# Component redundancy allocation in optimal cost preventive maintenance scheduling

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**Abstract**

This work presents a methodology to assess maintenance teams in the determination of the degree of redundancy that an active component must have in order to minimize its system life-cycle cost and in the definition of the corresponding maintenance plan schedule. The minimal required data are three average costs and one reliability function. It is useful in a system's design phase, since in this situation data is usually scarce or inaccurate, but can also be applied in the exploration phase. It is an adaptation of the classical Optimal Age Replacement method combined with a redundancy allocation problem. A set of simple illustrative examples covering a variety of operating conditions is presented, demonstrating quantitatively the applicability to a range of maintenance optimization decisions.

**Keywords**: life-cycle cost; maintenance; optimal age replacement; redundancy; reliability

## 1 Introduction

We denote here by component an element of hardware designed to provide a particular "simple" output, and by system an assembly or set of components working together to provide a "more complex" output. Redundancy is a way to increase a system's reliability and reduce its components failure consequences. Systems where the costs of failure are minimal, do not justify the use of redundancy and in some extreme cases, not even employing preventive replacement. In these cases the best solution is to repair after failure, usually known as run-to-failure.

For significant failure and repair costs the use of redundancy is recommended, especially if the component failure rate is decreasing or constant over time, since in this case, the preventive replacement does not reduce the risk of failure. Therefore, it is of key importance to know whether the cost increase due to purchasing the redundancy, is compensated by a reduction of failure and repair costs (Hsieh, 2005; Graves et al, 2000).

The practice of using redundancy revealed that system failures involving all or some of the redundant components collectively, can occur in a dependent way (i.e. the failure of one somehow influences and depends on the failure of the others), introducing a new failure mode in the system analysis known as Common Cause Failures [CCFs], which reduces the benefits of redundancy. A CCF is here understood as a (probabilistic) dependent failure (or unavailable state) of two or more components in a parallel arrangement, due to a shared cause during the system operation, resulting in the system shut-down. The CCF is the subset of all the possible dependent events that are not explicitly detailed in a plant model (Mosleh et al, 1988, 1998).

Additionally, also paramount in system maintenance, is to be able to establish a maintenance scheduling that in some way minimizes the costs or maximizes the reliability. In the present work the goal is to develop a simple methodology for estimating the optimal redundancy degree that an active component must have inside a system (i.e. solve the Redundancy Allocation Problem [RAP]), together with the respective maintenance scheduling, in order to minimize (in a probabilistic sense) the system's expected life-cycle cost (Optimal Age Replacement Problem [OARP]). Both the RAP and OARP are dealt with together. This represents an extension of the methodology proposed in (Senju, 1957), (Glasser, 1967) and (Geurts, 1983) to include the RAP. It is devised for a system (or a subsystem of a larger system or plant) built of several components in a parallel arrangement and aims to be of simple application, require the minimum data possible, and involve a cheap computational cost. It is interesting for employing in the system design phase, but hopefully also useful afterwards. The existence of CCFs is also accounted for. The necessary input consists of one reliability function and two average costs for each component, and an average system failure cost. When CCFs are to be included in the analysis, a CCF model must be selected and the necessary parameters provided.

Several strategies have been proposed to solve the OARP, with variations on how the repair processes and costs are included and on the number of maintenance operations to do, see for example, (Jardine, 1998),(Zhang and Jardine, 1998), (Usher et al, 1998), (Coolen-Schrijner and Coolen, 2004, 2007),(Coolen-Schrijner et al, 2006), (Chien and Sheu, 2006; Chien, 2008a,b). A multiplicity of optimization techniques (usually heuristic) to solve the RAP can be found in the literature, like tabu search (Kulturel-Konak et al, 2003; Ouzineba et al, 2008), ant colony (Nahas et al, 2007), dynamic programming (Yalaoui et al, 2005), penalty guided immune algorithm (Chen, 2006), Hooke-Jeeves pattern search (Wetter and Wright, 2003; Liu, 2006), and genetic algorithms (Levitin et al, 1998; Levitin and Lisnianski, 2000; Nahas et al, 2008; Bartholomew-Biggs et al, 2009). Most of these strategies for the RAP and OARP can be used in the framework proposed in the present work, however some of these require a considerable amount of knowledge about the features of the systems under analysis and are sometimes computationally demanding. Here a simpler version is preferred since the concern is the connection of the OARP and RAP ideas, and in a system's design phase, when there is a great uncertainty about the system failure data, the extra complexity of other possibly more accurate methodologies may not compensate the lack of quality input data.

## 2 Proposed methodology

Consider a generic system $\mathcal{S} = \{c_1, c_2, ..., c_n\}$ formed by a *redundant parallel arrangement* of $n$ components $c_i$ $i = 1, 2, ..., n$. If the system ordinary operation demands $k$ components sharing a common load, a system shut-down occurs when $n - k + 1)$ components fail. Let the following input data be available (roman subscripts are names and italicized subscripts are enumeration indexes):

- $C_{\mathrm{A}i}$ - **Acquisition cost**: average fixed cost that results from the possession of *one* component $c_i$ in a working system . For example, purchase price, operating permits, space, etc.

- $C_{\mathrm{P}i}$ - **Preventive repair cost**: average cost of repairing the defects of *one* component $c_i$ prior to a failure occurrence, including all materials and labour costs.

- $C_{\mathrm{F}}$ - **Failure repair cost**: average cost of in service failure occurrence. This cost includes the cost of completely repairing or acquiring $s = (n - k + 1)$ failed components, plus all the hazard costs $C_{\mathrm{H}}$ incurred from the system shut-down (loss of production, loss of image, labour, materials, etc.), so $C_{\mathrm{F}} = C_{\mathrm{H}} + \sum_{i=1}^{s} C_{\mathrm{P}i}$.

- Let $\mathsf{T}_{ci}$ be a continuous random variable denoting the time that component $c_i$ works without any failure. For each component $c_i$ (or for each different type of components, if components of the same type are considered equal), provide one function measuring the probability of component $c_i$ to be working until the time instant $t \in \mathbb{R}^{+}$, such as the failure probability distribution function $F_{\mathrm{c}i}(t)$, the failure probability density function $f_{\mathrm{c}i}(t)$, or the reliability distribution function $R_{\mathrm{c}i}(t)$.

## 2.1 System probable operating time

During the system life, several repair events will occur, identified in the sequence by the enumeration index $r = 1, 2, \ldots, m$. Let $\mathsf{T}_r$ be a continuous random variable ($\mathsf{T}_r \geq 0$) indicating the time the system operates until complete breakdown between repair event $(r-1)$ and $r$ ($r = 0$ denotes the reference system operational starting point), so for each $r$ it is adopted that $\mathsf{T}_r = 0$ when event $(r-1)$ occurs . The total operating time (life) of the system until repair $m$ is then $\mathsf{L} = \sum_{r=1}^{m} \mathsf{T}_r$. Denote by $t > 0$ the actual time elapsed after repair $(r-1)$ until some other repair or replacement $r$ is done. In the interval between events $(r-1)$ and $r$, the system failure probability distribution function $F_r(t) = \mathbb{P}\{\mathsf{T}_r \leq t\} = \mathbb{P}\left(\mathsf{T}_{\mathrm{c}1}^{(r)} \leq t,\ \mathsf{T}_{\mathrm{c}2}^{(r)} \leq t, \ldots, \mathsf{T}_{\mathrm{c}n}^{(r)} \leq t\right)$, and the reliability $R_r(t) = \mathbb{P}\{\mathsf{T}_r > t\}$, are related to the respective components life functions $F_{\mathrm{c}i}^{(r)}(t)$ (note that the superscript $(r)$ is used to index a quantity that may change with $r$). These failure functions $F_{\mathrm{c}i}^{(r)}(t)$ may differ from one repair $r$ to the following repair $(r+1)$, and the differences reflect the degree of perfection attained in the repair. If the repair is *perfect*, each component is assumed in an as good as new condition, and then these functions are evaluated from the life functions for new components. A component replacement is understood as a perfect repair, if the replacing component is new.

During the system operation the maintenance schedule is such that:

- if $\mathsf{T}_r > t$ then some preventive maintenance has to be made at $t$
- if $0 < \mathsf{T}_r \leq t$ then the system shuts down without having been repaired in that period and the system duration is $\mathsf{T}_r$.

Adopting the policy of repairing at time $t$ or after a failure at $\mathsf{T}_r$, the system usage can be expressed by:

$$u(\mathsf{T}_r, t) = \begin{cases} \mathsf{T}_r\,,\ 0 < \mathsf{T}_r \leq t \\ t\,,\ \mathsf{T}_r > t \end{cases} \tag{1}$$

The system expected operating time, which is a mean time between repairs (MTBR) is therefore:

$$\mathrm{MTBR}_r(t) = \mathbb{E}\left[u(\mathsf{T}_r, t)\right] = \int_{-\infty}^{+\infty} u(\tau, t) f_r(\tau) d\tau = t - \int_0^t F_r(\tau) d\tau = \int_0^t R_r(\tau) d\tau \tag{2}$$

Here $t$ is the operating time, but other counting unit that reflects a duration can be used such as cycles, energy consumption, production units, etc.

## 2.2 System operational cost

The system total acquisition cost depends on the redundancy specified (the subscript S is now added to denote a system total cost):

$$C_{\mathrm{AS}} = \sum_{i=1}^{n} C_{\mathrm{A}i} \tag{3}$$

in the general case of different but equivalent components (where diversity can be used to reduce common cause failures CCFs (Littlewood et al, 2001)), which simplifies to $C_{\mathrm{AS}} = nC_{\mathrm{A1}}$ for equal components .

The probability of having the system completely operational by time $t$ between repair events $(r-1)$ and $r$ is given by $R_r(t) = \mathbb{P}\{\mathsf{T}_r > t\}$, and therefore the system probable cost with preventive maintenance during this period is given by $C_{\mathrm{P}}^{(r)} = \left(\sum_{i=1}^{p} C_{\mathrm{P}i}^{(r)}\right) R_r(t)$, where $p$ is the number of components actually submitted to maintenance operations ($p \geq 1$ and $p = n$ if all the components are overhauled). Denoting now by $t_r$ the operational time between events $(r-1)$ and $r$, the system total probable cost with preventive maintenance until repair $m$ is:

$$C_{\mathrm{PS}} = \sum_{r=1}^{m} C_{\mathrm{P}}^{(r)} = \sum_{r=1}^{m} \left[ \left( \sum_{i=1}^{p} C_{\mathrm{P}i}^{(r)} \right) R_r(t_r) \right] \tag{4}$$

The average probable expenditure with failures during the system life is given by:

$$C_{\mathrm{FS}} = \sum_{r=1}^{m} C_{\mathrm{F}}^{(r)} \mathbb{P}\{\mathsf{T}_r \leq t_r\} = \sum_{r=1}^{m} C_{\mathrm{F}}^{(r)} F_r(t_r) \tag{5}$$

The overall system operational probable cost corresponds to the sum of all these parts $C = C_{\mathrm{AS}} + C_{\mathrm{PS}} + C_{\mathrm{FS}}$.

## 2.3 System cost optimization problem

In the well known optimal age replacement paradigm (Senju, 1957), the time $t^*$ for taking a preventive maintenance operation is found by minimizing the system average probable operational cost per unit time in the long run:

$$\Phi_{\mathrm{OARP}}(t) = \frac{C_{\mathrm{PS}} R_1(t) + C_{\mathrm{FS}} F_1(t)}{\mathbb{E}\left[u(\mathsf{T}_1)\right]} \tag{6}$$

In this work the same idea is pursued, but extending the original OARP by taking in account the system acquisition cost $C_{\mathrm{AS}}$, the redundancy $n$, the number of preventive repairs $m$ and possible adjustments to the preventive maintenance and failure costs with time.

### 2.3.1 The life-cycle cost optimization problem

In a system's design phase, the available data is in general incomplete and inaccurate, based on assumptions and rough estimations. In order to make inferences on the system operational behaviour, it is useful to have a system model that requires a small amount of data and is computationally cheap and simple. Helpful insights regarding the redundancy to select, the costs implied by maintenance and a possible maintenance schedule that minimizes the average system probable operational cost in the long run, can be obtained from the following optimization problem:

Find $\{n^*, m^*, t_1^*, t_2^*, \ldots, t_{m^*}^*\}$ that minimize the system average probable operational cost per unit time given by the terms of the sequence:

$$\Phi_m(n, t_1, t_2, \ldots, t_m) = \frac{C_{\text{AS}} + C_{\text{PS}} + C_{\text{FS}}}{\sum_{r=1}^{m} \text{MTBR}_r(t_r)} = \frac{\sum_{i=1}^{n} C_{\text{A}i} + \sum_{r=1}^{m} \left[ \left( \sum_{i=1}^{p} C_{\text{P}i}^{(r)} \right) R_r(t_r) + C_{\text{F}}^{(r)} \left(1 - R_r(t_r)\right) \right]}{\sum_{r=1}^{m} \int_0^{t_r} R_r(\tau) d\tau} \tag{7}$$

subject to:

$$n \geq k \, ; \; m > 0 \, ; \; t_r > 0 \, , \; r = 1, 2, \ldots, m \, ; \; 1 \leq p \leq n \tag{8}$$

where $p$ is the number of repaired components in the interval $r$ to keep the system operational (please note that if a system failure occurs, $C_{\text{F}}$ contains the cost of repairing the minimum number of components necessary for the system operation $s = (n - k + 1)$, so the remaining $(k - 1)$ components that may be repaired will contribute to the preventive repair cost). The denominator is the sum of the $\text{MTBR}_r$ for each interval $r$, which provides a time measure of the system probable usage.

Following the same idea other expressions can be devised and different constraints on $t_r$ and on the total cost $C$ can be assumed. The Eq.(7) is of general application and intended to be relatively simple to evaluate. All the system intricacy is embedded in the reliability functions $R_r(t_r)$, for which some specialized forms are provided for two classes of systems in subsection 2.4. More complex models can be more precise, but involve a greater amount of data and computation. If there is lack of quality data, extra work will not pay off.

### 2.3.2 The maintenance interval optimization problem

Considering a system in actual operation, it is useful to estimate the time for the next $m$ intended maintenance operations, and eventually the present adequate redundancy, that minimize the operational costs in the selected period. Let $n_0 \geq k$ be the system's actual implemented redundancy, a new redundancy $n$ can be estimated conducing to smaller costs, including the possibility of selling the extra redundancy, by changing the acquisition term $C_{\text{AS}}$ in Eq.(7) to:

$$C_{\text{AS}} = \sum_{i=n-n_0}^{\max(n-n_0,0)} \text{sign}(i) C_{\text{A}i} \ , \ C_{\text{A}0} = 0 \quad \Leftrightarrow \quad C_{\text{AS}} = \begin{cases} -\sum_{i=k+1}^{(k+n_0-n)} C_{\text{A}i} & n < n_0 \\ 0 & n = n_0 \\ \sum_{i=(n_0+1)}^{n} C_{\text{A}i} & n > n_0 \end{cases} \tag{9}$$

($\text{sign}(i) = 1$ if $i \geq 0$ and $\text{sign}(i) = -1$ if $i < 0$) and an index $i \leq n_0$ indicates that $C_{\text{A}i}$ is a selling benefit instead of a cost.

Incorporating Eq.(9) in Eq.(7) leads to the following optimization problem:

Find $\{n^*, m, t_1^*, t_2^*, \ldots, t_m^*\}$ the total number of redundant components and the time for the next repair, which minimize the system average operational cost per unit time:

$$\Phi(n, t_1, t_2, \ldots, t_m) = \frac{\sum_{i=n-n_0}^{\max(n-n_0,0)} \text{sign}(i) C_{\text{A}i} + \sum_{r=1}^{m} \left[ \left( \sum_{i=1}^{p} C_{\text{P}i}^{(r)} \right) R_r(t_r) + C_{\text{F}}^{(r)} \left(1 - R_r(t_r)\right) \right]}{\sum_{r=1}^{m} \int_0^{t_r} R_r(\tau) d\tau} \tag{10}$$

subject to:

$$n \geq k \ ; \ t_r - \hat{t}_r > 0 \ , \ r = 1, 2, \ldots, m \ ; \ 1 \leq p \leq n \tag{11}$$

where $\hat{t}_r > 0$ is a constant representing the minimum time that the system is prescribed to work during each maintenance interval. Other constraints can be considered.

As data is gathered for the system, better estimates and helpful insights can be obtained from Eq.(10), that can be used together with assessment methodologies such as, for example, the one presented by (Pérès and Noyes, 2003) , to assess the maintenance strategy being applied.

## 2.4 System reliability evaluation

Both Eq.(7) and Eq.(10) involve calculating the system reliability, which depends on the system details. The evaluation procedure for the cases of independent and dependent failures are presented next.

In the proposed framework the CCFs are accounted in Eq.(7) through the system reliability functions $R_r(t_r)$. To a great extent, the definitions and approach presented in (Mosleh et al, 1988, 1998) regarding CCFs, are adopted here. If a system with a configuration of $k$ out of $n$ is considered, then $R_r(t_r)$ must be derived from the partition of the failure event space of each component $c_i$ in independent failure events (I), explicitly modelled dependent failure events (E) and common cause failure events (C) (all the dependent events that are not detailed in the system failure mode graph). To illustrate the derivations necessary for Eq.(7), and without loss of generality, the simple case of $k = 2$ out of $n = 3$ is presented. Let $\mathcal{C}_i$ be the set of all the failure events of component $c_i$, which can be partitioned as $\mathcal{C}_i = \mathcal{C}_{i\mathrm{I}} \cup \mathcal{C}_{i\mathrm{E}} \cup \mathcal{C}_{i\mathrm{C}}$ by the disjoint subsets of the independent $\mathcal{C}_{i\mathrm{I}}$, the explicitly modelled $\mathcal{C}_{i\mathrm{E}}$ and the CC $\mathcal{C}_{i\mathrm{C}}$ failures, and $\bar{\mathcal{C}}_i$ its complement (no failure). Then the probability of failure of $c_1$ , for example, is:

$$\begin{gathered}\mathbb{P}(\mathcal{C}_1) = Q = \mathbb{P}\left\{\mathcal{C}_{1\mathrm{I}} \cup (\mathcal{C}_{1\mathrm{E}} \cap \mathcal{C}_{2\mathrm{E}}) \cup (\mathcal{C}_{1\mathrm{E}} \cap \mathcal{C}_{3\mathrm{E}}) \cup (\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}}) \cup (\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{3\mathrm{C}})\right\} = \\ = \mathbb{P}(\mathcal{C}_{1\mathrm{I}}) + \mathbb{P}(\mathcal{C}_{1\mathrm{E}} \cap \mathcal{C}_{2\mathrm{E}}) + \mathbb{P}(\mathcal{C}_{1\mathrm{E}} \cap \mathcal{C}_{3\mathrm{E}}) - \mathbb{P}(\mathcal{C}_{1\mathrm{E}} \cap \mathcal{C}_{2\mathrm{E}} \cap \mathcal{C}_{3\mathrm{E}}) + \\ + \mathbb{P}(\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}}) + \mathbb{P}(\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{3\mathrm{C}}) - \mathbb{P}(\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}} \cap \mathcal{C}_{3\mathrm{C}})\end{gathered} \tag{12}$$

and the probability of failure of the system is:

$$\begin{gathered}\mathbb{P}(\mathcal{S}) = \mathbb{P}\left\{(\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_1 \cap \mathcal{C}_3) \cup (\mathcal{C}_2 \cap \mathcal{C}_3)\right\} = \\ = \mathbb{P}\left\{(\mathcal{C}_1 \cap \mathcal{C}_2 \cap \bar{\mathcal{C}}_3) \cup (\mathcal{C}_1 \cap \mathcal{C}_3 \cap \bar{\mathcal{C}}_2) \cup (\mathcal{C}_2 \cap \mathcal{C}_3 \cap \bar{\mathcal{C}}_1) \cup (\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3)\right\} = \\ = \mathbb{P}\left\{[(\mathcal{C}_{1\mathrm{I}} \cap \mathcal{C}_{2\mathrm{I}}) \cup (\mathcal{C}_{1\mathrm{I}} \cap \mathcal{C}_{3\mathrm{I}}) \cup (\mathcal{C}_{2\mathrm{I}} \cap \mathcal{C}_{3\mathrm{I}})] \cup [(\mathcal{C}_{1\mathrm{E}} \cap \mathcal{C}_{2\mathrm{E}}) \cup (\mathcal{C}_{1\mathrm{E}} \cap \mathcal{C}_{3\mathrm{E}}) \cup (\mathcal{C}_{2\mathrm{E}} \cap \mathcal{C}_{3\mathrm{E}})] \right. \\ \left. [\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}}) \cup (\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{3\mathrm{C}}) \cup (\mathcal{C}_{2\mathrm{C}} \cap \mathcal{C}_{3\mathrm{C}})]\right\}\end{gathered} \tag{13}$$

where the following expressions replicate for analogous terms:

$$\begin{gathered}(\mathcal{C}_{1\mathrm{I}} \cap \mathcal{C}_{2\mathrm{I}}) = (\mathcal{C}_{1\mathrm{I}} \cap \mathcal{C}_{2\mathrm{I}} \cap \bar{\mathcal{C}}_3) \cup (\mathcal{C}_{1\mathrm{I}} \cap \mathcal{C}_{2\mathrm{I}} \cap \mathcal{C}_{3\mathrm{I}}) \\ (\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}}) = (\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}} \cap \bar{\mathcal{C}}_3) \cup (\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}} \cap \mathcal{C}_{3\mathrm{I}}) \cup (\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}} \cap \mathcal{C}_{3\mathrm{C}})\end{gathered} \tag{14}$$

If the probability terms involved in Eq.(12) are known for each component, then the system failure probability can be evaluated. This information is difficult to obtain and in practice approximate models have to be considered. (Lundteigen and Rausand, 2007; Stott et al, 2010) provide an overview of the consideration of CCFs in several industries.

### 2.4.1 Independent cause failures

This hypothesis can be a good approximation for some systems, and can also be used as a comparison to evaluate the impact of existing dependent failures. Without great loss of generality, if all the system components have the same failure distribution $\mathbb{P}(\mathcal{C}_i) = F_{ci}^{(r)}(t_r) = Q\ ;\ i = 1, \ldots, 3$, from Eq.(13) the system probability of failure for the case 2-out-of-3, is seen to be:

$$
\begin{aligned}
\mathbb{P}(\mathcal{S}^{(\mathrm{I})}) &= \mathbb{P}\left\{(\mathcal{C}_{1\mathrm{I}} \cap \mathcal{C}_{2\mathrm{I}}) \cup (\mathcal{C}_{1\mathrm{I}} \cap \mathcal{C}_{3\mathrm{I}}) \cup (\mathcal{C}_{2\mathrm{I}} \cap \mathcal{C}_{3\mathrm{I}})\right\} = \\
&= \mathbb{P}(\mathcal{C}_{1\mathrm{I}})\mathbb{P}(\mathcal{C}_{2\mathrm{I}}) + \mathbb{P}(\mathcal{C}_{1\mathrm{I}})\mathbb{P}(\mathcal{C}_{3\mathrm{I}}) + \mathbb{P}(\mathcal{C}_{2\mathrm{I}})\mathbb{P}(\mathcal{C}_{3\mathrm{I}}) - 2\mathbb{P}(\mathcal{C}_{1\mathrm{I}})\mathbb{P}(\mathcal{C}_{2\mathrm{I}})\mathbb{P}(\mathcal{C}_{2\mathrm{I}}) = \\
&= F_r(t_r) = 3Q^2 - 2Q^3
\end{aligned}
\tag{15}
$$

Just for the sake of completion, for $k$-out-of-$n$ the system reliability can be found on the literature to be ($R_\mathrm{c}$ is the components reliability):

$$
(1 - \mathbb{P}(\mathcal{S}^{(\mathrm{I})})) = R(t) = \sum_{p=k}^{n} \binom{n}{p} (R_\mathrm{c})^p \, (1 - R_\mathrm{c})^{(n-p)} \tag{16}
$$

### 2.4.2 Accounting for common cause failures

Since our objective is only to illustrate how Eq.(7) can be computed, to keep the presentation as simple as possible, we will consider further simplifications, such as that the components are similar (no diversity) and verify the symmetry assumption (Mosleh et al, 1988, 1998):

$$
\begin{gathered}
\mathbb{P}(\mathcal{C}_1) = \mathbb{P}(\mathcal{C}_2) = \mathbb{P}(\mathcal{C}_3) = Q \\
\mathbb{P}(\mathcal{C}_{1\mathrm{I}}) = \mathbb{P}(\mathcal{C}_{2\mathrm{I}}) = \mathbb{P}(\mathcal{C}_{3\mathrm{I}}) = Q_1 \\
\mathbb{P}\left\{(\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}}) - (\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}} \cap \mathcal{C}_{3\mathrm{C}})\right\} = Q_2 \\
\mathbb{P}(\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}}) = \mathbb{P}(\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{3\mathrm{C}}) = \mathbb{P}(\mathcal{C}_{2\mathrm{C}} \cap \mathcal{C}_{3\mathrm{C}}) = Q_2 + Q_3 \\
\mathbb{P}(\mathcal{C}_{1\mathrm{C}} \cap \mathcal{C}_{2\mathrm{C}} \cap \mathcal{C}_{3\mathrm{C}}) = Q_3
\end{gathered}
\tag{17}
$$

and that the explicit dependent failures are not included in the model $\mathbb{P}(\mathcal{C}_{i\mathrm{E}}) = 0$ , $i = 1, 2, 3$. The $\beta$ factor model (Fleming, 1975) is used for accounting the CCFs. Following (Mosleh et al, 1988, 1998), with the $\beta$ factor model $Q_3$ is a fraction of the single component probability of failure, and it is assumed that when there is a CCF all the components in the CC group fail, so:

$$
\begin{gathered}
Q_1 = (1 - \beta)\mathbb{P}(\mathcal{C}_1) \\
Q_2 = 0 \\
Q_3 = \beta\mathbb{P}(\mathcal{C}_1)
\end{gathered}
\tag{18}
$$

The system failure probability between repair events $(r - 1)$ and $r$ can be written as:

$$
\begin{gathered}
\mathbb{P}(\mathcal{S}) = 3(1-\beta)^2 Q^2 - 2(1-\beta)^3 Q^3 + \beta Q \Leftrightarrow \\
\Leftrightarrow R_r(t_r) = 1 - \left(3(1-\beta)^2 \left(F_{\mathrm{c}1}^{(r)}(t_r)\right)^2 - 2(1-\beta)^3 \left(F_{\mathrm{c}1}^{(r)}(t_r)\right)^3 + \beta \left(F_{\mathrm{c}1}^{(r)}(t_r)\right)\right)
\end{gathered}
\tag{19}
$$

where $\beta$ can also be made a $\beta_r$ and change between repairs (for other $k$ and $n$ another expression must be derived). However, from Eq.(19), and in contrast to Eq.(15), it is observed that for $Q = 1$ we have $\mathbb{P}(\mathcal{S}) = 3(1-\beta)^2 - 2(1-\beta)^3 + \beta < 1$ for any $\beta \in ]0, 1[$, which does not verify one of the main axioms of probability theory. This is so because the partition assumed in Eq.(18) mixes failure events that happen only when having a system of redundant components with all the failure events that may happen for a single component. Since $Q$ is the component total probability of failure, whether working alone or in a redundancy with any arrangement, $Q$ can not depend on the configuration of the system. Also CCFs just happen on redundant systems, and so any partitioning of probabilities of failure can only be made at the system level and depending on the system topology. Therefore in the examples we consider the partition between independent failures and CCFs, but taken at the system level (in our point of view a more coherent approach), substituting Eq.(19) by:

$$\begin{gathered}\mathbb{P}(\mathcal{S}) = (1-\beta)\mathbb{P}(\mathcal{S}^{(\mathrm{I})}) + \beta Q \Leftrightarrow \\ \Leftrightarrow \mathbb{P}(\mathcal{S}) = (1-\beta)\left(3Q^2 - 2Q^3\right) + \beta Q\end{gathered} \tag{20}$$

which can be shown to make the system always less reliable than Eq.(19) ($\forall n \in \mathbb{N}_1 \ \wedge \ \beta \in [0,1]$ ; $0 \leq (1-\beta)^n \leq 1-\beta$), within a small difference for most of $Q \in [0,1]$, so the computations are always on the safety side.

It is observed from both Eq.(19) and Eq.(20) that the $\beta$ model imposes a certain constraint in the evaluation of the redundancy, since it includes the term $\beta Q$ in the system probability of failure, which is not affected by the degree of redundancy.

## 2.5 Some comments

The formalism presented in sections 2.1 and 2.3 is related with Renewal Theory (Cox, 1967) and the limit theorems for the long run of renewal reward processes (Ross, 1997). Considering a failure or repair $r$ as a renewal, if $\mathsf{T}_r$ ($r = 1, 2, \ldots, m \ldots$) are all independent and identically distributed with distribution $F_r(t) = F(t)$ (with $t$ restarted from 0 after each renewal), the system restarts as brand new after each repair $r$ and there are no CCFs, then we have a renewal reward process and Eq.(7) represents an application of the sometimes called elementary renewal theorem for renewal reward processes, or limit theorem for renewal reward processes. By these theorems, for each $n$, $\lim\limits_{t \to +\infty} \Phi = {}^{\mathbb{E}[\text{Costs}]}/_{\mathbb{E}[\mathsf{T}_r]}$ , the average "reward" (cost) per unit time is the ratio between the mean cost and the mean time between failures. In the general case in which $\mathsf{T}_r$ are not identically distributed or independent, we do not have a strict renewal reward process, but Eq.(7) embodies the same idea of average of the renewal reward limit theorem and the strong law of large numbers, and in spite of not being under the conditions of those theorems, an analysis of the structure of Eq.(7) indicates that in the long run a definite limit value exists, as the examples in section 3 corroborate.

For positive costs ($C_{\mathrm{A}i} \geq 0$) $\Phi_m$ is always positive, singular $\Phi_m \to +\infty$ if $t_1 = 0$, approaches a constant value when all $t_r \to +\infty$, and in general is not convex. However, due to the fact that the numerator (total cost $C(n, t_1, \ldots)$) and denominator (sum of $\mathrm{MTBR}_r(t_r)$) of Eq.(7) are non decreasing functions of $t_r$, in general for each $n$ and $m$, or $\Phi_m$ has a global minimum for $t_r > 0$, or approaches a constant value for large $t_r$. For Eq.(10) the behaviour when $t_1 \to 0$ is different if some $C_{\mathrm{A}i} < 0$ because in this region $\Phi_m \to \pm\infty$ depending on the sign of the costs. To obtain a feasible solution (system operating for some time) it is essential to apply at least a constraint on $t_1$. The examples in section 3 will illustrate this description.

## 2.6 Optimization procedure

This procedure depends on the intended accuracy, the number of variables involved and software available. The optimization process is carried for each $n$ and $m$ only on the $m$ variables $t_r$, otherwise the solver would have to be able to deal with a mixture of integer and continuous variables. The following algorithm is used:

**Algorithm 1** Optimization procedure

```
For n = 1, 2, ...
  For m = 1, 2, 3, ...
    use the previous solution as initial estimate
    obtain t_r^{*(n,m)} that minimizes Φ_m(n, t_1, ..., t_m) using a solver for m continuous variables
  Select the best m solution for the present n
Select the best (n, m) solution
```

A single continuous variable optimization algorithm can also be used inside an iterative loop, to find the minimum for each $t_r$ fixing the others to a constant value and using the new value for the next iteration step, until convergence is achieved.

All the examples presented in section 3 were solved using the optimization subroutines in the software Mathematica, because its variable arithmetic precision allowed to evaluate the influences of the numerical computation over the proposed optimization problems (40 digits were used in the computations). The examples in sections 3.1 and 3.2 were also solved using a spreadsheet implementation in Excel, with this software available optimization solver, and also part of the results of the example in section 3.2 were obtained using the open source software Maxima. Essentially the same results were achieved with any of the software with no relevant accuracy problems.

# 3 Numerical examples and discussion

In this section several examples of application illustrating the proposed methodology are presented. These examples are made simple to be easily understood and reproducible. The presented methodology allows the use of any parametric or non parametric probability model. In practice the probability distributions that best fits the components failure as a function of time should be selected. The age replacement problem with Weibull distributions has been studied by (Glasser, 1967), and a comparison with condition based replacement was made by (Geurts, 1983). Because of its flexibility the Weibull probability distribution with two adjusting parameters $(\eta > 0, \alpha > 0)$ :

$$F(t) = 0\,,\, t \leq 0\ ;\ F(t) = 1 - \exp\left(-\left(\frac{t}{\eta}\right)^{\alpha}\right)\ ,\ t > 0 \tag{21}$$

was also selected for use in this section numerical examples. The constant $\eta$ (sometimes termed characteristic life) represents a time scale for $t$. The constant $\alpha$ parametrizes the evolution of the failure rate function with $t$ (Rausand and Høyland, 2004) (also called the shape parameter because its value defines the shape of the failure probability density function plot), $0 < \alpha < 1$ corresponds to a decreasing failure rate and $\alpha > 1$ to an increasing failure rate.

For simplification all the costs are nondimensionalized to a reference preventive repair cost $C_{\mathrm{P1}} = 1$ *monetary unit* and the times to the time scale of the Weibull distribution $\eta = 1$ *time unit*. The use of diversity is a preventive action to increase the reliability of a redundant system to CCFs, and can be accounted for with the proposed framework, but to simplify the presentation related with CCFs by using the $\beta$ model, the components are considered all equal. It is also assumed that explicitly modelled dependent failure events (denoted by $\mathcal{C}_{i\mathrm{E}}$ in section 2.4) do not occur.

## 3.1 System of equal components in a 1-out-of-n configuration

If a maintenance strategy of resetting the system reliability to a predefined level after each repair is considered (the failed/working items are all overhauled by equal, or replaced by new ones), the system reliability function is always the same for each $r$ and Eq.(7) simplifies to:

$$\Phi_m(n, t_1, t_2, \ldots, t_m) = \frac{\sum_{i=1}^{n} C_{\mathrm{A}i} + \sum_{r=1}^{m} \left[C_{\mathrm{P}} R(t_r) + C_{\mathrm{F}}\left(1 - R(t_r)\right)\right]}{\sum_{r=1}^{m} \int_0^{t_r} R(\tau) d\tau} \tag{22}$$

Since the components $c_i$ are assumed all equal and in equal status, for each $i = 1, 2, \ldots, n$

the failure functions $F_{ci}^{(r)}(t) = F_{c1}(t) = 1 - \exp\left(-(t/\eta)^{\alpha}\right)$, acquisition costs $C_{Ai} = C_{A1}$ and maintenance costs $C_{Pi} = C_{P1}$ are the same for each $i = 1, \ldots, m$, and $C_P = nC_{P1}$. The influence of the acquisition cost $C_{A1}$, the maintenance cost $C_{P1}$ and the hazard cost $C_H$, considering components with a decreasing failure rate $\alpha < 1$ or increasing failure rate $\alpha > 1$, on the optimal redundancy $n^*$ and maintenance schedule $t_r^*$, is analysed in two different cases.

### 3.1.1 Case 1 - System design for optimal redundancy and first time to repair assuming independent failures

In this case $m = 1$. The corresponding system reliability is $R(t_1) = 1 - (F_{c1}(t_1))^n$, and Eq.(22) renders:

$$\Phi_1(n, t_1) = \frac{n\left(C_{A1} + C_{P1}\right) + C_H\left(F_{c1}(t_1)\right)^n}{\int_0^{t_1} 1 - \left(F_{c1}(\tau)\right)^n d\tau} = \frac{n\left(C_{A1} + C_{P1}\right) + C_H\left(1 - \exp\left(-\left(t_1/\eta\right)^{\alpha}\right)\right)^n}{\int_0^{t_1} 1 - \left(1 - \exp\left(-\left(\tau/\eta\right)^{\alpha}\right)\right)^n d\tau} \tag{23}$$

where the cost of failure was written as $C_F = C_H + nC_{P1}$, since the shut-down implies $n$ failures. This is one of the simplest forms that Eq.(7) can take. The data used in this example is: $\eta = 1$, $\alpha = \{9/10, 12/10, 2\}$, $C_{P1} = 1$, $C_{A1} = \{1, 10\}$ and $C_H = \{3, 60, 120\}$, which provides results for cases where the acquisition and failure costs range from small to large values, and decreasing and increasing failure rates. The corresponding results are presented in Table 1 and Table 2. For each optimal value of $n$ and $t_1$ the corresponding value of $\Phi_1(n^*, t_1^*)$ is displayed, as well as the optimal values of $\Phi_1(n^* \pm 1, t_1)$, for which in general $t_1$ is found to be different from $t_1^*$. Since $\eta = 1$ sets the time unit, when $t > 70$ it is assumed in this and the following examples that $t \to +\infty$.

Table 1: Design of a $1\text{oo}n$ system for optimal redundancy and first time to repair when only independent failures are considered and the acquisition costs are $C_{A1} = 1$.

| $\alpha$ | $9/10$ | | | $12/10$ | | | $2$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $C_H$ | 3 | 15 | 60 | 3 | 15 | 60 | 3 | 15 | 60 |
| $n^*$ | 2 | 5 | 8 | 2 | 3 | 5 | 1 | 2 | 3 |
| $t_1^*$ | $\dashrightarrow +\infty$ | 2.362 | 1.446 | 6.124 | 0.995 | 0.846 | 0.865 | 0.624 | 0.584 |
| $\Phi_1(n^* - 1, t_1)$ | 4.752 | 9.695 | 15.65 | 5.315 | 11.52 | 16.12 | - | 11.08 | 13.33 |
| $\Phi_1(n^*, t_1^*)$ | 4.328 | 9.573 | 15.63 | 5.172 | 10.57 | 15.85 | 5.19 | 9.12 | 12.80 |
| $\Phi_1(n^* + 1, t_1)$ | 4.487 | 9.633 | 15.74 | 5.574 | 10.58 | 16.00 | 5.76 | 9.66 | 13.39 |
| $\Phi_1(n^*, +\infty)$ | 4.328 | 9.832 | 24.73 | 5.172 | 13.01 | 35.82 | 5.64 | 16.58 | 51.15 |
| $F(t_1^*)$ [%] | 100 | 54.46 | 10.22 | 99.97 | 25.02 | 5.45 | 52.65 | 10.39 | 2.416 |
| $\text{MTBR}_1(t_1^*)$ | 1.617 | 1.898 | 1.415 | 1.353 | 0.923 | 0.837 | 0.690 | 0.610 | 0.582 |
| MTBF | 1.617 | 2.543 | 3.073 | 1.353 | 1.615 | 1.954 | 0.886 | 1.146 | 1.290 |

The results in Table 1 and Table 2 indicate that for small hazard costs $C_H \leq 3$ and for components with a decreasing or approximately constant failure rate $\alpha \leq 12/10$, the optimal strategy is to consider run-to-failure or something close to that. For this case $\Phi_1$ approaches a constant value as $t_1 \to +\infty$ so that $\min \Phi_1(n, t_1) = \lim_{t_1 \to +\infty} \Phi_1(n, t_1)$, which is the global smaller value. The influence of the components failure distribution functions over the optimal times to repair, is greater than that of the hazard or acquisition costs, as can be seen for the different values of $\alpha$ considered. A rough estimate of the optimal time to repair is around half of the value of the $\mathrm{MTBF}$ or values less but close to $\mathrm{MTBR}$, when $\Phi_1$ has a global minimum for finite $t_1$ (no run-to-failure). The values of $t_1^*$ decrease with the increase of $\alpha$, since the components become less reliable with time, and with the increase of $C_{\mathrm{H}}$ because a failure would imply a large operational cost. In some situations a repair may be recommend for $F(t_1) < 10\%$.

Table 2: Design of a $1\mathrm{oo}n$ system for optimal redundancy and first time to repair when only independent failures are considered and the acquisition costs are $C_{\mathrm{A}1} = 10$.

| $\alpha$ | $^{9}/_{10}$ | | | $^{12}/_{10}$ | | | 2 | | |
|---|---|---|---|---|---|---|---|---|---|
| $C_{\mathrm{H}}$ | 15 | 60 | 120 | 15 | 60 | 120 | 15 | 60 | 120 |
| $n^*$ | 2 | 4 | 6 | 2 | 3 | 4 | 1 | 2 | 2 |
| $t_1^*$ | $\dashrightarrow +\infty$ | $\dashrightarrow +\infty$ | 1.926 | $\dashrightarrow +\infty$ | 1.203 | 1.019 | 0.912 | 0.693 | 0.554 |
| $\Phi_1(n^*-1, t_1)$ | 24.71 | 46.37 | 61.85 | 27.64 | 54.05 | 66.96 | - | 52.18 | 73.22 |
| $\Phi_1(n^*, t_1^*)$ | 22.88 | 45.17 | 61.55 | 27.34 | 51.32 | 65.66 | 27.36 | 45.86 | 55.69 |
| $\Phi_1(n^*+1, t_1)$ | 23.93 | 45.23 | 62.05 | 29.83 | 52.27 | 66.83 | 30.82 | 49.60 | 57.48 |
| $\Phi_1(n^*, +\infty)$ | 22.88 | 45.17 | 67.76 | 27.34 | 57.59 | 90.86 | 29.34 | 71.57 | 123.93 |
| $F(t_1^*)$ [%] | 100 | 100 | 33.96 | 100 | 36.28 | 16.82 | 56.48 | 14.52 | 7.003 |
| $\mathrm{MTBR}_1(t_1^*)$ | 1.617 | 2.302 | 1.734 | 1.353 | 1.067 | 0.978 | 0.712 | 0.669 | 0.546 |
| $\mathrm{MTBF}$ | 1.617 | 2.302 | 2.745 | 1.353 | 1.615 | 1.805 | 0.886 | 1.146 | 1.146 |

Regarding the redundancy $n$, it is observed that for $\alpha \leq {}^{12}/_{10}$ the redundancy is higher, since as each equipment kind of maintains its reliability with time, increasing the redundancy considerably lowers the system probability of failure. For larger values of $C_{\mathrm{H}}$ higher redundancies are obtained since in order to decrease the impact of the failure cost on the total cost the system has to be made more reliable. However, Table 2 shows that an increase of $C_{\mathrm{A}1}$ lowers the redundancy, because in this case this cost has greater impact than $C_{\mathrm{H}}$ over the system global cost.

#### 3.1.2 Case 2 - System design for optimal redundancy and first time to repair considering CCFs

This case is similar to the previous case but now considering the influence of CCFs using the $\beta$ model, as explained in section 2.4.2. The same data is used with $\beta = {}^{1}/_{10}$ added. The results are provided in Table 3. Globally it is observed that for the CCF model used and considering that a fraction of 10% of the failures are due to CCFs, the changes relative to the independent case are relatively small. A comparison of Table 3 and Table 2 indicates that the inclusion of CCFs tends to decrease the degree of redundancy, relative to the independent failures case, although that reduction is verified only in the case of $\alpha = {}^{12}/_{10}$ and $C_\mathrm{H} = 15$. The costs increase for the CCF case and the optimal times $t_1^*$ are slightly higher than for the independent case when $\alpha < 2$, but are smaller when $\alpha = 2$, in spite of the $\mathrm{MTBF}$ being always higher in the independent case. The explanation for this results is related with the system failure probability distribution function, which is now:

$$\mathbb{P}(\mathcal{S}) = \mathbb{P}\left\{(\mathcal{C}_{1\mathrm{I}} \cap \cdots \cap \mathcal{C}_{n\mathrm{I}}) \cup (\mathcal{C}_{1\mathrm{C}} \cap \cdots \cap \mathcal{C}_{n\mathrm{C}})\right\} = F(t_1) = (1-\beta)\, Q^n + \beta Q \;;\; Q = F_{\mathrm{c}1}(t_1) \quad (24)$$

indicating that the CCFs penalize the system reliability by reducing the benefit of the redundancy on the system reliability. Since having a redundancy implies an acquisition cost, if this cost is not amortized by an increase of reliability, it is cheaper to reduce the redundancy and run the system for a longer period between maintenances.

Table 3: Design of a $1\mathrm{oo}n$ system for optimal redundancy and first time to repair when CCFs are considered using the $\beta$ model with $\beta = {}^{1}/_{10}$ and $C_{\mathrm{A}1} = 10$.

| $\alpha$ | ${}^{9}/_{10}$ | | | ${}^{12}/_{10}$ | | | 2 | | |
|---|---|---|---|---|---|---|---|---|---|
| $C_\mathrm{H}$ | 15 | 60 | 120 | 15 | 60 | 120 | 15 | 60 | 120 |
| $n^*$ | 2 | 4 | 6 | 1 | 3 | 4 | 1 | 2 | 2 |
| $t_1^*$ | $\dashrightarrow +\infty$ | $\dashrightarrow +\infty$ | 2.161 | $\dashrightarrow +\infty$ | 1.272 | 1.065 | 0.912 | 0.692 | 0.548 |
| $\Phi_1(n^*-1, t_1)$ | 24.710 | 48.682 | 68.611 | - | 56.982 | 74.617 | - | 52.179 | 73.223 |
| $\Phi_1(n^*, t_1^*)$ | 23.706 | 47.764 | 68.251 | 27.640 | 54.982 | 73.698 | 27.365 | 48.509 | 60.439 |
| $\Phi_1(n^*+1, t_1)$ | 25.126 | 48.043 | 68.696 | 28.199 | 56.270 | 75.041 | 31.700 | 53.269 | 64.094 |
| $\Phi_1(n^*, +\infty)$ | 23.706 | 47.764 | 72.215 | 27.640 | 60.103 | 95.428 | 29.338 | 73.225 | 126.804 |
| $F(t_1^*)$ [%] | 100 | 100 | 46.274 | 100 | 43.374 | 23.6606 | 56.484 | 16.815 | 8.648 |
| $\mathrm{MTBR}_1(t_1^*)$ | 1.561 | 2.177 | 1.781 | 0.941 | 1.073 | 0.982 | 0.712 | 0.661 | 0.536 |
| MTBF | 1.561 | 2.177 | 2.576 | 0.941 | 1.547 | 1.719 | 0.886 | 1.120 | 1.120 |

## 3.2 Design of a redundant system in a 2-out-of-n configuration

This example represents a system design phase problem, where for a certain configuration, the redundancy to allocate and an approximate maintenance schedule to use, minimizing the average implementation and operational life-cycle costs, are to be found. The same global assumptions concerning the components and system failure distributions considered in example 3.1 are made here, but Eq.(7) is evaluated for $m$ repair times, assuming costs and failure distribution functions changing between repairs $r$. The maintenance and hazard costs are assumed increasing by a fraction of the original costs as $C_{\mathrm{P}1}^{(r)} = C_{\mathrm{P}1}^{(1)}\,(1 + \varepsilon_\mathrm{P}\,(r-1))$ ; $C_\mathrm{H}^{(r)} = C_\mathrm{H}^{(1)}\,(1 + \varepsilon_\mathrm{H}\,(r-1))$, and the Weibull failure distribution function time scale is reduced as $\eta_r = \eta\,(1 - \varepsilon(r-1))$, where $\varepsilon_{\mathrm{P}}, \varepsilon_{\mathrm{H}}, \varepsilon \in [0,1]$ (other variations with $r$ could be considered). The reduction in $\eta$ simulates imperfect repairs. The concrete numerical values used were $C_{\mathrm{A}i} = C_{\mathrm{A}1} = 15$, $C_{\mathrm{P}1}^{(1)} = 1$, $C_{\mathrm{H}}^{(1)} = \{60, 120\}$, $\varepsilon_{\mathrm{P}} = 1/20$, $\varepsilon_{\mathrm{H}} = 1/10$, $\varepsilon = 1/10$ and $C_{\mathrm{F}}^{(r)} = C_{\mathrm{H}}^{(r)} + sC_{\mathrm{P}1}^{(r)}$ with $s = (n-k+1)$.

### 3.2.1 Case 1- Life-cycle cost optimization considering only independent failures

For this specific case of $2\mathrm{oo}n$ configuration, the system failure distribution function is given by $F(t) = nQ^{(n-1)} - (n-1)Q^n$. The results obtained are presented in Table 4 and Table 5. Additional ("intermediate") optimal values for $n^* \pm 1$ and $m^* \pm 1$ are presented in Table 6 for comparison.

The Tables 4 and 5 include values for a time of nominal system complete failure defined here as $F_r(t_{99\%}) = 99/100$, and which gives an idea of the system probable life and quantify the meaning of a run-to-failure and of the $+\infty$ appearing in the tables (used here for values one order of magnitude or more, greater than $t_{99\%}$). For some situations, usually $m = 1$ (just one repair) when $\alpha < 1$, the corresponding ("intermediate") optimal solution is run-to-failure. However run-to-failure means that the system will only work in average during a period close to the $\mathrm{MTBF}$, or in extremely rare cases until a time close to $t_{99\%}$, which is much less than the duration of the system in operational conditions given by the problem "global" optimal solution, as a comparison of the different optimal values in the Tables 4 and 5 with the Table 6 shows.

Table 4: Design of a $2\mathrm{oo}n$ system for optimal life-cycle cost considering only independent failures, $C_{\mathrm{A}i} = 15, C_{\mathrm{H}} = 60$ .

| | $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| | $\eta_r$ | 1 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 |
| | $C_{\mathrm{P}1}^{(r)}$ | 1 | 1.05 | 1.10 | 1.15 | 1.20 | 1.25 | 1.30 | 1.35 |
| $\alpha$ | $C_{\mathrm{H}}^{(r)}$ | 60 | 66 | 72 | 78 | 84 | 90 | 96 | 102 |
| $9/10$ | $n^* = 11$, $m^* = 7$, $\Phi_{m^*}(n^*, t_1^*, \ldots, t_{m^*}^*) = 50.901$, $C(n^*, t_1^*, \ldots, t_{m^*}^*) = 341.639$ | | | | | | | | |
| | $t_r^*$ | 1.886 | 1.476 | 1.167 | 0.920 | 0.716 | 0.542 | 0.393 | - |
| | $t_r^*/\eta_r$ | 1.886 | 1.640 | 1.459 | 1.314 | 1.193 | 1.084 | 0.983 | - |
| | $\mathrm{MTBR}_r(t_r^*)$ | 1.692 | 1.380 | 1.118 | 0.895 | 0.702 | 0.536 | 0.390 | - |
| | $t_{99\%}^{(r)}$ | 5.010 | 4.509 | 4.008 | 3.507 | 3.006 | 2.505 | 2.004 | 1.503 |
| | $\mathrm{MTBF}_r$ | 2.202 | 1.981 | 1.761 | 1.541 | 1.321 | 1.101 | 0.881 | 0.660 |
| | $F_r(t_r^*)$ [%] | 41.80 | 29.38 | 20.69 | 14.51 | 10.05 | 6.795 | 4.405 | - |
| $2$ | $n^* = 5$, $m^* = 7$, $\Phi_{m^*}(n^*, t_1^*, \ldots, t_{m^*}^*) = 44.081$, $C(n^*, t_1^*, \ldots, t_{m^*}^*) = 137.873$ | | | | | | | | |
| | $t_r^*$ | 0.721 | 0.618 | 0.524 | 0.437 | 0.357 | 0.282 | 0.213 | - |
| | $t_r^*/\eta_r$ | 0.721 | 0.687 | 0.655 | 0.624 | 0.595 | 0.564 | 0.533 | - |
| | $\mathrm{MTBR}_r(t_r^*)$ | 0.711 | 0.612 | 0.520 | 0.435 | 0.355 | 0.282 | 0.213 | - |
| | $t_{99\%}^{(r)}$ | 1.850 | 1.665 | 1.480 | 1.295 | 1.110 | 0.925 | 0.740 | 0.555 |
| | $\mathrm{MTBF}_r$ | 1.095 | 0.985 | 0.876 | 0.766 | 0.657 | 0.547 | 0.438 | 0.328 |
| | $F_r(t_r^*)$ [%] | 9.118 | 6.981 | 5.327 | 4.029 | 2.997 | 2.172 | 1.507 | - |

The tables also include the total cost $C$ of the operation since in general there are budget objectives or constraints. It is interesting to note similar costs $C$ are attained for the solutions in Tables 4 and 5.

The optimal values found indicate different redundancies, but the same number of repairs $m = 7$, with the corresponding optimal times decreasing with $r$. This trend in $t_r^*$ and value for $m$ is a consequence of the reduction in $\eta$, for which the values of $t_r^*/\eta$ provide some insight, and the increase of the hazard cost $C_{\mathrm{H}}$, which imply maintaining increasing reliability with time to not penalize the global cost. Topically the values $t_r^*$ are close to $\mathrm{MTBF}_r/2$ and well estimated by the equation $t_r - \mathrm{MTBR}_r(t_r) = 0$, for the examples shown.

As for the previous example, the parameter $\alpha$ has a determinant influence over the optimal redundancy $n^*$. For $\alpha = 9/10$ redundancy compensates, but for $\alpha = 2$ the same redundancy $n^* = 5$ and almost the same solution is found for $C_{\mathrm{H}} = 60$ and $C_{\mathrm{H}} = 120$. The increasing failure rate dominates the effect of the costs in the solution.

Table 5: Design of a $2\mathrm{oo}n$ system for optimal life-cycle cost considering only independent failures, $C_{\mathrm{A}i} = 15, C_{\mathrm{H}} = 120$ .

| | $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $C_{\mathrm{H}}^{(r)}$ | 120 | 132 | 144 | 156 | 168 | 180 | 192 | 204 |
| $9/10$ | $n^* = 13$, $m^* = 7$, $\Phi_{m^*}(n^*, t_1^*, \ldots, t_{m^*}^*) = 59.990$, $C(n^*, t_1^*, \ldots, t_{m^*}^*) = 379.408$ | | | | | | | | |
| | $t_r^*$ | 1.567 | 1.298 | 1.069 | 0.870 | 0.696 | 0.540 | 0.401 | - |
| | $t_r^*/\eta_r$ | 1.567 | 1.442 | 1.336 | 1.243 | 1.160 | 1.080 | 1.003 | - |
| | $\mathrm{MTBR}_r(t_r^*)$ | 1.516 | 1.268 | 1.052 | 0.861 | 0.690 | 0.537 | 0.399 | - |
| | $t_{99\%}^{(r)}$ | 5.237 | 4.713 | 4.189 | 3.666 | 3.142 | 2.618 | 2.095 | 1.571 |
| | $\mathrm{MTBF}_r$ | 2.394 | 2.155 | 1.915 | 1.676 | 1.437 | 1.197 | 0.958 | 0.718 |
| | $F_r(t_r^*)$ [%] | 17.69 | 12.82 | 9.310 | 6.727 | 4.796 | 3.334 | 2.222 | - |
| 2 | $n^* = 5$, $m^* = 7$, $\Phi_{m^*}(n^*, t_1^*, \ldots, t_{m^*}^*) = 49.310$, $C(n^*, t_1^*, \ldots, t_{m^*}^*) = 136.559$ | | | | | | | | |
| | $t_r^*$ | 0.629 | 0.542 | 0.462 | 0.387 | 0.317 | 0.252 | 0.191 | - |
| | $t_r^*/\eta_r$ | 0.629 | 0.602 | 0.578 | 0.553 | 0.528 | 0.504 | 0.478 | - |
| | $\mathrm{MTBR}_r(t_r^*)$ | 0.625 | 0.540 | 0.460 | 0.386 | 0.316 | 0.252 | 0.191 | - |
| | $t_{99\%}^{(r)}$ | 1.850 | 1.665 | 1.480 | 1.295 | 1.110 | 0.925 | 0.740 | 0.555 |
| | $\mathrm{MTBF}_r$ | 1.095 | 0.985 | 0.876 | 0.766 | 0.657 | 0.547 | 0.438 | 0.328 |
| | $F_r(t_r^*)$ [%] | 4.197 | 3.235 | 2.485 | 1.893 | 1.418 | 1.035 | 0.724 | - |

For $C_{\mathrm{H}} = 120$ the optimal solution implies smaller probabilities of failure $F_r(t_r^*)$ and in general smaller times of operation than for $C_{\mathrm{H}} = 60$. The reliability must be maintained at a certain level or $C_{\mathrm{H}}$ penalizes the operational cost.

Table 6 indicates that there are very similar solutions. So there are several options that can be considered by the designer, which may be more appropriate with respect to some constraint or objective that was not included directly in the analysis.

Table 6: Design of a $2\text{oo}n$ system for optimal life-cycle cost considering only independent failures. Values for comparison between other cases of $n$ and $m$, $C_{\mathrm{A}i} = 15, C_{\mathrm{H}} = 60$ and $C_{\mathrm{H}} = 120$.

| $\alpha$ | $n$ | $m$ | $\Phi$ | $C$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ | $t_7$ | $t_8$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $C_{\mathrm{H}} = 60$ | | | | | | |
| $^9/_{10}$ | 2 | 1 | 186.82 | 91.00 | $\dashrightarrow +\infty$ | - | - | - | - | - | - | - |
| | 10 | 7 | 51.05 | 315.80 | 1.760 | 1.369 | 1.076 | 0.844 | 0.653 | 0.492 | 0.355 | - |
| | 11 | 1 | 106.74 | 235.00 | $\dashrightarrow +\infty$ | - | - | - | - | - | - | - |
| | 11 | 6 | 51.12 | 323.78 | 1.893 | 1.480 | 1.170 | 0.922 | 0.717 | 0.543 | - | - |
| | 11 | 8 | 51.48 | 360.91 | 1.903 | 1.487 | 1.174 | 0.925 | 0.719 | 0.544 | 0.395 | 0.266 |
| | 12 | 7 | 50.99 | 367.70 | 2.008 | 1.579 | 1.254 | 0.993 | 0.775 | 0.590 | 0.430 | - |
| 2 | 2 | 1 | 128.73 | 58.46 | 0.545 | - | - | - | - | - | - | - |
| | 4 | 7 | 44.58 | 115.32 | 0.614 | 0.520 | 0.436 | 0.359 | 0.289 | 0.226 | 0.168 | - |
| | 5 | 1 | 110.90 | 98.39 | 0.938 | - | - | - | - | - | - | - |
| | 5 | 6 | 44.58 | 130.26 | 0.723 | 0.620 | 0.525 | 0.438 | 0.358 | 0.283 | - | - |
| | 5 | 8 | 44.43 | 145.85 | 0.722 | 0.619 | 0.5247 | 0.438 | 0.357 | 0.283 | 0.214 | 0.150 |
| | 6 | 7 | 45.20 | 161.22 | 0.809 | 0.698 | 0.595 | 0.500 | 0.411 | 0.328 | 0.250 | - |
| | | | | | | $C_{\mathrm{H}} = 120$ | | | | | | |
| $^9/_{10}$ | 2 | 1 | 310.00 | 151.00 | $\dashrightarrow +\infty$ | - | - | - | - | - | - | - |
| | 12 | 7 | 60.02 | 355.28 | 1.471 | 1.214 | 0.997 | 0.810 | 0.645 | 0.499 | 0.369 | - |
| | 13 | 1 | 134.78 | 290.51 | 2.718 | - | - | - | - | - | - | - |
| | 13 | 6 | 60.46 | 359.20 | 1.573 | 1.302 | 1.072 | 0.873 | 0.697 | 0.541 | - | - |
| | 13 | 8 | 60.56 | 400.96 | 1.574 | 1.302 | 1.073 | 0.873 | 0.698 | 0.541 | 0.402 | 0.277 |
| | 14 | 7 | 60.16 | 405.01 | 1.659 | 1.377 | 1.137 | 0.929 | 0.744 | 0.579 | 0.431 | - |
| 2 | 2 | 1 | 178.55 | 61.19 | 0.375 | - | - | - | - | - | - | - |
| | 4 | 7 | 51.165 | 114.24 | 0.521 | 0.444 | 0.374 | 0.310 | 0.251 | 0.196 | 0.147 | - |
| | 5 | 1 | 126.073 | 97.30 | 0.789 | - | - | - | - | - | - | - |
| | 5 | 6 | 49.903 | 128.99 | 0.630 | 0.543 | 0.463 | 0.388 | 0.318 | 0.252 | - | - |
| | 5 | 8 | 49.679 | 144.48 | 0.630 | 0.543 | 0.462 | 0.387 | 0.316 | 0.252 | 0.191 | 0.135 |
| | 6 | 7 | 49.790 | 159.69 | 0.717 | 0.622 | 0.533 | 0.450 | 0.371 | 0.297 | 0.227 | - |

If it was considered that for each $r$ the repair was perfect (the repaired component is new or as good as a new one) and the costs were maintained equal and constant, this optimization problem would render as solution equal values $t_1 = t_2 = \ldots = t_m$ for each $n$.

#### 3.2.2 Case 2 - Life-cycle cost optimization considering CCFs

The same data of the previous case is used but considering now the CCFs modelled with the $\beta$ model and setting $\beta = {}^1/_{10}$. The corresponding results appear in Table 7, Table 8 and Table 9.

Table 7: Design of a $2\text{oo}n$ system for optimal life-cycle cost considering CCFs using the $\beta$ model with $\beta = 1/10, C_{\mathrm{A}i} = 15, C_{\mathrm{H}} = 60$ .

| | $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $C_{\mathrm{H}}^{(r)}$ | 60 | 66 | 72 | 78 | 84 | 90 | 96 | 102 |
| $9/10$ | $n^* = 11$, $m^* = 6$, $\Phi_{m^*}(n^*, t_1^*, \ldots, t_{m^*}^*) = 57.459$, $C(n^*, t_1^*, \ldots, t_{m^*}^*) = 363.309$ | | | | | | | | |
| | $t_r^*$ | 2.100 | 1.597 | 1.240 | 0.963 | 0.739 | 0.552 | - | - |
| | $t_r^*/\eta_r$ | 2.100 | 1.774 | 1.550 | 1.376 | 1.232 | 1.104 | - | - |
| | $\mathrm{MTBR}_r(t_r^*)$ | 1.712 | 1,389 | 1,118 | 0.889 | 0.692 | 0.523 | - | - |
| | $t_{99\%}^{(r)}$ | 5.039 | 4.535 | 4.031 | 3.527 | 3.023 | 2.519 | - | - |
| | $\mathrm{MTBF}_r$ | 2.087 | 1.878 | 1.669 | 1.461 | 1.252 | 1,403 | - | - |
| | $F_r(t_r^*)$ [%] | 55.58 | 40.67 | 30.17 | 22.69 | 17.25 | 13.23 | - | - |
| 2 | $n^* = 4$, $m^* = 7$, $\Phi_{m^*}(n^*, t_1^*, \ldots, t_{m^*}^*) = 48.759$, $C(n^*, t_1^*, \ldots, t_{m^*}^*) = 124.449$ | | | | | | | | |
| | $t_r^*$ | 0.623 | 0.524 | 0.436 | 0.356 | 0.284 | 0.218 | 0.159 | - |
| | $t_r^*/\eta_r$ | 0.623 | 0.582 | 0.545 | 0.509 | 0.473 | 0.436 | 0.398 | - |
| | $\mathrm{MTBR}_r(t_r^*)$ | 0.606 | 0.513 | 0.428 | 0.351 | 0.281 | 0.216 | 0.158 | - |
| | $t_{99\%}^{(r)}$ | 1.828 | 1.645 | 1.462 | 1.279 | 1.097 | 0.914 | 0.731 | 0.548 |
| | $\mathrm{MTBF}_r$ | 0.985 | 0.887 | 0.788 | 0.690 | 0.591 | 0.493 | 0.394 | 0.296 |
| | $F_r(t_r^*)$ [%] | 12.30 | 9.589 | 7.479 | 5.806 | 4.460 | 3.362 | 2.453 | - |

Globally it is observed that relatively to the independent failures case, the average costs increase and the number of repairs $m$ decreases or the redundancy $n$ decreases. For $\alpha = 9/10$ the redundancy is the same, the number of repairs decreases to $m^* = 6$, the optimal repair times $t_r^*$ increase and the probability of failure increases $F_r(t_r^*)$. For $\alpha = 2$ the optimal repair times $t_r^*$ decrease and the probability of failure increases $F_r(t_r^*)$. This is a consequence of using the $\beta$ model and Eq.(20), where the term $\beta Q$ penalizes the system reliability to a point where increasing the redundancy does not improve the reliability sufficiently to compensate the involved acquisition cost, as was also observed and explained in the first example 3.1. Other CCF models will affect the system reliability differently, but always implying a tendency for a reduction relative to the independent failures case.

Table 8: Design of a $2\text{oo}n$ system for optimal life-cycle cost considering CCFs using the $\beta$ model with $\beta = 1/10, C_{\mathrm{A}i} = 15, C_{\mathrm{H}} = 120$ .

| | $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $C_{\mathrm{H}}^{(r)}$ | 120 | 132 | 144 | 156 | 168 | 180 | 192 | 204 |
| 9/10 | $n^* = 13$, $m^* = 6$, $\Phi_{m^*}(n^*, t_1^*, \ldots, t_{m^*}^*) = 72.951$, $C(n^*, t_1^*, \ldots, t_{m^*}^*) = 434.240$ | | | | | | | | |
| | $t_r^*$ | 1.710 | 1.394 | 1.133 | 0.911 | 0.718 | 0.548 | - | - |
| | $t_r^*/\eta_r$ | 1.710 | 1.549 | 1.416 | 1.301 | 1.197 | 1.096 | - | - |
| | $\mathrm{MTBR}_r(t_r^*)$ | 1.547 | 1.285 | 1.058 | 0.859 | 0.681 | 0.523 | - | - |
| | $t_{99\%}^{(r)}$ | 5.250 | 4.725 | 4.200 | 3.675 | 3.150 | 2.625 | 2.100 | - |
| | $\mathrm{MTBF}_r$ | 2.260 | 2.034 | 1.808 | 1.582 | 1.356 | 1.130 | 0.904 | - |
| | $F_r(t_r^*)$ [%] | 29.582 | 22.958 | 18.177 | 14.637 | 11.954 | 9.871 | - | - |
| 2 | $n^* = 4$, $m^* = 6$, $\Phi_{m^*}(n^*, t_1^*, \ldots, t_{m^*}^*) = 58.813$, $C(n^*, t_1^*, \ldots, t_{m^*}^*) = 120.210$ | | | | | | | | |
| | $t_r^*$ | 0.526 | 0.444 | 0.370 | 0.302 | 0.240 | 0.184 | - | - |
| | $t_r^*/\eta_r$ | 0.526 | 0.493 | 0.463 | 0.431 | 0.400 | 0.368 | - | - |
| | $\mathrm{MTBR}_r(t_r^*)$ | 0.518 | 0.439 | 0.366 | 0.300 | 0.239 | 0.183 | - | - |
| | $t_{99\%}^{(r)}$ | 1.828 | 1.645 | 1.462 | 1.279 | 1.097 | 0.914 | 0.731 | - |
| | $\mathrm{MTBF}_r$ | 0.985 | 0.886 | 0.788 | 0.689 | 0.591 | 0.492 | 0.394 | - |
| | $F_r(t_r^*)$ [%] | 6.583 | 5.205 | 4.117 | 3.243 | 2.525 | 1.926 | - | - |

Table 9: Design of a $2\text{oo}n$ system for optimal life-cycle cost considering CCFs using the $\beta$ model. Comparison between other $n$ and $m$ values for $\beta = 1/10, C_{\mathrm{A}i} = 15, C_{\mathrm{H}} = 60$ and $C_{\mathrm{H}} = 120$ .

| $\alpha$ | $n$ | $m$ | $\Phi$ | $C$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ | $t_7$ | $t_8$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $C_{\mathrm{H}} = 60$ | | | | | | |
| 9/10 | 2 | 1 | 167.40 | 91.00 | $\dashrightarrow +\infty$ | - | - | - | - | - | - | - |
| | 10 | 6 | 57.59 | 337.54 | 1.968 | 1.485 | 1.145 | 0.884 | 0.674 | 0.499 | - | - |
| | 11 | 1 | 112.62 | 235.00 | $\dashrightarrow +\infty$ | - | - | - | - | - | - | - |
| | 11 | 5 | 58.24 | 340.00 | 2.128 | 1.613 | 1.250 | 0.970 | 0.743 | - | - | - |
| | 11 | 7 | 57.82 | 388.39 | 2.113 | 1.605 | 1.244 | 0.966 | 0.741 | 0.553 | 0.394 | - |
| | 12 | 6 | 57.57 | 389.10 | 2.229 | 1.704 | 1.330 | 1.038 | 0.801 | 0.601 | - | - |
| 2 | 2 | 1 | 125.03 | 58.84 | 0.568 | - | - | - | - | - | - | - |
| | 3 | 6 | 52.37 | 93.66 | 0.490 | 0.401 | 0.325 | 0.259 | 0.201 | 0.150 | - | - |
| | 4 | 1 | 108.53 | 84.01 | 0.839 | - | - | - | - | - | - | - |
| | 4 | 6 | 48.83 | 116.98 | 0.623 | 0.524 | 0.436 | 0.356 | 0.284 | 0.218 | - | - |
| | 4 | 8 | 49.50 | 132.15 | 0.626 | 0.527 | 0.438 | 0.358 | 0.285 | 0.219 | 0.160 | 0.106 |
| | 5 | 7 | 49.34 | 151.74 | 0.731 | 0.623 | 0.524 | 0.434 | 0.350 | 0.273 | 0.202 | - |
| | | | | | | $C_{\mathrm{H}} = 120$ | | | | | | |
| 9/10 | 2 | 1 | 277.78 | 151.00 | $\dashrightarrow +\infty$ | - | - | - | - | - | - | - |
| | 12 | 6 | 73.15 | 407.82 | 1.611 | 1.308 | 1.059 | 0.848 | 0.665 | 0.505 | - | - |
| | 13 | 1 | 143.92 | 303.15 | 3.018 | - | - | - | - | - | - | - |
| | 13 | 5 | 73.73 | 401.95 | 1.719 | 1.400 | 1.138 | 0.915 | 0.721 | - | - | - |
| | 13 | 7 | 73.70 | 468.54 | 1.719 | 1.400 | 1.138 | 0.914 | 0.721 | 0.550 | 0.399 | - |
| | 14 | 6 | 72.97 | 460.47 | 1.804 | 1.475 | 1.203 | 0.971 | 0.768 | 0.589 | - | - |
| 2 | 2 | 1 | 173.73 | 61.39 | 0.387 | - | - | - | - | - | - | - |
| | 3 | 6 | 64.89 | 94.41 | 0.392 | 0.323 | 0.262 | 0.209 | 0.162 | 0.121 | - | - |
| | 4 | 1 | 128.46 | 84.48 | 0.684 | - | - | - | - | - | - | - |
| | 4 | 5 | 60.05 | 112.48 | 0.530 | 0.447 | 0.372 | 0.304 | 0.242 | - | - | - |
| | 4 | 7 | 58.87 | 128.15 | 0.526 | 0.444 | 0.370 | 0.302 | 0.240 | 0.184 | 0.133 | - |
| | 5 | 6 | 58.96 | 148.50 | 0.635 | 0.543 | 0.457 | 0.378 | 0.304 | 0.235 | - | - |

### 3.3 Maintenance interval optimization problem

In this example the system in analysis uses a 3-out-of-$n_0$ configuration, where $n_0 = \{4, 5, 6, 10\}$ cases are studied. The same global assumptions present in the previous examples are maintained, but in this example it is investigated if the implemented redundancy is the one leading to the best average cost in the near future system operation. Only independent failures are considered. The redundancy can increase or decrease and Eq.(10) is used to evaluate the optimal times and average costs for the next $m = 2$ repairs, considering the time constraints:

$$t_1 > \frac{2}{3}\mathrm{MTBF}_1 \; ; \; t_2 > \frac{3}{5}\mathrm{MTBF}_2 \tag{25}$$

where $\mathrm{MTBF}_r$ is the medium time between failures corresponding to the failure distribution $F_r(t_r)$ during interval $r$. Other constraints could be considered such as $\hat{F}_r - F_r(t_r) \geq 0$. If the redundancy increases ($n > n_0$) then the acquisition cost is $C_{\mathrm{A}i} = 10$, otherwise if the redundancy is decreased ($n < n_0$) $C_{\mathrm{A}i} = 5$ and is considered a selling benefit, $i = 1, 2, \ldots, \max(n, n_0)$. Since for $n < n_0$ $C_{\mathrm{A}i}$ is affect by a negative sign in Eq.(10), the times $t_r$ must be constrained to be greater than a minimum feasible value, otherwise the optimal solution is to sell the complete system. The remaining data is: $C_{\mathrm{H}} = \{60, 120\}$, $\varepsilon_{\mathrm{P}} = \varepsilon_{\mathrm{H}} = 0$, $\varepsilon = 1/20$ ($\eta$ is reduced linearly with $r$) and $\alpha = 12/10$. The Table 10 and Table 11 present the optimal values found.

Table 10: Optimal redundancy and $m = 2$ maintenance schedule for a $3$oo$n$ system considering only independent failures and an initial redundancy of $n_0$ with $C_{\mathrm{H}} = 60$.

| $n_0$ | 4 | | 5 | | 6 | | 10 | |
|---|---|---|---|---|---|---|---|---|
| $m$ | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| $n^*$ | 6 | | 6 | | 6 | | 6 | |
| $\Phi_2$ | 50.163 | | 41.123 | | 32.083 | | 14.004 | |
| $C$ | 55.492 | | 45.492 | | 35.492 | | 15.492 | |
| $t_r^*$ | 0.627 | 0.537 | 0.627 | 0.537 | 0.627 | 0.537 | 0.627 | 0.537 |
| $\mathrm{MTBF}_r$ | 0.941 | 0.894 | 0.941 | 0.894 | 0.941 | 0.894 | 0.941 | 0.894 |
| $t_{99\%}$ | 2.123 | 2.017 | 2.123 | 2.017 | 2.123 | 2.017 | 2.123 | 2.017 |
| $F_r(t_r^*)[\%]$ | 23.167 | 17.337 | 23.167 | 17.337 | 23.167 | 17.337 | 23.167 | 17.337 |
| $n^*_{\mathrm{NC}}$ | 5 | | 4 | | 5 | | 8 | |
| $\Phi_{\mathrm{NC}2}$ | 47.553 | | 27.834 | | 18.815 | | 8.864 | |
| $C_{\mathrm{NC}}$ | 30.598 | | 5.382 | | 7.334 | | 7.383 | |
| $t^*_{\mathrm{NC}r}$ | 0.343 | 0.316 | 0.102 | 0.093 | 0.203 | 0.188 | 0.431 | 0.404 |

All the optimal times found correspond to active time constraints (the values of this constraint are relatively high) and the optimal redundancy can be greater or smaller than $n_0$. For $C_{\mathrm{H}} = 60$ the same redundancy is predicted for the different situations. For comparison a life-cycle optimization like the one presented in section 3.2.1, was performed using the present example's data without the time constraints, and the optimal redundancy for $C_{\mathrm{H}} = 60$ is found to be $n = 11$, involving $m = 12$ repairs at times $t_1 = 0.920$, $t_2 = 0.861$, $t_3 = 0.804$, $t_4 = 0.747$, $t_5 = 0.692$, $t_6 = 0.638$, $t_7 = 0.585$, $t_8 = 0.533$, $t_9 = 0.483$, $t_{10} = 0.433$, $t_{11} = 0.385$, $t_{12} = 0.338$, corresponding to an average cost of $\Phi = 39.914$ and a total cost of $C = 292.714$. The differences between the two problems are the acquisition costs, which here depend on $n_0$, and therefore are smaller, and the time constraints. The solutions are quite different. For $n_0 = 10$, using the time constraints in Eq.(25) for the first two repair intervals and analysing for different redundancies $n$ and number of repairs $m$, it was found that for $n < n_0$ $\min_{t_r} \Phi_m$ increases for each $n$ and $m$ and so the "global" minimum value happens for $n = 6$ and $m = 2$, the case in Table 10. If $n = n_0 = 10$ the there is a "global" minimum $\Phi_3 = 20.133$ for $m = 3$ corresponding to $t_1 = 0.889$, $t_2 = 0.760$ (time constraints are active) and $t_3 = 0.616$. For $n > n_0$ there are "global" minimal values for $\Phi_m$ but with higher values than for $n = n_0$, for instance for $n = 11$ and $m = 5$ $\Phi_5 = 22.845$ at $t_1 = 0.937$, $t_2 = 0.801$ (constraints are active), $t_3 = 0.698$, $t_4 = 0.648$ and $t_5 = 0.602$. For an initial redundancy of $n_0 = 10$ and based on the results obtained, if the system duration it to be greater than the time constraints, then the redundancy $n_0$ should be maintained.

Additionally the tables present the optimal values found for the non constrained case for comparison (subscript NC). In the non constrained case the optimal times are much smaller and the redundancy, and consequently the cost, are also smaller. Although not presented in the tables, for the non constrained case $m = 1$ and $n < n_0$ the cost is negative and $t_1 \to 0$, which is not feasible.

Table 11: Optimal redundancy and $m = 2$ maintenance schedule for a 3oo$n$ system considering only independent failures and an initial redundancy of $n_0$ with $C_{\mathrm{H}} = 120$.

| $n_0$ | 4 | | 5 | | 6 | | 10 | |
|---|---|---|---|---|---|---|---|---|
| $m$ | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| $n^*$ | 9 | | 8 | | 8 | | 9 | |
| $\Phi_2$ | 64.562 | | 57.546 | | 50.396 | | 28.276 | |
| $C$ | 97.859 | | 80.488 | | 70.488 | | 42.859 | |
| $t_r^*$ | 0.836 | 0.715 | 0.776 | 0.663 | 0.776 | 0.663 | 0.836 | 0.715 |
| $\mathrm{MTBF}_r$ | 1.254 | 1.191 | 1.164 | 1.106 | 1.164 | 1.106 | 1.254 | 1.191 |
| $t_{99\%}$ | 2.450 | 2.328 | 2.358 | 2.240 | 2.358 | 2.240 | 2.450 | 2.328 |
| $F_r(t_r^*)[\%]$ | 15.445 | 9.859 | 17.485 | 11.742 | 17.485 | 11.742 | 15.445 | 9.859 |
| $n_{\mathrm{NC}}^*$ | 6 | | 5 | | 5 | | 8 | |
| $\Phi_{\mathrm{NC}2}$ | 59.191 | | 37.835 | | 23.502 | | 10.093 | |
| $C_{\mathrm{NC}}$ | 44.010 | | 14.662 | | 7.232 | | 7.306 | |
| $t_{\mathrm{NC}r}^*$ | 0.390 | 0.362 | 0.202 | 0.187 | 0.160 | 0.148 | 0.374 | 0.351 |

# 4 Conclusions

The proposed procedure produces results considered feasible and realistic for the application in practical cases. It is a framework useful in assisting maintenance decisions, especially in a design phase, which is relatively simple to implement and requires minimum data. It is based solely on the idea of a probabilistic average cost and that may be its limitation in some situations. The set of examples presented demonstrates the type of results that can obtained and used in maintenance optimization operations.

The methodology can be extended to include predictions for multiple repairs, considering changes in the maintenance strategy after each repair (different scenarios). However this leads to a combinatorial optimization problem that can be very difficult or demanding to solve.

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