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On some third order nonlinear boundary value problems: Existence, location and multiplicity results

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Abstract

We prove an Ambrosetti–Prodi type result for the third order fully nonlinear equation

$$u'''(t) + f\big(t, u(t), u'(t), u''(t)\big) = sp(t)$$

with $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ and $p : [0,1] \to \mathbb{R}^+$ continuous functions, $s \in \mathbb{R}$, under several two-point separated boundary conditions. From a Nagumo-type growth condition, an *a priori* estimate on u'' is obtained. An existence and location result will be proved, by degree theory, for $s \in \mathbb{R}$ such that there exist lower and upper solutions. The location part can be used to prove the existence of positive solutions if a non-negative lower solution is considered. The existence, nonexistence and multiplicity of solutions will be discussed as s varies.

Keywords: Nagumo-type conditions; Lower and upper solution; Topological degree; Ambrosetti–Prodi problems

1. Introduction

In this paper we study the following third order fully nonlinear equation

$$u'''(t) + f\big(t, u(t), u'(t), u''(t)\big) = sp(t), \tag{E_s}$$

for $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ and $p : [0,1] \to \mathbb{R}^+$ continuous functions and s a real parameter, with several types of two-point boundary conditions.

If the boundary conditions are

$$u(0) = A, \qquad au'(0) - bu''(0) = B, \qquad cu'(1) + du''(1) = C, \tag{1}$$

for $a, b, c, d, A, B, C \in \mathbb{R}$ and $b, d \geqslant 0$ such that $a^2 + b > 0$ and $c^2 + d > 0$ an existence result is proved, for values of s such that there are lower and upper solutions to the problem (E_s)–(1).

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In Section 3 we consider boundary conditions

$$u(0)=0, \qquad au'(0)-bu''(0)=0, \qquad cu'(1)+du''(1)=0 \tag{2}$$

with $a,b,c,d \geqslant 0$ such that $a+b>0$, $c+d>0$ and proving that the existence of solutions for the problem (E_s)–(2) depends on s.

Considering, in (2), $b=d=0$ with $a,c>0$ the two-point boundary conditions are

$$u(0)=u'(0)=u'(1)=0, \tag{3}$$

an Ambrosetti–Prodi type result is obtained in Section 4. That is, we prove that there are $s_0, s_1 \in \mathbb{R}$ such that (E_s)–(3) has no solution if $s<s_0$, it has at least one solution if $s=s_0$ and (E_s)–(3) has at least two solutions for $s \in\,]s_0, s_1]$.

Equation (E_s) can be seen as a generalized model for various physical, natural or physiological phenomena such as the flow of a thin film of viscous fluid over a solid surface [1,12], the solitary waves solution of the Korteweg–de Vries equation [8] or the thyroid-pituitary interaction [3]. The problem (E_s)–(1) can model the static deflection of an elastic beam with linear supports at both endpoints.

The arguments used were suggested by several papers namely [4], applied to second order periodic problems [11], to third order three points boundary value problems [5–7], for two-point boundary value problems. In short, they make use of a Nagumo-type growth condition [10], the upper and lower solutions technique [2], and Leray–Schauder degree theory [9].

2. Preliminary results

In the following, $C([0,1])$ denotes the space of continuous functions with the norm

$$\|x\| = \max_{t\in[0,1]} \bigl|x(t)\bigr|.$$

Moreover, $C^k([0,1])$ denotes the space of real valued functions with continuous i-derivative in $[0,1]$, for $i=1,\dots,k$, equipped with the norm

$$\|x\|_{C^k} = \max_{0\leqslant i\leqslant k} \bigl\{\bigl|x^{(i)}(t)\bigr|:\ t\in[0,1]\bigr\}.$$

Some growth conditions on the nonlinearity of (E_s) will be assumed in the following. The first one is given by the next definition and provides also an *a priori* estimate for the second derivative of solutions u of (E_s), if some bounds on u and u' are verified.

Definition 1. A continuous function $g:[0,1]\times\mathbb{R}^3 \to \mathbb{R}$ is said to satisfy Nagumo-type condition in

$$E = \bigl\{(t,x,y,z)\in[0,1]\times\mathbb{R}^3:\ \gamma_0(t)\leqslant x\leqslant \Gamma_0(t),\ \gamma_1(t)\leqslant y\leqslant \Gamma_1(t)\bigr\},$$

with γ_0, γ_1, Γ_0 and Γ_1 continuous functions such that $\gamma_0(t)\leqslant\Gamma_0(t)$, $\gamma_1(t)\leqslant\Gamma_1(t)$, for every $t\in[0,1]$, if there exists a continuous function $h_E:\mathbb{R}_0^+ \to [k,+\infty[$, for some fixed $k>0$, such that

$$\bigl|g(t,x,y,z)\bigr| \leqslant h_E\bigl(|z|\bigr), \quad \forall (t,x,y,z)\in E, \tag{4}$$

with

$$\int_0^{+\infty} \frac{\xi}{h_E(\xi)}\,d\xi = +\infty. \tag{5}$$

If these assumptions hold for every $E\subset[0,1]\times\mathbb{R}^3$, given above, then g is said to satisfy Nagumo-type conditions.

Lemma 2. *Let $f:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ be a continuous function that satisfies Nagumo-type conditions* (4) *and* (5) *in*

$$E = \bigl\{(t,x,y,z)\in[0,1]\times\mathbb{R}^3:\ \gamma_0(t)\leqslant x\leqslant \Gamma_0(t),\ \gamma_1(t)\leqslant y\leqslant \Gamma_1(t)\bigr\}, \tag{6}$$

where $\gamma_0,\gamma_1,\Gamma_0,\Gamma_1$ are continuous functions. Then there is $r_*>0$ (depending only on the parameter s and on the functions p, h_E, γ_1 and Γ_1) such that every solution $u(t)$ of (E_s) *verifying*

$$\gamma_0(t) \leqslant u(t) \leqslant \Gamma_0(t), \qquad \gamma_1(t) \leqslant u'(t) \leqslant \Gamma_1(t)$$

for every $t \in [0,1]$, satisfies

$$\|u''\| < r_*.$$

Remark 1. We observe that r_* can be taken independent of s as long as s belongs to some bounded set.

Proof. Considering the non-negative number

$$\eta = \max\{\Gamma_1(1) - \gamma_1(0), \Gamma_1(0) - \gamma_1(1)\}$$

and $r > \eta$ such that

$$\int_\eta^r \frac{\xi}{h_E(\xi) + |s|\|p\|}\, d\xi \geqslant \max_{t\in[0,1]} \Gamma_1(t) - \min_{t\in[0,1]} \gamma_1(t),$$

then the proof follows from [5, Lemma 1], as (E_s) is a particular case of the equation there assumed. □

The appropriate definition of lower and upper-solutions for problem (E_s)–(1) is now given.

Definition 3. Consider $a, b, c, d, A, B, C \in \mathbb{R}$ such that $b, d \geqslant 0$, $a^2 + b > 0$ and $c^2 + d > 0$.

(i) A function $\alpha(t) \in C^3(]0,1[) \cap C^2([0,1])$ is a lower solution of (E_s)–(1) if

$$\alpha'''(t) + f(t, \alpha(t), \alpha'(t), \alpha''(t)) \geqslant sp(t), \quad \text{if } t \in]0,1[,$$

and

$$\alpha(0) \leqslant A, \quad a\alpha'(0) - b\alpha''(0) \leqslant B, \quad c\alpha'(1) + d\alpha''(1) \leqslant C.$$

(ii) A function $\beta(t) \in C^3(]0,1[) \cap C^2([0,1])$ is an upper solution of (E_s)–(1) if

$$\beta'''(t) + f(t, \beta(t), \beta'(t), \beta''(t)) \leqslant sp(t), \quad \text{if } t \in]0,1[,$$

and

$$\beta(0) \geqslant A, \quad a\beta'(0) - b\beta''(0) \geqslant B, \quad c\beta'(1) + d\beta''(1) \geqslant C.$$

For s such that there are upper and lower solutions of (E_s)–(1) with first derivative "well ordered," an existence result and some information concerning the location of the solution of (E_s)–(1) and its derivative are obtained.

Theorem 4. *Let $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Suppose that there are lower and upper solutions of* (E_s)–(1)*, $\alpha(t)$ and $\beta(t)$, respectively, such that, for $t \in [0,1]$,*

$$\alpha'(t) \leqslant \beta'(t)$$

and f satisfies Nagumo-type conditions (4) *and* (5) *in*

$$E_* = \{(t,x,y,z) \in [0,1] \times \mathbb{R}^3 \colon \alpha(t) \leqslant x \leqslant \beta(t),\ \alpha'(t) \leqslant y \leqslant \beta'(t)\}.$$

If f verifies

$$f(t, \alpha(t), y, z) \leqslant f(t,x,y,z) \leqslant f(t, \beta(t), y, z)), \tag{7}$$

for fixed $(t,y,z) \in [0,1] \times \mathbb{R}^2$ and $\alpha(t) \leqslant x \leqslant \beta(t)$, then ($E_s$)–(1) *has at least one solution $u(t) \in C^3([0,1])$ satisfying*

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \quad \alpha'(t) \leqslant u'(t) \leqslant \beta'(t), \quad \forall t \in [0,1].$$

Proof. Define the auxiliary continuous functions

$$\delta_0(t,x)=\begin{cases}\beta(t) & \text{if } x>\beta(t),\\ x & \text{if } \alpha(t)\leqslant x\leqslant\beta(t),\\ \alpha(t) & \text{if } x<\alpha^{(i)}(t),\end{cases}\tag{8}$$

$$\delta_1(t,y)=\begin{cases}\beta'(t) & \text{if } y>\beta'(t),\\ y & \text{if } \alpha'(t)\leqslant y\leqslant\beta'(t),\\ \alpha'(t) & \text{if } y<\alpha'(t),\end{cases}\tag{9}$$

and, for $\lambda\in[0,1]$, the modified problem composed, by

$$u'''(t)+\lambda f\big(t,\delta_0\big(t,u(t)\big),\delta_1\big(t,u'(t)\big),u''(t)\big)-u'(t)+\lambda\delta_1\big(t,u'(t)\big)=\lambda sp(t)\tag{10}$$

and the boundary conditions

$$\begin{aligned}&u(0)=\lambda A,\\ &u'(0)=\lambda\big[B-a\delta_1\big(0,u'(0)\big)+bu''(0)+\delta_1\big(0,u'(0)\big)\big],\\ &u'(1)=\lambda\big[C-c\delta_1\big(1,u'(1)\big)-du''(1)+\delta_1\big(1,u'(1)\big)\big].\end{aligned}\tag{11}$$

Taking $r_1>0$ such that, for every $t\in[0,1]$,

$$\begin{aligned}&-r_1\leqslant\alpha'(t)\leqslant\beta'(t)\leqslant r_1,\\ &sp(t)-f\big(t,\alpha(t),\alpha'(t),0\big)-r_1-\alpha'(t)<0,\\ &sp(t)-f\big(t,\beta(t),\beta'(t),0\big)+r_1-\beta'(t)>0\end{aligned}$$

and

$$\begin{aligned}&\big|B+(1-a)\beta'(0)\big|<r_1, \qquad \big|B+(1-a)\alpha'(0)\big|<r_1,\\ &\big|C+(1-c)\beta'(1)\big|<r_1, \qquad \big|C+(1-c)\alpha'(1)\big|<r_1\end{aligned}$$

the proof follows the arguments used in [5, Theorem 1]. So, only the following details due to a more general boundary conditions are included.

In Step 1 it is proved that every solution u of (10)–(11) satisfies $|u'(t)|<r_1$ and $|u(t)|<r_0$, for every $t\in[0,1]$ and $r_0:=r_1+|A|$, independently of λ.

In Step 2, the set

$$E_r=\big\{(t,x,y,z)\in[0,1]\times\mathbb{R}^3\colon\ |x|\leqslant r_0,\ |y|\leqslant r_1\big\}$$

and the function $F_\lambda: E_r\to\mathbb{R}$ given by

$$F_\lambda(t,x,y,z):=\lambda f\big(t,\delta_0(t,x),\delta_1(t,y),z\big)-y+\lambda\delta_1(t,y)$$

are considered. As $|F_\lambda(t,x,y,z)|\leqslant 2r_1+h_{E_*}(|z|)$ and

$$\int_0^{+\infty}\frac{z}{2r_1+h_{E_*}(z)}\,dz=+\infty$$

then F_λ satisfies a Nagumo-type condition in E_* and the assumptions of Lemma 2 are verified.

In Step 3 the nonlinear operator $\mathcal{N}_\lambda$ is defined by

$$\mathcal{N}_\lambda u=\big(-\lambda f\big(t,\delta_0\big(t,u(t)\big),\delta_1\big(t,u'(t)\big),u''(t)\big)+u'(t)-\lambda\delta_1\big(t,u'(t)\big)+\lambda sp(t),\lambda A,B_\lambda,C_\lambda\big)$$

with

$$\begin{aligned}&B_\lambda:=\lambda\big[B-a\delta_1\big(0,u'(0)\big)+bu''(0)+\delta_1\big(0,u'(0)\big)\big],\\ &C_\lambda:=\lambda\big[C-c\delta_1\big(1,u'(1)\big)-du''(1)+\delta_1\big(1,u'(1)\big)\big]\end{aligned}$$

and the Leray–Schauder degree is evaluated in the set

$$\Omega=\big\{x\in C^2\big([0,1]\big)\colon\ \|x\|<r_0,\ \|x'\|<r_1,\ \|x''\|<r_2\big\}. \qquad \square$$

Example. Consider the differential equation

$$u'''(t) + \left|u''(t)\right|^{\theta} - k\left[u'(t)\right]^{2n+1} + \left[u(t)\right]^{2m+1} = sp(t) \tag{12}$$

for $t \in [0,1]$, $\theta \in [0,2]$, $n,m \in \mathbb{N}$, $k > 0$, $s \in \mathbb{R}$ and $p : [0,1] \to \mathbb{R}^+$ a continuous function, with the boundary conditions

$$u(0) = 0, \quad au'(0) - bu''(0) = B, \quad cu'(1) + du''(1) = C, \tag{13}$$

for $B, C \in \mathbb{R}$, $a,b,c,d \geqslant 0$ with $a+b > 0$ and $c+d > 0$.

If a, c, B and C are such that $|B| \leqslant a$ and $|C| \leqslant c$ then functions $\alpha, \beta : [0,1] \to \mathbb{R}$ given by $\alpha(t) = -t$ and $\beta(t) = t$ are, respectively, lower and upper solutions of problem (12)–(13) for $|s| \leqslant \frac{k}{\|p\|}$. As

$$f(t,x,y,z) = |z|^{\theta} - ky^{2n+1} + x^{2m+1}$$

is continuous and verifies Nagumo-type assumptions (4) and (5) in

$$E = \left\{(t,x,y,z) \in [0,1] \times \mathbb{R}^3\colon\ |x| \leqslant t,\ |y| \leqslant 1\right\} \tag{14}$$

for $h_E(z) = k + 1 + |z|^{\theta}$ then, by Theorem 4, problem (12) has at least one solution $u(t)$ such that

$$-t \leqslant u(t) \leqslant t, \quad -1 \leqslant u'(t) \leqslant 1, \quad \forall t \in [0,1],$$

for $|s| \leqslant \frac{k}{\|p\|}$.

3. Existence and nonexistence results

A first discussion concerning the dependence on s of the existence and nonexistence of a solution will be given in the special case that $A = B = C = 0$ and $a,b,c,d \geqslant 0$ with $a+b > 0$, $c+d > 0$, that is, for (E_s)–(2). Lower and upper solutions definition for this problem are obtained considering in Definition 3 these restrictions.

Theorem 5. *Let $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying a Nagumo-type condition and such that*

(i) *for $(t,y,z) \in [0,1] \times \mathbb{R}^2$*

$$x_1 \geqslant x_2 \quad \Rightarrow \quad f(t,x_1,y,z) \geqslant f(t,x_2,y,z); \tag{15}$$

(ii) *for $(t,x,z) \in [0,1] \times \mathbb{R}^2$*

$$y_1 \geqslant y_2 \quad \Rightarrow \quad f(t,x,y_1,z) \leqslant f(t,x,y_2,z); \tag{16}$$

(iii) *there are $s_1 \in \mathbb{R}$ and $r > 0$ such that*

$$\frac{f(t,0,0,0)}{p(t)} < s_1 < \frac{f(t,x,-r,0)}{p(t)}, \tag{17}$$

for every $t \in [0,1]$ and every $x \leqslant -r$. Then there is $s_0 < s_1$ (with the possibility that $s_0 = -\infty$) such that

(1) *for $s < s_0$, (E_s)–(2) has no solution*;

(2) *for $s_0 < s \leqslant s_1$, (E_s)–(2) has at least one solution.*

Proof. *Step* 1. *There is $s^* < s_1$ such that (E_{s^*})–(2) has a solution.*

Defining

$$s^* = \max\left\{\frac{f(t,0,0,0)}{p(t)},\ t \in [0,1]\right\},$$

by (17), there exists $t^* \in [0,1]$ such that

$$\frac{f(t,0,0,0)}{p(t)} \leqslant s^* = \frac{f(t^*,0,0,0)}{p(t^*)} < s_1, \quad \forall t \in [0,1],$$

and, by the first inequality, $\beta(t) \equiv 0$ is an upper solution of (E_{s^*})–(2).

The function $\alpha(t) = -r\,t$ is a lower solution of (E_{s^*})–(2). In fact, as $\alpha(t) \geqslant -r$, $\alpha'(t) = -r$ and $\alpha''(t) = \alpha'''(t) \equiv 0$, then, by (17) and (15),

$$\alpha'''(t) = 0 > s_1 p(t) - f(t, -r, -r, 0) \geqslant s_1 p(t) - f(t, -rt, -r, 0) > s^* p(t) - f(t, -rt, -r, 0). \tag{18}$$

So, by Theorem 4, there is, at least a solution of (E_{s^*})–(2) with $s^* < s_1$.

Step 2. *If* (E_s)–(2) *has a solution for* $s = \sigma < s_1$, *then it has at least one solution for* $s \in [\sigma, s_1]$.

Suppose that (E_σ)–(2) has a solution $u_\sigma(t)$. For s such that $\sigma \leqslant s \leqslant s_1$,

$$u_\sigma'''(t) = \sigma p(t) - f\big(t, u_\sigma(t), u_\sigma'(t), u_\sigma''(t)\big) \leqslant sp(t) - f\big(t, u_\sigma(t), u_\sigma'(t), u_\sigma''(t)\big)$$

and so $u_\sigma(t)$ is an upper solution of (E_s)–(2) for every s such that $\sigma \leqslant s \leqslant s_1$.

For $r > 0$ given by (17) take $R \geqslant r$ large enough such that

$$u_\sigma'(0) \geqslant -R, \quad u_\sigma'(1) \geqslant -R \quad \text{and} \quad \min_{t\in[0,1]} u_\sigma(t) \geqslant -R. \tag{19}$$

Since, by (17) and (15), for $s \leqslant s_1$,

$$0 > s_1 p(t) - f(t, -R, -r, 0) \geqslant sp(t) - f(t, -Rt, -R, 0)$$

and $-aR \leqslant 0$, $-cR \leqslant 0$ then $\alpha(t) = -Rt$ is a lower solution of (E_s)–(2) for $s \leqslant s_1$.

To apply Theorem 4 the condition

$$-R \leqslant u_\sigma'(t), \quad \forall t \in [0,1], \tag{20}$$

must be verified. Suppose that (20) is not true. Therefore there is $t \in [0,1]$ such that $u_\sigma'(t) < -R$. Defining

$$\min_{t\in[0,1]} u_\sigma'(t) := u_\sigma'(t_0) \quad (< -R)$$

then, by (19), $t_0 \in\,]0,1[$, $u_\sigma''(t_0) = 0$, $u_\sigma'''(t_0) \geqslant 0$ and, by (16), (19) and (17), the following contradiction

$$\begin{aligned} 0 &\leqslant u_\sigma'''(t_0) = \sigma p(t_0) - f\big(t_0, u_\sigma(t_0), u_\sigma'(t_0), u''(t_0)\big) \\ &\leqslant \sigma p(t_0) - f\big(t_0, u_\sigma(t_0), -R, 0\big) \leqslant s_1 p(t_0) - f(t_0, -R, -R, 0) < 0 \end{aligned}$$

is obtained. So $-R \leqslant u_\sigma'(t)$, for every $t \in [0,1]$, and, by Theorem 4, problem (E_s)–(2) has at least a solution $u(t)$ for every s such that $\sigma \leqslant s \leqslant s_1$.

Step 3. *There is* $s_0 \in \mathbb{R}$ *such that*:

- *for* $s < s_0$, (E_s)–(2) *has no solution*;
- *for* $s \in\,]s_0, s_1]$, (E_s)–(2) *has at least a solution.*

Let $S = \{s \in \mathbb{R}\colon\ (E_s)\text{–(2) has at least a solution}\}$. As, by Step 1, $s^* \in S$ then $S \neq \emptyset$. Defining $s_0 = \inf S$, by Step 1, $s_0 \leqslant s^* < s_1$ and, by Step 2, (E_s)–(2) has at least a solution for $s \in\,]s_0, s_1]$ and (E_s)–(2) has no solution for $s < s_0$.

Observe that if $s_0 = -\infty$ then, by Step 2, (E_s)–(2) has a solution for every $s \leqslant s_1$. □

A variant of Theorem 5 can be obtained replacing, in (17), f by $-f$ and x by $-x$.

Theorem 6. *Let* $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ *be a continuous function satisfying a Nagumo-type condition and growth assumptions* (15) *and* (16). *If there are* $s_1 \in \mathbb{R}$ *and* $r > 0$ *such that*

$$\frac{f(t,0,0,0)}{p(t)} > s_1 > \frac{f(t,x,r,0)}{p(t)},$$

for every $t \in [0,1]$ *and every* $x \geqslant r$, *then there is* $s_0 > s_1$ (*with the possibility that* $s_0 = +\infty$) *such that*

(1) *for* $s > s_0$, (E_s)–(2) *has no solution*;
(2) *for* $s_0 > s \geqslant s_1$, (E_s)–(2) *has at least one solution.*

4. Multiplicity results

In the particular case of boundary conditions (1) where $b = d = A = B = C = 0$ and $a, c > 0$ is proved the existence of a second solution for problem (E_s)–(3) as a consequence of a non-null degree for the same operator in two disjoint sets.

The arguments are based on strict lower and upper solutions and some new assumptions on the nonlinearity.

Definition 7. Consider $\alpha, \beta : [0,1] \to \mathbb{R}$ such that $\alpha, \beta \in C^3(]0,1[) \cap C^2([0,1])$.

(i) $\alpha(t)$ is a strict lower solution of (E_s)–(3) if

$$\alpha'''(t) + f\big(t, \alpha(t), \alpha'(t), \alpha''(t)\big) > sp(t), \quad \text{if } t \in]0,1[,$$

and

$$\alpha(0) \leqslant 0, \quad \alpha'(0) < 0, \quad \alpha'(1) < 0. \tag{21}$$

(ii) $\beta(t)$ is a strict upper solution of (E_s)–(3) if

$$\beta'''(t) + f\big(t, \beta(t), \beta'(t), \beta''(t)\big) < sp(t), \quad \text{if } t \in]0,1[,$$

and

$$\beta(0) \geqslant 0, \quad \beta'(0) > 0, \quad \beta'(1) > 0.$$

Define the set $X = \{x \in C^2([0,1]):\ x(0) = x'(0) = x'(1) = 0\}$ and the operators $L : \operatorname{dom} L \to C([0,1])$, with $\operatorname{dom} L = C^3([0,1]) \cap X$, given by $Lu = u'''$ and, for $s \in \mathbb{R}$, $N_s : C^2([0,1]) \cap X \to C([0,1])$ given by

$$N_s u = f\big(t, u(t), u'(t), u''(t)\big) - sp(t).$$

For an open and bounded set $\Omega \subset X$, the operator $L + N_s$ is L-compact in $\overline{\Omega}$ [9]. Note that in $\operatorname{dom} L$ the equation $Lu + N_s u = 0$ is equivalent to problem (E_s)–(3).

The next result will be an important tool used to evaluate the Leray–Schauder topological degree.

Lemma 8. *Consider a continuous function $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ verifying a Nagumo-type condition and* (15). *If there are strict lower and upper solutions of* (E_s)–(3), *$\alpha(t)$ and $\beta(t)$, respectively, such that*

$$\alpha'(t) < \beta'(t), \quad \forall t \in [0,1], \tag{22}$$

then there is $\rho_2 > 0$ such that $d(L + N_s, \Omega) = \pm 1$ for

$$\Omega = \big\{x \in \operatorname{dom} L:\ \alpha(t) < x(t) < \beta(t),\ \alpha'(t) < x'(t) < \beta'(t),\ \|x''\| < \rho_2\big\}.$$

Remark 2. The set Ω can be taken the same for (E_s)–(3), independent of s, as long as α and β are strict lower and upper solutions for (E_s)–(3) and s belongs to a bounded set.

Proof. For the auxiliary functions δ_0, δ_1 defined in (8) and (9) consider the modified problem

$$\begin{cases} u'''(t) + F\big(t, u(t), u'(t), u''(t)\big) = sp(t), \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{23}$$

where $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is the continuous function given by

$$F(t, x, y, z) = f\big(t, \delta_0(t, x), \delta_1(t, y), z\big) - y + \delta_1(t, y)$$

and define the operator $F_s : C^2([0,1]) \cap X \to C([0,1])$ by

$$F_s u = F\big(t, u(t), u'(t), u''(t)\big) - sp(t).$$

With these definitions problem (23) is equivalent to the equation $Lu + F_s u = 0$ in $\operatorname{dom} L$. For $\lambda \in [0, 1]$ and $u \in \operatorname{dom} L$ consider the homotopy

$$H_\lambda u := Lu - (1-\lambda)u'' + \lambda F_s u$$

and take $\rho_1 > 0$ large enough such that, for every $t \in [0, 1]$,

$$-\rho_1 \leqslant \alpha'(t) < \beta'(t) \leqslant \rho_1,$$
$$sp(t) - f\big(t, \alpha(t), \alpha'(t), 0\big) - \rho_1 - \alpha'(t) < 0$$

and

$$sp(t) - f\big(t, \beta(t), \beta'(t), 0\big) + \rho_1 - \beta'(t) > 0.$$

Following the arguments referred in the proof of Theorem 4, there is $\rho_2 > 0$ such that every solution $u(t)$ of $H_\lambda u = 0$ satisfies $\|u'\| < \rho_1$ and $\|u''\| < \rho_2$, independently of $\lambda \in [0, 1]$. Defining

$$\Omega_1 = \big\{x \in \operatorname{dom} L\colon\ \|x'\| < \rho_1,\ \|x''\| < \rho_2\big\}$$

then, every solution u of $H_\lambda u = 0$ belongs to Ω_1 for every $\lambda \in [0, 1]$, $u \notin \partial\Omega_1$ and the degree $d(H_\lambda, \Omega_1)$ is well defined, for every $\lambda \in [0, 1]$.

For $\lambda = 0$ the equation $H_0 u = 0$, that is, the linear problem

$$\begin{cases} u'''(t) - u''(t) = 0, \\ u(0) = u'(0) = u'(1) = 0 \end{cases}$$

has only the trivial solution and, by degree theory, $d(H_0, \Omega_1) = \pm 1$. By the invariance under homotopy

$$\pm 1 = d(H_0, \Omega_1) = d(H_1, \Omega_1) = d(L + F_s, \Omega_1). \tag{24}$$

In the sequel it is proved that if $u \in \Omega_1$ is a solution of $Lu + F_s u = 0$ then $u \in \Omega$.

In fact, by (24), there is $u_1(t) \in \Omega_1$ solution of $Lu + F_s u = 0$. Assume, by contradiction, that there is $t \in [0, 1]$ such that $u_1'(t) \leqslant \alpha'(t)$ and define

$$\min_{t\in[0,1]} \big[u_1'(t) - \alpha'(t)\big] := u_1'(t_1) - \alpha'(t_1) \quad (\leqslant 0).$$

From (21) $t_1 \in]0, 1[$, $u_1''(t_1) - \alpha''(t_1) = 0$ and $u_1'''(t_1) - \alpha'''(t_1) \geqslant 0$. By (15), the following contradiction:

$$\begin{aligned} u_1'''(t_1) &= sp(t_1) - F\big(t_1, u_1(t_1), u_1'(t_1), u_1''(t_1)\big) \\ &= sp(t_1) - f\big(t_1, \delta_0\big(t_1, u_1(t_1)\big), \delta_1\big(t_1, u_1'(t_1)\big), u_1''(t_1)\big) + u_1'(t_1) - \delta_1\big(t_1, u_1'(t_1)\big) \\ &\leqslant sp(t_1) - f\big(t_1, \alpha(t_1), \alpha'(t_1), \alpha''(t_1)\big) + u_1'(t_1) - \alpha'(t_1) \\ &\leqslant sp(t_1) - f\big(t_1, \alpha(t_1), \alpha'(t_1), \alpha''(t_1)\big) < \alpha'''(t_1) \end{aligned}$$

is achieved. Therefore $u_1'(t) > \alpha'(t)$, for $t \in [0, 1]$. In a similar way it can be proved that $u_1'(t) < \beta'(t)$, for every $t \in [0, 1]$ and so $u_1 \in \Omega$.

As the equations $Lu + F_s u = 0$ and $Lu + N_s u = 0$ are equivalent on Ω then

$$d(L + F_s, \Omega_1) = d(L + F_s, \Omega) = d(L + N_s, \Omega) = \pm 1,$$

by (24) and the excision property of the degree. □

The main result is attained assuming that f is bounded from below and it satisfies some adequate condition of monotonicity-type which requires different "speeds" of growth.

Theorem 9. *Let $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function such that the assumptions of Theorem* 5 *are fulfilled. Suppose that there is $M > -r$ such that every solution u of* (E_s)–(3)*, with $s \leqslant s_1$, satisfies*

$$u'(t) < M, \quad \forall t \in [0, 1], \tag{25}$$

and there exists $m \in \mathbb{R}$ such that

$$f(t,x,y,z) \geqslant mp(t), \tag{26}$$

for every $(t,x,y,z) \in [0,1] \times [-r,|M|] \times [-r,M] \times \mathbb{R}$, with r given by (17)*. Then s_0, provided by Theorem* 5*, is finite and*

(1) *if $s < s_0$,* (E_s)–(3) *has no solution*;
(2) *if $s = s_0$,* (E_s)–(3) *has at least one solution.*

Moreover, let $M_1 := \max\{r, |M|\}$ and assume that there is $\theta > 0$ such that, for every $(t,x,y,z) \in [0,1] \times [-M_1, M_1]^2 \times \mathbb{R}$ and $0 \leqslant \eta \leqslant 1$,

$$f(t, x+\eta\theta, y+\theta, z) \leqslant f(t,x,y,z). \tag{27}$$

Then

(3) *for $s \in]s_0, s_1]$,* (E_s)–(3) *has at least two solutions.*

Proof. *Step* 1. *Every solution $u(t)$ of* (E_s)–(3)*, for $s \in]s_0, s_1]$, satisfies $-r < u'(t) < M$ and $-r < u(t) < |M|$, with r given by* (17) *and $t \in [0,1]$.*

For first condition, by (25), it will be enough to show that $-r < u'(t)$, for every $t \in [0,1]$ and for every solution u of (E_s)–(3), with $s \leqslant s_1$.

Suppose, by contradiction, that there are $s \in]s_0, s_1]$, a solution u of (E_s)–(3) and $t_2 \in [0,1]$ such that

$$u'(t_2) := \min_{t \in [0,1]} u'(t) \leqslant -r.$$

By (3), $t_2 \in]0,1[$, $u''(t_2) = 0$ and $u'''(t_2) \geqslant 0$. By (16),

$$0 \leqslant u'''(t_2) = sp(t_2) - f\bigl(t_2, u(t_2), u'(t_2), u''(t_2)\bigr) \leqslant s_1 p(t_2) - f\bigl(t_2, u(t_2), -r, 0\bigr).$$

If $u(t_2) < -r$, from (17) the following contradiction:

$$0 \leqslant s_1 p(t_2) - f\bigl(t_2, u(t_2), -r, 0\bigr) \leqslant s_1 p(t_2) - f(t_2, -r, -r, 0) < 0$$

is obtained. If $u(t_2) \geqslant -r$, from (15) and (17), the same contradiction is achieved. Then every solution u of (E_s)–(3), with $s_0 < s \leqslant s_1$, verifies

$$u'(t) > -r, \quad \forall t \in [0,1].$$

So, by (25), $-r < u'(t) < M$, for every $t \in [0,1]$. Integrating on $[0,t]$, we obtain

$$-r \leqslant -rt < u(t) < Mt \leqslant |M|, \quad \forall t \in [0,1].$$

Step 2. *The number s_0 is finite.*

Suppose that $s_0 = -\infty$, that is, by Theorem 5, for every $s \leqslant s_1$ problem (E_s)–(3) has at least a solution. Define $p_1 := \min_{t \in [0,1]} p(t) > 0$ and take s sufficiently negative such that

$$m - s > 0 \quad \text{and} \quad \frac{(m-s)p_1}{16} > M.$$

If $u(t)$ is a solution of (E_s)–(3), then, by (26),

$$u'''(t) = sp(t) - f\bigl(t, u(t), u'(t), u''(t)\bigr) \leqslant (s-m)p(t)$$

and, by (3), there is $t_3 \in]0,1[$ such that $u''(t_3) = 0$. For $t < t_3$

$$u''(t) = -\int_t^{t_3} u'''(\xi)\,d\xi \geqslant \int_t^{t_3} (m-s)p(\xi)\,d\xi \geqslant (m-s)(t_3 - t)p_1.$$

For $t \geqslant t_3$

$$u''(t) = \int_{t_3}^{t} u'''(\xi)\,d\xi \leqslant (s-m)(t-t_3)p_1.$$

Choose $I = [0, \frac{1}{4}]$, or $I = [\frac{3}{4}, 1]$, such that $|t_3 - t| \geqslant \frac{1}{4}$, for every $t \in I$. If $I = [0, \frac{1}{4}]$, then

$$u''(t) \geqslant \frac{(m-s)p_1}{4}, \quad \forall t \in I,$$

and if $I = [\frac{3}{4}, 1]$, then

$$u''(t) \leqslant \frac{(s-m)p_1}{4}, \quad \forall t \in I.$$

In the first case,

$$0 = \int_0^1 u''(t)\,dt = \int_0^{\frac{1}{4}} u''(t)\,dt + \int_{\frac{1}{4}}^1 u''(t)\,dt \geqslant \int_0^{\frac{1}{4}} \frac{(m-s)p_1}{4}\,dt - u'\left(\frac{1}{4}\right)$$
$$= \frac{1}{16}(m-s)p_1 - u'\left(\frac{1}{4}\right) > M - u'\left(\frac{1}{4}\right),$$

which is in contradiction with (25).

For $I = [\frac{3}{4}, 1]$ a similar contradiction is achieved. Therefore, s_0 is finite.

Step 3. *For* $s \in]s_0, s_1]$ (E_s)–(3) *has at least two solutions.*

As s_0 is finite, by Theorem 5, for $s_{-1} < s_0$, $(\mathrm{E}_{s_{-1}})$–(3) has no solution. By Lemma 2 and Remark 1, we can consider $\rho_2 > 0$ large enough such that the estimate $\|u''\| < \rho_2$ holds for every solution u of (E_s)–(3), with $s \in [s_{-1}, s_1]$.

Let $M_1 := \max\{r, |M|\}$ and define the set

$$\Omega_2 = \{x \in \operatorname{dom} L\colon\ \|x'\| < M_1,\ \|x''\| < \rho_2\}.$$

Then

$$d(L + N_{s_{-1}}, \Omega_2) = 0. \tag{28}$$

By Step 1, if u is a solution of (E_s)–(3), with $s \in [s_{-1}, s_1]$, then $u \notin \partial\Omega_2$. Defining the convex combination of s_1 and s_{-1} as $H(\lambda) = (1-\lambda)s_{-1} + \lambda s_1$ and considering the corresponding homotopic problems $(\mathrm{E}_{H(\lambda)})$–(3), the degree $d(L + N_{H(\lambda)}, \Omega_2)$ is well defined for every $\lambda \in [0, 1]$ and for every $s \in [s_{-1}, s_1]$. Therefore, by (28) and the invariance of the degree

$$0 = d(L + N_{s-1}, \Omega_2) = d(L + N_s, \Omega_2), \tag{29}$$

for $s \in [s_{-1}, s_1]$.

Let $\sigma \in]s_0, s_1] \subset [s_{-1}, s_1]$ and $u_\sigma(t)$ be a solution of (E_σ)–(3), which exists by Theorem 5. Take $\varepsilon > 0$ such that

$$|u'_\sigma(t) + \varepsilon| < M_1, \quad \forall t \in [0, 1]. \tag{30}$$

Then $\tilde{u}(t) := u_\sigma(t) + \varepsilon\, t$ is a strict upper solution of (E_s)–(3), with $\sigma < s \leqslant s_1$. In fact, by (27) with $\theta = \varepsilon$ and $\eta = t$, for such σ,

$$\begin{aligned}
\tilde{u}'''(t) = u'''_\sigma(t) &= \sigma p(t) - f\big(t, u_\sigma(t), u'_\sigma(t), u''_\sigma(t)\big) \\
&< sp(t) - f\big(t, u_\sigma(t), u'_\sigma(t), \tilde{u}''(t)\big) \\
&\leqslant sp(t) - f\big(t, u_\sigma(t) + \varepsilon t, u'_\sigma(t) + \varepsilon, \tilde{u}''(t)\big) \\
&= sp(t) - f\big(t, \tilde{u}(t), \tilde{u}'(t), \tilde{u}''(t)\big),
\end{aligned}$$

$$\tilde{u}(0) = 0, \qquad \tilde{u}'(0) = \tilde{u}'(1) = \varepsilon > 0.$$

Moreover $\alpha(t) := -r\, t$ is a strict lower solution of (E_s)–(3), for $s \leqslant s_1$. Indeed, by (17) and (15),

$$\alpha'''(t) = 0 > s_1 p(t) - f(t, -r, -r, 0) \geqslant sp(t) - f(t, -rt, -r, 0),$$
$$\alpha(0) = 0, \qquad \alpha'(0) = \alpha'(1) = -r < 0.$$

By Step 1, $-r < u'_\sigma(t)$ for every $t \in [0, 1]$ and therefore $-r < u'_\sigma(t) + \varepsilon$, $\forall t \in [0, 1]$, that is, $\alpha'(t) < \tilde{u}'(t)$. Integrating on $[0, t]$

$$\alpha(t) \leqslant \alpha(t) - \alpha(0) < \tilde{u}(t) - \tilde{u}(0) = \tilde{u}(t),$$

for every $t \in [0, 1]$.

Then, by (30), Lemma 8 and Remark 2, there is $\bar{\rho}_2 > 0$, independent of s, such that for

$$\Omega_\varepsilon = \{x \in \operatorname{dom} L\colon\ \alpha(t) < x(t) < \tilde{u}(t),\ \alpha'(t) < x'(t) < \tilde{u}'(t),\ \|x''\| < \bar{\rho}_2\}$$

the degree of $L + N_s$ in Ω_ε satisfies

$$d(L + N_s, \Omega_\varepsilon) = \pm 1, \quad \text{for } s \in\ ]\sigma, s_1]. \tag{31}$$

Taking ρ_2 in Ω_2 large enough such that $\Omega_\varepsilon \subset \Omega_2$, by (29), (30) and the additivity of the degree, we obtain

$$d(L + N_s, \Omega_2 - \overline{\Omega_\varepsilon}) = \mp 1, \quad \text{for } s \in\ ]\sigma, s_1]. \tag{32}$$

So, problem (E_s)–(3) has at least two solutions u_1, u_2 such that $u_1 \in \Omega_\varepsilon$ and $u_2 \in \Omega_2 - \overline{\Omega_\varepsilon}$, for $s \in\ ]s_0, s_1]$, since σ is arbitrary in $]s_0, s_1]$.

Step 4. *For $s = s_0$,* (E_s)–(3) *has at least one solution.*

Consider a sequence (s_m) with $s_m \in\ ]s_0, s_1]$ and $\lim s_m = s_0$. By Theorem 5, for each s_m, (E_{s_m})–(3) has a solution u_m. Using the estimates of Step 1, it is clear that $\|u_m\| < M_1$, $\|u'_m\| < M_1$ independently of m, and, by Remark 1, there is $\tilde{\rho}_2 > 0$ large enough such that $\|u''_m\| < \tilde{\rho}_2$, independently of m. Then sequences (u_m) and (u'_m), $m \in \mathbb{N}$, are bounded in $C([0, 1])$. By the Arzelà–Ascoli theorem, we can take a subsequence of (u_m) that converges in $C^2([0, 1])$ to a solution $u_0(t)$ of (E_{s_0})–(3).

Hence, there is at least one solution for $s = s_0$. □

A variant of Theorem 9 can be obtained replacing f by $-f$, x by $-x$ and y by $-y$.

Theorem 10. *Consider $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ a continuous function such that the assumptions of Theorem* 6 *are fulfilled. Suppose that there is $M > -r$ such that every solution u of* (E_s)–(3)*, with $s \geqslant s_1$, satisfies*

$$u'(t) > M, \quad \forall t \in [0, 1],$$

and there exists $m \in \mathbb{R}$ such that

$$f(t, x, y, z) \leqslant mp(t),$$

for every $(t, x, y, z) \in [0, 1] \times [-r, |M|] \times [-r, M] \times \mathbb{R}$. Then s_0 provided by Theorem 6 *is finite and*

(1) *if $s > s_0$,* (E_s)–(3) *has no solution*;
(2) *if $s = s_0$,* (E_s)–(3) *has at least one solution.*

Moreover, if condition (27) *holds then*

(3) *for $s \in [s_1, s_0[$,* (E_s)–(3) *has at least two solutions.*

Example. Consider a particular case of problem (12)–(13) with $n = m = 1$, $k = 4$, $b = d = B = C = 0$, $a, c > 0$ and $p(t) \equiv 1$, that is

$$(P) \quad \begin{cases} u'''(t) + |u''(t)|^\mu - 4(u'(t))^3 + (u(t))^3 = s, \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

with $\mu \in [0, 2]$. The function $f(t, x, y, z) = |z|^\mu - 4y^3 + x^3$ is continuous, verifies the Nagumo-type assumptions in E, given by (14), and monotonicity conditions (15) and (16). Consider s_1 and $r > 0$ large enough such that

$$0 < s_1 < f(t, x, -r, 0) = 4r^3 + x^3$$

holds for every $x \leqslant -r$. Therefore by Theorem 5 there is $s_0 < s_1$ such that (P) has no solution for $s < s_0$ (if $s_0 = -\infty$, (P) has a solution for every $s < s_1$) and for $s_0 < s \leqslant s_1$ problem (P) has at least a solution.

For r_* given by Lemma 2 define the set

$$E_1 = \{(t, x, y, z) \in [0, 1] \times \mathbb{R}^3 \colon |x| \leqslant 1,\ |y| \leqslant 1,\ |z| \leqslant r_*\} \subset E.$$

Therefore, following the arguments of the proof of Theorem 4, for $f : E_1 \to \mathbb{R}$ every solution u of (P) verifies $|u'(t)| \leqslant 1$ in $[0, 1]$ and condition (26) holds with $m = -(5 + r_*^\mu)$. Moreover, for $0 \leqslant \eta \leqslant 1$ and $\theta \geqslant \frac{5+\sqrt{29}}{2}$, the inequality

$$f(t, x + \eta\theta, y + \theta, z) = (x + \eta\theta)^3 - 4(y + \theta)^3 + |z|^\mu \leqslant f(t, x, y, z)$$

is verified for $(t, x, y, z) \in [0, 1] \times [-1, 1]^2 \times \mathbb{R}$. So, by Theorem 9, s_0 is finite and for $s_0 < s \leqslant s_1$ problem (P) has at least two solutions.

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