$$

if there is $h_E \in C\left(\mathbb{R}_0^+,]0,+\infty[\right)$*, verifying*

$$|f(x,y_0,...,y_{n-1})| \leq h_E(|y_{n-1}|),\ \forall (x,y_0,...,y_{n-1}) \in E, \tag{7.2.1}$$

with

$$\int_{\phi(r)}^{+\infty} \frac{\left|\phi^{-1}(s)\right|}{h_E(\left|\phi^{-1}(s)\right|)}\ ds > \max_{x\in I} M_{n-2}(x) - \min_{x\in I} m_{n-2}(x), \tag{7.2.2}$$

for $r \geq 0$ *such that*

$$r := \max\left\{M_{n-2}(1) - m_{n-2}(0), M_{n-2}(0) - m_{n-2}(1)\right\}. \tag{7.2.3}$$

The *a priori* estimation for the $(n-1)$ derivative is given by Lemma 1.2.2 now adapted to condition (7.2.2)

Next Lemma proves the existence and uniqueness of solution for a related problem of (7.1.1) – (7.1.2).

Lemma 7.2.2 *Consider* $\varphi : \mathbb{R} \to \mathbb{R}$ *an increasing homeomorphism such that* $\varphi(0) = 0$ *and* $\varphi(\mathbb{R}) = \mathbb{R}$, $p : I \to \mathbb{R}$ *such that* $p \in L^1(I)$, $A_i,\ B,\ C \in \mathbb{R}$, $i = 0,...,n-3$. *Then the problem*

$$\begin{cases} -\left(\varphi\left(u^{(n-1)}(x)\right)\right)' = p(x), \quad \textit{for a. e. } x \in I, \\ u^{(i)}(1) = A_i, \quad i = 0,...,n-3, \\ u^{(n-2)}(0) = B \\ u^{(n-2)}(1) = C \end{cases} \tag{7.2.4}$$

has a unique solution given by

$$u(x) = B + \int_0^x \varphi^{-1}\left(\tau_v - \int_0^s p(r)\ dr\right) ds$$

if $n = 2$, *and*

$$u(x) = \sum_{k=0}^{n-3} (-1)^k A_k \frac{(1-x)^k}{k!} + (-1)^n \int_x^1 \frac{(s-x)^{n-3}}{(n-3)!} v(s)\ ds, \tag{7.2.5}$$

if $n \geq 3$, *with*

$$v(x) := B + \int_0^x \varphi^{-1}\left(\tau_v - \int_0^s p(r)\ dr\right) ds$$

and $\tau_v \in \mathbb{R}$ *is the unique solution of the equation*

$$C - B = \int_0^1 \varphi^{-1}\left(\tau_v - \int_0^s p(r)\ dr\right) ds. \tag{7.2.6}$$

Proof. Defining $v(x) := u^{(n-2)}(x)$, from (7.2.4) we obtain the Dirichlet problem

$$-\left(\varphi\left(v'(x)\right)\right)' = p(x), \text{ for } a.e.\ x \in I \tag{7.2.7}$$

$$v(0) = B,\ v(1) = C. \tag{7.2.8}$$

Therefore, for some $\tau \in \mathbb{R}$,

$$v'(x) = \varphi^{-1}\left(\tau - \int_0^x p(r)\, dr\right)$$

and

$$v(x) = B + \int_0^x \varphi^{-1}\left(\tau - \int_0^x p(r)\, dr\right) ds. \tag{7.2.9}$$

Since φ^{-1} is increasing, we have

$$\begin{aligned} v_*(\tau) \quad : & \quad = B + \varphi^{-1}\left(\tau - \|p\|_1\right) \leq v(1) \\ & \leq \quad B + \varphi^{-1}\left(\tau + \|p\|_1\right) := v^*(\tau) \end{aligned}$$

for each $\tau \in \mathbb{R}$. Now $\varphi^{-1}(\mathbb{R}) = \mathbb{R}$ and the functions v_* and v^* are continuous and increasing, so $v_*(\mathbb{R}) = v^*(\mathbb{R}) = \mathbb{R}$. Thus there is a unique τ_v satisfying (7.2.6).

If $n = 2$ the proof is concluded. For $n \geq 3$, then repeatedly integrating (7.2.9) and applying the boundary conditions, we obtain (7.2.5). ■

In the sequel, it will be assumed that the continuous functions g_i : $(C(I))^{n-1} \times \mathbb{R} \to \mathbb{R},\ \ i = 0, ..., n-3$, and $g_{n-2}, g_{n-1} : (C(I))^{n-1} \times \mathbb{R}^2 \to \mathbb{R}$ have a different behavior as n is even or odd. More precisely:

(i) for n **even** it is said that the boundary functions verify assumption (N_1) if the following conditions hold:

- $g_j\left(y_0, y_1, ..., y_{n-1}\right)$, are nondecreasing on $y_0, y_2, ..., y_{n-2}$, and nonincreasing on $y_1, y_3, ..., y_{n-3}$, for j even such that $0 \leq j \leq n-4$;
- $g_k\left(y_0, y_1, ..., y_{n-1}\right)$, are nonincreasing on $y_0, y_2, ..., y_{n-2}$, and nondecreasing on $y_1, y_3, ..., y_{n-3}$, for k odd such that $1 \leq k \leq n-3$;
- $g_{n-2}\left(y_0, y_1, ..., y_{n-1}, y_n\right)$ is nondecreasing on $y_0, y_2, ..., y_{n-2}$ and y_n, and nonincreasing on $y_1, y_3, ..., y_{n-3}$;
- $g_{n-1}\left(y_0, y_1, ..., y_{n-1}, y_n\right)$ is nondecreasing on $y_0, y_2, ..., y_{n-2}$, and nonincreasing on $y_1, y_3, ..., y_{n-3}$ and y_n;

(ii) for n **odd** the boundary functions verify (N_2) if the following conditions hold:

- $g_j\left(y_0, y_1, ..., y_{n-1}\right)$, are nondecreasing on $y_0, y_2, ..., y_{n-3}$, and nonincreasing on $y_1, y_3, ..., y_{n-2}$, for j even such that $0 \leq j \leq n-3$;
- $g_k\left(y_0, y_1, ..., y_{n-1}\right)$, are nonincreasing on $y_0, y_2, ..., y_{n-3}$, and nondecreasing on $y_1, y_3, ..., y_{n-2}$, for k odd such that $1 \leq k \leq n-4$;
- $g_{n-2}\left(y_0, y_1, ..., y_{n-1}, y_n\right)$ is nonincreasing on $y_0, y_2, ..., y_{n-3}$, and nondecreasing on $y_1, y_3, ..., y_{n-2}$ and y_n;
- $g_{n-1}\left(y_0, y_1, ..., y_{n-1}, y_n\right)$ is nonincreasing on $y_0, y_2, ..., y_{n-3}$ and y_n, and nondecreasing on $y_1, y_3, ..., y_{n-2}$.

Noting by $AC(I)$, the set of absolutely continuous function on I, the functions used as lower and upper solutions are defined as it follows:

Definition 7.2.3 *Let $n \geq 2$. A function $\alpha \in C^{n-1}(I)$ such that $\phi\left(\alpha^{(n-1)}(x)\right) \in AC(I)$ is a lower solution of problem (7.1.1)-(7.1.2) if*

$$-\left(\phi\left(\alpha^{(n-1)}(x)\right)\right)' \leq f(x, \alpha(x), \alpha'(x), \ldots, \alpha^{(n-1)}(x)), \tag{7.2.10}$$

for $x \in]0,1[$,

(i) *for n* ***even***

$$\begin{aligned} &g_j\left(\alpha, \alpha', ..., \alpha^{(n-2)}, \alpha^{(j)}(1)\right) \geq 0, \text{ for } j \text{ even such that } 0 \leq j \leq n-4, \\ &g_k\left(\alpha, \alpha', ..., \alpha^{(n-2)}, \alpha^{(k)}(1)\right) \leq 0, \text{ for } k \text{ odd such that } 1 \leq k \leq n-3, \end{aligned} \tag{7.2.11}$$

(ii) *for n* ***odd***

$$\begin{aligned} &g_j\left(\alpha, \alpha', ..., \alpha^{(n-2)}, \alpha^{(j)}(1)\right) \leq 0, \text{ for } j \text{ even such that } 0 \leq j \leq n-3, \\ &g_k\left(\alpha, \alpha', ..., \alpha^{(n-2)}, \alpha^{(k)}(1)\right) \geq 0, \text{ for } k \text{ odd such that } 1 \leq k \leq n-4, \end{aligned} \tag{7.2.12}$$

and

(iii) *in* ***both cases***

$$\begin{aligned} &g_{n-2}\left(\alpha, \alpha', ..., \alpha^{(n-2)}, \alpha^{(n-2)}(0), \alpha^{(n-1)}(0)\right) \geq 0, \\ &g_{n-1}\left(\alpha, \alpha', ..., \alpha^{(n-2)}, \alpha^{(n-2)}(1), \alpha^{(n-1)}(1)\right) \geq 0. \end{aligned}$$

A function $\beta \in C^{n-1}(I)$ such that $\phi\left(\beta^{(n-1)}(x)\right) \in AC(I)$ is an upper solution of problem (7.1.1)-(7.1.2), if the reversed inequalities hold in each case.

The following version of the Schauder fixed point theorem will also be considered:

Theorem 7.2.4 *[87, Theorem 5.11]Let X be a normed vector space, and let $K \subset X$ be a non-empty, compact, and convex set. Then given any continuous mapping $f : K \to K$ there exists $x \in K$ such that $f(x) = x$.*

7.3 Existence and location theorem

The main result is an existence and location theorem, as it is usual in lower and upper solutions technique. However, in this case, the strips are bounded by well ordered lower and upper solutions and the corresponding derivatives, and in reversed order, as well. Therefore, for a more clear notation, it is defined the following functions:

$$\gamma_i(x) = \min_{x\in I}\left\{\alpha^{(i)}(x), \beta^{(i)}(x)\right\}, \ \Gamma_i(x) = \max_{x\in I}\left\{\alpha^{(i)}(x), \beta^{(i)}(x)\right\}, \tag{7.3.1}$$

for each $i = 0, \ldots, n-2$.

Theorem 7.3.1 *Let $f : I \times \mathbb{R}^n \to \mathbb{R}$ be a L^1-Carathéodory function. Assume that α and β are lower and upper solutions of problem (7.1.1)-(7.1.2), respectively, such that*

$$\begin{aligned} \alpha^{(n-2)}(x) &\leq \beta^{(n-2)}(x), \ \forall x \in [0,1], && (7.3.2)\\ (-1)^m\,\alpha^{(n-2-m)}(1) &\leq (-1)^m\,\beta^{(n-2-m)}(1), \ m = 1, \ldots, n-2, && (7.3.3)\end{aligned}$$

f satisfies Nagumo-type condition (7.2.1) in the set

$$E_* = \{(x, y_0, \ldots, y_{n-1}) \in I \times \mathbb{R}^n : \gamma_i(x) \leq y_i \leq \Gamma_i(x), \ i = 0, \ldots, n-2\},$$

and

$$\begin{aligned} f(x, \alpha(x), \ldots, \alpha^{(n-3)}(x), y_{n-2}, y_{n-1}) &\leq f(x, y_0, \ldots, y_{n-1}) \qquad (7.3.4)\\ &\leq f(x, \beta(x), \ldots, \beta^{(n-3)}(x), y_{n-2}, y_{n-1}),\end{aligned}$$

for fixed x, y_{n-2}, y_{n-1} and $\gamma_k(x) \leq y_k \leq \Gamma_k(x)$, $k = 0, ..., n-3$, $\forall x \in I$. Moreover, if n is even and the boundary functions verify (N_1), or n is odd, and the boundary functions satisfy (N_2), then problem (7.1.1)-(7.1.2) has at least a solution u such that

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x),$$

for $i = 0, \ldots, n-2$, *and*

$$-R \leq u^{(n-1)}(x) \leq R,$$

for every $x \in I$, *with*

$$R > \max \left\{ \begin{array}{c} \beta^{(n-2)}(1) - \alpha^{(n-2)}(0), \beta^{(n-2)}(0) - \alpha^{(n-2)}(1), \\ \|\alpha^{(n-1)}\|_\infty, \|\beta^{(n-1)}\|_\infty \end{array} \right\}. \tag{7.3.5}$$

Remark 7.3.2 *From the integration of (7.3.2) in* $[x, 1]$ *and conditions (7.3.3), the derivatives of lower and upper solutions will change order, that is, for every* $x \in I$,

$$\begin{array}{rcl} \alpha^{(n-3)}(x) & \geq & \beta^{(n-3)}(x), \\ \alpha^{(n-4)}(x) & \leq & \beta^{(n-4)}(x), \\ & \vdots & \\ \alpha(x) & \leq & \beta(x), \end{array}$$

if n *is even or, for* n *odd, the iteration will end with*

$$\alpha(x) \geq \beta(x), \text{ in } I.$$

As the relation between the lower and upper solutions depends on n, *and their derivatives can be well ordered or in reversed order, therefore this issue has not, for* $n \geq 3$, *the same relevance as it has for first and second order. As a consequence, the same can be said for the variation of the nonlinearity* f, *as it can be seen in (7.3.4).*

Proof. For $i = 0, ..., n-2$, consider the continuous truncations, ,

$$\delta_i(x, w) = \begin{cases} \Gamma_i(x) & , \quad w > \Gamma_i(x), \\ w & , \quad \gamma_i(x) \leq w \leq \Gamma_i(x), \\ \gamma_i(x) & , \quad w < \gamma_i(x), \end{cases} \tag{7.3.6}$$

where $\gamma_i(x)$ and $\Gamma_i(x)$ are given by (7.3.1), and, for R given by (7.3.5), the functions

$$\xi(z) = \max\{-R, \min\{z,\ R\}\} \tag{7.3.7}$$

and $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(y) = \begin{cases} \phi(y) & \text{if } \ |y| \le R, \\ \\ \frac{\phi(R)-\phi(-R)}{2R}\, y + \frac{\phi(R)+\phi(-R)}{2} & \text{if } \ |y| > R. \end{cases} \tag{7.3.8}$$

Define the modified problem composed by the differential equation

$$\begin{aligned} & -\left(\varphi\left(u^{(n-1)}(x)\right)\right)' \\ = \ & f\left(x, \delta_0(x,u), ..., \delta_{n-2}\left(x, u^{(n-2)}\right), \xi\left(\frac{d}{dx}\delta_{n-2}\left(x, u^{(n-2)}\right)\right)\right) \equiv F_u(x) \end{aligned} \tag{7.3.9}$$

and the boundary conditions, for $i = 0, ..., n-3$,

$$\begin{aligned} u^{(i)}(1) &= \delta_i\left(1, u^{(i)}(1) + g_i\left(u, ..., u^{(n-2)}, u^{(i)}(1)\right)\right), \\ u^{(n-2)}(0) &= \delta_{n-2}\begin{pmatrix} 0, u^{(n-2)}(0)+ \\ g_{n-2}\left(u, ..., u^{(n-2)}, u^{(n-2)}(0), u^{(n-1)}(0)\right) \end{pmatrix} \\ u^{(n-2)}(1) &= \delta_{n-2}\begin{pmatrix} 1, u^{(n-2)}(1)+ \\ g_{n-1}\left(u, ..., u^{(n-2)}, u^{(n-2)}(1), u^{(n-1)}(1)\right) \end{pmatrix}. \end{aligned} \tag{7.3.10}$$

A function $u \in C^{n-1}(I])$ such that $\left(\phi \circ u^{(n-1)}\right) \in AC(I)$ is a solution of problem (7.3.9)-(7.3.10) if it satisfies the above equalities.

Step 1- Every solution of problem (7.3.9)-(7.3.10) verifies in I

$$\gamma_i(x) \ \le \ u^{(i)}(x) \le \Gamma_i(x), \ \textit{for } i = 0, \ldots, n-2, \tag{7.3.11}$$

$$-R \ \le \ u^{(n-1)}(x) \le R. \tag{7.3.12}$$

Let u be a solution of (7.3.9)-(7.3.10).

For $i = n-2$ we have $\gamma_{n-2}(x) = \alpha^{(n-2)}(x)$ and $\Gamma_{n-2}(x) = \beta^{(n-2)}(x)$.

Assume, by contradiction, that the second inequality in (7.3.11) does not hold and define

$$\max_{x\in I} (u-\beta)^{(n-2)}(x) := (u-\beta)^{(n-2)}(x_0) > 0.$$

By (7.3.10), $u^{(n-2)}(0) \leq \beta^{(n-2)}(0)$ and $u^{(n-2)}(1) \leq \beta^{(n-2)}(1)$. So, $x_0 \in (0,1)$, $u^{(n-1)}(x_0) = \beta^{(n-1)}(x_0)$ and there is $\varepsilon > 0$ such that

$$u^{(n-2)}(x_0+\varepsilon) = \beta^{(n-2)}(x_0+\varepsilon)$$

and $u^{(n-2)}(x) > \beta^{(n-2)}(x)$ on $[x_0, x_0+\varepsilon)$.

On $(x_0, x_0+\varepsilon)$, by Definition 7.2.3, (7.3.4), (7.3.6), (7.3.7) and (7.3.5), the following inequality is achieved

$$\begin{aligned}
& -\left(\varphi\left(u^{(n-1)}(x)\right)\right)' \\
= \; & f\left(x, \delta_0(x,u), ..., \delta_{n-2}\left(x, u^{(n-2)}\right), \xi\left(\frac{d}{dx}\delta_{n-2}\left(x, u^{(n-2)}\right)\right)\right) \\
= \; & f\left(x, \delta_0(x,u), ..., \delta_{n-3}\left(x, u^{(n-3)}\right), \beta^{(n-2)}(x), \beta^{(n-1)}(x)\right) \\
\leq \; & f\left(x, \beta(x), ..., \beta^{(n-3)}, \beta^{(n-2)}(x), \beta^{(n-1)}(x)\right) \\
\leq \; & -\left(\phi\left(\beta^{(n-1)}(x)\right)\right)' = -\left(\varphi\left(\beta^{(n-1)}(x)\right)\right)',
\end{aligned}$$

therefore $u^{(n-1)}(x) \geq \beta^{(n-1)}(x)$ on $(x_0, x_0+\varepsilon)$, which is a contradiction with the definition of $[x_0, x_0+\varepsilon)$.

So $u^{(n-2)}(x) \leq \beta^{(n-2)}(x)$ for every $x \in I$. By analogous arguments it can be shown that $\alpha^{(n-2)}(x) \leq u^{(n-2)}(x)$ in I.

Integrating the inequalities

$$\alpha^{(n-2)}(x) \leq u^{(n-2)}(x) \leq \beta^{(n-2)}(x),$$

in $[x, 1]$, by (7.3.3) and (7.3.10), we obtain

$$\alpha^{(n-3)}(x) \geq u^{(n-3)}(x) \geq \beta^{(n-3)}(x).$$

Iterating this integration it can be proved that, for n even,

$$\alpha^{(j)}(x) \leq u^{(j)}(x) \leq \beta^{(j)}(x), \text{ for } j \text{ even such that } 0 \leq j \leq n-2,$$

and

$$\alpha^{(k)}(x) \geq u^{(k)}(x) \geq \beta^{(k)}(x), \text{ for } k \text{ odd such that } 1 \leq k \leq n-3.$$

For n odd

$$\alpha^{(k)}(x) \leq u^{(k)}(x) \leq \beta^{(k)}(x), \text{ for } k \text{ odd such that } 1 \leq k \leq n-2,$$

and

$$\alpha^{(j)}(x) \geq u^{(j)}(x) \geq \beta^{(j)}(x), \text{for } j \text{ even such that } 0 \leq j \leq n-3.$$

Therefore condition (7.3.11) holds for $i = 0, ..., n-2$.

From Lemma 4.4.1 and the definition of ξ, the right hand side of the equation (7.3.9) is a L^1 – function. Therefore, Lemma 1.2.2 can be applied, with the integral condition (7.2.2) and $m_j(x) = \gamma_j(x)$ and $M_j(x) = \Gamma_j(x)$, for $j = 0, ..., n-2$, that is, condition (7.3.12) holds.

Step 2 - Problem (7.3.9)-(7.3.10) has a solution $u_1(x)$.

Let $u \in C^{n-1}(I)$ be fixed. By Lemma 7.2.2, solutions of problem (7.3.9)-(7.3.10) are the fixed points of the operator

$$\begin{aligned} \mathcal{T}\, u(x) &= \sum_{k=0}^{n-3} (-1)^k \delta_k \left(1, u^{(k)}(1) + g_k \left(u, ..., u^{(n-2)}, u^{(k)}(1)\right)\right) \frac{(1-x)^k}{k!} \\ &\quad + (-1)^n \int_x^1 \frac{(s-x)^{n-3}}{(n-3)!} v_u(s)\ ds, \end{aligned}$$

with

$$\begin{aligned} v_u(x) \ &: = g_{n-2}\left(u, u', ..., u^{(n-2)}, u^{(n-2)}(0), u^{(n-1)}(0)\right) \\ &\quad + \int_0^x \varphi^{-1}\left(\tau_u - \int_0^s F_u(r)\ dr\right) ds \end{aligned}$$

and $\tau_u \in \mathbb{R}$ is the unique solution of the equation

$$\begin{aligned}
& g_{n-1}\left(u, ..., u^{(n-2)}, u^{(n-2)}(1), u^{(n-1)}(1)\right) \\
& -g_{n-2}\left(u, ..., u^{(n-2)}, u^{(n-2)}(0), u^{(n-1)}(0)\right) \\
= & \int_0^1 \varphi^{-1}\left(\tau_u - \int_0^s F_u(r)\ dr\right) ds. \qquad (7.3.13)
\end{aligned}$$

By (7.3.9), there is a function $\omega \in L^1(I)$ such that

$$\mid F_u(s) \mid \leq \omega(s) \text{ for a. e. } s \in [0,1] \text{ and for all } u \in C^{n-1}([0,1]),$$

and, by (7.3.13), there exists $L > 0$ such that

$$\mid \tau_u \mid \leq L \text{ for all } u \in C^{n-1}([0,1]).$$

So, we conclude that operator $\mathcal{T}(C^{n-1}(I))$ is bounded in $C^{n-1}(I)$ and, by Theorem 7.2.4, operator $\mathcal{T}$ has a fixed point u_1.

Step 3 - $u_1(x)$ is a solution of problem (7.1.1)-(7.1.2).

To show that this function $u_1(x)$ is a solution of the initial problem (7.1.1)-(7.1.2) by Step 1, it will be enough to prove that

$$\begin{aligned}
\gamma_i(1) & \leq u_1^{(i)}(1) + g_i\left(u_1, u_1', ..., u_1^{(n-2)}, u_1^{(i)}(1)\right) \qquad (7.3.14)\\
& \leq \Gamma_i(1), \quad i = 0, ..., n-3, \\
\alpha^{(n-2)}(0) & \leq u_1^{(n-2)}(0) + \qquad (7.3.15)\\
& \quad g_{n-2}\left(u_1, u_1', ..., u_1^{(n-2)}, u_1^{(n-2)}(0), u_1^{(n-1)}(0)\right) \\
& \leq \beta^{(n-2)}(0), \\
\alpha^{(n-2)}(1) & \leq u_1^{(n-2)}(1) + \\
& \quad g_{n-1}\left(u_1, u_1', ..., u_1^{(n-2)}, u_1^{(n-2)}(1), u_1^{(n-1)}(1)\right) \\
& \leq \beta^{(n-2)}(1).
\end{aligned}$$

Suppose that n is even.

Consider the case $i = 0$. Then, by (7.3.3), $\gamma_0(1) = \alpha(1)$ and $\Gamma_0(1) = \beta(1)$. Assume, by contradiction, that

$$u_1(1) + g_0\left(u_1, u_1', ..., u_1^{(n-2)}, u_1(1)\right) > \beta(1).$$

By (7.3.10), $u_1(1) = \beta(1)$, and, by (N_1) and Definition 7.2.3, the following contradiction is obtained

$$\begin{aligned} 0 &< g_0\left(u_1, u_1', ..., u_1^{(n-2)}, u_1(1)\right) = g_0\left(u_1, u_1', ..., u_1^{(n-2)}, \beta(1)\right) \\ &\leq g_0\left(\beta, \beta', ..., \beta^{(n-2)}, \beta(1)\right) \leq 0. \end{aligned}$$

Then

$$u_1(1) + g_0\left(u_1, u_1', ..., u_1^{(n-2)}, u_1(1)\right) \leq \beta(1)$$

and, with the same technique, it can be proved that

$$\alpha(1) \leq u_1(1) + g_0\left(u_1, u_1', ..., u_1^{(n-2)}, u_1(1)\right)$$

and the remaining inequalities in (7.3.14).

Considering that the first inequality of (7.3.15) does not hold, then, by (7.3.10), $u_1^{(n-2)}(0) = \alpha^{(n-2)}(0)$, and, by (7.3.11), $u_1^{(n-1)}(0) \geq \alpha^{(n-1)}(0)$. By monotone assumptions on g_{n-2}, it is obtained, by (7.2.11), this contradiction

$$\begin{aligned} 0 &> g_{n-2}\left(u_1, u_1', ..., u_1^{(n-2)}, u_1^{(n-2)}(0), u_1^{(n-1)}(0)\right) \\ &= g_{n-2}\left(u_1, u_1', ..., u_1^{(n-2)}, \alpha^{(n-2)}(0), u_1^{(n-1)}(0)\right) \\ &\geq g_{n-2}\left(\alpha, \alpha', ..., \alpha^{(n-2)}, \alpha^{(n-2)}(0), \alpha^{(n-1)}(0)\right) \geq 0. \end{aligned}$$

So, $\alpha^{(n-2)}(0) \leq u_1^{(n-2)}(0) + g\left(u_1, u_1', ..., u_1^{(n-2)}, u_1^{(n-2)}(0), u_1^{(n-1)}(0)\right)$ and, by a similar technique, it can be proved that

$$u^{(n-2)}(0) + g_{n-2}\left(u_1, u_1', ..., u_1^{(n-2)}, u_1^{(n-2)}(0), u_1^{(n-1)}(0)\right) \leq \beta^{(n-2)}(0).$$

Assuming that

$$\alpha^{(n-2)}(1) > u_1^{(n-2)}(1) + g_{n-1}\left(u_1, u_1', ..., u_1^{(n-2)}, u_1^{(n-2)}(1), u_1^{(n-1)}(1)\right),$$

by the same arguments we have

$$u_1^{(n-2)}(1) = \alpha^{(n-2)}(1) \text{ and } u_1^{(n-1)}(1) \leq \alpha^{(n-1)}(1).$$

Therefore, by the properties of g_{n-1}, it is achieved the contradiction

$$\begin{aligned} 0 &> g_{n-1}\left(u_1, u_1', ..., u_1^{(n-2)}, u_1^{(n-2)}(1), u_1^{(n-1)}(1)\right) \\ &= g_{n-1}\left(u_1, u_1', ..., u_1^{(n-2)}, \alpha^{(n-2)}(1), u_1^{(n-1)}(1)\right) \\ &\geq g_{n-1}\left(\alpha, \alpha', ..., \alpha^{(n-2)}, \alpha^{(n-2)}(1), \alpha^{(n-1)}(1)\right) \geq 0. \end{aligned}$$

The remained inequality can also be proved by the above technique.

For n odd the arguments are analogous, applying the monotone assumptions in (N_2) and the corresponding boundary conditions. ■

7.4 Examples

In this section three examples are presented to cover the cases where ϕ is not surjective, n is odd and n even. The boundary conditions are chosen not to get some physical meaning but only to emphasize the potentialities given by the functional dependence.

The existing results in the literature always assume that $\phi(\mathbb{R}) = \mathbb{R}$. In fact, due to the introduction of a truncation tool given by (7.3.8), this usual assumption is no longer assumed.

The following example illustrates this situation.

Example 7.4.1 *Let*

$$\phi(y) = \begin{cases} \arctan(y+5) - 125 & y < -5 \\ y^3 & -5 \le y \le 5 \\ \arctan(y-5) + 125 & y > 5, \end{cases}$$

the problem composed by the equation

$$(\phi(u'''(x)))' = u^{\frac{1}{3}}(x) - 2u''(x), \ x \in (0,1) \tag{7.4.1}$$

and the boundary conditions

$$\begin{gathered} u'\left(\tfrac{1}{2}\right) - 2u(0) = 0 \\ u(0) - 2u'(0) - 1 = 0 \\ u'''\left(\tfrac{1}{2}\right) - 3u''(0) = 0 \\ \int_0^1 u(s)\,ds + u'''\left(\tfrac{1}{4}\right) + u'''\left(\tfrac{3}{4}\right) - 10u''(1) = 0. \end{gathered} \tag{7.4.2}$$

The functions $\alpha(x) = -(x^2+x+1)$ *and* $\beta(x) = x^2+x+1$ *are lower and upper solutions, respectively of (7.4.1)-(7.4.2).*

It can be easily checked that this problem is a particular case of (7.1.1)-(7.1.2), with $n = 4$

$$f(x, y_0, y_1, y_2, y_3) = -y_0^{\frac{1}{3}} + 2y_2$$

and

$$\begin{aligned} g_0(y_0, y_1, y_2, y_3, y_4) &= y_1\left(\tfrac{1}{2}\right) - 2y_4 \\ g_1(y_0, y_1, y_2, y_3, y_4) &= y_0(0) - 2y_4 - 1 \\ g_2(y_0, y_1, y_2, y_3, y_4) &= y_3\left(\tfrac{1}{2}\right) - 3y_4 \\ g_3(y_0, y_1, y_2, y_3, y_4) &= \int_0^1 y_0(s)\,ds + y_3\left(\tfrac{1}{4}\right) + y_3\left(\tfrac{3}{4}\right) - 10y_4. \end{aligned}$$

Then by Theorem 7.3.1 the problem (7.4.1)-(7.4.2) has at least one non-trivial solution such that

$$\begin{gathered} -x^2 - x - 1 \le u(x) \le x^2 + x + 1 \\ -2x - 1 \le u'(x) \le 2x + 1 \\ -2 \le u''(x) \le 2, \ \textit{for } x \in (0,1). \end{gathered} \tag{7.4.3}$$

Example 7.4.2 *For $n = 3$ consider the problem composed by the equation*

$$\frac{u'''(x)}{1+(u''(x))^2} = (u(x))^3 + k\,(u'(x))^5 - \sqrt[3]{u''(x)+1} \tag{7.4.4}$$

and the boundary conditions

$$\begin{aligned} A\ u(1) &= \sum_{j=1}^{+\infty} a_j\ u\left(\xi_j\right) - \sum_{j=1}^{+\infty} b_j\ u'\left(\eta_j\right), \\ B\ u'(0) &= \max_{x\in[0,1]} u'(x) - \int_0^x u(t)dt + (u''(0))^{2p+1}, \\ C\ u'(1) &= \min_{x\in[0,1]} u'(x) - \max_{x\in[0,1]} u(x) - (u''(1))^{2q+1}, \end{aligned} \tag{7.4.5}$$

with $k, A, B, C \in \mathbb{R}$, $0 \le \xi_j, \eta_j \le 1$, $\forall j \in \mathbb{N}$, $p, q \in \mathbb{N}$ and $\sum_{j=1}^{+\infty} a_j$, $\sum_{j=1}^{+\infty} b_j$ are nonnegative and convergent series with sum $\overline{a}$ and $\overline{b}$, respectively.

This problem is a particular case of (7.1.1), (7.1.2), where $\phi(z) = \arctan z$ (remark that $\phi(\mathbb{R}) \neq \mathbb{R}$),

$$\begin{aligned} f\left(x, y_0, y_1, y_2\right) &= -y_0^3 - k\ y_1^5 + \sqrt[3]{y_2+1}, \\ g_0\left(z_1, z_2, z_3\right) &= \sum_{j=1}^{+\infty} a_j\ z_1\left(\xi_j\right) - \sum_{j=1}^{+\infty} b_j\ z_2\left(\eta_j\right) - Az_3, \\ g_1\left(z_1, z_2, z_3, z_4\right) &= \max_{x\in[0,1]} z_2 - \int_0^x z_1(t)dt + z_4^{2p+1} - Bz_3, \\ g_2\left(z_1, z_2, z_3, z_4\right) &= \min_{x\in[0,1]} z_2 - \max_{x\in[0,1]} z_1 - z_4^{2q+1} - Cz_3. \end{aligned}$$

The straight lines $\alpha(x) = 2 - x$ and $\beta(x) = x - 2$ are, respectively, lower and upper solutions of the problem (7.4.4), (7.4.5) for $k \ge 9$, $A \ge 2\overline{a} + \overline{b}$, $B \ge 3$ and $C \ge 3$. Therefore, by Theorem 7.3.1, there is a nontrivial solution, $u(x)$, of problem (7.4.4), (7.4.5), such that

$$\beta(x) = x - 2 \le u(x) \le 2 - x = \alpha(x)$$

and

$$\alpha'(x) = -1 \le u'(x) \le 1 = \beta'(x),\ \forall x \in [0,1].$$

Example 7.4.3 *In the case $n = 4$ it is considered the functional boundary value problem*

$$\begin{aligned} \left(u'''(x)^{2p+1}\right)' &= -\arctan(u(x)) + (u'(x))^3 - k\ (u''(x))^5 - |u'''(x)+1|^\theta, \\ A\ u(1) &= \max_{x\in[0,1]} u'(x) - \int_0^x u(t)dt, \\ B\ u'(1) &= \sum_{j=1}^{+\infty} a_j\ u''\left(\xi_j\right) \\ C\ \left(u''(0)\right)^3 &= -\max_{x\in[0,1]} u(x-\tau),\ \ (0<\tau\le x\le 1)\,, \\ D\ u''(1) &= u'\left(\max\{0, x-\varepsilon\}\right),\ \ (\varepsilon>0)\,, \end{aligned} \tag{7.4.6}$$

where $p \in \mathbb{N}$, $\theta \in [0,2]$, $k, A, B, C, D \in \mathbb{R}$, $0 \le \xi_j \le 1$, and $\sum_{j=1}^{+\infty} a_j$ $(a_j \ge 0)$ is convergent with sum $\overline{a}$.

The above problem verifies the assumptions of Theorem 7.3.1, with $\phi(z) = z^{2p+1}$, (in this case $\phi(\mathbb{R}) = \mathbb{R}$),

$$\begin{aligned} f\left(x, y_0, y_1, y_2, y_3\right) &= \arctan y_0 - y_1^3 + k\ y_2^5 + |y_3+1|^\theta, \\ g_0\left(z_1, z_2, z_3, z_4\right) &= A\ z_4 - \max_{x\in[0,1]} z_2 + \int_0^x z_1(t)dt, \\ g_1\left(z_1, z_2, z_3, z_4\right) &= B\ z_4 - \sum_{j=1}^{+\infty} a_j\ z_3\left(\xi_j\right), \\ g_2\left(z_1, z_2, z_3, z_4, z_5\right) &= C\ z_5^3 + \max_{x\in[0,1]} z_1(x-\tau), \\ g_3\left(z_1, z_2, z_3, z_4, z_5\right) &= D\ z_4 - z_2\left(\max\{0, x-\varepsilon\}\right). \end{aligned}$$

The functions $\alpha(x) = -(2-x)^2$ and $\beta(x) = (2-x)^2$ are, respectively, lower and upper solutions of the problem (7.4.6) for

$$\begin{aligned} k &\le -\frac{\pi}{4} - \frac{65}{2},\ \ A \le -\frac{19}{3}, \\ B &\le \overline{a},\ \ C \le -\frac{1}{8},\ \ D \le -2. \end{aligned}$$

So, by Theorem 7.3.1, there is a nontrivial solution, $u(x)$, of problem (7.4.6), *such that*

$$\begin{aligned}\alpha(x) &= -(2-x)^2 \le u(x) \le (2-x)^2 = \beta(x),\\ \beta'(x) &= 2x-4 \le u'(x) \le 4-2x = \alpha'(x)\end{aligned}$$

and

$$\alpha''(x) = -2 \le u''(x) \le 2 = \beta''(x),\ \forall x \in [0,1].$$

Chapter 8

Functional boundary value problems

8.1 Introduction

Until now we have dealt with problems with functional boundary conditions. Next chapters will consider functional boundary value problems, that is, problems where the functional dependence is allowed in the differential equation as well.

The first existence and location result will be discussed for the problem composed by the functional equation

$$u^{(iv)}(x) = f(x, u, u', u''(x), u'''(x)) \tag{8.1.1}$$

with $x \in [a, b]$, $f : [a, b] \times (C([a, b]))^2 \times \mathbb{R}^2 \to \mathbb{R}$ a L^1- Carathéodory function and the nonlinear functional boundary conditions

$$\begin{aligned} L_0(u, u', u'', u(a)) &= 0, \\ L_1(u, u', u'', u'(a)) &= 0, \\ L_2(u, u', u'', u''(a), u'''(a)) &= 0, \\ L_3(u, u', u'', u''(b), u'''(b)) &= 0, \end{aligned} \tag{8.1.2}$$

where L_i, $i = 0, 1, 2, 3$, are continuous functions to be defined later.

This type of functional boundary value problems has been studied in several works such as [15] for third order, [19] for fourth order and [16, 17] for higher order. As the functional dependence on the unknown function and its first derivative is allowed in the nonlinearity f these results not only improve the existing in the literature related to fourth order functional problems but they also generalize the results obtained in previous chapters. Moreover, the above problem cover several types of differential equations, such as, delay equations, integro-differential or equations with maxima or minima arguments, and many different boundary conditions, like Lidstone, separated, multipoint or non local conditions, among others.

The method used in this Chapter follows standard arguments in lower and upper solutions technique, combined with a stronger definition, which allows two features, not covered by the existing results:

- lower and upper functions can be considered with second order derivatives well ordered, or in reverse order, but eventually, with non-ordered first derivative and/or the unknown function (see Definition 8.2.1). If lower and upper solutions, and the corresponding derivatives, are "well ordered", the main theorem (Theorem 8.2.2), coincides with the classical theory.

  In the case of non-ordered lower and upper solutions the strips are defined by a pair of functions that are obtained by a perturbation of the initial lower and upper solutions. Therefore the set of admissible functions to be considered as lower and upper is hugely generalized.

- no monotone-type conditions are assumed on the nonlinearity f.

The second problem studied is composed by equation (8.1.1), with the nonlinear functional boundary conditions

$$\begin{aligned} L_0^*\left(u,u',u'',u''',u\left(a\right)\right) &= 0 \\ L_1^*\left(u,u',u'',u''',u'\left(a\right)\right) &= 0 \\ L_2^*\left(u,u',u'',u''',u''\left(a\right),u'''\left(a\right)\right) &= 0 \\ L_3^*\left(u,u',u'',u''',u''\left(b\right),u'''\left(b\right)\right) &= 0, \end{aligned} \tag{8.1.3}$$

where each L_i^*, $i = 0, .., 3$, allows a functional dependence on u''', too.

As far as we know, it is the first time where such dependence is considered together with a functional differential equation. In this way, problem (8.1.1),(8.1.3) is more ionteresting, not only from a theoretical point of view, but also because it can be applied to different real phenomena, such as periodic models that were not covered by the previous problems.

This problem is approached using similar techniques, as the boundary conditions are a generalization of (8.1.2), however the definition of the lower and upper solutions is considered as maximum/minimum in some adequate sets.

Section 8.5 contains, as example, a problem with Lidstone-type boundary conditions that could not be solved by the existing theory. In fact it includes an integro-differential equation and the existence and location results is obtained in presence of non-ordered lower and upper solutions and the corresponding first derivatives.

This last case is generalized for functional n^{th} order boundary value problem composed by the equation

$$u^{(n)}(x) = f(x, u, ..., u^{(n-3)}, u^{(n-2)}(x), u^{(n-1)}(x))$$

for $x \in [a, b]$, where $f : [a, b] \times (C([a, b]))^{(n-2)} \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function, and the boundary conditions

$$\begin{aligned} &\bar{L}_i^*(u, u', ..., u^{(n-1)}, u^{(i)}(a)) = 0, \quad i = 0, ..., n-2 \\ &\bar{L}_{n-1}^*(u, u', ..., u^{(n-1)}, u^{(n-2)}(b)) = 0, \end{aligned}$$

where $\bar{L}_i^*$, $i = 0, .., n-1$, are continuous functions verifying some conditions to be defined next.

Special emphasis should be put on the fact that the existence and location result obtained not only generalizes the previous problems studied on this chapter as it is also eliminates the monotone conditions assumed on the L_i, $i = 0, 1, 2, 3$, functions, given by (8.1.2), which were a usual condition for the known literature.

Last section includes an example where a periodic model is recreated due to the fact that boundary conditions (8.1.3) cover the periodic case, which was not the case of (8.1.2).

8.2 Fourth order functional problems

In this section it will be introduced the definitions and auxiliary results needed forward to construct some ordered functions on the basis of the not necessarily ordered lower and upper solutions of the problem (8.1.1)-(8.1.2).

Let it be considered a Nagumo-type growth condition, as defined in Definition 1.2.1 and Lemma 1.2.2, for $n = 4$ to set an *a priori* bound for the third derivative of the corresponding solutions.

The nonlinear part f will be a L^1-Carathéodory function and the functions L_i, $i = 0, 1, 2, 3$, in (8.1.2) verify the following monotonicity properties:

(P_0) $L_0,\ L_1 : (C([a,b]))^3 \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, nondecreasing in first, second and third variables;

(P_1) $L_2 : (C([a,b]))^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function, nondecreasing in first, second, third and fifth variables;

(P_2) $L_3 : (C([a,b]))^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function, nondecreasing in first, second and third variables and nonincreasing in the fifth one.

The main tool to obtain the location part is the lower and upper solutions method. However, in this case, they must be defined as a pair, which means that it is not possible to define them independently from each other. Moreover, it is pointed out that lower and upper functions, and the correspondent first derivatives, are not necessarily ordered.

To introduce "some order", it must be defined some auxiliary functions:

For any $\alpha,\ \beta \in W^{2,1}([a,b])$ define functions $\alpha_i,\ \beta_i : [a,b] \rightarrow \mathbb{R}$, $i = 0, 1$, as it follows:

$$\alpha_1(x) = \min\ \{\alpha'(a), \beta'(a)\} + \int_a^x \alpha''(s)\, ds, \tag{8.2.1}$$

$$\beta_1(x) = \max\ \{\alpha'(a), \beta'(a)\} + \int_a^x \beta''(s)\, ds, \tag{8.2.2}$$

$$\alpha_0(x) = \min\ \{\alpha(a), \beta(a)\} + \int_a^x \alpha_1(s)\, ds \tag{8.2.3}$$

$$\beta_0(x) = \max\ \{\alpha(a), \beta(a)\} + \int_a^x \beta_1(s)\, ds. \tag{8.2.4}$$

Definition 8.2.1 *The functions $\alpha, \beta \in W^{4,1}([a,b])$ are a pair of lower and upper solutions for problem (8.1.1)-(8.1.2) if $\alpha''(x) \leq \beta''(x)$, on $[a,b]$, and* *for all* $(v,w) \in A := [\alpha_0, \beta_0] \times [\alpha_1, \beta_1]$, *the following inequalities hold:*

$$\alpha^{(iv)}(x) \geq f(x, v, w, \alpha''(x), \alpha'''(x)), \text{ for a. e. } x \in [a,b], \tag{8.2.5}$$

$$\beta^{(iv)}(x) \leq f(x, v, w, \beta''(x), \beta'''(x)), \text{ for a. e. } x \in [a,b], \tag{8.2.6}$$

$$\begin{array}{rcccl} L_0(\alpha_0, \alpha_1, \alpha'', \alpha_0(a)) & \geq & 0 & \geq & L_0(\beta_0, \beta_1, \beta'', \beta_0(a)) \\ L_1(\alpha_0, \alpha_1, \alpha'', \alpha_1(a)) & \geq & 0 & \geq & L_1(\beta_0, \beta_1, \beta'', \beta_1(a)) \\ L_2(\alpha_0, \alpha_1, \alpha'', \alpha''(a), \alpha'''(a)) & \geq & 0 & \geq & L_2(\beta_0, \beta_1, \beta'', \beta''(a), \beta'''(a)) \\ L_3(\alpha_0, \alpha_1, \alpha'', \alpha''(b), \alpha'''(b)) & \geq & 0 & \geq & L_3(\beta_0, \beta_1, \beta'', \beta''(b), \beta'''(b)). \end{array} \tag{8.2.7}$$

Next existence and location theorem states sufficient conditions for, not only the existence of a solution u, but also to have information about the location of u, u', u'' and u'''.

Theorem 8.2.2 *Assume that there exists a pair* (α, β) *of lower and upper solutions of problem (8.1.1)-(8.1.2), respectively, such that conditions* (P_0), (P_1) *and* (P_2) *hold.*
If $f : [a,b] \times (C([a,b]))^2 \times \mathbb{R}^2 \to \mathbb{R}$ *is a* L^1-*Carathéodory function, satisfying a Nagumo-type condition in*

$$E_* = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3) \in [a,b] \times \mathbb{R}^4 : \alpha_0(x) \leq y_0 \leq \beta_0(x), \\ \alpha_1(x) \leq y_1 \leq \beta_1(x), \ \alpha''(x) \leq y_2 \leq \beta''(x) \end{array} \right\}, \tag{8.2.8}$$

then problem (8.1.1)-(8.1.2) has at least one solution u *such that*

$$\alpha_0(x) \leq u(x) \leq \beta_0(x), \quad \alpha_1(x) \leq u'(x) \leq \beta_1(x), \quad \alpha''(x) \leq u''(x) \leq \beta''(x),$$

for every $x \in [a,b]$, *and* $|u'''(x)| \leq K$, $\forall\, x \in [a,b]$, *where*

$$K = \max\{R, |\alpha'''(x)|, |\beta'''(x)|\} \tag{8.2.9}$$

and $R > 0$ *is given by Lemma 1.2.2 referred to the set* E_*.

Proof. Define the continuous functions

$$\begin{aligned} \delta_i(x, y_i) &= \max\{\alpha_i(x), \min\{y_i, \beta_i(x)\}\}, \text{ for } i = 0, 1, \quad (8.2.10) \\ \delta_2(x, y_2) &= \max\{\alpha''(x), \min\{y_2, \beta''(x)\}\} \end{aligned}$$

and

$$q(z) = \max\{-K, \min\{z, K\}\} \text{ for a.e. } z \in \mathbb{R}. \quad (8.2.11)$$

Consider the modified problem composed by the equation

$$u^{(iv)}(x) = f\left(x, \delta_0(\cdot, u(\cdot)), \delta_1(\cdot, u'(\cdot)), \delta_2(x, u''(x)), q\left(\frac{d}{dx}(\delta_2(x, u''))\right)\right) \quad (8.2.12)$$

and the boundary conditions

$$\begin{aligned} u(a) &= \delta_0(a, u(a) + L_0(u, u', u'', u(a))), \\ u'(a) &= \delta_1(a, u'(a) + L_1(u, u', u'', u'(a))), \\ u''(a) &= \delta_2(a, u''(a) + L_2(u, u', u'', u''(a), u'''(a))), \\ u''(b) &= \delta_2(b, u''(b) + L_3(u, u', u'', u''(b), u'''(b))). \end{aligned} \quad (8.2.13)$$

The proof applies typical steps of truncated problems by lower and upper solutions, as shown in previous chapters:

Step 1 - Every solution u of problem (8.2.12)-(8.2.13), satisfies

$$\alpha''(x) \le u''(x) \le \beta''(x), \ \alpha_1(x) \le u'(x) \le \beta_1(x), \ \alpha_0(x) \le u(x) \le \beta_0(x)$$

and $|u'''(x)| < K$, for every $x \in [a, b]$, with $K > 0$ given in (8.2.9).

Let u be a solution of the modified problem (8.2.12) – (8.2.13). Assume, by contradiction, that there exists $x \in [a, b]$ such that $\alpha''(x) > u''(x)$ and let $x_0 \in [a, b]$ be such that

$$\min_{x \in [a,b]} (u - \alpha)''(x) = (u - \alpha)''(x_0) < 0.$$

As, by (8.2.13), $u''(a) \geq \alpha''(a)$ and $u''(b) \geq \alpha''(b)$, then $x_0 \in (a,b)$. So, there is $(x_1, x_2) \subset (a,b)$ such that

$$u''(x) < \alpha''(x), \forall x \in (x_1, x_2), \quad (u-\alpha)''(x_1) = (u-\alpha)''(x_2) = 0. \quad (8.2.14)$$

Therefore, for all $x \in (x_1, x_2)$ it is satisfied that $\delta_2(x, u'') = \alpha''(x)$ and $\frac{d}{dx}\delta_2(x, u'') = \alpha'''(x)$. Now, since for all $u \in C^1([a,b])$ it is satisfied that $(\delta_0(\cdot, u), \delta_1(\cdot, u')) \in A$, we deduce that

$$\begin{aligned} u^{(iv)}(x) &= f\left(x, \delta_0(\cdot, u(\cdot)), \delta_1(\cdot, u'(\cdot)), \delta_2(x, u''), q\left(\frac{d}{dx}(\delta_2(x, u''))\right)\right) \\ &= f(x, \delta_0(\cdot, u(\cdot)), \delta_1(\cdot, u'(\cdot)), \alpha''(x), \alpha'''(x)) \\ &\leq \alpha^{(iv)}(x) \qquad \text{for a. e. } x \in (x_1, x_2). \end{aligned}$$

In consequence we deduce that function $(u-\alpha)'''$ is monotone nonincreasing on the interval (x_1, x_2). From the fact that $(u-\alpha)'''(x_0) = 0$, we know that $(u-\alpha)''$ is monotone nonincreasing too on (x_0, x_2), which contradicts the definitions of x_0 and x_2.

The inequality $u''(x) \leq \beta''(x)$, in $[a,b]$, can be proved in same way and, so,

$$\alpha''(x) \leq u''(x) \leq \beta''(x), \ \forall\, x \in [a,b]. \quad (8.2.15)$$

By (8.2.13) and (8.2.1), the following inequalities hold for every $x \in [a,b]$,

$$\begin{aligned} u'(x) &= u'(a) + \int_a^x u''(s)\, ds \\ &\geq \alpha_1(a) + \int_a^x \alpha''(s)\, ds = \min\ \{\alpha'(a), \beta'(a)\} + \int_a^x \alpha''(s)\, ds \\ &= \alpha_1(x). \end{aligned}$$

Analogously, it can be obtained $u'(x) \leq \beta_1(x)$, for $x \in [a,b]$.

On the other hand, by using (8.2.13), (8.2.3) and (8.2.4), the following inequalities are fulfilled:

$$\begin{aligned} u(x) &= u(a) + \int_a^x u'(s)\,ds \\ &\geq \alpha_0(a) + \int_a^x \alpha_1(s)\,ds = \min\ \{\alpha(a), \beta(a)\} + \int_a^x \alpha_1(s)\,ds \\ &= \alpha_0(x). \end{aligned}$$

The inequality $u(x) \leq \beta_0(x)$ for every $x \in [a,b]$ is deduced in the same way. The inequality for the third derivative is obtained from Lemma 1.2.2.

Step 2 - Problem (8.2.12)-(8.2.13) has at least one solution.

For $\lambda \in [0,1]$ let us consider the homotopic problem given by

$$u^{(iv)}(x) = \lambda f\left(x, \delta_0(\cdot, u(\cdot)), \delta_1(\cdot, u'(\cdot)), \delta_2(x, u''(x)), q\left(\frac{d}{dx}(\delta_2(x, u''(x)))\right)\right) \tag{8.2.16}$$

and the boundary conditions

$$\begin{aligned} u(a) &= \lambda\delta_0(a, u(a) + L_0(u, u', u'', u(a))) &&\equiv \lambda L_A, \\ u'(a) &= \lambda\delta_1(a, u'(a) + L_1(u, u', u'', u'(a))) &&\equiv \lambda L_B, \\ u''(a) &= \lambda\delta_2(a, u''(a) + L_2(u, u', u'', u''(a), u'''(a))) &&\equiv \lambda L_C, \\ u''(b) &= \lambda\delta_2(b, u''(b) + L_3(u, u', u'', u''(b), u'''(b))) &&\equiv \lambda L_D. \end{aligned} \tag{8.2.17}$$

Let us consider the norms in $C^3([a,b])$ and in $L^1([a,b]) \times \mathbb{R}^4$, respectively,

$$\|v\|_{C^3} := \max\left\{\|v\|_\infty, \|v'\|_\infty, \|v''\|_\infty, \|v'''\|_\infty\right\}$$

and

$$|(h, h_1, h_2, h_3, h_4)| := \max\left\{\|h\|_{L^1}, \max\left\{|h_1|, |h_2|, |h_3|, |h_4|\right\}\right\}.$$

Define the operators $\mathcal{L} : W^{4,1}([a,b]) \subset C^3([a,b]) \to L^1([a,b]) \times \mathbb{R}^4$ by $\mathcal{L}u = \left(u^{(iv)}, u(a), u'(a), u''(a), u''(b)\right)$ and, for $\lambda \in [0,1]$, $\mathcal{N}_\lambda : C^3([a,b]) \to L^1([a,b]) \times \mathbb{R}^4$ by

$$\mathcal{N}_\lambda u = \left(\begin{array}{c} \lambda f\left(x, \delta_0(\cdot, u(\cdot)), \delta_1(\cdot, u'(\cdot)), \delta_2(x, u''(x)), q\left(\frac{d}{dx}(\delta_2(x, u''(x)))\right)\right), \\ L_A, L_B, L_C, L_D \end{array}\right)$$

Since L_0, L_1, L_2 and L_3 are continuous and f is a L^1- Carathéodory function, then, from Lemma 4.4.1, $\mathcal{N}_\lambda$ is continuous. Moreover, as $\mathcal{L}^{-1}$ is compact, it can be defined the completely continuous operator $\mathcal{T}_\lambda : C^3([a,b]) \to C^3([a,b])$ by $\mathcal{T}_\lambda u = \mathcal{L}^{-1}\mathcal{N}_\lambda(u)$.

As $\mathcal{N}_\lambda u$ is bounded in $L^1([a,b]) \times \mathbb{R}^4$ and uniformly bounded in $C^3([a,b])$, we have that any solution of the problem *(8.2.16)-(8.2.17)*, verifies $\|u\|_{C^3} \leq \|\mathcal{L}^{-1}\| \, |\mathcal{N}_\lambda(u)| \leq \bar{K}$, for some $\bar{K} > 0$ independent of λ.

In the set $\Omega = \left\{u \in C^3([a,b]) : \|u\|_{C^3} < \bar{K} + 1\right\}$ the degree $d(\mathcal{I} - \mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0,1]$ and, by the invariance under homotopy,

$$d(\mathcal{I} - \mathcal{T}_1, \Omega, 0) = d(\mathcal{I} - \mathcal{T}_0, \Omega, 0) = \pm 1.$$

So by degree theory, the equation $x = \mathcal{T}_1(x)$ has at least one solution, that is, the problem *(8.2.12)-(8.2.13)* has at least one solution in Ω.

<u>Step 3</u> - *Every solution u of problem (8.2.12)-(8.2.13) is a solution of (8.1.1)-(8.1.2).*

Let u be a solution of the modified problem (8.2.12)-(8.2.13). By previous steps, function u fulfills equation (8.1.1). So, it will be enough to prove the inequalities:

$$\begin{array}{rcl}
\alpha_0(a) \leq & u(a) + L_0(u, u', u'', u(a)) & \leq \beta_0(a), \\
\alpha_1(a) \leq & u'(a) + L_1(u, u', u'', u'(a)) & \leq \beta_1(a), \\
\alpha''(a) \leq & u''(a) + L_2(u, u', u'', u''(a), u'''(a)) & \leq \beta''(a), \\
\alpha''(b) \leq & u''(b) + L_3(u, u', u'', u''(b), u'''(b)) & \leq \beta''(b),
\end{array}$$

applying the arguments suggested in, for instance, Step 3 of Theorem 7.3.1, with $n = 4$. ■

As a corollary the following result for multipoint boundary value problems holds:

Corollary 8.2.3 *Assume that there exist $\alpha,\ \beta \in W^{4,1}([a,b])$ satisfying the following inequalities:*

$$\alpha''(x) \leq \beta''(x), \quad \text{for all } x \in [a,b],$$
$$\alpha^{(iv)}(x) - f\left(x, v, w, \alpha''(x), \alpha'''(x)\right) \geq 0 \geq \beta^{(iv)}(x) - f\left(x, v, w, \beta''(x), \beta'''(x)\right)$$

for a. e. $x \in [a,b]$ and all $(v,w) \in A$.

If f is a L^1-Carathéodory function, satisfying a Nagumo-type condition in E_*, then problem

$$\begin{aligned}
u^{(iv)}(x) &= f\left(x, u, u', u''(x), u'''(x)\right) \quad \text{for a. e. } t \in [a,b], \\
u(a) &= \sum_{i=1}^{m_1^0} a_i^0\, u(\xi_i^0) + \sum_{i=1}^{m_2^0} b_i^0\, u'(\rho_i^0) + \sum_{i=1}^{m_3^0} c_i^0\, u''(\zeta_i^0), \\
u'(a) &= \sum_{i=1}^{m_1^1} a_i^1\, u(\xi_i^1) + \sum_{i=1}^{m_2^1} b_i^1\, u'(\rho_i^1) + \sum_{i=1}^{m_3^1} c_i^1\, u''(\zeta_i^1), \\
u''(a) &= \sum_{i=1}^{m_1^2} a_i^2\, u(\xi_i^2) + \sum_{i=1}^{m_2^2} b_i^2\, u'(\rho_i^2) + \sum_{i=1}^{m_3^2} c_i^2\, u''(\zeta_i^2) + c\, u'''(a), \\
u''(b) &= \sum_{i=1}^{m_1^3} a_i^3\, u(\xi_i^3) + \sum_{i=1}^{m_2^3} b_i^3\, u'(\rho_i^3) + \sum_{i=1}^{m_3^3} c_i^3\, u''(\zeta_i^3) - d\, u'''(b),
\end{aligned}$$

with $m_k^j \in \mathbb{N}$ for $k = 1,2,3$ and $j = 0,1,2,3$, $a \leq \xi_1^j < \xi_2^j < ... < \xi_{m_k^j}^j \leq b$, $a \leq \rho_1^j < \rho_2^j < ... < \rho_{m_k^j}^j \leq b$, $a \leq \zeta_1^j < \zeta_2^j < ... < \zeta_{m_k^j}^j \leq b$, and c, d, a_i^j, b_i^j and c_i^j non-negative constants, has at least one solution u such that $\alpha_0(x) \leq u(x) \leq \beta_0(x)$, $\alpha_1(x) \leq u'(x) \leq \beta_1(x)$, $\alpha''(x) \leq u''(x) \leq \beta''(x)$, for every $x \in [a,b]$.

Proof. The proof is a direct consequence of Theorem 8.2.2. In this case it is enough to define the following functions:

$$
\begin{aligned}
L_0(u,v,w,z) &= -z+\sum_{i=1}^{m_1^0} a_i^0\, u(\xi_i^0)+\sum_{i=1}^{m_2^0} b_i^0\, v(\rho_i^0)+\sum_{i=1}^{m_3^0} c_i^0\, w(\zeta_i^0),\\
L_1(u,v,w,z) &= -z+\sum_{i=1}^{m_1^1} a_i^1\, u(\xi_i^1)+\sum_{i=1}^{m_2^1} b_i^1\, v(\rho_i^1)+\sum_{i=1}^{m_3^1} c_i^1\, w(\zeta_i^1),\\
L_2(u,v,w,z,p) &= -z+\sum_{i=1}^{m_1^2} a_i^2\, u(\xi_i^2)+\sum_{i=1}^{m_2^2} b_i^2\, v(\rho_i^2)+\sum_{i=1}^{m_3^2} c_i^2\, w(\zeta_i^2)+c\,p,\\
L_3(u,v,w,z,p) &= -z+\sum_{i=1}^{m_1^3} a_i^3\, u(\xi_i^3)+\sum_{i=1}^{m_2^3} b_i^3\, v(\rho_i^3)+\sum_{i=1}^{m_3^3} c_i^3\, w(\zeta_i^3)-d\,p.
\end{aligned}
$$

■

8.3 Functional problems including the periodic case

It is easy to check that functions (8.1.2) do not cover the periodic boundary conditions. To overcome this "gap" it is considered in this section the functions (8.1.3), where L_i^*, $i=0,..,3$, verify the following monotonicity properties:

(P_0^*) L_0^*, $L_1^*: (C([a,b]))^3\times\mathbb{R}\to\mathbb{R}$ are continuous functions, nondecreasing in first, second and third variables;

(P_1^*) $L_2^*: (C([a,b]))^3\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function, nondecreasing in first, second, third and sixth variables;

(P_2^*) $L_3^*: (C([a,b]))^3\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function, nondecreasing in first, second and third variables and nonincreasing in the sixth one.

In this case the definition of lower and upper solution is as it follows:

Definition 8.3.1 *The functions $\alpha, \beta \in W^{4,1}([a,b])$ are a pair of lower and upper solutions for problem (8.1.1),(8.1.3), respectively, if $\alpha''(x) \leq \beta''(x)$, on $[a,b]$, and for all $(v,w) \in A := [\alpha_0, \beta_0] \times [\alpha_1, \beta_1]$, where α_i, β_i, $i = 0, 1$, are given by (8.2.1)-(8.2.4), the following inequalities hold:*

$$\alpha^{(iv)}(x) \geq f(x, v, w, \alpha''(x), \alpha'''(x)), \text{ for a. e. } x \in [a,b], \tag{8.3.1}$$

$$\beta^{(iv)}(x) \leq f(x, v, w, \beta''(x), \beta'''(x)), \text{ for a. e. } x \in [a,b], \tag{8.3.2}$$

$$\begin{gathered}
\min_{\|z\| \leq K} L_0^*(\alpha_0, \alpha_1, \alpha'', z, \alpha_0(a)) \geq 0 \geq \max_{\|z\| \leq K} L_0^*(\beta_0, \beta_1, \beta'', z, \beta_0(a)) \\
\min_{\|z\| \leq K} L_1^*(\alpha_0, \alpha_1, \alpha'', z, \alpha_1(a)) \geq 0 \geq \max_{\|z\| \leq K} L_1^*(\beta_0, \beta_1, \beta'', z, \beta_1(a)) \\
\min_{\|z\| \leq K} L_2^*(\alpha_0, \alpha_1, \alpha'', z, \alpha''(a), \alpha'''(a)) \geq 0 \\
\geq \max_{\|z\| \leq K} L_2^*(\beta_0, \beta_1, \beta'', z, \beta''(a), \beta'''(a)) \\
\min_{\|z\| \leq K} L_3^*(\alpha_0, \alpha_1, \alpha'', z, \alpha''(b), \alpha'''(b)) \geq 0 \\
\geq \max_{\|z\| \leq K} L_3^*(\beta_0, \beta_1, \beta'', z, \beta''(b), \beta'''(b)).
\end{gathered} \tag{8.3.3}$$

with K given by (8.2.9).

The existence and location result is as follows.

Theorem 8.3.2 *Assume that there exists a pair (α, β) of lower and upper solutions of problem (8.1.1),(8.1.3), such that conditions (P_0^*), (P_1^*) and (P_2^*) hold.*
If $f : [a,b] \times (C([a,b]))^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function, satisfying a Nagumo-type condition in E_* defined in (8.2.8), then problem (8.1.1),(8.1.3) has at least one solution u such that

$$\alpha_0(x) \leq u(x) \leq \beta_0(x), \quad \alpha_1(x) \leq u'(x) \leq \beta_1(x), \quad \alpha''(x) \leq u''(x) \leq \beta''(x),$$

for every $x \in [a,b]$, and $|u'''(x)| \leq K$, $\forall\, x \in [a,b]$, with K as in (8.2.9).

Proof. With the truncations defined in (8.2.10) and (8.2.8) consider the modified problem composed by (8.2.12) and the boundary conditions

$$\begin{aligned} u(a) &= \delta_0\left(a, u(a) + L_0^*\left(u, u', u'', u''', u(a)\right)\right), \\ u'(a) &= \delta_1\left(a, u'(a) + L_1^*\left(u, u', u'', u''', u'(a)\right)\right), \\ u''(a) &= \delta_2\left(a, u''(a) + L_2^*\left(u, u', u'', u''', u''(a), u'''(a)\right)\right), \\ u''(b) &= \delta_2\left(b, u''(b) + L_3^*\left(u, u', u'', u''', u''(b), u'''(b)\right)\right). \end{aligned} \tag{8.3.4}$$

The proof will follow the same process as in Theorem 8.2.2, in the first two Steps and this part is omitted.

As to the boundary conditions it will be enough to prove that:

$$\begin{array}{lcl} \alpha_0(a) \leq & u(a) + L_0^*\left(u, u', u'', u''', u(a)\right) & \leq \beta_0(a), \\ \alpha_1(a) \leq & u'(a) + L_1^*\left(u, u', u'', u''', u'(a)\right) & \leq \beta_1(a), \\ \alpha''(a) \leq & u''(a) + L_2^*\left(u, u', u'', u''', u''(a), u'''(a)\right) & \leq \beta''(a), \\ \alpha''(b) \leq & u''(b) + L_3^*\left(u, u', u'', u''', u''(b), u'''(b)\right) & \leq \beta''(b). \end{array}$$

Assume

$$u(a) + L_0^*\left(u, u', u'', u(a)\right) > \beta_0(a). \tag{8.3.5}$$

Then, by (8.3.4), $u(a) = \beta_0(a)$ and, by (P_0^*) and previous steps, it is obtained the following contradiction with (8.3.5):

$$\begin{aligned} u(a) + L_0\left(u, u', u'', u''', u(a)\right) &\leq \beta_0(a) + L_0^*\left(\beta_0, \beta_1, \beta'', u''', \beta_0(a)\right) \\ &\leq \beta_0(a) + \max_{\|z\|_\infty < K} L_0^*\left(\beta_0, \beta_1, \beta'', z, \beta_0(a)\right) \\ &\leq \beta_0(a). \end{aligned}$$

Applying similar arguments it can be proved that

$$\alpha_0(a) \leq u(a) + L_0^*\left(u, u', u'', u''', u(a)\right)$$

and therefore

$$\alpha_1(a) \leq u'(a) + L_1^*\left(u, u', u'', u''', u'(a)\right) \leq \beta_1(a).$$

For the third case assume, again by contradiction, that

$$u''(a) + L_2^*(u, u', u'', u''(a), u'''(a)) > \beta''(a). \tag{8.3.6}$$

By (8.3.4), $u''(a) = \beta''(a)$ and, as $u''(x) \leq \beta''(x)$ in $[a,b]$, then $u'''(a) \leq \beta'''(a)$ and, by (P_1^*) and (8.3.1), it is achieved this contradiction with (8.3.6):

$$\begin{aligned}
& u''(a) + L_2(u, u', u'', u''', u''(a), u'''(a)) \\
\leq\; & \beta''(a) + L_2^*(\beta_0, \beta_1, \beta'', u''', \beta''(a), \beta'''(a)) \\
\leq\; & \beta''(a) + \max_{\|z\|_\infty < K} L_2^*(\beta_0, \beta_1, \beta'', z, \beta''(a), \beta'''(a)) \\
\leq\; & \beta''(a).
\end{aligned}$$

The same technique yields the two last inequalities. ■

To generalize the previous technique to any order $n \geq 2$, it is considered the equation

$$u^{(n)}(x) = f(x, u, ..., u^{(n-3)}, u^{(n-2)}(x), u^{(n-1)}(x)) \tag{8.3.7}$$

for a.e. $x \in [a,b]$, where $f : [a,b] \times (C([a,b]))^{(n-2)} \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function, and the boundary conditions

$$\begin{aligned}
&\bar{L}_i^*(u, u', ..., u^{(n-1)}, u^{(i)}(a)) = 0, \quad i = 0, ..., n-2 \\
&\bar{L}_{n-1}^*(u, u', ..., u^{(n-1)}, u^{(n-2)}(b)) = 0,
\end{aligned} \tag{8.3.8}$$

where $\bar{L}_i^*$, $i = 0, .., n-1$, are continuous functions to be precise.

As before, it is pointed out that lower and upper functions, and the correspondent first derivatives, do not need to be ordered.

The "order required" is given by the auxiliary functions:

For any $\alpha,\ \beta \in W^{n-2,1}([a,b])$ let $\alpha_i,\ \beta_i : [a,b] \to \mathbb{R},\ i = 0,...,n-3,$ defined as it follows:

$$\begin{array}{r}
\alpha_{n-3}(x) = \min\ \left\{\alpha^{(n-3)}(a), \beta^{(n-3)}(a)\right\} + \int_a^x \alpha^{(n-2)}(s)\, ds, \\
\beta_{n-3}(x) = \max\ \left\{\alpha^{(n-3)}(a), \beta^{(n-3)}(a)\right\} + \int_a^x \beta^{(n-2)}(s)\, ds, \\
\alpha_i(x) = \min\ \left\{\alpha^{(i)}(a), \beta^{(i)}(a)\right\} + \int_a^x \alpha_{i+1}(s)\, ds, \\
\beta_i(x) = \max\ \left\{\alpha^{(i)}(a), \beta^{(i)}(a)\right\} + \int_a^x \beta_{i+1}(s)\, ds,
\end{array}, \tag{8.3.9}$$

for $i = 0,...,n-4$.

Definition 8.3.3 *The functions $\alpha, \beta \in W^{n-2,1}([a,b])$ are a pair of lower and upper solutions for problem (8.3.7)-(8.3.8) if $\alpha^{(n-2)}(x) \leq \beta^{(n-2)}(x)$, on $[a,b]$, and for all $(v_0,...,v_{n-3}) \in A_* := [\alpha_0, \beta_0] \times ... \times [\alpha_{n-3}, \beta_{n-3}]$, and all $(w_1, w_2) \in B := \left[\alpha^{(n-2)}, \beta^{(n-2)}\right] \times [-K, K]$, with K given by (8.2.9), the following inequalities hold,for a. e. $x \in [a,b]$,*

$$\alpha^{(n)}(x) \geq f\left(x, v_0, ..., v_{n-3}, \alpha^{(n-2)}(x), \alpha^{(n-1)}(x)\right), \tag{8.3.10}$$

$$\beta^{(n)}(x) \leq f\left(x, v_0, ..., v_{n-3}, \beta^{(n-2)}(x), \beta^{(n-1)}(x)\right), \tag{8.3.11}$$

and

$$\begin{array}{r}
\bar{L}_i^*(v_0, ..., v_{n-3}, w_1, w_2, \alpha_i(a)) \geq 0 \\
\bar{L}_i^*(v_0, ..., v_{n-3}, w_1, w_2, \beta_i(a)) \leq 0,\ \text{for } i = 0, ..., n-2, \\
\bar{L}_{n-1}^*\left(v_0, ..., v_{n-3}, w_1, w_2, \alpha^{(n-2)}(b)\right) \geq 0 \\
\bar{L}_{n-1}^*\left(v_0, ..., v_{n-3}, w_1, w_2, \beta^{(n-2)}(b)\right) \leq 0
\end{array} \tag{8.3.12}$$

The main result is given by next theorem:

Theorem 8.3.4 *Assume that there exists a pair of lower and upper solutions, (α, β) of problem (8.3.7)-(8.3.8).*

If $f : [a,b] \times (C([a,b]))^{(n-2)} \times \mathbb{R}^2 \to \mathbb{R}$ *is a* L^1-*Carathéodory function, satisfying a Nagumo-type condition in*

$$E^* = \left\{ \begin{array}{c} (x, y_0, ..., y_{n-1}) \in [a,b] \times \ \mathbb{R}^{n-1} : \alpha_i(x) \le y_i \le \beta_i(x), \ i = 0, ..., n-3 \\ \alpha^{(n-2)}(x) \le y_{n-2} \le \beta^{(n-2)}(x) \end{array} \right\},$$

then problem (8.3.7)-(8.3.8) has at least a solution u *such that*

$$\begin{aligned} \alpha_i(x) &\le u^{(i)}(x) \le \beta_i(x), \ i = 0, ..., n-3, \\ \alpha^{(n-2)}(x) &\le u^{(n-2)}(x) \le \beta^{(n-2)}(x), \end{aligned}$$

and $\left|u^{(n-1)}(x)\right| \le K$, $\forall x \in [a,b]$, *where* K *is defined by (8.2.9).*

The proof is similar to Theorem 8.3.2 with the obvious modifications for order n.

8.4 Examples

Example 8.4.1 *This example shows a problem composed by an integro differential equation with separated boundary conditions, which solvability is proved in presence of non-ordered lower and upper solutions, which was not possible in the current literature. This example does not model any particular problem arising in real phenomena, but the aim is to emphasize the strength of the developed theory in this chapter, by showing what kind of problems it can deal with.*

Consider, for $x \in [0,1]$, *the fourth order equation*

$$u^{(iv)}(x) = \int_0^x u(s)\,ds + \max_{x\in[0,1]} \{u'(x)\} + 2u''(x) - (u'''(x)+1)^{\frac{2}{3}} \qquad (8.4.1)$$

with the boundary conditions

$$u(0) = u''(0) = u''(1) = 0, \quad u(1) = \delta\, u'(1), \qquad (8.4.2)$$

where $\delta \geq 9/2$. *The functions*

$$\alpha(x) = -x^2 + \frac{x}{2} + \frac{3}{4} \text{ and } \beta(x) = x^2 - \frac{x}{2} - \frac{3}{4}$$

are, respectively, lower and upper solutions for the problem (8.4.1)–(8.4.2), according Definition 8.2.1, with

$$\begin{aligned} \alpha_1(x) &= -2x - \frac{1}{2}, \quad \alpha_0(x) = -x^2 - \frac{x}{2} - \frac{3}{4}, \\ \beta_1(x) &= 2x + \frac{1}{2}, \quad \beta_0(x) = x^2 + \frac{x}{2} + \frac{3}{4}. \end{aligned}$$

It can easily be checked that problem (8.4.1)–(8.4.2) is a particular case of (8.1.1)-(8.1.2) for

$$f(x, y_0, y_1, y_2, y_3) = \int_0^x y_0(s)ds + \max_{x\in[0,1]} \{y_1(x)\} + 2\, y_2(x) - (y_3(x) + 1)^{\frac{2}{3}},$$

$$\begin{aligned} L_0(z_1, z_2, z_3, z_4) &= -z_4, \; L_1(z_1, z_2, z_3, z_4) = z_1(1) - \delta\, z_4, \\ L_2(z_1, z_2, z_3, z_4, z_5) &= -z_4, \; L_3(z_1, z_2, z_3, z_4, z_5) = -z_4. \end{aligned}$$

As the continuous function f *verifies Nagumo conditions in*

$$E_* = \left\{ (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \begin{array}{c} -x^2 - \frac{x}{2} - \frac{3}{4} \leq y_0 \leq x^2 + \frac{x}{2} + \frac{3}{4} \\ -2x - \frac{1}{2} \leq y_1 \leq 2x + \frac{1}{2} \\ -2 \leq y_2 \leq 2 \end{array} \right\},$$

for $h_{E_*}(y_3) = \frac{47}{6} + (y_3 + 1)^{\frac{2}{3}}$, *then, by Theorem 8.2.2, there is a nontrivial solution* u *for problem (8.4.1)–(8.4.2) such that*

$$\begin{aligned} -x^2 - \frac{x}{2} - \frac{3}{4} &\leq u(x) \leq x^2 + \frac{x}{2} + \frac{3}{4}, \\ -2x - \frac{1}{2} &\leq u'(x) \leq 2x + \frac{1}{2}, \\ -2 &\leq u''(x) \leq 2, \end{aligned}$$

for all $x \in [0,1]$.

Example 8.4.2 *This example emphsizes the difference between Definition 8.2.1 and Definition 8.3.3. In fact a periodic problem is considered for an equation, where f has a functional dependence on the second derivative, which was not covered by Definition 8.2.1 and Theorem 8.2.2. To study the existence of periodic solutions for these functional fourth order fully differential equations we have to consider lower and upper solutions defined as in Definition 8.3.3.*

Consider

$$u^{(iv)}(x) = \int_0^x u(s)\,ds + \max_{x\in[0,1]}\{u'(x)\} + \min_{x\in[0,1]} u''(x) - |u'''(x)+1|^{\theta}, \quad (8.4.3)$$

for $\theta \in [0,2]$ with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(1), \; i = 0,1,2,3. \quad (8.4.4)$$

The functions

$$\alpha(x) = -\frac{x^3}{6} - 12x^2 + 20x - 1 \text{ and } \beta(x) = \frac{x^3}{3} + 12x^2 + 1$$

are a pair of lower and upper solutions, respectively, for the problem (8.4.3)–(8.4.4), according Definition 8.3.3, where

$$\begin{array}{rclrcl} \alpha_1(x) & = & -\frac{x^2}{2} - 24x, & \alpha_0(x) & = & -\frac{x^3}{6} - 12x^2 - 1, \\ \beta_1(x) & = & x^2 + 24x + 20, & \beta_0(x) & = & \frac{x^3}{3} + 12x^2 + 20x + 1. \end{array}$$

The above problem is a particular case of problem (8.1.1),(8.1.3) defining

$$f(x, y_0, y_1, y_2, y_3) = \int_0^x y_0(s)ds + \max_{x\in[0,1]}\{y_1(x)\} + \min_{x\in[0,1]} y_2(x) - |y_3(x)+1|^{\theta},$$

and

$$\begin{array}{rcl} \bar{L}_0^*(z_1, z_2, z_3, z_4, z_5) & = & z_1 - z_5, \\ \bar{L}_1^*(z_1, z_2, z_3, z_4, z_5) & = & z_2 - z_5, \\ \bar{L}_2^*(z_1, z_2, z_3, z_4, z_5, z_6) & = & z_3 - z_5, \\ \bar{L}_3^*(z_1, z_2, z_3, z_4, z_5, z_6) & = & z_4 - z_6. \end{array}$$

As the continuous function f verifies Nagumo conditions in

$$E_* = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \\ -\frac{x^3}{6} - 12x^2 - 1 \leq y_0 \leq \frac{x^3}{3} + 12x^2 + 20x + 1 \\ -\frac{x}{2}^2 - 24x \leq y_1 \leq x^2 + 24x + 20 \\ -x - 24 \leq y_2 \leq 2x + 24 \end{array} \right\},$$

for $h_{E_*}(y_3) = \frac{1847}{12} + |y_3 + 1|^\theta$, *then, by Theorem 8.3.4, there is a nontrivial periodic solution u for problem (8.4.3) – (8.4.4) such that*

$$\begin{aligned} -\frac{x^3}{6} - 12x^2 - 1 &\leq u(x) \leq \frac{x^3}{3} + 12x^2 + 20x + 1, \\ -\frac{x}{2}^2 - 24x &\leq u'(x) \leq x^2 + 24x + 20, \\ -x - 24 &\leq u''(x) \leq 2x + 24, \end{aligned}$$

for all $x \in [0,1]$.

Remark that despite α and β are not ordered, the auxiliary functions α_0, β_0 are well ordered. (see Figure 8.4.1)

8.5 The Lidstone case

In this section it is presented a technique to functional boundary value problems, which allows more generalized results in Lidstone problems, overcoming condition (2.3.1).

Consider now the problem given by the equation

$$u^{(iv)}(x) + f(x, u, u', u''(x), u'''(x)) = sp(x) \tag{8.5.1}$$

along with the boundary conditions (2.1.2). Next Definition and Theorem allow the introduction of some functional depends in the equation.

Let it now be considered the following definition for lower and upper solutions:

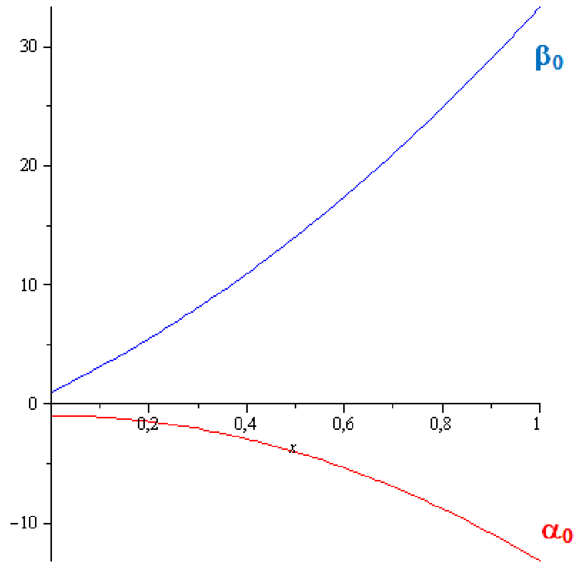

Figure 8.4.1: The auxiliary functions α_0, β_0 are well ordered.

Definition 8.5.1 *Functions* $\alpha, \beta \in C^4\left(]0,1[\right) \cap C^2\left([0,1]\right)$ *are a pair of lower and upper solutions of the problem (8.5.1),(2.1.2), respectively, if the following conditions are satisfied:*

(i) $\alpha^{(iv)}(x) + f(x, v, w, \alpha''(x), \alpha'''(x)) \geq sp(x)$, *for every* $(v, w) \in [\alpha_0, \beta_0] \times [\alpha_1, \beta_1]$ *and every* $x \in [0,1]$

where

$$\alpha_0(x) = \int_0^x \alpha_1(s)\, ds$$
$$,$$
$$\alpha_1(x) = \alpha'(x) - \alpha'(0) - \int_0^1 \int_0^x |\beta''(s)|\, ds dx$$

(ii) $\alpha''(0) \leq 0, \quad \alpha''(1) \leq 0,$

(iii) $\beta^{(iv)}(x) + f(x, v, w, \beta''(x), \beta'''(x)) \leq sp(x)$, *for every* $(v, w) \in [\alpha_0, \beta_0] \times [\alpha_1, \beta_1]$ *and every* $x \in [0,1]$

where

$$\begin{array}{c}\beta_0(x) = \int_0^x \beta_1(s)\, ds \\ \beta_1(x) = \beta'(x) - \beta'(0) + \int_0^1 \int_0^x |\alpha''(s)|\, ds dx\end{array},$$

(iv) $\beta''(0) \geq 0, \ \ \beta''(1) \geq 0.$

The existence and location result does not include the monotonicity type conditions (2.3.1) or (2.4.4) on f.

Theorem 8.5.2 *Suppose that there is a pair of lower and upper solutions of the problem (8.5.1),(2.1.2), $\alpha(x)$ and $\beta(x)$, respectively verifying*

$$\alpha''(x) \leq \beta''(x), \ \forall x \in [0,1].$$

Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying the one-sided Nagumo conditions (2.2.1), or (2.2.2), and (2.2.3) in

$$E_* = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \alpha_0(x) \leq y_0 \leq \beta_0(x), \\ \alpha_1(x) \leq y_1 \leq \beta_1(x), \ \alpha''(x) \leq y_2 \leq \beta''(x) \end{array} \right\}.$$

Then the problem (8.5.1),(2.1.2) has at least a solution $u(x) \in C^4([0,1])$, satisfying

$$\alpha_i(x) \leq u^{(i)}(x) \leq \beta_i(x), \ \textit{for } i = 0, 1, \ \forall x \in [0,1],$$

and

$$\alpha''(x) \leq u''(x) \leq \beta''(x), \ \forall x \in [0,1].$$

Proof. The proof is similar to Theorem 2.4.2. The only change occurs in the following consideration:

Suppose, by contradiction, that there is $x \in [0,1]$ such that $\alpha''(x) > u_1''(x)$ and define

$$\min_{x \in [0,1]} [u_1''(x) - \alpha''(x)] := u_1''(x_1) - \alpha''(x_1) < 0.$$

If $x_1 \in]0,1[$, then $u_1'''(x_1) = \alpha'''(x_1)$ and $u^{(iv)}(x_1) \geq \alpha^{(iv)}(x_1)$.

By Definition 8.5.1 it is obtained the contradiction:

$$\begin{aligned}
\alpha^{(iv)}(x_1) &\leq u_1^{(iv)}(x_1) \\
&= sp(x_1) - f(x_1, \delta_0(x_1, u), \delta_1(x_1, u'), \alpha''(x_1), \alpha'''(x_1)) \\
&\quad + u''(x_1) - \alpha''(x_1) \\
&< sp(x_1) - f(x_1, v, w, \alpha''(x_1), \alpha'''(x_1)) \leq \alpha^{(iv)}(x_1),
\end{aligned}$$

for every $(v,w) \in [\alpha_0, \beta_0] \times [\alpha_1, \beta_1]$. ■

In this case condition (i) in Definition 8.5.1 allows the elimination of the monotonicity type condition on f,(2.4.4), and introduces some functional dependence.

Next example illustrates the case of a function f that was not covered by Theorem 2.3.1, but can now be approached using Definition 8.5.1 and Theorem 8.5.2, generalizing the range of admissible lower and upper solutions.

Example 8.5.3 *For $x \in [0,1]$ consider the functional differential equation*

$$u^{(iv)}(x) + \int_0^x u(s)\,ds - \max_{x\in[0,1]} u'(x) - (u''(x))^3 - |u'''(x) + 1|^k = sp(x) \quad (8.5.2)$$

with $k \in [0,2]$, along with the boundary conditions (2.1.2)

The functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ given by

$$\alpha(x) = -x^2 + \tfrac{1}{2}$$

$$\beta(x) = x^2 - \tfrac{1}{2}$$

are lower and upper solutions, respectively, of problem (8.5.1),(2.1.2) verifying (2.4.3) with the auxiliary functions given by Definition 8.5.1

$$\alpha_0(x) = -x^2 - x,$$
$$\alpha_1(x) = -2x - 1,$$

and

$$\beta_0(x) = x^2 + x,$$
$$\beta_1(x) = 2x + 1,$$

for

$$\frac{-9}{\min\limits_{x\in[0,1]} p(x)} \leq s \leq \frac{5}{\max\limits_{x\in[0,1]} p(x)}$$

The function

$$f(x, y_0, y_1, y_2, y_3) = \int_0^x y_0(s)\, ds - \max_{x\in[0,1]} (y_1) - (y_2)^3 - |y_3 + 1|^k$$

is continuous, verifies conditions (2.2.3) and (2.2.1) in

$$E = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3) \in [a,b] \times \mathbb{R}^5 : \alpha_i \leq y_i \leq \beta_i, \ i = 0,1 \\ \alpha'' \leq y_2 \leq \beta'' \end{array} \right\}.$$

By Theorem 8.5.2 there is a non trivial solution $u(x)$ of problem (8.5.2)-(2.1.2), such that

$$-x^2 - x \leq u(x) \leq x^2 + x$$
$$-2x - 1 \leq u'(x) \leq 2x + 1$$
$$-2 \leq u''(x) \leq 2$$

8.6 Periodic oscillations of the axis of a satellite

In [7] it is considered the boundary value problem

$$\left\{ \begin{array}{c} (1 + \epsilon \cos x)\, u''(x) - 2\epsilon \sin(x)\, .u'(x) + \lambda \sin(u(x)) = 4\epsilon \sin(x) \\ u(0) = u(2\pi) \\ u'(0) = u'(2\pi), \end{array} \right. \tag{8.6.1}$$

for $x \in [0, 2\pi]$.

This problem models the periodic oscillations of the axis of a satellite in the plane of the elliptic orbit around its centre of mass. In this model

ϵ represents the ecentricity of the ellipse $(0 \leq \epsilon < 1)$ and λ is a parameter related with the inertia of the satellite.

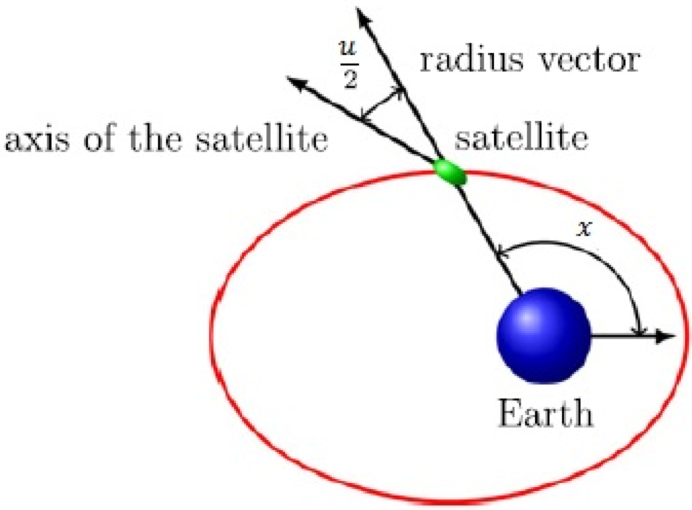

Figure 8.6.1: Oscillation of the axis of a satellite

From the mathematical point of view it is interesting to study for which values of the parameters in the (ϵ, λ) plane the problem (8.6.1) has a solution. Several authors, such as, [51, 71] have studied this problem and obtained different results for different combinations of these parameters. Beletskii, [8], has formulated this problem for $|\lambda| \leq 3$. In [13] existence results are obtained for the values of the parameter, $\lambda \leq -4$.

Using the lower and upper solution method and applying this example as a particular case of problem (8.3.7)-(8.3.8) for $n = 2$, with the suitable adaptations, we obtain different values of the parameters.

The functions

$$\alpha(x) = \frac{\pi}{2} \text{ and } \beta(x) = \frac{3\pi}{2}$$

are lower and upper solutions for the problem (8.6.1), according Definition 8.3.3, for $\lambda \geq 4$ and $0 \leq \epsilon < 1$

$$f(x, y_0, y_1) = \frac{4\epsilon \sin(x) + 2\epsilon \sin(x)\, y_1 - \lambda \sin(y_0)}{1 + \epsilon \cos x}$$

and

$$\begin{aligned}
\bar{L}_0^*(z_1, z_2, z_3) &= z_1(2\pi) - z_3(0), \\
\bar{L}_1^*(z_1, z_2, z_3) &= z_3(2\pi) - z_3(0).
\end{aligned}$$

As the continuous function f verifies Nagumo conditions in

$$E_* = \left\{ (x, y_0, y_1) \in [0, 2\pi] \times \mathbb{R}^2 : \frac{\pi}{2} \leq y_0 \leq \frac{3\pi}{2} \right\},$$

then, by Theorem 8.3.4, with $n = 2$, there is a nontrivial solution u for problem (8.6.1) such that

$$\frac{\pi}{2} \leq u(x) \leq \frac{3\pi}{2},$$

for $\lambda \geq 4$ and for all $x \in [0, 2\pi]$, $0 \leq \epsilon < 1$.

These are in fact different values of the parameter than the ones formulated in [7, 8] for the same problem.

Chapter 9

Extremal solutions to fourth order problems

9.1 Introduction

The aim of this chapter is to present sufficient conditions to the existence of extremal solutions for the functional fourth order boundary value problem composed by the equation

$$u^{(iv)}(x) = f(x, u, u''(x), u'''(x)) \tag{9.1.1}$$

for a.e. $x \in [a, b]$, where $f : [a, b] \times C([a, b]) \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function, and the boundary conditions

$$\begin{aligned} L_0(u, u'', u(a)) &= 0 \\ L_1(u, u'', u(b)) &= 0 \\ L_2(u, u'', u''(a), u'''(a)) &= 0 \\ L_3(u, u''(b), u'''(b)) &= 0 \end{aligned} \tag{9.1.2}$$

where L_i, $i = 0, .., 3$, satisfy some adequate conditions and are allowed to be discontinuous on some of their variables.

By extremal solutions we mean the existence of a maximal solution, that is, a solution which is greater or equal than any other solution, and a minimal solution with a similar sense. The existence of extremal solutions has been studied for several types of problems and in different fields, as it can be seen, for instance, in [3, 12, 16, 18, 20, 22, 52, 72, 81, 91, 99, 102]. Functional boundary value problems include a large number of differential equations and many types of boundary conditions, as discussed previously. However, on this Chapter, it is the first time where the existence of extremal solutions is obtained to fourth order problems with functional dependence in every boundary conditions. Moreover, it is remarked that boundary conditions (9.1.2) include the Lidstone case. As such these results provide extremal solutions for Lidstone boundary value problems as well.

A key point in this work is a second order auxiliary problem, obtained from (9.1.1), (9.1.2) by a reduction of order, where it is applied a standard Nagumo condition and a previous result, from [17], to have the existence of extremal solutions.

The fourth order problem is studied by adding to the previous problem two algebraic equations, to which it applies a sharp version of the Bolzano's theorem, given in [32]. Combining this technique with the non-ordered lower and upper solutions technique developed in the previous chapters, allows to define a convenient integral operator, which has a least and a greatest fixed points, as it is given in [53]. Through this technique it is obtained an existence and location result for the extremal solutions.

There are still issues that worthy further research in these type of problems, such as:

- A generalization of these results to higher order problems
- Conditions for the inclusion of functional dependence in the second and

third derivative on the differential equation (9.1.1)

- Inclusion of functional dependence of the first derivative on the boundary conditions (9.1.2)

9.2 Definitions and auxiliary results

Throughout this chapter it will be assumed the following hypothesis :

(S$_1$) $f : [a, b] \times C([a, b]) \times \mathbb{R}^2 \to \mathbb{R}$ is such that for every $u \in C([a, b])$, the function $f_u : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ defined as $f_u(x, y, z) := f(x, u, y, z)$ is a L^1-Carathéodory function, that is, $f_u(x, \cdot, \cdot)$ is a continuous function for a.e. $x \in [a, b]$; $f_u(\cdot, y, z)$ is measurable for $(y, z) \in \mathbb{R}^2$; and for every $M > 0$ there is a real-valued function $\psi_M \in L^1([a, b])$ such that

$$|f_u(x, y, z)| \leq \psi_M(x), \text{ for } a.e.\ x \in [a, b]$$

and for every $(y, z) \in \mathbb{R}^2$ with $|y| \leq M$, $|z| \leq M$.

(S$_2$) $L_0, L_1 : (C([a, b]))^2 \times \mathbb{R} \to \mathbb{R}$ are nonincreasing in the first variable and nondecreasing in the second one.

(S$_3$) $L_2 : (C([a, b]))^2 \times \mathbb{R}^2 \to \mathbb{R}$ is nonincreasing in the first variable and nondecreasing in the second and fourth variables. Moreover, for every $u \in C(I)$ given, $L_2(u, v_n, x_n, y_n) \to L_2(u, v, x, y)$ whenever $\{v_n\} \to v$ in $C(I)$ and $\{(x_n, y_n)\} \to (x, y)$ in $\mathbb{R}^2$.

(S$_4$) $L_3 : C([a, b]) \times \mathbb{R}^2 \to \mathbb{R}$ is nondecreasing in the first and third variables. Moreover, for every $u \in C(I)$ given, $L_3(u, x_n, y_n) \to L_3(u, x, y)$ whenever $\{(x_n, y_n)\} \to (x, y)$ in $\mathbb{R}^2$.

Remark 9.2.1 *Notice that some continuities are allowed in the two first variables of function f, in the first variable of functions L_2 and L_3 and in all the variables of L_0 and L_1.*

The preliminary results are related to some second order boundary value problems for which it will be assumed that conditions (S_1) and (S_2) hold.

Let $v \in W^{4,1}([a,b])$ be a fixed function and denote by (P_v) the problem composed by the equation

$$y''(x) = f(x, v, y(x), y'(x)) \tag{9.2.1}$$

and the boundary conditions

$$\begin{aligned} L_2(v, y, y(a), y'(a)) &= 0 \\ L_3(v, y(b), y'(b)) &= 0. \end{aligned} \tag{9.2.2}$$

Definition 9.2.2 *A function $y \in W^{2,1}([a,b])$ is a solution of (P_v) if it satisfies conditions (9.2.1) and (9.2.2).*

For this second-order auxiliary problem we define as lower and upper solutions the functions that verify the following conditions:

Definition 9.2.3 *A function $\zeta : [a,b] \to \mathbb{R}$, $\zeta \in W^{2,1}([a,b])$, is said to be a lower solution of problem (P_v) if:*

(i) $\zeta''(x) \geq f(x, v, \zeta(x), \zeta'(x))$;

(ii) $L_2(v, \zeta, \zeta(a), \zeta'(a)) \geq 0$ *and* $L_3(v, \zeta(b), \zeta'(b)) \leq 0$.

A function $\eta \in W^{2,1}([a,b])$ is said to be an upper solution to the problem (P_v) if the reversed inequalities hold.

A Nagumo-type growth condition, assumed on the nonlinear part, will be an important tool to set *a priori* bounds for solutions of some differential equations.

Definition 9.2.4 *Consider* $\Gamma_i, \gamma_i \in L^1([a,b])$, $i = 0,1$, *such that,* $\Gamma_i(x) \geq \gamma_i(x)$, $\forall x \in [a,b]$, *and the set*

$$F = \left\{(x, y_0, y_1) \in [a,b] \times \mathbb{R}^3 : \gamma(x) \leq y_0 \leq \Gamma(x)\right\}.$$

A function $f : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ *is said to verify a Nagumo-type condition in* F *if there exists* $\varphi \in C([0,+\infty), (0,+\infty))$ *such that*

$$|f(x, y_0, y_1)| \leq \varphi(|y_1|),$$

for every $(x, y_0, y_1) \in F$, *and*

$$\int_r^{+\infty} \frac{s}{\varphi(s)} ds > \max_{x\in[a,b]} \Gamma(x) - \min_{x\in[a,b]} \gamma(x)$$

where r *is given by*

$$r := \max\left\{\frac{\Gamma(b) - \gamma(a)}{b-a}, \frac{\Gamma(a) - \gamma(b)}{b-a}\right\}.$$

Standard arguments, as the ones followed in Lemma 1.2.2, allow to obtain an *a priori* bound for the solutions of the differential equation (9.2.1).

Lemma 9.2.5 *There exists* $R > 0$, *depending only on* φ, γ *and* Γ, *such that for every* L^1-*Carathéodory function* $f : I \times C([a,b]) \times \mathbb{R}^2 \to \mathbb{R}$ *satisfying a Nagumo-type condition in E, and every solution* y_v *of (9.2.1) such that*

$$\gamma(x) \leq y_v(x) \leq \Gamma_1(x), \ \forall\, x \in I,$$

we have that $\|y_v'\| < R$.

Remark that problem (P_v) can be considered as a particular case of the following second order problem presented in [17]:

$$\begin{gathered} -\left(\phi(u'(t))\right)' = g(t, v, u(t), u'(t)), \quad for\ a.e.t \in [a,b], \\ l_1(v, u, u(a), u(b), u'(a)) = 0, \\ l_2(v, u(a), u(b), u'(b)) = 0, \end{gathered} \tag{P}$$

which has the corresponding lower and upper solutions:

Definition 9.2.6 *A function $\gamma \in W^{2,1}([a,b])$ is a lower solution of (P) if:*

(i) $-(\phi(\gamma'(t))' \geq g(t, v, \gamma(t), \gamma'(t))$;

(ii) $l_1(v, \gamma, \gamma(a), \gamma(b), \gamma'(a)) \geq 0$ *and* $l_2(v, \gamma(a), \gamma(b), \gamma'(b)) \leq 0.$

A function $\delta \in W^{2,1}([a,b])$ is an upper solution of (P) for the reversed inequalities.

For the above problem and definitions it is obtained the following result, from [17]

Theorem 9.2.7 *[17, Theorem 3.2] Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing and continuous function and assumptions (S_1),(S_3) and (S_4) hold.*
If there are lower and upper solutions of (P), γ, δ respectively, such that $\gamma \leq \delta$ and f satisfies a Nagumo-type growth condition in

$$E_{\gamma\delta} = \left\{(x, y_0, y_1, y_2) \in [a,b] \times \mathbb{R}^3 : \gamma(x) \leq y_1 \leq \delta(x)\right\}$$

then (P) has extremal solutions in $[\gamma, \delta]$.

In the proof of the main result it is applied the following version of the Bolzano Theorem:

Lemma 9.2.8 *[32, Lemma 2.3] Let $a, b \in \mathbb{R}$, $a \leq b$, and $h : \mathbb{R} \to \mathbb{R}$ be such that either $h(a) \geq 0 \geq h(b)$ and*

$$\lim_{z \to x^-} \sup\ h(z) \leq h(x) \leq \lim_{z \to x^+} \inf\ h(z), \text{ for all } x \in [a,b],$$

or $h(a) \leq 0 \leq h(b)$ and

$$\lim_{z \to x^-} \inf\ h(z) \geq h(x) \geq \lim_{z \to x^+} \sup\ h(z), \text{ for all } x \in [a,b].$$

Then there exists $c_1, c_2 \in [a,b]$ such that $h(c_1) = 0 = h(c_2)$ and if $h(c) = 0$ for some $c \in [a,b]$ then $c_1 \leq c \leq c_2$, i.e., c_1 and c_2 are, respectively, the least and the greatest of the zeros of h in $[a,b]$.

Sufficient conditions for the existence of extremal fixed points of some operator, will be given by the following result:

Lemma 9.2.9 *[53, Theorem 1.2.2] Let Y be a subset of an ordered metric space $(X, \leq)$, $[a,b]$ a non empty ordered interval in Y, and $T : [a,b] \to [a,b]$ a nondecreasing mapping. If $\{Tx_n\}$ converges in Y whenever $\{x_n\}$ is a monotone sequence in $[a,b]$, then there exists x_* the least fixed point of T in $[a,b]$ and x^* is the greatest one. Moreover*

$$x_* = \min\{y \mid Ty \leq y\} \ \ \textit{and}\ x^* = \max\{y \mid Ty \geq y\}.$$

9.3 Extremal solutions to fourth-order problems

Lower and upper solutions technique used in this work allows the non-ordered case, that is, lower and upper solution do not need to be ordered. In fact, we apply some auxiliary functions "to get some order".

For $\alpha, \beta \in W^{2,1}([a,b])$, with $\alpha'' \leq \beta''$ *a.e.* on $[a,b]$, we define the functions $\alpha_0, \beta_0 : [a,b] \to \mathbb{R}$ by

$$\beta_0(x) = A_0 \frac{b-x}{b-a} + A_1 \frac{x-a}{b-a} + \int_a^b G(x,s)\,\beta''(x)\,ds, \tag{9.3.1}$$

and

$$\alpha_0(x) = B_0 \frac{b-x}{b-a} + B_1 \frac{x-a}{b-a} + \int_a^b G(x,s)\,\alpha''(x)\,ds,$$

where $A_1, A_2, B_1, B_2 \in \mathbb{R}$ are given by

$$\begin{aligned} A_0 &= \min\{\alpha(a), \beta(a)\},\ B_0 = \max\{\alpha(a), \beta(a)\}, \\ A_1 &= \min\{\alpha(b), \beta(b)\},\ B_1 = \max\{\alpha(b), \beta(b)\}, \end{aligned}$$

and G is the Green's function associated to the Dirichlet problem

$$y''(x) = 0, \quad \text{a. a. } x \in [a,b], \qquad y(a) = y(b) = 0.$$

By standard computations, it is well known that such function is defined by

$$G(x,s) = \frac{1}{b-a} \begin{cases} (a-s)(b-x), & \text{if} \quad a \le x \le s \le b, \\ \\ (a-x)(b-s), & \text{if} \quad a \le s \le x \le b. \end{cases}$$

In particular it is non-positive on $[a,b] \times [a,b]$ and, as a consequence, $\beta_0 \le \alpha_0$ in $[a,b]$.

Lower and upper solutions for the fourth order problem (9.1.1)-(9.1.2) are based on the previous auxiliary functions and so they must be defined as a pair:

Definition 9.3.1 *The functions $\alpha, \beta \in W^{4,1}([a,b])$ are a pair of lower and upper solutions of the problem (9.1.1)-(9.1.2) if the following conditions hold:*

$$\begin{aligned} \alpha^{(iv)}(x) &\ge f(x, \alpha_0, \alpha''(x), \alpha'''(x)), \quad a.\ e.\ \ x \in [a,b], \\ 0 &\le L_0(\alpha_0, \alpha'', \alpha_0(a)), \\ 0 &\le L_1(\alpha_0, \alpha'', \alpha_0(b)), \\ 0 &\le L_2(\alpha_0, \alpha'', \alpha''(a), \alpha'''(a)), \\ 0 &\ge L_3(\alpha_0, \alpha''(b), \alpha'''(b)). \end{aligned}$$

and

$$\begin{aligned} \beta^{(iv)}(x) &\le f(x, \beta_0, \beta''(x), \beta'''(x)), \quad a.\ e.\ \ x \in [a,b], \\ 0 &\ge L_0(\beta_0, \beta'', \beta_0(a)), \\ 0 &\ge L_1(\beta_0, \beta'', \beta_0(b)), \\ 0 &\ge L_2(\beta_0, \beta'', \beta''(a), \beta'''(a)), \\ 0 &\le L_3(\beta_0, \beta''(b), \beta'''(b)). \end{aligned}$$

To obtain the main result one needs the following hypothesis on the functions L_0 and L_1:

(S$_5$) For every $(v,u,x) \in [\alpha_0, \beta_0] \times [\alpha'', \beta''] \times [A_i, B_i]$, $i = 0, 1$, the following property holds:

$$\limsup_{z \to x^+} L_i(v,u,z) \leq L_i(v,u,x) \leq \liminf_{z \to x^-} L_i(v,u,z).$$

The main result is given by the following theorem:

Theorem 9.3.2 *Assume that conditions (S_1)-(S_5) hold and $f(x,.,y_0,y_1)$ is nondecreasing for a.e. $x \in [a,b]$ and all $(y_0,y_1) \in \mathbb{R}^2$.*

If there is a pair of lower and upper solutions of (9.1.1)-(9.1.2), α and β, respectively, such that

$$\alpha''(x) \leq \beta''(x) \text{ for every } x \in \ [a,b],$$

and f satisfies a Nagumo type growth condition in the set

$$E_{\alpha,\beta} := \left\{(x,y_0,y_1) \in [a,b] \times \mathbb{R}^2 : \alpha''(x) \leq y_0 \leq \beta''(x)\right\},$$

then problem (9.1.1) – (9.1.2) has extremal solutions in the set

$$S \equiv \left\{u \in C^2([a,b]) : u \in [\beta_0, \alpha_0] \quad \text{and} \quad u'' \in [\alpha'', \beta'']\right\}.$$

Proof. Let $v \in [\beta_0, \alpha_0]$ be fixed.

Consider the second-order problem (P_v). As α and β are, respectively, lower and upper solutions of problem (9.1.1) – (9.1.2), then the monotonicity assumptions on function f with respect to its second variable implies that α'' and β'' are lower and upper solutions of (P_v), respectively, according to Definition 9.2.3. In consequence problem (P_v) has extremal solutions in $[\alpha'', \beta'']$ for all $v \in [\beta_0, \alpha_0]$.

Denote by y_v the minimal solution of (P_v) in $[\alpha'', \beta'']$.

By (S_2) and Definition 9.3.1, we have for $i = 0, 1$.

$$L_i (v, y_v, B_i) \geq L_i (\alpha_0, \alpha'', B_i) \geq 0, \tag{9.3.2}$$

and

$$L_i (v, y_v, A_i) \leq L_i (\beta_0, \beta'', A_i) \leq 0. \tag{9.3.3}$$

From condition (S_5) one can apply Lemma 9.2.8 to obtain that equations

$$L_i (v, y_v, z) = 0, \quad i = 0, 1,$$

have greatest zeros in $[A_i, B_i]$, denoted by w_v if $i = 0$ and z_v when $i = 1$.

Define, for each $x \in [a, b]$, the operator T by

$$Tv(x) = \int_a^b G(x,s)\, y_v(s)\, ds + w_v \frac{b-x}{b-a} + z_v \frac{x-a}{b-a}.$$

It follows from the definition of T, α_0 and β_0 that $T([\beta_0, \alpha_0]) \subset [\beta_0, \alpha_0]$.

To analyze the monotonicity of T, consider $v_1, v_2 \in [\beta_0, \alpha_0]$ such that $v_1 \leq v_2$ and let y_{v_1} and y_{v_2} be the corresponding minimal solutions of (P_{v_1}) and (P_{v_2}) in $[\alpha'', \beta'']$, respectively. Therefore, by the assumptions on f,

$$y''_{v_1}(x) = f\left(x, v_1, y_{v_1}(x), y'_{v_1}(x)\right) \leq f\left(x, v_2, y_{v_1}(x), y'_{v_1}(x)\right)$$

and, by (S_3) and (S_4),

$$0 = L_2\left(v_1, y_{v_1}, y_{v_1}(a), y'_{v_1}(a)\right) \geq L_2\left(v_2, y_{v_1}, y_{v_1}(a), y'_{v_1}(a)\right),$$

$$0 = L_3\left(v_1, y_{v_1}(b), y'_{v_1}(b)\right) \leq L_3\left(v_2, y_{v_1}(b), y'_{v_1}(b)\right).$$

So, y_{v_1} is an upper solution of (P_{v_2}). As $\alpha'' \leq y_{v_1} \leq \beta''$, then, by Theorem 9.2.7, there are extremal solutions for the problem (P_{v_2}) in $[\alpha'', y_{v_1}]$. In particular the least solution y_{v_2} of (P_{v_2}) in $[\alpha'', y_{v_1}]$ is the least solution of (P_{v_2}) in $[\alpha'', \beta'']$.

Therefore, $y_{v_1} \geq y_{v_2}$ and, by (S$_2$),

$$L_i\left(v_2, y_{v_2}, w\right) \leq L_i\left(v_1, y_{v_2}, w\right) \leq L_i\left(v_1, y_{v_1}, w\right), \ \forall w \in \mathbb{R}, \quad i = 0, 1. \quad (9.3.4)$$

In consequence $w_{v_1} \leq w_{v_2}$ and $z_{v_1} \leq z_{v_2}$.

Therefore $Tv_1 \leq Tv_2$, that is, the operator T is nondecreasing in $[\beta_0, \alpha_0]$.

Consider now a monotone sequence $\{v_n\}_n$ in $[\beta_0, \alpha_0]$. Therefore the sequence $\{Tv_n\}_n$ is monotone too and, since

$$(Tv_n)''(x) = y_{v_n}(x) \in [\alpha''(x), \beta''(x)], \quad x \in [a, b],$$

one can easily verify that it is bounded in $C^2([a, b])$. So, applying Ascoli-Arzelá theorem, $\{Tv_n\}_n$ is convergent in $C([a, b])$.

Therefore T sends monotone sequences into convergent ones and, by Lemma 9.2.9, T has a greatest fixed point in $[\beta_0, \alpha_0]$, denoted by v^*, satisfying

$$v^* = \max\{v \in [\beta_0, \alpha_0] : v \leq Tv\}. \quad (9.3.5)$$

It is immediate to verify that $v^* \in S$ and it is a solution of problem (9.1.1)-(9.1.2).

Let us see that v^* is actually the maximal solution of problem (9.1.1)-(9.1.2) in the set S.

Consider v an arbitrary solution of problem (9.1.1)-(9.1.2) in $[\beta_0, \alpha_0]$, with $v'' \in [\alpha'', \beta'']$. From Theorem 9.2.7 we have that $v(a) \leq w_v$ in $[\beta_0(a), \alpha_0(a)]$ and $v(b) \leq z_v$ in $[\beta_0(b), \alpha_0(b)]$.

Since $v'' = y$, with y a solution of (P_v) in $[\alpha'', \beta'']$, and y_v is the minimal solution of (P_v) in $[\alpha'', \beta'']$, then $v'' \geq y_v$ and we deduce that $v \leq T\,v$. Thus, by (9.3.5), $v \leq v^*$ and so v^* is the greatest solution of (9.1.1)-(9.1.2) in S.

The existence of the least solution can be proved using analogous arguments and obvious changes in the operator T. ■

9.4 Example

Consider, for $x \in [0,1]$, the fourth order equation

$$u^{(iv)}(x) = \max_{x\in[0,1]} \left(\int_0^x u(s)\,ds \right) + \lambda \left(u''(x)\right)^3 - \left(u'''(x)+1\right)^{\frac{2}{3}} \tag{9.4.1}$$

along with the functional boundary conditions

$$\begin{aligned} -\max_{x\in[0,1]} u(x) + u(0) &= 0 \\ \min_{x\in[0,1]} u''(x) + \delta u(1) &= 0 \\ u''(0) &= 0 \\ u''(1) &= 0. \end{aligned} \tag{9.4.2}$$

This problem is a particular case of (9.1.1)-(9.1.2) with

$$f(x, y_0, y_1, y_2) = \max_{x\in[0,1]} \left(\int_0^x y_0(s)\,ds \right) + \lambda y_1^3 - (y_2+1)^{\frac{2}{3}},$$

$$\begin{aligned} L_0(z_1, z_2, z_3) &= -\max_{x\in[0,1]} z_1 + z_3, \\ L_1(z_1, z_2, z_3) &= \min_{x\in[0,1]} z_2 + \delta z_3, \\ L_2(z_1, z_2, z_3, z_4) &= -z_3, \\ L_3(z_1, z_2, z_3) &= z_2. \end{aligned}$$

The functions

$$\alpha(x) = -\frac{x^2}{2} - x + 1 \text{ and } \beta(x) = \frac{x^2}{2} + x - 1$$

are a pair of lower and upper solutions, respectively, of problem (9.4.1)-(9.4.2), with

$$\begin{gathered} A_0 = -1,\ B_0 = 1,\ A_1 = -\frac{1}{2},\ B_1 = \frac{1}{2}, \\ \alpha_0(x) = 1 - \frac{x^2}{2} \text{ and } \beta_0(x) = \frac{x^2}{2} - 1 \end{gathered}$$

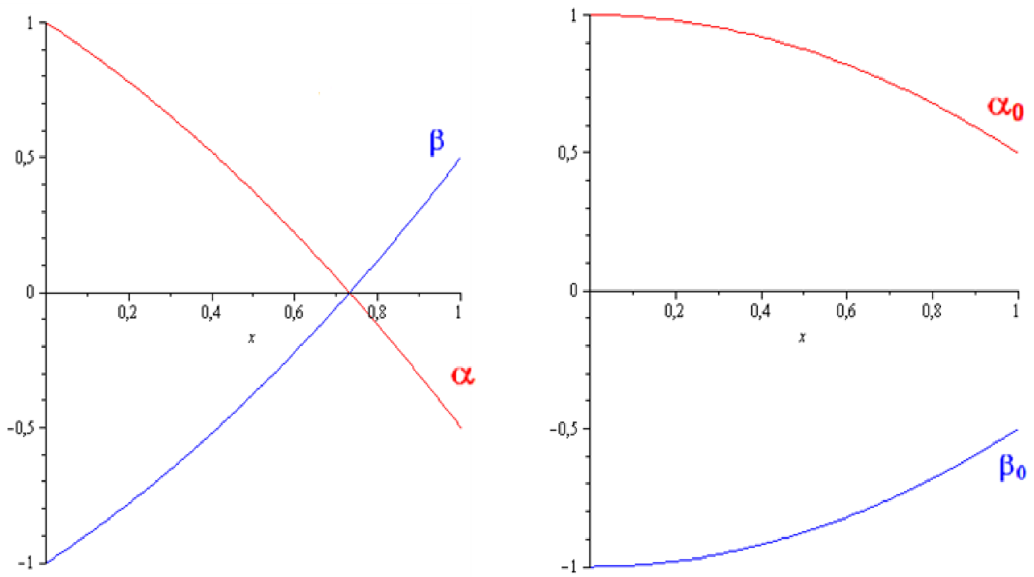

Figure 9.4.1: Despite α and β are non ordered there are extremal solutions in the set $[\beta_{0,}\alpha_0]$.

for 1$\leq$ $\lambda < \infty$ and $\delta \geq 2$.

As the continuous function f verifies a Nagumo type growth condition, according Definition 9.2.4, in

$$E = \{(x, y_1, y_2) : -1 \leq y_1 \leq 1\}$$

with $\varphi(y_1) = 1 + |\lambda| + |y_1 + 1|^{\frac{2}{3}}$, then, by Theorem 9.3.2 the problem (9.4.1)-(9.4.2) has extremal solutions in $[\beta_0, \alpha_0]$.

As one can see by Figure 9.4.1, despite the fact that the lower and upper solutions α and β are not ordered, the auxiliar functions α_0 and β_0 are ordered, but in the "reversed" way.

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