On vanishing dissipative-dispersive perturbations of hyperbolic conservation laws

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Abstract: In presence of linear diffusion and non-positive dispersion, we prove well-posedness of the nonlinear conservation equation $u_t + f(u)_x = \varepsilon u_{xx} - \delta(u_{xx}^2)_x$. Then, as the right-hand perturbations vanish, we prove convergence of the previous solutions to the entropy weak solution of the hyperbolic conservation law $u_t + f(u)_x = 0$.

Key–Words: diffusion, viscosity, capillarity, dissipation, dispersion, KdV equation, Burgers' equation, hyperbolic conservation laws, entropy measure-valued solutions

1 Introduction

We consider the initial value problem

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta(u_{xx}^2)_x \qquad (1)$$

$$u(x, 0) = u_0(x). \qquad (2)$$

When $\delta = 0$ we reduce to the (generalized) Burgers' equation and the approximate solutions $u^{\varepsilon,0}$ converge to the solution of the inviscid Burgers' equation (this is the *vanishing viscosity method*, see, e.g., Whitham [13] or Kružkov [6])

$$u_t + f(u)_x = 0 \qquad (3)$$

$$u(x, 0) = u_0(x). \qquad (4)$$

On the other hand, when $\varepsilon = 0$, if we consider the flux function $f(u) = u^2$ and the linear dispersion δu_{xxx} we obtain the Korteweg-de Vries equation. The approximate solutions $u^{0,\delta}$ do not converge in a strong topology, Lax-Levermore [7]). So, as parameters ε and δ vanish, we are concerned with singular limits and to ensure convergence it is necessary a dominant dissipation regime.

The pioneer study of these singular limits was given by Schonbek [11] about (generalized) Korteweg-de Vries-Burgers equation

$$u_t + f(u)_x = \varepsilon\, u_{xx} - \delta\, u_{xxx}.$$

In the case of a convex flux function $f(u)$, she proved the convergence under the condition that $\delta = o(\varepsilon^2)$, while the sharp condition should be, according to Perthame-Ryzhic [10], $\delta = o(\varepsilon^1)$.

See also the analogy between the singular limit for the Korteweg-de Vries-Burgers equation and the hydrodynamic limit of the kinetic Boltzmann equation for a rarefied gas to the continuum Euler equations of compressible gas dynamics as the Knudsen number approaches zero in "From Boltzmann to Euler: Hilbert?s 6th problem revisited", Slemrod [12].

LeFloch-Natalini [8] proved the convergence in the case of a nonlinear viscosity function β and linear capillarity

$$u_t + f(u)_x = \varepsilon\, \beta(u_x)_x - \delta\, u_{xxx}.$$

Then, Correia-LeFloch [4] improved the estimates in Schonbek [11] and LeFloch-Natalini [8] and for the first time treated the multidimensional equation, but still in the case of a nonlinear viscosity function and linear capillarity. In fact, there, the dominant dissipation regime is also assured by the nonlinear viscosity. In our case, we consider the reverse situation.

In general for $\varepsilon = 0$, like for the Korteweg-de Vries equation, the divergent behaviour is expected, as we are considering "pure-dispersive equations". But, Brenier-Levy [3] considered the fully nonlinear equation

$$u_t + f(u)_x = -\delta(u_{xx}^2)_x$$

as a nonlinear generalization of the Korteweg-de Vries equation. Such nonlinear dispersion significantly affects the dispersive behaviour of the solutions that differs completely from the linear case. In particular, Brenier and Levy conjectured that for strictly convex flux functions f we have convergence when $\delta = o(\varepsilon^1)$.

In this work we show first that the initial value problem (1)-(2) is well-posed and then we prove the convergence to the initial value problem (5)-(6). So, our proof of convergence is not formal. To obtain the well-posedness, a condition which links the dispersion and the dissipation is needed. It can be written as, for all u_0 sufficiently smooth initial data,

$$||u_0||_{H^4} \leq C\varepsilon/\delta.$$

And the vanishing dissipation-dispersion limit is obtained when $\delta \ll \varepsilon$.

The paper is organized as follows. In section 2, we prove the well-posedness of the perturbed initial value problem. Then section 3 deals with the hyperbolic limit as ε, δ go to zero.

2 Well-posedness

We prove here that the initial value problem (1)-(2) is well-posed.

2.1 Regularized equation

To compute the well-posedness of the initial value problem, we consider the fourth order regularization, with $\mu > 0$

$$u_t + f(u)_x + \delta(u_{xx}^2)_x - \varepsilon u_{xx} + \mu u_{xxxx} = 0 \quad (5)$$

$$u(x,0) = u_0(x). \quad (6)$$

The solution of the linearized equation

$$S_t u(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{i\xi x - \varepsilon\xi^2 t - \mu\xi^4 t} d\xi,$$

satisfies the following regularization property. For $r, s \geq 0$ and $u \in H^s()$, we have

$$||S_t u||_{r+s} \leq C_r \left(1 + \left(\frac{1}{2\mu|t|}\right)^{r/2}\right)^{1/2} ||u||_s.$$

Proposition 1 *Let $s > 5/2$. Assume that $|f(u)| = \mathcal{O}(|u|^{\beta+1})$ with $\beta \geq 1$. Then there exists $T_\mu > 0$, depending on μ, such that*

$$\phi(u)(t) := S_t u_0 - \int_0^t S_{t-\tau}(f(u)_x - \delta(-u_{xx}^2)_x)(\tau) d\tau, \quad (7)$$

is a contraction mapping on the closed ball

$$\overline{B}(T_\mu) = \{u \in \mathcal{C}([0, T_\mu]; H^s(\mathbb{R})) : ||u(t) - u_0||_s \leq c||u_0||_s\}.$$

Proof: Let $u, v \in \overline{B}(T_\mu)$. We have

$$\phi(u)(t) - \phi(v)(t) = \int_0^t S_{t-\tau}((f(u)_x - f(v)_x) + \delta((-u_{xx}^2)_x - (-v_{xx}^2)_x))(\tau) d\tau.$$

On one hand, we write

$$||S_{t-\tau}(f(u)_x - f(v)_x)(\tau)||_s = ||S_{t-\tau}(f(u)_x - f(v)_x)(\tau)||_{(s-1)+1} \leq C_1 \left(1 + \left(\frac{1}{2\mu(t-\tau)}\right)^{1/2}\right)^{1/2} ||f(u)_x - f(v)_x||_{s-1},$$

and, the Sobolev embedding implies with $s > 1/2$

$$||f(u)_x - f(v)_x||_{s-1} = ||f(u) - f(v)||_s \leq C||u_0||^\beta ||u - v||_s.$$

On the other hand, it gets thanks to the Sobolev embedding with $s > 5/2$ and using $u^2 - v^2 = (u+v)(u-v)$

$$\left|\left|S_{t-\tau}(u_{xx}^2)_x - (v_{xx}^2)_x)(\tau)\right|\right|_s \leq C||u_0||_s \left(1 + \left(\frac{1}{2\mu(t-\tau)}\right)^{3/2}\right)^{1/2} ||u - v||_s,$$

We deduce

$$\sup_{t\in[0,T]} ||\phi(u)(t) - \phi(v)(t)||_s \leq C(\mu, ||u_0||_{H^s}, T) \sup_{t\in[0,T]} ||u - v||_s,$$

and we choose $T > 0$ so that ϕ is a contraction mapping in $\overline{B}(T_\mu)$. □

First of all, we have to prove that the time T can be chosen independently of μ.

Theorem 2 *Let $s > 5/2$. Assume that $|f^{(i)}(u)| = \mathcal{O}(|u|^{\beta+1-i})$, for $0 \leq i \leq 2$ and $\beta \geq 1$. Let $u_0 \in H^4(\mathbb{R})$ with $||u_0||_4 \leq \varepsilon/(2\delta)$. Then there exists a time T, independent of μ, such that there exists a unique solution $u \in \mathcal{C}([-T, T]; H^s\mathbb{R})$ of the initial value problem (5)-(6).*

Proof: Multiplying the equation (5) by $\sum_{i=0}^{4} \partial_x^{2i} u$ and integrating over space give

$$\frac{1}{2}\frac{d}{dt}||u(t)||_4^2 + \mu \sum_{i=2}^{6} \int_{-\infty}^{+\infty} (\partial_x^i u)^2 dx =$$
$$\sum_{i=0}^{4} \int_{-\infty}^{+\infty} (-1)^{i+1} (\partial_x^{2i} u) f(u)_x dx$$
$$+ \sum_{i=0}^{4} \int_{-\infty}^{+\infty} \delta(-1)^i (\partial_x^{2i} u)(-u_{xx}^2)_x$$
$$+\varepsilon(-1)^i (\partial_x^{2i} u) u_{xx} dx = \mathrm{I} + \mathrm{II}.$$

Using the Sobolev embedding and the Gagliardo-Nirenberg inequality[9], we obtain

$$\mathrm{I} \leq C_s ||u||_s^{\beta+2},$$

and

$$\mathrm{II} = \int_{-\infty}^{+\infty} u_{5x}^2 \left(-\varepsilon + \frac{14\delta}{5} u_{xxx}\right) dx + O(||u||_4^3).$$

We deduce that, for $\beta > 0$,

$$\frac{d}{dt}||u(t)||_4^2 + \mu \sum_{i=2}^{6} ||(\partial_x^i u)||_{L^2}^2 = \mathcal{O}(||u||_4^{\beta+2})$$
$$+ \int_{-\infty}^{+\infty} u_{5x}^2 \left(-\varepsilon + \frac{14\delta}{5} u_{xxx}\right) dx. \quad (8)$$

Thanks to $||u||_\infty \leq \sqrt{2}||u||_{L^2}^{1/2} ||u_x||_{L^2}^{1/2}$, it gets

$$||u_{0,xxx}||_\infty \leq \sqrt{2}||u_0||_4 \leq \frac{\varepsilon}{\sqrt{2}\delta},$$

and

$$\int_{-\infty}^{+\infty} u_{0,5x}^2 \left(-\varepsilon + \frac{14\delta}{5} u_{0,xxx}\right) dx \leq 0.$$

Since $||u(t)||_4^2 \leq y(t)$ where y is solution of

$$y'(t) = 2C(t)^{(p+2)/2}$$
$$y(0) = ||u_0||_4^2,$$

we can choose $T > 0, (T = 1/(2p||u_0||_4^p))$ such $||u(t)||_4 \leq \varepsilon/(2\delta)$ for all $t \leq T$. □

2.2 Regularization limit

Theorem 3 *Let $u_0 \in H^4(\mathrm{IR})$ with $||u_0||_4 \leq \varepsilon/(2\delta)$. There exists $T > 0$, inversely proportional to $||u_0||_4$, such that there exists a unique solution $u \in \mathcal{C}([-T, T], H^4(\mathrm{IR}))$ of the initial value problem* (1)-(2).
Moreover, there exists $C > 0$ such that the solutions u and v, with u_0 and v_0 as initial datum respectively, satisfy for $|t| \leq T$,

$$||u(t) - v(t)||_4 \leq C||u_0 - v_0||_4.$$

Proof: To obtain the limit as μ goes to zero, we show that the solution $(u^\mu(t))_\mu$ is a Cauchy sequence for $t \in [0, T]$. Let $\mu, \nu \geq 0$, and u^μ, v^ν be the respective solution of (5)-(6). We have, for $t \in [0, T]$,

$$\begin{aligned}\partial_t ||u^\mu - v^\nu||^2 &= 2 < u - v, u_t - v_t > \\ &= -2 < u - v, f(u)_x - f(v)_x > \\ &\quad + 2\delta < u - v, g(u_x)_{xx} \\ &\quad - g(v_x)_{xx} > \\ &\quad + 2\varepsilon < u - v, u_{xx} - v_{xx} > \\ &\quad - < u - v, \mu u_{xxxx} - \nu v_{xxxx} >\end{aligned}$$

and it comes for $z_i = (1 - \lambda_i)u + \lambda_i v, i = 1, 2$ and $\lambda \in (0, 1)$

$$\begin{aligned}\partial_t ||u^\mu - v^\nu||^2 &= -2 \int_{-\infty}^{+\infty} \frac{(u - v)^2}{2} f'(z_1)_x dx \\ &\quad -2 \int_{-\infty}^{+\infty} ((u - v)_x)^2 (\varepsilon \\ &\quad -\delta z_{2,xxx}) dx \\ &\quad -\mu \int_{-\infty}^{+\infty} ((u - v)_{xx})^2 dx \\ &\quad +(\mu - \nu) \int_{-\infty}^{+\infty} (u - v)_{xx} v_{xx} dx \\ &\leq \left| \int_{-\infty}^{+\infty} (u - v)^2 f'(z_1)_x dx \right| \\ &\quad +|\mu - \nu| \left| \int_{-\infty}^{+\infty} (u - v)_{xx} v_{xx} dx \right| \\ &\leq C_M ||u^\mu - v^\nu||^2 + C_M |\mu - \nu|,\end{aligned}$$

because $||u^\mu(t)||_s$ and $||v^\nu(t)||_s$ are uniformly bounded. We conclude using the Gronwall lemma [2, 5]. □

Remark 4

- *The time T, proportional to $1/||u_0||_4$, is also the time well-posedness of the purely hyperbolic initial value problem.*
- *The constraint $||u_0||_4 \leq \varepsilon/(2\delta)$ is not so restrictive, δ being chosen very small compared to ε to obtain the hyperbolic limit.*

3 Convergence

Theorem 5 (Main theorem) *Let $\varepsilon > 0$, $\delta = o(\varepsilon^{5/2})$ and $f : \mathrm{IR} \to \mathrm{IR}$ be a convex flux function satisfying $f''(u) \leq C(1 + |u|^\beta)$, where $0 \leq \beta < 1/2$. Then, setting $u = u_{\varepsilon,\delta}$ the solution of* (1)-(2)*, the family of solutions $\{u_{\varepsilon,\delta}\}$ converges to the entropy solution of* **(??)**-**(??)**.

3.1 A priori Estimates

Multiply (1) by a function $\eta'(u)$ and let $q' = \eta' f'$ be the derivative of a new flux function:

$$\begin{aligned}\eta(u)_t + q(u)_x &= \varepsilon\,(\eta'(u)\,u_x)_x - \varepsilon\,\eta''(u)\,u_x^2 \\ &\quad - \delta\,(\eta'(u)\,u_{xx}^2)_x \\ &\quad - \delta\,\eta''(u)\,u_x\,u_{xx}^2\,. \end{aligned} \tag{9}$$

Integrate over $\mathbb{R} \times [0,t]$ with $\eta(u) = |u|^{\alpha+1}$. The conservative terms vanish and we obtain the

Lemma 6 *Let $\alpha \geq 1$. Each solution of* (1) *satisfies for $t \in [0,T]$*

$$\begin{aligned}&\int_{\mathbb{R}} |u(t)|^{\alpha+1}\,dx + (\alpha+1)\,\alpha\,\varepsilon \int_0^t\int_{\mathbb{R}} |u|^{\alpha-1} \\ &u_x^2\,dxds - (\alpha+1)\,\alpha\,\delta \int_0^t\int_{\mathbb{R}} |u|^{\alpha-1}\,u_x\,u_{xx}^2\,dxds \\ &= \int_{\mathbb{R}} |u_0|^{\alpha+1}\,dx\,. \end{aligned} \tag{10}$$

Usually, taking $\alpha = 1$ in (10), we deduce the a priori L^2 first energy estimates. It is not the case here, unless the factor δu_x of u_{xx}^2 is always negative.

We use now the multipliers $(q+2)(|u_x|^q u_x)_x$ and $(q+2)(u_x^{q+1})_x$ to obtain

$$\begin{aligned}&\int_{\mathbb{R}} |u_x(t)|^{q+2}\,dx \\ &+\varepsilon\,(q+2)(q+1)\int_0^t\int_{\mathbb{R}} |u_x|^q\,u_{xx}^2\,dxds \\ &= \int_{\mathbb{R}} |u_0'|^{q+2}\,dx - (q+1)\int_0^t\int_{\mathbb{R}} u_x\,|u_x|^{q+2} \\ &f''(u)\,dxds - \frac{1}{3}\delta\,(q+2)(q+1)\,q\int_0^t\int_{\mathbb{R}} u_x \\ &|u_x|^{q-2}\,u_{xx}^4\,dxds\,, \end{aligned} \tag{11}$$

$$\begin{aligned}&\int_{\mathbb{R}} u_x(t)^{q+2}\,dx \\ &+\varepsilon\,(q+2)(q+1)\int_0^t\int_{\mathbb{R}} u_x^q\,u_{xx}^2\,dxds \\ &= \int_{\mathbb{R}} (u_0')^{q+2}\,dx - (q+1)\int_0^t\int_{\mathbb{R}} u_x^{q+3} \\ &f''(u)\,dxds - \frac{1}{3}\delta\,(q+2)(q+1)\,q\int_0^t\int_{\mathbb{R}} u_x^{q-1} \\ &u_{xx}^4\,dxds\,. \end{aligned} \tag{12}$$

We restrict to odd q and $\delta > 0$, we add (12) to (11). We abbreviate as $\mathcal{U}^+$ (analogously for $\mathcal{U}^-$) the $\{(x,t) \in \mathbb{R} \times [0,T] : \delta\,u_x > 0\}$ or their section by $t = s$ as ${\mathcal{U}_s}^+$. We obtain:

Lemma 7 *Let q be a odd number, then each solution of* (1) *satisfies for $t \in [0,T]$*

$$\begin{aligned}&\int_{{\mathcal{U}_t}^+} |u_x(t)|^{q+2}\,dx \\ &+\varepsilon\,(q+2)(q+1)\int_0^t\int_{{\mathcal{U}_s}^+} |u_x|^q\,u_{xx}^2\,dxds \\ &+\frac{1}{3}\delta\,(q+2)(q+1)\,q\int_0^t\int_{{\mathcal{U}_s}^+} |u_x|^{q-1}\,u_{xx}^4\,dxds \\ &+(q+1)\int_0^t\int_{{\mathcal{U}_s}^+} |u_x|^{q+3}\,f''(u)\,dxds \\ &= \int_{{\mathcal{U}_0}^+} |u_0'|^{q+2}\,dx\,. \end{aligned} \tag{13}$$

Actually, Lemmas 6 and 7 together will solve our problem.

Proposition 8 *Let $\varepsilon > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a convex flux function. Then, setting $u = u_{\varepsilon,\delta}$ the solution of* (1)*, the family of solutions $\{u_{\varepsilon,\delta}\}$ satisfy the estimate*

$$\begin{aligned}&\int_{\mathbb{R}} |u(t)|^{\alpha+1}\,dx + \varepsilon\int_0^t\int_{\mathbb{R}} |u|^{\alpha-1}\,u_x^2\,dxds \\ &+\delta\int_0^t\int_{\mathbb{R}} |u|^{\alpha-1}\,|u_x|\,u_{xx}^2\,dxds \leq C_0, \end{aligned} \tag{14}$$

for all $\frac{7}{5} \leq \alpha < 3$.

When $\delta \leq k\,\varepsilon$, then the family of solutions $\{u_{\varepsilon,\delta}\}$ satisfy the estimate (14) *for $\alpha = 1$, i.e.,*

$$\begin{aligned}&\int_{\mathbb{R}} u(t)^2\,dx + \varepsilon\int_0^t\int_{\mathbb{R}} u_x^2\,dxds \\ &+\delta\int_0^t\int_{\mathbb{R}} |u_x|\,u_{xx}^2\,dxds \leq C_0. \end{aligned} \tag{15}$$

If in addition $f''(u) \leq C(1 + |u|^{\beta})$, where $0 \leq \beta < 1/2$, and $\delta \leq k\,\varepsilon$, then $\{u_{\varepsilon,\delta}\}$ satisfy

$$\begin{aligned}&\int_{\mathbb{R}} u_x(t)^2\,dx + \varepsilon\int_0^t\int_{\mathbb{R}} u_{xx}^2\,dxds \\ &\leq C_0 + C_0\,\delta^{-1/2}\epsilon^{-1/4} \end{aligned} \tag{16}$$

Proposition 9 *Let $\varepsilon > 0$, $\delta = o(\varepsilon^{5/2})$ and $f : \mathbb{R} \to \mathbb{R}$ be a convex flux function satisfying $f''(u) \leq C(1+|u|^{\beta})$, where $0 \leq \beta < 1/2$. Then, setting $u = u_{\varepsilon,\delta}$ the solution of* (1)*, the family solutions $\{u_{\varepsilon,\delta}\}$ satisfy*

(a): $\{\varepsilon\,u_x^2\}$ is bounded in $L^1(\Omega)$.

(b): $\{\varepsilon\,u_x\} \to 0$ when $\varepsilon \to 0$, in $L^2(\Omega)$.

(c): $\{\delta\,u_x^-\,u_{xx}^2\}$, where $u_x^- = max(0, -u_x)$, is bounded in $L^1(\Omega)$.

(d): $\{\delta\,u_x^+\,u_{xx}^2\} \to 0$, where $u_x^+ = max(0, u_x)$ when $\varepsilon \to 0$ in $L^1(\Omega)$.

(e): $\{\delta\,u_{xx}^2\} \to 0$ when $\varepsilon \to 0$, in $L^1(\Omega)$.

Proof: The statements $(a), (b)$ and (c) are obtained thanks to (15). Now, (d) is obtained from (13) with $q = 1$ since

$$\delta \int_0^t \int_{\mathcal{U}_s{}^+} u_x\, u_{xx}^2\, dxds \qquad (17)$$
$$\leq \frac{\delta}{\varepsilon} \left(\varepsilon \int_0^t \int_{\mathcal{U}_s{}^+} u_x\, u_{xx}^2\, dxds \right) \leq C_0\, \frac{\delta}{\varepsilon}.$$

Finally, (e) is obtained thanks to (9) since,

$$\delta \int_0^t \int_{\mathrm{I\!R}} u_{xx}^2\, dxds \qquad (18)$$
$$\leq \delta^{\frac{1}{2}}\, \varepsilon^{-5/4} (\delta^{\frac{1}{2}}\, \varepsilon^{5/4} \int_0^t \int_{\mathrm{I\!R}} u_{xx}^2\, dxds)$$
$$\leq C_0\, \sqrt{\delta \varepsilon^{-5/2}}.$$

□

3.2 Convergence proof

Definition 10 *Assume that* $u_0 \in L^1(\mathrm{I\!R}) \cap L^q(\mathrm{I\!R})$ *and* $f \in \mathcal{C}(\mathrm{I\!R})$ *satisfies the growth condition, for some* $m \in [0, q)$

$$|f(u)| \leq \mathcal{O}(|u|^m) \text{ as } |u| \to \infty. \qquad (19)$$

A Young measure ν *is called an entropy measure-valued (e.m.-v.) solution to* (1)-(2) *if for all* $k \in \mathrm{I\!R}$

$$\langle \nu, |u - k| \rangle_t + \langle \nu, sgn(u - k)(f(u) - f(k)) \rangle_x \leq 0 \qquad (20)$$

in the sense of distributions on $\mathrm{I\!R} \times (0, T)$ *and for all compact set* $K \subseteq \mathrm{I\!R}$

$$\lim_{t \to 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)|\, \rangle\, dxds = 0. \qquad (21)$$

Lemma 11 *Let* $\{u_n\}_{n \in \mathrm{I\!N}}$ *be a bounded sequence in* $L^\infty((0, T); L^q(\mathrm{I\!R}))$. *Then there exists a subsequence denoted by* $\{\tilde{u}_n\}_{n \in \mathrm{I\!N}}$ *and a weakly-*$\star$ *measurable mapping* $\nu : \mathrm{I\!R} \times (0, T) \to \mathrm{Prob}(\mathrm{I\!R})$ *such that, for all functions* $g \in \mathcal{C}(\mathrm{I\!R})$ *satisfying* (19), $\langle \nu_{(x,t)}, g \rangle$ *belongs to* $L^\infty((0, T); L^{q/m}_{loc}(\mathrm{I\!R}))$ *and the following limit representation holds:*

$$\iint_{\mathrm{I\!R} \times (0,T)} \langle \nu_{(x,t)}, g \rangle\ \phi(x, t)\ dxdt \qquad (22)$$
$$= \lim_{n \to \infty} \iint_{\mathrm{I\!R} \times (0,T)} g(\tilde{u}_n(x, t))\, \phi(x, t)\ dxdt$$

for all $\phi \in L^1(\mathrm{I\!R} \times (0, T)) \cap L^\infty(\mathrm{I\!R} \times (0, T))$.

Conversely, given ν, *there exists a sequence* $\{u_n\}$ *satisfying the same conditions as above and such that* (22) *holds for any* g *satisfying* (19).

For details on the setting of e.m.-v. solutions see, e.g., Correia and LeFloch [4] and references therein.

Proof of Theorem 5 [Main theorem]: We begin proving (20). We use the L^q bound given by (14) of Proposition 8 and we apply the Young measure representation theorem in this L^q space (i.e., formula (22) of Lemma 11) to show that ν satisfies (20): by a standard regularization of $\mathrm{sgn}(u - k)(f(u) - f(k))$ and $|u - k|$ ($k \in \mathrm{I\!R}$), which satisfies the growth condition (19), we see it is sufficient to show that there exists a bounded measure $\mu \leq 0$ such that

$$\eta(u)_t + q(u)_x \longrightarrow \mu \quad \text{in } \mathcal{D}'(\mathrm{I\!R} \times (0, T))$$

for an arbitrary convex function η (we assume η', η'', η''' to be bounded functions on $\mathrm{I\!R}$).

$$\begin{aligned} \eta(u)_t + q(u)_x &= \varepsilon\, (\eta'(u)\, u_x)_x - \varepsilon\, \eta''(u)\, u_x^2 \\ &\quad - \delta\, (\eta'(u)\, u_{xx}^2)_x \\ &\quad + \delta\ \eta''(u)\, u_x\, u_{xx}^2\,. \end{aligned} \qquad (23)$$

We rewrite the last formula in the form

$$\eta(u)_t + q(u)_x = \mu_1 + \mu_2 + \mu_3 + \mu_4$$

where,

$$\begin{aligned} \mu_1 : &= \varepsilon\, (\eta'(u)\, u_x)_x; \\ \mu_2 : &= -\, \varepsilon\, \eta''(u)\, u_x^2; \\ \mu_3 : &= -\delta\, (\eta'(u)\, u_{xx}^2)_x; \\ \mu_4 : &= \delta\ \eta''(u)\, u_x\, u_{xx}^2. \end{aligned}$$

$$\begin{aligned} | < \mu_1, \theta > | &\leq \varepsilon \int_0^T \int_{\mathrm{I\!R}} |\theta_x\, \eta'(u)\, u_x|\, dxds \\ &\leq \varepsilon \int_0^T \int_{\mathrm{I\!R}} |\theta_x\, u_x|\, dxds \\ &\leq C ||\theta_x||_{L^2} ||\varepsilon\, u_x||_{L^2}. \end{aligned}$$

$$\begin{aligned} | < \mu_2, \theta > | &\leq \varepsilon \int_0^T \int_{\mathrm{I\!R}} |\theta\, \eta''(u)\, u_x^2|\, dxds \\ &\leq C ||\theta||_{L^\infty} ||\varepsilon\, u_x^2||_{L^1}. \end{aligned}$$

Since η is convexe, for a non negative function θ we have

$$< \mu_2, \theta > \ = \ -\varepsilon \int_0^T \int_{\mathrm{I\!R}} \theta\, \eta''(u)\, u_x^2\, dxds \leq 0.$$

$$\begin{aligned} | < \mu_3, \theta > | &\leq \delta \int_0^T \int_{\mathrm{I\!R}} |\theta_x\, \eta'(u)\, u_{xx}^2|\, dxds \\ &\leq C\, \delta \int_0^T \int_{\mathrm{I\!R}} |\theta_x\, u_{xx}^2|\, dxds \\ &\leq C ||\theta_x||_{L^2} ||\delta\, u_{xx}^2||_{L^2}. \end{aligned}$$

Now, we can decompose μ_4 in the form

$$\mu_4 = \mu_{41} + \mu_{42},$$

where,

$$\begin{aligned}\mu_{41} : &= \delta\, \eta''(u)\, u_x^+\, u_{xx}^2;\\ \mu_{42} : &= -\delta\, \eta''(u)\, u_x^-\, u_{xx}^2.\end{aligned}$$

Then we have

$$\begin{aligned}|<\mu_{41}, \theta>| &\leq \delta \int_0^T \int_{\rm I\!R} |\theta\, \eta''(u)\, u_x^+ u_{xx}^2|\, dxds\\ &\leq C\,\delta \int_0^T \int_{\rm I\!R} |\theta\, u_x^+\, u_{xx}^2|\, dxds\\ &\leq C||\theta||_{L^\infty} ||\delta\, u_x^+\, u_{xx}^2||_{L^1}.\end{aligned}$$

Thus, thanks to (d) of Proposition 9, when $\varepsilon \to 0$ $< \mu_{41}, \theta > \to 0$. Also, thanks to (c) of Proposition 9 we get

$$\begin{aligned}|<\mu_{42}, \theta>| &\leq \delta \int_0^T \int_{\rm I\!R} |\theta\, \eta''(u)\, u_x^- u_{xx}^2|\, dxds\\ &\leq C\,\delta \int_0^T \int_{\rm I\!R} |\theta\, u_x^-\, u_{xx}^2|\, dxds\\ &\leq C||\theta||_{L^\infty} ||\delta\, u_x^-\, u_{xx}^2||_{L^1},\\ &\leq C\end{aligned}$$

and

$$< \mu_{42}, \theta > \;=\; -\delta \int_0^T \int_{\rm I\!R} \theta\, \eta''(u)\, u_x^- u_{xx}^2\, dxds \leq 0$$

In order to prove (21) we can follow the argument as in Correia-LeFloch [4]. □

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References:

[1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation, *Geom. Funct. Anal.* 3, 1993, pp. 209–262.

[2] J. L. Bona and R. Smith, The initial-value problem for the Korteweg-de Vries equation, *Philos. Trans. Roy. Soc. London Ser. A278* 1287, 1975, pp. 555–601.

[3] Y. Brenier and D. Levy, Dissipative behavior of some fully non-linear KdV-type equations, *Physica D* 137, 2000, pp. 277-294.

[4] J. M. C. Correia and P. G. LeFloch, Nonlinear diffusive-dispersive limits for multidimensional conservation laws, *Advances in Partial Differential Equations and Related Areas (Beijing, 1997),* World Sci. Publ., River Edge, NJ, 1998, pp. 103–123 (or http://arxiv.org/abs/0810.1880).

[5] R. J. Iório, Jr., KdV, BO and friends in weighted Sobolev spaces (Tokyo, 1989), *Functional-Analytic Methods for Partial Differential equations, Lecture Notes in Math., vol. 1450,* Springer-Verlag, Berlin, 1990, pp. 104–121.

[6] S. N. Kružkov, First order quasilinear equations in several independent variables, *Mat. Sb.* 81, 1970, pp. 285; *Math. USSR Sb.* 10, 1970, pp. 217.

[7] P. D. Lax and C. D. Levermore, The small dispersion limit of the Korteweg-de Vries equation, *Comm. Pure Appl. Math.* 36, 1983, I, p.253, II, p.571, III, p.809.

[8] P. G. LeFloch and R. Natalini, Conservation laws with vanishing nonlinear diffusion and dispersion, *Nonlinear Analysis* 36, 1999, pp. 213–230

[9] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.* 41, 1988, pp. 890–907.

[10] B. Perthame and L. Ryzhik, Moderate Dispersion in Conservation Laws with Convex Fluxes, *Comm. Math. Sci.* 5(2), 2007, pp. 473–484.

[11] M. Schonbek, Convergence of Solutions to Nonlinear Dispersive Equations, *Comm. Part. Diff. Equa.* 7(8), 1982, pp. 959-1000.

[12] M. Slemrod, From Boltzmann to Euler: Hilbert's 6th problem revisited, *Computers and Mathematics with Applications* 65, 2013, pp. 1497-1501.

[13] G. B. Whitham, *Linear and Nonlinear Waves,* Pure & Appl. Math., Wiley-Interscience Publ., New York, 1974.