Analysis of new situations for quasiconvexity versus rank-one convexity in 2×2 and other dimensions

PhD Thesis by
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Universidade de Évora, Portugal

In partial fulfillment of the requirements for the European degree of Doctor of Philosophy in Mathematics

This thesis does not include the critiques and suggestions formulated by the jury

Departamento de Matemática
Universidade de Évora
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I wish to thank Prof. Pablo Pedregal and Prof. António Ornelas.

To my family

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Abstract

It is well-known that quasiconvexity is a fundamental concept for vector problems in the Calculus of Variations. Its main necessary condition is rank-one convexity. Still today it is not known whether it is also sufficient or not, when the target space of deformations is m=2 (in the general case).

We introduce a method to find, in a systematic way, rank-one convex polynomials. We show how it works in several examples. It can also be applied to convexity along general cones.

An alternative proof is provided for the well-known quadratic case of quasiconvexity, which does not use the Plancherel formula. An application to the case of 4th degree homogeneous polynomials is shown.

We also explore an attempt to disprove the implication from rank-one convexity to quasiconvexity for 2x2 symmetric matrices, using the viewpoint of laminates and homogeneous gradient Young measures.

Análise de novas situações para a quasiconvexidade versus convexidade característica-1 em dimensão 2×2 e outras dimensões

Resumo

É bem conhecido que a quasiconvexidade é um conceito fundamental para problemas vectoriais do Cálculo das Variações. A sua principal condição necessária é a convexidade característica-1. Ainda hoje não é conhecido se é ou não suficiente, quando o espaço alvo das deformações é m=2 (no caso geral).

Introduzimos um método para determinar, de uma forma sistemática, polinómios convexos característica-1. Mostramos como funciona em diversos exemplos. Pode também ser aplicado à convexidade ao longo de cones gerais.

Providenciamos uma demonstração alternativa para o bem conhecido caso quadrático da quasiconvexidade, que não utiliza a fórmula de Plancherel. Apresentamos uma aplicação para o caso dos polinómios homogéneos de grau 4.

Exploramos também uma tentativa para refutar a implicação da convexidade característica-1 para a quasiconvexidade nas matrizes 2x2 simétricas, sob o ponto de vista dos laminados e das medidas de Young gradiente homogéneas.

Extended abstract

It is well-known that quasiconvexity is a fundamental concept for vector problems in the Calculus of Variations. One important related convexity condition is rank-one convexity, which is a necessary condition. Still today it is not known if this type of convexity implies or not quasiconvexity, when the target space of deformations is m=2 (in the general case). Our work aim at contributing for a better understanding of this outstanding problem.

Rank-one convexity, though a more manageable concept, is not easy to check on explicit examples. Indeed, deciding when a given function is or is not rank-one convex is not an easy task. In Chapter 2, we provide a new method to determine (at least in some specific situations) the rank-one convexity of functions of a particular structure, but not only restricted to homogeneous polynomials. We show how it works in several examples, exploring both classical examples and new ones. An interpretation of this results in terms of laminates is also presented, and it seems to be more promising in terms of applying these ideas to other kinds of convexity. Our ideas can also be applied to convexity along general cones as, for example, the characteristic cone associated to quasiconvexity for second order gradients (called 2-quasiconvexity).

It is known for a long time that in the quadratic case, quasiconvexity is equivalent to rank-one convexity. The classic (and only known) proof makes use of Fourier transforms and the Plancherel formula and so it cannot be applied to other cases. We provide, in Chapter 3, an alternative proof for this well-known case, which does not make use of Plancherel formula and so, in principle, it can be used in other cases, especially with polynomials. This has further interest nowadays, as we now know that one can approximate quasiconvex functions by quasiconvex polynomials ([21]). Using this new approach, we derive necessary and sufficient conditions for quasiconvexity at the origin for fourth degree homogeneous polynomials. We also make an application to the case of 2-quasiconvexity at the origin for the same kind of polynomials. The ideas here contained can also be applied to homogeneous polynomials of any even-degree.

In Chapter 4 it is explored the problem of the equivalence between quasiconvexity and rank-one convexity in the case of 2×2 symmetric matrices from the viewpoint of probability measures, that is, to know if every homogeneous gradient Young measure (supported in the space of 2×2 symmetric matrices) is a laminate. We follow the approach of [36], using distinct first moments, including the one there used (the origin). We were not able to find a counterexample, and several difficulties involved are shown through the text. A characterization of the set of laminates in a precise class is obtained.

Resumo alargado

É bem conhecido que a quasiconvexidade é um conceito fundamental para problemas vectoriais do Cálculo das Variações. Uma importante condição de convexidade relacionada é a convexidade característica-1, que é uma condição necessária. Ainda hoje não é conhecido se este tipo de convexidade implica ou não a quasiconvexidade, quando o espaço alvo das deformações é m=2 (no caso geral). O nosso trabalho aspira a contribuir para um melhor entendimento deste problema extraordinário.

A convexidade característica-1, embora parecendo um conceito mais manejável, não é fácil de verificar em exemplos concretos. De facto, decidir quando uma dada função é ou não convexa característica-1 não é uma tarefa fácil. No Capítulo 2, providenciamos um novo método para determinar (pelo menos em algumas situações específicas) a convexidade característica-1 de funções com uma estrutura particular, mas não restrita apenas a polinómios homogéneos. Mostramos como funciona em diversos exemplos, explorando exemplos clássicos e novos. É também apresentada uma interpretação destes resultados em termos de laminados, que parece ser mais promissora em termos de aplicação destas ideias a outros tipos de convexidade. As nossas ideias podem também ser aplicadas à convexidade ao longo de cones gerais como, por exemplo, o cone característico associado à quasiconvexidade para segundos gradientes (chamada 2-quasiconvexidade).

É conhecido há muito tempo que no caso quadrático, a quasiconvexidade é equivalente à convexidade característica-1. A demonstração clássica (e única conhecida) utiliza transformadas de Fourier e a fórmula de Plancherel e, consequentemente, não pode ser aplicada a outros casos. Providenciamos, no Capítulo 3, uma demonstração alternativa para o bem conhecido caso quadrático da quasiconvexidade, que não utiliza a fórmula de Plancherel e poderá então, em princípio, ser utilizada noutros casos, especialmente com polinómios. Este facto tem interesse acrescido hoje em dia, dado que sabemos agora que é possível aproximar funções quasiconvexas com polinómios quasiconvexos ([21]). Usando esta nova abordagem, deduzimos condições necessárias e suficientes para a quasiconvexidade na origem para polinómios homogéneos de quarto grau. Apresentamos também uma aplicação ao caso da 2-quasiconvexidade na origem para o mesmo tipo de polinómios. As ideias aqui contidas podem também ser aplicadas a polinómios homogéneos de qualquer grau par.

No Capítulo 4 é explorado o problema da equivalência entre a quasiconvexidade e a convexidade característica-1 no caso das matrizes 2×2 simétricas, do ponto de vista das medidas de probabilidade, isto é, o saber

se qualquer medida de Young gradiente homogénea (suportada no espaço das matrizes 2×2 simétricas) é um laminado. Seguimos a abordagem de [36], usando distintos primeiros momentos, incluindo o utilizado nesse artigo (a origem). Não fomos capazes de encontrar tal contra-exemplo, e as diferentes dificuldades envolvidas são mostradas ao longo do texto. Para terminar obtemos uma caracterização do conjunto dos laminados numa classe precisa.

Chapter 1

Introduction

In the framework of nonlinear elasticity ([2],[9],[38]) we are interested in proving the existence of equilibrium configurations for elastic bodies under prescribed environmental conditions. Let m and N be, for the moment, either 2 or 3 and $\Omega \subset \mathbb{R}^N$ be a bounded regular open set, representing the body whose deformation we want to study. The equilibrium configuration must satisfy

$$-\text{div } \sigma(x, \nabla u) = f(x, u), \ \ x \in \Omega \tag{1.1}$$

where $u : \Omega \longrightarrow \mathbb{R}^m$ represents the displacement fields (assumed to be smooth enough), which should also satisfy some boundary conditions over $\partial\Omega$, $f : \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ are the external forces and $\sigma : \Omega \times \mathbb{M}^{m\times N} \longrightarrow \mathbb{M}^{m\times N}$ gives the internal stress. Assuming that the elastic material is in fact hyperelastic, there exists a function $\widetilde{\varphi} : \Omega \times \mathbb{M}^{m\times N} \longrightarrow \mathbb{R}$, differentiable with respect to $\nabla u \in \mathbb{M}^{m\times N}$ such that

$$\sigma_{ij}(x, \nabla u) = \frac{\partial \widetilde{\varphi}}{\partial(\nabla u)_{ij}}(x, \nabla u), \ \ x \in \Omega.$$

This equation is called the stress-strain relation and it represents the constitutive assumption made on the material at position $x \in \Omega$. It corresponds to the generalization of Hooke's law ([20]). If in addition there exists a function $\widetilde{f}$ such that

$$\frac{\partial \widetilde{f}}{\partial u}(x, u) = f(x, u), \ \ x \in \Omega,$$

then the equilibrium configurations are extremals of the total energy functional

$$I(u) = \int_\Omega \varphi(x, u, \nabla u)\, dx,$$

where

$$\varphi(x, u, \nabla u) = \tilde{\varphi}(x, \nabla u) - \tilde{f}(x, u),$$

u satisfying the same boundary conditions assumed above. Another way of saying this is that the Euler-Lagrange system associated with the functional I is exactly (1.1). In particular, minimizers of the total energy satisfying the boundary conditions will be (weak) solutions of the equilibrium equations. For simplicity, we will consider

$$\varphi = \varphi(\nabla u).$$

The central problem in the Calculus of Variations is to show the existence of minimizers of energy functionals of the type

$$I(u) = \int_\Omega \varphi(\nabla u(x))\, dx$$

among competing fields $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^m$ complying with boundary conditions over $\partial\Omega$ ([12]). Here Ω is supposed to be a bounded, regular domain (i.e. Lipschitz), and feasible fields u belong to suitable Sobolev classes related to the growth properties of the density φ at infinity. The integrand $\varphi : \mathbb{M}^{m\times N} \to \mathbb{R}$ is assumed to be continuous. More specific assumptions are necessary to deal with problems in non-linear elasticity ([9]).

The crucial property on φ to ensure existence of solutions through the direct method ([12]) is quasiconvexity. One such density φ is said to be quasiconvex if

$$\varphi(\xi) \leq \int_D \varphi(\xi + \nabla v(x))\, dx \tag{1.2}$$

for some unitary domain D ($|D| = 1$), for any matrix $\xi \in \mathbb{M}^{m\times N}$, and every test field v in D. It turns out that this concept is independent of the domain. This property on φ is equivalent to the weak lower semicontinuity of the functional I above with respect to weak convergence of Lipschitz fields. This was established by Morrey in [28].

In the scalar case, when $m = 1$ or $N = 1$, quasiconvexity reduces to plain convexity, but it is not so in the fully vector case $N, m > 1$. The concept of quasiconvexity is hard to grasp and analyze due to its non-local character expressed in the inequality (1.2) above ([23]). So a principal issue has been to find more manageable necessary and sufficient conditions for it.

A main necessary condition is rank-one convexity. An integrand φ as before is rank-one convex if it satisfies the typical convexity inequality along rank-one matrices

$$\varphi(t\xi_1 + (1-t)\xi_0) \leq t\varphi(\xi_1) + (1-t)\varphi(\xi_0),\ \operatorname{rank}(\xi_1 - \xi_0) \leq 1, t \in [0,1].$$

If φ is smooth, this convexity condition is equivalent to the Legendre-Hadamard condition (or ellipticity condition) ([2])

$$A^T \nabla^2 \varphi(\xi) A \geq 0, \ \text{rank}(A) \leq 1.$$

On the other hand, a sufficient condition is polyconvexity. φ is polyconvex if it can be written in the form

$$\varphi(\xi) = \phi(M(\xi))$$

where $M(\xi)$ is the vector of all minors of ξ, and ϕ is a convex function of all its arguments. Polyconvexity has played a major role in existence theorems in non-linear elasticity ([2]). A lot of effort has been dedicated to establishing the differences among these three convexity concepts. All three are different and counterexamples of various forms have been found through the years (see [1], [13], [14], [41], [44]). Perhaps one of the most interesting examples is the one in [1],[14], as with the help of a single real parameter c, characterizes the different notions of convexity. For $\varphi : \mathbb{M}^{2\times 2} \to \mathbb{R}$, define

$$\varphi(\xi) = |\xi|^4 - c\,|\xi|^2 \det \xi.$$

Then

$$\begin{array}{lcl} \varphi \text{ is convex} & \Leftrightarrow & |c| \leq \frac{4}{3}\sqrt{2} \\ \varphi \text{ is polyconvex} & \Leftrightarrow & |c| \leq 2 \\ \varphi \text{ is quasiconvex} & \Leftrightarrow & |c| \leq 2 + \varepsilon, \ \varepsilon > 0 \\ \varphi \text{ is rank-one convex} & \Leftrightarrow & |c| \leq \frac{4}{\sqrt{3}}. \end{array}$$

Unfortunately, it is not known if $2 + \varepsilon = \frac{4}{\sqrt{3}}$.

The equivalence between rank-one convexity and quasiconvexity is the one that has stood unsolved longer. Morrey himself ([29]) stated that "it is an unsolved problem to prove or disprove the theorem that every rank-one convex function of ∇u is quasiconvex." In his seminal paper [28], he conjectured (informally) that "... after a great deal of experimentation, the writer is inclined to think that there is no condition of the type discussed, which involves φ and only a finite number of its derivatives, and which is both necessary and sufficient for quasiconvexity in the general case." So, usually, Morrey's conjecture is stated by saying that rank-one convexity does not imply quasiconvexity.

In the special class of quadratic forms, it was known long ago ([45],[46], although implicitly known earlier), and not difficult to see through Fourier analysis, that these two kinds of convexity are equivalent. In the general case, evidence in favor of the equivalence and against it started to pile up (see [3] for a very nice account of the situation until 1986) until the conclusive counterexample of Šverák ([42]). As far as we can tell, there is no essentially new counterexample, and this one is only valid for $m \geq 3$ so that rank-one convexity does not imply quasiconvexity in this situation. Further attempts to extend the counterexample for $m = 2$ have failed ([39],[4]). As rank-one convexity and polyconvexity are invariant under transposition (that is, if $\varphi(\xi)$ is rank-one convex (resp. polyconvex) then $\widetilde{\varphi}(\xi) = \varphi(\xi^T)$ is rank-one convex (resp. polyconvex), where ξ^T denotes the transpose of ξ), one might think of adapting the counterexample of Šverák to the case were $m \geq 2$, $N \geq 3$. However, this is not possible, as quasiconvexity revealed to be not invariant under transposition ([24],[31]).

Some additional evidence against the equivalence can be found in [34], while evidence in favor is contained in [8] and [30]. The problem remains open for $m = 2$.

The question whether rank-one convexity implies quasiconvexity can be restated in terms of laminates and homogeneous gradient Young measures: is every homogeneous gradient Young measure a laminate? This question seems to be, unfortunately, as hard as the previous one ([36],[4]). Laminates can be understood, at least conceptually, in a constructive way ([35]). The basic idea comes from the (H_k) conditions ([11]): a set of pairs $\{(\lambda_i, A_i)\}_{1\leq i\leq k}$ where $\lambda_i > 0$, $\sum_i \lambda_i = 1$, $A_i \in \mathbb{M}^{m\times N}$ satisfies the (H_k) condition if

1. when $k = 2$, then $\operatorname{rank}\{A_1 - A_2\} \leq 1$;
2. when $k \geq 2$, then, up to a permutation, $\operatorname{rank}\{A_1 - A_2\} \leq 1$ and if, for every $2 \leq i \leq k-1$, we define

$$\begin{cases} \theta_1 = \lambda_1 + \lambda_2 & B_1 = \frac{\lambda_1 A_1 + \lambda_2 A_2}{\lambda_1 + \lambda_2} \\ \theta_i = \lambda_{i+1} & B_i = A_{i+1} \end{cases}$$

then $(\theta_i, B_i)_{1\leq i\leq k-1}$ satisfy (H_{k-1}).

Then a laminate is the weak-* limit in the sense of measures of sequences of finite order laminates, that is, convex combinations of Dirac masses supported in sets of points verifying (H_k) conditions

$$\sum_{i=1}^{k} \lambda_i \delta_{A_i} \overset{*}{\rightharpoonup} \mu.$$

Laminates can be characterized as the probability measures μ (with support on a compact set $K \in \mathbb{M}^{m\times N}$) for which Jensen's inequality

$$\varphi\left(\int_K A\,d\mu(A)\right) \leq \int_K \varphi(A)\,d\mu(A)$$

holds for every rank-one convex function φ (see [35]), while homogeneous gradient Young measures are the probability measures characterized by Jensen's inequality for quasiconvex functions ([22]). The homogeneous gradient Young measures can be defined as the probability measures μ for which there is a sequence $(u_j) \subset W^{1,\infty}(\Omega, \mathbb{R}^m)$ such that

$$u_j \stackrel{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(\Omega, \mathbb{R}^m),$$

(∇u_j) generates the Young measure μ in the sense that

$$\varphi(\nabla u_j) \stackrel{*}{\rightharpoonup} \int_K \varphi(A)\,d\mu(A) \text{ in } L^\infty(\Omega),$$

whenever φ is continuous. For simplicity, we will omit the term "homogeneous". The Riemann-Lebesgue lemma is one interesting (nontrivial) example where we can determine explicitly the underlying gradient Young measure, and several versions can be found in [37]. We include one here for convenience of the reader.

Lemma 1 *Let $\Omega = (0,1)^N$ and $u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$, $u - u_F \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)$, where u_F is the affine Lipschitz function $u_F(x) = Fx$ for $x \in \Omega$. There exists a sequence (u_j) bounded in $W^{1,\infty}(\Omega, \mathbb{R}^m)$, $u_j - u_F \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)$, such that the Young measure associated with (∇u_j) is homogeneous and defined by*

$$\langle \mu, \varphi \rangle = \int_\Omega \varphi(\nabla u(x))\,dx$$

for any continuous φ.

A polyconvex measure is a probability measure for which Jensen's inequality holds for every polyconvex function ([35]). It turns out that polyconvex measures can also be characterized as the probability measures that commute with the minors of the matrices

$$M\left(\int_K A\,d\mu(A)\right) = \int_K M(A)\,d\mu(A).$$

All these 3 classes of probability measures (laminates, gradient Young measures, polyconvex measures) with fixed first moment

$$\xi = \int_K A \, d\mu(A)$$

form convex sets. We also have that, for a fixed first moment, the class of laminates is a subset of the class of gradient Young measures, which in turn is a subset of the class of polyconvex measures.

The quasiconvexity condition (1.2) can also be formulated as

$$\varphi(\xi) \leq \int_{(0,1)^N} \varphi(\xi + w(x)) \, dx$$

for all $\xi \in \mathbb{R}^d$ and all $w \in C^\infty_{per}((0,1)^N, \mathbb{R}^d)$[1] such that $\operatorname{curl} w = 0$ and $\int_{(0,1)^N} w(x) \, dx = 0$, $d = m \times N$ ([17]). In the setting of continuum mechanics and electromagnetism more general linear partial differential equations than $\operatorname{curl} w = 0$ appear, which are physically relevant ([43]). It was then introduced ([10]) the concept of $\mathcal{A}$-quasiconvexity (see also [18]): consider a collection of linear operators $A^{(i)} \in \operatorname{Lin}(\mathbb{R}^d, \mathbb{R}^l)$, $i = 1, ..., N$ and define

$$\mathcal{A}v := \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i}, \quad v : \mathbb{R}^N \to \mathbb{R}^d,$$

$$\mathbf{A}(w) := \sum_{i=1}^N A^{(i)} w_i \in \operatorname{Lin}(\mathbb{R}^d, \mathbb{R}^l), \quad w \in \mathbb{R}^N,$$

$\operatorname{Lin}(X, Y)$ is the vector space of linear mappings from the vector space X into the vector space Y and where we assume that $\mathcal{A}$ satisfies the constant rank property ([33]): there exists $p \in \mathbb{N}$ such that

$$\operatorname{rank} \mathbf{A}(w) = p,$$

for all $w \in S^{N-1}$, the unit sphere of $\mathbb{R}^N$. Then φ is $\mathcal{A}$-quasiconvex if

$$\varphi(\xi) \leq \int_{(0,1)^N} \varphi(\xi + w(x)) \, dx \tag{1.3}$$

for all $\xi \in \mathbb{R}^d$ and all $w \in C^\infty_{per}((0,1)^N, \mathbb{R}^d)$ such that $\mathcal{A}(w) = 0$ and $\int_{(0,1)^N} w(x) \, dx = 0$. This is the necessary and sufficient condition for (sequential) weak lower semicontinuity of

$$I(v) = \int_{(0,1)^N} \varphi(v(x)) \, dx,$$

[1] that is, w belongs to $C^\infty(\mathbb{R}^N, \mathbb{R}^d)$ and is $(0,1)^N$-periodic.

along sequences that satisfy $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty((0,1)^N, \mathbb{R}^d)$ and $\mathcal{A}v_n = 0$, where $\varphi : \mathbb{R}^d \to [0,+\infty)$ is again assumed continuous. The interesting necessary condition now is that φ must be convex along the characteristic cone ([33],[43])

$$\Lambda := \bigcup_{w \in S^{N-1}} \ker \mathbf{A}(w).$$

Some important examples included in this general framework, besides the case $\mathcal{A}v = \operatorname{curl} v = 0$, are

(a) Divergence free fields:

$$\mathcal{A}v = \operatorname{div} v = 0,$$

where $v : (0,1)^N \to \mathbb{R}^N$, ([37])

(b) Maxwell's equations:

$$\mathcal{A}\begin{pmatrix} m \\ h \end{pmatrix} = \begin{pmatrix} \operatorname{div}(m+h) \\ \operatorname{curl} h \end{pmatrix} = 0,$$

where $m : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetization and $h : \mathbb{R}^3 \to \mathbb{R}^3$ is the induced magnetic field ([43]).

(c) Higher order gradients: replacing the target space $\mathbb{R}^d$ by an appropriate finite dimensional vector space E_s^m of m-tuples of symmetric linear maps on $\mathbb{R}^N$, it is possible to find a first order linear partial differential operator $\mathcal{A}$ such that $v \in C^\infty_{per}((0,1)^N, E_s^m)$, $\mathcal{A}v = 0$ and $\int_{(0,1)^N} v(x)\, dx = 0$ if and only if there exists $\psi \in C^\infty_{per}((0,1)^N, \mathbb{R}^m)$ such that $v = \nabla^s\psi$, where $\nabla^s\psi = \nabla^{s-1}(\nabla\psi)$ with $\nabla^1\psi = \nabla\psi$ ([18]). In this case (1.3) reduces to the s-quasiconvexity notion introduced by Meyers in [26].

We will be especially interested in the particular case of second-order gradients: a function $\varphi : \mathbb{M}^{N\times N}_{sym} \to \mathbb{R}$ is said to be 2-quasiconvex if

$$\varphi(\xi) \leq \int_D \varphi(\xi + \nabla^2 v(x))\, dx,$$

for any $\xi \in \mathbb{M}^{N\times N}_{sym}$ and every $v \in C^\infty_c(D, \mathbb{R})$, where $\mathbb{M}^{N\times N}_{sym}$ denote the space of $\mathbb{M}^{N\times N}$-symmetric matrices and $|D| = 1$ (the choice of the domain is irrelevant, [26]). In [15] (generalizing a result of [32]) it was proved that 2-quasiconvexity reduces to quasiconvexity for symmetric matrices or, to be more precise, that each 2-quasiconvex function is the restriction of a quasiconvex function to the space of symmetric matrices. Nevertheless, we will see why we are interested in it.

Chapter 2

Finding new families of rank-one convex polynomials

In this chapter we introduce a method to find, in a systematic way, rank-one convex polynomials. We show how it works in several examples, including both new and classical ones. It can also be applied to convexity along general cones.

2.1 Introduction

Two important convexity conditions related with quasiconvexity are polyconvexity and rank-one convexity. Although quasiconvexity is harder to verify, notice that these two other types of convexity, though more manageable, are also not easy to check on explicit examples ([19]). In particular, rank-one convexity is an appealing property as it is like the usual convexity. But deciding when a given function is or is not rank-one convex is not an easy task. Our aim is to provide a way to determine (at least in some specific situations) the rank-one convexity of functions of a particular structure.

Our method can be applied to the following situation. Let

$$\varphi_i : \mathbb{M}^{m\times N} \to \mathbb{R}, \quad i = 1, 2,$$

be two polynomials such that

- The combination
$$\varphi(\xi) = \varphi_1(\xi) - c\,\varphi_2(\xi), \tag{2.1}$$
for any constant $c \in \mathbb{R}$, is coercive with superlinear growth;

- φ_1 is strictly convex.

The basic important problem we would like to address is

Problem 1 *Determine the range of the constant c so that $\varphi(\xi)$ is rank-one convex.*

In the case $c = 1$, the rank-one convexity of φ (supposed smooth) is then equivalent to

$$A^T \nabla^2 \varphi_1(\xi) A - A^T \nabla^2 \varphi_2(\xi) A \geq 0,$$

for every $A \in \Lambda$, $\xi \in \mathbb{M}^{m \times N}$, where Λ is the rank-one cone

$$\Lambda = \{a \otimes n, \ a \in \mathbb{R}^m, \ n \in \mathbb{R}^N, \ |n| = 1\},$$

or to

Proposition 1 *Let φ be as before. Then φ is rank-one convex if and only if*

$$\frac{A^T \nabla^2 \varphi_2(\xi) A}{A^T \nabla^2 \varphi_1(\xi) A} \leq 1, \ \ A \in \Lambda, \ \xi \in \mathbb{M}^{m \times N}.$$

For a general parameter c, it is then possible to determine the range of this constants for which the corresponding family of functions are rank-one convex. In fact, by Proposition 1, we have that (2.1) is rank-one convex if and only if

$$\frac{c\, A^T \nabla^2 \varphi_2(\xi) A}{A^T \nabla^2 \varphi_1(\xi) A} \leq 1, \ \ A \in \Lambda, \ \xi \in \mathbb{M}^{m \times N}.$$

If

$$\frac{1}{c_-} \left(resp\, \frac{1}{c_+} \right) = \inf_{A \in \Lambda, \, \xi \in \mathbb{M}^{m \times N}} (resp \ \sup) \, \frac{A^T \nabla^2 \varphi_2(\xi) A}{A^T \nabla^2 \varphi_1(\xi) A},$$

then it is easy to derive

Theorem 1 *Let*

$$\varphi = \varphi_1 - c\, \varphi_2,$$

where φ_i are smooth and φ_1 is strictly convex Then φ is rank-one convex if and only if

1. *$c \in [c_-, c_+]$, in case φ_2 is neither rank-one convex nor rank-one concave (alternatively, we can write: $A^T \nabla^2 \varphi_2(\xi) A$ attains both positive and negative values);*

2. $c \in (-\infty, c_+]$, *in case*

$$\inf_{A\in\Lambda,\,\xi\in\mathbb{M}^{m\times N}} \frac{A^T\nabla^2\varphi_2(\xi)A}{A^T\nabla^2\varphi_1(\xi)A} = 0;$$

3. $c \in [c_-, +\infty)$, *in case*

$$\sup_{A\in\Lambda,\,\xi\in\mathbb{M}^{m\times N}} \frac{A^T\nabla^2\varphi_2(\xi)A}{A^T\nabla^2\varphi_1(\xi)A} = 0.$$

Remark 1 *We will make the assumption that if $\frac{1}{c_-} = -\infty$ (resp $\frac{1}{c_+} = +\infty$) then $c_- = 0$ (resp $c_+ = 0$).*

Though the proof of this result is straightforward in these terms, it is quite remarkable that these optimal constants can be computed explicitly in specific examples, as we show in Section 2.3.

Before that, we also provide an appropriate description of this theorem in terms of laminates. This seems interesting as this strategy looks more promising for other situations like polyconvexity and, even, quasiconvexity. The proof of this theorem from this viewpoint can be found in Section 2.4.

2.2 Alternative route: laminates

We know that laminates are the class of probability measures which play a fundamental role with respect to rank-one convexity through duality with Jensen's inequality ([35]). In this section it is presented the result of the previous one, from the viewpoint of laminates. We think that this gives further insight into the problem, especially because it is more easily visualized. To state the main result in terms of laminates requires some notation.

Let $\mathcal{L}(\xi_0)$ denote the set of laminates with first moment ξ_0. Consider the linear mapping

$$T : \mathcal{L}(\xi_0) \mapsto \mathbb{R}^2, \quad T(\mu) = \left(\int \varphi_1(\xi)\,d\mu(\xi), \int \varphi_2(\xi)\,d\mu(\xi)\right).$$

It is clear that $T(\mathcal{L}(\xi_0))$ is a convex set in $\mathbb{R}^2$. If (x, y) designate usual coordinates in $\mathbb{R}^2$, and we put

$$x_0 = \varphi_1(\xi_0), \quad y_0 = \varphi_2(\xi_0),$$

we know, due to convexity of φ_1, that

$$T(\mathcal{L}(\xi_0)) \subset \{(x,y) \in \mathbb{R}^2 : x \geq x_0\}.$$

Even more, because of strict convexity of φ_1, the intersection of $T(\mathcal{L}(\xi_0))$ with the vertical line $x = x_0$ is the unique point (x_0, y_0). Then solving Problem 1 is equivalent to determining the best constants c_-, c_+ so that

$$T(\mathcal{L}(\xi_0)) \subset C\left((x_0, y_0), \frac{1}{c_-}, \frac{1}{c_+}\right),$$

for every $\xi_0 \in \mathbb{M}^{m\times N}$, where $C((\overline{x},\overline{y}), c_1, c_2)$ is the cone in $\mathbb{R}^2$ defined by

$$C((\overline{x},\overline{y}), c_1, c_2) = \{(x,y) \in \mathbb{R}^2 : c_1(x-\overline{x}) + \overline{y} \leq y \leq c_2(x-\overline{x}) + \overline{y}, x \geq \overline{x}\},$$

$c_1 < 0 < c_2$. For $s \in [0,1]$, we consider our basic first-order laminates

$$\mu_s = \frac{1}{2}\delta_{\xi_0+sA} + \frac{1}{2}\delta_{\xi_0-sA},$$

for A of rank one. Finally, consider the plane curve

$$\begin{aligned}\sigma^{(A,\xi_0)}(s) &= T(\mu_s)\\ &= \left(\frac{1}{2}\varphi_1(\xi_0+sA) + \frac{1}{2}\varphi_1(\xi_0-sA), \frac{1}{2}\varphi_2(\xi_0+sA) + \frac{1}{2}\varphi_2(\xi_0-sA)\right).\end{aligned}$$

Λ stands for the cone of rank-one matrices.

Theorem 2 *Let φ be as in Theorem 1 and*

$$\frac{1}{c_-}\left(resp\ \frac{1}{c_+}\right) = \inf_{A\in\Lambda,\xi_0\in\mathbb{M}^{m\times N}}(resp\ \sup)\frac{\ddot{\sigma}_2^{(A,\xi_0)}(0)}{\ddot{\sigma}_1^{(A,\xi_0)}(0)}.$$

Then φ is rank-one convex if and only if

1. *$c \in [c_-, c_+]$, if $\ddot{\sigma}_2$ attains both positive and negative values;*
2. *$c \in (-\infty, c_+]$, if*
$$\inf_{A\in\Lambda,\xi_0\in\mathbb{M}^{m\times N}}\frac{\ddot{\sigma}_2^{(A,\xi_0)}(0)}{\ddot{\sigma}_1^{(A,\xi_0)}(0)} = 0;$$
3. *$c \in [c_-, +\infty)$, if*
$$\sup_{A\in\Lambda,\xi_0\in\mathbb{M}^{m\times N}}\frac{\ddot{\sigma}_2^{(A,\xi_0)}(0)}{\ddot{\sigma}_1^{(A,\xi_0)}(0)} = 0.$$

Remark 2 *Obviously, we have that*

$$\ddot{\sigma}_i^{A,\xi_0}(0) = A^T \nabla^2 \varphi_i(\xi_0) A,$$

where

$$rank(A) \leq 1.$$

2.3 Examples

We now want to solve the problem

$$\inf_{A,\xi_0 \in \mathbb{M}^{m\times N}} (resp\ \sup) \frac{A^T \nabla^2 \varphi_2(\xi_0) A}{A^T \nabla^2 \varphi_1(\xi_0) A}$$

subject to the restriction

$$\text{rank}(A) \leq 1.$$

To fix ideas, consider the minimization problem as a partial double minimization problem. If we minimize first in $A \in \mathbb{M}^{m\times N}$, the above quotient is always a quotient of two expressions which are homogeneous of degree two in A, where

$$A^T \nabla^2 \varphi_1(\xi_0) A > 0.$$

So, we can consider the equivalent problem

$$\min_{A \in \mathbb{M}^{m\times N}} A^T \nabla^2 \varphi_2(\xi_0) A$$

subject to the restrictions

$$\begin{cases} A^T \nabla^2 \varphi_1(\xi_0) A = 1 \\ \\ A, \text{ rank-one.} \end{cases}$$

In the particular case of 2×2 matrices, we can replace the rank-one condition on A by the more quantitative condition $A^T DA = \det A$. Anyhow, this minimum is attained since the function to minimize is continuous, and the domain is the intersection between a compact set and a closed set.

Let us stick to the 2×2 situation for the sake of this short discussion. If α, β are Lagrange multipliers, we put

$$L(A, \alpha, \beta) = A^T \nabla^2 \varphi_2(\xi_0) A - \alpha(A^T \nabla^2 \varphi_1(\xi_0) A - 1) - \beta A^T DA.$$

From first-order optimality conditions, if A is a critical point of the objective function, one obtains

$$A^T \nabla^2 \varphi_2(\xi_0) A = \alpha,$$

where α can be recovered from solving the following system

$$\begin{cases} (\nabla^2 \varphi_2(\xi_0) - \alpha \nabla^2 \varphi_1(\xi_0) - \beta D) A = 0 \\ A^T \nabla^2 \varphi_1(\xi_0) A = 1 \\ A^T D A = 0. \end{cases}$$

α will be a function of ξ_0, and to finish, we would have to compute the infimum with respect to the variable $\xi_0 \in \mathbb{M}^{2\times 2}$. In the case where the φ_i's are polynomials, the above system of equations is indeed a parametric system of polynomial equations, where ξ_0 is the parameter, and A, α, β are the variables to solve for. There exist several algorithms which deal with the problem of describing the solutions of these systems in terms of the parameters, such as comprehensive Gröbner bases ([48]), triangular sets decomposition ([47]) and rational parametrizations ([40]). There also exist more recent developments ([25], [49]). The description of the generic solutions of these systems is in general difficult and is beyond the scope of this work. Here we will deal with a simple example, whose system can be solved with several recent symbolic mathematical softwares.

For a more general situation, we can replace the matrix A by $a \otimes n$ even under the constraints $|a| = |n| = 1$. In this case, we would have to solve the problem

$$\inf_{\xi_0} \min_{a,n} n \otimes a \nabla^2 \varphi_2(\xi_0) a \otimes n$$

subject to the constraint

$$n \otimes a \nabla^2 \varphi_2(\xi_0) a \otimes n = 1.$$

We can then use optimality conditions to make some progress in the calculations. However, one has to keep track of the dependence on a and ξ_0 when solving the minimization problem for n. In general, it is not so easy to compute the range for the constant c through this approach.

In the case of 4th degree homogeneous polynomials, we can easily overcome these difficulties. For this special situation, we can take advantage of the fact that $A^T \nabla^2 \varphi_i(\xi_0) A$ is also quadratic in ξ_0. More explicitly, and keeping in mind its special structure, we can write

$$A^T \nabla^2 \varphi_i(\xi_0) A = \xi_0^T M_i(A) \xi_0,$$

where $M_i(A)$, for $i = 1, 2$, is a matrix whose entries only depend on $A \in \Lambda$. This is a huge advantage, as in this case we can perform first the minimization in ξ_0, and then in A, avoiding in this way to include the additional rank-one restriction, but still dealing with quadratic problems. We want hence to compute

$$\min_{A\in\Lambda}\left(\min_{\xi_0\in\mathbb{M}^{m\times N}}\frac{\xi_0^T M_2(A)\xi_0}{\xi_0^T M_1(A)\xi_0}\right).$$

To evaluate the first minimum, we can now fix

$$\xi_0^T M_1(A)\xi_0 = 1,$$

and calculate

$$\min_{\xi_0}\xi_0^T M_2(A)\xi_0$$

subject to this restriction. Notice that this minimum is attained, as the smallest eigenvalue of $\nabla^2\varphi_1(A)$ is strictly positive. If α is a Lagrange multiplier, we put

$$L(\xi,\alpha) = \xi_0^T M_2(A)\xi_0 - \alpha(\xi_0^T M_1(A)\xi_0 - 1),$$

and from first-order optimality conditions, if ξ_0 is a critical point, one obtains

$$\xi_0^T M_2(A)\xi_0 = \alpha,$$

where α are the solutions of

$$\det(M_2(A) - \alpha M_1(A)) = 0.$$

Notice that in this case this condition is a necessary and sufficient condition for the existence of minimizers.

α will be a function of A, and to finish we have to compute the minimum with respect to this variable $A \in \mathbb{M}^{m\times N}$ with $\operatorname{rank}(A) \leq 1$.

2.3.1 Classical examples

We deal first with some classical examples ([1], [12], [14]).

Example 1

$$\varphi : \mathbb{M}^{2\times 2} \to \mathbb{R},$$

given by

$$\varphi(\xi) = |\xi|^4 - c|\xi|^2\det\xi.$$

If $A \in \mathbb{M}^{2\times 2}$ *is such that* $|A| = 1$, *by putting*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get here that

$$M_2(A) = \begin{pmatrix} 2ad & bd-ac & cd-ab & \frac{1}{2}+a^2+d^2 \\ bd-ac & -2bc & -\frac{1}{2}-b^2-c^2 & ab-cd \\ cd-ab & -\frac{1}{2}-b^2-c^2 & -2bc & ac-bd \\ \frac{1}{2}+a^2+d^2 & ab-cd & ac-bd & 2ad \end{pmatrix}$$

and

$$M_1(A) = \begin{pmatrix} 2+4a^2 & 4ab & 4ac & 4ad \\ 4ab & 2+4b^2 & 4bc & 4bd \\ 4ac & 4bc & 2+4c^2 & 4cd \\ 4ad & 4bd & 4cd & 2+4d^2 \end{pmatrix}.$$

To obtain the values of α *we have to solve the equation*

$$\det(M_2(A) - \alpha M_1(A)) = 0.$$

But if we now perform the substitution

$$A = (\cos\theta_1, \sin\theta_1) \otimes (\cos\theta_2, \sin\theta_2),$$

with $\theta_1, \theta_2 \in [0, 2\pi]$, *the above equation becomes*

$$\frac{9}{16} - 12\alpha^2 + 48\alpha^4 = 0,$$

and the maximum and the minimum values are, respectively, $\alpha = \frac{\sqrt{3}}{4}$ *and* $\alpha = -\frac{\sqrt{3}}{4}$. *So,* φ *is rank-one convex if and only if*

$$c \in \left[-\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right].$$

In the case of convexity, it is known ([1]) that φ *is convex if and only if*

$$c \in \left[-\frac{4\sqrt{2}}{3}, \frac{4\sqrt{2}}{3}\right].$$

Example 2

$$\varphi : \mathbb{M}^{2\times 2} \to \mathbb{R},$$

given by

$$\varphi(\xi) = |\xi|^4 - c(\det \xi)^2.$$

If we proceed as in the previous example, and put

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for $A \in \mathbb{M}^{2\times 2}$ *with* $|A| = 1$, $M_1(A)$ *will be the same as before, and*

$$M_2(A) = \begin{pmatrix} 2d^2 & -2cd & -2bd & 2ad \\ -2cd & 2c^2 & 2bc & -2ac \\ -2bd & 2bc & 2b^2 & -2ab \\ 2ad & -2ac & -2ab & 2a^2 \end{pmatrix}.$$

For

$$A = (\cos\theta_1, \sin\theta_1) \otimes (\cos\theta_2, \sin\theta_2),$$

with $\theta_1, \theta_2 \in [0, 2\pi]$, *we have*

$$\det(M_2(A) - \alpha M_1(A)) = 384\alpha^3(-1 + 2\alpha) = 0,$$

and so, the maximum value of α *is* $\frac{1}{2}$ *and the minimum is* 0. *Regarding the minimum value of* α, *it was expected, as* φ_2 *is polyconvex.*

In this case, it is clear that φ *is rank-one convex if and only if*

$$c \in (-\infty, 2].$$

The range for the constant c for which the corresponding φ *is convex is given by*

$$c \in [-4, 2].$$

2.3.2 New examples

We now present some other examples to stress our main result.

Example 3 *For*

$$\varphi : \mathbb{M}^{2\times 2} \to \mathbb{R},$$

put

$$\varphi(\xi) = |\xi|^4 - c\ (\ tr\ \xi\ )^4.$$

where $tr\ \xi$ *represents the trace of the matrix* ξ. *For*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $A \in \mathbb{M}^{2\times 2}$, $|A| = 1$, $M_1(A)$ *is given above, and*

$$M_2(A) = \begin{pmatrix} 12(a+d)^2 & 0 & 0 & 12(a+d)^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12(a+d)^2 & 0 & 0 & 12(a+d)^2 \end{pmatrix}.$$

In the rank-one directions

$$A = (\cos\theta_1, \sin\theta_1) \otimes (\cos\theta_2, \sin\theta_2),$$

where $\theta_1, \theta_2 \in [0, 2\pi]$, *we have*

$$\det(M_2(A) - \alpha M_1(A)) = 0 \Leftrightarrow$$

$$\begin{aligned}\Leftrightarrow 768\alpha^3(-4 + 2\cos(\theta_2)^2 - 16\cos(\theta_1)^2\cos(\theta_2)^4 + 2\cos(\theta_2)^4 + 2\cos(\theta_1)^2 + \\ +2\cos(\theta_1)^4 + 8\cos(\theta_2)^2\cos(\theta 1)^2 - 4\cos(\theta_1)\cos(\theta_2)\sin(\theta_1)\sin(\theta_2) + \\ +16\cos(\theta_1)^4\cos(\theta_2)^4 - 8\cos(\theta_1)^3\cos(\theta_2)\sin(\theta_1)\sin(\theta_2) + \\ +16\cos(\theta_1)^3\cos(\theta_2)^3\sin(\theta_1)\sin(\theta_2) - 8\cos(\theta_1)\cos(\theta_2)^3\sin(\theta_1)\sin(\theta_2) + \\ -16\cos(\theta_2)^2\cos(\theta_1)^4 + \alpha) = 0.\end{aligned}$$

Consequently the maximum value for α *is* 4. *Regarding the minimum value of* α, *notice that* φ_2 *is convex and so* φ *is rank-one convex if and only if*

$$c \in \left(-\infty, \frac{1}{4}\right].$$

φ *is convex if and only if*

$$c \in \left(-\infty, \frac{2}{9}\right].$$

Example 4 *An example with a non-homogeneous polynomial.*

$$\varphi : \mathbb{M}^{2\times 2} \to \mathbb{R},$$

defined by

$$\varphi(\xi) = (\ tr\ \xi\ )^4 + |\xi|^2 - c(\ tr\ \xi\ )^3.$$

For

$$\xi = \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

we have

$$\nabla^2\varphi_1(\xi) = \begin{pmatrix} 12(x+w)^2+2 & 0 & 0 & 12(x+w)^2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 12(x+w)^2 & 0 & 0 & 12(x+w)^2+2 \end{pmatrix}$$

and

$$\nabla^2\varphi_2(\xi) = \begin{pmatrix} 6(x+w) & 0 & 0 & 6(x+w) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6(x+w) & 0 & 0 & 6(x+w) \end{pmatrix}.$$

In addition, for

$$A = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

and

$$D = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix},$$

the first order necessary conditions will be the parametric system of polynomial equations

$$\begin{cases} (6x+6w-12\alpha(x+w)^2-\beta)a+(6x+6w-\alpha(12(x+w)^2+2))d=0 \\ -2\alpha b+\beta c=0 \\ \beta b-2\alpha c=0 \\ (6x+6w-\alpha(12(x+w)^2+2))a+(6x+6w-12\alpha(x+w)^2-\beta)d=0 \\ (a(12(x+w)^2+2)+12d(x+w)^2)a+2b^2+2c^2+(12a(x+w)^2+d(12(x+w)^2+2))d=1 \\ ad-bc=0 \end{cases}$$

which give us the real solutions

$$\alpha = 0, \ \alpha = \frac{3(x+w)}{6(x+w)^2+1},$$

which in turn provide the range of the constant c to be

$$c \in \left[-\frac{2\sqrt{6}}{3}, \frac{2\sqrt{6}}{3}\right].$$

For convexity, we have

$$c \in \left[-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right].$$

Example 5 *An example for* 2×3 *matrices.*

$$\varphi : \mathbb{M}^{2\times 3} \to \mathbb{R},$$

given by

$$\varphi(\xi) = |\xi|^4 - c|\xi|^2(\xi^1_{2\times 2} + \xi^2_{2\times 2} + \xi^3_{2\times 2}),$$

where $\xi^j_{2\times 2}$, $j = 1, 2, 3$ *represents the* 2×2 *minor that is obtained from* ξ, *by removing the* j *column. If* $A \in \mathbb{M}^{2\times 3}$ *with* $|A| = 1$ *we set*

$$A = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}.$$

We have

$$\frac{M_1(A)}{2} = \begin{pmatrix} 4a^2+2 & 4ba & 4ca & 4da & 4ea & 4fa \\ 4ba & 4b^2+2 & 4cb & 4db & 4eb & 4fb \\ 4ca & 4cb & 4c^2+2 & 4dc & 4ec & 4fc \\ 4da & 4db & 4dc & 4d^2+2 & 4ed & 4fd \\ 4ea & 4eb & 4ec & 4ed & 4e^2+2 & 4fe \\ 4fa & 4fb & 4fc & 4fd & 4fe & 4f^2+2 \end{pmatrix}$$

and

$$\frac{M_2(A)}{2} = \begin{pmatrix} 3ad+3af-bc-be+cf-de & -ca-ea+db+fb & -ba+af+dc+cf & a^2-ea+d^2+fd+\frac{1}{2} & -ba-ad+de+fe & a^2+ca+fd+f^2+\frac{1}{2} \\ -ca-ea+db+fb & ad+af-3bc-3be+cf-de & -\frac{1}{2}-b^2+fb-c^2-ec & ba-be-dc-de & -\frac{1}{2}-b^2-db-ec-e^2 & ba+bc-cf-fe \\ -ba+af+dc+cf & -\frac{1}{2}-b^2+fb-c^2-ec & ad+af-3bc-be+3cf-de & ca-db-ec+fd & -bc-be-dc+fe & \frac{1}{2}+ca-fb+c^2+f^2 \\ a^2-ea+d^2+fd+\frac{1}{2} & ba-be-dc-de & ca-db-ec+fd & 3ad+af-bc-be+cf-3de & -\frac{1}{2}+ea-db-d^2-e^2 & ad+af+dc-fe \\ -ba-ad+de+fe & -\frac{1}{2}-b^2-db-ec-e^2 & -bc-be-dc+fe & -\frac{1}{2}+ea-db-d^2-e^2 & ad+af-bc-3be+cf-3de & ea-fb+ec-fd \\ a^2+ca+fd+f^2+\frac{1}{2} & ba+bc-cf-fe & \frac{1}{2}+ca-fb+c^2+f^2 & ad+af+dc-fe & ea-fb+ec-fd & ad+3af-bc-be+3cf-de \end{pmatrix}.$$

For

$$A = (\cos\theta_1, \sin\theta_1) \otimes (\cos\theta_2 \sin\theta_3, \sin\theta_2 \sin\theta_3, \cos\theta_3),$$

$\theta_1, \theta_2 \in [0, 2\pi]$, $\theta_3 \in [0, \pi]$, *we have*

$$\begin{aligned} \alpha^2(21 &- 12\sin\theta_3\cos\theta_3\cos\theta_2 + 12\sin\theta_2\sin\theta_3\cos\theta_3 - 12\sin\theta_2\cos\theta_3^2\cos\theta_2 + \\ &+12\sin\theta_2\cos\theta_2 + 64\alpha^2\sin\theta_2\cos\theta_3^2\cos\theta_2 + 64\alpha^2\sin\theta_3\cos\theta_3\cos\theta_2 + \\ &-64\alpha^2\sin\theta_2\cos\theta_2 - 64\alpha^2\sin\theta_2\sin\theta_3\cos\theta_3 - 160\alpha^2 + 256\alpha^4) = 0. \end{aligned}$$

The roots α are

$$\alpha = \pm\sqrt{\frac{7\tan^2\theta_3 + 7 + 4\sin\theta_2\tan\theta_3 - 4\cos\theta_2\tan\theta_3 + 4\sin\theta_2\tan^2\theta_3\cos\theta_2}{16(\tan^2\theta_3+1)}},$$

$$\alpha = 0,\ \alpha = \pm\frac{\sqrt{3}}{4},$$

and consequently the maximum and minimum values for α are $\alpha = \frac{3}{4}$ and $\alpha = -\frac{3}{4}$ respectively (obtained from maximizing and minimizing, respectively, the above quotients in θ_2, θ_3) so, in this case we have φ rank-one convex if and only if

$$c \in \left[-\frac{4}{3}, \frac{4}{3}\right].$$

Remark 3 1. *In this case it is harder to compute the constants for convexity than for rank-one convexity, following this approach. In fact, we were not able to recover those constants.*

2. *As rank-one convexity is invariant under transposition, one can trivially compute the constants for the* 3×2 *example implicitly given by example 5.*

2.4 Main proof

This section is devoted to the proof of Theorem 2.

Proof. We will use the characterization of rank-one convexity through Jensen's inequality for laminates ([35]) so that we are interested in determining the exact range for the constant c so that Jensen's inequality holds for every laminate and φ in (2.1). The key point is that we can control the slope of the secants that pass through the image of the barycenter by the slope of its tangents through zero. In this terminology, secants are related, somehow, to quasiconvexity whereas tangents at the origin reflect rank-one convexity.

We divide the proof in several steps.

Step 1. If μ is a laminate, then by definition ([35]), there exists a sequence of sets of pairs $\{(\lambda_i^k, A_i^k)\}_{1\leq i\leq k}$, verifying the (H_k) condition ([12]) such that

$$\mu_k = \sum_i \lambda_i^k \delta_{A_i^k} \overset{*}{\rightharpoonup} \mu$$

in the sense of measures. So if

$$\varphi\left(\int \xi \, d\mu_k(\xi)\right) \leq \int \varphi(\xi)\, d\mu_k(\xi),$$

holds for all k and for some value of c, then by taking weak-* limits on both sides of the above inequality (φ is, in particular, continuous), we have

$$\varphi\left(\int \xi \, d\mu(\xi)\right) \leq \int \varphi(\xi)\, d\mu(\xi), \ \forall \mu \in \mathcal{L}$$

for the same value of c.

Step 2. We will now prove that it suffices to use first-order laminates to determine the range of c. We argue, in particular, that building finite-order laminates recursively from first-order laminates does not reduce the range of the constant c.

Our hypothesis is that c is such that

$$\varphi\left(\int \xi \, d\mu(\xi)\right) \leq \int \varphi(\xi) \, d\mu(\xi) \tag{2.2}$$

for every

$$\mu = \lambda\delta_{A_1} + (1-\lambda)\delta_{A_2}, \text{ with } \operatorname{rank}(A_1 - A_2) \leq 1;$$

and we want to prove that, for the same value of c, we have

$$\varphi\left(\int \xi \, d\mu_N(\xi)\right) \leq \int \varphi(\xi) \, d\mu_N(\xi), \tag{2.3}$$

for every finite-order laminate

$$\mu_N = \sum_{i=1}^{N} \lambda_i \delta_{A_i}.$$

We proceed by induction (keep in mind that the value c is fixed but arbitrary). For $N = 2$, (2.3) is just (2.2). Suppose now that (2.3) holds for every probability measure associated with (H_{N-1}) conditions. Then, if $\{(\lambda_i^N, A_i^N)\}_{1\leq i\leq N}$ satisfies the (H_N) condition, we can assume, without loss of generality, that $\operatorname{rank}(A_1 - A_2) \leq 1$ (we drop the superindex for simplicity), and by the induction hypothesis, we have

$$\begin{aligned}\int \varphi(\xi)\, d\mu_N(\xi) &= \sum_{i=1}^{N} \lambda_i \varphi(A_i) = (\lambda_1+\lambda_2)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\varphi(A_1) + \frac{\lambda_2}{\lambda_1+\lambda_2}\varphi(A_2)\right) + \\ &+ \sum_{i=3}^{N} \lambda_i\varphi(A_i) \geq (\lambda_1+\lambda_2)\varphi\left(\frac{\lambda_1}{\lambda_1+\lambda_2}A_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}A_2\right) + \sum_{i=3}^{N}\lambda_i\varphi(A_i) \geq \\ &\geq \varphi\left(\sum_{i=1}^{N}\lambda_i A_i\right) = \varphi\left(\int \xi\, d\mu_N(\xi)\right).\end{aligned}$$

In fact, notice that we can further simplify the situation (since φ is continuous), because (2.2) holds for a value c if and only if

$$\varphi\left(\int \xi \, d\mu(\xi)\right) \leq \int \varphi(\xi) \, d\mu(\xi) \tag{2.4}$$

for every

$$\mu = \frac{1}{2}\delta_{A_1} + \frac{1}{2}\delta_{A_2}, \text{ with } \operatorname{rank}(A_1 - A_2) \leq 1,$$

holds for the same value of c.

After a change of variables, we can write down this measure as

$$\mu = \frac{1}{2}\delta_{\xi_0+A} + \frac{1}{2}\delta_{\xi_0-A},$$

where $\mathrm{rank}(A) \leq 1$. For $s \in [0,1]$, we can take

$$\mu = \mu_s = \frac{1}{2}\delta_{\xi_0+sA} + \frac{1}{2}\delta_{\xi_0-sA}$$

with $\mathrm{rank}(A) = 1$ and $|A| \leq 1$ (for $|A| > 1$ just use the fact that $\xi \in \mathbb{M}^{m\times N}$ is arbitrary and that φ is continuous). By dealing with this class of measures (which will play the role of "generators"), we can determine the exact range for the constant c that we are interested in.

Step 3. For $s \in [0,1]$, consider

$$\mu_s = \frac{1}{2}\delta_{\xi_0+sA} + \frac{1}{2}\delta_{\xi_0-sA},$$

and the corresponding plane curve

$$\sigma^{(A,\xi_0)}(s) = T(\mu_s)$$

with end-points

$$(\varphi_1(\xi_0), \varphi_2(\xi_0))$$

and

$$\left(\frac{1}{2}\varphi_1(\xi_0 + A) + \frac{1}{2}\varphi_1(\xi_0 - A), \frac{1}{2}\varphi_2(\xi_0 + A) + \frac{1}{2}\varphi_2(\xi_0 - A)\right).$$

If σ and μ_s are defined as above, then finding all c's such that

$$\int \varphi(\xi)\, d\mu_s(\xi) \geq \varphi\left(\int \xi\, d\mu_s(\xi)\right),$$

is equivalent to finding all c's for which we have

$$\sigma_1^{(A,\xi_0)}(s) - \sigma_1^{(A,\xi_0)}(0) \geq c(\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0)),$$

for every $\xi_0 \in \mathbb{M}^{m\times N}$, $A \in \Lambda$ with $|A| \leq 1$, $s \in [0,1]$; or, if we consider $c > 0$ (the other case is similar), that

$$\frac{1}{c} \geq \frac{\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0)}{\sigma_1^{(A,\xi_0)}(s) - \sigma_1^{(A,\xi_0)}(0)}$$

for every $\xi_0 \in \mathbb{M}^{m\times N}$, $A \in \Lambda$ with $|A| \leq 1$, $s \in (0,1]$. If

$$\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0) \leq 0,$$

then $c > 0$, and we do not have any additional constraint. Otherwise, we can set

$$\sup_{A,\xi_0,s\in(0,1]} \frac{\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0)}{\sigma_1^{(A,\xi_0)}(s) - \sigma_1^{(A,\xi_0)}(0)} = \frac{1}{c_+} \leq \frac{1}{c}.$$

Since

$$\sigma_i(s) = \frac{1}{2}\varphi_i(\xi_0 + sA) + \frac{1}{2}\varphi_i(\xi_0 - sA),$$

it follows

$$\dot{\sigma_i}(0) = 0,$$

thus it is obvious that

$$\sup_{A,\xi_0,s\in(0,1]} \frac{\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0)}{\sigma_1^{(A,\xi_0)}(s) - \sigma_1^{(A,\xi_0)}(0)} \geq \sup_{A,\xi_0} \frac{\ddot{\sigma_2}^{(A,\xi_0)}(0)}{\ddot{\sigma_1}^{(A,\xi_0)}(0)}. \tag{2.5}$$

Step 4. To finish the proof, we have to show that the equality holds. First we will suppose that the supremum on the left side of (2.5) (and where we can suppose $s \geq r > 0$, otherwise there is nothing to prove) is indeed a maximum and that a strict inequality holds

$$\frac{1}{c_+} = \max_{A,\xi_0,s\in(0,1]} \frac{\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0)}{\sigma_1^{(A,\xi_0)}(s) - \sigma_1^{(A,\xi_0)}(0)} = \frac{\sigma_2^{(A^*,\xi_0^*)}(s^*) - \sigma_2^{(A^*,\xi_0^*)}(0)}{\sigma_1^{(A^*,\xi_0^*)}(s^*) - \sigma_1^{(A^*,\xi_0^*)}(0)} >$$

$$> \sup_{A,\xi_0} \frac{\ddot{\sigma_2}^{(A,\xi_0)}(0)}{\ddot{\sigma_1}^{(A,\xi_0)}(0)}.$$

Then there has to be a point $t \in (0, s^*)$ such that

$$\frac{\sigma_2^{(A^*,\xi_0^*)}(s^*) - \sigma_2^{(A^*,\xi_0^*)}(t)}{\sigma_1^{(A^*,\xi_0^*)}(s^*) - \sigma_1^{(A^*,\xi_0^*)}(t)} =$$

$$= \frac{\frac{1}{2}(\varphi_2(\xi_0^* - s^*A^*) + \varphi_2(\xi_0^* + s^*A^*)) - \frac{1}{2}(\varphi_2(\xi_0^* - tA^*) + \varphi_2(\xi_0^* + tA^*))}{\frac{1}{2}(\varphi_1(\xi_0^* - s^*A^*) + \varphi_1(\xi_0^* + s^*A^*)) - \frac{1}{2}(\varphi_1(\xi_0^* - tA^*) + \varphi_1(\xi_0^* + tA^*))} > \frac{1}{c_+}.$$

But because $\xi_0^* - tA^*$ and $\xi_0^* + tA^*$ can be regarded as new barycenters of first-order laminates, it is clear, by definition of $\frac{1}{c_+}$, that

$$\frac{\frac{s^*+t}{2s^*}\varphi_2((\xi_0^* - tA^*) - (s^* - t)A^*) + \frac{s^*-t}{2s^*}\varphi_2((\xi_0^* - tA^*) + (s^* + t)A^*) - \varphi_2(\xi_0^* - tA^*)}{\frac{s^*+t}{2s^*}\varphi_1((\xi_0^* - tA^*) - (s^* - t)A^*) + \frac{s^*-t}{2s^*}\varphi_1((\xi_0^* - tA^*) + (s^* + t)A^*) - \varphi_1(\xi_0^* - tA^*)} \leq \frac{1}{c_+}$$

and

$$\frac{\frac{s^*-t}{2s^*}\varphi_2((\xi_0^*+tA^*)-(s^*+t)A^*)+\frac{s^*+t}{2s^*}\varphi_2((\xi_0^*+tA^*)+(s^*-t)A^*)-\varphi_2(\xi_0^*+tA^*)}{\frac{s^*-t}{2s^*}\varphi_1((\xi_0^*+tA^*)-(s^*+t)A^*)+\frac{s^*+t}{2s^*}\varphi_1((\xi_0^*+tA^*)+(s^*-t)A^*)-\varphi_1(\xi_0^*+tA^*)} \leq \frac{1}{c_+}.$$

From here and because φ_1 is strictly convex (and so, in the above fractions both denominators are strictly positive), it is trivial to obtain

$$\frac{\sigma_2^{(A^*,\xi_0^*)}(s^*)-\sigma_2^{(A^*,\xi_0^*)}(t)}{\sigma_1^{(A^*,\xi_0^*)}(s^*)-\sigma_1^{(A^*,\xi_0^*)}(t)} \leq \frac{1}{c_+},$$

which contradicts the above strict inequality, leading to the desired conclusion, that is

$$\max_{A,\xi_0,s\in(0,1]}\frac{\sigma_2^{(A,\xi_0)}(s)-\sigma_2^{(A,\xi_0)}(0)}{\sigma_1^{(A,\xi_0)}(s)-\sigma_1^{(A,\xi_0)}(0)} = \sup_{A,\xi_0}\frac{\ddot{\sigma_2}^{(A,\xi_0)}(0)}{\ddot{\sigma_1}^{(A,\xi_0)}(0)} = \frac{1}{c_+}.$$

Now it remains to prove the case where we have a genuine supremum on the left side of (2.5). This can only happen if the supremum is obtained by taking $|\xi| \to \infty$. Suppose

$$\frac{1}{c_+} = \sup_{A,\xi,s\in(0,1]}\frac{\sigma_2^{(A,\xi)}(s)-\sigma_2^{(A,\xi)}(0)}{\sigma_1^{(A,\xi)}(s)-\sigma_1^{(A,\xi)}(0)} = \lim_{|\xi|\to\infty}\max_{A,s\in(0,1]}\frac{\sigma_2^{(A,\xi)}(s)-\sigma_2^{(A,\xi)}(0)}{\sigma_1^{(A,\xi)}(s)-\sigma_1^{(A,\xi)}(0)} >$$
$$> \sup_{A,\xi}\frac{\ddot{\sigma_2}^{(A,\xi)}(0)}{\ddot{\sigma_1}^{(A,\xi)}(0)} = \lim_{s\to 0}\sup_{A,\xi}\frac{\sigma_2^{(A,\xi)}(s)-\sigma_2^{(A,\xi)}(0)}{\sigma_1^{(A,\xi)}(s)-\sigma_1^{(A,\xi)}(0)}.$$

Then there exists $\delta > 0$ such that

$$\sup_{A,\xi}\frac{\ddot{\sigma_2}^{(A,\xi)}(0)}{\ddot{\sigma_1}^{(A,\xi)}(0)} = \frac{1}{c_+} - 3\delta.$$

We also have that for each $\varepsilon > 0$, there exists $k = k(\varepsilon) \in \mathbb{R}^+$ such that for $|\xi| \geq k(\varepsilon)$,

$$\max_{A,s\in(0,1]}\frac{\sigma_2^{(A,\xi)}(s)-\sigma_2^{(A,\xi)}(0)}{\sigma_1^{(A,\xi)}(s)-\sigma_1^{(A,\xi)}(0)} > \frac{1}{c_+} - \varepsilon.$$

We take $\varepsilon = \delta$, and for ξ such that $|\xi| > k(\delta)$ one has

$$\max_{A,s\in(0,1]}\frac{\sigma_2^{(A,\xi)}(s)-\sigma_2^{(A,\xi)}(0)}{\sigma_1^{(A,\xi)}(s)-\sigma_1^{(A,\xi)}(0)} > \frac{1}{c_+} - \delta > \sup_{A,\xi}\frac{\ddot{\sigma_2}^{(A,\xi)}(0)}{\ddot{\sigma_1}^{(A,\xi)}(0)} = \frac{1}{c_+} - 3\delta.$$

As for such ξ

$$\sup_A \frac{\ddot{\sigma_2}^{(A,\xi)}(0)}{\ddot{\sigma_1}^{(A,\xi)}(0)} \leq \frac{1}{c_+} - 3\delta,$$

then for each A there must exist a point $t \in (0,1)$ for which

$$\lim_{s\to t} \frac{\sigma_2^{(A,\xi)}(s) - \sigma_2^{(A,\xi)}(t)}{\sigma_1^{(A,\xi)}(s) - \sigma_1^{(A,\xi)}(t)} > \frac{1}{c_+} - \delta.$$

Using again the fact that $\xi - tA$ and $\xi + tA$ can be regarded as new barycenters of first order laminates, one has

$$\lim_{s\to t} \frac{\frac{s+t}{2s}\varphi_2((\xi - tA) - (s-t)A) + \frac{s-t}{2s}\varphi_2((\xi - tA) + (s+t)A) - \varphi_2(\xi - tA)}{\frac{s+t}{2s}\varphi_1((\xi - tA) - (s-t)A) + \frac{s-t}{2s}\varphi_1((\xi - tA) + (s+t)A) - \varphi_1(\xi - tA)} \leq$$
$$\leq \frac{1}{c_+} - 3\delta$$

and

$$\lim_{s\to t} \frac{\frac{s-t}{2s}\varphi_2((\xi + tA) - (s+t)A) + \frac{s+t}{2s}\varphi_2((\xi + tA) + (s-t)A) - \varphi_2(\xi + tA)}{\frac{s-t}{2s}\varphi_1((\xi + tA) - (s+t)A) + \frac{s+t}{2s}\varphi_1((\xi + tA) + (s-t)A) - \varphi_1(\xi + tA)} \leq$$
$$\leq \frac{1}{c_+} - 3\delta.$$

Consequently, there exists $r_1 > 0$ such that for each $s \in B(t, r_1)$

$$\frac{\frac{s+t}{2s}\varphi_2((\xi - tA) - (s-t)A) + \frac{s-t}{2s}\varphi_2((\xi - tA) + (s+t)A) - \varphi_2(\xi - tA)}{\frac{s+t}{2s}\varphi_1((\xi - tA) - (s-t)A) + \frac{s-t}{2s}\varphi_1((\xi - tA) + (s+t)A) - \varphi_1(\xi - tA)} \leq$$
$$\leq \frac{1}{c_+} - 2\delta$$

and a $r_2 > 0$ such that for each $s \in B(t, r_2)$

$$\frac{\frac{s-t}{2s}\varphi_2((\xi + tA) - (s+t)A) + \frac{s+t}{2s}\varphi_2((\xi + tA) + (s-t)A) - \varphi_2(\xi + tA)}{\frac{s-t}{2s}\varphi_1((\xi + tA) - (s+t)A) + \frac{s+t}{2s}\varphi_1((\xi + tA) + (s-t)A) - \varphi_1(\xi + tA)} \leq$$
$$\leq \frac{1}{c_+} - 2\delta.$$

For each $s \in B(t, r)$, where $r = \min\{r_1, r_2\}$ and noticing that φ_1 is strictly convex, one can get

$$\frac{\sigma_2^{(A,\xi)}(s) - \sigma_2^{(A,\xi)}(t)}{\sigma_1^{(A,\xi)}(s) - \sigma_1^{(A,\xi)}(t)} \leq \frac{1}{c_+} - 2\delta,$$

which is absurd. ∎

Chapter 3

Quasiconvexity: the quadratic case revisited, and some consequences for fourth-degree polynomials

In this chapter we provide an alternative proof for the well-known equivalence between quasiconvexity and rank-one convexity in the quadratic case. Our proof avoids the Plancherel formula. Some consequences and some new ideas for the case of 4th degree homogeneous polynomials are shown.

3.1 The quadratic case

A well-known result is the following

Theorem 3 *Let $\varphi : \mathbb{M}^{m\times N} \to \mathbb{R}$ be a quadratic form. Then φ is quasiconvex if and only if is rank-one convex.*

The proof of this result is known for a long time ([45],[46], although implicitly known earlier). Nevertheless, all known proofs until now use Fourier transforms and the Plancherel formula, and so they cannot be applied to other than the quadratic case. We propose an alternative proof, which does not make use of Plancherel formula, and so we hope in this way to gain more insight about this outstanding problem of if rank-one convexity implies quasiconvexity when $m = 2$.

Proof.

Step 1. Notice that a quadratic form $\varphi : \mathbb{M}^{m\times N} \to \mathbb{R}$ can always be written as

$$\varphi(\xi) = \xi^T A\,\xi - \xi^T B\,\xi,$$

with $A, B \in \mathbb{M}^{(m\times N)\times(m\times N)}$ symmetric matrices and A is positive definite.

For φ as above, its quasiconvexity is equivalent to

$$\int_Q (\xi+\nabla u(x))^T A\,(\xi+\nabla u(x)) - (\xi+\nabla u(x))^T B\,(\xi+\nabla u(x))\,dx \geq \xi^T A\,\xi - \xi^T B\,\xi \Leftrightarrow$$

$$\Leftrightarrow \int_Q \nabla^T u(x)\,A\,\nabla u(x)\,dx \geq \int_Q \nabla^T u(x)\,B\,\nabla u(x)\,dx \Leftrightarrow$$

$$\Leftrightarrow 1 \geq \frac{\int_Q \nabla^T u(x)\,B\,\nabla u(x)\,dx}{\int_Q \nabla^T u(x)\,A\,\nabla u(x)\,dx},$$

for every $u \in C_0^\infty(Q, \mathbb{R}^m)$ since, by the divergence theorem,

$$\int_Q \nabla u(x)\,dx = 0,$$

where $Q = (0,1)^N$. This last inequality can be rewritten as

$$1 \geq \sup_{\nabla u} \frac{\int_Q \nabla^T u(x)\,B\,\nabla u(x)\,dx}{\int_Q \nabla^T u(x)\,A\,\nabla u(x)\,dx},$$

for $u \in C_0^\infty(Q, \mathbb{R}^m)$.

Step 2. We want now to solve the (infinite dimensional) problem of finding

$$\sup_{\nabla u} \frac{\int_Q \nabla^T u(x)\,B\,\nabla u(x)\,dx}{\int_Q \nabla^T u(x)\,A\,\nabla u(x)\,dx},$$

where A is positive definite. However, we can reduce this infinite dimensional problem to a finite dimensional one (but now with an infinite number of variables), by expanding u in a Fourier series. If $u \in C_0^\infty(Q, \mathbb{R}^m)$, we put

$$u(x) = \sum_{k\in Z^N} c_k\, e^{2\pi i k.\,x},\ c_k = \int_Q u(x)\, e^{-2\pi i k.\,x}\,dx.$$

Notice that, although we take $k \in Z^N$ in the summations, we are thinking only in expansions with a finite (but arbitrary) number of terms. In this way it is straightforward to justify all the computations done, and then to achieve the conclusion for any $u \in C_0^\infty(Q, \mathbb{R}^m)$ is just a limit procedure to obtain expansions with all $k \in Z^N$ ([50]), preserving in this way the required inequality of quasiconvexity. The same assumption is made through the following sections of this chapter in all the computations involving Fourier expansions.

Now, $\int_Q \nabla^T u(x) A \nabla u(x)\, dx$ is equal to

$$
\begin{aligned}
-4\pi^2 \int_Q \sum_{j\in Z^N} (c_j \otimes j)^T e^{2\pi i j.x} A \sum_{k\in Z^N} (c_k \otimes k) e^{2\pi i k.x}\, dx &= \\
= -4\pi^2 \sum_{j\in Z^N} \sum_{k\in Z^N} j \otimes c_j\ A\ c_k \otimes k \underbrace{\int_Q e^{2\pi i (j+k).x}\, dx}_{=0\ if\ j+k\neq 0} &= \\
= -4\pi^2 \sum_{k\in Z^N} -k \otimes c_{-k}\ A\ c_k \otimes k &= \\
= 4\pi^2 \sum_{k\in Z^N} k \otimes \overline{c}_k\ A\ k \otimes c_k&,
\end{aligned}
$$

where $\overline{c}_k$ denotes the complex conjugate of c_k. We can ignore any multiplicative constants, as they will appear both in the numerator and denominator, so we can take

$$
\int_Q \nabla^T u(x) A \nabla u(x)\, dx = \sum_{k\in Z^N} k \otimes \overline{c}_k\ A\ c_k \otimes k.
$$

If we put

$$
c_k = (c_k^1, c_k^2, ..., c_k^m),\ k = (k_1, k_2, ..., k_N),
$$

then

$$
c_k \otimes k = (c_k^1, c_k^2, ..., c_k^m) \otimes (k_1, k_2, ..., k_N) = \begin{pmatrix} c_k^1 k_1 \\ \cdots \\ c_k^1 k_N \\ c_k^2 k_1 \\ \cdots \\ c_k^m k_N \end{pmatrix},
$$

$$k \otimes \overline{c}_k = \begin{pmatrix} \overline{c}_k^1 k_1 & \dots & \overline{c}_k^1 k_N & \overline{c}_k^2 k_1 & \dots & \overline{c}_k^m k_N \end{pmatrix}$$

and, if we consider

$$A = \begin{pmatrix} A_1^1 & A_2^1 & \dots & A_N^1 & A_{N+1}^1 & \dots & A_{2N}^1 & \dots & A_{mN}^1 \\ A_2^1 & A_2^2 & \dots & A_N^2 & A_{N+1}^2 & \dots & A_{2N}^2 & \dots & A_{mN}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_N^1 & A_N^2 & \dots & A_N^N & A_{N+1}^N & \dots & A_{2N}^N & \dots & A_{mN}^N \\ A_{N+1}^1 & A_{N+1}^2 & \dots & A_{N+1}^N & A_{N+1}^{N+1} & \dots & A_{2N}^{N+1} & \dots & A_{mN}^{N+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{2N}^1 & A_{2N}^2 & \dots & A_{2N}^N & A_{2N}^{N+1} & \dots & A_{2N}^{2N} & \dots & A_{mN}^{2N} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{mN}^1 & A_{mN}^2 & \dots & A_{mN}^N & A_{mN}^{N+1} & \dots & A_{mN}^{2N} & \dots & A_{mN}^{mN} \end{pmatrix}$$

then

$$\int_Q \nabla^T u(x) A \nabla u(x)\, dx = \sum_{k \in Z^N} k \otimes \overline{c}_k \; A \; c_k \otimes k = \sum_{k \in Z^N} \overline{c}_k^T \; A_k \; c_k,$$

for

$$A_k = \begin{pmatrix} \alpha_1^1(k) & \alpha_2^1(k) & \dots & \alpha_m^1(k) \\ \alpha_2^1(k) & \alpha_2^2(k) & \dots & \alpha_m^2(k) \\ \dots & \dots & \dots & \dots \\ \alpha_m^1(k) & \alpha_m^2(k) & \dots & \alpha_m^m(k) \end{pmatrix},$$

where

$$\alpha_p^p(k) = \sum_{r=1}^{N} (k_r)^2 A_{r+(p-1)N}^{r+(p-1)N} + 2 \sum_{\substack{r,s=1 \\ r<s}}^{N} k_r k_s A_{s+(p-1)N}^{r+(p-1)N},\ p = 1, \dots, m,$$

$$\alpha_p^q(k) = \sum_{r=1}^{N}(k_r)^2 A_{r+(p-1)N}^{r+(q-1)N} + \sum_{\substack{r,s=1 \\ r<s}}^{N} k_r k_s \left(A_{s+(p-1)N}^{r+(q-1)N} + A_{r+(p-1)N}^{s+(q-1)N}\right),$$

$$p, q = 1, ..., m, \ q < p.$$

But

$$\sum_{k\in Z^N} \overline{c}_k^T \, A_k \, c_k = \sum_{k\in Z^N} (Re\, c_k - i\, Im\, c_k)^T \, A_k \, (Re\, c_k + i\, Im\, c_k) =$$

$$= \sum_{k\in Z^N} X_k^T \, \widetilde{A}_k \, X_k,$$

for

$$X_k = \begin{pmatrix} Re\, c_k \\ \\ Im\, c_k \end{pmatrix}, \ \widetilde{A}_k = \begin{pmatrix} A_k & 0 \\ \\ 0 & A_k \end{pmatrix}.$$

Similarly, we have

$$\int_Q \nabla^T u(x)\, B\, \nabla u(x)\, dx = \sum_{k\in Z^N} k \otimes \overline{c}_k \; B \; c_k \otimes k = \sum_{k\in Z^N} X_k^T \, \widetilde{B}_k \, X_k.$$

Step 3. In order to determine

$$\sup_{\nabla u} \frac{\int_Q \nabla^T u(x)\, B\, \nabla u(x)\, dx}{\int_Q \nabla^T u(x)\, A\, \nabla u(x)\, dx},$$

we must first find the

$$\max_{X=(X_k)_{k\in Z^N}} \frac{\sum_{k\in Z^N} X_k^T \, \widetilde{B}_k \, X_k}{\sum_{k\in Z^N} X_k^T \, \widetilde{A}_k \, X_k},$$

which is the maximum of the quotient of two expressions homogeneous of degree two in X. Instead of computing

$$\max_{X=(X_k)_{k\in Z^N}} \frac{\sum_{k\in Z^N} X_k^T \, \widetilde{B}_k \, X_k}{\sum_{k\in Z^N} X_k^T \, \widetilde{A}_k \, X_k},$$

we can consider the equivalent problem

$$\max_{X=(X_k)_{k\in Z^N}} \sum_{k\in Z^N} X_k^T \widetilde{B}_k X_k$$

subject to

$$\sum_{k\in Z^N} X_k^T \widetilde{A}_k X_k = 1,$$

where $\widetilde{A}_k$ is positive definite for each $k \neq 0$.

If λ is a Lagrange multiplier, we put

$$L(X,\lambda) = \sum_{k\in Z^N} X_k^T \widetilde{B}_k X_k - \lambda \left(\sum_{k\in Z^N} X_k^T \widetilde{A}_k X_k - 1 \right).$$

The first-order necessary conditions will then tell us that

$$\nabla L = 0 \Leftrightarrow \begin{cases} (\widetilde{B}_k - \lambda \widetilde{A}_k) X_k = 0 \text{ for each } k \in Z^N, \\ \sum_{k\in Z^N} X_k^T \widetilde{A}_k X_k = 1. \end{cases} \tag{3.1}$$

If $X = (X_k)_{k\in Z^N}$ is a critical point, then it is easy to derive

$$\sum_{k\in Z^N} X_k^T \widetilde{B}_k X_k = \lambda.$$

Also from the above, for having solutions of this system of equations, it is necessary (and also sufficient, as $\widetilde{A}_k$ is positive definite for $k \neq 0$) to have

$$\det(\widetilde{B}_j - \lambda \widetilde{A}_j) = 0,$$

for some j and some λ. The solutions λ of this equation will be denoted by λ_j. We put $X_j = w_j \neq 0$ and suppose, without loss of generality, that $X_k = 0$ for $k \neq j$. Then

$$k \neq j \Rightarrow (\widetilde{B}_k - \lambda_j \widetilde{A}_k) X_k = (\widetilde{B}_k - \lambda_j \widetilde{A}_k) 0 = 0.$$

As $w_j \in \ker(\widetilde{B}_j - \lambda_j \widetilde{A}_j)$,

$$(\widetilde{B}_j - \lambda_j \widetilde{A}_j) X_j = (\widetilde{B}_j - \lambda_j \widetilde{A}_j) w_j = 0.$$

With respect to the last equation of (3.1), we have

$$\sum_{k\in Z^N} X_k^T\ A_k\ X_k = w_j^T\ A_j\ w_j = c^2 > 0.$$

By setting

$$s_j = \frac{1}{c} w_j = \frac{1}{c} X_j,$$

and

$$s_k = 0,\ k \neq j,$$

we have that $((s_k)_{k\in Z^N}, \lambda_j)$ is a critical point of the problem, with associated cost equal to λ_j.

The above proves that the problem of finding the supremum of the solutions of

$$\max_{X=(X_k)_{k\in Z^N}} \sum_{k\in Z^N} X_k^T\ \widetilde{B}_k\ X_k$$

subject to

$$\sum_{k\in Z^N} X_k^T\ \widetilde{A}_k\ X_k = 1,$$

is equivalent to determining

$$\sup_{j\in Z^N} \lambda_j,$$

where λ_j are the solutions of

$$\det(\widetilde{B}_j - \lambda_j \widetilde{A}_j) = 0,\ j \in Z^N.$$

Step 4. Suppose, by hypothesis, that φ is rank one convex. Since φ is quadratic, making the decomposition $\varphi = \varphi_1 - \varphi_2$, with φ_1 strictly convex and putting $\nabla^2\varphi_1(\xi) = A$, $\nabla^2\varphi_2(\xi) = B$, we get

$$1 \geq \frac{n\otimes a\ B\, a\otimes n}{n\otimes a\ A\, a\otimes n},$$

for every $a \in \mathbb{R}^m$ and $n \in \mathbb{R}^N$, or, which is the same,

$$1 \geq \max_{a\in\mathbb{R}^m, n\in\mathbb{R}^N} \frac{n\otimes a\ B\, a\otimes n}{n\otimes a\ A\, a\otimes n}.$$

We can incorporate, in the above maximum, the dependence on n within the matrices A (in the spirit of the above case for quasiconvexity), thus obtaining

$$n \otimes a\, A\, a \otimes n = a^T A_n\, a$$

with

$$A_n = \begin{pmatrix} \alpha_1^1(n) & \ldots & \alpha_m^1(n) \\ \ldots & \ldots & \ldots \\ \alpha_m^1(n) & \ldots & \alpha_m^m(n) \end{pmatrix},$$

for α_j^j, α_j^l defined as before (but now as functions of $n \in \mathbb{R}^N$). Similarly we can use the same reasoning with B

$$n \otimes a\, B\, a \otimes n = a^T B_n\, a.$$

We now want to compute

$$\max_{n \in \mathbb{R}^N} \max_{a \in \mathbb{R}^m} \frac{a^T B_n\, a}{a^T A_n\, a}.$$

Since

$$\max_{a \in \mathbb{R}^m} \frac{a^T B_n\, a}{a^T A_n\, a}$$

is the quotient of two expressions homogeneous of degree two in a, we consider the equivalent problem

$$\max_{a \in \mathbb{R}^m} a^T B_n\, a$$

subject to

$$a^T A_n\, a = 1.$$

If λ is a Lagrange multiplier, we put

$$L(a, \lambda) = a^T B_n\, a - \lambda \left(a^T A_n\, a - 1\right).$$

The first-order necessary conditions will then tell us that

$$\nabla L = 0 \Leftrightarrow \begin{cases} (B_n - \lambda A_n)\, a = 0 \\ a^T A_n\, a = 1. \end{cases}$$

Once again, if $a \in \mathbb{R}^m$ is a critical point, then

$$a^T B_n a = \lambda.$$

The above system has solutions if and only if $\det(B_n - \lambda A_n) = 0$. The solutions λ of this equation will be denoted by λ_n.

Thus we have shown that the problem

$$\max_{n\in\mathbb{R}^N} \max_{a\in\mathbb{R}^m} \frac{a^T B_n a}{a^T A_n a},$$

is equivalent to determining

$$\max_{n\in\mathbb{R}^N} \lambda_n,$$

where the λ_n are the solutions of

$$\det(B_n - \lambda_n A_n) = 0, \; n \in \mathbb{R}^N.$$

The conclusion from this step is that, φ is rank-one convex if and only if

$$1 \geq \max_{n\in\mathbb{R}^N} \lambda_n.$$

Step 5. Observe that

$$1 \geq \max_{a\in\mathbb{R}^m, n\in\mathbb{R}^N} \frac{n\otimes a \; B \, a\otimes n}{n\otimes a \; A \, a\otimes n} = \max_{n\in\mathbb{R}^N} \lambda_n \geq \sup_{j\in Z^N} \lambda_j,$$

because the solutions λ_j of

$$\det(\widetilde{B}_j - \lambda_j \widetilde{A}_j) = 0, \; j \in Z^N,$$

are contained in the set of solutions λ_n of

$$\det(B_n - \lambda_n A_n) = 0, \; n \in \mathbb{R}^N.$$

Consequently, a quadratic φ is quasiconvex if and only if it is rank-one convex. ∎

3.2 Quasiconvexity for 4th degree homogeneous polynomials

We now try to apply the ideas of the previous section to the case of fourth-degree homogeneous polynomials. Let $\varphi_1, \varphi_2 : \mathbb{M}^{m\times N} \to \mathbb{R}$ be homogeneous polynomials of degree four, with φ_1 strictly convex and take

$$\varphi(\xi) = \varphi_1(\xi) - c\varphi_2(\xi), \ c \in \mathbb{R}.$$

In order to determine the values of c for which the corresponding φ is quasiconvex, we want to determine the extrema of the function

$$\frac{\int_Q \varphi_2(\xi + \nabla u(x)) - \varphi_2(\xi)\,dx}{\int_Q \varphi_1(\xi + \nabla u(x)) - \varphi_1(\xi)\,dx},$$

for every $\xi \in \mathbb{M}^{m\times N}$ and for every $u \in C_c^\infty(Q, \mathbb{R}^m)$ where, for convenience, we take $Q = (-\pi, \pi)^N$. In the case of $\xi = 0$, this will be answered by Theorem 4. For checking the quasiconvexity at the origin, the above quotient is much simpler

$$\frac{\int_Q \varphi_2(\nabla u(x))\,dx}{\int_Q \varphi_1(\nabla u(x))\,dx},$$

as $\varphi_i(0) = 0$. We can reduce this infinite-dimensional problem to a finite dimensional one, as we did in the quadratic case, by taking the Fourier expansion of u. Furthermore, we can expand u as a Fourier series with imaginary coefficients, with the help of the following lemma.

Lemma 2 $\varphi : \mathbb{M}^{m\times N} \to \mathbb{R}$ *is quasiconvex if and only if*

$$\int_{(-\pi,\pi)^N} \varphi(\xi + \nabla u(x))\,dx \geq \int_{(-\pi,\pi)^N} \varphi(\xi)\,dx \tag{3.2}$$

for each $\xi \in M^{m\times N}$ *and* $u \in C_c^\infty((-\pi,\pi)^N, \mathbb{R}^m)$ *such that* $u(-x) = -u(x)$, $x \in (-\pi,\pi)^N$.

Proof. We only need to prove the "if" part. Suppose, by hypothesis, that φ verifies (3.2). We want to prove that φ is quasiconvex. For this purpose, consider an arbitrary $\xi \in M^{m\times N}$ and take for domain the set $\Omega = (0,\pi) \times (-\pi,\pi)^{N-1}$. So one must verify the inequality of the definition of quasiconvexity for every $u \in C_c^\infty(\Omega, \mathbb{R}^m)$. Define the "odd" extension of

u to $Q = (-\pi, \pi)^N$ by

$$U(x) = \begin{cases} u(x), \ x \in \Omega \\ 0, \ x \in \{0\} \times (-\pi, \pi)^{N-1} \\ -u(-x), \ x \in \Omega' := (-\pi, 0) \times (-\pi, \pi)^{N-1}. \end{cases}$$

In particular, the following properties hold: $U \in C_c^\infty((-\pi, \pi)^N, \mathbb{R})$ with $U(-x) = -U(x)$, $\nabla U(-x) = \nabla U(x)$, $x \in (-\pi, \pi)^N$. Then

$$2 \int_\Omega \varphi(\xi + \nabla u(x))\, dx = \int_\Omega \varphi(\xi + \nabla U(x))\, dx + \int_\Omega \varphi(\xi + \nabla U(x))\, dx =$$

$$= \int_\Omega \varphi(\xi + \nabla U(x))\, dx + \int_{\Omega'} \varphi(\xi + \nabla U(-y))\, dy = \int_{(-\pi,\pi)^N} \varphi(\xi + \nabla U(x))\, dx \geq$$

$$\geq \int_{(-\pi,\pi)^N} \varphi(\xi)\, dx = 2 \int_\Omega \varphi(\xi)\, dx.$$

■

By the above lemma one can take, without loss of generality, $u \in C_c^\infty(Q, \mathbb{R}^m)$ with $u(-x) = -u(x)$, $x \in Q$. Then

$$u(x) = \sum_{j \in Z^N} c_j\, e^{ij.x}, \ C_j = \frac{1}{(2\pi)^N} \int_Q u(x) e^{-ij.x} dx,$$

and so, in particular we have

$$\nabla u(x) = \sum_{j \in Z^N} iC_j \otimes j \ e^{ij.x},$$

with

$$C_{-j} = -C_j, \ j \in Z^N.$$

Since C_j is purely imaginary, iC_j is real and so we take as variables $c_j = iC_j$, which are real.

The problem is now to find the extrema (now in the c_j, at the end we must then compute the extrema with respect to the $j \in Z^N$) of

$$\frac{\int_Q \varphi_2 \left(\sum\limits_{j \in Z^N} c_j \otimes j \ e^{ij.x} \right) dx}{\int_Q \varphi_1 \left(\sum\limits_{j \in Z^N} c_j \otimes j \ e^{ij.x} \right) dx}.$$

As the above quotient is the quotient of two homogeneous expressions of degree 4 in c_j, we can consider the equivalent problem of computing the extrema of

$$\int_Q \varphi_2 \left(\sum_{j \in Z^N} c_j \otimes j \ e^{ij.x} \right) dx$$

subject to

$$\int_Q \varphi_1 \left(\sum_{j \in Z^N} c_j \otimes j \ e^{ij.x} \right) dx = 1.$$

If λ is a Lagrange multiplier and $C = (c_j)_{j \in Z^N}$, we write

$$L(C, \lambda) = \int_Q \varphi_2 \left(\sum_{j \in Z^N} c_j \otimes j \ e^{ij.x} \right) dx + \\ - \lambda \left(\int_Q \varphi_1 \left(\sum_{j \in Z^N} c_j \otimes j \ e^{ij.x} \right) dx - 1 \right).$$

In order to obtain the first-order necessary conditions one has to compute

$$\frac{\partial}{\partial c_j^p} \int_Q \varphi_i \left(\sum_{k \in Z^N} c_k \otimes k \ e^{ik.x} \right) dx =$$

$$= \int_Q \nabla \varphi_i \left(\sum_{k \in Z^N} c_k \otimes k \ e^{ik.x} \right) \frac{\partial}{\partial c_j^p} \left(\sum_{k \in Z^N} c_k \otimes k \ e^{ik.x} \right) dx =$$

$$= \int_Q \nabla \varphi_i \left(\sum_{k \in Z^N} c_k \otimes k \, e^{ik.x} \right) (0, ..., 1, ..., 0) \otimes j \ (e^{ij.x} + e^{-ij.x}) \, dx,$$

where $p = 1, ..., m$, $c_j = (c_j^1, ..., c_j^m)$ and $(0, ..., 1, ..., 0)$ above means that all the coordinates are zero except the p-th one.

Since φ_i are homogeneous polynomials of degree four, one can write

$$\varphi_1(\nabla u) = A_1(\nabla u, \nabla u, \nabla u, \nabla u), \ \varphi_2(\nabla u) = A_2(\nabla u, \nabla u, \nabla u, \nabla u),$$

where A_1, A_2 are 4th order (totally symmetric) tensors with m^4N^4 constant coefficients, and so the last equality above becomes

$$4\int_Q A_i\left(\sum_{k\in Z^N} c_k\otimes k\ e^{ik.x}, \sum_{l\in Z^N} c_l\otimes l\ e^{il.x}, \sum_{m\in Z^N} c_m\otimes m\ e^{im.x}\right)$$
$$(0,...,1,...,0)\otimes j\ (e^{ij.x}+e^{-ij.x})\,dx =$$

$$= 4\int_Q \sum_{k,l,m\in Z^N} A_i\left(c_k\otimes k, c_l\otimes l, c_m\otimes m\right)\ e^{i(k+l+m).x}$$
$$(0,...,1,...,0)\otimes j\ (e^{ij.x}+e^{-ij.x})\,dx =$$

$$= 4\sum_{k,l,m\in Z^N} A_i\left(c_k\otimes k, c_l\otimes l, c_m\otimes m, (0,...,1,...,0)\otimes j\right)$$
$$\underbrace{\int_Q e^{i(k+l+m).x}(e^{ij.x}+e^{-ij.x})\,dx}_{=1 \text{ if } j+k+l+m=0 \text{ or } j-k-l-m=0} =$$

$$= 4\sum_{k,l\in Z^N}\left(A_i\left((0,...,1,...,0)\otimes j, c_k\otimes k, c_l\otimes l, c_{j+k+l}\otimes(j+k+l)\right)+\right.$$
$$\left.+A_i\left((0,...,1,...,0)\otimes j, c_k\otimes k, c_l\otimes l, c_{j-k-l}\otimes(j-k-l)\right)\right),$$

where $p = 1, ..., m$.

With respect to the last equation of the set of first-order conditions, one obtains

$$\int_Q \varphi_1\left(\sum_{k\in Z^N} c_k\otimes k\ e^{ik.x}\right)\,dx - 1 =$$
$$= \int_Q \sum_{j,k,l,m\in Z^N} A_1\left(c_j\otimes j, c_k\otimes k, c_l\otimes l, c_m\otimes m\right)\ e^{i(j+k+l+m).x}\,dx - 1 =$$
$$= \sum_{j,k,l\in Z^N} A_1\left(c_j\otimes j, c_k\otimes k, c_l\otimes l, c_{j+k+l}\otimes(j+k+l)\right) - 1 = 0.$$

We are now interested in incorporating $j, k, l, j+k+l$ inside the matrices A_1 and A_2, before writing the optimality conditions. That can be done using a procedure similar to the one used in the quadratic case, but now using the formulas two times, as here we deal with fourth order tensors, instead of second order tensors. For a fourth order tensor

$$A=\begin{pmatrix} A_{1,1}^{1,1} & A_{1,2}^{1,1} & \ldots & A_{1,mN}^{1,1} & A_{2,1}^{1,1} & \ldots & A_{mN,mN}^{1,1} \\ A_{1,2}^{1,1} & A_{1,2}^{1,2} & \ldots & A_{1,mN}^{1,2} & A_{2,1}^{1,2} & \ldots & A_{mN,mN}^{1,2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{1,mN}^{1,1} & A_{1,mN}^{1,2} & \ldots & A_{1,mN}^{1,mN} & A_{2,1}^{1,mN} & \ldots & A_{mN,mN}^{1,mN} \\ A_{2,1}^{1,1} & A_{2,1}^{1,2} & \ldots & A_{2,1}^{1,mN} & A_{2,1}^{2,1} & \ldots & A_{mN,mN}^{2,1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{mN,mN}^{1,1} & A_{mN,mN}^{1,2} & \ldots & A_{mN,mN}^{1,mN} & A_{mN,mN}^{2,1} & \ldots & A_{mN,mN}^{mN,mN} \end{pmatrix},$$

we put

$$A\left(c_j \otimes j\ , c_k \otimes k\ , c_l \otimes l\ , c_{j+k+l} \otimes (j+k+l)\right) =$$

$$= l \otimes c_l \begin{pmatrix} j \otimes c_j\, A_1^1\, c_k \otimes k & \ldots & j \otimes c_j\, A_{mN}^1\, c_k \otimes k \\ \ldots & \ldots & \ldots \\ j \otimes c_j\, A_{mN}^1\, c_k \otimes k & \ldots & j \otimes c_j\, A_{mN}^{mN}\, c_k \otimes k \end{pmatrix} c_{j+k+l} \otimes (j+k+l),$$

$$A_b^a = \left(A_{b,d}^{a,c}\right)_{c,d=1,\ldots,mN},\quad a,b=1,\ldots,mN.$$

First we deal with

$$j \otimes c_j\, A_b^a\, c_k \otimes k.$$

For

$$c_j = (c_j^1, \ldots, c_j^m),\ j = (j_1, \ldots, j_N),$$

$$c_j \otimes j = \begin{pmatrix} c_j^1 j_1 \\ \ldots \\ c_j^1 j_N \\ c_j^2 j_1 \\ \ldots \\ c_j^m j_N \end{pmatrix}.$$

Then we have

$$j \otimes c_j\, A_b^a\, c_k \otimes k = c_j^T A_b^a(j,k)\, c_k,$$

where

$$A_b^a(j,k) = \begin{pmatrix} \alpha_{b,1}^{a,1}(j,k) & \dots & \alpha_{b,m}^{a,1}(j,k) \\ \dots & \dots & \dots \\ \alpha_{b,1}^{a,m}(j,k) & \dots & \alpha_{b,m}^{a,m}(j,k) \end{pmatrix},$$

$$\alpha_{a,p}^{b,p}(j,k) = \sum_{r=1}^{N} j_r k_r A_{b,r+(p-1)N}^{a,r+(p-1)N} + \sum_{\substack{r,s=1 \\ r<s}}^{N} (j_r k_s + j_s k_r) A_{b,s+(p-1)N}^{a,r+(p-1)N},$$

$$p = 1, \dots, m, \quad (3.3)$$

$$\alpha_{b,p}^{a,q}(j,k) = \sum_{r=1}^{N} j_r k_r A_{b,r+(p-1)N}^{a,r+(q-1)N} + \sum_{\substack{r,s=1 \\ r<s}}^{N} j_r k_s A_{b,s+(p-1)N}^{a,r+(q-1)N} + j_s k_r A_{b,r+(p-1)N}^{a,s+(q-1)N},$$

$$q < p, \quad q,p = 1, \dots, m, \quad (3.4)$$

$$\alpha_{b,p}^{a,q}(j,k) = \sum_{r=1}^{N} j_r k_r A_{b,r+(q-1)N}^{a,r+(p-1)N} + \sum_{\substack{r,s=1 \\ r<s}}^{N} j_r k_s A_{b,r+(q-1)N}^{a,s+(p-1)N} + j_s k_r A_{b,s+(q-1)N}^{a,r+(p-1)N},$$

$$q > p, \quad q,p = 1, \dots, m. \quad (3.5)$$

So we have

$$A\,(c_j \otimes j\ , c_k \otimes k\ , c_l \otimes l\ , c_{j+k+l} \otimes (j+k+l)) =$$

$$= l \otimes c_l \begin{pmatrix} c_j^T A_1^1(j,k)\, c_k & \dots & c_j^T A_{mN}^1(j,k)\, c_k \\ \dots & \dots & \dots \\ c_j^T A_{mN}^1(j,k)\, c_k & \dots & c_j^T A_{mN}^{mN}(j,k)\, c_k \end{pmatrix} c_{j+k+l} \otimes (j+k+l) =$$

$$= c_l^T \begin{pmatrix} c_j^T A_1^1(j,k,l,j+k+l)\, c_k & \dots & c_j^T A_m^1(j,k,l,j+k+l)\, c_k \\ \dots & \dots & \dots \\ c_j^T A_1^m(j,k,l,j+k+l)\, c_k & \dots & c_j^T A_m^m(j,k,l,j+k+l)\, c_k \end{pmatrix} c_{j+k+l},$$

where

$$A_p^p(j,k,l,j+k+l) = \sum_{r=1}^{N} l_r(j_r+k_r+l_r) A_{r+(p-1)N}^{r+(p-1)N}(j,k) +$$

$$+ \sum_{\substack{r,s=1 \\ r<s}}^{N} (l_r(j_s+k_s+l_s) + l_s(j_r+k_r+l_r)) A_{s+(p-1)N}^{r+(p-1)N}(j,k), \ p=1,...,m,$$

$$A_p^q(j,k,l,j+k+l) = \sum_{r=1}^{N} l_r(j_r+k_r+l_r) A_{r+(p-1)N}^{r+(q-1)N}(j,k) +$$

$$+ \sum_{\substack{r,s=1 \\ r<s}}^{N} l_r(j_s+k_s+l_s) A_{s+(p-1)N}^{r+(q-1)N}(j,k) + l_s(j_r+k_r+l_r) A_{r+(p-1)N}^{s+(q-1)N}(j,k),$$

$$q<p, \ q,p=1,...,m,$$

$$A_p^q(j,k,l,j+k+l) = \sum_{r=1}^{N} l_r(j_r+k_r+l_r) A_{r+(q-1)N}^{r+(p-1)N}(j,k) +$$

$$+ \sum_{\substack{r,s=1 \\ r<s}}^{N} l_r(j_s+k_s+l_s) A_{r+(q-1)N}^{s+(p-1)N}(j,k) + l_s(j_r+k_r+l_r) A_{s+(q-1)N}^{r+(p-1)N}(j,k),$$

$$q>p, \ q,p=1,...,m.$$

To achieve this formulas one just has to make a second iteration with the formulas (3.3), (3.4), (3.5) and use the distributivity of the matrix multiplication with respect to its sum. Finally, we have

$$A\,(c_j \otimes j\ , c_k \otimes k\ , c_l \otimes l\ , c_{j+k+l} \otimes (j+k+l)) = A^{j,k,l,j+k+l}(c_j, c_k, c_l, c_{j+k+l}),$$

for

$$A^{j,k,l,j+k+l} = \begin{pmatrix} A_1^1(j,k,l,j+k+l) & \dots & A_m^1(j,k,l,j+k+l) \\ \dots & \dots & \dots \\ A_1^m(j,k,l,j+k+l) & \dots & A_m^m(j,k,l,j+k+l) \end{pmatrix}.$$

If we apply this method to A_1 and A_2 and simplify the notation by introducing

$$a_{j,k,l,j+k+l} = A_1^{j,k,l,j+k+l},$$
$$b_{j,k,l,j+k+l} = A_2^{j,k,l,j+k+l},$$

the first-order optimality conditions will then be

$$\left\{\begin{array}{l}\sum_{k,l\in Z^N} (b_{j,k,l,j+k+l} - \lambda\, a_{j,k,l,j+k+l})((1,0,...0), c_k, c_l, c_{j+k+l})+ \\ +(b_{j,k,l,j-k-l} - \lambda\, a_{j,k,l,j-k-l})((1,0,...,0), c_k, c_l, c_{j-k-l})) = 0, \\ \qquad\qquad\qquad \dots \qquad\qquad\qquad \text{for each } j\in Z^N \\ \sum_{k,l\in Z^N} (b_{j,k,l,j+k+l} - \lambda\, a_{j,k,l,j+k+l})((0,...,0,1), c_k, c_l, c_{j+k+l})+ \\ +(b_{j,k,l,j-k-l} - \lambda\, a_{j,k,l,j-k-l})((0,...,0,1), c_k, c_l, c_{j-k-l})) = 0, \\ \sum_{j,k,l\in Z^N} a_{j,k,l,j+k+l}(c_j, c_k, c_l, c_{j+k+l}) = 1. \end{array}\right. \tag{3.6}$$

If $C = (c_j)_{j\in Z^N}$ is a critical point, we can multiply

$$\sum_{k,l\in Z^N} (b_{j,k,l,j+k+l} - \lambda\, a_{j,k,l,j+k+l})((0,...,1,...,0), c_k, c_l, c_{j+k+l})+$$
$$+(b_{j,k,l,j-k-l} - \lambda\, a_{j,k,l,j-k-l})((0,...,1,...,0), c_k, c_l, c_{j-k-l})) = 0$$

by c_j^p and sum in $p = 1, ..., m$, thus obtaining

$$\sum_{k,l\in Z^N} (b_{j,k,l,j+k+l} - \lambda\, a_{j,k,l,j+k+l})(c_j, c_k, c_l, c_{j+k+l})+$$
$$+(b_{j,k,l,j-k-l} - \lambda\, a_{j,k,l,j-k-l})(c_j, c_k, c_l, c_{j-k-l})) = 0.$$

Then summing in $j \in Z^N$ gives

$$2 \sum_{j,k,l\in Z^N} (b_{j,k,l,j+k+l} - \lambda\, a_{j,k,l,j+k+l})\; c_j c_k c_l c_{j+k+l} = 0,$$

and using the last equation from (3.6) leads to

$$\sum_{j,k,l\in Z^N} b_{j,k,l,j+k+l}(c_j, c_k, c_l, c_{j+k+l}) = \lambda$$

and then to

$$\int_Q \varphi_2 \left(\sum_{k \in Z^N} c_k \otimes k\, e^{ik.x} \right) dx = \lambda.$$

We are now entitled to formulate the following

Theorem 4 *Let $\varphi_1, \varphi_2 : \mathbb{M}^{m\times N} \to \mathbb{R}$ be homogeneous polynomials of degree four, with φ_1 strictly convex and consider*

$$\varphi(\xi) = \varphi_1(\xi) - c\,\varphi_2(\xi).$$

Then φ is quasiconvex at zero if and only if

1. $c \in [c_-, c_+]$, *if* $\frac{1}{c_+}\frac{1}{c_-} < 0$;
2. $c \in (-\infty, c_+]$, *if* $\frac{1}{c_-} = 0$;
3. $c \in [c_-, +\infty)$, *if* $\frac{1}{c_+} = 0$,

with

$$\frac{1}{c_-}\left(resp\,\frac{1}{c_+}\right) = \inf \lambda\,(resp\ \sup \lambda),$$

where the values of λ can be obtained as the solutions of (3.6).

Unfortunately, the set of equations (3.6) does not provide us with better understanding of the problem than those given by itself, even if we consider deformations with just a few terms. This is the major reason why we proceed to the case of the second gradients, where the c_j are scalars.

3.3 The case of the second gradients

Suppose that $u \in C_c^\infty(Q, \mathbb{R})$ where we set $Q = (-\pi, \pi)^N$. Let $\varphi_1, \varphi_2 : \mathbb{M}^{N\times N}_{sym} \to \mathbb{R}$ be homogeneous polynomials of degree four, with φ_1 strictly convex and take

$$\varphi(\xi) = \varphi_1(\xi) - c\,\varphi_2(\xi),\ c \in \mathbb{R}.$$

In order to determine the values of c for which the corresponding φ is 2-quasiconvex, we want to determine the extrema of the function

$$\frac{\int_Q \varphi_2(\xi + \nabla^2 u(x)) - \varphi_2(\xi)\,dx}{\int_Q \varphi_1(\xi + \nabla^2 u(x)) - \varphi_1(\xi)\,dx},$$

for every $\xi \in \mathbb{M}^{N\times N}_{sym}$ and for every $u \in C^\infty_c(Q,\mathbb{R})$. We will study the case $\xi = 0$, as in the gradient case. For checking the 2-quasiconvexity of φ at the origin, one must check

$$\frac{\int_Q \varphi_2(\nabla^2 u(x))\,dx}{\int_Q \varphi_1(\nabla^2 u(x))\,dx},$$

as $\varphi_i(0) = 0$. We can now expand u as a Fourier series with real coefficients, with the help of the following lemma, whose proof is similar to the gradient case.

Lemma 3 $\varphi : \mathbb{M}^{N\times N}_{sym} \to \mathbb{R}$ *is 2-quasiconvex if and only if*

$$\int_{(-\pi,\pi)^N} \varphi(\xi + \nabla^2 u(x))\,dx \geq \int_{(-\pi,\pi)^N} \varphi(\xi)\,dx \tag{3.7}$$

for each $\xi \in M^{N\times N}_{sym}$ *and* $u \in C^\infty_c((-\pi,\pi)^N,\mathbb{R})$ *such that* $u(-x) = u(x),\ x \in (-\pi,\pi)^N$.

By the above lemma one can take, without loss of generality, $u \in C^\infty_c(Q,\mathbb{R})$ with $u(-x) = u(x),\ x \in Q$. Then

$$u(x) = \sum_{j\in Z^N} c_j\, e^{ij.x},\ c_j = \frac{1}{(2\pi)^N}\int_Q u(x)e^{-ij.x}dx,$$

and so, in particular we have

$$\nabla^2 u(x) = -\sum_{j\in Z^N} j\otimes j\, c_j\, e^{ij.x},$$

with

$$c_{-j} = c_j,\ j \in Z^N.$$

The problem is now to find the extrema (now in the c_j) of

$$\frac{\int_Q \varphi_2\left(\sum_{j\in Z^N} j\otimes j\, c_j\, e^{ij.x}\right)\,dx}{\int_Q \varphi_1\left(\sum_{j\in Z^N} j\otimes j\, c_j\, e^{ij.x}\right)\,dx},$$

or, as we did in the previous section, to find the extrema of

$$\int_Q \varphi_2\left(\sum_{j\in Z^N} j\otimes j\, c_j\, e^{ij.x}\right)\,dx$$

subject to

$$\int_Q \varphi_1 \left(\sum_{j \in Z^N} j \otimes j \, c_j \, e^{ij.x} \right) dx = 1.$$

If λ is a Lagrange multiplier and $C = (c_j)_{j \in Z^N}$, we write

$$L(C, \lambda) = \int_Q \varphi_2 \left(\sum_{j \in Z^N} j \otimes j \, c_j \, e^{ij.x} \right) dx +$$

$$- \lambda \left(\int_Q \varphi_1 \left(\sum_{j \in Z^N} j \otimes j \, c_j \, e^{ij.x} \right) dx - 1 \right).$$

In order to obtain the first-order necessary conditions one has to compute

$$\frac{\partial}{\partial c_j} \int_Q \varphi_i \left(\sum_{k \in Z^N} k \otimes k \, c_k \, e^{ik.x} \right) dx =$$

$$= \int_Q \nabla \varphi_i \left(\sum_{k \in Z^N} k \otimes k \, c_k \, e^{ik.x} \right) \frac{\partial}{\partial c_j} \left(\sum_{k \in Z^N} k \otimes k \, c_k \, e^{ik.x} \right) dx =$$

$$= \int_Q \nabla \varphi_i \left(\sum_{k \in Z^N} k \otimes k \, c_k \, e^{ik.x} \right) j \otimes j \, (e^{ij.x} + e^{-ij.x}) \, dx.$$

Because φ_i are homogeneous polynomials of degree four, we can write

$$\varphi_1(\nabla^2 u) = A_1(\nabla^2 u, \nabla^2 u, \nabla^2 u, \nabla^2 u), \ \ \varphi_2(\nabla^2 u) = A_2(\nabla^2 u, \nabla^2 u, \nabla^2 u, \nabla^2 u),$$

where A_1, A_2 are 4th order (totally symmetric) tensors with N^8 constant coefficients, and so the last equality above becomes

$$4 \int_Q A_i \left(\sum_{k \in Z^N} k \otimes k \, c_k \, e^{ik.x}, \sum_{l \in Z^N} l \otimes l \, c_l \, e^{il.x}, \sum_{m \in Z^N} m \otimes m \, c_m \, e^{im.x} \right)$$
$$j \otimes j \, (e^{ij.x} + e^{-ij.x}) \, dx =$$

$$= 4 \int_Q \sum_{k,l,m \in Z^N} A_i \, (k \otimes k, l \otimes l, m \otimes m) \, c_k c_l c_m \, e^{i(k+l+m).x}$$
$$j \otimes j \, (e^{ij.x} + e^{-ij.x}) \, dx =$$

$$= 4 \sum_{k,l,m \in Z^N} A_i \left(k \otimes k, l \otimes l, m \otimes m, j \otimes j\right) c_k c_l c_m$$

$$\underbrace{\int_Q e^{i(k+l+m).x} (e^{ij.x} + e^{-ij.x})\, dx}_{=1 \text{ if } j+k+l+m=0 \text{ or } j-k-l-m=0} =$$

$$= 4 \sum_{k,l \in Z^N} \left(A_i \left(j \otimes j, k \otimes k, l \otimes l, (j+k+l) \otimes (j+k+l)\right) c_k c_l c_{j+k+l} + \right.$$
$$\left. + A_i \left(j \otimes j, k \otimes k, l \otimes l, (j-k-l) \otimes (j-k-l)\right) c_k c_l c_{j-k-l} \right).$$

With respect to the last equation of the set of first-order optimality conditions, one obtains

$$\int_Q \varphi_1 \left(\sum_{k \in Z^N} k \otimes k\, c_k\, e^{ik.x} \right) dx - 1 =$$
$$= \int_Q \sum_{j,k,l,m \in Z^N} A_1 \left(j \otimes j, k \otimes k, l \otimes l, m \otimes m\right) c_j c_k c_l c_m\, e^{i(j+k+l+m).x}\, dx - 1 =$$
$$= \sum_{j,k,l \in Z^N} A_1 \left(j \otimes j, k \otimes k, l \otimes l, (j+k+l) \otimes (j+k+l)\right) c_j c_k c_l c_{j+k+l} - 1 = 0$$

Consequently, simplifying the notation and writing

$$a_{j,k,l,j+k+l} = A_1 \left(j \otimes j, k \otimes k, l \otimes l, (j+k+l) \otimes (j+k+l)\right),$$
$$b_{j,k,l,j+k+l} = A_2 \left(j \otimes j, k \otimes k, l \otimes l, (j+k+l) \otimes (j+k+l)\right),$$

the desired set of equations will read

$$\begin{cases} \sum\limits_{k,l \in Z^N} \left(b_{j,k,l,j+k+l} - \lambda\, a_{j,k,l,j+k+l}\right) c_k c_l c_{j+k+l} + \\ +\left(b_{j,k,l,j-k-l} - \lambda\, a_{j,k,l,j-k-l}\right) c_k c_l c_{j-k-l}) = 0, \text{ for each } j \in Z^N, \\ \\ \sum\limits_{j,k,l \in Z^N} a_{j,k,l,j+k+l}\, c_j c_k c_l c_{j+k+l} = 1. \end{cases} \tag{3.8}$$

If $C = (c_j)_{j \in Z^N}$ is a critical point, we can multiply

$$\sum_{k,l \in Z^N} \left(b_{j,k,l,j+k+l} - \lambda\, a_{j,k,l,j+k+l}\right) c_k c_l c_{j+k+l} +$$
$$+ \left(b_{j,k,l,j-k-l} - \lambda\, a_{j,k,l,j-k-l}\right) c_k c_l c_{j-k-l}) = 0$$

by c_j and sum in $j \in Z^N$, thus obtaining

$$2 \sum_{j,k,l \in Z^N} \left(b_{j,k,l,j+k+l} - \lambda\, a_{j,k,l,j+k+l}\right) c_j c_k c_l c_{j+k+l} = 0.$$

Then using the last equation from (3.8) leads to

$$\sum_{j,k,l \in Z^N} b_{j,k,l,j+k+l}\, c_j c_k c_l c_{j+k+l} = \lambda$$

and then to

$$\int_Q \varphi_2 \left(\sum_{k \in Z^N} k \otimes k\, c_k\, e^{ik.x} \right) dx = \lambda.$$

We are now entitled to state the following

Theorem 5 *Let $\varphi_1, \varphi_2 : \mathbb{M}^{N \times N}_{sym} \to \mathbb{R}$ be homogeneous polynomials of degree four, with φ_1 strictly convex and consider*

$$\varphi(\xi) = \varphi_1(\xi) - c\,\varphi_2(\xi).$$

Then φ is 2-quasiconvex at zero if and only if

1. $c \in [c_-, c_+]$, *if* $\frac{1}{c_+}\frac{1}{c_-} < 0$;
2. $c \in (-\infty, c_+]$, *if* $\frac{1}{c_-} = 0$;
3. $c \in [c_-, +\infty)$, *if* $\frac{1}{c_+} = 0$,

with

$$\frac{1}{c_-} \left(resp\, \frac{1}{c_+} \right) = \inf \lambda \,(resp\ \sup \lambda),$$

where the values of λ can be obtained as the solutions of (3.8).

In general it is hard to solve (3.8), because the equations are extremely connected and determined as they share its variables, and so it is not possible to simplify the problem as one can do in the quadratic case. Nevertheless, one can consider, in some particular cases, Fourier expansions of u with just a few terms, aiming to understand better the details involved.

3.4 The classical examples for N=2

In this case $j = (j_1, j_2) \in Z^2$ and we will consider competing deformations with just a few terms.

3.4.1 One term

In this case we consider $c_j = c_{-j} \neq 0,\ c_k = 0$ for $k \neq j, -j$. Notice that (3.8) is now

$$\begin{cases} 6(b_{j,j,j,j} - \lambda\, a_{j,j,j,j})\, c_j^3 = 0, \\ 6\, a_{j,j,j,j}\, c_j^4 = 1. \end{cases}$$

For

$$\varphi(\xi) = \varphi_1(\xi) - c\,\varphi_2(\xi) = |\xi|^4 - c\,|\xi|^2 \det \xi$$

and

$$\varphi(\xi) = \varphi_1(\xi) - c\,\varphi_2(\xi) = |\xi|^4 - c\,(\det \xi)^2,$$

one has

$$a_{j,j,j,j} = (j_1^2 + j_2^2)^4,$$
$$b_{j,j,j,j} = 0,$$

and then $\lambda = 0$. Consequently, this case is not interesting.

3.4.2 Two terms

For this case we consider $c_j, c_k \neq 0,\ c_{-j} = c_j,\ c_{-k} = c_k$ (with $k \neq \alpha j$, otherwise it will lead to $\lambda = 0$), $c_l = 0$ for $l \neq j, -j, k, -k$. The first order necessary conditions will now be

$$\begin{cases} 6(b_{j,j,j,j} - \lambda\, a_{j,j,j,j})\, c_j^3 + 12(b_{j,j,k,k} - \lambda\, a_{j,j,k,k})\, c_j c_k^2 = 0 \\ 12(b_{j,j,k,k} - \lambda\, a_{j,j,k,k})\, c_j^2 c_k + 6(b_{k,k,k,k} - \lambda\, a_{k,k,k,k})\, c_k^3 = 0 \\ 6\, a_{j,j,j,j}\, c_j^4 + 24\, a_{j,j,j,j}\, c_j^2 c_k^2 + 6\, a_{k,k,k,k}\, c_k^4 = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (b_{j,j,j,j} - \lambda\, a_{j,j,j,j})\, c_j^2 + 2(b_{j,j,k,k} - \lambda\, a_{j,j,k,k})\, c_k^2 = 0 \\ 2(b_{j,j,k,k} - \lambda\, a_{j,j,k,k})\, c_j^2 + (b_{k,k,k,k} - \lambda\, a_{k,k,k,k})\, c_k^2 = 0 \\ 6\, a_{j,j,j,j}\, c_j^4 + 24\, a_{j,j,j,j}\, c_j^2 c_k^2 + 6\, a_{k,k,k,k}\, c_k^4 = 1 \end{cases}$$

In the case of

$$\varphi_2(\xi) = (\det \xi)^2 \quad \text{or} \quad \varphi_2(\xi) = |\xi|^2 \det \xi,$$

this system is equivalent to

$$\begin{cases} -\lambda\, a_{j,j,j,j}\, c_j^2 + 2(b_{j,j,k,k} - \lambda\, a_{j,j,k,k})\, c_k^2 = 0 \\ 2(b_{j,j,k,k} - \lambda\, a_{j,j,k,k})\, c_j^2 - \lambda\, a_{k,k,k,k}\, c_k^2 = 0 \\ 6\, a_{j,j,j,j}\, c_j^4 + 24\, a_{j,j,j,j}\, c_j^2 c_k^2 + 6\, a_{k,k,k,k}\, c_k^4 = 1. \end{cases}$$

Notice that using $\varphi_1 = |\xi|^4$,

$$a_{j,j,k,k} = \frac{1}{3}(j_1^2 + j_2^2)^2(k_1^2 + k_2^2)^2 + \frac{2}{3}(j_1k_1 + j_2k_2)^4$$

and

$$b_{j,j,k,k} = \frac{1}{3}(j_1k_1 + j_2k_2)^2(j_2k_1 - j_1k_2)^2,\ \varphi_2(\xi) = |\xi|^2 \det \xi$$

$$b_{j,j,k,k} = \frac{1}{6}(j_1k_2 - j_2k_1)^4,\ \varphi_2(\xi) = (\det \xi)^2$$

are non-negative, and so λ must be non-negative also, otherwise the above system is impossible.

Furthermore, we have to impose

$$\lambda^2 a_{j,j,j,j}a_{k,k,k,k} + 4(b_{j,j,k,k} - \lambda\, a_{j,j,k,k})^2 = 0 \Leftrightarrow$$

$$\Leftrightarrow \lambda = \frac{2b_{j,j,k,k}}{\sqrt{a_{j,j,j,j}a_{k,k,k,k}} + 2a_{j,j,k,k}},$$

again by the non-negativity of λ. To compute the extrema of λ, one has only to compute the maximum because $\lambda \geq 0$, and 0 is attained.

With

$$\varphi_2(\xi) = (\det \xi)^2,$$

one has

$$\sup_{j,k} \lambda = \sup_{j,k} \frac{(j_1k_2 - j_2k_1)^4}{5(j_1^2 + j_2^2)^2(k_1^2 + k_2^2)^2 + 4(j_1k_1 + j_2k_2)^4}.$$

In the above quotient, j, k can be taken such that $|j| = |k| = 1$, leading to

$$\frac{(j_1k_2 - j_2k_1)^4}{5 + 4(j_1k_1 + j_2k_2)^4},$$

and then

$$\sup_{j,k} \lambda = \sup_{j,k} \frac{(j_1k_2 - j_2k_1)^4}{5 + 4(j_1k_1 + j_2k_2)^4} = \frac{1}{5}.$$

This maximum is attained in the initial fraction by taking any 2 orthonormal vectors j, k.

For

$$\varphi_2(\xi) = |\xi|^2 \det \xi,$$

one has to compute

$$\sup_{j,k} \lambda = \sup_{j,k} \frac{2(j_1k_1 + j_2k_2)^2(j_1k_2 - j_2k_1)^2}{5(j_1^2 + j_2^2)^2(k_1^2 + k_2^2)^2 + 4(j_1k_1 + j_2k_2)^4}.$$

In this quotient, one can again take $j = (j_1, j_2),\ k = (k_1, k_2)$ with $|j| = |k| = 1$ and then get

$$\sup_{j,k} \lambda = \sup_{j,k} \frac{2\langle j, k\rangle^2 \langle j, \widetilde{k}\rangle^2}{5 + 4\langle j, k\rangle^4},$$

where $\widetilde{k} = (-k_2, k_1)$. As j, k are unit vectors and $k, \widetilde{k}$ are orthogonal, this can be further simplified into

$$\max_{\theta \in [0,2\pi)} \frac{2\cos^2(\theta)(1 - \cos^2(\theta))}{5 + 4\cos^4(\theta)} = \max_{x \in [0,1]} \frac{2x(1-x)}{5 + 4x^2} = -\frac{1}{4} + \frac{3\sqrt{5}}{20}.$$

As we easily observe, the quotient

$$\frac{\int_Q \varphi_2(\nabla^2 u(x))\, dx}{\int_Q \varphi_1(\nabla^2 u(x))\, dx}$$

is obviously positive in the case were

$$\varphi_1 = |\xi|^4,\ \varphi_2 = (\det \xi)^2,$$

but it surely takes both positive and negative values when

$$\varphi_2 = |\xi|^2 \det \xi.$$

One might be tempted to try to find a counterexample for $N = 2$, but notice that we are restricted to check the 2-quasiconvexity of φ at 0. First of all, we need to know for which values of $c \in \mathbb{R}$ the corresponding φ is convex along its characteristic cone. Then, if the smallest value obtained is zero, this will provide the desired conclusion (if one has an example of a periodic deformation with more than two terms, which is easy).

The characteristic cone ([18]) associated with 2-quasiconvexity is

$$\Lambda = \{a \otimes a,\ a \in \mathbb{R}^N\}.$$

The determination of which values of c provide functions φ that are convex along the directions of Λ can be done with the techniques developed in [5], applied to this particular case. It is then easy to conclude that

$$\varphi : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{R}$$

is convex along Λ if and only if

$$c \in \left[-\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right],$$

that is, the maximum value for λ (with at most two terms, considering all possible $\xi \in \mathbb{M}^{2\times 2}_{sym}$ and not only $\xi = 0$) is $\frac{\sqrt{3}}{4}$ and the minimum is $-\frac{\sqrt{3}}{4}$. It could seem quite surprising that the values are exactly the same here, but in fact the computations in [5] for the classical example of [14] shows that the extrema are attained, e.g., for

$$\xi = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix},$$

which is, in particular, a symmetric matrix, and for any direction, provided that it is of rank-one (that is, in particular one can choose to take a matrix in Λ).

We have then to proceed to computations with deformations of three or more terms, if we want to study the possibility of finding a counterexample.

3.4.3 Three terms

In this case we consider three non-zero terms $c_j, c_k, c_l \neq 0$. As $N = 2$, we know that one of the j, k, l must be a linear combination of the other two. We must analyze several subcases. In all the subcases we get $\lambda \geq 0$ from the necessary conditions (3.8), except one, which will be treated below. For example, in the subcase of j, k, l, with $j \neq \alpha k, k \neq \alpha l, l \neq \alpha j$, we have

$$\begin{cases} -\lambda a_{j,j,j,j} c_j^2 + 2(b_{j,j,k,k} - \lambda a_{j,j,k,k}) c_k^2 + 2(b_{j,j,l,l} - \lambda a_{j,j,l,l}) c_l^2 = 0 \\ 2(b_{j,j,k,k} - \lambda a_{j,j,k,k}) c_j^2 - \lambda a_{k,k,k,k} c_k^2 + 2(b_{k,k,l,l} - \lambda a_{k,k,l,l}) c_l^2 = 0 \\ 2(b_{j,j,l,l} - \lambda a_{j,j,l,l}) c_j^2 + 2(b_{k,k,l,l} - \lambda a_{k,k,l,l}) c_k^2 - \lambda a_{l,l,l,l} c_l^2 = 0 \\ 6a_{j,j,j,j} c_j^4 + 24 a_{j,j,k,k} c_j^2 c_k^2 + 24 a_{j,j,l,l} c_j^2 c_l^2 + 6 a_{k,k,k,k} c_k^4 + \\ \qquad\qquad + 24 a_{k,k,l,l} c_k^2 c_l^2 + 6 a_{l,l,l,l} c_l^4 = 1, \end{cases}$$

and in the other subcases we have at least one equation of the kind of the first 3 equations of this system, which implies that $\lambda \geq 0$, as stated (with the above mentioned exception). Despite the fact that this system looks harmless, the computations involved to solve it become too hard to find the exact solutions, as we did in the previous cases with less terms, and the same happens in the other subcases, in general.

The exception in terms of the positivity of λ is the subcase were $l = 2j+k$ for $k \neq \alpha j$, $\alpha \in \mathbb{R}$. In this subcase it is possible to obtain negative values of λ (for $\varphi_2(\xi) = |\xi|^2 \det \xi$), which means that for deformations with three terms, the quotient

$$\frac{\int_Q \varphi_2(\nabla^2 u(x))\, dx}{\int_Q \varphi_1(\nabla^2 u(x))\, dx}$$

attain negative values. An example of such a deformation is

$$u(x_1, x_2) = -\frac{1}{2}\left(e^{i(1,0)\cdot(x_1,x_2)} + e^{-i(1,0)\cdot(x_1,x_2)}\right) +$$
$$-\frac{1}{5}\left(e^{i(0,1)\cdot(x_1,x_2)} + e^{-i(0,1)\cdot(x_1,x_2)}\right) + \frac{1}{20}\left(e^{i(2,1)\cdot(x_1,x_2)} + e^{-i(2,1)\cdot(x_1,x_2)}\right),$$

and the corresponding value attained,

$$\lambda = -\frac{2944}{61971}.$$

We recall that this does not provide any counterexample, because the minimum value attained by λ with at most two terms (if the first moment is not fixed) is

$$\lambda_{\text{min}} = -\frac{\sqrt{3}}{4}.$$

Chapter 4

On the characterization of laminates for 2×2 symmetric gradients

In this chapter we explore the problem of the equivalence between rank-one convexity and quasiconvexity for 2×2 symmetric matrices from the viewpoint of probability measures, that is, we search for the existence of a gradient Young measure that is not a laminate, following the approach of [36]. As a by-product, we have reached a characterization of a couple of laminates, by using the concept of 3-edge-laminate.

4.1 Introduction

The question whether rank-one convexity implies quasiconvexity can be restated in terms of laminates and (homogeneous) gradient Young measures: is every (homogeneous) gradient Young measure a laminate?

Several authors have tried to answer the above question about the equivalence between quasiconvexity and rank-one convexity when $m = 2$ (see e.g. [34], [36], [39]) without success. The interested reader may find general reviews on this subject in [37]. A wider reference is [12].

Here we follow the attempt of Pedregal [36] to adapt the approach of Šverák [42] to the space of 2×2 symmetric matrices, which uses measures supported on the 8 vertices of the cube $[-1,1]^3$. Pedregal attempted the following strategy: to generate a point Q^- in the set of gradient Young measures, as extreme in this set as possible, with the aim of showing the impossibility of generating such Q^- as a laminate. Gradient Young measures with barycenter $(0,0,0)$ and supported on the vertices of the $[-1,1]^3$ cube were used in his attempt.

Our contribution aims at analyzing what we believe to be one of the best choices to find a counterexample in the case of 2×2 symmetric matrices. Our initial aim was to try the same strategy of [36], with other barycenters. We have considered not only $(0,0,0)$ but also, for several reasons (mostly to keep the symmetry between the x and y coordinates, which translates into symmetry of the different sets of measures), $(\frac{1}{3},\frac{1}{3},0)$ and $(\frac{1}{2},\frac{1}{2},0)$. But, after several attempts, we have succeeded, unfortunately, to generate also their corresponding laminates.

This is why we have then changed our focus, towards the problem of characterizing exactly the laminates involved. The characterization we have reached in Theorem 6 below concerns a precise class of laminates, which we call 3-edge-laminates (see Definition 3). They seem to generate (through convexity) all the laminates; but we were unable to prove this. It is amazingly difficult, in general, to prove rigorously (in concrete examples) that a given gradient Young measure or polyconvex measure is not a laminate. Similarly, if we fix an arbitrary polyconvex measure then it looks equally difficult to prove that it is not a gradient Young measure. All the computations done do not seem to relieve our doubts: they just reinforce our feeling that the relationship between rank-one convexity and quasiconvexity is not at all trivial or superficial; and (beyond the question of being able to find or not a counterexample) that both concepts, of laminates and gradient Young measures, are not yet well understood.

The organization of this chapter is as follows. In Section 4.2 we explain in detail our initial aim. This is complemented by Section 4.3, which exhibits sets of points generating the laminates mentioned before. As to Section 4.4, it deals with sets of polyconvex measures; while Section 4.5 concerns sets of gradient Young measures. These four sections constitute the first part of this chapter, which is a kind of preparation for the second, and main, part. This one starts with Section 4.6, where after some preliminaries we reach Theorem 6, characterizing, in a precise sense, the extreme points of the 3 sets of laminates (i.e. those corresponding to the barycenter $(a,a,0)$ with $a=0$, $a=\frac{1}{3}$, $a=\frac{1}{2}$) along their edges, which is our main result. Finally, Section 4.7 describes computational experiments designed to confirm, via a different route, such characterization.

4.2 Statement of the conclusions which we have reached

Since we generalize the work of [36], in a sense, by considering different barycenters, we use the same notations, namely the ones of its section 4. Thus we consider Lipschitz deformations $u : \mathbb{R}^2 \to \mathbb{R}^2$ of the type

$$u(x) = \nabla\varphi(x) + P_0'x$$

where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is $[0,1]^2$-periodic; with u the sum of 3 plane-waves along directions $(1,0), (0,1)$ and $(1,1)$ respectively, and

$$P_0' = \begin{pmatrix} \alpha_1 + \alpha_3 & \alpha_3 \\ \alpha_3 & \alpha_2 + \alpha_3 \end{pmatrix}, \quad P_0 = (\alpha_1, \alpha_2, \alpha_3) \in [-1,1]^3.$$

These Lipschitz deformations have gradients represented, pointwise, by symmetric matrices

$$\begin{pmatrix} x+z & z \\ z & y+z \end{pmatrix},$$

hence by vectors (x, y, z), assuming only 8 different values. These values are the vertices of the 3-dimensional cube $[-1,1]^3$. We also have laminates, gradient Young measures and polyconvex measures supported on the above 8 vertices. These are characterized by their barycenter P_0 and by the weights a, b, c on 3 vertices of the cube. Thus we represent them as (compact convex) sets in (a, b, c)-space.

To simplify the presentation, instead of probability measures we use here measures with total mass $= 576 = 24^2$, so that our relevant vector measures become triples (a, b, c) of integer numbers, with few exceptions (which involve the numbers 64.8, $74.(6) = 74 + \frac{2}{3}$, 76.5, $106.(6) = 106 + \frac{2}{3}$, $157.(09) = 157 + \frac{1}{11}$, 158.4); we thus avoid writing lots of cumbersome fractions.

Let us start by presenting the result in [36]. There the barycenter is $P_0 = (0,0,0)$, and (as shown below in Section 4.4) the set of obtainable polyconvex measures having weights on the vertices of the cube $[-1,1]^3$ which we denote as follows (see figure 4.1):

$$a \mapsto (1,1,1),\ b \mapsto (-1,1,1),\ c \mapsto (1,-1,1),\ d = 288 - a - b - c \mapsto (-1,-1,1),$$
$$\bar{a} = 432 - 3a - b - c \mapsto (1,1,-1),\ \bar{b} = 2a + c - 144 \mapsto (-1,1,-1),$$
$$\bar{c} = 2a + b - 144 \mapsto (1,-1,-1),\ \bar{d} = 144 - a \mapsto (-1,-1,-1).$$

Using this notation, the polyconvex measures constitute the polyhedron in

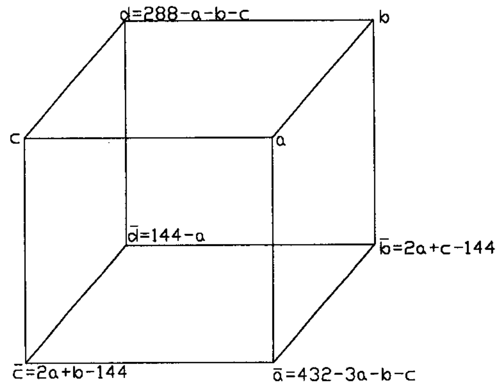

Figure 4.1: weights on the vertices of the cube $[-1,1]^3$ for $P_0 = (0,0,0)$.

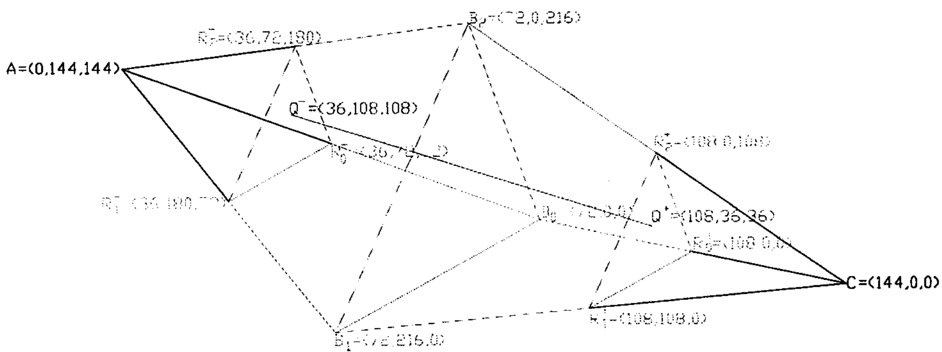

Figure 4.2: sets of measures in (a,b,c)-space for $P_0 = (0,0,0)$.

(a,b,c)-space which is the convex hull of its vertices:

$$A = (0,144,144),\ B_0 = (72,0,0),\ B_1 = (72,216,0),\ B_2 = (72,0,216),\\ C = (144,0,0).$$

(This 3-dimensional solid is easily visualized: B_0, B_1, B_2 are the vertices of a vertical triangle which is the common basis of two opposite pyramids having vertex at A, C respectively, see figure 4.2.)

Inside this set of polyconvex measures we have the corresponding set of gradient Young measures obtained in [36] from the Riemann-Lebesgue lemma (see, e.g., [35]) for periodic gradients, which is a segment (see Section

4.5 below). For the barycenter zero, the extremities of this segment are

$$Q^- = (36, 108, 108), \ Q^+ = (108, 36, 36).$$

We prefer, however, in order to simplify further the geometric picture of the relationship between gradient Young measures and laminates, to present each one of these sets of polyconvex measures in (a, b, c)-space through its intersection with the bisector plane $b = c$. For example, the edge B_1B_2 – with extremities $(a, b, c) = (72, 216, 0)$ and (a, c, b) – is thus represented by its point of intersection with the bisector plane: $B = (72, 108)$. In this way the above polyhedron (which is the set of polyconvex measures) becomes represented by a polygon, the convex hull of its 4 vertices:

$$A = (0, 144), \ B_0 = (72, 0), \ B = (72, 108), \ C = (144, 0),$$

see figure 4.3. (Notice: in this figure, and also in the next ones, Q_+^- and Q_0^- represent the points of intersection of the vertical line through Q^- with the boundary of the above polygon; similarly for Q_+^+ and Q_0^+. The reader should not pay attention, for the moment, to the points in these figures which are denoted using the letter R, namely R^-, $R_0^-, \ldots$; indeed, these points will be the subject of Section 4.6 below.) Notice that the set of gradient Young measures mentioned above is contained in the bisector plane, with its extremities being now represented by $Q^- = (36, 108)$ and $Q^+ = (108, 36)$. As to the set of laminates, its intersection with the bisector plane – as happens with the set of polyconvex measures – coincides with its orthogonal projection into this plane.

We now present some definitions to simplify the notation.

In the search for a counterexample, one important question is how to obtain all the gradient Young measures which can be generated directly from the Riemann-Lebesgue lemma (i.e. not indirectly through laminates).

Definition 1 *$\mathcal{D}$ will denote the class of Lipschitz deformations $u : \mathbb{R}^2 \to \mathbb{R}^2$ of the type*

$$u(x) = \nabla\varphi(x) + P_0'x,$$

where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is $[0,1]^2$-periodic, which is the sum of 3 plane-waves in directions $(1,0), (0,1)$ and $(1,1)$ respectively, with

$$P_0' = \begin{pmatrix} \alpha_1 + \alpha_3 & \alpha_3 \\ \alpha_3 & \alpha_2 + \alpha_3 \end{pmatrix}, \ P_0 = (\alpha_1, \alpha_2, \alpha_3) \in [-1, 1]^3.$$

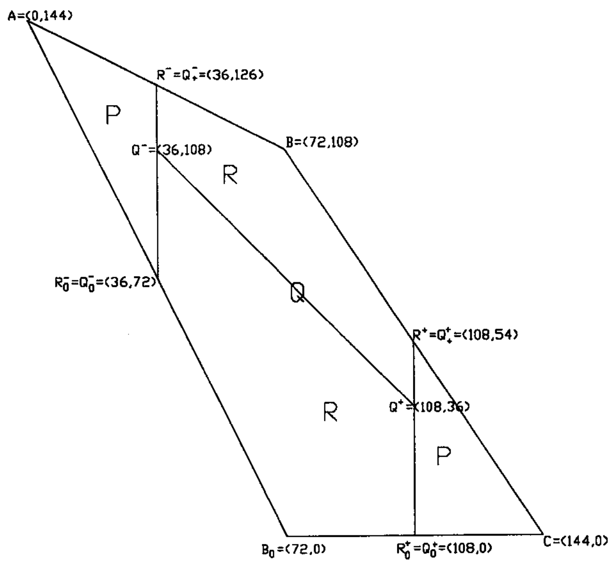

Figure 4.3: P, R-polygons and Q-segment for $P_0 = (0, 0, 0)$.

Definition 2 *$\mathcal{D}_\chi$ denotes the set of deformations $u(\cdot)$ in the class $\mathcal{D}$ which can be expressed as*

$$u_1(x, y) = \int_0^x \chi_1(t - \delta_1)\, dt + \int_0^{x+y} \chi_3(t - \delta_3)\, dt,$$

$$u_2(x, y) = \int_0^y \chi_2(t - \delta_2)\, dt + \int_0^{x+y} \chi_3(t - \delta_3)\, dt,$$

with $\delta_i \in (0, 1)$ *and*

$$\chi_i(s) := \begin{cases} 1, & s \in (0, s_i), \\ -1, & s \in (s_i, 1), \end{cases}$$

extended periodically to $\mathbb{R}$, *where*

$$s_i := \frac{1}{2}\, (1 + \alpha_i)\,.$$

The class $\mathcal{D}_\chi$ is just the natural generalization of the parametrized form of the deformations appearing in [36], for a barycenter $P_0 = (\alpha_1, \alpha_2, \alpha_3)$. We were unable to write down a more general expression for the deformations in the class $\mathcal{D}$, capable of yielding more extreme gradient Young measures, namely outside of the Q-segment, which we define below.

Definition 3 *We call 3-edge-laminate to any third order laminate supported on edges of the* $[-1,1]^3$ *cube, which lies on an edge of the closed convex hull of the set of all laminates.*

Definition 4 *For each fixed barycenter, the intersection of the bisector plane* $b = c$*:*

(a) *with the set of gradient Young mesures (obtained through the Riemann-Lebesgue lemma with the deformations* $u \in \mathcal{D}_\chi$*) is denoted by Q-segment;*

(b) *with the set of 3-edge-laminates is denote by R-polygon;*

(c) *with the set of polyconvex measures is denoted by P-polygon.*

Remark 4 *Notice that in the case of polyconvex measures, we can generate all such measures.*

Therefore, in trying to reach the answer "no" (to the question starting the introduction), the aim would be: to show that the extremities Q^-, Q^+ of the Q-segment could not be reached by laminates. However, for the barycenter $(0,0,0)$ such aim was frustrated in [36, proposition 4.1], showing that the measures $Q_0^- = (36, 72)$ and $Q_+^- = (36, 126)$ are indeed laminates, so that Q^- belongs to the set of laminates. The same happens with Q^+: just apply symmetry.

We proceed now to present our own work concerning the other barycenters P_0. With

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$$

one obtains the following weights on the vertices of the cube $[-1,1]^3$:

$$a \mapsto (1,1,1),\ b \mapsto (-1,1,1),\ c \mapsto (1,-1,1),\ d = 288 - a - b - c \mapsto (-1,-1,1),$$
$$\overline{a} = 640 - 3a - b - c \mapsto (1,1,-1),\ \overline{b} = 2a + c - 256 \mapsto (-1,1,-1),$$
$$\overline{c} = 2a + c - 256 \mapsto (1,-1,-1),\ \overline{d} = 160 - a \mapsto (-1,-1,-1).$$

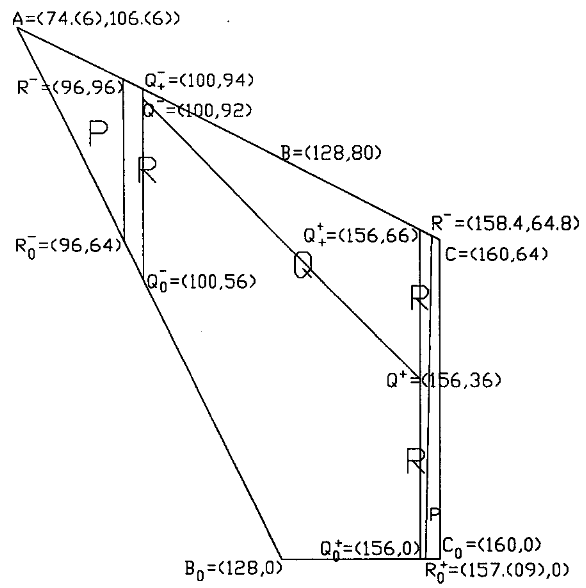

Figure 4.4: P, R-polygons and Q-segment for $P_0 = (\frac{1}{3}, \frac{1}{3}, 0)$.

Using this notation, the corresponding set of polyconvex measures has extreme points:

$$A = (74.(6), 106.(6), 106.(6)), \; B_0 = (128, 0, 0), \; B_1 = (128, 160, 0),$$
$$B_2 = (128, 0, 160), \; C_0 = (160, 0, 0), \; C_1 = (160, 128, 0), \; C_2 = (160, 0, 128)$$

(yielding again two opposite pyramids, but now a vertical plane cuts a triangular face, in the second pyramid, with vertices C_0, C_1, C_2); so that the corresponding P-polygon is the convex hull of its vertices

$$A = (74.(6), 106.(6)), B_0 = (128, 0), B = (128, 80), C_0 = (160, 0), C = (160, 64),$$

see figure 4.4. On the other hand (directly) by the Riemann-Lebesgue lemma, we were able to obtain no more than the Q-segment with extremities

$$Q^- = (100, 92), \; Q^+ = (156, 36).$$

Thus, concerning the barycenter $P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$, our aim was to show these Q^-, Q^+ to be out of reach of laminates; but it got frustrated, when we came to the conclusion (see Proposition 2) that one may indeed obtain the laminates

$$Q_0^- = (100, 56),\ Q_+^- = (100, 94),\ Q_0^+ = (156, 0),\ Q_+^+ = (156, 66),$$

hence the corresponding Q-segment.

Finally, with

$$P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right),$$

we denote the weights on the vertices of the cube $[-1, 1]^3$ by:

$$a \mapsto (1,1,1),\ b \mapsto (-1,1,1),\ c \mapsto (1,-1,1),\ d = 288 - a - b - c \mapsto (-1,-1,1),$$
$$\bar{a} = 756 - 3a - b - c \mapsto (1,1,-1),\ \bar{b} = 2a + c - 324 \mapsto (-1,1,-1),$$
$$\bar{c} = 2a + b - 324 \mapsto (1,-1,-1),\ \bar{d} = 180 - a \mapsto (-1,-1,-1).$$

The vertices of the corresponding set of polyconvex measures are then:

$$A = (120, 84, 84),\ B_0 = (162, 0, 0),\ B_1 = (162, 126, 0),\ B_2 = (162, 0, 126),$$
$$C_0 = (180, 0, 0),\ C_1 = (180, 108, 0),\ C_2 = (180, 0, 108)$$

(yielding again: two opposite pyramids with the second one cut by a vertical plane); so that the P-polygon is the convex hull of its vertices

$$A = (120, 84),\ B_0 = (162, 0),\ B = (162, 63), C_0 = (180, 0),\ C = (180, 54),$$

see figure 4.5. As to the Q-segment, it has now extremities

$$Q^- = (144, 72),\ Q^+ = (180, 36);$$

which, again, are convex combinations of the following laminates (see Proposition 2)

$$Q_0^- = (144, 36),\ Q_+^- = (144, 72),\ Q_0^+ = (180, 0),\ Q_+^+ = (180, 54).$$

4.3 Presenting the sets of points which generate Q_0^-, Q_+^-, Q_0^+ and Q_+^+

To describe these structures, we use the same notations which were used in [36]. And as there, instead of providing the required sets of pairs (verifying

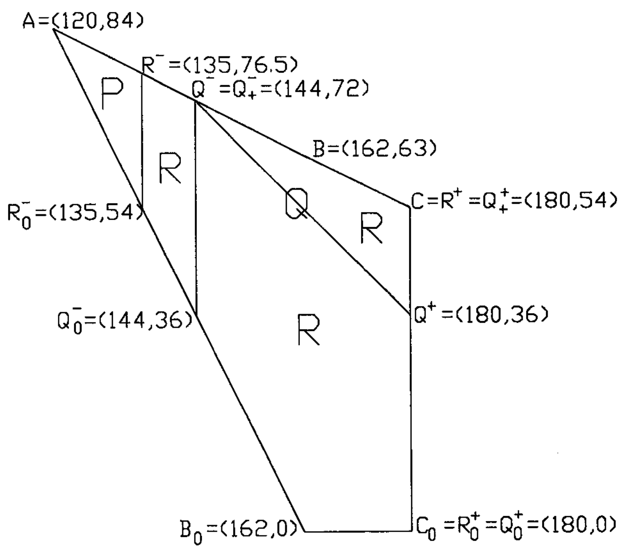

Figure 4.5: P, R-polygons and Q-segment for $P_0 = (\frac{1}{2}, \frac{1}{2}, 0)$.

some (H_k) condition) which generate a specific laminate, we provide a set of points from which one can obtain univocally the mentioned set of pairs. A set of points which gives Q_0^- for the barycenter zero is ([36])

$$P_0 = (0,0,0),\ \ P_1 = \left(-\frac{1}{2}, 1, 1\right),\ \ P_2 = \left(\frac{1}{10}, -\frac{1}{5}, -\frac{1}{5}\right),\ \ P_3 = \left(1, -\frac{5}{7}, 1\right),$$
$$P_4 = \left(-\frac{1}{11}, -\frac{1}{11}, -\frac{5}{11}\right),\ \ P_5 = (1,1,-1),\ \ P_6 = (-1,-1,0).$$

(Notice that these sets of points are not unique in general.) Each segment P_1P_2, P_3P_4, P_5P_6 has rank-one direction. This means, e.g. for P_1P_2, that if one writes $P_2 - P_1 = (x, y, z)$ then the determinant of $P_2 - P_1$, given (as one easily checks) by $xy + xz + yz$, is zero.

Starting, as explained in Section 4.2, with the weight 576 from the barycenter $P_0 = (0,0,0)$, the above set of points generates (as in (a) below) the measure $(36, 72, 72)$. Similarly one reaches the measures

$$(36, 180, 72),\ (36, 72, 180),\ (108, 0, 0),\ (108, 108, 0),\ (108, 0, 108).$$

This shows that, for this barycenter, the following points indeed belong (as

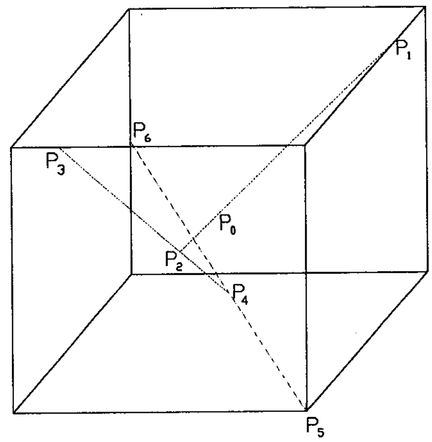

Figure 4.6: example of a laminate for $P_0 = (0, 0, 0)$.

mentioned above) to the intersection of the laminate with the bisector plane:

$$Q_0^- = (36, 72),\ Q_+^- = (36, 126),\ Q_0^+ = (108, 0),\ Q_+^+ = (108, 54).$$

This is a result of [36], included in the next

Proposition 2 *The following points belong to the R-polygon generated by starting with the weight* 576 *from the barycenter* P_0 *:*

$Q_0^- = (36, 72)$, $Q_+^- = (36, 126)$, $Q_0^+ = (108, 0)$, $Q_+^+ = (108, 54)$ *for* $P_0 = (0, 0, 0)$;

$Q_0^- = (100, 56)$, $Q_+^- = (100, 94)$, $Q_0^+ = (156, 0)$, $Q_+^+ = (156, 66)$ *for* $P_0 = (\frac{1}{3}, \frac{1}{3}, 0)$;

$Q_0^- = (144, 36)$, $Q_+^- = (144, 72)$, $Q_0^+ = (180, 0)$, $Q_+^+ = (180, 54)$ *for* $P_0 = (\frac{1}{2}, \frac{1}{2}, 0)$.

Proof. (a) The proof of this proposition consists in exhibiting, for each measure, an adequate set of points. To find such measure one finds weights p_1, p_2 adequate to have $p_1 + p_2 = p_0 := 576$ and $p_1 P_1 + p_2 P_2 = p_0 P_0$; then proceeds in a similar way until the weight $p_0 = p_1 + p_3 + p_5 + p_6$ has been thus distributed into weights p_1, p_3, p_5, p_6 on points $P_1, P_3, P_5, P_6 \in \partial[-1, 1]^3$. Finally, one distributes these weights p_1, p_3, p_5, p_6 into appropriate weights on the vertices of this cube, obtaining in particular the total weights a on $(1,1,1)$, b on $(-1,1,1)$, c on $(1,-1,1)$. The measure (a,b,c) of the laminate is thus obtained, and hence also the measure (a,c,b) and the point $(a, \frac{b+c}{2})$ of the intersection of the set of laminates with the bisector plane.

(b) If one computes in this manner the measure associated with the set of points

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \; P_1 = \left(1, 1, -\frac{1}{3}\right), \; P_2 = \left(-\frac{3}{29}, -\frac{3}{29}, \frac{19}{87}\right),$$
$$P_3 = \left(-\frac{23}{41}, 1, 1\right), \; P_4 = \left(\frac{1}{65}, -\frac{431}{1105}, \frac{1}{65}\right),$$
$$P_5 = \left(1, -\frac{15}{17}, 1\right), \; P_6 = \left(-1, \frac{2}{17}, -1\right)$$

then the result is $(a,b,c) = (100,132,56)$, hence $(a,c,b) = (100,56,132)$ and $(a, \frac{b+c}{2}) = (100,94)$.

Similarly for the other measures associated to this barycenter, as follows: the measure $(100,56,56)$, hence the point $(100,56)$, is generated by

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \; P_1 = \left(1, -\frac{1}{15}, 1\right), \; P_2 = \left(\frac{29}{157}, \frac{199}{471}, -\frac{35}{157}\right),$$
$$P_3 = \left(-\frac{11}{53}, 1, 1\right), \; P_4 = \left(\frac{25}{89}, \frac{25}{89}, -\frac{791}{1513}\right),$$
$$P_5 = \left(1, 1, \frac{15}{17}\right), \; P_6 = \left(-1, -1, \frac{2}{17}\right);$$

the measure $(156,0,0)$, hence the point $(156,0)$, is generated by

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \; P_1 = \left(\frac{19}{21}, -1, -1\right), \; P_2 = \left(\frac{347}{1293}, \frac{209}{431}, \frac{49}{431}\right),$$
$$P_3 = \left(-\frac{89}{129}, 1, -1\right), \; P_4 = \left(\frac{151}{369}, \frac{151}{369}, \frac{3787}{13653}\right),$$
$$P_5 = \left(1, 1, -\frac{2}{111}\right), \; P_6 = \left(-1, -1, \frac{109}{111}\right);$$

and the measure $(156,132,0)$, hence the point $(156,66)$, is generated by

$$P_0 = \left(\frac{1}{3},\frac{1}{3},0\right),\; P_1 = \left(-1,\frac{19}{21},-1\right),\; P_2 = \left(\frac{583}{1497},\frac{463}{1497},\frac{21}{499}\right),$$
$$P_3 = \left(\frac{32}{33},-1,-1\right),\; P_4 = \left(\frac{1}{12},1,\frac{29}{49}\right),\; P_5 = \left(1,1,\frac{29}{49}\right),$$
$$P_6 = \left(-1,1,\frac{29}{49}\right).$$

Similarly for the measures associated to the barycenter $P_0 = (\frac{1}{2},\frac{1}{2},0)$: the measure $(144,36,36)$, hence the point $(144,36)$, is generated by

$$P_0 = \left(\frac{1}{2},\frac{1}{2},0\right),\; P_1 = \left(\frac{1}{6},1,1\right),\; P_2 = \left(\frac{19}{34},\frac{7}{17},-\frac{3}{17}\right),\; P_3 = \left(1,\frac{1}{11},1\right),$$
$$P_4 = \left(\frac{9}{19},\frac{9}{19},-\frac{23}{57}\right),\; P_5 = \left(1,1,-\frac{2}{3}\right),\; P_6 = \left(-1,-1,\frac{1}{3}\right);$$

the measure $(144,36,108)$, hence the point $(144,72)$, is generated by

$$P_0 = \left(\frac{1}{2},\frac{1}{2},0\right),\; P_1 = \left(1,1,-\frac{1}{4}\right),\; P_2 = \left(0,0,\frac{1}{4}\right),\; P_3 = \left(-\frac{3}{7},1,1\right),$$
$$P_4 = \left(\frac{1}{11},-\frac{7}{33},\frac{1}{11}\right),\; P_5 = \left(-1,\frac{1}{3},-1\right),\; P_6 = \left(1,-\frac{2}{3},1\right);$$

the measure $C_0 = (180,0,0)$, hence the point $C_0 = (180,0)$, is generated by

$$P_0 = \left(\frac{1}{2},\frac{1}{2},0\right),\; P_1 = \left(1,-\frac{1}{2},-1\right),\; P_2 = \left(\frac{5}{11},\frac{13}{22},\frac{1}{11}\right),$$
$$P_3 = \left(-\frac{1}{5},1,-1\right),\; P_4 = \left(\frac{7}{13},\frac{7}{13},\frac{3}{13}\right),$$
$$P_5 = (-1,-1,1),\; P_6 = (1,1,0);$$

and the measure $C_1 = (180,108,0)$, hence the point $C = (180,54)$, is generated by

$$P_0 = \left(\frac{1}{2},\frac{1}{2},0\right),\; P_1 = \left(-\frac{1}{2},1,-1\right),\; P_2 = \left(1,\frac{1}{4},\frac{1}{2}\right),\; P_3 = \left(1,1,\frac{1}{2}\right),$$
$$P_4 = \left(1,-1,\frac{1}{2}\right).$$

As to the barycenter zero, see the paragraph before the statement of the proposition. ∎

The reader should be aware of the fact that what is difficult here is not to prove Proposition 2, but to state it.

Notice also that all the sets of points presented in this paper have all their odd points (i.e. P_1, P_3, P_5, P_6) on edges of the $[-1,1]^3$ cube: this makes sense since what matters is to find extreme laminates.

4.4 The characterization of the sets of polyconvex measures

We want to determine the set of polyconvex measures supported on the vertices of the cube $[-1,1]^3$, with barycenter $P_0 = (\alpha_1\,,\ \alpha_2\,,\ \alpha_3)$ (for simplicity we again take the total weight$= p^2$, instead of 1). Denote again by a, b, c, d the weights thus generated on the four upper vertices,

$$(1,1,1),\ (-1,1,1),\ (1,-1,1),\ (-1,-1,1);$$

and by $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ the weights generated on its four lower vertices,

$$(1,1,-1),\ (-1,1,-1),\ (1,-1,-1),\ (-1,-1,-1).$$

Defining now the parameters

$$s_i := \frac{1}{2}\ (1+\alpha_i)\,,$$

$$\gamma := p^2\ \left[\left(s_1 - \frac{1}{2}\right)\ \left(s_2 - \frac{1}{2}\right) +\ s_1 s_3\ +\ s_2 s_3\right],$$

one easily checks that the set of possible weights associated to polyconvexity can be thus represented as the set in (a,b,c)-space described by the restrictions

$$a\ \geq\ 0\,, b\ \geq\ 0\,, c\ \geq\ 0\,,$$

$$d := p^2 s_3 - a - b - c \geq 0$$

$$\overline{a} := \gamma + \frac{p^2}{2}\ \left(s_1 + s_2 - \frac{1}{2}\right) - 3a\ - b\ - c\ \geq 0$$

$$\overline{b} := -\gamma + \frac{p^2}{2}\ \left(\frac{1}{2} - s_1 + s_2\right) + 2a\ + c\ \geq 0$$

$$\overline{c} := -\gamma + \frac{p^2}{2} \left(\frac{1}{2} + s_1 - s_2\right) + 2a \ + b \ \geq 0$$

$$\overline{d} := \gamma - \frac{p^2}{2} \left(s_1 + s_2 + 2\, s_3 - \frac{3}{2}\right) - a \ \geq 0.$$

As we saw in the introduction (for each fixed barycenter) the set of polyconvex measures contains both the set of gradient Young measures and the set of laminates. Consequently, the P-polygon contains both the Q-segment and the R-polygon.

4.5 The characterization of the gradient Young measures generated by $\mathcal{D}_\chi$ deformations

Each deformation $u(\cdot) \in \mathcal{D}_\chi$ generates (as described in the proof below) weights a, b, c, d on the 4 upper vertices of the cube $[-1, 1]^3$

$$(1,1,1),\ (-1,1,1),\ (1,-1,1),\ (-1,-1,1)$$

and weights $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ on the 4 lower vertices,

$$(1,1,-,1),\ (-1,1,-1),\ (1,-1,-1),\ (-1,-1,-1).$$

Or, in other words, each such deformation $u(\cdot)$ generates a gradient Young measure with barycenter P_0, represented by the triple (a, b, c), consisting of the weights generated on the first 3 of these vertices; which may be compared with laminates (a, b, c) having weights a, b, c generated, on the vertices

$$(1,1,1),\ (-1,1,1),\ (1,-1,1),$$

by sets of points contained in the cube $[-1, 1]^3$ and having barycenter P_0. In particular, if one fixes $P_0 := (\alpha, \alpha, 0)$, with $\alpha = 0, \frac{1}{3}, \frac{1}{2}$, then examples of such sets of points appear in the proof of Proposition 2.

In the next proposition we consider gradient Young measures not as probability measures but as measures having total mass$= p^2$. This is convenient to avoid many cumbersome fractions when treating concrete examples, as above with $p = 24$.

Proposition 3 *The gradient Young measures generated by deformations $u(\cdot) \in \mathcal{D}_\chi$ are all the points of a segment, namely the convex hull of its extremities Q^-, Q^+:*

$$Q^- := (a^-, d_2 - a^-, d_3 - a^-), \qquad Q^+ := (a^+, d_2 - a^+, d_3 - a^+),$$

with

$$d_2 := p^2 \; s_2 \; s_3 \,, \qquad d_3 := p^2 \; s_1 \; s_3 \,, \qquad s_i := \frac{1}{2} \; (1 + \alpha_i) \,,$$

$$a^- := p^2 \left(\frac{[s_1 + s_2 + s_3 - 1]^+}{2} \right)^2 , \; a^+ := p^2 \left[s_1 \; s_2 - \left(\frac{[s_1 + s_2 - s_3]^+}{2} \right)^2 \right],$$

where

$$[x]^+ := \max\{0, x\}.$$

Proof. To compute the weights a, b, c, d and $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ generated by this general deformation $u(\cdot)$, one has to compute the areas of the corresponding regions (denoted by the same letters $a, b, c, \ldots$) determined on the square $[0, p]^2$ by the lines

$$x = p \; \delta_1, \qquad x = p \; \delta_1 + p \; s_1, \qquad y = p \; \delta_2, \qquad y = p \; \delta_2 + p \; s_2,$$

$$x + y = p \; \delta_3, \; x + y = p \; \delta_3 + p \; s_3, \; x + y = p + p \; \delta_3, \; x + y = p + p \; \delta_3 + p \; s_3.$$

One easily checks, geometrically, that

$$\overline{a} = p^2 \; s_1 \; s_2 \; - a, \qquad \overline{b} = p^2 \, (1 - s_1) \, s_2 \; - b, \qquad \overline{c} = p^2 \; s_1 \, (1 - s_2) - c$$

and

$$d = p^2 \; s_3 - a - b - c, \qquad \overline{d} = p^2 \; [(1 - s_1) \; (1 - s_2) - s_3] + a + b + c.$$

On the other hand, we must have

$$b = p^2 \; s_2 \; s_3 - a, \qquad c = p^2 \; s_1 \; s_3 - a.$$

In this way one expresses the coordinates $b, c, d, \ldots$ as affine functions of a (dependent on the chosen P_0). Therefore the gradient Young measures generated by deformations $u(\cdot) \in \mathcal{D}_\chi$ form a segment; and to characterize it and thus end the proof, we only need to obtain its extreme values. But these are obtained by plugging in the extreme values a^-, a^+ of a, whose expressions are those stated above. ■

Remark 5 *The results of the sections 4.4, 4.5 can be extended from the cube* $[-1,1]^3$ *to a rectangular parallelepiped*

$$[-A_1, A_1] \times [-A_2, A_2] \times [-A_3, A_3],$$

where $A_1, A_2, A_3 \in (0, +\infty)$.

4.6 The characterization of the 3-edge-laminates

One easily checks that the three P-polygons considered in Section 4.2 all have the same form, their only difference being that the vertices C_0, C collapse, in the case of the barycenter zero, into the unique vertex C. (One may also observe the following: for the other 2 barycenters, if one extends the edges B_0C_0, BC then they meet at the point $(288, 0)$ which is, however, out of reach for the polyconvex measures.

Proposition 4 *For each one of the above* 3 *barycenters, the points* B_0, B *of the P-polygon always belong to the corresponding R-polygon.*

We leave the proof of this proposition to the interested reader; it is similar to the proof of Proposition 2, but here it involves only the discovery of three first order laminates and two second order laminates.

But the main aim of this section is the determination - for the intersection of the bisector plane with each one of the three sets of laminates - of the extreme points R_0^-, R^- along the edges E_0^-, E^- of the P-polygon (i.e. those joining the vertices B_0A, BA); and of the extreme points R_0^+, R^+ along the edges E_0^+, E^+ (i.e. B_0C_0, BC assuming, in case $P_0 = (0,0,0)$, $C_0 := C$).

Then what we do below is the determination of the extreme points R_0^-, R_1^-, R_2^- (respectively R_0^+, R_1^+, R_2^+) along the edges E_0^-, E_1^-, E_2^- (respectively E_0^+, E_1^+, E_2^+) of the convex hull of the set of all 3-edge-laminates. We believe these are all the extreme points of the set of general laminates, together with B_0, B_1, B_2; but were unable to prove it.

What is remarkable here is that, for some barycenters, the Q-segment is entirely contained in the interior of the corresponding R-polygon, hence does not reach at least its boundary, as one would expect. This is what happens for the barycenter $\left(\frac{1}{3}, \frac{1}{3}, 0\right)$; while for $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ the lower value of the coordinate a along the laminate is strictly smaller than its lower value along the Q-segment. This situation is unfortunate for the search of counterexamples, but we were unable to improve it, as was remarked in Section 4.2.

Theorem 6 *The extreme points of the intersection of the bisector plane with the convex hull of the set of 3-edge-laminates are, besides B_0, B:*

for the barycenter $P_0 = (0,0,0)$,

$R_0^- = (36,72)$, $\quad R^- = (36,126)$, $\quad R_0^+ = (108,0)$, $\quad R^+ = (108,54)$;

for the barycenter $P_0 = \left(\frac{1}{3},\frac{1}{3},0\right)$,

$R_0^- = (96,64)$, $\quad R^- = (96,96)$, $\quad R_0^+ = (157.(09),0)$, $\quad R^+ = (158.4,64.8)$;

and, for the barycenter $P_0 = \left(\frac{1}{2},\frac{1}{2},0\right)$,

$R_0^- = (135,54)$, $R^- = (135,76.5)$, $R_0^+ = C_0 = (180,0)$, $R^+ = C = (180,54)$.

Proof.

(a) For the barycenter $P_0 = (0,0,0)$ the points R_0^-, R^-, R_0^+, R^+ are generated as listed in the paragraph before Proposition 2.

For the barycenter $P_0 = \left(\frac{1}{3},\frac{1}{3},0\right)$ the measure $R_0^- = (96,64,64)$, hence the point $R_0^- = (96,64)$, is generated by

$$P_0 = \left(\frac{1}{3},\frac{1}{3},0\right), \; P_1 = \left(-\frac{1}{15},1,1\right), \; P_2 = \left(\frac{25}{57},\frac{3}{19},-\frac{5}{19}\right),$$
$$P_3 = \left(1,-\frac{3}{13},1\right), \; P_4 = \left(\frac{3}{11},\frac{3}{11},-\frac{7}{11}\right),$$
$$P_5 = (-1,-1,0), \; P_6 = (1,1,-1);$$

while $R_1^- = (96,128,64)$, hence $R_2^- = (96,64,128)$ and the point $R^- = (96,96)$, is generated by

$$P_0 = \left(\frac{1}{3},\frac{1}{3},0\right), \; P_1 = \left(1,-\frac{1}{15},1\right), \; P_2 = \left(\frac{3}{19},\frac{25}{57},-\frac{5}{19}\right),$$
$$P_3 = \left(1,1,-\frac{3}{5}\right), \; P_4 = \left(-\frac{1}{2},0,0\right),$$
$$P_5 = (0,-1,-1), \; P_6 = (-1,1,1).$$

Still for the barycenter $P_0 = \left(\frac{1}{3},\frac{1}{3},0\right)$, the measure $R_0^+ = (157.(09),0,0)$, hence the point $R_0^+ = (157.(09),0)$, is generated by

$$P_0 = \left(\frac{1}{3},\frac{1}{3},0\right), \; P_1 = \left(\frac{19}{21},-1,-1\right), \; P_2 = \left(\frac{47}{117},\frac{29}{59},\frac{7}{59}\right),$$
$$P_3 = \left(-\frac{2}{3},1,-1\right), \; P_4 = \left(\frac{7}{17},\frac{7}{17},\frac{5}{17}\right),$$
$$P_5 = (1,1,0), \; P_6 = (-1,-1,1);$$

while $R_1^+ = (158.4, 129.6, 0)$, hence $R_2^+ = (158.4, 0, 129.6)$ and the point $R^+ = (158.4, 64.8)$, is generated by

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \; P_1 = \left(\frac{19}{21}, -1, -1\right), \; P_2 = \left(\frac{101}{339}, \frac{47}{113}, \frac{7}{113}\right),$$
$$P_3 = (-1, 1, -1), \; P_4 = \left(\frac{139}{301}, \frac{103}{301}, \frac{59}{301}\right),$$
$$P_5 = \left(1, -\frac{7}{11}, -1\right), \; P_6 = \left(\frac{1}{10}, 1, 1\right).$$

For the barycenter $P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ the measure $R_0^- = (135, 54, 54)$, hence the point $R_0^- = (135, 54)$, is generated by

$$P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \; P_1 = \left(\frac{1}{6}, 1, 1\right), \; P_2 = \left(\frac{37}{62}, \frac{11}{31}, -\frac{9}{31}\right), \; P_3 = \left(1, \frac{1}{21}, 1\right),$$
$$P_4 = \left(\frac{17}{37}, \frac{17}{37}, -\frac{27}{37}\right), \; P_5 = (-1, -1, 0), \; P_6 = (1, 1, -1);$$

while $R_1^- = (135, 99, 54)$, hence $R_2^- = (135, 54, 99)$ and the point $R^- = (135, 76.5)$, is generated by

$$P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \; P_1 = \left(1, 1, -\frac{1}{4}\right), \; P_2 = \left(-\frac{1}{9}, -\frac{1}{9}, \frac{11}{36}\right), P_3 = \left(-\frac{7}{13}, 1, 1\right),$$
$$P_4 = \left(\frac{1}{21}, -\frac{11}{21}, \frac{1}{21}\right), \; P_5 = (-1, 0, -1), \; P_6 = (1, -1, 1).$$

Finally, for the barycenter $P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$, the measures $R_0^+ = C_0$, $R^+ = C$ are generated as indicated in the proof of Proposition 2, see figure 4.5.

(b) It remains only to show that these measures are extreme, in the sense explained just before the statement of this theorem.

Consider the barycenter $P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$. One wishes to show that the measure $(96, 128, 64)$ is R_1^-, namely the extreme point along the segment which is the convex hull of $(74 + \frac{2}{3}, 106 + \frac{2}{3}, 106 + \frac{2}{3})$ and $(128, 160, 0)$ (i.e. along the edge E_1^- of the corresponding set of polyconvex measures).

Parametrize the part of this edge E_1^- having $a < 96$:

$$(a, b, c) = (a, 32 + a, 256 - 2\,a), \qquad a \in \left[74 + \frac{2}{3}, 96\right).$$

For each a, the weights obtained on the remaining vertices of the cube are:

$$d = 0 \mapsto (-1, -1, 1), \; \overline{a} = 352 - 2a \mapsto (1, 1, -1), \; \overline{b} = 0 \mapsto (-1, 1, -1),$$
$$\overline{c} = 3a - 224 \mapsto (1, -1, -1), \; \overline{d} = 160 - a \mapsto (-1, -1, -1).$$

Our aim is to show that there exists no set of points generating weights $(a, b, c) \in E_1^-$ having $a \in [74 + \frac{2}{3}, 96)$. We begin by choosing the edges of the cube upon which one could place each one of the 4 points P_1, P_3, P_5, P_6. Since $d = 0 = \bar{b}$, only the edges $S_{ab}, S_{a\bar{a}}, S_{c\bar{c}}, S_{\bar{a}\ \bar{c}}, S_{\bar{c}\bar{d}}$ may be used. (Here S_{ab}, say, is the edge of the cube which joins the vertices holding weights a, b; i.e. $S_{ab} = co\ \{(1,1,1), (-1,1,1)\}$.)

We begin by choosing an edge to hold P_1, so that P_0P_1 is rank-one. We have two possibilities:

$$- \text{ either } (b_1) \quad (P_1 \in S_{ab} \text{ or } P_1 \in S_{\bar{c}\bar{d}})\,;$$

$$- \text{ or else } (b_2) \quad (P_1 \in S_{a\bar{a}} \text{ or } P_1 \in S_{ac})\,.$$

Then it suffices to convince oneself that none of these choices works, by exploring wisely all the available possibilities. Indeed, each one of them leads to a situation in which one of the restrictions to apply simply turns out to be impossible to satisfy.

For the other edges, one shows similarly that the extreme points on the edges are the ones shown in part (a) above. ■

4.7 A computational attempt to characterize the 3 sets of laminates

After having computed the above extreme values, the following question comes naturally to one's mind: are the vertical segments $S^- := [R_0^-, R^-]$, $S^+ := [R_0^+, R^+]$ extreme in the intersection of the bisector plane with the corresponding set of laminates, in each case? (Or, more precisely, considering the 3-dimensional picture and using the same notation as in the proof of Theorem 6: are the vertical triangles $T^- := co\{R_0^-, R_1^-, R_2^-\}$ and $T^+ := co\{R_0^+, R_1^+, R_2^+\}$ extreme faces of the laminate?) If one could ensure this, then the intersection of the set of laminates with the bisector plane, in each case, would become completely characterized as the convex hull of the 3 vertical segments S^-, S^+ and $S := \{B_0, B\}$. (Or, in the 3-dimensional picture: then each set of laminates would be exactly the convex hull of the 3 vertical triangles T^-, T^+ and $T := co\{B_0, B_1, B_2\}$, see figure 4.2.)

To show the plausibility of this conjecture, we have tried to characterize the extreme values of the first coordinate a (the weight on the vertex $(1,1,1)$ of the $[-1,1]^3$ cube), in each one of the above laminates, independently of the weights on the other vertices of the $[-1,1]^3$ cube. Or, in other words, to find the extreme values of the coordinate a, regardless of restricting attention to edges of the corresponding set of laminates. To avoid any bias coming from wishful thinking, we have constructed (in a personal computer) exact samples of all the possible third (at most) order laminates. Since all the corresponding sets of points constructed here have all their odd points on edges of the $[-1,1]^3$ cube, in trying to construct a third order laminate starting from one of the chosen barycenters ($(a,a,0)$ with $a=0$, $a=\frac{1}{3}$, $a=\frac{1}{2}$), the choices one has to make, concerning each odd point (namely P_1, P_3, P_5 or P_6) lead to less than a dozen possibilities. On the contrary, concerning each even point (i.e. P_2 or P_4) the possibilities are, instead, all the points of a straight-line segment, which we call an even segment; and our strategy has been to divide each such segment into $n=100$ pieces, all with equal length. In this way we have generated blindly many hundreds of thousands of different third order laminates for each barycenter.

(Notice: the word "exact" is used above in the following sense: the coordinates are represented as quotients of integers with 16 decimal digits. Thus the sets of points we have generated have exact coordinates and exact weight-distributions, hence yield exact - i.e. not approximate - points of the corresponding laminates. In a first attempt we have represented all the coordinates, of the sets of points in our computer, by 64-bit double-precision real numbers. However, since errors tended to accumulate in an explosive way, we have soon shifted towards an exact representation of coordinates as quotients of integers with 16 decimal digits. Thus the sets of points we have generated have exact coordinates and exact weight-distributions, hence yield exact - i.e. not approximate - points of the corresponding laminates.)

One might also wonder whether by using fourth order laminates it would be possible to obtain a more extreme value of a, namely a value not reachable with third order laminates only. In order to try and discard such possibility we have also generated fourth order laminates on the computer. But since we have, in this case, 3 even segments instead of 2, we had to reduce the number n of divisions from 100 to just 30, due to memory limitations.

The computations thus performed tend to indicate that it is sufficient to consider third order laminates. However, even with such a small n we have run into problems, frequently, due to an explosive propagation of errors. Indeed, in many cases, when computing the last points P_7, P_8 of the set of points corresponding to a fourth order laminate, the integers involved exceeded the largest integer available (even if we have been careful to cancel out all common factors in the numerator and denominator of all fractions representing all points and weights). In many other cases, such excess occurred not in the computation of P_7, P_8, which were still exact, but in the computation of the corresponding weights p_7, p_8. We have also tried to compute P_7, P_8, p_7, p_8 using approximate double-precision real numbers (instead of exact quotients of integers), starting from the exact values of the other points, P_0 to P_6; but again errors tended to accumulate explosively.

The conclusions we have reached from all these computations simply confirmed all the expectations we had from the start, coming from our other method described above. That is: these computational experiments just reinforced our confidence on the validity of the conclusions of Section 4.2.

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