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Extremal solutions to fourth order discontinuous functional boundary value problems

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In this paper, given $f : [a, b] \times C([a, b]) \times \mathbf{R}^2 \to \mathbf{R}$ a L^1-Carathéodory function, it is considered the functional fourth order equation

$$u^{(iv)}(x) = f\big(x, u, u''(x), u'''(x)\big), \quad \text{a.a.} \quad x \in [a, b],$$

together with the nonlinear functional boundary conditions

$$\begin{aligned} L_0\big(u, u'', u(a)\big) &= 0, \\ L_1\big(u, u'', u(b)\big) &= 0, \\ L_2\big(u, u'', u''(a), u'''(a)\big) &= 0, \\ L_3\big(u, u''(b), u'''(b)\big) &= 0. \end{aligned}$$

Here L_i, $i = 0, 1, 2, 3$, satisfy some adequate monotonicity assumptions and are not necessarily continuous functions. It will be proved an existence and location result in presence of non ordered lower and upper solutions.

1 Introduction

In this paper it is considered the following functional fourth order boundary value problem composed by the equation

$$u^{(iv)}(x) = f\big(x, u, u''(x), u'''(x)\big) \tag{1.1}$$

for a.a. $x \in [a, b]$, where $f : [a, b] \times C([a, b]) \times \mathbf{R}^2 \to \mathbf{R}$ is a L^1-Carathéodory function, and the boundary conditions

$$\left.\begin{aligned} L_0\big(u, u'', u(a)\big) &= 0, \\ L_1\big(u, u'', u(b)\big) &= 0, \\ L_2\big(u, u'', u''(a), u'''(a)\big) &= 0, \\ L_3\big(u, u''(b), u'''(b)\big) &= 0, \end{aligned}\right\} \tag{1.2}$$

where L_i, $i = 0, \ldots, 3$, satisfy some adequate conditions and are allowed to be discontinuous on some of their variables.

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The existence of extremal solutions has been studied for several types of problems and in different fields, as it can be seen, for instance, in [1], [2], [4], [6], [7], [9], [11], [13], [16]–[19]. Functional boundary value problems include a large number of differential equations and many types of boundary conditions, as it is discriminated and exemplified in [5], [8], [14]. The functional dependence in the differential equation along with such general boundary conditions, allow to study the existence of maximal and minimal solutions even to classic problems such as Lidstone, multipoint, nonlocal, . . . , coupled with fourth-order integro-differential equations, with delay, advanced, maximum or minimum arguments, . . . For these and others fourth order boundary value problems, as well as for a huge variety of applications in beam theory and suspension bridges, among others, there were not, to the best of our knowledge, results to obtain the existence of extremal solutions allowing the functional dependence in every boundary conditions.

A key point in this work is a second order auxiliary problem, obtained from (1.1)–(1.2) by a reduction of order, where it is applied a standard Nagumo condition and a previous result, from [3], to have the existence of extremal solutions.

The fourth order problem is studied by adding to the previous problem two algebraic equations, to which it applies an advanced version of the Bolzano's theorem, given in [10]. An important tool is a non-ordered lower and upper solutions technique, meaning that lower and upper solutions have not to be necessarily ordered, to define a convenient integral operator, which has a least and a greatest fixed points, as it is given in [12].

The paper is developed as follows: in Section 2 we introduce the main concepts that we will use, in Section 3, to deduce the existence of extremal solutions of problem (1.1)–(1.2). We finalize the paper in Section 4 with an example in which we apply the obtained results.

2 Definitions and auxiliary results

In this section it will be introduced the notations and definitions needed forward together with some auxiliary functions and results useful to obtain the main result.

In the following, $W^{m,1}([a,b])$ denotes the usual Sobolev Spaces in $[a,b]$, that is, the subset of $\mathcal{C}^{m-1}([a,b])$ functions, whose $(m-1)^{th}$ derivative is absolutely continuous in $[a,b]$ and the m^{th} derivative belongs to $L^1([a,b])$.

Throughout this paper the following hypothesis will be assumed:

(H1) $f:[a,b]\times\mathcal{C}([a,b])\times\mathbf{R}^2\to\mathbf{R}$ is such that for every $u\in\mathcal{C}([a,b])$, the function $f_u:[a,b]\times\mathbf{R}^2\to\mathbf{R}$ defined as $f_u(x,y,z):=f(x,u,y,z)$ is a L^1-Carathéodory function, that is, $f_u(x,\cdot,\cdot)$ is a continuous function for a.a. $x\in[a,b]$; $f_u(\cdot,y,z)$ is measurable for $(y,z)\in\mathbf{R}^2$; and for every $M>0$ there is a real-valued function $\psi_M\in L^1([a,b])$ such that

$$\left|f_u(x,y,z)\right|\le\psi_M(x),\quad \text{for a.a.}\quad x\in[a,b]$$

and for every $(y,z)\in\mathbf{R}^2$ with $|y|\le M$, $|z|\le M$.

(H2) $L_0, L_1:(\mathcal{C}([a,b]))^2\times\mathbf{R}\to\mathbf{R}$ are nonincreasing in the first variable and nondecreasing in the second one.

(H3) $L_2:(\mathcal{C}([a,b]))^2\times\mathbf{R}^2\to\mathbf{R}$ is non increasing in the first variable and nondecreasing in the second and fourth variables. Moreover, for every $u\in C(I)$ given, $L_2(u,v_n,x_n,y_n)\to L_2(u,v,x,y)$ whenever $\{v_n\}\to v$ in $C(I)$ and $\{(x_n,y_n)\}\to(x,y)$ in $\mathbf{R}^2$.

(H4) $L_3:\mathcal{C}([a,b])\times\mathbf{R}^2\to\mathbf{R}$ is nondecreasing in the first and third variables. Moreover, for every $u\in C(I)$ given, $L_3(u,x_n,y_n)\to L_3(u,x,y)$ whenever $\{(x_n,y_n)\}\to(x,y)$ in $\mathbf{R}^2$.

Remark 2.1 Notice that some continuities are allowed in the two first variables of the function f, in the first variable of the functions L_2 and L_3 and in all the variables of L_0 and L_1.

The preliminary results are related to some second order boundary value problems for which it will be assumed that the conditions $(H1)$ and $(H2)$ hold.

Let $v\in W^{4,1}([a,b])$ be a fixed function and denote by (P_v) the problem composed by the equation

$$y''(x)=f_v\big(x,y(x),y'(x)\big)\equiv f\big(x,v,y(x),y'(x)\big),\quad \text{a.a.}\quad x\in[a,b],\tag{2.1}$$

and the boundary conditions

$$L_2\big(v, y, y(a), y'(a)\big) = 0, \quad L_3\big(v, y(b), y'(b)\big) = 0. \tag{2.2}$$

Definition 2.2 A function $y_v \in W^{2,1}([a, b])$ is a *solution of problem* (P_v) if it satisfies conditions (2.1) and (2.2).

For this second-order auxiliary problem we define as lower and upper solutions the functions that verify the following conditions:

Definition 2.3 A function $\zeta : [a, b] \to \mathbf{R}$, $\zeta \in W^{2,1}([a, b])$, is said to be a *lower solution of problem* (P_v) if:

(i) $\zeta''(x) \geq f\big(x, v, \zeta(x), \zeta'(x)\big)$, a.a. $x \in [a, b]$;
(ii) $L_2\big(v, \zeta, \zeta(a), \zeta'(a)\big) \geq 0$ and $L_3\big(v, \zeta(b), \zeta'(b)\big) \leq 0$.
A function $\eta \in W^{2,1}([a, b])$ is said to be an *upper solution to the problem* (P_v) if the reversed inequalities hold.

A Nagumo-type growth condition, assumed on the nonlinear part, will be an important tool to set a priori bounds for solutions of some differential equations.

Definition 2.4 Consider $\Gamma, \gamma \in L^1([a, b])$, such that, $\Gamma(x) \geq \gamma(x), \forall x \in [a, b]$, and the set

$$E = \left\{(x, y_0, y_1) \in [a, b] \times \mathbf{R}^2 : \gamma(x) \leq y_0 \leq \Gamma(x)\right\}.$$

A function $f : [a, b] \times C[a, b] \times \mathbf{R}^2 \to \mathbf{R}$ is said to *verify a Nagumo-type condition in* E if there exists $\varphi \in C\left([0, +\infty), (0, +\infty)\right)$ such that

$$|f_v(x, y_0, y_1)| \leq \varphi(|y_1|),$$

for every $(x, y_0, y_1) \in E$ and all $v \in [\gamma, \Gamma]$, with

$$\int_r^{+\infty} \frac{s}{\varphi(s)}\, ds > \max_{x\in[a,b]} \Gamma(x) - \min_{x\in[a,b]} \gamma(x).$$

Here r is given by

$$r := \max\left\{\frac{\Gamma(b) - \gamma(a)}{b - a}, \frac{\Gamma(a) - \gamma(b)}{b - a}\right\}.$$

Standard arguments (see for example [15]) give us the following a priori estimation on the first derivative of the solutions of problem (P_v).

Lemma 2.5 *There exists $R > 0$, depending only on φ, γ and Γ, such that for every L^1-Carathéodory function $f : I \times C([a, b]) \times \mathbf{R}^2 \to \mathbf{R}$ satisfying a Nagumo-type condition in E, and every solution y_v of* (2.1) *such that*

$$\gamma(x) \leq y_v(x) \leq \Gamma(x), \quad \forall\, x \in I,$$

we have that $\left\|y_v'\right\| < R$.

Considering (P_v) as a particular case of problem (3.1)–(3.3) presented in [3], we deduce the following result as a consequence of [3, Theorem 3.2].

Theorem 2.6 *Assume that the assumptions $(H1)$, $(H3)$ and $(H4)$ hold.*
If there are lower and upper solutions of (P_v) such that $\gamma \leq \delta$ and f satisfies a Nagumo-type growth condition in

$$E_{\gamma\delta} = \left\{(x, y_0, y_1) \in [a, b] \times \mathbf{R}^2 : \gamma(x) \leq y_0 \leq \delta(x)\right\},$$

then (P_v) has extremal solutions in $[\gamma, \delta]$.

In the proof of the main result of this paper, it will be applied the following version of the Bolzano's Theorem:

Lemma 2.7 ([10, Lemma 2.3]) *Let $c, d \in$ R, $c \le d$, and $h : \mathbf{R} \to \mathbf{R}$ be such that either $h(c) \ge 0 \ge h(d)$ and*

$$\limsup_{z\to x^-} h(z) \le h(x) \le \liminf_{z\to x^+} h(z), \quad \textit{for all} \quad x \in [c, d],$$

or $h(c) \le 0 \le h(d)$ and

$$\liminf_{z\to x^-} h(z) \ge h(x) \ge \limsup_{z\to x^+} h(z), \quad \textit{for all} \quad x \in [c, d].$$

Then there exist $c_1, c_2 \in [c, d]$ such that $h(c_1) = 0 = h(c_2)$ and if $h(c_3) = 0$ for some $c_3 \in [c, d]$ then $c_1 \le c_3 \le c_2$, i.e., c_1 and c_2 are, respectively, the least and the greatest of the zeros of h in $[c, d]$.

The following fixed point theorem will also be needed in the proof of the main result.

Lemma 2.8 ([12, Theorem 1.2.2]) *Let Y be a subset of an ordered metric space $(X, \le)$, $[p, q]$ a nonempty ordered interval in Y, and $T : [p, q] \to [p, q]$ a nondecreasing mapping. If $\{Tx_n\}$ converges in Y whenever $\{x_n\}$ is a monotone sequence in $[p, q]$, then there exists x_* the least fixed point of T in $[p, q]$ and x^* is the greatest one. Moreover*

$$x_* = \min\{y \mid Ty \le y\} \quad \textit{and} \quad x^* = \max\{y \mid Ty \ge y\}.$$

3 Extremal solutions to fourth-order problem

This section is devoted to prove the existence of extremal solutions of the problem (1.1)–(1.2). To this end we use the lower and upper solutions technique. In this case we consider the non-ordered case, that is, lower and upper solution do not need to be ordered.

In fact, we apply some auxiliary functions "to get some order".

For $\alpha, \beta \in W^{2,1}([a, b])$, with $\alpha'' \le \beta''$ a.a. on $[a, b]$, we define the functions $\alpha_0, \beta_0 : [a, b] \to \mathbf{R}$ by

$$\beta_0(x) = A_0\frac{b-x}{b-a} + A_1\frac{x-a}{b-a} + \int_a^b G(x, s)\beta''(x)ds, \tag{3.1}$$

and

$$\alpha_0(x) = B_0\frac{b-x}{b-a} + B_1\frac{x-a}{b-a} + \int_a^b G(x, s)\alpha''(x)ds,$$

where $A_1, A_2, B_1, B_2 \in \mathbf{R}$ are given by

$$A_0 = \min\left\{\alpha(a), \beta(a)\right\}, \quad B_0 = \max\left\{\alpha(a), \beta(a)\right\},$$
$$A_1 = \min\left\{\alpha(b), \beta(b)\right\}, \quad B_1 = \max\left\{\alpha(b), \beta(b)\right\},$$

and G is the Green's function associated to the Dirichlet problem

$$y''(x) = 0, \quad \text{a.a.} \quad x \in [a, b], \quad y(a) = y(b) = 0.$$

By standard computations, it is well known that such function follows the expression

$$G(x, s) = \frac{1}{b-a}\begin{cases} (a-s)(b-x), \text{ if } a \le x \le s \le b, \\ (a-x)(b-s), \text{ if } a \le s \le x \le b. \end{cases}$$

In particular it is non-positive on $[a, b] \times [a, b]$ and, as a consquence, $\beta_0 \le \alpha_0$ in $[a, b]$.

Lower and upper solutions for the fourth order problem (1.1)–(1.2) are defined with the previous auxiliary functions:

Definition 3.1 The functions $\alpha, \beta \in W^{4,1}([a,b])$ are a *pair of lower and upper solutions*, respectively, of the problem (1.1)–(1.2) if the following conditions hold:

$$\begin{aligned}
\alpha^{(iv)}(x) &\geq f\big(x, \alpha_0, \alpha''(x), \alpha'''(x)\big), \quad \text{a.a.} \quad x \in [a,b],\\
0 &\leq L_0\big(\alpha_0, \alpha'', \alpha_0(a)\big),\\
0 &\leq L_1\big(\alpha_0, \alpha'', \alpha_0(b)\big),\\
0 &\leq L_2\big(\alpha_0, \alpha'', \alpha''(a), \alpha'''(a)\big),\\
0 &\geq L_3\big(\alpha_0, \alpha''(b), \alpha'''(b)\big),\\
\beta^{(iv)}(x) &\leq f\big(x, \beta_0, \beta''(x), \beta'''(x)\big), \quad \text{a.a.} \quad x \in [a,b],\\
0 &\geq L_0\big(\beta_0, \beta'', \beta_0(a)\big),\\
0 &\geq L_1\big(\beta_0, \beta'', \beta_0(b)\big),\\
0 &\geq L_2\big(\beta_0, \beta'', \beta''(a), \beta'''(a)\big),\\
0 &\leq L_3\big(\beta_0, \beta''(b), \beta'''(b)\big).
\end{aligned}$$

To obtain the main reult one needs the following hypothesis on the functions L_0 and L_1:

(H5) For every $(v, u, x) \in [\alpha_0, \beta_0] \times [\alpha'', \beta''] \times [A_i, B_i]$, $i = 0, 1$, the following property holds:

$$\limsup_{z \to x^+} L_i(v, u, z) \leq L_i(v, u, x) \leq \liminf_{z \to x^-} L_i(v, u, z).$$

The main result is given by the following theorem:

Theorem 3.2 *Assume that conditions* (H1)–(H5) *hold and $f(x, ., y_0, y_1)$ is nondecreasing for a.a. $x \in [a,b]$ and all $(y_0, y_1) \in \mathbf{R}^2$.*

If there is a pair of lower and upper solutions of (1.1)–(1.2)*, α and β, respectively, such that*

$$\alpha''(x) \leq \beta''(x) \quad \textit{for every} \quad x \in [a,b],$$

and f satisfies a Nagumo type growth condition in the set

$$E_{\alpha,\beta} := \left\{(x, y_0, y_1) \in [a,b] \times \mathbf{R}^2 : \alpha''(x) \leq y_0 \leq \beta''(x)\right\},$$

then problem (1.1)–(1.2) *has extremal solutions in the set*

$$S \equiv \left\{u \in C^2([a,b]) : u \in [\beta_0, \alpha_0] \textit{ and } u'' \in \left[\alpha'', \beta''\right]\right\}.$$

Proof. Let $v \in [\beta_0, \alpha_0]$ be fixed. Consider the second-order ordinary problem (P_v). As α and β are, respectively, lower and upper solutions of problem (1.1)–(1.2), then the monotonicity assumptions on function f with respect to its second variable implies that α'' and β'' are lower and upper solutions of (P_v), respectively, according to Definition 2.3. In consequence problem (P_v) has extremal solutions in $\left[\alpha'', \beta''\right]$ for all $v \in [\beta_0, \alpha_0]$.
Denote by y_v the minimal solution of (P_v) in $\left[\alpha'', \beta''\right]$.
By (H2) and Definition 3.1, we have for $i = 0, 1$.

$$L_i(v, y_v, B_i) \geq L_i\big(\alpha_0, \alpha'', B_i\big) \geq 0, \tag{3.2}$$

and

$$L_i(v, y_v, A_i) \leq L_i\big(\beta_0, \beta'', A_i\big) \leq 0. \tag{3.3}$$

From condition (H5) one can apply Lemma 2.7 to obtain that equations

$$L_i(v, y_v, z) = 0, \quad i = 0, 1,$$

have greatest zeros in $[A_i, B_i]$, denoted by w_v if $i = 0$ and z_v when $i = 1$

Define, for each $x \in [a, b]$, the operator T by

$$Tv(x) = \int_a^b G(x,s) y_v(s) ds + w_v \frac{b-x}{b-a} + z_v \frac{x-a}{b-a}.$$

It follows from the definition of T, α_0 and β_0 that $T([\beta_0, \alpha_0]) \subset [\beta_0, \alpha_0]$.

To analyze the monotonicity of T, consider $v_1, v_2 \in [\beta_0, \alpha_0]$ such that $v_1 \le v_2$ and let y_{v_1} and y_{v_2} be the corresponding minimal solutions of (P_{v_1}) and (P_{v_2}) in $[\alpha'', \beta'']$, respectively. Therefore, by the assumptions on f,

$$y''_{v_1}(x) = f\big(x, v_1, y_{v_1}(x), y'_{v_1}(x)\big) \le f\big(x, v_2, y_{v_1}(x), y'_{v_1}(x)\big)$$

and, by (H3) and (H4),

$$0 = L_2\big(v_1, y_{v_1}, y_{v_1}(a), y'_{v_1}(a)\big) \ge L_2\big(v_2, y_{v_1}, y_{v_1}(a), y'_{v_1}(a)\big),$$
$$0 = L_3\big(v_1, y_{v_1}(b), y'_{v_1}(b)\big) \le L_3\big(v_2, y_{v_1}(b), y'_{v_1}(b)\big).$$

So, y_{v_1} is an upper solution of (P_{v_2}). As $\alpha'' \le y_{v_1} \le \beta''$, then, by Theorem 2.6, there are extremal solutions for the problem (P_{v_2}) in $[\alpha'', y_{v_1}]$. In particular the least solution y_{v_2} of (P_{v_2}) in $[\alpha'', y_{v_1}]$ is the least solution of (P_{v_2}) in $[\alpha'', \beta'']$.

Therefore, $y_{v_1} \ge y_{v_2}$ and, by (H_2),

$$L_i(v_2, y_{v_2}, w) \le L_i(v_1, y_{v_2}, w) \le L_i(v_1, y_{v_1}, w), \quad \forall w \in \mathbf{R}, \quad i = 0, 1. \tag{3.4}$$

In consequence $w_{v_1} \le w_{v_2}$ and $z_{v_1} \le z_{v_2}$.

Therefore

$$Tv_1 \le Tv_2,$$

that is, the operator T is nondecreasing in $[\beta_0, \alpha_0]$.

Consider now a monotone sequence $\{v_n\}_n$ in $[\beta_0, \alpha_0]$. Therefore the sequence $\{Tv_n\}_n$ is monotone too and, since

$$(Tv_n)''(x) = y_{v_n}(x) \in \big[\alpha''(x), \beta''(x)\big], \quad x \in [a, b],$$

one can easily verify that it is bounded in $\mathcal{C}^2([a, b])$. So, applying Ascoli-Arzéla theorem, $\{Tv_n\}_n$ is convergent in $\mathcal{C}([a, b])$.

Therefore T sends monotone sequences into convergent ones and, by Lemma 2.8, T has a greatest fixed point in $[\beta_0, \alpha_0]$, denoted by v^*, satisfying

$$v^* = \max\big\{v \in [\beta_0, \alpha_0] : v \le Tv\big\}. \tag{3.5}$$

It is immediate to verify that $v^* \in S$ and it is a solution of problem (1.1)–(1.2). Let's see that v^* is actually the maximal solution of problem (1.1)–(1.2) in the set S.

Let v be an arbitrary solution of problem (1.1)–(1.2) in $[\beta_0, \alpha_0]$, with $v'' \in [\alpha'', \beta'']$. From Theorem 2.6 we have that $v(a) \le w_v$ in $[\beta_0(a), \alpha_0(a)]$ and $v(b) \le z_v$ in $[\beta_0(b), \alpha_0(b)]$.

Since $v'' = y$, with y a solution of (P_v) in $[\alpha'', \beta'']$, and y_v is the minimal solution of (P_v) in $[\alpha'', \beta'']$, then $v'' \ge y_v$ and we deduce that $v \le T\,v$. Thus, by (3.5), $v \le v^*$ and so v^* is the greatest solution of (1.1)–(1.2) in S.

The existence of the least solution can be proved using analogous arguments and obvious changes in the operator T. □

4 Example

Consider, for $x \in [0, 1]$, the fourth order equation

$$u^{(iv)}(x) = \max_{x \in [0,1]} \left(\int_0^x u(s) ds \right) + \lambda \big(u''(x)\big)^3 - \big(u'''(x) + 1\big)^{\frac{2}{3}} \tag{4.1}$$

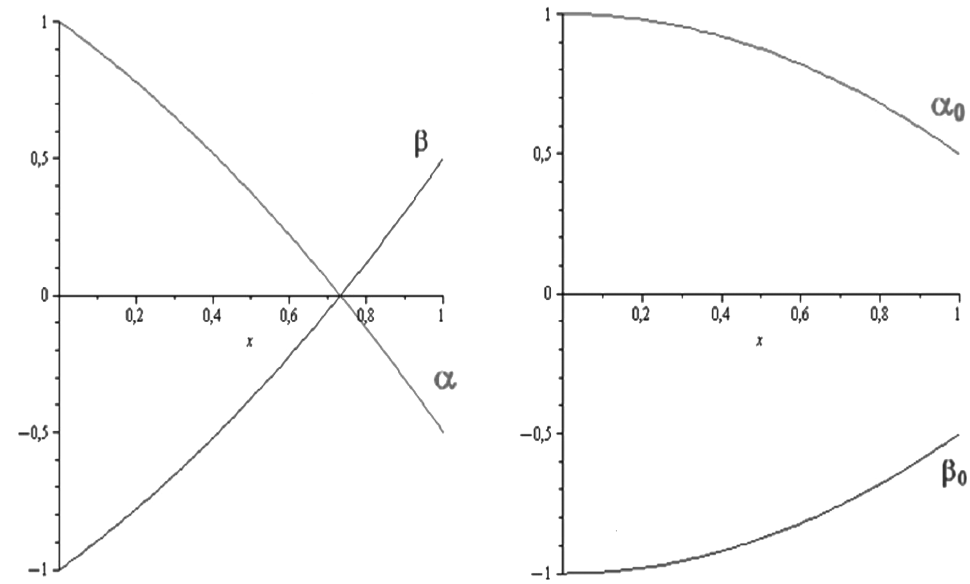

Fig. 1 Despite α and β are non ordered there are extremal solutions in the set $[\beta_0, \alpha_0]$.

along with the functional boundary conditions

$$\begin{aligned} -\max_{x\in[0,1]} u(x) + u(0) &= 0, \\ \min_{x\in[0,1]} u''(x) + \delta u(1) &= 0, \\ u''(0) &= 0, \\ u''(1) &= 0. \end{aligned} \tag{4.2}$$

This problem is a particular case of (1.1)–(1.2) with

$$\begin{aligned} f(x, y_0, y_1, y_2) &= \max_{x\in[0,1]} \left(\int_0^x y_0(s)ds \right) + \lambda y_1^3 - (y_2+1)^{\frac{2}{3}}, \\ L_0(z_1, z_2, z_3) &= -\max_{x\in[0,1]} z_1 + z_3, \\ L_1(z_1, z_2, z_3) &= \min_{x\in[0,1]} z_2 + \delta z_3, \\ L_2(z_1, z_2, z_3, z_4) &= -z_3, \\ L_3(z_1, z_2, z_3) &= z_2. \end{aligned}$$

The functions

$$\alpha(x) = -\frac{x^2}{2} - x + 1 \quad \text{and} \quad \beta(x) = \frac{x^2}{2} + x - 1$$

are, respectively, lower and upper solutions to the problem (4.1)–(4.2), with

$$A_0 = -1, \quad B_0 = 1, \quad A_1 = -\frac{1}{2}, \quad B_1 = \frac{1}{2},$$

$$\alpha_0 \quad (x) = 1 - \frac{x^2}{2} \quad \text{and} \quad \beta_0(x) = \frac{x^2}{2} - 1$$

for $1 \leq \lambda < \infty$ and $\delta \geq 2$.

As one can see in Figure 1, despite the fact that the lower and upper solutions α and β are not ordered, the auxiliar functions α_0 and β_0 are well ordered.

As the continuous function f verifies a Nagumo type growth condition, according Definition 2.4, in

$$E = \{(x, y_1, y_2) : -1 \leq y_1 \leq 1\}$$

with $\varphi(y_2) = 1 + |\lambda| + |y_2 + 1|^{\frac{2}{3}}$, then, by Theorem 3.2 the problem (4.1)–(4.2) has extremal solutions in $[\beta_0, \alpha_0]$.

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