Existência de minimizantes para integrais n-dimensionais não-convexos

PhD Thesis

Luís Manuel Balsa Bicho

Dissertação apresentada na Universidade de Évora
para a obtenção do grau de Doutor em Matemática
sob orientação do Prof. António Ornelas

Esta Tese inclui as críticas e sugestões feitas pelo júri.

Departamento de Matemática
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I wish to thank Prof. António Ornelas.

To my family

Contents

Existence of minimizers for n-dimensional nonconvex integrals

Abstract

First it is proved the *existence* of *minimizers* for the *multiple integral*

$$\int_\Omega \ell^{**}\left(u\left(x\right), \rho_1\left(x, u(x)\right) \nabla u\left(x\right) \right)\, \rho_2\left(x, u(x)\right)\, d\,x \text{ on } W^{1,1}_{u_\partial}\left(\Omega\right),$$

where $\Omega \subset \mathbb{R}^d$ is *open bounded*, $u : \Omega \to \mathbb{R}$ is in the *Sobolev space* $u_\partial(\cdot) + W^{1,1}_0(\Omega)$, with *boundary* data $u_\partial(\cdot) \in W^{1,1}(\Omega) \cap C^0\left(\overline{\Omega}\right)$; and $\ell : \mathbb{R}\times\mathbb{R}^d \to [0,\infty]$ is *superlinear* $\mathcal{L}\otimes\mathcal{B} - measurable$ with $\rho_1(\cdot,\cdot)$, $\rho_2(\cdot,\cdot) \in C^0\left(\Omega\times\mathbb{R}\right)$ both > 0 and $\ell^{**}(\cdot,\cdot)$ only has to be *lsc* at $(\cdot, 0)$. The case $\int_\Omega L^{**}\left(x, u(x), \nabla u(x)\right)$ is also treated, though with less natural hypotheses, but still allowing $L(x,\cdot,\xi)$ $non-lsc$ for $\xi \neq 0$;

Lastly it is proved the *existence* of *uniformly continuous radially symmetric minimizers* $u_A(x) = U_A\left(|x|\right)$ for the *multiple integral*

$$\int_{B_R} L^{**}\left(u(x), |\,D\,u(x)\,|\right)\, d\,x$$

on a *ball* $B_R \subset \mathbb{R}^d$, among the *vector Sobolev functions* $u(\cdot)$ in $A + W^{1,1}_0\left(B_R, \mathbb{R}^m\right)$, using a *convex lsc* $L^{**} : \mathbb{R}^m\times\mathbb{R} \to [0,\infty]$ with $L^{**}\left(S,\cdot\right)$ *even* and *superlinear*; but while in the *scalar* $m = 1$ case we only need one more hypothesis: $\exists \min L^{**}\left(\mathbb{R}, 0\right)$, in the *vectorial* $m > 1$ case $L^{**}(\cdot,\cdot)$ also has to satisfy a *geometric* constraint, which we call $quasi - scalar$; the simplest example being the *biradial* case $L^{**}\left(|\,u(x)\,|, |\,D\,u(x)\,|\right)$.

Existência de minimizantes para integrais n-dimensionais não-convexos

Resumo

Primeiro demonstra-se a existência de minimizantes para o integral múltiplo

$$\int_{\Omega} \ell^{**}\left(u\,(x)\,, \rho_1\,(x, u(x))\,\nabla u\,(x)\,\right)\ \rho_2\,(x, u(x))\ \ d\,x \text{ on } W^{1,1}_{u_\partial}\,(\Omega)\,,$$

onde $\Omega \subset \mathbb{R}^d$ é aberto e limitado, $u : \Omega \to \mathbb{R}$ pertence ao espaço de Sobolev $u_\partial(\cdot) + W^{1,1}_0\,(\Omega)$, $u_\partial(\cdot) \in W^{1,1}\,(\Omega) \cap C^0\left(\overline{\Omega}\right)$; $\ell : \mathbb{R}{\times}\mathbb{R}^d \to [0, \infty]$ é superlinear $\mathcal{L}{\otimes}\mathcal{B}-$mensurável, $\rho_1(\cdot,\cdot)$, $\rho_2(\cdot,\cdot) \in C^0\,(\Omega{\times}\mathbb{R})$ ambos > 0 e $\ell^{**}(\cdot,\cdot)$ é apenas *sci* em $(\cdot, 0)$. Também se considera o caso $\int_\Omega L^{**}\,(x, u(x), \nabla u(x)\,)$, embora com hipóteses mais complexas, mas é igualmente possível ter $L(x,\cdot,\xi)$ não-*sci* para $\xi \neq 0$;

Por último demonstra-se a existência de minimizantes radialmente simétricos, i.e. $u_A(x) = U_A\,(\,|x|\,)$, uniformemente contínous para o integral múltiplo

$$\int_{B_R} L^{**}\left(\,u(x), |\,D\,u(x)\,|\,\right)\ \ d\,x$$

na bola $B_R \subset \mathbb{R}^d$, $u : \Omega \to \mathbb{R}^m$ pertence ao espaço de Sobolev $A + W^{1,1}_0\,(\,B_R, \mathbb{R}^m\,)$, $L^{**} : \mathbb{R}^m{\times}\mathbb{R} \to [0, \infty]$ é convexa, *sci* e superlinear, $L^{**}\,(\,S, \cdot\,)$ é par; note-se também que enquanto no caso escalar, $m = 1$, apenas necessitamos de mais uma hipótese: $\exists \min L^{**}\,(\,\mathbb{R}, 0\,)$, no caso vectorial, $m > 1$, $L^{**}\,(\cdot,\cdot)$ também tem de satisfazer uma restrição geométrica, a qual chamamos $quasi-escalar$; sendo o exemplo mais simples de uma função $quasi-escalar$ o caso biradial $L^{**}\,(\,|\,u(x)\,|\,, |\,D\,u(x)\,|\,)$.

Chapter 1

Notations and introduction

1.1 Main notations

$\nabla u(x) :=$ *gradient vector* of $u(\cdot)$ at *point* x;

$\langle (x_1, ..., x_d), (y_1, ..., y_d) \rangle := \sum_{i=1}^{d} x_i y_i$;

$(x_n) \rightharpoonup x$ means that the *sequence* (x_n) *converges weakly* to x;

$(x_n) \nearrow x$ means that the *sequence* (x_n) *converges* to x and *increases*;

$\alpha_d :=$ *Hausdorff measure*, in dimension $d-1$, of the *unit sphere* of $\mathbb{R}^d$;

$\partial B_R := \{ x \in \mathbb{R}^d : |x| = R \}$;

$\partial f(\xi) :=$ classic *subdifferential* of $f(\cdot)$ at *point* ξ;

$\partial^0 L(S, 0) :=$ the *minimal norm element* of $\partial L(\cdot, 0)$ at S;

$\Omega :=$ *open* and *bounded subset* of $\mathbb{R}^d$;

$AC :=$ *absolutely continuous*;

$AC_{loc} :=$ *locally absolutely continuous*;

$B_R := \{ x \in \mathbb{R}^d : |x| \leq R \}$;

$\mathcal{B}-$measurable $:=$ *Borel measurable*;

$\overline{C} :=$ *smallest closed set* that contains C;

$C^0(\Omega) :=$ the *set* of all *continuous functions* $u : \Omega \to \mathbb{R}$;

$C_b^0(\Omega) :=$ the *set* of all *continuous bounded functions* $u : \Omega \to \mathbb{R}$;

$C_c^1(\Omega)$:= *set* of all *functions* $u : \Omega \to \mathbb{R}$ with *continuous derivative* and *compact support*;

$co\, C$:= *smallest convex set* that contains C;

$div\,(f_1, ..., f_d)(x_1, ..., x_d) := \frac{\partial f_1}{\partial x_1}(x_1, ..., x_d) + ... + \frac{\partial f_d}{\partial x_d}(x_1, ..., x_d)$;

$D\, u(x)$:= *Jacobian matrix* of $u(\cdot)$ at *point* x;

$epi\, f(\cdot)$:= *epigraph* of $f(\cdot)$;

$ext\, C$:= *set* of all *extreme points* of the *convex set* C (an *extreme point* of a *convex set* C in a *real vector space* is a *point* in C which does not lie in any *open line segment joining two points* of C);

$f^{**}(\cdot)$:= the *greatest convex function less* than or *equal* to $f(\cdot)$;

$int\, C$:= *interior* of the *set* C;

L^p and L^p_{loc}, $1 \leq p \leq \infty$, represent the usual *Lebesgue spaces*;

lsc := *lower semicontinuous*;

$\mathcal{L}-$measurable := *Lebesgue measurable*;

$S^d := \{ x \in \mathbb{R}^d \, : \, |x| = 1 \}$;

$U \subset\subset V$, means that the *set* U is *compactly contained* in the *set* V, i.e. $\overline{U} \subset V$ and $\overline{U}$ is a *compact set*;

usc := *upper semicontinuous*;

$w - lsc$:= *weak lower semicontinuous*;

$W^{1,p}$, $W^{1,p}_{loc}$ and $W^{1,p}_0$, $1 \leq p \leq \infty$, represent the usual *Sobolev spaces*;

$W^{1,p}_{u_\partial} := u_\partial(\cdot) + W^{1,p}_0$;

$W^{1,p}_{A} := A + W^{1,p}_0$.

1.2 Introduction

Here the main purpose is to study the *existence* of *minimizers* for the *integral functional*:

$$F\,(\,u(\cdot)\,) := \int_{\Omega} L\,(x, u\,(x)\,, D\,u\,(x))\;\,d\,x \qquad \text{on} \quad W^{1,p}_{u_\partial}\,(\,\Omega,\,\mathbb{R}^m\,)\,, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is an *open bounded set*; the *boundary* condition

$$u_\partial(\cdot) \in W^{1,p}\,(\,\Omega,\,\mathbb{R}^m\,) \quad \text{and} \qquad W^{1,p}_{u_\partial}\,(\,\Omega,\,\mathbb{R}^m\,) = u_\partial(\cdot) + W^{1,p}_0\,(\,\Omega,\,\mathbb{R}^m\,)\,,$$

with $W^{1,p}\,(\,\Omega,\,\mathbb{R}^m\,)$ and $W^{1,p}_0\,(\,\Omega,\,\mathbb{R}^m\,)$ the usual *Sobolev spaces*, $L\,:\,\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times d} \to [0,\infty]$ and $D\,u\,(x)$ represents the *Jacobian matrix* of $u(\cdot)$ at *point* x.

Convex Case First we will considerer $L\,(\,x, s, \cdot\,)$ *convex*.

Definition 1.2.1. *Let X be a vectorial space, a function $f : X \to \mathbb{R} \cup \{\infty\}$ is said convex if*

$$f\,(\,\lambda x + (\,1-\lambda\,)\;y\,) \leq \lambda\,f(x) + (\,1-\lambda\,)\;f(y)$$

$\forall\,\lambda \in (0,1)$ *and* $\forall\,x, y$ *such that* $f(x),\; f(y) < \infty\,.$

In this case the two main ingredients are the *coercivity* and the *lower semicontinuity* of the *functional* $F(\cdot)$.

Definition 1.2.2. *Let X be a vectorial space and $f\,:\,X \to \mathbb{R} \cup \{\infty\}$ a function.*

a) $f(\cdot)$ is said sequentially coercive if for each $t \in \mathbb{R}$ $\;\exists\,K_t \subseteq X$ closed and sequentially compact such that

$$\{\,x \in X \;\;:\;\; f(x) \leq t\,\} \subseteq K_t;$$

b) $f(\cdot)$ is said sequentially lower semicontinuous $(\,lsc\,)$ if

$$f(x) \leq \liminf_{n\to\infty} f\,(\,x_n\,) \qquad\qquad \forall\,(x_n) \to x;$$

c) $f(\cdot)$ is said weak sequentially lower semicontinuous ($w - lsc$) if

$$f(x) \leq \liminf_{n \to \infty} f\,(\,x_n\,) \qquad\qquad \forall\,(x_n) \rightharpoonup x.$$

A direct consequence of the definitions is that whenever the *function* $f(\cdot)$ is *coercive* and *lsc* then $\exists\, u_{\min} \in X$ such that

$$f\,(\,u_{\min}\,) = \min_{x \in X} f(x).$$

In particular to ensure that our *functional* $F(\cdot)$ is *coercive*, when $p = 1$, due to e.g. [Ces, 10.3.i], it is only necessary to impose to the *lagrangian* $L(\cdot, \cdot, \cdot)$ a *superlinear* growth, i.e.,

$$\exists\, \theta : (0, \infty) \to [0, \infty) \;\; : \;\; \begin{cases} L(x, s, \xi) \geq \theta\,(\,|\xi|\,) & \forall\, x, s, \xi \\ \\ \frac{\theta(r)}{r} \to \infty \quad \text{as} & r \to \infty, \end{cases} \tag{1.2}$$

or equivalently

$$\frac{\inf L\,(\Omega, \mathbb{R}^m, \partial B_r)}{r} \to \infty \qquad \text{as} \;\; r \to \infty\,, \tag{1.3}$$

where $\partial B_r := \left\{ \xi \in \mathbb{R}^{m \times d} \; : \; |\xi| = r \right\}$, while for $p > 1$ we need to impose to $L(\cdot, \cdot, \cdot)$ a *polynomial* growth, i.e.,

$$L(x, s, \xi) \geq c_1\,|\xi|^p + c_2 \qquad (\,c_1 > 0,\; c_2 \in \mathbb{R}\,)\,. \tag{1.4}$$

On the other hand to obtain the *lsc* of $F(\cdot)$ we need to appeal, e.g., to results like the next one:

Proposition 1.2.1. (*See [Iof].*)

Let $L : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to [0, \infty]$ be $\mathcal{L} \otimes \mathcal{B} \otimes \mathcal{B}-$mensurable with

a) $L\,(\,x, s, \cdot\,)$ convex, $\forall\, s$ and for a.e. (almost every) $x \in \Omega$,

b) $L\,(\,x, \cdot, \cdot\,)$ lsc, for a.e. $x \in \Omega$.

Then the functional $F(\cdot)$ is $w - lsc$ on $W^{1,p}_{u_\partial}\,(\,\Omega,\, \mathbb{R}^m\,)$.

Similar results had been proved by many authors, an example being the following proposition:

Proposition 1.2.2. *(See [Amb, th. 4.14].)*

Let $L : \Omega\times\mathbb{R}\times\mathbb{R}^d \to [0,\infty]$ *satisfies:*

i) $L(\cdot,\cdot,\cdot)$ *is* $\mathcal{B}\otimes\mathcal{L}\otimes\mathcal{B}-$*measurable;*

ii) $L(x,s,\cdot)$ *is convex* $\forall\, x,s$*;*

iii) $L(\cdot,\cdot,0)$ *is Borel,* $L(x,s,0) < \infty\ \forall\, x,s$*, and* $L(x,\cdot,0)$ *is lsc for a.e.* $x\in\Omega$*;*

iv) $L(\cdot,s,\cdot) - L(\cdot,s,0)$ *is lsc* $\forall\, s$*;*

v) $\exists\, m : \Omega\times\mathbb{R}\to\mathbb{R}^d$ *such that*

v) a) $m(\cdot,s)$ *is continuous* $\forall\, s$*,* $m(x,\cdot)$ *is measurable* $\forall\, x$ *and*

$$m(x,s)\in\partial L(x,s,0)\ \ \forall\, x,s\,;$$

v) b) for each open $\Omega' \subset\subset \Omega$*, the function*

$$M^{\Omega'}(s) := \sup\{|m(x,s)|\,:\,x\in\Omega'\}\in L^1_{loc}(\mathbb{R})\,;$$

v) c) for each $\mathrm{V}\subset\subset\mathbb{R}$*, the family*

$$\{m(\cdot,s)\,:\,s\in\mathrm{V}\}\qquad \textit{is equicontinuous in } C^0\left(\Omega',\mathbb{R}^d\right).$$

Then $F(\cdot)$ *is* $w-lsc$ *on* $W^{1,1}(\Omega)$*.*

Clearly if $L(\cdot,\cdot,\cdot)$ satisfies the *coercivity* condition (1.2) plus, e.g., the conditions of proposition 1.2.2, then $\exists\, u_{min}(\cdot)\in W^{1,1}_{u_\partial}(\Omega)$ such that

$$F\left(u_{min}(\cdot)\right) = \min_{u(\cdot)\in W^{1,1}_{u_\partial}(\Omega)} F\left(u(\cdot)\right).$$

Nonconvex Case Here we drop the condition $L\,(\,x, s, \cdot\,)$ *convex.*

Due to the lack of *convexity* of $L\,(\,x, s, \cdot\,)$, in general, the *functional* $F(\cdot)$ will be non $w-lsc$ on $W^{1,p}_{u_\partial}$. The general framework in this case is to consider the *relaxed functional* of $F(\cdot)$:

$$\overline{F}(\cdot) := \sup\{\,G(\cdot)\;:\;G(\cdot)\leq F(\cdot)\;\;\text{and}\;\;G(\cdot)\;\;\text{is}\;\;w-lsc\;\;\text{on}\;\;W^{1,p}_{u_\partial}\,\}\,;$$

an important property of $\overline{F}(\cdot)$ is that when $F(\cdot)$ is *coercive* then

$$\inf\,F(\cdot) = \min\,\overline{F}(\cdot).$$

A relevant question related to $\overline{F}(\cdot)$ is to know when it admits an integral representation, the answer to this question is found, e.g., on the next proposition :

Proposition 1.2.3. (*See [Mcl, th. 2.3].*)

Let $L\,:\,\Omega\times\mathbb{R}\times\mathbb{R}^d \to [0,\infty]$ *be a Carathéodory function satisfying the growth condition* (1.4)*, for some* $p > 1$*, plus*

$$L(x, s, \xi) \leq g\,(\,x, |s|\,, |\xi|\,)\,,$$

with $g(\cdot,\cdot,\cdot)$ *increasing with respect to* $|s|$ *and* $|\xi|$ *and locally integrable with respect to* x*.*

then

$$\overline{F}\,(\,u(\cdot)\,) = \int_\Omega L^{**}\,(\,x, u(x), \nabla\,u(x)\,)\;\,d\,x \qquad\qquad \forall\,u(\cdot)\in W^{1,\infty}\,(\,\Omega\,)\;,$$

where $L^{**}(x, s, \xi)$ *is the greatest function less than or equal to* $L(x, s, \xi)$ *and convex with respect to* ξ *.*

A consequence of last proposition is that in some cases we may apply the results of the *convex* case to the *relaxed functional* $\overline{F}(\cdot)$, obtaining this way more information about the original *functional* $F(\cdot)$.

In this thesis we study various issues of the problem of *minimizing* the *integral* in (1.1), a more detailed presentation of those issues is made on each chapter.

Chapter 2

Existence of scalar minimizers

2.1 Introduction

Here is considered the *problem* of *existence* of *minimizers* for *integrals* of the type

$$\int_\Omega L\left(x, u\left(x\right), \nabla u\left(x\right)\right)\; d\,x \qquad \text{on} \;\; W^{1,p}_{u_\partial}\left(\Omega\right), \tag{2.1}$$

namely on the *Sobolev space*

$$W^{1,p}_{u_\partial}\left(\Omega\right) := \left\{ u(\cdot) \in W^{1,p}\left(\Omega\right) \;:\; \left(u - u_\partial\right)(\cdot) \in W^{1,p}_0\left(\Omega\right) \right\} \,,$$

where the *boundary* data $u_\partial(\cdot)$ is any given *function* in $W^{1,p}\left(\Omega\right) \cap C^0\left(\overline{\Omega}\right)$, $p \geq 1$, $\Omega \subset \mathbb{R}^d$ *open bounded.*

Our main result deals with *existence* in the *convex* case: we denote in this way the case where $L(\cdot,\cdot,\cdot)$ *equals* its *bipolar* $L^{**}(\cdot,\cdot,\cdot)$ defined as usual, e.g. $convexify - close$ the *epigraphs*:

$$epi\, L^{**}(x,s,\cdot) := \overline{co}\, epi\, L(x,s,\cdot). \tag{2.2}$$

It proves *existence*, in this case, under quite general hypotheses, allowing namely $L^{**}(x,\cdot,\xi)$ $non - lsc$ for $\xi \neq 0$. In the special "$factor$" case

$$L^{**}(x,s,\xi) = \ell^{**}\left(s, \rho_1(x,s) \;\; \xi\,\right) \;\; \rho_2(x,s)\,, \tag{2.3}$$

our hypotheses are simpler and may look more natural.

Notice that $L^{**}(x,s,\xi\,) = \infty$ or $L(x,s,\xi\,) = \infty$ are freely allowed here, while where $L^{**}(x,s,\xi\,)$ is *finite* one may even have an *empty* $subdifferential$ of $L^{**}(x,s,\cdot\,)$ at ξ (even at $\xi = 0$); in particular, (2.1) may be the $variational\ reformulation$ of general *state* & *gradient constrained optimal control problems* involving *multiple integrals* and $implicit\ first-order\ nonsmooth\ scalar\ partial\ differential\ inclusions.$ (Following the modern but classical tradition of [Roc & Wet], we use here the symbol ∞ where some authors use $+\infty$.)

Indeed, $L(\cdot,\cdot,\cdot)$ is only assumed to satisfy the following extremely weak

Basic Hypotheses for the general case convexified of (2.1):

(H_1) $L\,:\,\Omega{\times}\mathbb{R}{\times}\mathbb{R}^d \to [0,\infty]$ has (where $\mathcal{B} = Borel$, $\mathcal{L} = Lebesgue$ and $lsc = lower\ semicontinuous$):

$$L(\cdot,\cdot,\cdot)\quad \mathcal{B}{\otimes}\mathcal{L}{\otimes}\mathcal{B}-measurable \quad\text{and}\quad L(x,s,\cdot)\quad lsc \qquad \forall\, x,s;$$

(H_2) $L(\cdot,\cdot,\cdot)$ has at least *superlinear growth*, namely

$$\frac{\inf L\,(\Omega,\mathbb{R},\partial B_r)}{r} \to \infty \qquad \text{as}\quad r\to\infty\,, \tag{2.4}$$

where $\partial B_r := \big\{\xi\in\mathbb{R}^d\,:\,|\xi| = r\big\}$; while for $p>1$ we need to impose

$$L(x,s,\xi) \geq c_1\,|\xi|^p + c_2 \qquad (\,c_1>0,\ c_2\in\mathbb{R}\,)\,; \tag{2.5}$$

(H_3) the *bipolar* $L^{**}(\cdot,\cdot,\cdot)$ of $L(\cdot,\cdot,\cdot)$ is

$$approximable \quad\text{by}\quad equi-integrable\ slopes \quad\text{at}\quad zero\ gradient. \tag{2.6}$$

The precise definition of (H_3), named so after [Orn] due to (2.40)+(2.43)+(2.47), being rather technical, is postponed to the section 2.4; however, in the *autonomous* case it is simply the hypothesis (2.7) + (2.8) + (2.9) of [Orn], where the *autonomous* $d=1$ case is treated: for each $i\in\mathbb{N}$,

$$\begin{gathered}\exists\ m_i(\cdot)\in L^1_{loc}\left(\mathbb{R},\mathbb{R}^d\right) \quad \& \\ \exists\ (\varphi_i(s)) \nearrow L^{**}(s,0)\quad \forall\, s\quad\text{with}\quad \varphi_i:\mathbb{R}\to[0,i]\quad lsc,\end{gathered} \tag{2.7}$$

for which

$$L^{**}(s,\xi) \geq \varphi_i(s) + \langle\, m_i(s),\xi\,\rangle \quad \forall s,\xi,i\,. \tag{2.8}$$

Another situation in which (H_3) may be expressed very simply is the general *nonautonomous* case (2.1) whenever $0 \in \partial L^{**}(x,s,0)$ (e.g. $L^{**}(x,s,\cdot)$ *even* $\forall\, x,s$); or, more generally, whenever

$$\exists\, m(\cdot) \in L^1_{loc}\left(\mathbb{R},\mathbb{R}^d\right) \;:\; m(s) \in \partial L^{**}(x,s,0) \quad \forall\, x,s\,. \tag{2.9}$$

In such case, (H_3) reduces (trivially) to:

$$\begin{gathered} L^{**}(\cdot,\cdot,0) \;\; \textit{Borel} \;\; \text{with} \\ L^{**}(x,\cdot,0) \;\;\&\;\; L^{**}(\cdot,s,\cdot) - L^{**}(\cdot,s,0) \;\; lsc \quad \forall\, x,s\,. \end{gathered} \tag{2.10}$$

(Notice that, above and elsewhere, our *subdifferential* symbol ∂ always refers to the *gradient variable.*)

The hypothesis (H_3) also takes a very simple form in the *nonautonomous* special factorized case (2.3), namely our basic hypotheses (H_1), (H_2) and (H_3) are satisfied by $L(\cdot,\cdot,\cdot)$ whenever (2.2) and (2.3) hold true for some $\ell(\cdot,\cdot)$ satisfying the next more natural hypotheses.

Basic Hypotheses for the special convex factorized case (2.3):

$\left(H_1'\right)$ $\ell : \mathbb{R}\times\mathbb{R}^d \to [0,\infty]$ is $\mathcal{L}\otimes\mathcal{B}-$ *measurable* with $\ell(s,\cdot)$ *lsc* $\forall\, s$,

$$\begin{cases} \rho_1(\cdot,\cdot),\; \rho_2(\cdot,\cdot) \in C^0_b\,(\Omega\times\mathbb{R}) \quad (\textit{cont. bounded functions}) \\ \inf\, \rho_1(\Omega\times\mathbb{R}) > 0,\; \inf\, \rho_2(\Omega\times\mathbb{R}) > 0\,; \end{cases} \tag{2.11}$$

$\left(H_2'\right)$ *superlinear growth*:

$$\frac{\inf\, \ell\,(\mathbb{R},\partial B_r)}{r} \to \infty \qquad \text{as} \;\; r \to \infty\,, \tag{2.12}$$

where $\partial B_r := \left\{v \in \mathbb{R}^d \;:\; |v| = r\right\}$; while for $p > 1$ we need to impose

$$\ell(s,v) \geq c_1\,|v|^p + c_2 \qquad (\,c_1 > 0,\;\; c_2 \in \mathbb{R}\,)\,; \tag{2.13}$$

$$\left(H_3'\right) \quad \ell^{**}(\cdot,\cdot) \;\text{ is }\; lsc \;\text{ at }\; \textit{zero gradient} \;\; (\text{i.e. at } (s,0)\;\forall\, s)\,. \tag{2.14}$$

In the *convex* case, our proof uses Ambrosio's *lower semicontinuity* result [Amb, th. 4.14], a generalization of a result obtained by DeGiorgi, Buttazzo & Dal Maso ([DG & But & DM]). While the last paper dealt with *finite* values of $L^{**}(s, \xi)$ only, [Amb] generalized it to allow *nonautonomous lagrangians* and ∞ values for $L^{**}(x, s, \xi)$ where $\xi \neq 0$. However, while [Amb] imposed $L^{**}(x, s, 0) < \infty$ and $\partial L^{**}(x, s, 0) \neq \emptyset$, we avoid such hypotheses. Moreover $\left(H_3^{'}\right)$ appears here for the first time.

We wish to underline properly this novelty : while usually one assumes $\ell^{**}(\cdot, \cdot)$ *lsc* at (s, v) $\forall s, v$; here we need it only at $v = 0$, in $\left(H_3^{'}\right)$. On the other hand, in the weaker hypothesis (H_3) (see (2.6) and (2.40) to (2.47)) there are no *joint lower semicontinuity* conditions imposed on the *lagrangian*. We even allow $L^{**}(x, \cdot, \xi)$ to be neither *lsc* nor *Borel*, for $\xi \neq 0$.

After treating the *convex* case, we also prove *existence* of *minimizers* for the general (*nonconvex*) case, in which we have to assume the *existence* of a *well − behaved relaxed minimizer*, i.e. a *minimizer* of the *convexified integral*, associated to $L^{**}(\cdot, \cdot, \cdot)$. But in some cases the *boundary* data $u_\partial(\cdot)$ itself may satisfy this trivially. Moreover, we have to assume a strong extra hypothesis. Indeed, the "*nonconvexity points*" ξ should all *lie* in $d - dim$ "*faces*", as expressed formally in (2.24) + (2.26) + (2.29) + (2.31) + (2.32); each such *face* being allowed to *move continuously* with (x, s) provided it remains *inside* a *hyperplane*; and each such *hyperplane*, in turn, may *move*, with x only, its *intersection* with the *vertical axis*, and it may *move*, with s only, the "*signed − length*" of its *slope*, as is expressed formally in the *equality* (2.32) below. Moreover, there should *exist*, roughly speaking, some *function* in $W^{1,1}_{u_\partial} \cap C^0$ having *gradients* in the *interior* of those *faces*, i.e. the *set* $\mathcal{U}_i^{v\,int}$ in (2.28) should be *nonempty*; so that we can guarantee that the *nonconvexity set* $\mathcal{U}_i^{v\,ext}$ in (2.34) is *nonempty* also, as (2.38) ensures. (See the precise hypotheses in (2.22) to (2.32).)

Nevertheless, in the *nonconvex calculus* of *variations* involving *multiple integrals* and *highly discontinuous lagrangians* $L(\cdot, \cdot, \cdot)$, namely having $L(x, \cdot, \xi)$ just $\mathcal{L} -$ *measurable* for $\xi \neq 0$, assuming ∞ values freely, our *existence* result for (2.1) is new.

For a historical review of initial motivations, *problems* and *results* of the *Nonconvex Calculus* of *Variations* see [Mcl], while for a more recent overview consult [Dac & Mcl].

Several previous results in the *nonconvex* case may be seen now as proving *existence* of *minimizers* in special cases, as compared with our own theorem 2.3.1 below: [Mas & Sch 1], [Mas & Sch 2], [Cell 1], [Cell 2], [Fri], [Cela & Pe 1], [Cela & Pe 2], [Fon & Fus & Mcl], [Zag 1], [Cela & Cup & Gui], [Zag 2], ... The paper [Mcl], but specially the papers [Cell 1] and [Cell 2], show the need of imposing *affinity* to $L^{**}(x, s, \cdot)$ on *nonconvexity regions*, even when x, s are absent. Most of these *existence* results do not consider general dependence $L(x, s, \xi)$ as we do here in $(H_1) +$ (2.6), or in $(H_1^{'}) +$ (2.14). Indeed, [Cell 1], [Cell 2], [Fri] treat the case $L(\xi)$; [Cela & Pe 1] treats the *sum* case $L(s, \xi) = f(s) + g(\xi)$ and [Cela & Pe 2] treats the *product* case $L(s, \xi) = f(s)\, g(\xi)$; while [Cela & Cup & Gui] deals with $L(s, \xi)$, [Fon & Fus & Mcl] and [Zag 1] with $L(x, \xi)$.

Our general case $L(x, s, \xi)$ is considered only (to my knownledge) in [Mas & Sch 1], [Mas & Sch 2] and [Zag 2], under much heavier hypotheses. Indeed, in the best of these results, [Zag 2], $L(\cdot, \cdot, \cdot)$ and $L^{**}(\cdot, \cdot, \cdot)$ are *continuous* and, at *each point* where $L^{**}(x_0, s_0, \xi_0) < L(x_0, s_0, \xi_0)$, the following must hold: there *exists* a *ball* $B_{r_0}(\xi_0)$, a *neighborhood* U of (x_0, s_0), a *function* $q(\cdot) \in C^0(\Omega)$; and a *function* $m(\cdot) \in C^0\left(\Omega, \mathbb{R}^d\right)$ having *divergence* *div* $m(\cdot)$ a *positive Radon measure* (possibly after *multiplying* it by -1) satisfying:

$$L^{**}(x, s, \xi) \geq q(x) + \langle\, m(x), \xi\, \rangle, \quad \forall\, (x, s) \in U,\ \ \forall\, \xi \in \mathbb{R}^d,$$

$$L^{**}(x, s, \xi) = q(x) + \langle\, m(x), \xi\, \rangle, \quad \forall\, (x, s, \xi) \in U \times \overline{B_{r_0}(\xi_0)}.$$

Moreover, [Zag 2] assumes the *set* $\mathcal{M}$ of *relaxed minimizers* to be *non* $-$ *empty* and, specially, the following heavy hypotheses, completely avoided in our (H_1) + (2.6): $\mathcal{M}$ must be *sequentially strongly compact* in $L^1(\Omega)$ and must consist *entirely* of *continuous* a.e. *differentiable functions*. In contrast, we only assume, in our (2.22) below, *existence* of *one well* $-$ *behaved relaxed minimizer*. Notice also that, more generally, *continuity* of $L(\cdot, \cdot, \cdot)$, $L^{**}(\cdot, \cdot, \cdot)$, with *finite* values, is a common assumption to all of these papers, while we here allow ∞ values freely.

Theorems 2.2.1 and 2.2.2 below (*existence* in the *convex* case) are the main results of this chapter; while theorem 2.3.1 (the *nonconvex* case) is just a simple application of [DBla & Pian 1], based on *Baire category*, a method pioneered, for *existence* of *solutions* of *differential inclusions*, 3 decades ago in [Cell 3]; and blossoming namely in [Mar & Orn], [Gon & Orn]; and later in the much more complete and well-known book [Dac & Mcl], which deals in particular also with *scalar problems* which could be dealt with via our theorem 2.3.1 below, under weaker assumptions than [Dac & Mcl], namely in what regards *regularity* of $L(\cdot,\cdot,\xi)$ in our $(H_1)+(2.6)$ or even in our (2.10). Notice also that the *scalar dependence* on the *state* s of the *affinity slope* in (2.32) seems to be new.

2.2 Existence of relaxed minimizers

Consider the *Basic Hypotheses for the general case convexified of* (2.1), $(H_1)+(H_2)+(H_3)$, with (2.6) more precisely defined in (2.40) to (2.47), due to its technically heavy machinery.

Theorem 2.2.1. *Let $L : \Omega \times \mathbb{R} \times \mathbb{R}^d \to [0,\infty]$ be a function satisfying the basic hypotheses $(H_1)+(H_2)+(H_3)$, see* (2.4) *to* (2.6).

Then there exist minimizers for the convexified integral

$$\int_\Omega L^{**}\left(x, u\left(x\right), \nabla u\left(x\right)\right)\, d\,x \qquad on \;\; W^{1,p}_{u_\partial}\left(\Omega\right), \tag{2.15}$$

for any $u_\partial(\cdot)$ as in (2.1).

Remark 2.2.1. *Existence of minimizers for* (2.15) *also holds true provided one imposes $(H_1)+(H_2)+(H_3)$ on $L^{**}(\cdot,\cdot,\cdot)$ instead; or provided one replaces $(H_1)+(H_3)$ by the following hypothesis:*

$$(H_1'') \quad L(\cdot,\cdot,\cdot) \quad is \quad \mathcal{L}\otimes\mathcal{B}\otimes\mathcal{B}-measurable \quad \& \quad L(x,\cdot,\cdot) \quad is \;\; lsc \;\; \forall\, x.$$

Indeed (see [Iof]) this is true even in the vectorial case, when $L : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to [0,\infty]$.

Theorem 2.2.2. *Let $\ell : \mathbb{R}\times\mathbb{R}^d \to [0,\infty]$ be a function satisfying the special basic hypotheses $\left(H_1'\right)+\left(H_2'\right)+\left(H_3'\right)$, see* (2.11) *to* (2.14)*; also sufficing $\left(H_1'\right)+\left(H_2'\right)+\left(H_3'\right)$ to hold true for $\ell^{**}(\cdot,\cdot)$ itself instead.*

Then there exist minimizers for the convexified integral

$$\int_\Omega \ell^{**}\left(u\left(x\right), \rho_1(x,u(x))\,\nabla u\left(x\right)\right)\;\; \rho_2(x,u(x))\; d\,x \qquad on \;\; W^{1,p}_{u_\partial}\left(\Omega\right), \tag{2.16}$$

for any $u_\partial(\cdot)$ as in (2.1).

We prove first theorem 2.2.1, using the definition of (H_3) presented in (2.40) to (2.47). Then we prove theorem 2.2.2 by showing, namely, that $\left(H_3'\right) \Rightarrow\ (H_3)$ in case (2.3).

Proof of theorem 2.2.1. a) Assuming (H_3), i.e. the hypotheses (2.40) to (2.47), define, for each $i \in \mathbb{N}$,

$$f_i(x,s,\xi) := \begin{cases} L^{**}(x,s,\xi) & \text{for } \xi \neq 0 \\ \\ \varphi_i(x,s) & \text{for } \xi = 0 \end{cases} \tag{2.17}$$

and $f_i^{**}(\cdot,\cdot,\cdot)$ by $epi\, f_i^{**}(x,s,\cdot) := \overline{co}\, epi\, f_i(x,s,\cdot)\ \ \forall x,s.$

Let us fix $i \in \mathbb{N}$ and use the next

Proposition 2.2.1. (*This is [Amb, th. 4.14] adapted to our notation.*)

Consider the integral

$$\int_\Omega f_i^{**}\left(x,u\left(x\right),\nabla u\left(x\right)\right)\ d\,x \qquad on \quad W^{1,1}_{u_\partial}\left(\Omega\right), \tag{2.18}$$

where $f_i^{**} : \Omega{\times}\mathbb{R}{\times}\mathbb{R}^d \to [0,\infty]$ *satisfies:*

i) $f_i^{**}(\cdot,\cdot,\cdot)$ *is* $\mathcal{B}{\otimes}\mathcal{L}{\otimes}\mathcal{B}$*-measurable;*

ii) $f_i^{**}(x,s,\cdot)$ *is convex* $\forall\, x,s;$

iii) $f_i^{**}(\cdot,\cdot,0)$ *is Borel,* $f_i^{**}(x,s,0) < \infty\ \forall\, x,s,$ *and* $f_i^{**}(x,\cdot,0)$ *is lsc for a.e.* $x \in \Omega$*;*

iv) $f_i^{**}(\cdot,s,\cdot) - f_i^{**}(\cdot,s,0)$ *is lsc* $\forall\, s$;

v) $\exists\, m_i : \Omega{\times}\mathbb{R} \to \mathbb{R}^d$ *such that*

v) a) $m_i(\cdot,s)$ *is continuous* $\forall\, s$, $m_i(x,\cdot)$ *is measurable* $\forall\, x$ *and*

$$m_i(x,s) \in \partial f_i^{**}(x,s,0)\ \ \forall\, x,s\,;$$

v) b) see (2.47)*;*

v) c) see (2.46)*.*

Then

$$\textit{the integral}\ \ (2.18)\ \ \textit{is}\ \ w-lsc\ \ \textit{on}\ \ W^{1,1}(\Omega)\ . \tag{2.19}$$

Let us check the validity of the hypotheses of proposition 2.2.1 for our $f_i^{**}(\cdot,\cdot,\cdot)$ in (2.17). Indeed,

i) is satisfied due to (H_1)+(2.4) + (2.41);

ii) holds by definition of $f_i^{**}(\cdot,\cdot,\cdot)$;

iii) holds true, due to (2.41) + (2.42);

iv) is satisfied by $f_i(\cdot,\cdot,\cdot)$, using (2.42); hence, by (2.4), it is also satisfied by $f_i^{**}(\cdot,\cdot,\cdot)$;

v) a) is satisfied, due to (2.17)+(2.40)+(2.44)+(2.45), for $f_i(\cdot,\cdot,\cdot)$, so, by (2.4), it is valid also for $f_i^{**}(\cdot,\cdot,\cdot)$;

v) b) is satisfied, due to (2.47);

v) c) is satisfied, using (2.46).

Therefore (2.19) holds true. Take a *minimizing sequence* $\left(u_i^k(\cdot)\right)_k$ for (2.18); then, by *superlinearity*, one may pass to a *sublimit* $u_i(\cdot) \in W^{1,p}_{u_\partial}(\Omega)$ of this *sequence*, i.e. $u_i(\cdot)$ is the *limit* of a *subsequence* of $\left(u_i^k(\cdot)\right)_k$, necessarily a *minimizer* of (2.18).

b) **Claim 1** *The following pointwise increasing convergence holds true:*

$$(\, f_i^{**}(x,s,\xi)\,) \nearrow L^{**}(x,s,\xi) \qquad \forall\, x,s,\xi\,. \tag{2.20}$$

Indeed, as in the proof of [Orn], one easily checks that $f_i^{**}(\cdot,\cdot,\cdot)$ satisfies the same *growth* condition (2.4) as $L^{**}(\cdot,\cdot,\cdot)$ does, hence

$$\bigcap_{i=1}^{\infty} \mathit{epi}\, f_i^{**}(x,s,\cdot) \subseteq \mathit{epi}\, L^{**}(x,s,\cdot)$$

i.e. (2.20) holds true.

Again, by the *equi − superlinearity* condition (2.4), satisfied by the whole *sequence* ($f_i^{**}(\cdot,\cdot,\cdot)$), the *sequence* $(u_i(\cdot))$ (where $u_i(\cdot)$ *minimizes* (2.18)) has itself a *weak* $W^{1,1}(\Omega)$ *convergent subsequence.* Denote by $v(\cdot) \in W_{u_\partial}^{1,1}(\Omega)$ its *weak limit.*

Claim 2 $v(\cdot)$ *minimizes the integral* (2.15).

Indeed, for each $k \leq i$ and each $u(\cdot) \in W_{u_\partial}^{1,1}(\Omega)$,

$$\begin{aligned}
&\int_\Omega f_k^{**}(x, v(x), \nabla v(x))\, dx \\
&\qquad \leq \liminf_{i\to\infty} \int_\Omega f_k^{**}(x, u_i(x), \nabla u_i(x))\, dx \\
&\qquad \leq \liminf_{i\to\infty} \int_\Omega f_i^{**}(x, u_i(x), \nabla u_i(x))\, dx \\
&\qquad \leq \liminf_{i\to\infty} \int_\Omega f_i^{**}(x, u(x), \nabla u(x))\, dx \\
&\qquad \leq \int_\Omega L^{**}(x, u(x), \nabla u(x))\; dx\,,
\end{aligned} \tag{2.21}$$

by (2.19) + (2.20). Hence, by the Fatou lemma and (2.21),

$$\begin{aligned}
&\int_\Omega L^{**}(x, v(x), \nabla v(x))\; dx \\
&\qquad \leq \liminf_{k\to\infty} \int_\Omega f_k^{**}(x, v(x), \nabla v(x))\, dx \\
&\qquad \leq \int_\Omega L^{**}(x, u(x), \nabla u(x))\; dx\,,
\end{aligned}$$

thus proving claim 2 and theorem 2.2.1, for $p = 1$.

c) In case $p > 1$ the reasoning is similar.

□

Proof of theorem 2.2.2. To prove this theorem (with $L^{**}(\cdot,\cdot,\cdot)$ defined by (2.3)), one easily checks that $\left(H_1^{'}\right) \Rightarrow\ (H_1)$ and $\left(H_2^{'}\right) \Rightarrow\ (H_2)$. Finally, using lemma 2.4.1 (with proof beginning in (2.48)), also $\left(H_3^{'}\right) \Rightarrow\ (H_3)$, thus proving theorem 2.2.2. □

2.3 Existence of true minimizers

For our *nonconvex* result, and besides the basic hypotheses already used in the *convex* case $-$ namely either $(H_1)+(H_2)+(H_3)$ or else $\left(H_1'\right)+\left(H_2'\right)+\left(H_3'\right)$ in the special factorized case (2.3) $-$ we need to impose the following extra hypothesis, whose intuitive meaning has already been roughly explained above (see the beginning of page 25) :

Extra Nonconvexity Hypothesis for the general case (2.1) :

$$(H_4) \qquad \begin{cases} \exists \; \textit{minimizer } v(\cdot) \text{ for the } \textit{convexified integral } (2.15) \\ \qquad \text{which is } \textit{well}-\textit{behaved}, \text{ in the sense that the} \\ \text{following hypotheses (2.23) } \textit{to } \text{(2.32) are satisfied.} \end{cases} \tag{2.22}$$

Namely :

$$\exists \; \textit{open sets} \quad \Omega_i \subset \Omega, \quad \textit{pairwise disjoint}, \text{ such that} \tag{2.23}$$

$$L^{**}\left(\cdot, v(\cdot), \nabla v(\cdot)\right) = L\left(\cdot, v(\cdot), \nabla v(\cdot)\right) \qquad \text{a.e. on} \quad \Omega \setminus \bigcup_{i=1}^{\infty} \Omega_i\,, \tag{2.24}$$

i.e. *outside* of the *open sets* Ω_i where $\nabla\, v(\cdot)$ is *inside* the "*faces*" $F_i(\cdot,\cdot)$, with $v(\cdot)$ the *well $-$ behaved relaxed minimizer* assumed to *exist* in (2.22) for which we have, at each $i \in \mathbb{N}$:

$$\begin{cases} \exists \; F_i : \Omega{\times}\mathbb{R} \to \mathbb{R}^d, \text{ a } \textit{continuous bounded multifunction} \\ \qquad \text{with } \textit{compact} \text{ values} \quad F_i(x,s) \neq \emptyset \qquad \forall x, s\,; \end{cases} \tag{2.25}$$

$$\nabla v(x) \in co\, F_i(x, v(x)) \qquad \text{for a.e. } x \in \Omega_i\,; \tag{2.26}$$

$$\text{the } \textit{set } \; \mathcal{U}_i^{v\, int} \; \text{ defined in (2.28) is } \textit{nonempty} : \tag{2.27}$$

$$\mathcal{U}_i^{v\,int} := \{u(\cdot) \in \Sigma_i^v \;:\; \nabla u(\cdot) \in int\,co\,F_i(\cdot, u(\cdot)) \quad \text{a.e. on} \quad \Omega_i\} \tag{2.28}$$

where $\Sigma_i^v := C^0\left(\overline{\Omega}_i\right) \cap W_v^{1,1}(\Omega_i)$;

$$L^{**}(x, s, \xi) = L(x, s, \xi) \quad \forall\, \xi \in ext\,co\,F_i(x, s) \quad \forall\, x, s\,; \tag{2.29}$$

$$\exists\, q_i(\cdot) \in L^1(\Omega_i)\,, \quad \exists\, m_i \in \mathbb{R}^d \quad \text{and} \quad \exists\, \alpha_i(\cdot) \in L^\infty_{loc}(\mathbb{R}) \;: \tag{2.30}$$

$$L^{**}(x, s, \xi) \geq q_i(x) + \langle\, \alpha_i(s)\, m_i\,,\, \xi\,\rangle \quad \forall\, s, \xi \quad \text{for a.e. } x \in \Omega_i, \tag{2.31}$$

$$\begin{cases} L^{**}(x, s, \xi) = q_i(x) + \langle\, \alpha_i(s)\, m_i\,,\, \xi\,\rangle \\ \forall \xi \in F_i(x, s) \quad \forall\, s \quad \text{and for a.e.} \quad x \in \Omega_i\,. \end{cases} \tag{2.32}$$

Define also

$$\mathcal{U}_i^v := \Big\{u(\cdot) \in W_v^{1,1}(\Omega_i) \;:\; \nabla u(\cdot) \in co\,F_i(\cdot, u(\cdot)) \;\; \text{a.e. on } \Omega_i\Big\}\,, \tag{2.33}$$

and

$$\mathcal{U}_i^{v\,ext} := \{w(\cdot) \in \mathcal{U}_i^v \;:\; \nabla w(\cdot) \in ext\,co\,F_i(\cdot, w(\cdot)) \;\; \text{a.e. on } \Omega_i\}\,. \tag{2.34}$$

We recall that $int\,co\,F_i(x, u(x))$ is the *interior*, in $\mathbb{R}^d$, of the *convex hull* *co* of $F_i(x, u(x))$; while $ext\,co\,F_i(x, w(x))$ means the *set* of *extreme points* of this *compact convex set*.

Remark 2.3.1. *Given the weakness of our hypotheses on $L(\cdot, s, \xi)$ namely possible Borel dependence at $\xi \neq 0$ in (H_1), and (2.42) or (2.10), one might consider the possibility of weakening the continuity of $F_i(\cdot, s)$ in (2.25); however [DBla & Pian 2, section 4] presents a counterexample to proposition 2.3.1 below, using a $F_i(\cdot, s)$ usc but not lsc.*

Recall again the *Basic Hypotheses for the general case* (2.1), $(H_1)+(H_2)+(H_3)$, with (2.6) more precisely defined in (2.40) to (2.47), due to its technically heavy machinery, together with this new extra *nonconvexity* hypothesis (H_4) in (2.22).

Theorem 2.3.1. *Let $L : \Omega\times\mathbb{R}\times\mathbb{R}^d \to [0,\infty]$ satisfy the basic hypotheses $(H_1)+(H_2)+(H_3)$ plus the extra hypothesis (H_4) (see (2.22) to (2.32)).*

Then the nonconvex integral (2.1) has minimizers, for $p = 1$.

Proof of theorem 2.3.1. a) Fix $i \in \mathbb{N}$ and define, using (2.30),

$$Q_i := \int_{\Omega_i} q_i(x)\; d\,x$$

and

$$\beta_i(s) := \int_0^s \alpha_i(\sigma)\; d\,\sigma.$$

For $u(\cdot) \in \mathcal{U}_i^v$, we have (see [DCic & Leo] and the references therein)

$$\begin{aligned} &\beta_i(u(\cdot)) \in W^{1,1}_{\beta_i(v)}(\Omega_i) \qquad\qquad \& \\ &\nabla\beta_i(u(x)) = \alpha_i(u(x))\,\nabla u(x) \quad \text{ a.e. on } \ \Omega_i\,; \end{aligned} \tag{2.35}$$

hence, by (2.32)+(2.35),

$$L^{**}(x, u(x), \nabla u(x)) = q_i(x) + \langle m_i, \nabla\beta_i(u(x))\rangle \quad \text{ a.e. on } \ \Omega_i\,,$$

and, for each $u(\cdot) \in \mathcal{U}_i^v$,

$$\int_{\Omega_i} L^{**}(x, u(x), \nabla u(x))\; d\,x = Q_i + \int_{\Omega_i} \langle m_i, \nabla\beta_i(u(x))\rangle\; d\,x. \tag{2.36}$$

But, on the other hand,

$$(\, \beta_i(u(\cdot)) - \beta_i(v(\cdot))\,) \in W_0^{1,1}\,(\Omega_i)$$

yields, by the *divergence* theorem,

$$\int_{\Omega_i} \langle m_i, \nabla\beta_i(u(x))\rangle \, dx = \int_{\Omega_i} \langle m_i, \nabla\beta_i(v(x))\rangle \, dx \quad \forall u(\cdot) \in \mathcal{U}_i^v \, . \tag{2.37}$$

Therefore applying (2.27) and, using the *sets* defined in (2.28) and (2.34),

Proposition 2.3.1. (*See [DBla & Pian 1, th. 4.2] together with* (2.28) *and* (2.34).)
Let $F_i : \overline{\Omega}_i \times \mathbb{R} \to \mathbb{R}^d$ *be a continuous bounded multifunction with compact values and let* Ω_i *be an open set in* $\mathbb{R}^d$*, with* $\overline{\Omega}_i \neq \mathbb{R}^d$*. Assume* (2.22) *to* (2.32)*.*
Then

$$\mathcal{U}_i^{v\,int} \neq \emptyset \quad \Rightarrow \quad \mathcal{U}_i^{v\,ext} \neq \emptyset \, . \tag{2.38}$$

we obtain some $w_i(\cdot) \in \mathcal{U}_i^{v\,ext}$; and since $v(\cdot)$, $w_i(\cdot) \in \mathcal{U}_i^v$, by (2.29) + (2.36) + (2.37),

$$\int_{\Omega_i} L(x, w_i(x), \nabla w_i(x)) \, dx = \int_{\Omega_i} L^{**}(x, v(x), \nabla v(x)) \, dx. \tag{2.39}$$

b) Finally defining

$$w(x) := \begin{cases} w_i(x) & \text{for } x \in \Omega_i, \ i = 1, 2, ... \\ \\ v(x) & \text{for } x \in \Omega \setminus \bigcup_{i=1}^{\infty} \Omega_i \, , \end{cases}$$

we get $w(\cdot) \in W^{1,1}_{u_\partial}(\Omega)$ and, by (2.24) + (2.39),

$$\int_{\Omega} L(x, w(x), \nabla w(x)) \, dx = \int_{\Omega} L^{**}(x, v(x), \nabla v(x)) \, dx.$$

□

2.4 Approximation by equi-integrable slopes at zero gradient

Let us start by defining precisely the hypothesis (H_3), mentioned in (2.6), namely:

$$(H_3) \quad L^{**}(x,s,\xi) \geq \varphi_i(x,s) + \langle\, m_i(x,s),\, \xi\,\rangle \qquad \forall x,s,\xi,i\,; \tag{2.40}$$

where:

$(H_{3.1})$ for each $i \in \mathbb{N}$, $\varphi_i : \Omega{\times}\mathbb{R} \to [0,i]$ satisfies:

$$\varphi_i(\cdot,\cdot) \quad \text{is a} \quad \textit{Borel function}, \tag{2.41}$$

$$\varphi_i(x,\cdot) \quad \text{for a.e. } x \quad \& \quad L^{**}(\cdot,s,\cdot) - \varphi_i(\cdot,s) \;\; \forall\, s \quad \text{are} \;\; \textit{lsc}\,; \tag{2.42}$$

and this *sequence increases* and *converges pointwise*:

$$(\varphi_i(x,s)) \nearrow L^{**}(x,s,0) \qquad \forall\, x,s\,; \tag{2.43}$$

$(H_{3.2})$ for each $i \in \mathbb{N}$, $m_i : \Omega{\times}\mathbb{R} \to \mathbb{R}^d$ satisfies:

$$m_i(x,\cdot) \quad \text{is} \quad \mathcal{L}-\textit{measurable} \qquad \text{for a.e. } x\,, \tag{2.44}$$

$$m_i(\cdot,s) \in C^0\left(\Omega,\mathbb{R}^d\right) \qquad \forall s\,, \tag{2.45}$$

and considering, for each $\mathrm{V} \subset\subset \mathbb{R}$ and each *open* $\Omega' \subset\subset \Omega$, the *family* $\{m_i(\cdot,s)\,:\, s \in \mathrm{V}\}$ and its *sup*, i.e.

$$M_i^{\Omega'}(s) := \sup\left\{|m_i(x,s)|\,:\, x \in \Omega'\right\},$$

then

$$\{m_i(\cdot,s)\,:\, s \in \mathrm{V}\} \qquad \text{is} \quad \textit{equicontinuous} \quad \text{in} \quad C^0\left(\Omega',\mathbb{R}^d\right) \tag{2.46}$$

and

$$M_i^{\Omega'}(\cdot) \in L^1_{loc}(\mathbb{R})\,. \tag{2.47}$$

As stated after (2.16), we wish to prove now that whenever the $lagran-gian$ $L^{**}(\cdot,\cdot,\cdot)$ in (2.15) assumes the special form (2.3), for some $\ell(\cdot,\cdot)$ satisfying $\left(H_1^{'}\right)+\left(H_2^{'}\right)+\left(H_3^{'}\right)$ (see (2.11) to (2.14)), then $(H_1)+(H_2)+(H_3)$ (see (2.4) to (2.6)) are also satisfied, hence the result of theorem 2.2.1 is applicable. Indeed, since $\left(H_1^{'}\right) \Rightarrow (H_1)$ and $\left(H_2^{'}\right) \Rightarrow (H_2)$ are straightforward, we only need to prove that $\ell^{**}(\cdot,\cdot)$ *lsc* at $(\cdot,0)$ *implies* $L^{**}(\cdot,\cdot,\cdot)$ *approximable* by $equi-integrable$ *slopes* at *zero gradient*, namely the next

Lemma 2.4.1. $\left(H_3^{'}\right) \Rightarrow (H_3)$*, under* (2.3)*.*

Proof of lemma 2.4.1. Using (2.3), by $\left(H_3^{'}\right)$+(2.12) one may show, as in the proof of [Orn], that

$$\begin{cases} \exists\, \mu_i(\cdot) \in L^1_{loc}\left(\mathbb{R},\mathbb{R}^d\right) \qquad (\text{ namely } \ |\mu_i(\cdot)| \leq i\,)\,, \\ \\ \exists\, \psi_i : \mathbb{R} \to [0,i] \quad \ lsc \quad \text{ for which } \quad (\psi_i(s)) \nearrow \ell^{**}(s,0) \end{cases} \tag{2.48}$$

satisfying

$$\ell^{**}(s,v) \geq \psi_i(s) + \langle\, \mu_i(s), v\rangle \qquad \forall s,v,i\,. \tag{2.49}$$

Then, setting

$$\varphi_i(x,s) := \rho_2(x,s)\,\psi_i(s)$$

and

$$m_i(x,s) := \rho_1(x,s)\,\rho_2(x,s)\,\mu_i(s),$$

we get, using (2.49),

$$\begin{aligned} &L^{**}(x,s,\xi) \\ &\qquad = \ell^{**}(s,\rho_1(x,s)\,\xi\,)\,\rho_2(x,s) \\ &\qquad \geq \rho_2(x,s)\,\psi_i(s) + \langle\, \rho_1(x,s)\,\rho_2(x,s)\,\mu_i(s),\,\xi\,\rangle \\ &\qquad = \varphi_i(x,s) + \langle\, m_i(x,s),\,\xi\,\rangle, \end{aligned}$$

i.e. (2.40). By (2.48) + (2.11), $\varphi_i(\cdot)$ is *Borel*, $\varphi_i(x,\cdot) = \rho_2(x,\cdot)\,\psi_i(\cdot)$ is *lsc*,

$$L^{**}(x,s,\xi) - \varphi_i(x,s) = \rho_2(x,s)\,[\,\ell^{**}(s,\rho_1(x,s)\,\xi) - \psi_i(s)\,]\,,$$

hence

$$L^{**}(\cdot, s, \cdot) - \varphi_i(\cdot, s) = \rho_2(\cdot, s) \left[\ell^{**}(s, \rho_1(\cdot, s) \cdot) - \psi_i(s) \right] \quad \text{is } \; lsc \; \forall s \, ;$$

and since we also have

$$(\varphi_i(x, s)) = \rho_2(x, s) \, (\psi_i(s)) \nearrow \rho_2(x, s) \, \ell^{**}(s, 0) = L^{**}(x, s, 0) \qquad \forall x, s,$$

$(H_{3.1})$ holds true.

On the other hand, for each $(x, s) \in (\Omega' {\times} \mathbb{R})$,

$$\begin{aligned} &|m_i(x, s)| \\ &\qquad = \rho_1(x, s) \, \rho_2(x, s) \, |\mu_i(s)| \\ &\qquad \leq i \; \sup \left(\rho_1 \rho_2\right) (\Omega' {\times} \mathbb{R}) \, , \end{aligned}$$

hence $M_i^{\Omega'}(\cdot) \in L^1_{loc}(\mathbb{R})$; while

$$m_i(\cdot, s) = \mu_i(s) \, \rho_1(\cdot, s) \, \rho_2(\cdot, s) \in C^0 \left(\Omega, \mathbb{R}^d\right)$$

and, since $|\mu_i(s)| \leq i \;\; \forall \, s, i$ and $(\rho_1 \rho_2)(\cdot, \cdot)$ is *uniformly continuous* on $(\Omega' {\times} \mathrm{V})$, the *family* of *functions*

$$\{ \, m_i(\cdot, s) \; : \; s \in \mathrm{V} \, \} = \{ \, \mu_i(s) \rho_1(\cdot, s) \rho_2(\cdot, s) \; : \; s \in \mathrm{V} \, \}$$

is *equicontinuous* in $C^0 \left(\Omega', \mathbb{R}^d\right)$.

Therefore also $(H_{3.2})$ holds true, and lemma 2.4.1 is proved. $\square$

Chapter 3

Radial symmetry in convex case : regularity of scalar minimizers

3.1 Introduction

Recall that in several different areas of *physics* and *engineering* (namely *nonlinear elasticity*, *fluid dynamics*, *syntectic materials*, *shape optimization*, *liquid crystals*, ..., see e.g. [Bau & Phi], [Cell & Orn], [Goo & Koh & Rey], [Koh & Str], [Kro & Kie], [Mcl], [Orn & Ped], [Tah]) often *physical* or *geometrical problems* are reduced to adequate *mathematical models* involving the *minimization* of an *integral* of the *calculus* of *variations*. Since in general this problems are typically *vectorial*, as a first step we consider, in this chapter, only the *scalar* case; namely the problem of minimization of the *integral* :

$$\int_{B_R} L^{**}\left(|x| , u(x), |\nabla u(x)| \right) \, dx \quad \text{on} \quad W_A^{1,1}\left(B_R \right), \tag{3.1}$$

($|\nabla u(x)|$ = *euclidian norm* of the *gradient* vector) with *lagrangian* $L^{**} : [0,R]\times\mathbb{R}\times\mathbb{R} \to [0,\infty]$ having at least $L^{**}(r,\cdot,\cdot)$ *convex lsc* (*lower semicontinuous*); and where the *class* of *functions* in competition is the *Sobolev space* $W_A^{1,1} := A + W_0^{1,1}\left(B_R \right)$ of those $u(\cdot)$ defined on the *ball* $B_R := \left\{ x \in \mathbb{R}^d : |x| < R \right\}$ and taking *constant value* $A \in \mathbb{R}$ along its *boundary* ∂B_R.

In these problems, (beyond *existence*) *simplicity* of the *solution* is often a highly desirable goal; in particular often the aim is to prove *existence* of a *radial* (or *radially symmetric*) *minimizer* $u_A(\cdot)$, i.e. one whose *values* $u_A(x)$ are *constant* along each *spherical layer* of B_R, namely

$$u_A(x) = U_A\left(|x| \right) \quad \text{on} \quad \overline{B}_R, \tag{3.2}$$

for an adequate "*profile*" $U_A : [0,R] \to \mathbb{R}$ as simple as possible.

In the literature one finds several theoretical results in which *radial minimizers* have been obtained, under increasingly weaker hypotheses, see e.g. [Cell & Per], [Cra 1], [Cra 2], [Cra & Mal 1], [Cra & Mal 2], [Kro], [Kro & Kie]. In general, (3.2) is reached via a *symmetric rearrangement* (i.e. *averaging* over each *spherical layer* of B_R, see (3.36) & (3.35)), applied to some *minimizer* $u(\cdot)$ of (3.1) in order to obtain a new *radial minimizer* $u_A(\cdot)$.

However, while all of the above papers assume *separation* of *state* & *gradient variables*,

$$L^{**}\left(r, s, |v| \right) = g\left(r, s \right) + h^{**}\left(r, |v| \right), \tag{3.3}$$

here we avoid it, under joint convexity ; namely we consider any *lagrangian* $L^{**}(\cdot)$ of the special form

$$L^{**}\,(\,r, s, |v|\,) = \ell^{**}\,(\,s, |v| \quad \rho_1(r)\,) \quad . \quad \rho_2(r), \tag{3.4}$$

i.e. we want to *minimize* the *integral*

$$\int_{B_R} \ell^{**}(\,u(x), |\,\nabla\, u(x)\,| \quad \rho_1(\,|x|\,)\,) \;\;.\;\; \rho_2(\,|x|\,)\, d\,x \quad \text{on} \;\; W_A^{1,1}\,(\,B_R\,)\,, \tag{3.5}$$

where $\ell^{**} : \mathbb{R}{\times}\mathbb{R} \to [0, \infty]$ is any *convex lsc superlinear function*, namely :

$$\frac{\inf\; \ell^{**}\,(\,\mathbb{R}, \lambda\,)}{\lambda} \to \infty \qquad \text{as} \;\; \lambda \to \infty\,, \tag{3.6}$$

but we need no *growth* in case the above *integral* is known to have *minimum.*

Notice also that our *minimizers* are, unlike those of previous papers, *bounded* & *uniformly continuous* near the *center* 0 of the *ball* B_R, under a very weak extra hypothesis (3.16).

Remarkably, (3.5) may also be seen as the *calculus* of *variations* reformulation of a *distributed parameter scalar optimal control problem.* Indeed, *state* & *gradient pointwise constraints* are, in a sense, built-in, since $\ell^{**}(s, v) = \infty$ is freely allowed (see e.g. [Ces]).

It is admissible any $A \in \mathbb{R}$ for the *constant boundary value* assumed by the competing *functions* $u(\cdot)$ along $\partial\, B_R$, in contrast with the above papers, which set $A \equiv 0$; notice that our *minimizing profile* $U_A(\cdot)$ *depends* on the *position* of the *point* A relative to *graph* $\ell^{**}(\cdot, 0)$. Indeed, e.g. if $\ell^{**}\,(\,A, 0\,) \;\leq\; \ell^{**}\,(s, 0) \;\; \forall\, s$ then obviously the *constant function* $\equiv A$ *minimizes* the *integral* (3.5).

Useful properties of *regularity* of *minimizers* are here obtained in case $\exists\, \min \ell^{**}\,(\,\mathbb{R}, 0\,)$: $U_A(\cdot)$ has to be *monotone* & *AC* (*absolutely continuous*) with $\ell^{**}\,(\,U_A(\cdot), 0\,)$ *increasing* along $[\,0, R\,]$, so that again the choice of A determines the behaviour of *minimizers* :

$$sign\; U_A'(\cdot) = sign\; \partial\, \ell^{**}\,(\,A, 0\,)\,.$$

3.2 Radial minimizing surfaces and deformations

Here is our *existence* result :

Theorem 3.2.1. (*Existence of radial minimizers*)
Let

$$\ell^{**} : \mathbb{R}^m \times \mathbb{R} \to [0, \infty] \quad \textit{be convex lsc with} \quad \ell^{**}(S, \cdot) \quad \textit{even} \quad \forall\, S, \tag{3.7}$$

and let

$$\rho_1,\, \rho_2 \,:\, [\,0, R\,] \to [c_0, c_\infty] \subset (0, \infty) \quad \textit{be Borel measurable.} \tag{3.8}$$

Then :

$$\exists \textit{ radial minimizer } \; u_A(x) = U_A(\,|x|\,) \tag{3.9}$$

for the multiple integral

$$\int_{B_R} \ell^{**}(\,u(x), |\,D\,u(x)\,| \quad \rho_1(\,|x|\,)\,) \quad . \quad \rho_2(\,|x|\,)\; d\,x \quad \textit{on} \;\; W_A^{1,1}\,(\,B_R,\, \mathbb{R}^m\,) \tag{3.10}$$

if and only if

$$\textit{the integral} \;\; (3.10) \;\; \textit{has minimum}; \tag{3.11}$$

if and only if

$$\exists \textit{ minimizer } \; z_A(\cdot) \tag{3.12}$$

for its associated single integral (using (3.30) & (3.31))

$$\int_0^a \ell^{**}(\,z(t), |\,z'(t)\,| \quad \rho(t)\,)\; d\,t \qquad \textit{on} \;\; \mathcal{Z}_A^{0,a}, \tag{3.13}$$

where $a > 0$ & $\rho(\cdot)$ & $\gamma(\cdot)$ *are adequately chosen and define the new space*

$$\mathcal{Z}_A^{0,a} := \left\{ z(\cdot) \in W_{loc}^{1,1}\,(\,(0, a], \mathbb{R}^m\,) : \begin{array}{c} z(a) = A \qquad \& \\ \\ |z'(\cdot)| \;\; \gamma(\cdot)^{d-1} \in L^1(0, a) \end{array} \right\}. \tag{3.14}$$

Moreover

$$\ell^{**}(\cdot) \;\; \textit{superlinear} \quad (\,\textit{as in} \;\; (3.6)\,) \qquad \Rightarrow \qquad (3.9)\ \&\ (3.12). \tag{3.15}$$

Here is the main result of this section, about *regularity* of *minimizers* in the *scalar* $m = 1$ case:

Theorem 3.2.2. (*Regularity in scalar* $m = 1$ *case under extra hypothesis*) *Assume* (3.7) & (3.8) & (*either* (3.11) *or* (3.6)) *plus the extra hypothesis*

$$m = 1 \qquad \& \qquad \exists \min \ell^{**}(\mathbb{R}, 0) . \tag{3.16}$$

Then

$$u_A(\cdot) \ \ \textit{in} \ \ (3.9) \qquad \textit{is uniformly continuous} \tag{3.17}$$

while

$$U_A(\cdot) \ \& \ z_A(\cdot) \qquad \textit{in} \ \ (3.9) \ \& \ (3.12) \ \ \textit{are AC monotone with} \tag{3.18}$$

$$\ell^{**}(U_A(\cdot), 0) \ \ \& \ \ \ell^{**}(z_A(\cdot), 0) \qquad \textit{both increasing.} \tag{3.19}$$

Finally we present a quite trivial *regularity* result for the *vectorial* $m \geq 1$ case. Define:

$$\Sigma_A := \{ S \in \mathbb{R}^m \ : \ \ell^{**}(S, 0) \leq \ell^{**}(A, 0) \} , \tag{3.20}$$

$$q_A := \frac{1}{c_0} \sup \left\{ \frac{|\xi|}{c_1 + c_2 |S|} \ : \ S \in \Sigma_A \ \ \& \ \ \ell^{**}(S, \xi) < \infty \right\} , \tag{3.21}$$

and

$$q := \frac{1}{c_0} \sup \left\{ \frac{|\xi|}{c_1 + c_2 |S|} \ : \ \ell^{**}(S, \xi) < \infty \right\} . \tag{3.22}$$

(Here c_1 & c_2 are any *constants* > 0, while c_0 appears in (3.8).)

Theorem 3.2.3. (*Lipschitz regularity in vector* $m \geq 1$ *case of optimal control or design, under extra hypothesis*)

Assume (3.7) & (3.8) & (*either* (3.11) *or* (3.6)) *plus the extra hypothesis*

$$q < \infty \qquad\qquad (\, see \quad (3.22)\,). \tag{3.23}$$

Then

$$u_A(\cdot) \ \& \ U_A(\cdot) \quad in \quad (3.9) \quad are \quad Lipschitz\ continuous \tag{3.24}$$

while

$$z_A(\cdot) \quad in \quad (3.12) \quad is \quad AC \quad on \quad [0,a] \quad \& \quad locally\ Lipschitz \quad on \quad (0,a]. \tag{3.25}$$

Moreover,

$$m = 1 \quad \& \quad q_A < \infty \quad (\, see \quad (3.21)\ \&\ (3.20)\,) \quad \Rightarrow \quad (3.24)\ \&\ (3.25). \tag{3.26}$$

Remark 3.2.1. *More precisely, we prove below that, under the hypotheses of th. 3.2.2 or 3.2.3,*

$$U_A\,(\,[\,0,R_A\,]\,) = \{\,B\,\} = z_A\,(\,[0,b]\,) \qquad (\, possibly \quad R_A = 0 = b\,) \tag{3.27}$$

where

$$b := \max\{\, t \in [0,a] \ :\ \ell^{**}\,(\,z_A(t),0\,) = \min\, \ell^{**}\,(\,z_A\,(\,[0,a]\,)\,,0\,)\,\}\,, \tag{3.28}$$

$$B := z_A(b), \quad R_A := \gamma(b), \quad z_A(t) = U_A \circ \gamma(t) \qquad \forall\, t\,; \tag{3.29}$$

and, in (3.13) & (3.14),

$$\rho(t) := \rho_1\,(\,\gamma(t)\,)\ \rho_2\,(\,\gamma(t)\,)\ \gamma(t)^{d-1} \qquad \forall\, t\,, \tag{3.30}$$

using

$$\gamma : [0,a] \to [\,0,R\,]\,,\ r = \gamma(t),\ the\ inverse\ function\ of \tag{3.31}$$

$$r \mapsto t = \gamma^{-1}(r) := \int_0^r \rho_2(\tau) \quad \tau^{d-1}\ d\,\tau \in [0,a]\,, \tag{3.32}$$

with

$$a := \int_0^R \rho_2(r) \quad r^{d-1}\ d\,r\,. \tag{3.33}$$

Proof.

(a) Let us assume that

$$\text{the } \textit{integral} \ (3.10) \ \text{ is } \textit{finite} \ \text{ at some } \ u(\cdot) \in W_A^{1,1}\,(\,B_R,\,\mathbb{R}^m\,)\,. \tag{3.34}$$

To each such competing $u(\cdot)$, as in (3.34), associate the corresponding:

$$\text{“}\textit{radial symmetrization}\text{”} \qquad \overline{u}(x) := U\,(\,|x|\,)\,, \tag{3.35}$$

using the

$$\text{“}\textit{radial plane cut mean profile}\text{”} \qquad U(r) := \frac{1}{\alpha_d}\int_{S^d} u\,(\,\omega\; r\,)\; d\omega, \tag{3.36}$$

where α_d is the *Hausdorff measure*, in dimension $d-1$, of the *unit sphere* $S^d := \left\{\, x \in \mathbb{R}^d \;:\; |x| = 1 \,\right\}$.

Claim 1

$$\int_{B_R} |\,D\,\overline{u}(x)\,| \; dx \leq \int_{B_R} |\,D\,u(x)\,| \; dx. \tag{3.37}$$

Indeed (e.g. as in [Cell & Per])

$$\nabla\overline{u}_i(x) = \frac{1}{\alpha_d}\,\frac{x}{|x|}\int_{S^d}\langle\,\nabla u_i(\,\omega\;\;|x|\,),\,\omega\,\rangle\; d\omega\,; \tag{3.38}$$

and since

$$U_i'(r) = \frac{1}{\alpha_d}\int_{S^d}\langle\,\nabla u_i(\,\omega\; r\,),\,\omega\,\rangle\; d\omega,$$

hence

$$U'(r) = \frac{1}{\alpha_d}\int_{S^d} D\,u(\,\omega\; r\,)\;\omega\; d\omega\,, \tag{3.39}$$

one gets $|\,\nabla\overline{u}_i(x)\,| = |\,U_i'(\,|x|\,)\,|$ hence

$$|\,D\,\overline{u}(x)\,| = |\,U'(\,|x|\,)\,| \leq \frac{1}{\alpha_d}\int_{S^d} |\,D\,u(\,\omega\;\;|x|\,)\,| \; d\omega. \tag{3.40}$$

Therefore

$$\int_{B_R} |\,D\,\overline{u}(x)\,|\;\;d\,x = \int_{B_R} |\,U'(\,|x|\,)\,|\;\;d\,x = \alpha_d \int_0^R |\,U'(r)\,| \quad r^{d-1}\;d\,r \le$$

$$\le \int_0^R \int_{S^d} |\,D\,u(\,\omega\;r\,)\,|\;\;d\,\omega \quad r^{d-1}\;d\,r = \int_{B_R} |\,D\,u(x)\,|\;\;d\,x,$$

through a *radial* $-$ *spherical change* of *variables* (see e.g. [Yeh, th. 26.19, 26.20]), thus proving (3.37), i.e. claim 1.

Claim 2

Radial symmetrization lowers the value of the integral (3.41)

$$I_\psi\,(\,u(\cdot)\,) := \int_{B_R} \psi\,(\,|x|\,,\,u(x),\,|\,D\,u(x)\,|\,)\;\;d\,x \quad \text{on} \quad W_A^{1,1}\,(\,B_R,\,\mathbb{R}^m\,) \tag{3.42}$$

i.e., with $\overline{u}(\cdot)$ *obtained from* $u(\cdot)$ *via* (3.35) & (3.36),

$$I_\psi\,(\,\overline{u}(\cdot)\,) \le I_\psi\,(\,u(\cdot)\,) \qquad \forall\,u(\cdot) \in W_A^{1,1}\,(\,B_R,\,\mathbb{R}^m\,)\,, \tag{3.43}$$

for any general lagrangian

$$\psi : [\,0,R\,] \times \mathbb{R}^m \times \mathbb{R} \to [0,\infty] \qquad \mathcal{L}\otimes\mathcal{B}\otimes\mathcal{B} - \textit{measurable} \tag{3.44}$$

$$\textit{having} \quad \psi\,(t,\cdot,\cdot) \quad \textit{convex lsc} \quad \textit{and} \quad \psi\,(t,S,\cdot) \quad \textit{even.} \tag{3.45}$$

Indeed, we have (generalizing arguments from [Cell & Per]), by (3.40),

$$I_\psi\,(\,\overline{u}(\cdot)\,) = \int_{B_R} \psi\,(\,|x|\,,\,\overline{u}(x),\,|\,D\,\overline{u}(x)\,|\,)\;\;d\,x$$

$$= \int_{B_R} \psi\left(\,|x|\,,\,\frac{1}{\alpha_d}\int_{S^d} u\,(\,\omega\;\;|x|\,)\;\;d\,\omega,\,|\,D\,\overline{u}(x)\,|\,\right)\;d\,x$$

$$\le \int_{B_R} \psi\left(\,|x|\,,\,\frac{1}{\alpha_d}\int_{S^d} (\,u\,(\,\omega\;\;|x|\,)\,,\,|\,D\,u\,(\,\omega\;\;|x|\,)\,|\,)\;\;d\,\omega\right)\;d\,x$$

$$\le \int_{B_R} \frac{1}{\alpha_d}\int_{S^d} \psi\,(\,|x|\,,\,(\,u\,(\,\omega\;\;|x|\,)\,,\,|\,D\,u\,(\,\omega\;\;|x|\,)\,|\,))\;\;d\,\omega\;d\,x\,,$$

by (3.45) & *Jensen inequality* (valid also for *convex lsc functions* assuming the ∞ *value*, see e.g. [Dac]); so that, again by *radial − spherical changes* of *variables* and setting

$$\phi_u(x) := \psi\,(\,|x|\,, u(x),\, |\,D\,u(x)\,|\,)\,, \qquad \overline{\phi}_u(r) := \frac{1}{\alpha_d}\int_{S^d}\phi_u(\,\omega\ r\,)\ d\,\omega\,, \tag{3.46}$$

$$I_\psi\,(\,\overline{u}(\cdot)\,) \le \int_{B_R}\frac{1}{\alpha_d}\int_{S^d}\phi_u\,(\,\omega\ \ |x|\,)\ \ d\,\omega\ d\,x =: \int_{B_R}\overline{\phi}_u\,(\,|x|\,)\ \ d\,x$$

$$= \alpha_d\int_0^R\overline{\phi}_u\,(r)\qquad r^{d-1}\ d\,r := \ \alpha_d\int_0^R\frac{1}{\alpha_d}\int_{S^d}\phi_u(\,\omega\ r)\ d\,\omega\qquad r^{d-1}\ d\,r$$

$$= \int_0^R\int_{S^d}\phi_u(\,\omega\ r)\ d\,\omega\qquad r^{d-1}\ d\,r = \int_{B_R}\phi_u(x)\ d\,x$$

$$:= \int_{B_R}\psi\,(\,|x|\,,\,u(x),\,|\,D\,u(x)\,|\,)\ \ d\,x =:\ I_\psi\,(\,u(\cdot)\,)$$

by (3.42) & (3.46); thus proving (3.43), i.e. claim 2.

Naturally, in particular with

$$\psi\,(\,t, S, r\,) := \ell^{**}\,(\,S,\ r\qquad \rho_1(t)\,)\qquad .\qquad \rho_2(t)\,, \tag{3.47}$$

one gets, from (3.41) & (3.43),

radial symmetrization lowers the *value* of the *integral* (3.10), i.e.

$$\int_{B_R}\ell^{**}(\,\overline{u}(x), |\,\nabla\,\overline{u}(x)\,|\quad \rho_1(\,|x|\,)\,)\ \ .\ \ \rho_2(\,|x|\,)\ d\,x \le \tag{3.48}$$

$$\le \int_{B_R}\ell^{**}(\,u(x), |\,\nabla\,u(x)\,|\quad \rho_1(\,|x|\,)\,)\ \ .\ \ \rho_2(\,|x|\,)\ d\,x, \tag{3.49}$$

with $\overline{u}(\cdot)$ the *radial symmetrization* of $u(\cdot)$, as in (3.35) & (3.36).

Claim 3 *The problem of minimizing the general integral* (3.42) *is equivalent to the problem of minimizing the general integral*

$$\int_0^R \psi\,(\,r, U(r), |\,U'(r)\,|\,) \quad r^{d-1}\,d\,r \qquad on\ \ \mathcal{U}_A^{0,R}, \tag{3.50}$$

$$\mathcal{U}_A^{0,R} := \left\{ U(\cdot) \in W_{loc}^{1,1}\,(\,(0,R]\,,\,\mathbb{R}^m\,) : \begin{array}{c} U(\,R\,) = A \qquad \& \\ r \to r^{d-1}\ \ |\,U'(r)\,| \in L^1(\,0,R\,) \end{array} \right\}. \tag{3.51}$$

Indeed, defining by (3.36) the *profile* $U(\cdot)$ associated to each given $u(\cdot) \in W_A^{1,1}\,(\,B_R,\,\mathbb{R}^m\,)$, one gets $U(\cdot) \in \mathcal{U}_A^{0,R}$ (as seen above, after (3.40)); while conversely, defining by (3.35) the *mean* $\overline{u}(\cdot)$ associated to each given *profile* $U(\cdot) \in \mathcal{U}_A^{0,R}$, one gets $\overline{u}(\cdot) \in W_A^{1,1}\,(\,B_R,\,\mathbb{R}^m\,)$ (again as after (3.40)). Therefore:

$$u(\cdot) \in W_A^{1,1}\,(\,B_R,\,\mathbb{R}^m\,) \quad \Rightarrow \quad \overline{u}(\cdot) \in W_A^{1,1}\,(\,B_R,\,\mathbb{R}^m\,) \quad \Leftrightarrow \quad U(\cdot) \in \mathcal{U}_A^{0,R}. \tag{3.52}$$

Moreover, by (3.35) & (3.40),

$$\begin{aligned} &\int_{B_R} \psi\,(\,|x|\,, \overline{u}(x), |\,D\,\overline{u}(x)\,|\,)\ d\,x \\ &= \int_{B_R} \psi(\,|x|\,, U(\,|x|\,), |\,U'(\,|x|\,)\,|\,)\,d\,x \\ &= \alpha_d \int_0^R \psi(\,r, U(r), |\,U'(r)\,|\,) \quad r^{d-1}\ d\,r, \end{aligned}$$

so that (3.42) & (3.50) are, by (3.43), *equivalent*. This proves claim 3.

A special case of the *equivalence* proved in claim 3 is, naturally, again using $\psi(\cdot)$ as in (3.47), *equivalence* between the *problems* of *minimizing* the *integral* (3.10) and of *minimizing* the *integral*

$$\int_0^R \ell^{**}\,(\,U(r), |\,U'(r)\,| \quad \rho_1(r)\,) \quad . \quad \rho_2(r)\,r^{d-1}\,d\,r \quad \text{on}\ \ \mathcal{U}_A^{0,R}. \tag{3.53}$$

Our next step consists in changing this integral (3.53) into another equivalent and still more convenient form.

Claim 4 *There exists a bijective change of variable* $\gamma : [0,a] \to [\,0,R\,]$ *(defined in* (3.31) *) such that, setting* $z(t) := U\,(\,\gamma(t)\,)$ *for each* $U(\cdot) \in \mathcal{U}_A^{0,R}$ *as in* (3.51)*, and defining* $\rho(t)$ *as in* (3.30)*, then the problem of minimizing the integral* (3.10) *(or* (3.53) *) is equivalent to the problem of minimizing the associated single integral* (3.13)*.*

Indeed, to begin with, since, by (3.8),

$$\rho_2 : [\,0,R\,] \to [\,c_0, c_\infty\,] \subset (0,\infty) \qquad \text{is } \textit{Borel measurable}, \tag{3.54}$$

$$0 < c_0 \;\; \tau^{d-1} \le \rho_2(\tau) \;\; \tau^{d-1} \le c_\infty \;\; R^{d-1} < \infty \qquad \forall\,\tau \in (\,0,R\,), \tag{3.55}$$

in particular

$$\tau \mapsto \rho_2(\tau) \quad \tau^{d-1} \qquad \text{is } \textit{Lebesgue integrable}, \tag{3.56}$$

defining the new *function*

$$\gamma^{-1} : [\,0,R\,] \to [0,a]\,, \qquad \gamma^{-1}(r) := \int_0^r \rho_2(\tau) \quad \tau^{d-1}\; d\,\tau \tag{3.57}$$

we get, by [Leo, 3.31] or [Yeh, 13.17 & 13.15],

$$\gamma^{-1}(\cdot) \;\text{ is }\; AC \;\;\&\;\; \gamma^{-1\prime}(r) = \rho_2(r) \;\; r^{d-1} \in \big(\,0,\; c_\infty \;\; R^{d-1}\,\big] \subset (0,\infty) \tag{3.58}$$

for a.e. $r \in [\,0,R\,]$; $\gamma^{-1}(\cdot)$ is even *Lipschitz*, by (3.58). Since $\gamma^{-1}(\cdot)$ also *increases strictly* and has, by (3.33), $\gamma^{-1}\,(\,[\,0,R\,]\,) = [\,0,a\,]$, its *inverse function*

$$\gamma : [0,a] \to [\,0,R\,] \qquad \text{is } \textit{continuous strictly increasing} \text{ onto} \tag{3.59}$$

$$\& \qquad \gamma\,(\,\mathcal{N}\,) \qquad \text{is a } \textit{null set} \text{ for each } \textit{null set } \mathcal{N} \text{ in } [0,a]. \tag{3.60}$$

In fact, if $E_0 := \gamma\,(\,\mathcal{N}_0\,)$ had *positive measure*, for some *null set* $\mathcal{N}_0$, then $\gamma^{-1}\,(\,E_0\,) = \mathcal{N}_0$ would be a *null set* hence $\gamma^{-1\prime}(r) = \rho_2(r) \;\; r^{d-1}$ would, by [Leo, 3.45], be *zero* a.e. on E_0, absurd (see (3.58)). Such absurd proves Lusin's condition (3.60); which, together with (3.59), shows that, by [Yeh, th. 13.8] (see also [Spa]),

$$\gamma : [0,a] \to [\,0,R\,] \qquad \text{is} \quad \textit{AC strictly increasing.} \tag{3.61}$$

Moreover, $\gamma(\cdot)$ is *locally Lipschitz* on $(0,a]$: for each $\varepsilon \in (\,0,R\,)$,

$$\gamma^{-1\,\prime}(r) = \rho_2(r) \quad r^{d-1} \in \left[\,c_0\,\varepsilon^{\,d-1},\ c_\infty\ R^{\,d-1}\right] \subset (0,\infty) \qquad \text{a.e. on} \quad [\,\varepsilon,R\,],$$

so that, a.e. on $[\,\gamma^{-1}(\varepsilon),a\,]$,

$$\gamma'(t) = \frac{1}{\rho_2\,(\,\gamma(t)\,)\ \gamma(t)^{d-1}} \in \left[\,\frac{R^{\,1-d}}{c_\infty},\frac{\varepsilon^{1-d}}{c_0}\,\right] \subset (0,\infty). \tag{3.62}$$

Defining, for each $U(\cdot) \in \mathcal{U}_A^{0,R}$ (recall (3.53) & (3.51)),

$$z : [0,a] \to \mathbb{R}^m, \qquad\qquad z(t) := U\,(\,\gamma(t)\,)\,, \tag{3.63}$$

then, since $U(\cdot)$ is *locally AC* on $(\,0,R\,]$ and (3.61) & (3.62) hold true, by [Leo, 3.50 & 3.51],

$$\begin{gathered} z(\cdot) \quad \text{is} \quad \textit{locally AC} \qquad\qquad \& \\ z'(t) = U'\,(\,\gamma(t)\,)\ \frac{1}{\rho_2\,(\,\gamma(t)\,)\ \gamma(t)^{d-1}} \qquad \text{a.e. on} \quad (0,a]\,; \end{gathered} \tag{3.64}$$

hence changing the *variable* of *integration* we obtain, by (3.62) & (3.64), due to (3.61) & [Leo, 3.57],

$$\begin{aligned} &\int_0^R |\,U'(r)\,| \quad r^{d-1}\ d\,r = \\ &= \int_0^a \frac{|\,U'\,(\,\gamma(t)\,)\,|}{\rho_2\,(\,\gamma(t)\,)}\,dt = \int_0^a |\,z'(t)\,| \quad \gamma(t)^{d-1}\ d\,t\,, \end{aligned} \tag{3.65}$$

so that, by (3.51) & (3.54) & (3.14), $z(\cdot) \in \mathcal{Z}_A^{0,a}$:

$$U(\cdot) \in \mathcal{U}_A^{0,R} \quad \Rightarrow \quad |\,U'\,(\,\gamma(\cdot)\,)\,| \in L^1(0,a) \quad \Rightarrow \quad |\,z'(\cdot)\,|\ \gamma(\cdot)^{d-1} \in L^1(0,a).$$

Conversely, for any $z(\cdot) \in \mathcal{Z}_A^{0,a}$, setting $U(r) := z\,(\,\gamma^{-1}(r)\,)$, one similarly gets, again by [Leo, 3.50 & 3.51], that the first *integral* in (3.65) is *finite* and

$$U(\cdot) \in W^{1,1}_{loc}\,(\,(0,R],\,\mathbb{R}^m\,)\,, \qquad \text{i.e.} \quad U(\cdot) \in \mathcal{U}_A^{0,R}. \tag{3.66}$$

Moreover, the above *change* of *variables* also gives, by (3.63) & (3.64) & (3.62) & (3.47) & (3.30),

$$\int_0^R \psi\,(\,r, U(r), |\,U'(r)\,|\,) \quad r^{d-1}\,dr$$

$$= \int_0^a \psi\,\big(\,\gamma(t), z(t), |\,z'(t)\,| \quad \rho_2(\gamma(t)\,)\;\gamma(t)^{d-1}\big) \quad . \quad \frac{1}{\rho_2\,(\,\gamma(t)\,)}\,dt$$

$$= \int_0^a \ell^{**}(\,z(t), |\,z'(t)\,| \quad \rho(t)\,)\,dt,$$

i.e. the *integral* (3.13). Thus the proof of claim 4 is complete, since this *integral* $\int_0^R \psi\,(\,r, U(r), |\,U'(r)\,|\,) \quad r^{d-1}\,dr$, with $\psi(\cdot)$ as in (3.47), is (as proved above, just before (3.53)) *equivalent* to the integral (3.10).

(b) Thus the trivial th. 3.2.1 is proved; and we prove now the *regularity* properties for *minimizers* stated in theorems 3.2.2 & 3.2.3.

Let

$$z_A(\cdot) \in \mathcal{Z}_A^{0,a} \qquad \textit{minimize} \ \text{the} \ \textit{integral} \ (3.13). \tag{3.67}$$

Claim 5 *Recalling* (3.67) & Σ_A *defined in* (3.20),

$$\Sigma_A \cap z_A\,(\,(0,a]\,) \ \ \textit{bounded} \qquad \Rightarrow \qquad b \ \text{ is well defined in } \ (3.28) \tag{3.68}$$

since

$$\exists\; i_a := \min\, \ell^{**}\,(\,z_A\,(\,[0,a\,]\,,0\,)\,. \tag{3.69}$$

Indeed, clearly there exists a *minimizing sequence* $(S_k) \subset \Sigma_A \cap z_A\,(\,(0,a]\,)$, i.e. $(\,\ell^{**}\,(\,S_k, 0\,)\,) \searrow i_a$; and one may assume that

$$(\,S_k\,) \quad \textit{converges} \quad \text{to some} \quad B_a \in \Sigma_A \cap \overline{z_A\,(\,(0,a]\,)}\,, \tag{3.70}$$

since this *set* is *compact*. Thus (defining, if needed, the *value* of $z_A(\cdot)$ at $t = 0$) one may also assume that

$$\exists\, b_a \in [0,a] \;:\; \; B_a = z_A(b_a) \quad \& \quad \ell^{**}\,(\,z_A(b_a), 0\,) = i_a\,. \tag{3.71}$$

Indeed, this is the same as saying that whenever

$$\ell^{**}\,(\,z_A(t), 0\,) > \inf\, \ell^{**}\,(\,z_A\,(\,(0,a]\,)\,, 0\,) \qquad \forall\, t \in (0,a] \tag{3.72}$$

one may define (or change) $z_A(\cdot)$ at $t = 0$ so as to become

$$\ell^{**}\,(\,z_A(\cdot), 0\,) \qquad\qquad lsc \quad \text{on} \quad [0,a] \qquad \& \tag{3.73}$$

$$\ell^{**}\,(\,z_A(0), 0\,) = i_a \qquad\qquad \& \qquad\qquad b_a = 0\,; \tag{3.74}$$

so that (3.69) & (3.71) become true (even if $z_A(\cdot)$ is/becomes *discontinuous* at $t = 0$). This proves (3.69) hence (3.68) & claim 5.

Claim 6 *Recalling* $q_{_A}$ & q *defined respectively in* (3.21) & (3.22)

$$q < \infty \quad \text{or} \quad \Sigma_A \textit{ bounded} \quad \Rightarrow \quad b \text{ is well defined in } (3.28), \tag{3.75}$$

$$m = 1 \quad \& \quad q_{_A} < \infty \quad \Rightarrow \quad b \text{ is well defined in } (3.28). \tag{3.76}$$

Indeed, due to (3.68), the *implication* (3.75) is obvious in case Σ_A is *bounded*. Let us prove now (3.75) in case $q < \infty$: since, by (3.67) & (3.34) & claim 4, $U_A(r) := z_A\,(\,\gamma^{-1}(r)\,)$ satisfies

$$\int_0^R \ell^{**}\,(\,U_A(r), |\,U_A'(r)\,| \quad \rho_1(r)\,) \quad . \quad \rho_2(r)\; r^{d-1}\; d\,r < \infty,$$

so that

$$\ell^{**}\,(\,U_A(r), |\,U_A'(r)\,| \quad \rho_1(r)\,) < \infty \quad \text{for a.e.} \quad r \in [\,0, R\,]\,;$$

and since, by the definition (3.22) of $q < \infty$, this *implies*

$$
\begin{array}{c}
|\, U_A'(r)\,| \leq \; q \; \left(c_1 + c_2 \,|\, U_A(r)\,|\, \right) \leq \; \left(1 + c_2 \,\right) q \; |\, U_A(r)\,| \\
\text{a.e. where} \qquad\qquad |\, U_A(r)\,| \geq c_1 \, ,
\end{array}
\tag{3.77}
$$

setting $\alpha(r) := \log\left(\,|\, U_A(r)\,|\,\right)$ we have $|\,\alpha'(r)\,| \leq (1 + c_2\,)\, q$ where $\alpha(r) \geq \log(c_1)$; and since $\alpha(r) = \alpha\,(\,R\,) + \int_R^r \alpha'(\tau)\; d\,\tau$, then certainly $\alpha(r) \leq \log\left(\,1 + c_1\,\right) + \log\left(\,1 + |\,A\,|\,\right) + (1 + c_2\,)\, q\, R$, so that $U_A\left(\,(\,0, R\,]\,\right) = z_A\left(\,(0, a]\,\right)$ must be *bounded* and (3.75) follows from (3.68).

Such arguments also prove (3.76), since

$$
m = 1 \quad \Rightarrow \quad z_A\left(\,(0, a]\,\right) \subset \Sigma_A \quad (\,\text{see} \;\; (3.20)\; \&\; (3.21)\; \&\; (3.22)\,). \tag{3.78}
$$

In fact, the *open set*

$$
\mathcal{O} := \{\, t \in (0, a) \; : \; z_A(t) \in \mathbb{R} \backslash \Sigma_A \,\} \qquad\qquad \text{is} \;\; \textit{empty}, \tag{3.79}
$$

because: if (t_1, t_2) is *one* of its *maximal open intervals*, and one defines a new *function* $\widetilde{z}_A(\cdot) \in \mathcal{Z}_A^{0,a}$ by

$$
\widetilde{z}_A(t) := \begin{cases} z_A(t) & \text{for} \;\; t \notin [t_1, t_2] \\ A & \text{for} \;\; t \in [t_1, t_2], \end{cases}
$$

then

$$
\int_{t_1}^{t_2} \ell^{**}\left(\, \widetilde{z}_A(t), |\, \widetilde{z}_A'(t)\,| \quad \rho(t)\,\right)\, d\,t = \int_{t_1}^{t_2} \ell^{**}\left(\, A, 0\,\right)\, d\,t <
$$

$$
< \int_{t_1}^{t_2} \ell^{**}\left(\, z_A(t), 0\,\right)\, d\,t \leq \int_{t_1}^{t_2} \ell^{**}\left(\, z_A(t), |\, z_A'(t)\,| \quad \rho(t)\,\right)\, d\,t,
$$

in contradiction with (3.67). Such contradiction proves (3.79), hence (3.78) & claim 6.

Claim 7 *One may assume that*

$$
b \geq 0 \;\; \text{well defined in} \;\; (3.28) \quad \Rightarrow \quad z_A(\cdot) \equiv B := z_A(b) \;\; \text{on} \;\; [0, b] \tag{3.80}
$$

and

$$\ell^{**}(B,0) < \ell^{**}(z_A(\cdot),0) \quad \text{on} \quad (b,a]\,. \tag{3.81}$$

Indeed, one may set

$$\widetilde{z}_A(t) := \begin{cases} B := z_A(b) & \text{for } t \in [0,b] \\ \\ z_A(t) & \text{for } t \in [b,a], \end{cases} \tag{3.82}$$

obtaining, by (3.67) & (3.14), $\widetilde{z}_A(\cdot) \in \mathcal{Z}_A^{0,a}$ and (by (3.7) & (3.28) & (3.29))

$$\int_0^b \ell^{**}(\widetilde{z}_A(t), |\,\widetilde{z}_A'(t)\,| \quad \rho(t))\ dt = \int_0^b \ell^{**}(B,0)\ dt \leq$$

$$\leq \int_0^b \ell^{**}(z_A(t),0)\ dt < \int_0^b \ell^{**}(z_A(t), |\,z_A'(t)\,| \quad \rho(t))\ dt$$

(thus contradicting (3.67)) unless

$$\ell^{**}(z_A(\cdot), |\,z_A'(\cdot)\,| \quad \rho(\cdot)) \equiv \ell^{**}(B,0) \quad \text{on} \quad [0,b]\,. \tag{3.83}$$

Thus (3.83) holds true and $\widetilde{z}_A(\cdot)$ also *minimizes* the *integral* (3.13). Moreover, $\widetilde{z}_A(\cdot)$ satisfies (3.80).

On the other hand, (3.28) $\Rightarrow$ (3.81); so that (3.80) & (3.81) & claim 7 are proved.

Claim 8

$$m = 1 \quad \& \quad \exists \min \ell^{**}(\mathbb{R},0) \quad \Rightarrow \quad b \ \text{ is well defined in } \ (3.28). \tag{3.84}$$

Indeed, setting, by (3.16),

$$\Sigma_{\min} := \{\, s \in \mathbb{R} \,:\, \ell^{**}(s,0) = \min \ell^{**}(\mathbb{R},0)\,\} \tag{3.85}$$

and

$$C := \ \text{the } \textit{point} \text{ of } \Sigma_{min} \text{ closer to } A, \tag{3.86}$$

one may assume

$$z_A\,(\,(0,a]\,) \subset co\,\{\,C, A\,\} \quad \textit{bounded.} \tag{3.87}$$

In fact, unless $\ell^{**}\,(\,C, 0\,) = \ell^{**}\,(\,A, 0\,)$ (in which case $b = a$), since $\ell^{**}(\cdot, 0)$ *increases strictly* near A & outside of $co\{\,C, A\,\}$, as the *distance* from C *increases*, then, by (3.78), $z_A(t)$ cannot go out of $co\{\,C, A\,\}$ through A; while, by the same reasoning used to prove (3.80), going out of $co\{\,C, A\,\}$ through C brings no advantage to $z_A(t)$, since $\ell^{**}(\cdot, 0)$, hence $\ell^{**}(\cdot, \cdot)$, will not *decrease* further (see (3.80)).

Thus we may assume (3.87); hence (3.84) follows from (3.68).

Claim 9 *One may assume*

$$(3.16) \;\Rightarrow\; b \text{ is well defined in } (3.28) \;\Rightarrow\; z_A\,(\,[0,a]\,) \subset co\,\{\,B, A\,\}\,. \tag{3.88}$$

Indeed, by (3.84), b is well defined in (3.28); and, by (3.80) & (3.87) & (3.28) & (3.29), $z_A(\cdot)$ will not go out of $co\,\{\,B, A\,\}$.

Claim 10

$$(3.16) \qquad \Rightarrow \qquad z_A(\cdot) \quad \textit{is monotone and in } W^{1,1}\,(\,[0,a]\,), \tag{3.89}$$

$$\ell^{**}\,(\,z_A(\cdot), 0\,) \quad \textit{is minimal on} \quad [0,b] \quad \& \quad \textit{increases on} \quad [b,a]\,. \tag{3.90}$$

Let us prove this in case, say, $B < A$, since the case $B = A$ is obvious. Whenever one finds a *maximal interval* $[\,s^-, s^+\,]$ for which $B \le s^- \le s^+ \le A$ (recall (3.88)) and

$$T(s) := \{\,t \in [b,a] \;:\; w_A(t) = s\,\} \quad \text{has more than one} \quad \textit{point} \quad \forall\, s \in \left[\,s^-, s^+\,\right] \tag{3.91}$$

then, setting

$$t^- := \min T\left(\,s^-\,\right) \qquad \& \qquad t^+ := \max T\left(\,s^-\,\right)$$

and

$$\widetilde{z}_A(t) := \begin{cases} z_A(t) & \text{on } [\,0, t^-\,] \\ s^- & \text{on } [\,t^-, t^+\,] \\ z_A(t) & \text{on } [\,t^+, a\,]\,, \end{cases}$$

one gets $\widetilde{z}_A(\cdot) \in \mathcal{Z}_A^{0,a}$ and (unless $\widetilde{z}_A(\cdot) = z_A(\cdot)$)

$$\int_{t^-}^{t^+} \ell^{**}(\, z_A(t), |\, z'_A(t)\,| \quad \rho(t)\,)\; dt \geq \int_{t^-}^{t^+} \ell^{**}(\, z_A(t), 0)\; dt >$$

$$> \int_{t^-}^{t^+} \ell^{**}(s^-, 0)\; dt = \int_{t^-}^{t^+} \ell^{**}(\, \widetilde{z}_A(t), |\, \widetilde{z}'_A(t)\,| \quad \rho(t)\,)\; dt\,,$$

since $\ell^{**}(\cdot, 0)$ (being *convex*) *increases strictly* on $[\,B, A\,]$.

But since this *inequality* $>$ contradicts (3.67), one must have $\widetilde{z}_A(\cdot) = z_A(\cdot)$, i.e.

$$(3.91) \qquad \Rightarrow \qquad z_A(\cdot) \equiv s^- = s^+ \qquad \text{on} \quad \left[\,t^-, t^+\,\right]; \tag{3.92}$$

which (applied to each such *interval* $[\,s^-, s^+\,]$) means that $z_A(\cdot)$ must satisfy the *monotonicity* property

$$z_A(t) = \min\, z_A\,(\,[t, a]\,) \qquad \forall\, t \in [0, a]\,, \tag{3.93}$$

in particular, by (3.72) & (3.74), we will have

$$z_A(\cdot) \qquad \textit{monotone} \qquad \text{and in} \;\; C^0\,(\,[0, a]\,); \tag{3.94}$$

and since the Lusin property (3.60) with $\gamma(\cdot)$ replaced by $z_A(\cdot)$ holds true (by (3.14)), also (3.89) must hold, due to [Yeh, th. 13.8].

As to (3.90), it follows from (3.28) & (3.89) & (3.80) & (3.81) & (3.7). This completes the proof of (3.89) & (3.90) & claim 10 & (3.17) & (3.18) & (3.19), since $u_A(x) = U_A(\,|x|\,) = z_A\,(\,\gamma^{-1}\,(\,|x|\,)\,)$ & $\gamma^{-1}(\cdot)$ is *Lipschitz increasing*: by (3.64) & (3.62) & [Leo, 3.57],

$$\int_0^a |\, z'_A(t)\,|\;\, dt = \int_0^a |\, U'_A\,(\,\gamma(t)\,)\,| \quad \gamma'(t)\; dt = \int_0^R |\, U'_A(r)\,|\;\, dr < \infty. \tag{3.95}$$

Claim 11

$$\begin{gathered} q < \infty \quad in \quad (3.22) \ \Rightarrow \ U_A(\cdot) \ \text{ is } \ Lipschitz \ \text{ and, a.e. on } [\,0,R\,]\,, \\ |\,U_A'(r)\,| \le (\,1+c_1\,)\,(\,1+c_2\,)\,(\,1+|\,A\,|\,)\ q\ \exp[\,(\,1+c_2\,)\ q\ R\,]\,. \end{gathered} \tag{3.96}$$

Indeed, by (3.67) & claim 4, $U_A(\cdot) := z_A\,(\,\gamma^{-1}(\cdot)\,) \in \mathcal{U}_A^{0,R}$ (recall (3.51)); while, on the other hand, by the computations above performed (after (3.77)) we have: $|\,U_A'(r)\,| \le \ (\,1+c_2\,)\,q\,c_1$ where $|\,U_A(r)\,| \le c_1$; and, elsewhere, $|\,U_A'(r)\,| \le \ (1+c_2\,)\ q\ \ |\,U_A\,(r)\,|$ hence (3.96). This proves (3.24) & (3.25), since $z_A(t) = U_A\,(\,\gamma(t)\,)$ and (3.61) & (3.62) & (3.95) hold true.

Similarly, (3.26) holds true, due to (3.90); and also th. 3.2.3 is proved. $\square$

Chapter 4

Radial symmetry in convex case: regularity of vectorial minimizers in quasi-scalar case

4.1 Introduction

In last chapter we have proved *existence* of a *radial* (or *radially symmetric*) *minimizer* $u^0_A(x) = U^0_A(|x|)$ for the *convex vectorial multiple integral*

$$\int_{B_R} L^{**}\left(u(x), |\,D\,u(x)\,| \quad \rho_1(|x|)\right) \;\; . \;\; \rho_2(|x|)\;\, d\,x \;\text{ on } W^{1,1}_A\,(B_R\,,\mathbb{R}^m)\,, \tag{4.1}$$

where the *lagrangian*

$$L^{**} : \mathbb{R}^m \times \mathbb{R} \to [0,\infty] \qquad \text{is } \textit{convex lsc} \text{ with } L^{**}(S,\cdot) \;\; \textit{even} \tag{4.2}$$

$$\text{and} \quad \rho_1 \,\&\, \rho_2 : [\,0,R\,] \to [\,c_0, c_\infty\,] \subset (0,\infty) \quad \text{are } \textit{Borel measurable}\,, \tag{4.3}$$

e.g. $\rho_1(\cdot) \equiv 1 \equiv \rho_2(\cdot)$; while the class of *functions* in competition is the usual *Sobolev space*

$$W^{1,1}_A := A + W^{1,1}_0\,(B_R,\,\mathbb{R}^m) \tag{4.4}$$

of those $u(\cdot)$ taking the *constant value* $A \in \mathbb{R}^m$ along the *boundary* $\partial\,B_R$ of the *ball* $B_R := \left\{ x \in \mathbb{R}^d \,:\, |x| < R \right\}$; and $|\,D\,u(x)\,|$ is the *euclidian norm* of the $m \times d$ – *gradient matrix*.

It is quite helpful, e.g. for *engineering design*, to guarantee nice *regularity properties* which these *minimizers* must *necessarily* satisfy, besides belonging to $W^{1,1}_A$ and being *radial*; namely to grant some specific *geometrical behaviour* of the *optimal radial "profile" curve* $U_A : [\,0,R\,] \to \mathbb{R}^m$; but even reinforcing *superlinearity* into e.g. p – *growth* with $p = 7/6$, generic *radial functions* $u(x) = U(|x|)$ in $W^{1,p}_A(B_R,\,\mathbb{R}^m)$ are *Hölder continuous* (essentially in $C^{0,1/7}_{loc}(B_R \setminus \{0\},\,\mathbb{R}^m)$) but may turn wildly *discontinuous* as $|x| \to 0$, to the point of *mapping* arbitrarily small *balls* $B(0,\varepsilon)$ onto the whole of $\mathbb{R}^m$! A simple and striking example is

$$u(x) := |x|^{-1/4} \left| \sin\left(|x|^{-1/4}\right) \right| \left(\cos\left(|x|^{-1/4}\right), \sin\left(|x|^{-1/4}\right) \right), \tag{4.5}$$

in which $U(\,(0,1/i\,)\,) = \mathbb{R}^2 \quad \forall\, i \in \mathbb{N}$. (Clearly $|\,U(r)\,| \sim r^{-1/4}$ & $|\,U'(r)\,|^p \;\; r^{d-1} \sim r^{-3/4}\, r^{d-2}$ both belong to $L^1(0,R)$, while $|\,U'(r)\,| \sim r^{-3/2}$ does not.)

In contrast with (4.5), our *minimizer* $u_A(\cdot)$ (in both cases: $m = 1$, i.e. *scalar* $u_A(\cdot)$, in th. 3.2.2; and $m > 1$, i.e. *vectorial* $u_A(\cdot)$, in th. 4.3.1

below) is *uniformly continuous* and has *tame profile* $U_A(\cdot)$ (namely : with *increasing level* $L^{**}(U_A(\cdot), 0)$ and each *coordinate* $U_{A\,i}(\cdot)$ *mapping null sets* to *null sets*).

We need no *growth* hypotheses on $L^{**}(S, \cdot)$, sufficing the knowledge of *existence* of *minimum* for the *integral* (4.1); which is automatic for *superlinear* $L^{**}(\cdot, \cdot)$:

$$\frac{\inf\, L^{**}(\mathbb{R}^m, \lambda)}{\lambda} \to \infty \qquad \text{as} \quad \lambda \to \infty. \tag{4.6}$$

We freely allow $L^{**}(S, v) = \infty$, so that implicity included is the possibility of imposing *state* and *gradient pointwise constraints* at will, e.g. under the form of *partial differential equations* or *inclusions* (in explicit or implicit form), so that e.g. *optimal control problems* are also (theoretically) included in our *optimization problem* for the *integral* (4.1). Provided, of course, their *variational* reformulation has the form (4.1).

Recall that the general hypothesis (4.2) is anyway needed, in order to apply *Jensen inequality* so as to reach *radial minimizers*; while the generic hypothesis

$$\exists\ \min L^{**}(\mathbb{R}^m, 0) \tag{4.7}$$

holds true not only whenever $L^{**}(\cdot, 0)$ has *bounded sublevel sets*

$$\Sigma_A := \{ S \in \mathbb{R}^m : \quad L^{**}(S, 0) \le L^{**}(A, 0) \}, \tag{4.8}$$

but also in case its set of *minimizers* is *unbounded*, e.g. an $half - space$.

Since in the *scalar* $m = 1$ case, or in *optimal control problems* with allowed *gradients* constrained to *grow* at most *linearly* with *states*, the possibility of wild behaviour of *minimizers* (as displayed in example (4.5)) is excluded (see theorems 3.2.2 & 3.2.3 respectively), our aim here is to expand the scope of applicability of such nice *regularity* of *minimizers* into the *vectorial* $m > 1$ case. Indeed, while in *optimal control* it may be reasonable to impose $q < \infty$ in th. 3.2.3 when $m > 1$, for *calculus* of *variations* problems this seems rather artificial. On the other hand, to impose $q < \infty$, instead of $q_A < \infty$, in the *vector* case, seemed to make not much sense, intuitively.

We have succeeded in proving such extension into the *vectorial* $m > 1$ case by adding only three mild extra hypotheses. First, we ask that

$$\text{the } \mathit{subdifferential} \text{ of } L^{**}(\cdot, 0) \text{ at } A \text{ be } \mathit{nonempty}. \tag{4.9}$$

Second, and defining

$$\partial^0 L^{**}(S,0) := \begin{cases} \text{the } \textit{minimal norm element} \text{ of the} \\ \\ \text{subdifferential } \textit{of} \;\; L^{**}(\cdot,0) \;\; \text{at} \;\; S, \end{cases} \tag{4.10}$$

we ask that $\exists \mu_L > 0$ for which, at any $S \in \mathbb{R}^m$ having $\partial L^{**}(S,0) \neq \emptyset$,

$$\partial^0 L^{**}(S,0) \neq 0 \qquad \Rightarrow \qquad \left| \partial^0 L^{**}(S,0) \right| \geq \mu_L > 0. \tag{4.11}$$

(While (4.9) holds quite generally true in real-life applications, we feel, clearly the effect of (4.11) is to reinforce (4.7), due to (4.2), by imposing a mild *geometrical* restriction on approaching *min points*: the *slope* of $L^{**}(\cdot,0)$ cannot approach *smoothly zero*, on the contrary it must jump *nonsmoothly* to *zero*.)

Third, an hypothesis which (by being trivially true in the *scalar* or *radial* case) was hidden: for any $S \,\&\, S'$ in $\mathbb{R}^m$,

$$\begin{aligned} &\inf L^{**}(\mathbb{R}^m,0) < L^{**}(S,0) = L^{**}(S',0) \leq L^{**}(A,0) \quad \Rightarrow \\ &\Rightarrow |\partial^0 L^{**}(S,0)| = |\partial^0 L^{**}(S',0)| \;\;\&\;\; L^{**}(S,v) = L^{**}(S',v) \;\forall v. \end{aligned} \tag{4.12}$$

Definition 4.1.1. *Under* (4.2) & (4.10), *we call*

$$L^{**}(\cdot,\cdot) \qquad \textit{quasi} - \textit{scalar} \qquad \textit{whenever} \;\; (4.12) \;\; \textit{is satisfied.} \tag{4.13}$$

While the last *equality* in (4.12) trivially holds true whenever $L^{**}(S,v) = \ell^{**}(L^{**}(S,0),v)$ for some $\ell^{**}(\cdot,\cdot)$ having $\ell^{**}(p,0) = p \;\; \forall p$, on the contrary the preceding one imposes a more serious intrinsic *geometric* constraint on *graph* $L^{**}(\cdot,0)$, namely on its *level sets* (which seemingly should be C^1, as happens in case $L^{**}(S,0) = dist_C(S)$).

Reciprocally, for any such *quasi − scalar* $L^{**}(\cdot,\cdot)$ one may find a corresponding $\ell^{**}(\cdot,\cdot)$ as above, namely satisfying $L^{**}(S,v) = \ell^{**}(L^{**}(S,0),v)$ for any v, and for any S having $\inf L^{**}(\mathbb{R}^m,0) < L^{**}(S,0) \leq L^{**}(A,0)$. Indeed, just set $\ell^{**}(p,v) := L^{**}(S,v)$, at any $p \in (\inf L^{**}(\mathbb{R}^m,0), L^{**}(A,0)]$, picking any *point* S on the *level set* $L^{**}(\cdot,0)^{-1}(p)$.

Notice that the property expressed in (4.12), of *graph* $L^{**}(\cdot, 0)$ having *constant slope* along each *level set*, is satisfied e.g. in case

$$L^{**}(S, 0) := \ (\mathit{signed})\ \mathit{distance} \ \text{ from } \ S \ \text{ to a set } \ C;$$

in particular whenever $L^{**}(S, 0) = |S|$ or $L^{**}(\cdot, 0)$ is *affine*; or whenever $m = 1$ (using $L^{**}(s, 0) = s$ and the *set* $(-\infty, 0)$). Another example: given any *open convex set* Ω, the *cube* of the (*signed*) *distance* to $\partial\Omega$. (The *signed distance* to $\partial\Omega$ becomes *negative* inside Ω, *equal* to *minus* the *distance* to its *boundary*.)

Besides the *nonautonomous vectorial multiple integral* (4.1), we also deal here with the problems of *minimizing* three other auxiliary *nonautonomous scalar* and *vectorial single integrals*, for which we prove both *existence* of *minimizers* (which was not known before, we feel) and *regularity* as well.

4.2 Statement of the integrals to be minimized and their spaces of functions in competition

Starting from a given

$$u_A^0(\cdot) \quad \textit{minimizer} \quad \text{of the} \quad \textit{integral} \quad (4.1), \tag{4.14}$$

we construct below, under the basic hypotheses (4.2) & (4.3) & (4.7) & (4.9) & (4.11) & (4.12), and recalling the notation $W_A^{1,1}$ in (4.4), a *radial minimizer* $u_A(\cdot)$ for the *vectorial convex multiple integral*

$$\int_{B_R} L^{**}\left(u(x), |\, D\, u(x)\,| \quad \rho_1\left(|x|\right)\right) \quad . \quad \rho_2\left(|x|\right)\, d\, x \quad \text{on} \quad W_{A\nearrow}^{1,1}, \tag{4.15}$$

$$W_{A\nearrow}^{1,1} := \left\{ u(\cdot) \in W_A^{1,1} \cap C^0\left(\overline{B}_R, \mathbb{R}^m\right) \; : \; \begin{array}{c} \exists\, U(\cdot) \in \mathcal{U}_{A\nearrow}^{0,R} \;\; \text{with} \\ \\ u(x) = U\left(|x|\right) \; \forall\, x \end{array} \right\}, \tag{4.16}$$

$$\mathcal{U}_{A\nearrow}^{0,R} := \left\{ U(\cdot) \in W^{1,1}\left([\,0, R\,], \mathbb{R}^m\right) : \begin{array}{c} L^{**}\left(U(\cdot), 0\right) \quad \textit{increases} \\ \\ \& \qquad U(\,R\,) = A \end{array} \right\}. \tag{4.17}$$

However, besides this main pair of *vectorial convex multiple integrals* (4.1) & (4.15), we also consider below the problems of *minimizing* the following three pairs of auxiliary *single integrals* (the second in each pair being, again, the same *integral* but defined over a more *regular class* of *functions* in competition). First such pair :

$$\alpha_d \int_0^a L^{**}(\, z(t), |\, z'(t)\,| \quad \rho(t)\,)\; d\, t \quad \text{on} \quad Z_A^{0,a} \tag{4.18}$$

$$\alpha_d \int_0^a L^{**}(\, z(t), |\, z'(t)\,| \quad \rho(t)\,)\; d\, t \quad \text{on} \quad Z_{A\nearrow}^{0,a}\,; \tag{4.19}$$

where α_d is the *Hausdorff measure* in *dimension* $d-1$ of the *unit sphere* $S^d := \left\{\, x \in \mathbb{R}^d \;:\; \; |x| = 1 \,\right\}$,

$$a := \int_0^R \rho_2(r) \quad r^{d-1}\; d\, r \quad \& \quad \rho(t) := \rho_1\left(\gamma(t)\right)\; \rho_2\left(\gamma(t)\right)\; \gamma(t)^{d-1}, \tag{4.20}$$

$$\gamma : [0,a] \to [\,0,R\,]\,,\ r=\gamma(t), \qquad \text{being the } \ \textit{inverse function} \ \text{ of } \tag{4.21}$$

$$\gamma^{-1}(\cdot):\ r \mapsto t=\gamma^{-1}(r) := \int_0^r \rho_2(\alpha) \quad \alpha^{d-1}\, d\,\alpha\,; \tag{4.22}$$

and where

$$Z_A^{0,a} := \left\{ z(\cdot) \in W_{loc}^{1,1}\,(\,(0,a],\, \mathbb{R}^m\,) : \begin{array}{c} |\,z'(\cdot)\,|\ \ \gamma(\cdot)^{d-1} \in L^1(0,a) \\ \\ \& \qquad z(a)=A \end{array} \right\}, \tag{4.23}$$

$$Z_{A\nearrow}^{0,a} := \left\{ z(\cdot) \in W^{1,1}\,(\,[0,a],\, \mathbb{R}^m\,) : \begin{array}{c} L^{**}\,(\,z(\cdot),0\,) \quad \textit{increases} \\ \\ \& \qquad z(a)=A \end{array} \right\}. \tag{4.24}$$

Second such pair of auxiliary *single integrals*:

$$\alpha_d \int_0^a \ell_A^{**}\,(\,w(t),\, g_A\,(\,w(t)\,)\ |\,w'(t)\,| \quad \rho(t)\,)\ d\,t \ \text{ on } \ \mathcal{W}_A^{0,a} \tag{4.25}$$

$$\alpha_d \int_0^a \ell_A^{**}\,(\,w(t),\, g_A\,(\,w(t)\,)\ |\,w'(t)\,| \quad \rho(t)\,)\ d\,t \ \text{ on } \ \mathcal{W}_{A\nearrow}^{0,a}\,, \tag{4.26}$$

with adequately defined $g_A(\cdot)$ and $\ell_A^{**}(\cdot,\cdot)$ (which the reader may already peep-in at (4.82) & (4.87)) and

$$\mathcal{W}_A^{0,a} := \left\{ w(\cdot) \in W_{loc}^{1,1}\,(\,(0,a]\,) : \begin{array}{c} \exists\ z(\cdot) \in Z_A^{0,a}\ :\ w(\cdot) = L^{**}\,(\,z(\cdot),0\,) \\ \\ \& \qquad w'(\cdot)\ \gamma(\cdot)^{d-1} \in L^1(0,a) \end{array} \right\}, \tag{4.27}$$

$$\mathcal{W}_{A\nearrow}^{0,a} := \left\{ w(\cdot) \in W^{1,1}\,(\,[0,a]\,) : \begin{array}{c} w(\cdot) \quad \textit{increases} \\ \\ \&\ \ w(a) = L^{**}\,(\,A,0\,) \end{array} \right\}. \tag{4.28}$$

Third such pair of auxiliary *single integrals*:

$$\alpha_d \int_0^R L^{**}\,(\,U(r),|\,U'(r)\,| \quad \rho_1(r)\,) \quad . \quad \rho_2(r)\, r^{d-1}\ d\,r \quad \text{on} \quad \mathcal{U}_A^{0,R} \tag{4.29}$$

$$\alpha_d \int_0^R L^{**}\,(\,U(r),|\,U'(r)\,| \quad \rho_1(r)\,) \quad . \quad \rho_2(r)\, r^{d-1}\ d\,r \quad \text{on} \quad \mathcal{U}_{A\nearrow}^{0,R}\,; \tag{4.30}$$

where $\mathcal{U}^{0,R}_{A\nearrow}$ has been defined in (4.17), while

$$\mathcal{U}^{0,R}_{A} := \left\{ U(\cdot) \in W^{1,1}_{loc}\,(\,(\,0,R\,]\,,\,\mathbb{R}^m\,) : \begin{array}{c} r \to |\,U'(r)\,|\ \ r^{d-1} \in L^1(0,R) \\ \\ \& \qquad U(\,R\,) = A \end{array} \right\}. \tag{4.31}$$

4.3 Radially monotone minimizing deformations for vectorial quasi-scalar convex integrals

Here is the *existence* and *regularity* result of this section:

Theorem 4.3.1. *Assume* (4.2) & (4.3) & (4.7) & (4.9) & (4.11) & (4.12) *together with*

$$\begin{cases} \textit{either} \ \ (4.6) \quad \textit{or} \quad \exists \ \textit{minimum} \ \ \textit{for} \ \ (4.1) \quad \textit{or} \\ \\ \qquad\qquad (4.18) \qquad \textit{or} \qquad (4.29) \end{cases} \tag{4.32}$$

$$\textit{or else} \quad \exists \ \textit{minimum} \quad \textit{for} \ \ (4.25). \tag{4.33}$$

Then

$$\exists \ \textit{radial} \quad u_A(x) = U_A\,(\,|x|\,) \quad \textit{minimizing} \ \ \textit{both} \ \ (4.1) \ \ \& \ \ (4.15) \tag{4.34}$$

$$\& \quad \exists \ U_A(r) \ \ \textit{minimizing} \ \ \textit{both} \ \ (4.29) \ \ \& \ \ (4.30) \tag{4.35}$$

$$\& \quad \exists \ z_A(t) \ \ \textit{minimizing} \ \ \textit{both} \ \ (4.18) \ \ \& \ \ (4.19) \tag{4.36}$$

$$\& \quad \exists \ w_A(t) \ \ \textit{minimizing} \ \ \textit{both} \ \ (4.25) \ \ \& \ \ (4.26). \tag{4.37}$$

Moreover: the minimum value for all these integrals is the same; the following equivalences hold true

$$(4.32) \quad \Leftrightarrow \quad (4.33) \quad \Leftrightarrow \quad (4.34) \quad \Leftrightarrow \quad (4.35) \quad \Leftrightarrow \quad (4.36) \quad \Leftrightarrow \quad (4.37); \tag{4.38}$$

and the minimizers in (4.34) *to* (4.37) *are related by the equalities*

$$u_A(x) = U_A\,(\,|x|\,) = z_A\left(\gamma^{-1}\,(\,|x|\,)\right) \qquad \& \qquad w_A(t) = L^{**}\,(\,z_A(t),\,0\,) \tag{4.39}$$

$$z_A(t) = Q_A\,(\,w_A(t)\,) \qquad \& \qquad U_A(r) = Q_A\left(\,w_A\left(\gamma^{-1}(r)\right)\right) \tag{4.40}$$

(*with* $\gamma^{-1}(\cdot)$ & $Q_A\,(\cdot)$ *defined in* (4.22) & (4.86)); *and* (*recall* (3.21), *with* L^{**} *instead of* ℓ^{**}, & (4.8))

$$m \geq 1 \ \ \& \ \ q_A < \infty \qquad \Rightarrow \qquad U_A(\cdot) \ \ \& \ \ u_A(\cdot) \ \ \textit{are Lipschitz}. \tag{4.41}$$

Proof. Under (4.2) & (4.3) th. 3.2.1 implies the following:

Proposition 4.3.1.

$$\exists \ a \quad radial \ minimizer \quad u_A^0(x) = U_A^0\,(\,|x|\,) \quad for \quad (4.1) \tag{4.42}$$

$$if \ and \ only \ if \tag{4.43}$$

$$the \quad integral \quad (4.1) \quad has \quad minimum \tag{4.44}$$

$$if \ and \ only \ if \tag{4.45}$$

$$\exists \ a \quad minimizer \quad U_A^0(r) \quad for \quad (4.29) \tag{4.46}$$

$$if \ and \ only \ if \tag{4.47}$$

$$\exists \ a \quad minimizer \quad z_A^0(t) \quad for \quad (4.18). \tag{4.48}$$

Moreover the minimum value for these integrals is the same and

$$U_A^0(\cdot) = z_A^0\left(\gamma^{-1}(\cdot)\right) \qquad with \qquad \gamma^{-1}(\cdot) \quad Lipschitz \ increasing \tag{4.49}$$

(*see* (4.22) & (4.3)), *in the sense that: by picking a minimizer* $U_A^0(\cdot)$ *for* (4.29) (*resp.* $z_A^0(\cdot)$ *for* (4.18)) *and applying the formula* (4.49) *one gets a minimizer* $z_A^0(\cdot)$ *for* (4.18) (*resp.* $U_A^0(\cdot)$ *for* (4.29)).

Thus, by (4.43) & (4.45) & (4.47), all of the *implications*

$$(4.42) \quad \Leftrightarrow \quad (4.32) \quad \Leftrightarrow \quad (4.46) \quad \Leftrightarrow \quad (4.48) \quad \Rightarrow \quad (4.36) \tag{4.50}$$

are proved, except for (4.48) $\Rightarrow$ (4.36); the proof of which will be our first and crucial step, accomplished in (4.115), after due preliminaries. As to our second step, it will (in (4.120) & (4.132) & (4.137) & (4.143) respectively) consist in establishing the further *implications*

$$(4.36) \qquad \Leftrightarrow \qquad (4.37) \tag{4.51}$$

$$(4.33) \qquad \Rightarrow \qquad (4.37) \tag{4.52}$$

$$(4.36) \quad \Rightarrow \quad (4.35) \quad \Rightarrow \quad (4.34) \quad \Rightarrow \quad (4.32) \tag{4.53}$$

hence the *equivalences* in (4.38), thus proving th. 4.3.1.

First step : So, to start such first and crucial step, take (4.48), namely a

$$z_A^0(\cdot) \in Z_A^{0,a} \quad \textit{minimizer} \quad \text{of} \quad (4.18), \tag{4.54}$$

define its "*lsc optimal level*"

$$\begin{aligned} w_A^0(0) &:= \inf\, L^{**}\left(z_A^0\left((0,a] \right), 0 \right) \quad \& \\ w_A^0(t) &:= L^{**}\left(z_A^0\,(t)\,, 0 \right) \quad \text{for} \quad t \in (0,a] \end{aligned} \tag{4.55}$$

and its "*increasing continuous optimal level*"

$$w_A(t) := \min\, w_A^0\left([t,a] \right) \qquad \forall\, t \in [0,a]. \tag{4.56}$$

Then clearly

$$0 \le w_A(\cdot) \le w_A^0(\cdot) \in W^{1,1}_{loc}\left((0,a] \right), \tag{4.57}$$

by (4.55) & (4.56) & (4.23) & (4.2), since $L^{**}(\cdot,0)$ is *convex* hence *locally Lipschitz* there; while since $w_A(\cdot) \in C^0\left([0,a] \right)$ & $w_A(\cdot)$ *increases* & also $w_A(\cdot) \in W^{1,1}_{loc}\left((0,a] \right)$ (due to remaining *constant* where it *differs* from $w_A^0(\cdot)$), by [Yeh, th. 13.8] we have :

$$w_A(\cdot) \in W^{1,1}\left([0,a] \right) \qquad \& \qquad w_A(\cdot) \quad \text{increases}\,; \tag{4.58}$$

and setting

$$p_A^{\min} := w_A(0) \qquad\qquad \& \qquad\qquad p_A^{\max} := w_A(a) \tag{4.59}$$

we have

$$\begin{aligned} 0 \le w_A(0) = p_A^{\min} &= \min\, w_A\left([0,a] \right) = \\ = \min\, w_A^0\left([0,a] \right) &= \inf\, L^{**}\left(z_A^0\left((0,a] \right), 0 \right) \le w_A(\cdot) \qquad \& \end{aligned} \tag{4.60}$$

$$w_A(\cdot) \le w_A(a) = p_A^{\max} = L^{**}\left(A, 0 \right) = \max\, w_A\left([0,a] \right). \tag{4.61}$$

Define

$$b := \max\,\{\, t \in [0,a] \;:\; \quad w_A(t) = w_A(0) \,\} \qquad (\, b \in [0,a] \,) \tag{4.62}$$

$$a' := \min\,\{\, t \in [0,a] \;:\; w_A(t) = w_A(a) \,\} \qquad (\, b < a' \in [0,a] \,). \tag{4.63}$$

Indeed, $b < a'$ because, excluding the trivial case $p_A^{\min} = p_A^{\max}$, we assume, in this proof,

$$p_A^{\min} \qquad < \qquad p_A^{\max}\,; \tag{4.64}$$

and clearly

$$\begin{gathered} w_A(\cdot) \equiv p_A^{\min} \quad \text{on} \quad [0,b] \quad \& \quad w_A\,(\,(\,b,a'\,)\,) = \left(p_A^{\min}, p_A^{\max}\right) \quad \& \\ w_A(\cdot) \equiv p_A^{\max} \qquad \text{on} \quad [\,a',a\,]\,. \end{gathered} \tag{4.65}$$

Set

$$\Sigma_A^{<} := \left\{\,S \in \mathbb{R}^m \;: \qquad p_A^{\min} \le L^{**}\,(\,S, 0\,) < p_A^{\max}\,\right\} \tag{4.66}$$

and, using the notation (4.10), define the *vector orthogonal* to the *level sets* and *pointing downwards*:

$$V_A : \Sigma_A^{<} \to \mathbb{R}^m, \qquad\qquad V_A\,(\,S\,) := -\,\partial^0\,L^{**}\,(\,S, 0\,)\;. \tag{4.67}$$

Since we are assuming (4.9),

$$\partial^0\,L^{**}\,(\,A, 0\,) \qquad \textit{exists} \;\text{ and is }\; \neq 0 \qquad (\,\textit{see}\;\;(4.10)\;\&\;(4.64)\;\&\;(4.61)\,). \tag{4.68}$$

Then, by (4.2) & (4.68) & [Cell & Vor] , there *exists* a *unique solution*, in $W^{1,2}\,(\,(0,\infty),\,\mathbb{R}^m\,)$, to the *ordinary differential equation*

$$\sigma_A'(\tau) = V_A\,(\,\sigma_A(\tau)\,) \quad \text{ for a.e. } \;\tau \in \left[\,0, \tau_A^0\,\right] \quad \& \;\; \sigma_A(0) = A, \tag{4.69}$$

with

$$\tau_A^0 := \min\left\{\,\tau \in (0,\infty)\;: \quad L^{**}\,(\,\sigma_A(\tau), 0\,) = p_A^{\min}\,\right\}; \tag{4.70}$$

so that, setting

$$p_A : \left[\,0, \tau_A^0\,\right] \to \left[\,p_A^{\min},\, p_A^{\max}\,\right], \qquad\qquad p_A(\tau) := L^{**}\,(\,\sigma_A(\tau), 0\,)\,, \tag{4.71}$$

we get

$$-\,p_A'(\tau) = \left|\,\partial^0\,L^{**}\,(\,\sigma_A(\tau), 0\,)\,\right|^2 = |\,V_A\,(\,\sigma_A(\tau)\,)\,|^2 > 0 \text{ a.e. on } \left[\,0, \tau_A^0\,\right] \tag{4.72}$$

$$p_A(\cdot) \;\text{ is }\; \textit{decreasing convex} \qquad (\,\text{since }\; p_A'(\cdot) \;\; \textit{increases}\,) \tag{4.73}$$

$$p_A(0) = p_A^{\max} \qquad\qquad \& \qquad\qquad p_A\left(\,\tau_A^0\,\right) = p_A^{\min} \tag{4.74}$$

$$L^{**}\left(\,\sigma_A\left(\,\tau_A^0\,\right), 0\,\right) = p_A^{\min} = w_A(0) = w_A^0(0) \qquad (\,\text{see also }\;(4.60)\,). \tag{4.75}$$

In particular, by (4.69) & (4.72), $|\,\sigma_A'(\cdot)\,| \;=| \,V_A\,(\,\sigma_A(\cdot)\,)\,| \;=\; |\,p_A'(\cdot)\,|^{1/2}$ *decreases*, hence

$$|\,\sigma_A'(\cdot)\,| \leq |\,p_A'(0)\,|^{1/2} = \left|\,\partial^0\,L^{**}\,(\,A,0\,)\,\right| < \infty \quad \& \quad \sigma_A(\cdot) \;\text{ is } \textit{Lipschitz}. \tag{4.76}$$

On the other hand, to check that

$$\tau_A^0(\cdot) \in (0,\infty) \qquad \text{is well-defined by} \quad (4.70), \tag{4.77}$$

recall (4.64) and notice that since, by (4.72) & (4.11), $p_A'(\tau) \leq -\mu_L^2$ on $(\,0,\,\tau_A^0\,)$, we would have $p_A(\tau) \leq 0 \leq p_A^{\min}$ whenever $\tau \geq p_A^{\max}\,\mu_L^{-2} < \infty$.

Clearly $p_A(\cdot)$ has, due to (4.77) & (4.71) & (4.72), *continuous inverse*

$$\begin{gathered} \tau_A : \left[\,p_A^{\min}, p_A^{\max}\,\right] \subset [0,\infty) \to [\,0,\tau_A^0\,] \subset [0,\infty), \\ \tau = \tau_A(p) \qquad \Leftrightarrow \qquad p = p_A(\tau), \end{gathered} \tag{4.78}$$

$$-\frac{1}{\mu_L^2} \leq \tau_A'(p) = \frac{1}{p_A'\,(\,\tau_A(p)\,)} = \frac{-1}{|\,\partial^0\,L^{**}\,(\,\sigma_A\,(\,\tau_A(p)\,)\,,0\,)\,|^2} < 0 \tag{4.79}$$

$$\tau_A'(\cdot) \;\; \textit{increases} \quad \& \quad \tau_A(\cdot) \;\text{ is }\; \textit{Lipschitz convex decreasing} \tag{4.80}$$

$$\tau_A\left(p_A^{\min}\right) = \tau_A^0 \qquad \& \qquad \tau_A\,(\,p_A^{\max}\,) = 0. \tag{4.81}$$

We are finally in position to define the $g_A(\cdot)$ appearing in (4.25) & (4.26):

$$g_A : \left[\,p_A^{\min},\, p_A^{\max}\,\right] \to (\,0,\infty\,)\,, \quad g_A(p) := \frac{1}{|\,\partial^0\,L^{**}\,(\,\sigma_A\,(\,\tau_A(p)\,)\,,0\,)\,|} \tag{4.82}$$

for $p > p_A^{\min}$, with $g_A\left(p_A^{\min}\right) := 1/\mu_L$, so that

$$g_A(p)^2 = -\,\tau_A'(p) \qquad \& \qquad g_A(\cdot) \qquad \textit{decreases} \qquad \&$$

$$0 < \frac{1}{|\,\partial^0\,L^{**}\,(\,A,0\,)\,|} \leq g_A(p) = |\,\tau_A'(p)\,|^{1/2} \leq \frac{1}{\mu_L} < \infty \quad (\,\text{due to } (4.11)\,) \tag{4.83}$$

$$\& \qquad g_A(\cdot) \qquad \text{is } \textit{bounded} \;\text{ away from } \; 0 \;\;\&\;\; \infty \tag{4.84}$$

$$\& \qquad \tau_A(\cdot) \;\text{ and }\; p_A(\cdot) \qquad \text{are both } \; \textit{Lipschitz}\,; \tag{4.85}$$

to define the other *functions* also needed for (4.25) & (4.26) (see (4.69) & (4.78)),

$$Q_A : \left[p_A^{\min}, p_A^{\max} \right] \to \mathbb{R}^m, \quad Q_A(p) := \sigma_A \left(\tau_A(p) \right) \tag{4.86}$$

$$\ell_A^{**} : \left[p_A^{\min}, p_A^{\max} \right] \times \mathbb{R} \to [0, \infty], \quad \ell_A^{**}(p, v) := L^{**} \left(Q_A(p), v \right); \tag{4.87}$$

and to define our new *minimizer* of (4.18) (recall (4.54) & see (4.116)) by :

$$z_A(t) := Q_A \left(w_A(t) \right) \qquad \text{(using (4.56) \& (4.86)).} \tag{4.88}$$

Clearly

$$Q_A(\cdot) \text{ is } \textit{Lipschitz} \quad \& \quad z_A(\cdot) \in W^{1,1}\left([0, a], \mathbb{R}^m \right), \tag{4.89}$$

by (4.86) & (4.76) & (4.85) & (4.88) & (4.58); while by (4.88) & (4.86) & (4.71) & (4.78),

$$L^{**}\left(z_A(t), 0 \right) = L^{**}\left(\sigma_A \left(\tau_A \left(w_A(t) \right) \right), 0 \right) = p_A \left(\tau_A \left(w_A(t) \right) \right) = w_A(t) \tag{4.90}$$

so that, by (4.90) & (4.57) & (4.55) & (4.2),

$$\begin{aligned} &L^{**}\left(z_A(\cdot), 0 \right) = w_A(\cdot) \le w_A^0(\cdot) = L^{**}\left(z_A^0(\cdot), 0 \right) \le \\ &\le L^{**}\left(z_A^0(\cdot), |\, z_A^{0\prime}(\cdot) \,| \quad \rho(\cdot) \right) \qquad \text{on } (0, a\,]; \end{aligned} \tag{4.91}$$

and, by (4.88) & (4.86) & (4.69) & (4.79) & (4.82) & (4.67),

$$z_A'(t) = -\frac{V_A\left(z_A(t) \right)}{|\, V_A\left(z_A(t) \right) |} \; g_A\left(w_A(t) \right) \; w_A'(t) \quad \text{a.e. on } (\, b, a' \,) \tag{4.92}$$

hence

$$|\, z_A'(t) \,| = g_A\left(w_A(t) \right) \; w_A'(t) \qquad \text{a.e. on } (\, b, a' \,). \tag{4.93}$$

Moreover, by (4.87) & (4.86) & (4.71) & (4.78), for any $p \in \left[p_A^{\min}, p_A^{\max} \right]$,

$$\ell_A^{**}(p, 0) := L^{**}\left(Q_A(p), 0 \right) = p_A \left(\tau_A(p) \right) = p. \tag{4.94}$$

By (4.57) & (4.58) & (4.54) & (4.23) & (4.89) & (4.62) & (4.63) one may define the *set*

$$\mathcal{T}_+ := \left\{ t \in (\, b, a' \,) \; : \; \exists\, w_A'(t) > 0 \;\&\; \exists\, w_A^{0\prime}(t) \;\&\; \exists\, z_A^{0\prime}(t) \;\&\; \exists\, z_A'(t) \right\}, \tag{4.95}$$

whose crucial properties are condensed in the next

Claim 1 *Extending*

$$w_A(t) = w_A^0(t) := p_A^{\min} \quad \textit{for} \;\; t<0 \qquad \& \\ w_A(t) = w_A^0(t) := p_A^{\max} \quad \textit{for} \;\; t>a, \tag{4.96}$$

then :

$$L^{**}\left(z_A^0(t), 0 \right) = w_A^0(t) = w_A(t) = L^{**}\left(z_A(t), 0 \right) \qquad \forall\, t \in \mathcal{T}_+ \tag{4.97}$$

$$w_A^0(t) < w_A^0(t+h) \qquad\qquad \forall\, t \in \mathcal{T}_+ \qquad\qquad \forall\, h>0 \tag{4.98}$$

$$w_A(t) < w_A(t+h) \qquad\qquad \forall\, t \in \mathcal{T}_+ \;\; \forall\, h>0 \tag{4.99}$$

$$\forall\, t \in \mathcal{T}_+ \;\; \exists\,(\,h_k\,) \searrow 0 \;: \qquad w_A(t+h_k) = w_A^0(t+h_k) \tag{4.100}$$

$$0 < w_A'(t) = w_A^{0\,\prime}(t) = \frac{|\,z_A'(t)\,|}{g_A\,(\,w_A(t)\,)} \qquad\qquad \forall\, t \in \mathcal{T}_+ \tag{4.101}$$

$$\forall\, t \in \mathcal{T}_+ \;\; \exists\, \delta>0 \;: \qquad w_A^0(t-h) < w_A^0(t) \qquad\qquad \forall\, h \in (0,\delta) \tag{4.102}$$

$$w_A(t-h) < w_A(t) \qquad\qquad \forall\, t \in \mathcal{T}_+ \;\; \forall\, h>0 \tag{4.103}$$

$$t \in \mathcal{T}_+ \quad \Rightarrow \quad 0 < w_A'(t) = w_A^{0\,\prime}(t) = |\,z_A'(t)\,| \;/\; g_A\,(\,w_A(t)\,) \;\leq \\ \leq\; |\,z_A^{0\,\prime}(t)\,| \;.\; |\,\partial^0 L^{**}\,(\,z_A(t),0\,)\,| \;\leq\; |\,z_A^{0\,\prime}(t)\,| \;.\; |\,\partial^0 L^{**}\,(\,A,0\,)\,| \tag{4.104}$$

$$0 < |\,z_A'(t)\,| \leq \left|\,z_A^{0\,\prime}(t)\,\right| \qquad\qquad \forall\, t \in \mathcal{T}_+ \tag{4.105}$$

$$L^{**}\,(\,z_A(t), |\,z_A'(t)\,| \quad \rho(t)\,) \leq L^{**}\left(z_A^0(t), \left|\,z_A^{0\,\prime}(t)\,\right| \quad \rho(t) \right) \quad \forall\, t \in \mathcal{T}_+ \tag{4.106}$$

$$0 = w_A'(t) = |\,z_A'(t)\,| \leq \left|\,z_A^{0\,\prime}(t)\,\right| \qquad\qquad a.e. \;\; on \;\; [\,0,a\,] \setminus \mathcal{T}_+ \tag{4.107}$$

$$L^{**}\left(z_A^0(t), 0 \right) = w_A^0(t) = w_A(t) = L^{**}\,(\,z_A(t), 0\,) \qquad \forall\, t \in [0,a] \tag{4.108}$$

$$L^{**}\,(\,z_A^0(t), |\,z_A^{0\,\prime}(t)\,| \quad \rho(t)\,) = \\ = L^{**}\,(\,z_A(t), |\,z_A'(t)\,| \quad \rho(t)\,) \quad a.e. \;\; on \;\; [\,0,a\,] \tag{4.109}$$

$$L^{**}\left(z_A^0(t), \left|\,z_A^{0\,\prime}(t)\,\right| \quad \rho(t) \right) = L^{**}\left(z_A^0(t), 0 \right) \quad a.e. \;\; on \;\; [\,0,a\,] \setminus \mathcal{T}_+ \tag{4.110}$$

$$L^{**}\,(\,S, v\,) > L^{**}\,(\,S, 0\,) \qquad \forall\, S \in \Sigma_A^< \;\; \forall\, v>0 \qquad \Rightarrow \\ \Rightarrow \qquad |\,z_A^{0\,\prime}(\cdot)\,| = 0 \qquad\qquad a.e. \;\; on \;\; [\,0,a\,] \setminus \mathcal{T}_+\,. \tag{4.111}$$

Indeed, by (4.91), the denial of (4.97), i.e. $w_A^0(t) > w_A(t)$, would imply, by (4.56) & (4.57),

$$\exists\,\delta > 0 \,: \quad w_A(t) = w_A(t+h) = w_A(t+\delta) = w_A^0(t+\delta) \qquad \forall\, h \in (0,\delta) \quad (4.112)$$

hence $t \notin \mathcal{T}_+$, by (4.95), thus proving (4.97), by (4.91). On the other hand, denying (4.98) we would get, by (4.97) & (4.57) & (4.91),

$$\exists\,\delta > 0 \,: \quad w_A(t) = w_A^0(t) \geq w_A^0(t+\delta) \geq w_A(t+\delta) \geq w_A(t)$$

so that these coincide and again (4.112) holds and $t \notin \mathcal{T}_+$, which proves (4.98); while (4.99) is still easier to prove. Denying (4.100) would yield, by (4.57),

$$\exists\,\delta_1 > 0 \,: \qquad\qquad w_A(t+h) < w_A^0(t+h) \qquad\qquad \forall\, h \in (0,\delta_1)$$

so that, by (4.56), $\exists\,\delta \geq \delta_1 > 0$ for which again (4.112) holds & $t \notin \mathcal{T}_+$. Thus (4.100) is proved. Moreover, since, for $t \in \mathcal{T}_+$, by (4.95) & (4.97) & (4.100),

$$0 < w_A'(t) = \lim\, \frac{w_A\,(\,t+h_k\,) - w_A(t)}{h_k} = \lim\, \frac{w_A^0\,(\,t+h_k\,) - w_A^0(t)}{h_k} = w_A^{0\prime}(t);$$

and (4.101) is proved, by (4.93). On the other hand, denial of (4.102) would imply

$$\exists\,(\,h_k\,) \searrow 0 \,: \qquad\qquad w_A^0(t) - w_A^0\,(\,t-h_k\,) \leq 0$$

so that, by (4.101), we would reach an absurd proving (4.102):

$$0 < w_A'(t) = w_A^{0\prime}(t) = \lim\, \frac{w_A^0(t) - w_A^0\,(\,t-h_k\,)}{h_k} \leq 0.$$

As to (4.103), it is still easier to prove.

Consider now the *inequality* associated to the fact of $\partial^0 L^{**}\,(\,z_A^0(t), 0\,)$ being in the *subdifferential* of $L^{**}(\cdot,0)$ at $z_A^0(t)$ (recall (4.10)), namely

$$L^{**}\left(\,z_A^0\,(\,t-h\,)\,,0\,\right) \geq$$

$$\geq\, L^{**}\left(\,z_A^0(t), 0\,\right) + \left\langle\,\partial^0 L^{**}\left(\,z_A^0(t), 0\,\right), z_A^0\,(\,t-h\,) - z_A^0(t)\,\right\rangle;$$

together with (4.102) & (4.55), it yields some $\delta > 0$ for which

$$0 < w_A^0(t) - w_A^0\,(\,t-h\,) \leq \left\langle\,\partial^0 L^{**}\left(\,z_A^0(t), 0\,\right), z_A^0(t) - z_A^0\,(\,t-h\,)\,\right\rangle \leq$$

$$\leq \qquad \left|\,\partial^0 L^{**}\left(z_A^0(t), 0 \right)\right| \;.\; \left| z_A^0(t) - z_A^0\,(\,t-h\,)\right| \qquad \forall\, h \in (0,\delta)$$

so that (4.95) & (4.101) *implies* (4.104). Moreover, by (4.97) & (4.12) & (4.82) & (4.88),

$$L^{**}\,(\,z_A(\cdot), |\,z_A'(\cdot)\,| \quad \rho(\cdot)\,) = L^{**}\left(z_A^0(\cdot), |\,z_A'(\cdot)\,| \quad \rho(\cdot) \right) \quad \text{on } \mathcal{T}_+ \quad (4.113)$$

$$\frac{1}{g_A\,(\,w_A(\cdot)\,)} = \left|\,\partial^0 L^{**}\,(\,z_A(\cdot), 0\,)\right| = \left|\,\partial^0 L^{**}\left(z_A^0(\cdot), 0 \right)\right| \quad \text{on } \mathcal{T}_+ \quad (4.114)$$

so that, by (4.113) & (4.104), we have proved (4.105). On the other hand, by (4.105) & (4.113) & (4.2), we get (4.106). Finally, a.e. on $[0,a] \setminus (\,b, a'\,)$ we have, by (4.65) & (4.88), $w_A'(\cdot) = 0 = |\,z_A'(\cdot)\,|$; while on $(\,b, a'\,) \setminus \mathcal{T}_+$, by (4.95) & (4.93), $\exists\, w_A'(t) = 0 = |\,z_A'(t)\,|$, proving (4.107). Thus claim 1 is proved, except for (4.108) to (4.111) which will be proved below (in (4.118)).

Claim 2

$$(4.48) \qquad \Rightarrow \qquad (4.36) \qquad\qquad (4.115)$$

namely: having assumed (4.48)*, by taking a* $z_A^0(\cdot)$ *minimizer of* (4.18)*, in* (4.54)*; and having constructed from it a new* $z_A(\cdot)$ *by the formula* (4.88) *(using* $z_A^0(\cdot)$ *through* (4.55) & (4.56) *), we now claim* (4.36)*, namely that this*

$$z_A(\cdot) \qquad \textit{minimizes both } (4.18) \;\&\; (4.19). \qquad (4.116)$$

Indeed, by (4.89) & (4.90) & (4.58) & (4.61) & (4.86) & (4.81) & (4.69) & (4.24) & (4.23),

$$z_A(\cdot) \in Z_{A\nearrow}^{0,a} \qquad \subset \qquad Z_A^{0,a}. \qquad (4.117)$$

On the other hand, by (4.107) & (4.91),

$$\int_{[0,a]\setminus\mathcal{T}_+} L^{**}\,(\,z_A(t), |\,z_A'(t)\,| \quad \rho(t)\,)\;dt = \int_{[0,a]\setminus\mathcal{T}_+} L^{**}\,(\,z_A(t), 0\,)\;dt =$$

$$= \int_{[0,a]\setminus\mathcal{T}_+} w_A(t)\;dt \leq \int_{[0,a]\setminus\mathcal{T}_+} w_A^0(t)\;dt = \int_{[0,a]\setminus\mathcal{T}_+} L^{**}\left(z_A^0(t), 0 \right)\;dt \leq$$

$$\leq \int_{[0,a]\setminus\mathcal{T}_+} L^{**}\left(z_A^0(t), \left| z_A^{0\prime}(t) \right| \quad \rho(t) \right)\;dt\,;$$

and adding this *inequality* to the *inequality* (4.106), by (4.54) & (4.117) the proof of (4.116), hence of (4.115) and claim 2 and (4.50), is complete.

Recalling the final comment before (4.115), let us return now our attention back to the statements (4.108) to (4.111). To begin with, by (4.91) & (4.97) the proof of (4.108) reduces to showing that

$$w_A^0(t) \leq w_A(t) \qquad\qquad \forall\, t \in [0,a] \setminus \mathcal{T}_+ \,; \tag{4.118}$$

but denial of (4.118) would yield $w_A(\cdot) < w_A^0(\cdot)$ along a *nonempty open interval* $\subset (0,a] \setminus \mathcal{T}_+$ hence the first *inequality* in the preceding paragraph would be *strict*, in contradiction with (4.54) & (4.117). Exactly the same would happen if we did not have (4.109), by (4.106) together with (a.e. on $[0,a] \setminus \mathcal{T}_+$, by (4.107) & (4.108)) :

$$L^{**}\left(z_A(t), |z_A'(t)|\ \rho(t)\right) = L^{**}\left(z_A^0(t), 0\right) \leq L^{**}\left(z_A^0(t), \left|z_A^{0\prime}(t)\right|\ \rho(t)\right). \tag{4.119}$$

But the same reasoning also proves (4.110), hence (4.111) and claim 1.

Second step : Having thus proved (4.50) (in the paragraph before (4.118)), we now proceed to prove the *implication*

$$(4.36) \qquad\qquad \Rightarrow \qquad\qquad (4.37) \tag{4.120}$$

in (4.51), by showing that our $w_A(\cdot)$, as in (4.90), satisfies

$$w_A(\cdot) \in \mathcal{W}_{A\nearrow}^{0,a} \qquad \subset \qquad \mathcal{W}_A^{0,a} \tag{4.121}$$

$$w_A(\cdot) \qquad\qquad \textit{minimizes} \ \text{ both } \ (4.25) \ \& \ (4.26). \tag{4.122}$$

Indeed, by (4.27) & (4.28) & (4.58) & (4.117) & (4.61) & (4.90), obviously $w_A(\cdot)$ *belongs* to both *spaces* in (4.121); while, on the other hand, picking any generic $w_1(\cdot)$ in $\mathcal{W}_{A\nearrow}^{0,a}$ then exactly the same arguments as above (namely in (4.88) to (4.117), with $w_A(\cdot)$ replaced by this $w_1(\cdot)$) yield a corresponding $z_1(\cdot) := Q_A\,(\,w_1(\cdot)\,)$ in $Z_{A\nearrow}^{0,a} \subset Z_A^{0,a}$ (see (4.88) & (4.117)), thus showing that such generic $w_1(\cdot)$ also *belongs* to $\mathcal{W}_A^{0,a}$, see (4.27), hence proving the general *inclusion* in (4.121).

To complete the proof of (4.122) assume, by contradiction, the *existence* of some

$$w_0^0(\cdot) \in \mathcal{W}_A^{0,a} \qquad \text{for which} \qquad (\,\text{recalling } \ (4.82) \ \& \ (4.87)\,) \tag{4.123}$$

$$\begin{aligned} &\textstyle\int_0^a \ell_A^{**}\,(\,w_0^0(t),\, g_A\,(\,w_0^0(t)\,)\;\; {w_0^0}'(t) \quad \rho(t)\,)\;\, d\,t < \\ &< \textstyle\int_0^a \ell_A^{**}\,(\,w_A(t),\, g_A\,(\,w_A(t)\,)\;\; w_A'(t) \quad \rho(t)\,)\;\, d\,t. \end{aligned} \tag{4.124}$$

Then, redefining (recall (4.27) & (4.55))

$$w_0^0(0) \;\text{ to become }\; := \inf w_0^0\,(\,(0,a]\,) \tag{4.125}$$

and setting (recall (4.56) & (4.88))

$$w_0(t) := \min w_0^0\,(\,[t,a]\,) \quad \text{for} \quad t \in [0,a] \tag{4.126}$$

$$z_0(t) := Q_A\,(\,w_0(t)\,) \quad \text{for} \quad t \in [0,a], \tag{4.127}$$

one, by using the same arguments as above, would reach (as in (4.117) & (4.93) & (4.57))

$$z_0(\cdot) \in Z_{A\nearrow}^{0,a} \qquad \subset \qquad Z_A^{0,a} \tag{4.128}$$

$$|\,z_0'(t)\,| = g_A\,(\,w_0(t)\,)\;\; w_0'(t) \qquad\qquad \& \qquad\qquad w_0^0(t) \le w_0(t). \tag{4.129}$$

Moreover, reasoning as after (4.117) but with $z_A(\cdot)$ & $w_A(\cdot)$ & $w_A^0(\cdot)$ & $z_A^0(\cdot)$ replaced by $z_0(\cdot)$ & $w_0(\cdot)$ & $w_0^0(\cdot)$ & $z_0^0(\cdot)$, one reaches, by (4.97) & (4.101) & (4.107) & (4.94) & (4.129), the *inequality*:

$$\begin{aligned} &\textstyle\int_0^a \ell_A^{**}\,(\,w_0(t),\, g_A\,(\,w_0(t)\,)\;\; w_0'(t) \quad \rho(t)\,)\;\, d\,t \;\le \\ &\le\; \textstyle\int_0^a \ell_A^{**}\,(\,w_0^0(t),\, g_A\,(\,w_0^0(t)\,)\;\; {w_0^0}'(t) \quad \rho(t)\,)\;\, d\,t. \end{aligned} \tag{4.130}$$

Therefore, by (4.127) & (4.129) & (4.87) & (4.124),

$$\int_0^a L^{**}\,(\,z_0(t),|\,z_0'(t)\,| \quad \rho(t)\,)\;\, d\,t =$$

$$= \int_0^a L^{**}\,(\,Q_A\,(\,w_0(t)\,)\,,\, g_A\,(\,w_0(t)\,)\;\; w_0'(t) \quad \rho(t)\,)\;\, d\,t =$$

$$= \int_0^a \ell_A^{**}\,(\,w_0(t),\, g_A\,(\,w_0(t)\,)\;\; w_0'(t) \quad \rho(t)\,)\;\, d\,t \le$$

$$\le \int_0^a \ell_A^{**}\left(\,w_0^0(t),\, g_A\left(\,w_0^0(t)\,\right)\;\; {w_0^0}'(t) \quad \rho(t)\right)\;\, d\,t <$$

$$< \int_0^a \ell_A^{**}\left(w_A(t),\, g_A\left(w_A(t)\right) \;\; w_A'(t) \quad\; \rho(t)\right) \, dt = \\ = \int_0^a L^{**}\left(z_A(t), \left| z_A'(t) \right| \quad\; \rho(t)\right) \, dt, \tag{4.131}$$

applying again (4.87) & (4.88) & (4.93), and thus contradicting (4.116), by (4.128). Such absurd denies the possibility of *existence* of a $w_0^0(\cdot)$ as in (4.123) & (4.124) and proves (4.122), due to (4.121), hence (4.120), i.e. the *implication* (4.36) $\Rightarrow$ (4.37) in (4.51).

To complete the proof of (4.51) we now prove the opposite *implication*

$$(4.37) \qquad \Rightarrow \qquad (4.36). \tag{4.132}$$

Taking a

$$w_A(t) \in \mathcal{W}_{A\nearrow}^{0,a} \qquad \textit{minimizer} \;\text{ to both }\; (4.25)\ \&\ (4.26) \tag{4.133}$$

and setting $z_A(t) := Q_A\left(w_A(t)\right)$, as in (4.88), we now claim that $z_A(\cdot) \in Z_{A\nearrow}^{0,a} \subset Z_A^{0,a}$, i.e. (see (4.23) & (4.24))

$$z_A(\cdot) \in W^{1,1}\left([0,a],\, \mathbb{R}^m \right) \;\&\; z_A(a) = A \;\&\; L^{**}\left(z_A(\cdot), 0 \right) \; \textit{increases}. \tag{4.134}$$

Indeed, since $w_A(\cdot) \in W^{1,1}\left([0,a] \right)$ & $w_A(\cdot)$ *increases* & $w_A(a) = L^{**}\left(A, 0 \right)$, by (4.28); and since, by (4.89), $Q_A(\cdot)$ in (4.86) is *Lipschitz*, we have $z_A(\cdot) \in W^{1,1}\left([0,a],\, \mathbb{R}^m \right)$. On the other hand, by (4.59) & (4.86) & (4.81) & (4.69), $z_A(a) = Q_A\left(w_A(a)\right) = \sigma_A\left(\tau_A\left(p_A^{\max}\right)\right) = \sigma_A(0) = A$; and, by (4.90), $L^{**}\left(z_A(\cdot), 0\right) = L^{**}\left(Q_A\left(w_A(\cdot)\right), 0\right) = w_A(\cdot)$ *increases.*

Thus (4.134) is proved; and to complete the proof of (4.116) assume, by contradiction, that

$$\exists\; z_0^0(\cdot) \in Z_A^{0,a} \qquad \text{for which} \tag{4.135}$$

$$\int_0^a L^{**}\left(z_0^0(t), \left| z_0^{0\prime}(t) \right| \quad\; \rho(t)\right) \, dt \;<\; \int_0^a L^{**}\left(z_A(t), \left| z_A'(t) \right| \quad\; \rho(t)\right) \, dt. \tag{4.136}$$

Then, defining, as in (4.55),

$$w_0^0(t) := L^{**}\left(z_0^0(t), 0\right) \;\text{ for }\; t \in (0,a] \;\;\&\;\; w_0^0(0) := \inf L^{**}\left(z_0^0\left((0,a] \right), 0\right),$$

obtain from $w_0^0(t)$ the new *functions* $w_0(\cdot)$ & $z_0(\cdot)$ as in (4.126) & (4.127), hence satisfying (4.128) & (4.129); so that, by (4.87) & (4.127) & (4.129) & (4.136),

$$\int_0^a \ell_A^{**}\left(w_0(t), g_A\left(w_0(t)\right) \;\; w_0'(t) \quad\; \rho(t)\right) \, dt \;=$$

$$= \int_0^a L^{**}\,(\,Q_A\,(\,w_0(t)\,)\,, g_A\,(\,w_0(t)\,)\;\; w_0'(t) \quad \rho(t)\,)\;\; d\,t\; =$$

$$= \int_0^a L^{**}\,(\,z_0(t), |\,z_0'(t)\,|\;\; \rho(t)\,)\;\; d\,t \;<\; \int_0^a L^{**}\,(\,z_A(t), |\,z_A'(t)\,|\;\; \rho(t)\,)\;\; d\,t =$$

$$= \int_0^a L^{**}\,(\,Q_A\,(\,w_A(t)\,)\,, g_A\,(\,w_A(t)\,)\;\; w_A'(t) \quad \rho(t)\,)\;\; d\,t\; =$$

$$= \int_0^a \ell_A^{**}\,(\,w_A(t), g_A\,(\,w_A(t)\,)\;\; w_A'(t) \quad \rho(t)\,)\;\; d\,t,$$

a contradiction to (4.133) showing that no such $z_0^0(\cdot)$ as in (4.135) & (4.136) can be; and proving (4.116), hence (4.132) & (4.51), due to (4.117).

To prove (4.52), i.e.

$$(4.33) \qquad \Rightarrow \qquad (4.37), \tag{4.137}$$

assume (4.33), namely

$$\exists\; w_A^0(\cdot) \in \mathcal{W}_A^{0,a} \quad \textit{minimizing} \;\; (4.25). \tag{4.138}$$

Since, by (4.27), $\exists\; z_A^0(\cdot) \in Z_A^{0,a}$ with $L^{**}\,(\,z_A^0(\cdot), 0\,) = w_A^0(\cdot)$, one may redefine $w_A^0(0)$ as in (4.55), and define $w_A(\cdot)$ & $z_A(\cdot)$ as in (4.56) & (4.88). We claim that :

$$w_A(\cdot) \in \mathcal{W}_{A\nearrow}^{0,a} \qquad \subset \qquad \mathcal{W}_A^{0,a} \tag{4.139}$$

$$0 \le w_A(t) = w_A^0(t) \quad \& \quad 0 < w_A'(t) = w_A^{0\prime}(t), \qquad \forall\, t \in \mathcal{T}_+ \tag{4.140}$$

$$\begin{aligned} &\ell_A^{**}\,(\,w_A(t), g_A\,(\,w_A(t)\,)\;\; w_A'(t) \quad \rho(t)\,) \le \\ &\le\; \ell_A^{**}\,(\,w_A^0(t), g_A\,(\,w_A^0(t)\,)\;\; w_A^{0\prime}(t) \;\; \rho(t)\,) \quad \text{for a.e.} \;\; t \in [0,a] \setminus \mathcal{T}_+ \end{aligned} \tag{4.141}$$

$$w_A(t) \;\; \textit{minimizes} \;\; \text{both} \;\; (4.25) \;\; \& \;\; (4.26). \tag{4.142}$$

Indeed, one may prove (4.139) as in (4.121); while (4.140) follows as in (4.97) & (4.104); and, finally, to prove (4.141) (and noticing that the proof of (4.108) requires the use of (4.54), see (4.118)) we have, a.e. on $[0,a] \setminus \mathcal{T}_+$: $0 = w_A'(t) \le |\,w_A^{0\prime}(t)\,|$, by (4.107) hence, by (4.94) & (4.57),

$$\ell_A^{**}\,(\,w_A(t), g_A\,(\,w_A(t)\,)\;\; w_A'(t) \quad \rho(t)\,) = \ell_A^{**}\,(\,w_A(t), 0\,) = w_A(t) \le w_A^0(t)\; =$$

$$= \ell_A^{**}\left(w_A^0(t), 0\right) \leq \ell_A^{**}\left(w_A^0(t), g_A\left(w_A^0(t)\right)\ w_A^{0\prime}(t) \quad \rho(t)\right),$$

so that (4.141) holds true. Since the *inequality* in (4.141) also holds true (trivially) on $\mathcal{T}_+$, by (4.140), we get (4.142), by (4.138) & (4.139), thus proving (4.137), i.e. (4.52).

Finally, let us prove (4.53), i. e.

$$(4.36) \quad \Rightarrow \quad (4.35) \quad \Rightarrow \quad (4.34) \quad \Rightarrow \quad (4.32). \tag{4.143}$$

Obviously (4.36) $\Rightarrow$ (4.35) also: taking a *minimizer* $z_A(\cdot)$ for both (4.18) & (4.19), then $z_A(\cdot) \in Z_{A\nearrow}^{0,a}$; and setting $U_A(r) := z_A\left(\gamma^{-1}(r)\right)$ one gets (since $\gamma^{-1}(\cdot)$ is *Lipschitz increasing*, see (4.49)) $U_A(\cdot) \in \mathcal{U}_{A\nearrow}^{0,a} \subset \mathcal{U}_A^{0,a}$ (recall (4.24) & (4.17) & (4.31) & [Bic & Orn 3, (95)]); while $U_A(\cdot)$ *minimizes* (4.29), by the comment after (4.49), so that it also *minimizes* (4.30).

Similarly (4.35) $\Rightarrow$ (4.34): taking a *minimizer* $U_A(\cdot)$ for both (4.29) & (4.30) then $U_A(\cdot) \in \mathcal{U}_{A\nearrow}^{0,a}$; and setting $u_A(x) := U_A\left(|x|\right)$ one gets $u_A(\cdot) \in W_{A\nearrow}^{1,1}$ (see (4.16) & (4.17)); while, by (4.43) & (4.45), $u_A(\cdot)$ *minimizes* (4.1), so that obviously it also *minimizes* (4.15).

Trivially (4.34) $\Rightarrow$ (4.32): *existence* of a *minimizer* for (4.1) *implies* at least *existence* of *minimum* for (4.1).

This proves (4.143), i. e. (4.53) hence, by (4.50) & (4.51) & (4.52), also (4.38) & th. 4.3.1. □

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