Nonlinear Hyperbolicity

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Matemática e Física de Aerossóis
UÉvora, May 23, 2013

Outline

Abstract: In this presentation we attempt to stress two points of view on hyperbolic conservation laws: *modelization* and *analytical theory*. And, how they are sensitively related. While appliers are concerned with reliability, integrity or failure of solutions, mathematicians are concerned with non uniqueness, selection of physically relevant solutions or entropy criteria.
In the modeling process, within simplifications, some "spurious terms" are usually discarded from the equations and so, in order to address uniqueness, a crucial information is lost. We discuss here the relevant dissipative or dispersive effect of some of those small scale terms (zero singular limits).

Keywords: hyperbolic conservation law; shock wave; entropy weak solution; measure-valued solution; dissipation; dispersion; diffusion; capilllarity; Burgers equation; KdV-type equation

Hyperbolicity & Real World

$$w_{tt} - (\sigma(w_x))_x = 0\,, \qquad \text{(nonlin. wave eq.)}$$

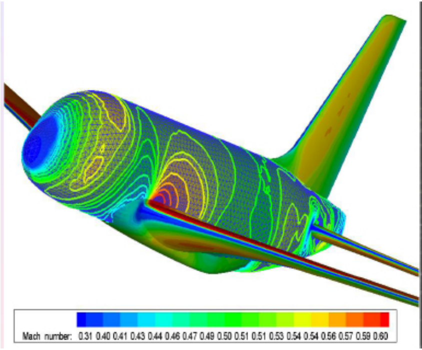
$u := w_t\,,$
$v := w_x\,,$
$p(v) := -\sigma(v)\,,$

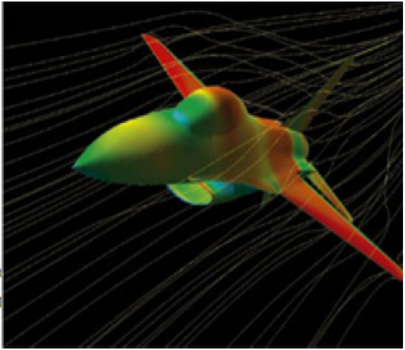
$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = 0 \end{cases} \iff \begin{bmatrix} v \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ p'(v) & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}_x = 0 \qquad (p\text{-system})$$

Real eigenvalues? $\lambda_1 := -\sqrt{-p'(v)} < \lambda_2 := \sqrt{-p'(v)}$ (speed...)

Nonlinear World:

Discontinuities. . .

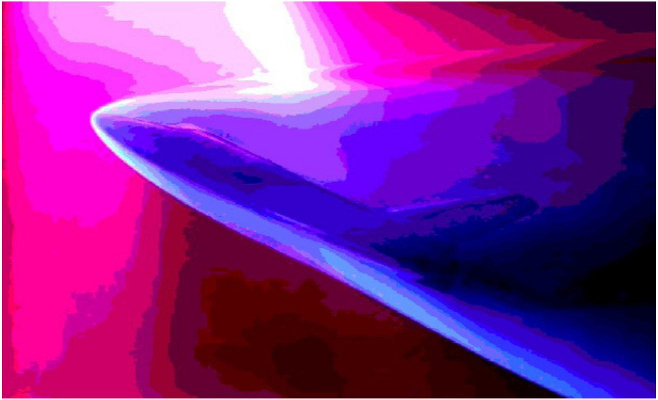




The transonic regime issues:

- control of vibrations and
- shocks strength magnitude

. . . & Irreversibility



NASA: Visible shocks at the nose in the windtunnel test

Conservation Laws:

$$\partial_t u + \mathrm{div}_{\vec{x}} \left(\overbrace{f(u) - \underbrace{\varepsilon\, b(u, \nabla u) - \delta\, \partial_{\xi}\, c(u, \nabla u)}_{\text{pertubation} \quad \mathcal{P}_{\varepsilon,\delta}(u; f, b, c)}}^{\text{transport + viscosity + capillarity}} \right) = 0$$

- 'hyperbolic': finite speed of propagation;
- 'divergence form': via modelization of "physical closed systems",
- sources; anisotropy; $\xi \in \{t, x_1, \ldots, x_n\}$:
  - $\xi = t$ (the time): **gBBM-Burgers**;
  - $\xi = x_k$ (one space variable): **gKdV-Burgers**.

Nonlinear Hyperbolic Conservation Laws

- Same simplified equation:

$$\partial_t u + \mathrm{div}_{\vec{x}} f(u) = 0,$$

if we consider the

  - ε,b-viscosity (with diffusive, dissipative effect),
  - δ,c-capillarity (with oscillatory, dispersive effect),

as a

  - "spurious small scale mechanisms",

or at

  - the formal "zero viscosity-capillarity limit" ($\varepsilon, \delta \to 0$):

Singular Limits

N.B. Well-posedness of the (time-evolution) Cauchy problem means that this equation must be hyperbolic and, because it is nonlinear, it develops discontinuities ("shocks") in finite time: the solutions are not unique.

So: how can we select the physically relevant solution?

As the ε,δ-parameters tend to zero and according to the balance of ε,δ-strengths and the growth ratio of *b*-viscosity and *c*-capillarity, we can have:

- *classical-entropy* solutions;
- *nonclassical-entropy* solutions;
- no limit at all.

And, we proved a 15 years old conjecture: some "pure capillarity" ($\varepsilon \equiv 0$ or KdV-like) equations converge (have a dissipative behaviour).

Paradox

What are the “spurious” $b, c = ???$

A+

Mathematical issues concern:

- the behaviour and selection of the right models/solutions;
- the proof of a “vanishing viscosity-capillarity method”.

Physical issues concern:

- Suggestions ?...

Obrigado!

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