Braumann, C. A. (2007). Itô versus Stratonovich calculus in random population growth. *Mathematical Biosciences* **206**: 81-107.
doi:10.1016/j.mbs.2004.09.002
http://dx.doi.org/10.1016/j.mbs.2004.09.002
ISSN: 0025-5564

Abstract

The context is the general stochastic differential equation (SDE) model $dN/dt = N(g(N) + \sigma\varepsilon(t))$ for population growth in a randomly fluctuating environment. Here, $N = N(t)$ is the population size at time t, $g(N)$ is the 'average' *per capita* growth rate (we work with a general almost arbitrary function g), and $\sigma\varepsilon(t)$ is the effect of environmental fluctuations ($\sigma > 0$, $\varepsilon(t)$ standard white noise). There are two main stochastic calculus used to interpret the SDE, Itô calculus and Stratonovich calculus. They yield different solutions and even qualitatively different predictions (on extinction, for example). So, there is a controversy on which calculus one should use.

We will resolve the controversy and show that the real issue is merely semantic. It is due to the informal interpretation of $g(x)$ as being an (unspecified) 'average' *per capita* growth rate (when population size is x). The implicit assumption usually made in the literature is that the 'average' growth rate is the same for both calculi, when indeed this rate should be defined in terms of the observed process. We prove that, when using Itô calculus, $g(N)$ is indeed the arithmetic average growth rate $R_a(x)$ and, when using Stratonovich calculus, $g(N)$ is indeed the geometric average growth rate $R_g(x)$. Writing the solutions of the SDE in terms of a well-defined average, $R_a(x)$ or $R_g(x)$, instead of an undefined 'average' $g(x)$, we prove that the two calculi yield exactly the same solution. The apparent difference was due to the semantic confusion of taking the informal term 'average growth rate' as meaning the same average.

Keywords: Population growth; Itô calculus; Stratonovich calculus; Random environments; Stochastic differential equations