A Combined Double Sampling and Predetermined Sampling Intervals $\overline{X}$ Control Chart

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1. Introduction

Several studies have shown that we can improve the statistical performance of control charts by changing their parameters during the production process. Such charts can be classified, from an implementation point of view, into two categories: charts with parameters that are fixed but not constant for the duration of the monitoring operation, and charts for which at least one of the parameters is allowed to change in real time, taking into account current sample information. The latter are called adaptive charts and the previous are called control charts with predetermined parameters. Reynolds *et al*. (1988) proposed the variable sampling interval (VSI) control chart, where the position of the sample mean determines the time that the next sample is analyzed. Banerjee and Rahim (1988), assuming that the system lifetime follows a Weibull distribution and adopting an economic approach, analyzed a model in which sampling interval is a predetermined parameter. Prabhu *et al*. (1993) and Costa (1994) proposed the variable sample size (VSS) control chart with two possible sample sizes, where the operating rules resemble closely those of VSI chart. Prabhu *et al*. (1994) proposed the variable sample size and sampling interval (VSSI) control chart that alternates between a long sampling interval with a small sample size and a short sampling interval with a large sample size. Costa (1999) proposed the variable parameter (VP) control chart, where the three parameters (sample size, sampling interval, and control limit coefficient) are variable.

Daudin (1992) proposed a double sampling (DS) control chart characterized by two sample sizes where a second sample is analyzed only if the first is not enough to make a decision about the state of the process. Since this chart has a good performance, Carot *et al*. (2002) combine it with the VSI control chart (DSVSI) and concludes that the sensitivity to shifts in the mean compared with the other control charts increase. Rodrigues Dias (2002) presents a predetermined sampling intervals (PSI) control chart in which the sampling intervals are obtained on the basis of the cumulative system hazard rate. The subjacent assumption is the same as in Banerjee and Rahim (1988), but this new approach allows for statistical comparison with other sampling schemes, which is a novel feature. Its statistical performance is better when the probability of the shift being detected decreases and when the average number of samples analyzed in the in-control state decreases. Such results become more marked as failure rate increases.

In this paper we present, analyze and compare to other charts, a new control chart where the double sampling method is combined with the PSI. In general, we can draw the conclusion that this method is globally more efficient than the others sampling schemes considered, in terms of adjusted average time to signal (AATS) and average number of inspected items (ANI). Sometimes the reductions in AATS are very marked, particularly when used in systems with a strongly increasing failure rate distribution.

2. Design of the DSPSI $\bar{X}$ control chart

Let us consider T as a random variable that represents the time before the occurrence of an assignable cause, with reliability function *R(t)* and cumulative hazard rate function *H(t)*. Rodrigues Dias (2002) proposed a new methodology (PSI method) where the sampling instants t_k are obtained in such a way that the cumulative hazard rate between any two consecutive inspections is constant ($H(t_k) = \Delta H,\ k = 1,2,\ldots,$) that is, the probability of a process shift during a sampling interval, given that no shift occurs until the start of the interval, is constant for all intervals. According to this methodology, sampling instants are given by

$$t_k = R^{-1}\left[\exp\left(-k\Delta H\right)\right];\ t_0 = 0 \qquad (1)$$

This expression enables sampling instants to be obtained for any system with a known invertible reliability function or if the reliability inverses function can be obtained by numerical methods. It is based on the intuitively simple idea that less frequent sampling should be carried out when the hazard rate is low and, conversely, more frequent sampling should be carried out when the hazard rate is high. According to this method, if failure rate increases (or decreases) then sampling intervals decrease (or increase), as it can be seen in Rodrigues Dias (1987). Infante (2004) and Rodrigues Dias and Infante (preprint) have studied its statistical properties and compare its performance to other sampling schemes.

In this paper, it is assumed that the quality characteristic of the production process is normally distributed, with mean μ_0 and standard deviation σ_0. Let us consider U_1 as the standardized mean of the first sample and U_2 as the standardized weighted mean of the two samples. The DS procedure consists in taken a first sample of size n_1 and:

a) If $|U_1| \le W$, we conclude the process is in control (*W* is the threshold limit);
b) If $|U_1| > L_1$, we conclude the process is out of control;
c) If $W < |U_1| \le L_1$, a second sample of size $n_2 > n_1$ is taken and
- if $|U_2| \le L_2$, we conclude the process is in control;
- if $|U_2| > L_2$, we conclude the process is out of control;

We propose to combine the DS method with the PSI method. In the new DSPSI method, the instants, at which the samples are taken from the process, are schedule at the beginning of the monitoring process according to the system lifetime distribution (using PSI method), and the samples are taken in each predetermined instant using the double sampling method.

We consider the case that at some time in the future, as a result of the occurrence of an assignable cause, the mean shifts to $\mu_1=\mu_0 \pm \lambda\sigma_0$, $\lambda>0$. The adjusted average time to signal has the same expression that AATS for the PSI chart (Rodrigues Dias, 2002):

$$\begin{aligned} AATS = {} & \left[1-\exp\left(-\Delta H\right)\right] \sum_{k=1}^{\infty} t_k \exp\left[-\left(k-1\right)\Delta H\right] - E(T) + \\ & + \left[1-\exp\left(-\Delta H\right)\right] \frac{q}{q-\exp\left(-\Delta H\right)} \sum_{k=1}^{\infty}\left(t_{k+1}-t_k\right)\left[q^k - \exp\left(-k\Delta H\right)\right] \end{aligned} \qquad (2)$$

In (2) sample instants t_k are given by (1) and q is the probability of the shift not being detected. Since we use the DS sample procedure to take the samples, q is given by (Daudin (1992))

$$\begin{aligned} q = {} & 1 - \Phi\left(W + \lambda\sqrt{n_1}\right) + \Phi\left(-W + \lambda\sqrt{n_1}\right) - \\ & - \int_{Z\in I} \left\{ \Phi\left[\frac{1}{\sqrt{n_2}}\left(\sqrt{n_1+n_2}\,L_2 + \lambda\left(n_1+n_2\right) - \sqrt{n_1}\,z\right)\right]\phi(z)\right\} dz + \\ & + \int_{z\in I} \left\{ \Phi\left[\frac{1}{\sqrt{n_2}}\left(-\sqrt{n_1+n_2}\,L_2 + \lambda\left(n_1+n_2\right) - \sqrt{n_1}\,z\right)\right]\phi(z)\right\} dz \end{aligned} \qquad (3)$$

where $I = \left[-L_1 + \lambda\sqrt{n_1}, -W + \lambda\sqrt{n_1}\right[\cup \left]W + \lambda\sqrt{n_1}, L_1 + \lambda\sqrt{n_1}\right]$, $\Phi(.)$ is the standard normal distribution function and ϕ(.) is the standard normal probability density function.

The average sample size is given by

$$E(N \mid \lambda) = n_1 + n_2\left[\Phi\left(L_1 + \lambda\sqrt{n_1}\right) - \Phi\left(W + \lambda\sqrt{n_1}\right) + \Phi\left(-W + \lambda\sqrt{n_1}\right) - \Phi\left(-L_1 + \lambda\sqrt{n_1}\right)\right] \quad (4)$$

and, the average number of sample items is

$$ANI = E(N \mid \lambda)/q\ . \quad (5)$$

3. Comparison with other adaptive sampling schemes

In order to compare the statistical performance of the DSPSI $\bar{X}$ with other $\bar{X}$ charts in terms of AATS and ANI, the usual procedure is to match their in-control performances. This can be accomplished by designing the charts in such a way that, during the in-control period, they have the same rate of inspected items and the same average number of false alarms. To obtain n_1, n_2, W, L_1 and L_2 we follow the guidelines in Daudin (1992) and Carot *et al*. (2002). However, the parameters used in this paper were not selected to optimize its efficiency at detecting a particular shift. We only wanted to show some potentialities of this sampling scheme. The comparisons are made setting n_1=3, n_2=9, W=1.22, L_1=4.1 and L_2=2.875 which produces a chart that matches the in-control performance of the standard Shewhart chart with n_0=5, L_0=3 and d_0=1. To obtain an average sample interval equal to d_0, when the process is in control, we can obtain ΔH using the approximation $\Delta H \cong P/E(T)$ given by Rodrigues Dias (1987). This is an important relationship because, it enables ΔH to be obtained very simply, it does not depend on lifetime distribution, and it is an excellent approximation.

Let us represent by $AATS_1$ and by $AATS_2$ the adjusted average time to signal using the DSPSI chart and other chart, respectively, and by ANI_1 and ANI_2 the average number of items inspected using the DSPSI chart and other chart, respectively. To compare charts statistical performances, we can use:

$$Q_{AATS} = \frac{AATS_2 - AATS_1}{AATS_2} \times 100\% \qquad Q_{ANI} = \frac{ANI_2 - ANI_1}{ANI_2} \times 100\% \quad (6)$$

Those quantities measure the relative variation in AATS and in ANI when using the DSPSI control chart instead of another control chart. Results obtained for a range of possible mean shifts and two values for the shape parameter β of the Weibull distribution are presented in Table 1. All charts considered are comparable in control with n_0=5, L_0=3, d_0=1 and $E(T)$=1000.

4. Discussion and concluding remarks

We conclude that this new method provides a valuable alternative to adaptive sampling schemes when the aim is to detect different mean shifts. In fact, there is a substantial reduction on the adjusted average time to signal when we use the DSPSI control chart in the great majority of the cases considered here. This reduction has become more accentuate when the shape parameter increase: for example, when β=4 (results not presented in Table 1) the DSPSI method has better performance than the other methods for all values of λ. On the other hand, it is always more efficient in terms of AATS than the standard control chart, even for systems that have a decreasing failure rate with β=0.8 (not in Table 1), which is not the case for any other adaptive scheme. We have compared it to other VSI, VSS, VSSI and VP features and the results are very similar. If the average number of samples analyzed in the in-control state decreases, differences become more marked. For example, considering a system with β=3, and a shift of size λ=0.75, the VP chart has lesser AATS (Q_{AATS}=-32.3%) but if $E(T)$=100, is the DSPSI charts that has lesser AATS (Q_{AATS}=1,3%). The reduction in the AATS accomplishes a reduction in the average number of sampling items when compared to VSS and VSSI for all shifts, to the periodic schemes (standard chart and VSI) for small shifts (λ<1) and to the VP chart for moderate to large shifts (λ>1).

The DSPSI chart has the drawback of the DS chart in that the procedure for deciding if the process is under control is somewhat more complex than with some other charts. The fact that the second sample is contiguous to the first sample implies that it must be feasible to collect, analyze, and measures the samples in a tiny period. However, if the distribution of system lifetime is known, what usually happens (at least approximately), then at the beginning of the sampling process we can settle the instants at which the samples are taken from the process. This is a great advantage in terms of quality management over the others adaptive sampling methods. Further investigation is required to give some guidelines to select the parameters for different situations and to confirm these conclusions under different practical situations.

Table 1 - Q_{AATS} and Q_{ANI} for different adaptive sampling schemes and different mean shifts (λ).

Sampling Schemes		λ									
		0.125	**0.250**	**0.375**	**0.500**	**0.750**	**1.000**	**1.250**	**1.500**	**2.000**	**3.000**
Standard Chart	Q_{AATS} (β=2)	49,7	60,8	66,8	70,3	72,4	69,4	60,5	44,9	12,3	1,5
(n_0=5; d_0=1; L_0=3)	Q_{AATS} (β=3)	62,1	68,5	72,1	74,2	75,0	71,7	62,8	47,9	16,6	6,2
VSI	Q_{AATS} (β=2)	48,2	55,8	56,9	53,5	36,6	25,1	33,3	42,1	48,4	49,5
(d_1=0.05; d_2=2; W=0.65)	Q_{AATS} (β=3)	61,0	64,6	63,7	59,5	42,5	30,5	37,3	45,2	50,9	51,9
	Q_{ANI}	26,3	46,8	54,7	56,7	49,4	28,0	-8,1	-49,2	-77,1	15,4
VSS	Q_{AATS} (β=2)	47,3	40,3	9,9	-15,0	15,5	46,5	58,1	59,8	51,7	16,9
(n_1=2; n_2=25; W=1.50)	Q_{AATS} (β=3)	60,3	52,1	24,3	0,0	23,4	50,4	60,6	62,0	54,0	20,9
	Q_{ANI}	27,6	37,7	27,6	17,4	23,1	39,2	46,2	46,2	42,5	41,9
VSSI	Q_{AATS} (β=2)	46,9	43,0	18,9	-13,0	4,3	42,9	56,0	57,5	49,8	33,5
(d_1=0.05; d_2=1.38; n_1=1; n_2=15;	Q_{AATS} (β=3)	61,3	61,8	52,4	37,6	26,6	39,9	48,9	50,2	37,8	8,3
W=1.06)	Q_{ANI}	27,4	42,0	39,5	32,0	20,0	17,9	20,1	22,2	28,6	49,4
VP	Q_{AATS} (β=2)	24,5	-4,1	-37,4	-52,2	2,1	43,1	56,2	57,8	50,4	35,7
(d_1=0.05; d_2=1.38; n_1=1; n_2=15;	Q_{AATS} (β=3)	43,1	16,5	-15,5	-32,3	11,3	47,2	58,9	60,1	52,8	38,8
W_1=1.07; W_2=1.05; L_1=6; L_2=2.6)	Q_{ANI}	-3,7	-9,3	-13,2	-13,4	-1,9	13,4	23,6	29,1	41,2	73,6
DSVSI	Q_{AATS} (β=2)	29,4	15,7	3,3	-6,1	-5,4	16,7	34,7	43,2	48,5	49,5
(d_1=0.05; d_2=2; n_1=3; n_2=9;	Q_{AATS} (β=3)	46,8	32,4	18,7	7,8	4,5	22,7	38,6	46,2	51,0	51,9
W=0.65; L_1=4.1; L_2=2.875)											

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