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On the Synchronisation of Elections
A differential Games Approach

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Resumo:

The paper offers an analysis of the issues related to the election dates synchronisation between two countries. The first purpose of the paper is to analyse the circumstances in which a government of a single country, considered to be a small economy, has incentives, or not, to synchronise the domestic election dates with the election dates (not necessarily determined in an endogenous way) of a country performing the role of an 'anchor', considered to be a big economy.
To achieve this purpose, the paper uses an asymmetric version of MILLER and SALMON's (1990) model in order to derive the optimal domestic electoral period length, which, in this sense, can be said to be endogenously determined. The second main purpose of the paper is to re-analyse the situation being studied by considering that the foreign government also determines its election dates in an optimal way, this leading to a differential game played by the two incumbents from which incentives to totally synchronise the election dates may result. The paper shows that the interests of both economies in what concerns the existing electoral period length in the other economy are not always compatible.

Palavras-chave: Differential games, Electoral business cycles, Election dates, Mandates durations, Synchronisation of elections

Classificação JEL: C73, E32, E61, F42

1 Introduction

In November 17, 1997, Prof. Mervyn King, in a lecture at the European University Institute on The Political Economy of EMU, stressed that, the third stage of Economic and Monetary Union (EMU) would show more pronounced business cycles even though cooperation was to be facilitated with synchronised cycles; see KING (1998). The European Commission also recognised that:

> "*If countries (...) experience de-synchronised business cycles, giving up national monetary policy may prove costly.*", in EUROPEAN COMMISSION (1997), p. 26.

Despite this clear concern about the importance of business cycle synchronisation, little research has been undertaken on the importance of temporal horizons for business cycles synchronisation and, to the best of my knowledge, almost none has been done on the impact of the synchronisation of election dates; two exceptions are KAYSER (1998) and SAPIR and SEKKAT (1999).[1] Even before those these two references, the following question was being made:

> "*Does international cooperation or coordination of economic policies become easier or harder when domestic elections across countries are synchronised? Take, for instance, the usual way of determining a cooperative solution (between two countries). This is obtained through the minimisation of a global weighted loss function:*
>
> $$V^C = wV + (1-w)\,V^*,$$
>
> *where V and V^* are the domestic loss functions and w and $(1-w)$ are weights that depend on the bargaining power of the governments. Clearly, when the (two) governments have distinct time horizons these weights can*

[1] SAPIR and SEKKAT (1999) present a model where employment, X_t, depends on a domestic inflation surprise, on the competence of the incumbent government μ, and also on the *degree of openness* of the economy, as follows:

$$X_t = (\pi_t - \pi_t^e) + (\mu_t + \mu_{t-1}) + \beta\,(\pi_t^* - \pi_t^{*e}),$$

where $|\beta| < 1$ measures the extent in which foreign unanticipated inflation influences domestic employment.

The model explores the situation where joint decisions may not be taken when players possess different electoral calendars. As, in the European Union (EU) case, countries are economically interdependent – which causes coordination problems arising from spillover effects resulting from domestic-oriented electoral policies – but politically independent, the authors suggest the adoption of a single election date in the EU.

(cooperatively) evolve in time. Does this increase the probability of cooperation?".[2]

In order to fill part of this gap in the literature, the paper formalises some of the interactions between inter-national and inter-temporal problems of policy coordination through the analysis of the implications of the *synchronisation* (*or not*) *of election dates* on international policy cooperation. Specifically, this paper adds to the literature by computing the solution of a differential game in a model *a la* MILLER and SALMON (1990), where governments face elections at possibly distinct moments of time.[3] That said, the rest of the paper is structured as follows. It starts with a simplified version of MILLER and SALMON's (1990) model in order to focus and introduce the analysis. Section 3 then considers an asymmetric version of the model. Sub-section 3.1 offers the development of the full model such that, when the two economies are of equal size and structure, the model collapses into MILLER and SALMON's (1990) model. After that, it is straightforward to introduce a difference in the size of the economies, which is considered in sub-section 3.2. Sub-section 3.3 considers the optimal choice of the election date from the viewpoint of the home country considered as a small economy where its government faces an endogenous timing of elections problem. Section 4 presents a possible solution for the differential game. Section 5 concludes.

2 A Simplified Version of MILLER and SALMON's (1990) Model

In order to analyse the possible implications of different electoral term lengths, let us start by considering a simplified *finite horizon* version of the model discussed in MILLER and SALMON (1990). The use of this model allows us to study the implications for *international policy coordination* when governments may have distinct time horizons, *i.e.* of possibly non-synchronised national elections implications, which is the main goal of the paper.

MILLER and SALMON (1990) consider a dynamic model where countries are linked by trade and perfectly mobile capital flows. Forward-looking private sector behaviour in the foreign exchange market and, in particular, in the government's future interest rate policies influence present outcomes. As such, the reaction of the forward-looking private sector may make it impossible to observe the welfare improvement that certainly results from cooperation on economic policies, that is, from the internalisation of the externalities generated by the (monetary) policies. This

[2] CALEIRO (1996), pp. 11-12.

[3] In CALEIRO (2000) a *difference games case* is considered precisely to study how distinct electoral period lengths may influence the benefits from international policy coordination.

is the reason why coordination may not pay and, as shown in MILLER and SALMON (1990), this happens when the initial core inflation rates are different and/or when the shocks affecting national policies are relatively uncorrelated.[4]

On this basis, let us consider the following 'block' of MILLER and SALMON's (1990) model:

$$\begin{aligned} y &= -\gamma r + \delta c + \eta y^*, && (1)\\ i &= \phi y + \sigma \frac{dc}{dt} + \pi, && (2)\\ \pi &= \xi \phi z + \xi \sigma c, && (3) \end{aligned}$$

where

$y :=$ output measured from the 'natural rate';
$r :=$ real consumer rate of interest;
$c :=$ competitiveness of the economy defined as the real price of foreign goods;
$y^* :=$ overseas output;
$i :=$ inflation;
$\pi :=$ 'core' inflation;
$z :=$ integral of past output, *i.e.* $\frac{dz}{dt} = y$;
γ, δ, η, ϕ, σ, and $\xi :=$ parameters.

Equation (1) can be view as a reduced form equation of the interdependence between output and aggregate demand solved for y. Equation (2) explains inflation as the result of demand pressure, changes in the real exchange rate reflected in changes in competitiveness and of some 'core' inflation. Equation (3) explains 'core' inflation as a weighted sum of a backward-looking component, z, and a forward-looking component c.

It is straightforward to show that, under the simplification $\gamma = \phi = 0$, the above model can be reduced to

$$\begin{aligned} y &= \delta c + \eta y^* \\ \frac{dc}{dt} &= \frac{1}{\sigma} i - \xi c. \end{aligned}$$

[4] In MILLER *et al.* (1991), the influence of discounting on those results is studied. See also MILLER and SALMON (1985a).

Before proceeding, a short note on this simplified 'model' is worthwhile.[5] First of all, given the assumption that $\gamma = 0$, this obviously means that output is not affected by the interest rate.[6] As a consequence, the policy variable cannot be the interest rate, as indeed MILLER and SALMON (1990) consider. To preserve consistency with the models more recently used, it is the inflation rate that will be used as the policy variable. The need to consider exogenous the inflation rate then justifies the second assumption, that is $\phi = 0$.

Let us then assume that the incumbent government manipulates the inflation rate in order to maximise its popularity on election's eve ($t = T$) which depends (symmetrically) on inflation, i, and on unemployment *via* aggregate output, y. On this basis, we may then formulate the optimal control problem of the incumbent government as follows:

$$\max_i W = -\frac{1}{2}\int_0^T (\beta i^2 + y^2)dt$$

subject to

$$\frac{dc}{dt} = \frac{1}{\sigma}i - \xi c. \tag{4}$$

The Hamiltonian corresponding to the optimal political programme is

$$\mathcal{H} = -\frac{1}{2}\left(\beta i^2 + (\delta c + \eta y^*)^2\right) + \lambda\left(\frac{1}{\sigma}i - \xi c\right),$$

where λ is a co-state variable associated with the competitiveness restriction. Because the votes on election day arising from a marginal change in competitiveness must be zero, then $\lambda(T) = 0$.

The first-order conditions associated with this programme are:

$$\frac{\partial \mathcal{H}}{\partial i} = -\beta i + \lambda\frac{1}{\sigma} \stackrel{!}{=} 0, \qquad \text{(output equation)} \tag{5}$$

$$\frac{\partial \mathcal{H}}{\partial c} = -\frac{d\lambda}{dt} \Rightarrow \frac{d\lambda}{dt} = \delta^2 c + \xi\lambda + \delta\eta y^*. \qquad \text{(state equation)} \tag{6}$$

The output and state equations derived from the first-order conditions, together with the 'restriction' $\frac{dc}{dt} = (1/\sigma)i - \xi c$ and the transversality condition $\lambda(T) = 0$,

[5] Please note that we will discuss below the model without the 'restrictive'/simplifying assumptions that result in the 'model' under consideration.

[6] This assumption thus makes it impossible to derive the interest rate path from the solution of the 'model'.

constitute the 'inputs' for PSREM[7], which will be used to perform some simulations.[8]

2.1 Simulation results

Let us parameterise the model as in MILLER and SALMON (1990), *i.e.* consider $\delta = 1/2$, $\sigma = 0.1$, $\beta = \xi = 1$ and perform two simulations ($T = 2.5,\ T = 5.0$) considering a constant 'shock' given by a negative value for $\eta y^* = -0.5$.[9] The results can be summarised in figure 1.

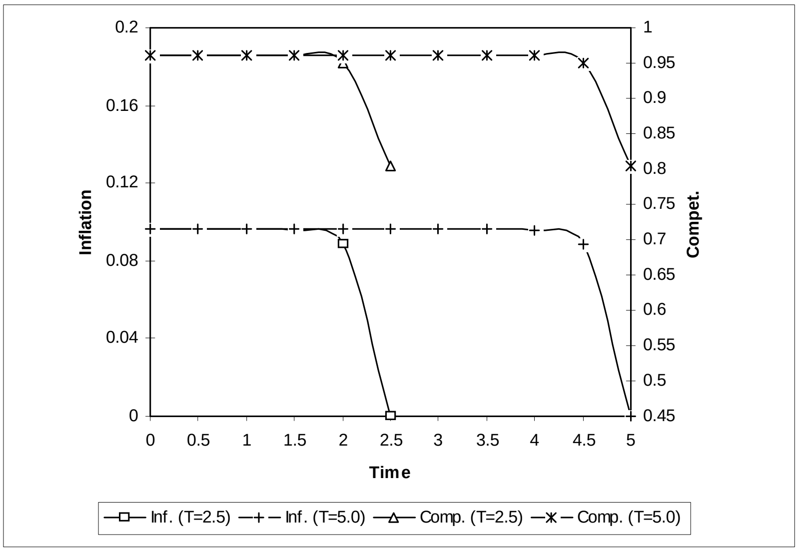

Figure 1: The PSREM simulation results

For these two finite time horizons we can observe that, *until very near the election date*, both inflation and competitiveness are fixed at what can be derived to be their 'steady-state' values.[10]Due to the particular form of agents' behaviour, the economy will stay at the 'steady-state' path as long as it can, exactly until the election introduces the incentive to deviate from that path; or, in other words, when the co-state

[7] An acronym for Policy Simulation with Rational Expectations Models; see MARKINK and Van der PLOEG (1989).

[8] The PSREM input file that was written is available upon request.

[9] Although certainly open to criticism, this assumption has to be made in order to proceed with the analysis. Even so, it should be highlighted that different types of 'shocks' can easily be considered and, as we will see, the possible consequences of varying overseas output y^* (probably related with domestic y) can also be anticipated. A more formal analysis of this issue is carried out in the next section.

[10] This terminology, which is used by PSREM, means the values that would be obtained in an infinite horizon case, that is when $T \to \infty$.

variable path has to change in order to fulfil the transversality condition at T. Thus we get exactly the same behaviour of inflation *at the beginning* of the term, no matter what the electoral term length is; and exactly the same behaviour near the end of every *finite* term. In fact, very near the election date, inflation is decreased sharply to zero, this being the result from *ex-ante* benefits and e*x-post* costs.

In addition, we can say that, as the electoral period length increases, higher inflation arises at the cost of a better result in terms of competitiveness.[11] Furthermore, as we might expect, when the electoral period length is infinite, the cycle disappears – inflation and competitiveness will always be equal to the initial value, which is always the same in every finite term – but the best result in terms of competitiveness is achieved with a higher rate of inflation, given that inflation and competitiveness will not decrease as happens in finite horizon cases.

As it is well known, the presence of forward-looking rational agents induces optimal solution paths characterised by sudden changes of the relevant (jump) economic variables. To some extent, this is in agreement with the results above presented. In fact, one could observe that the optimally determined variables present a sudden change in their trajectory as the economy reaches the time horizon. We would like to conclude by discussing a possible generalisation of those results, taking into account that, besides the existence of *forward-looking behaviour*, we are dealing with *finite horizon* models.

Given that the *forward-looking* nature of agents induces the features in the economic timing of elections discussed above, it seems important to clarify the intuition as to why this behaviour implies the conclusions it does. This, by itself, justifies a short discussion of *uniqueness and convergence* of the solution path, but the intrinsic *finite horizon* nature of the model justifies, even more, that we spend some time analysing the role of *terminal conditions* on the optimal solution paths.

Plainly, a system is said to be *globally stable* if it converges to an equilibrium no matter what the initial conditions, which, in many cases, seem to be a 'desirable' characteristic of the system. However, when the system is (partly) driven by free expectational variables, that is, variables which are free to take on any value at time t, then global stability is no longer a 'desirable' characteristic given that, in this case, there can be an infinity of solution paths. Hence, one way of eliminating the degree of indeterminacy in agents' expectations is to consider *saddle-path stable* models. In this case, there is only one convergent path to the equilibrium which can be identified by agents under the assumption of perfect foresight. Thus, in order to ensure convergence to the equilibrium or, in other words, if we 'rule out speculative bubbles', an additional

[11] Note that the unconstrained desired value for aggregate demand will be $y = 0$. But $\eta y^* = -0.5$ which implies that, because $\delta = 0.5$, the desired value of c will be 1. Obviously, the more distant c is from 1 the worse is the result.

condition is usually imposed on the model which implies that the forward-looking variables 'jump' in the initial moment so as to put the system on its saddle-path.[12] To sum up, the need to determine a unique solution requires the imposition of a boundary condition compatible with the convergence to the long-run equilibrium.

Besides this type of boundary condition, in finite horizon models one has to consider another set of 'boundary' conditions, the so-called *terminal conditions*. Generally speaking, this kind of condition imposes at the final period T that the system reach some state y_T which would be the value taken in period T by the solution path of the infinite time horizon case. In more specific terms, these terminal conditions impose that the finite period optimal solution path y_t, $(t = 1, ..., T)$, coincide with the infinite time solution path for the first T periods.

As it is known, in a finite horizon model, the solution paths depend on the expected values for the instruments past the final date T. Thus, at time T, the terminal condition corresponding to the assumption of constant values for x after period T, that is $x_t = \bar{x}$ for $t > T$, will be

$$\begin{aligned} z_T &= a_{12}^{-1}\left((\lambda_1 - a_{11})\, y_T + ((\lambda_1 - a_{11})\,\lambda_1 - a_{12}\lambda_2) \sum_{i=0}^{\infty} \lambda_2^{-i-1} \bar{x} \right) \\ &= \frac{\lambda_1 - a_{11}}{a_{12}} y_T + \frac{(\lambda_1 - a_{11})\,\lambda_1 - a_{12}\lambda_2}{a_{12}\,(\lambda_2 - 1)} \bar{x}. \end{aligned}$$

In the literature, there have been suggestions regarding these terminal conditions. MINFORD *et al.* (1979) proposed that beyond some date the endogenous variables assume long-run equilibrium values.[13] HALL and HENRY (1988) argue that, in practice, it becomes impossible to solve a model with certain kinds of forward-looking equations with anything else than *ad hoc* fixed terminal conditions, such as long-run equilibrium values as proposed by MINFORD *et al.* (1979). But since these *ad hoc* conditions will be, in general, inconsistent with the true solution, then its imposition leads to distortions which are the less problematic the more the forward-looking root is below one; see BLAKE and WESTAWAY (1995). This is so, because this forward-looking root acts as a discount rate of the future, in the sense that near future becomes more important (than far future) to explain the current value of the forward-looking variable as this root decreases in absolute value.

In order to minimise those possible distortions induced by the imposition of the terminal conditions, authors agree that a sufficiently distant terminal date should be

[12] In mathematical terms, this additional condition consists on setting to zero the coefficients of the unstable or divergent roots.

[13] In MINFORD *et al.*'s (1979) opinion, using appropriate terminal conditions one can also solve the solution uniqueness problem when the imposition of transversality conditions, as described above, do not ensure the existence of a unique solution.

chosen so that the solution path over the period $[1, T]$ is not significantly changed. This provide immediately a 'test' on the influence of terminal conditions, because by simulating the dynamic system over the period of interest $[1, T]$, given distinct terminal dates and terminal conditions kinds – equilibrium, *etc.* –, one can verify whether the solution path is significantly altered or not.

Moreover, as clearly pointed out by BLAKE and WESTAWAY (1995), the consideration of a finite horizon has another implication. As the economic system becomes closer to the final period T, the (short-run) gains from sharp changes in economic policy increase relatively more than the (long-run) costs derived by those sharp changes in policy. Thus we should observe, near the final period T, significant changes in the control variables which, in turn, can affect *all* the solution path if agents can anticipate them, as in the forward-looking models. Thus, by exploring the *forward-looking* nature of agents, government can design, and specially delay, those sharp changes in policy in order to affect the solution path in a desired way. Clearly, as T becomes larger, the more those sharp changes will be delayed and the sharper they might be, if discounting reduces the costs in manipulating the instruments.

To conclude, in the case of *finite horizon(s)* models with forward-looking behaviour, one should expect that near the election(s) day the optimal trajectories change considerably, which may generally impose difficulties for international policy coordination, except if these final periods are determined and coordinated also in an optimal way.

Taking into consideration the results reported above, what are the *predictable consequences on international policy coordination*? With synchronised elections, *i.e.* with elections taking place at the same moment in every country, cooperation on coordination will be easier most of the time, but very near the elections, the sharp changes in domestic policy will, almost certainly, not be compatible with the other player's objectives *unless*, almost tautologically, those sharp changes help the other player to win the elections, that is, act as external 'disturbances' but, fortunately, well correlated with the optimal solution paths for the domestic economy. Interestingly enough, one can quote MILLER and SALMON (1990), p. 569:

> "*coordination may or may not pay depending on the correlation of disturbances facing the two countries*".

In order to make this point clearer, let us recall that from (5) the optimal solution inflation path will be:

$$i(t) = \frac{1}{\beta\sigma}\lambda(t). \tag{7}$$

Given the transversality condition $\lambda(T) = 0$, we know that, at least on the election day, inflation will be at its most unconstrained favourable level, that is, zero. Moreover, in order to be fixed at this steady-state level *during all the term*, one has to have $\lambda(t) = 0, \forall t \in [0, T]$. However, a necessary condition for this to happen is that $\frac{d\lambda}{dt} = 0$, which, as we know from (6), depends crucially on the external output y^* behaviour. If, for some reason, $y^* = 0$, then $c = 0$ would be compatible with a constant value for λ fixed at zero and, *via* (4), would also result in $\frac{dc}{dt} = 0$ when $i = 0$. Everything would be compatible at the first-best values, which is no surprise, given the lack of disturbances acting as 'noise' in the optimal electoral programme.

In our case, it is easy to verify that, as we abandon the hypothesis of an exogenous *non-null* value for ηy^*, a welfare improving solution can be obtained if we allow overseas output y^* to be correlated with domestic output y. Considering the usual case of two 'identical' economies, then inflation can be manipulated to maintain competitiveness at its first-best value, which leads to $y = 0$ and $y^* = 0$ in each country which, in turn, is the intersection of the two player's reaction function. Clearly, this 'positive' correlation in these two 'identical' economies is likely to occur when the elections take place at the same time in both countries. This does not mean that a relaxation of the symmetry and no long-run conflict of objective assumptions will still make this conjecture true. This issue will be analysed in the following section.

3 An Asymmetric Version of MILLER and SALMON's (1990) Model

Considering two possibly asymmetric economies, the model would be as follows. The home economy is described by

$$y = -\gamma r + \delta c + \eta y^* \qquad \text{(Aggregate demand)} \tag{8}$$

$$i = \phi y + \sigma \frac{dc}{dt} + \pi \qquad \text{(Phillips curve)} \tag{9}$$

$$\pi = \xi \phi z + \xi \sigma c \qquad \text{(Core inflation)} \tag{10}$$

where

$$\frac{dz}{dt} = y. \qquad \text{(Accumulation)} \tag{11}$$

The policy-maker aims to minimise an undiscounted stream of quadratic costs arising from fluctuations in output and core inflation through the choice of real interest rates, that is

$$\min_r V \equiv \frac{1}{2} \int_0^\infty \left(\beta \pi^2 + y^2\right) dt.$$

A similar framework is valid for the foreign economy such that

$$
\begin{aligned}
y^* &= -\gamma^* r^* - \delta^* c + \eta^* y \qquad \text{(Aggregate demand)} \qquad (12)\\
i^* &= \phi^* y^* - \sigma^* \frac{dc}{dt} + \pi^* \qquad \text{(Phillips curve)} \qquad (13)\\
\pi^* &= \xi^* \phi^* z^* - \xi^* \sigma^* c \qquad \text{(Core inflation)} \qquad (14)
\end{aligned}
$$

where

$$\frac{dz^*}{dt} = y^*. \qquad \text{(Accumulation)} \qquad (15)$$

The foreign policy-maker has the following objective

$$\min_{r^*} V^* \equiv \frac{1}{2} \int_0^\infty \left(\beta^* {\pi^*}^2 + {y^*}^2 \right) dt.$$

Besides the spillover effects at the demand level, an arbitrage condition establishing the connection between the two economies is assumed:

$$\mathsf{E}\left[\frac{dc}{dt}\right] = r - r^*. \qquad (16)$$

In this problem there are three state variables z, z^* and c, each one associated with a co-state variable λ_z, λ_{z^*} and λ_c. As shown by COHEN and MICHEL (1988), the time consistent solutions can be obtained from the time inconsistent ones if the corresponding Hamiltonian does not include the co-state variable λ_c, as it is assumed that the real exchange rate c has a stable relation with the two other state variables z and z^* as follows:

$$c = \theta_1 z + \theta_2 z^*,$$

where θ_1 and θ_2 are to be chosen in a way that consistency is obtained.

The private sector rational expectations about the real exchange rate will depend upon the strength of policy response. In the symmetric case, if χ designates a measure of the policy feedback of output in response to inflation, it can be shown that the rational expectation about the real exchange rate of θ will be given by

$$\theta = \frac{1+\eta}{(\gamma + 2\delta\chi^{-1})}.$$

As we will later assume an asymmetric version of the model in which one of the economies is not influenced, at the domestic demand level, by the other economy's demand, a plausible solution to be considered is the non-cooperative one. Thus, we proceed with the determination of the time consistent Nash solution.

In the non-cooperative solution, the two policy-makers set policy independently. In fact, this is a plausible behaviour when one of them belongs to a country which is not influenced, at the demand level, by the other. This justifies our choice in what concerns the solution under analysis.

3.1 The Nash time consistent solutions

As mentioned above, time consistency is obtained dropping c from the Hamiltonians which, assuming the open-loop case, are then defined as follows; see MILLER and SALMON (1990), p. 557:

$$\mathcal{H} = \frac{1}{2}\left(\beta\pi^2 + y^2\right) + \lambda_z \underbrace{\frac{dz}{dt}}_{y}, \tag{17}$$

$$\mathcal{H}^* = \frac{1}{2}\left(\beta^*{\pi^*}^2 + {y^*}^2\right) + \lambda^*_{z^*} \underbrace{\frac{dz^*}{dt}}_{y^*}. \tag{18}$$

For this problem the first-order conditions are the following; the mathematical details are in the Appendix.

$$\frac{\partial \mathcal{H}}{\partial r} = (y + \lambda_z)\left(-\frac{\gamma}{1-\eta\eta^*}\right) \stackrel{!}{=} 0, \tag{19}$$

$$\frac{\partial \mathcal{H}^*}{\partial r^*} = (y^* + \lambda^*_{z^*})\left(-\frac{\gamma^*}{1-\eta\eta^*}\right) \stackrel{!}{=} 0. \tag{20}$$

The previous first-order conditions can be expressed equivalently as[14]

$$\frac{\partial \mathcal{H}}{\partial y} = y + \lambda \stackrel{!}{=} 0 \Rightarrow y = -\lambda, \tag{21}$$

$$\frac{\partial \mathcal{H}^*}{\partial y^*} = y^* + \lambda^* \stackrel{!}{=} 0 \Rightarrow y^* = -\lambda^*. \tag{22}$$

As $\frac{dz}{dt} = y$ and $\frac{dz^*}{dt} = y^*$ we have

$$\frac{dz}{dt} = -\lambda, \tag{23}$$

$$\frac{dz^*}{dt} = -\lambda^*. \tag{24}$$

[14] To simplify the notation, let us use λ and λ^* to designate, respectively, λ_z and $\lambda^*_{z^*}$.

Moreover

$$-\frac{d\lambda}{dt} = \frac{\partial \mathcal{H}}{\partial z} \Rightarrow \frac{d\lambda}{dt} = -\beta\xi^2 (\phi + \sigma\theta)^2 z - \beta\xi^2 (\phi + \sigma\theta) \sigma\theta^* z^* \tag{25}$$

$$-\frac{d\lambda^*}{dt} = \frac{\partial \mathcal{H}^*}{\partial z^*} \Rightarrow \frac{d\lambda^*}{dt} = -\beta^*\xi^{*^2} (\sigma^*\theta^* - \phi^*)^2 z^* - \beta^*\xi^{*^2} (\sigma^*\theta^* - \phi^*) \sigma^*\theta z \tag{26}$$

Finally,

$$\frac{dc}{dt} = \frac{\gamma^* + \gamma\eta^*}{\gamma\gamma^*}\lambda - \frac{\gamma + \gamma^*\eta}{\gamma\gamma^*}\lambda^* + \frac{\gamma^*\delta + \gamma\delta^*}{\gamma\gamma^*}c. \tag{27}$$

The first-order conditions (23) , (24) , (25) , (26), (27) can be then expressed as

$$\begin{bmatrix} \frac{dz}{dt} & \frac{dz^*}{dt} & \frac{dc}{dt} & \frac{d\lambda}{dt} & \frac{d\lambda^*}{dt} \end{bmatrix}^{\mathrm{T}} = \mathbf{A} \begin{bmatrix} z & z^* & c & \lambda & \lambda^* \end{bmatrix}^{\mathrm{T}} \tag{28}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \frac{\gamma^*\delta+\gamma\delta^*}{\gamma\gamma^*} & \frac{\gamma^*+\eta^*\gamma}{\gamma\gamma^*} & -\frac{\gamma+\eta\gamma^*}{\gamma\gamma^*} \\ -\beta\xi^2 (\phi + \sigma\theta_1)^2 & -\beta\xi^2 (\phi + \sigma\theta_1) \sigma\theta_2 & 0 & 0 & 0 \\ -\beta^*\xi^{*^2} (\sigma^*\theta_2 - \phi^*) \sigma^*\theta_1 & -\beta^*\xi^{*^2} (\sigma^*\theta_2 - \phi^*)^2 & 0 & 0 & 0 \end{bmatrix}.$$

It is straightforward to verify that, when the two economies are symmetric such that $\gamma = \gamma^*$, $\delta = \delta^*$, $\eta = \eta^*$, $\beta = \beta^*$, $\xi = \xi^*$, $\phi = \phi^*$, $\sigma = \sigma^*$ and $\theta_1 = \theta$, $\theta_2 = -\theta$, the system (28) collapses into the one derived in MILLER and SALMON (1990).

3.2 The small economy *versus* the big economy case

Let us now suppose that the home country represents a small open economy while the foreign one is a big economy. If this is the case, it is plausible to assume that the home economy is of negligible size in what concerns its spillover effects on the foreign economy demand, that is, $\eta^* = 0$. Moreover, if one considers that further to the European and Monetary Union, the exchange rate of the single currency is completely predetermined independently of the domestic policy-makers actions, then $\theta_1 = \theta_2 = 0$ is also a plausible hypothesis to be assumed; see MILLER and SALMON (1985a), p. 194. In this context, the previous system (28) can be reduced to

$$\begin{bmatrix} \frac{dz}{dt} & \frac{dz^*}{dt} & \frac{dc}{dt} & \frac{d\lambda}{dt} & \frac{d\lambda^*}{dt} \end{bmatrix}^{\mathrm{T}} = \mathbf{B} \begin{bmatrix} z & z^* & c & \lambda & \lambda^* \end{bmatrix}^{\mathrm{T}}, \tag{29}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \frac{\gamma^*\delta+\gamma\delta^*}{\gamma\gamma^*} & \frac{1}{\gamma} & -\frac{\gamma+\eta\gamma^*}{\gamma\gamma^*} \\ -\beta\xi^2\phi^2 & 0 & 0 & 0 & 0 \\ 0 & \beta^*{\xi^*}^2{\phi^*}^2 & 0 & 0 & 0 \end{bmatrix}.$$

Let us then consider that (domestic) voters take into account the evolution of unemployment, y_t, and inflation, π_t, such that the accumulated (net) popularity at the election date, T, is

$$V_T = -\frac{1}{2}\int_0^T \left(\beta\pi^2 + y^2\right) dt. \tag{30}$$

We may then formulate the optimal control problem of the domestic government as follows:

$$\max \ V_T = -\frac{1}{2}\int_0^T \left(\beta\pi^2 + y^2\right) dt, \tag{31}$$

subject to the economic model governing the two economies (8)-(16).

The foreign government possesses a similar programme, that is

$$\max \ V^*_{T^*} = -\frac{1}{2}\int_0^{T^*} \left(\beta^*{\pi^*}^2 + {y^*}^2\right) dt,$$

where T^* corresponds to the foreign economy election date.

3.3 The optimal degree of election dates synchronisation

We are now in a position to derive the optimal domestic electoral period length $\tilde{T}$, which, in this sense, can be said to be endogenously determined; see BALKE (1991), CHAPPELL and PEEL (1979), ELLIS and THOMA (1991), GINSBURGH and MICHEL (1983) and LÄCHLER (1982). Taking into account that this corresponds to an *open final time problem* (see TAKAYAMA, 1994, pp. 464-465 and/or LÉONARD and LONG, 1992, p. 241), to solve for $\tilde{T}$ requires that

$$\sup \mathcal{H}\left(y\left(\tilde{T}\right), \pi\left(\tilde{T}\right), \lambda\left(\tilde{T}\right), \tilde{T}\right) = 0, \tag{32}$$

where

$$\mathcal{H} = -\frac{1}{2}\left(\beta\pi^2 + y^2\right) + \lambda \underbrace{\frac{dz}{dt}}_{y}.$$

As the foreign demand $y^*(t)$ trajectory will 'mirror' the co-state $\lambda^*(t)$ trajectory, the fulfilment of the transversality condition will assure that, on the foreign economy election date T^*, the aggregate demand will be at its 'unconstrained' maximum, *i.e.*

$y^{*}(T^{*}) = 0$. This result, in turn, will be obtained when the foreign interest rate is used such that

$$r(T^{*}) = -\frac{\delta^{*}}{\gamma^{*}} c(T^{*}), \tag{33}$$

that is, for a given domestic electoral period length, the foreign interest rate is uniquely determined by (33) as there are, by assumption, no spillover demand effects. Moreover, a possible incompatibility of this policy with a zero core inflation at T^{*} is excluded given that there is not necessarily an optimality in T^{*}.[15]

Concerning the domestic economy, the aggregate demand $y(t)$ trajectory will also 'mirror' the co-state $\lambda(t)$ trajectory. Given the transversality condition $\lambda(T) = 0$, (32) will then be fulfilled if $\pi\left(\tilde{T}\right) = 0$. In words, the optimal election date will then be the one where the domestic government achieves also the best 'unconstrained' value for the core inflation. This, in turn, implies that $z(t)$ has to follow a trajectory such that

$$z\left(\tilde{T}\right) = -\frac{\sigma}{\phi} c(T). \tag{34}$$

The combination of (33) and (34) will give us the optimal domestic period length $\tilde{T}$ as a function of the foreign electoral period length T^{*} in an *implicit form* resulting from the solution of the system (29). In order to illustrate this, let us consider next the same (symmetric) parameterisation as considered in MILLER and SALMON (1990), that is $\beta = \phi = \xi = 1, \gamma = \delta = \frac{1}{2}, \eta = \frac{1}{3}, \sigma = \frac{1}{10}$.

The solution of (29), given the transversality conditions $\lambda(T) = \lambda^{*}(T^{*}) = 0$, gives us quite cumbersome expressions, especially the solution for the real exchange rate trajectory; see the Appendix. Despite this difficulty, it is, however, straightforward to see that the higher z_0 the higher will be $z(t)$ at the election date, while an increase in the electoral period length results in a decrease in $z(T)$. In fact, as T goes from 0 (continuous elections case) to ∞ (social welfare case), $z(T)$ goes from $z(0)$ to 0.

Let us proceed with the consideration of a balanced initial situation characterized by $c_0 = 1$ and an equal initial inflation, *v.g.* $\pi_0 = \pi_0^{*} = 10\%$. Figure 2 gives us the relation between the domestic inflation rate, *at the election date*, as a function of the domestic electoral period length T for distinct values of the foreign electoral period length T^{*}.

[15] The case where T^{*} is also chosen as an *optimal* electoral period length will be discussed further below.

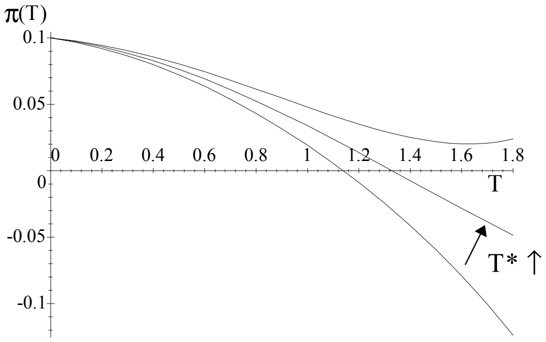

Figure 2: Inflation on the election day

An increase in the foreign electoral period length leads to an increase in the domestic inflation rate on the election day. This is so because an increase in the foreign electoral period length will create, *via* (34), an increase in the time response needed to 'remove' the effect of an appreciated real exchange rate at the core inflation. Hence, for a *limited* increase in T^*, the domestic policy-makers would find it optimal to increase the domestic electoral period length T in order to make it possible to obtain a zero inflation at the election date. However, for a sufficiently higher T^*, it may be not possible to obtain a zero level of inflation at the election date, as the previous figure also shows. This amounts to saying that, for T^* belonging to certain intervals, there is no first-best domestic electoral period length. However, given the periodic characteristics of the solutions, it may be possible to obtain, again, a (first-best) optimal electoral period length for higher values of the foreign electoral period lengths. In fact it is possible to obtain a zero domestic inflation level at a given election date $\tilde{T}$ for distinct values of T^*; see figure 3.

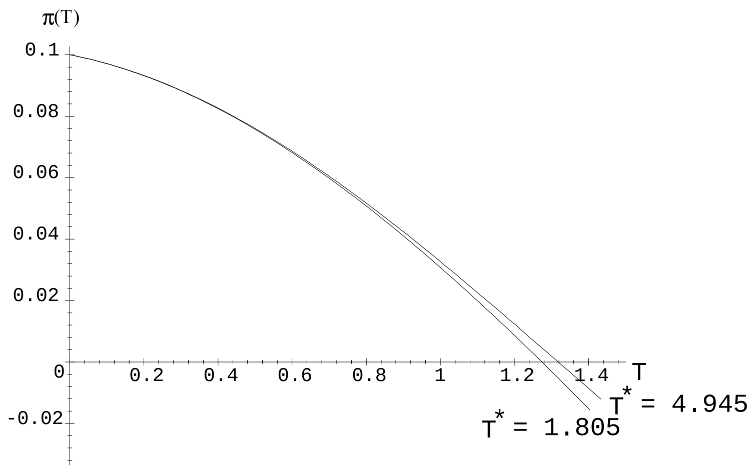

Figure 3: (Almost) the same domestic inflation for distinct foreign terms

The previous fact is also evident from figures 4.1 and 4.2, which show the implicit relation between the optimal electoral period length, $\tilde{T}$, and the foreign one, T^*.

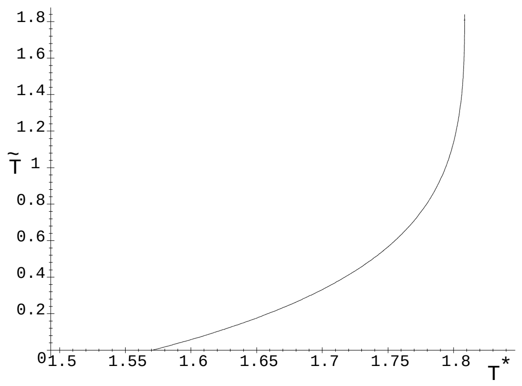

Figure 4.1: The domestic optimal *vs.* the foreign electoral period lengths

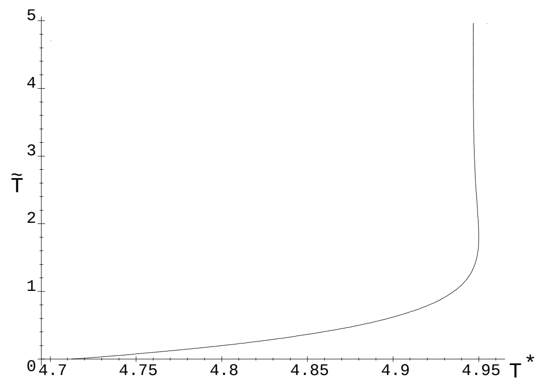

Figure 4.2: The domestic optimal *vs.* the foreign electoral period lengths

As is obvious, as T^* decreases there is a rapid increase in the optimal electoral period length such that the perfect synchronisation of election occurs when $\tilde{T} = T^* \simeq 2$ or $\tilde{T} = T^* \simeq 5$.

To sum up, for a given foreign electoral period length, T^*, within a certain interval, an increase in the optimal domestic electoral period length, $\tilde{T}$, should be observed as T^* also increases. This direct relationship between $\tilde{T}$ and T^* can be explained by the augmented time response of domestic policy needed in order to obtain the first-best optimal core inflation value on the domestic election day. Naturally, the fact that the initial core inflation values are the same for both countries is, to a certain extent, crucial. In fact, one should confirm that this direct relationship between the electoral period lengths in both countries should be observed once *the initial core inflations in both countries are of the same sign*. Hence, let us proceed to consider a case where $\pi_0 = -\pi_0^* = -10\%$. This case is illustrated by figure 5.

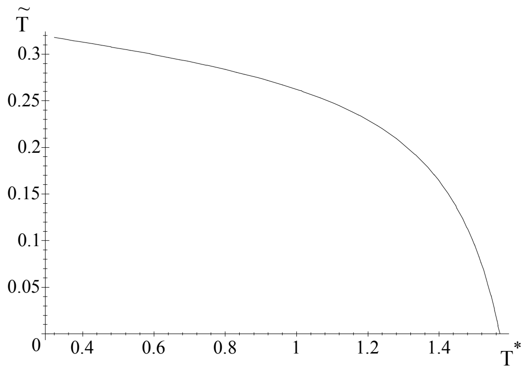

Figure 5: A case where $\pi_0 \neq \pi_0^*$

As expected, the optimal electoral period length for the domestic economy is inversely related to the foreign electoral period length. One would tentatively argue that, as in MILLER and SALMON (1990), the equality or not (in our case, in terms of the signs) of initial core inflation values is shown to be of decisive importance. In fact, also as happens with MILLER and SALMON's (1990) conclusions about when coordination pays, one has to admit that, *at first sight*, it may seem quite unsatisfactory to obtain conclusions about the degree of electoral synchronisation which depend on some specific initial conditions. However, as MILLER and SALMON (1990) clearly point out, this 'dependence' is simply a reflection of the deterministic nature of the analysis. Using some results obtained by LEVINE and CURRIE (1987), it is, in fact, possible to generalize the obtained conclusions by performing a stochastic interpretation of the results.[16] In this sense, the situation where both countries start with the same rate of core inflation – as illustrated in figures 4.1 and 4.2 – corresponds to stochastic inflation shocks perfectly (and positively) correlated whereas the situation

[16]In technical terms, this is based on the fact that for some deterministic environment characterised by a set of initial conditions, it is possible to consider an appropriate correlation matrix for stochastic shocks leading to the same expected cost.

where one country starts with a rate of core inflation that is symmetric to the one corresponding to the other country's initial rate of core inflation – as illustrated in figure 5 – corresponds to stochastic inflation shocks perfectly (and negatively) correlated; see MILLER *et al.* (1991), p. 153.

To sum up, one may tentatively add to MILLER and SALMON's (1990) conclusion that "*coordination may or may not pay depending on the correlation of the disturbances facing the two countries*" by saying that this correlation is also decisive for inferring the (optimal) degree of electoral synchronisation in the sense that the way stochastic shocks impinging on inflation are correlated is also the way the small economy electoral period length should be correlated with the electoral period length of the other economy.

4 The Solution for the Differential Game

Given our scientific objectives, that is, the study of (im)perfect synchronisation of time horizons (*v.g.* elections) this leads us to a rather interesting problem. To the best of my knowledge, the existing differential games models *always* consider that players possess the same time horizon, that is, ∞ or some finite value T; see, *inter alia*, MILLER and SALMON (1985a,1985b,1990). However, in order to analyse the implications of non-synchronised elections one must consider that governments possess distinct time horizons and this induces the following problem.

Suppose that some (vector) of state variables, y, has the following law of motion:

$$\frac{dy}{dt} \equiv \dot{y} = \mathbf{A}y + \mathbf{B}_1 x_1 + \mathbf{B}_2 x_2, \tag{35}$$

where x_1 and x_2 denote, respectively, the control variables of player 1 and player 2 and $\mathbf{A}$, $\mathbf{B}_1$, $\mathbf{B}_2$ are matrices of appropriate dimensions. Furthermore, suppose that each player wants to maximise the following criterion:

$$J_i = -\frac{1}{2}\int_0^{T_i} (y - \hat{y}_i)' \, \mathbf{Q}_i \, (y - \hat{y}_i) + (x_i - \hat{x}_i)' \, \mathbf{R}_i \, (x_i - \hat{x}_i) \, dt, \qquad i = 1, 2, \tag{36}$$

where $\hat{y}_i$ and $\hat{x}_i$ represent desired/bliss values and $\mathbf{Q}_i \geq 0$ and $\mathbf{R}_i > 0$ are symmetric matrices of weights. As usual, the (non-cooperative) maximisation of (36) taking into account (35) will be obtained through the maximisation of the Hamiltonians defined as follows:

$$\mathcal{H}_i = -\frac{1}{2}\left((y - \hat{y}_i)' \, \mathbf{Q}_i \, (y - \hat{y}_i) + (x_i - \hat{x}_i)' \, \mathbf{R}_i \, (x_i - \hat{x}_i)\right) + \lambda_i \, (t) \, (\mathbf{A}y + \mathbf{B}_1 x_1 + \mathbf{B}_2 x_2) \, ,$$

where $\lambda_i \, (t)$ are (vectors of) co-state variables.

Hence, the Nash equilibrium for this game will be the solution of the following set of first-order conditions:[17]

$$\frac{\partial \mathcal{H}_1}{\partial x_1} = 0 \Rightarrow -\mathbf{R}_1 \left(\tilde{x}_1 - \hat{x}_1\right) + \mathbf{B}_1^{\prime}\tilde{\lambda}_1 = 0 \Leftrightarrow \tilde{x}_1 = \hat{x}_1 + \mathbf{R}_1^{-1}\mathbf{B}_1^{\prime}\tilde{\lambda}_1 \tag{37}$$

$$\frac{\partial \mathcal{H}_1}{\partial \lambda_1} = \mathbf{A}\tilde{y} + \mathbf{B}_1\tilde{x}_1 + \mathbf{B}_2\tilde{x}_2 = \dot{\tilde{y}} \tag{38}$$

$$-\frac{\partial \mathcal{H}_1}{\partial y} = \mathbf{Q}_1 \left(\tilde{y} - \hat{y}_1\right) - \mathbf{A}^{\prime}\tilde{\lambda}_1 = \dot{\tilde{\lambda}}_1 \tag{39}$$

$$\frac{\partial \mathcal{H}_2}{\partial x_2} = 0 \Rightarrow -\mathbf{R}_2 \left(\tilde{x}_2 - \hat{x}_2\right) + \mathbf{B}_2^{\prime}\tilde{\lambda}_2 = 0 \Leftrightarrow \tilde{x}_2 = \hat{x}_2 + \mathbf{R}_2^{-1}\mathbf{B}_2^{\prime}\tilde{\lambda}_2 \tag{40}$$

$$\frac{\partial \mathcal{H}_2}{\partial \lambda_2} = \mathbf{A}\tilde{y} + \mathbf{B}_1\tilde{x}_1 + \mathbf{B}_2\tilde{x}_2 = \dot{\tilde{y}} \tag{41}$$

$$-\frac{\partial \mathcal{H}_2}{\partial y} = \mathbf{Q}_2 \left(\tilde{y} - \hat{y}_2\right) - \mathbf{A}^{\prime}\tilde{\lambda}_2 = \dot{\tilde{\lambda}}_2 \tag{42}$$

Plugging (37) and (40) into (38) or (41) we obtain

$$\dot{\tilde{y}} = \mathbf{A}\tilde{y} + \mathbf{B}_1\hat{x}_1 + \mathbf{B}_1\mathbf{R}_1^{-1}\mathbf{B}_1^{\prime}\tilde{\lambda}_1 + \mathbf{B}_2\hat{x}_2 + \mathbf{B}_2\mathbf{R}_2^{-1}\mathbf{B}_2^{\prime}\tilde{\lambda}_2,$$

which, in conjugation with (39) and (42), leads us to the following system of linear differential equations:

$$\begin{bmatrix} \dot{\tilde{y}} \\ \dot{\tilde{\lambda}}_1 \\ \dot{\tilde{\lambda}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_1\mathbf{R}_1^{-1}\mathbf{B}_1^{\prime} & \mathbf{B}_2\mathbf{R}_2^{-1}\mathbf{B}_2^{\prime} \\ \mathbf{Q}_1 & -\mathbf{A}^{\prime} & 0 \\ \mathbf{Q}_2 & 0 & -\mathbf{A}^{\prime} \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1\hat{x}_1 + \mathbf{B}_2\hat{x}_2 \\ -\mathbf{Q}_1\hat{y}_1 \\ -\mathbf{Q}_2\hat{y}_2 \end{bmatrix}. \tag{43}$$

This system (43) has to be solved given some initial conditions $y(0) = y_0$ and some transversality conditions concerning the co-state variables which, in this case, would be to have $\tilde{\lambda}_i = 0$ at the horizons T_i. Now, when both players possess the same horizon, that is, when $T_1 = T_2$, the solution of (43) offers no particular difficulty except, of course, computational ones. However, when $T_1 \neq T_2$, which is (also) our interesting case, the solution of (43) given $y(0) = y_0$, $\tilde{\lambda}(T_1) = 0$ and $\tilde{\lambda}(T_2) = 0$ would be wrong if it simply corresponded to the assumption that the co-state path $\lambda_1(T)$ for $(T_1, T_2]$ – if, say, $T_2 > T_1$ – is just the continuation of the one determined for the interval $[0, T_1]$. This is basically wrong, because at T_1 player 1 will re-optimise his criterion and, thus, there is certainly a 'jump' on $\tilde{\lambda}_1$. Moreover, we cannot assume that the co-state variable $\tilde{\lambda}_1$ takes the same evolution in $(T_1, T_2]$ as it did in the first

[17] The ˜ indicates the optimal values.

$T_2 - T_1$ first periods of the interval $[0, T_1]$, because the initial conditions are different, that is, $y_0 \neq y_{T_1}$.

We can, however, solve the problem if the time horizons are determined endogenously, as will be shown next.

4.1 The (endogenous) optimal time solution

As the main problem is to choose the optimal timing of elections, *from the governments' point of view*, the correct way of attacking the problem is by considering a *free end terminal problem* or, in other words, the determination of the (optimal) horizons T_i endogenously. This imposes, see, *inter alia*, LÉONARD and LONG (1992) or TAKAYAMA (1994), the (additional) condition that

$$\sup_{x_i} \mathcal{H}_i \left(y_i \left(\tilde{T}_i \right), x_i \left(\tilde{T}_i \right), \lambda_i \left(\tilde{T}_i \right), \tilde{T}_i \right) = 0, \tag{44}$$

where the Hamiltonians are given by[18]

$$\mathcal{H}_i = -\frac{1}{2} \left((y - \hat{y}_i)' \mathbf{Q}_i (y - \hat{y}_i) + (x_i - \hat{x}_i)' \mathbf{R}_i (x_i - \hat{x}_i) \right) + \lambda_i (t) \left(\mathbf{A} y + \mathbf{B}_1 x_1 + \mathbf{B}_2 x_2 \right).$$

Fulfilled the transversality conditions $\lambda_i = 0$ at the horizons T_i, we then have that, to fulfil condition (44),

$$\tilde{y} = \hat{y}_i \textbf{ and } \tilde{x}_i = \hat{x}_i. \tag{45}$$

1. If $\hat{y}_1 = \hat{y}_2$, then to make (45) possible, *necessarily*, $T_1 = T_2 = \tilde{T}$, that is, a perfect synchronisation of elections. **But**, in order for a solution $\tilde{T}$ to exist:

2. $$\tilde{x}_1 \left(\tilde{T} \right) = \hat{x}_1 \left(\tilde{T} \right) \textbf{ and } \tilde{x}_2 \left(\tilde{T} \right) = \hat{x}_2 \left(\tilde{T} \right). \tag{46}$$

In fact, the fulfilment of (46) is guaranteed by the transversality conditions $\tilde{\lambda}_i \left(\tilde{T} \right) = 0$, see equations (37) and (40).

We can then conjecture that when both governments happen to possess the same desired value for the state variables, *i.e.* $\hat{y}_1 = \hat{y}_2$, the solution of (43) is possible to obtain, given the initial conditions $y(0) = y_0$ and some transversality conditions $\tilde{\lambda}_i \left(\tilde{T} \right) = 0$. The determination of $\tilde{T}$ comes, then, indirectly from the solution of the system (43).[19] Let us apply this to MILLER and SALMON's (1990) model.

[18] LÉONARD and LONG (1992), p. 241, offer the proof that, in fact, $\frac{\partial J_i}{\partial T} = \sup_{x_i} \mathcal{H}_i (.)$, where J_i are given by (36). It is, thus, evident that it is exactly the best horizon that it is being chosen as it is the one that maximizes the criteria J_i.

[19] On the contrary, if $\hat{y}_1 \neq \hat{y}_2$, the existence of a solution is not guaranteed.

In this case, let us consider that both economies are of the same size such that a differential game is appropriate to describe the situation. Using the same parameterisation as in MILLER and SALMON (1990) one can obtain the solution given the perfect synchronisation of elections, *i.e.* $T = T^*$; see the Appendix. Figures 6.1 and 6.2 show the relationships between the *optimal electoral period lengths*, that is, those corresponding to a zero inflation rate on the election day, and the *initial inflation rates*, which we assume to be equal.[20]

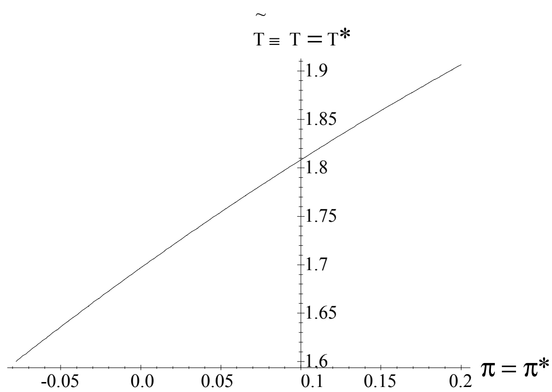

Figure 6.1: The optimal (*domestic*) electoral synchronisation

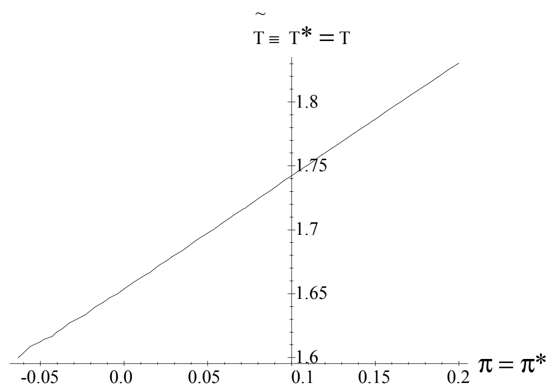

Figure 6.2: The optimal (*foreign*) electoral synchronisation

An increase in the initial core inflation corresponds to an increase in the optimal electoral period length. Given the equality in the initial inflation rates, the optimal electoral period length in one of the economies, given that in the other economy the

[20] As pointed out at the end of the previous section, the importance of the 'initial conditions' for inflation rates must be made relative to the deterministic nature of the analysis.

same electoral period length is in practice (not necessarily the optimal one), increases as the initial inflation rate increases. Figure 6.1 shows the optimal electoral synchronisation *from the viewpoint of the domestic economy*, whereas Figure 6.2 shows the same *from the viewpoint of the foreign economy.*[21] As is clear from the two figures, the interests of both economies in what concerns the existing electoral period length in the other economy are not always compatible, which is due to the evolution of the exchange rate. This is not to say that there is no possible electoral period length $\tilde{T}$ corresponding to the optimal one for *both* economies. In fact, considering the previous two figures together, one can verify that, for some electoral period length $\tilde{T}$, both economies would find it optimal to possess national electoral period lengths equal to $\tilde{T}$. Figure 7 shows this fact.

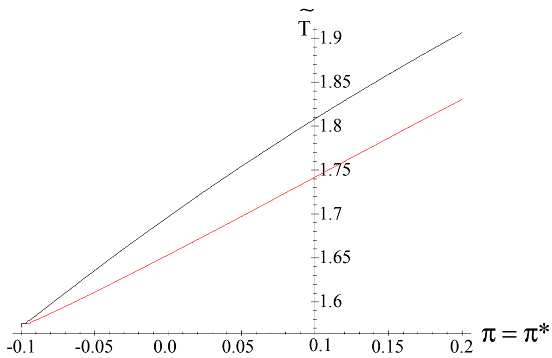

Figure 7: The optimal electoral synchronisation

5 Conclusions

The first purpose of the paper was to analyse how the government of a *small* open economy can determine the optimal degree of election dates synchronisation with those existing in a *big* economy. To achieve this purpose, the paper used an asymmetric version of MILLER and SALMON's (1990) model in order to derive the optimal domestic electoral period length $\tilde{T}$, which, in this sense, can be said to be endogenously determined. This being said, the analysis performed in this paper should be viewed as relevant to the study of the circumstances in which a government of a single country (taken as a representative agent) has incentives (or not) to synchronise the domestic election dates with the election dates (not necessarily determined in an endogenous way) of a country performing the role of an 'anchor'.

[21] In a sense, one can view the lines represented in both figures as *reaction curves* giving the optimal responses in terms of the national electoral lengths.

As a first conclusion, the paper has shown how crucial are the initial conditions in what concerns inflation to the determination of the kind of relationship that should exist between the domestic election period length and the foreign one. This direct relationship between $\tilde{T}$ and T^* can be explained by the augmented time response of domestic policy needed in order to obtain the first-best optimal core inflation value on the domestic election day. At first sight, it may seem quite unsatisfactory to obtain conclusions about the degree of electoral synchronisation which depend on some specific initial conditions. However, as MILLER and SALMON (1990) clearly point out, this 'dependence' is simply a reflection of the deterministic nature of the analysis. Using results obtained by LEVINE and CURRIE (1987), one could tentatively generalise the obtained conclusions by performing a stochastic interpretation of the results. In this sense, the situation where both countries start with the same rate of core inflation corresponds to stochastic inflation shocks perfectly (and positively) correlated, whereas the situation where one country starts with a rate of core inflation that is symmetric to the one corresponding to the other country's initial rate of core inflation corresponds to stochastic inflation shocks perfectly (and negatively) correlated.

The second main purpose of the paper was to re-analyse the situation being studied by considering that the foreign government also determine its election dates in an optimal way, this leading to a differential game played by the two incumbents from which incentives to totally synchronise the election dates may result. As was shown, the interests of both economies in what concerns the existing electoral period length in the other economy are not always compatible, which is due to the evolution of the exchange rate. This is not to say that there is no possible electoral period length corresponding to the optimal one for *both* economies. In fact, one could verify in what circumstances both economies would find it optimal to possess the same national electoral period lengths.

6 Appendix – Mathematical Details

From the equations expressing the domestic demands,

$$\begin{aligned} y &= -\gamma r + \delta c + \eta y^* \\ y^* &= -\gamma^* r^* - \delta^* c + \eta^* y, \end{aligned}$$

one can obtain

$$
\begin{aligned}
y &= -\frac{\gamma}{1-\eta\eta^*}r - \frac{\eta\gamma^*}{1-\eta\eta^*}r^* - \frac{\delta+\eta\delta^*}{1-\eta\eta^*}c \\
y^* &= -\frac{\gamma^*}{1-\eta\eta^*}r^* - \frac{\eta^*\gamma}{1-\eta\eta^*}r - \frac{\delta^*-\eta^*\delta}{1-\eta\eta^*}c.
\end{aligned}
$$

In terms of the interest rates,

$$
\begin{aligned}
r &= -\frac{y-\delta c-\eta y^*}{\gamma} \\
r^* &= -\frac{y^*+\delta^* c-\eta^* y}{\gamma^*},
\end{aligned}
$$

such that the arbitrage condition $\mathsf{E}\left[\frac{dc}{dt}\right] = r - r^*$ can be expressed as

$$
\frac{dc}{dt} = -\frac{\gamma^*+\gamma\eta^*}{\gamma\gamma^*}y + \frac{\gamma+\gamma^*\eta}{\gamma\gamma^*}y^* + \frac{\gamma^*\delta+\gamma\delta^*}{\gamma\gamma^*}c,
$$

or

$$
\frac{dc}{dt} = \frac{\gamma^*+\gamma\eta^*}{\gamma\gamma^*}\lambda - \frac{\gamma+\gamma^*\eta}{\gamma\gamma^*}\lambda^* + \frac{\gamma^*\delta+\gamma\delta^*}{\gamma\gamma^*}c
$$

given that

$$
y = -\lambda
$$

and

$$
y^* = -\lambda^*.
$$

From the equations concerning the core inflations,

$$
\begin{aligned}
\pi &= \xi\phi z + \xi\sigma c \\
\pi^* &= \xi^*\phi^* z^* - \xi^*\sigma^* c,
\end{aligned}
$$

one can obtain

$$
\begin{aligned}
\pi &= \xi\left(\phi+\sigma\theta_1\right)z + \xi\sigma\theta_2 z^* \\
\pi^* &= \xi^*\left(\phi^*-\sigma^*\theta_2\right)z^* - \xi^*\sigma^*\theta_1 z,
\end{aligned}
$$

given that

$$c = \theta_1 z + \theta_2 z^*.$$

From the previous equations one can easily derive

$$\begin{aligned}
-\frac{d\lambda}{dt} &= \frac{\partial \mathcal{H}}{\partial z} = \beta\xi\left(\phi + \sigma\theta_1\right)\pi \Rightarrow \\
\frac{d\lambda}{dt} &= -\beta\xi^2\left(\phi + \sigma\theta_1\right)^2 z - \beta\xi^2\left(\phi + \sigma\theta_1\right)\sigma\theta_2 z^*; \\
-\frac{d\lambda^*}{dt} &= \frac{\partial \mathcal{H}^*}{\partial z^*} = \beta^*\xi^*\left(\phi^* - \sigma^*\theta_2\right)\pi^* \Rightarrow \\
\frac{d\lambda^*}{dt} &= -\beta^*\xi^{*^2}\left(\sigma^*\theta_2 - \phi^*\right)^2 z^* - \beta^*\xi^{*^2}\left(\sigma^*\theta_2 - \phi^*\right)\sigma^*\theta_1 z.
\end{aligned}$$

The parameterisation considered in MILLER and SALMON (1990) results in the following system of differential equations:

$$\begin{aligned}
\frac{dz}{dt} &= -\lambda \\
\frac{dz^*}{dt} &= -\lambda^* \\
\frac{dc}{dt} &= 2c + 2\lambda - \frac{8}{3}\lambda^* \\
\frac{d\lambda}{dt} &= -z \\
\frac{d\lambda^*}{dt} &= z^*,
\end{aligned}$$

which, after considering the transversality conditions $\lambda\left(T\right) = \lambda^*\left(T^*\right) = 0$ and the initial conditions $z\left(0\right) = z_0$, $z^*\left(0\right) = z_0^*$, $c\left(0\right) = c_0$, leads to the following solutions:

$$
\begin{aligned}
z(t) &= \frac{e^{T-t}+e^{t-T}}{e^{T}+e^{-T}} z_0 \\
z^*(t) &= \frac{\sin T^* \sin t + \cos T^* \cos t}{\cos T^*} z_0^* \\
c(t) &= \frac{n(t)}{15\left(e^{T}+e^{-T}\right)\cos T^*} \\
\lambda(t) &= \frac{e^{T-t}-e^{t-T}}{e^{T}+e^{-T}} z_0 \\
\lambda^*(t) &= \frac{\cos T^* \sin t - \sin T^* \cos t}{\cos T^*} z_0^*,
\end{aligned}
$$

where

$$
\begin{aligned}
n(t) = {} & 10\left(3e^{t-T}+e^{2t+T}-3e^{2t-T}-e^{T-t}\right) z_0 \cos T^* - 8\left(e^{2t+T}+e^{2t-T}\right) z_0^* \cos T^* \\
& +15\left(e^{2t+T}+e^{2t-T}\right) c_0 \cos T^* + 16\left(e^{2t+T}+e^{2t-T}\right) z_0^* \sin T^* \\
& +8\left(e^{T}+e^{-T}\right)\left(\sin t \sin T^* + 2\sin t \cos T^* - 2\cos t \sin T^* + \cos t \cos T^*\right) z_0^*.
\end{aligned}
$$

The solution of the system where both economies are of the same size such that $\eta = \eta^* = \frac{1}{3}$ and a perfect synchronisation of elections is imposed *a priori*, *i.e.* $T = T^* = \tau$ is:

$$
\begin{aligned}
z(t) &= \frac{e^{\tau-t}+e^{t-\tau}}{e^{\tau}+e^{-\tau}} z_0 \\
z^*(t) &= \frac{\sin \tau \sin t + \cos \tau \cos t}{\cos \tau} z_0^* \\
c(t) &= \frac{p(t)}{45\left(e^{\tau}+e^{-\tau}\right)\cos \tau} \\
\lambda(t) &= \frac{e^{\tau-t}-e^{t-\tau}}{e^{\tau}+e^{-\tau}} z_0 \\
\lambda^*(t) &= \frac{\cos \tau \sin t - \sin \tau \cos t}{\cos \tau} z_0^*,
\end{aligned}
$$

where

$$
\begin{aligned}
p(t) = {} & 40\left(3e^{t-\tau}+e^{2t+\tau}-3e^{2t-\tau}-e^{\tau-t}\right) z_0 \cos \tau + 45\left(e^{2t+\tau}+e^{2t-\tau}\right) c_0 \cos \tau \\
& -24\left(e^{2t+\tau}+e^{2t-\tau}\right) z_0^* \cos \tau + 48\left(e^{2t+\tau}+e^{2t-\tau}\right) z_0^* \sin \tau \\
& +24\left((\sin t \sin \tau) e^{\tau} + (\sin t \sin \tau) e^{-\tau} + (\cos t \cos \tau) e^{\tau} + (\cos t \cos \tau) e^{-\tau}\right) z_0^* \\
& +48\left((\sin t \cos \tau) e^{\tau} + (\sin t \cos \tau) e^{-\tau} - (\cos t \sin \tau) e^{\tau} - (\cos t \sin \tau) e^{-\tau}\right) z_0^*.
\end{aligned}
$$

References

[1] BALKE, Nathan S. (1991), "Partisanship Theory, Macroeconomic Outcomes, and Endogenous Elections", *Southern Economic Journal*, **57**, **No. 4**, April, 920-935.

[2] BLAKE, Andrew P., and Peter F. WESTAWAY (1995), "An Analysis of the Impact of Finite Horizons on Macroeconomic Control", *Oxford Economic Papers*, **47**, **No. 1**, January, 98-116.

[3] CALEIRO, António (1996), "Business Cycles and Elections: A Possible Application to the Portuguese Economy?", *June Paper*, mimeo, Department of Economics, European University Institute, Florence.

[4] CALEIRO, António (2000), "Dynamic Interactions of Time Horizons in a Two Country Model: An electoral business cycles approach", forthcoming in *Proceedings of the 3rd International Workshop on European Economy*, CEDIN, Instituto Superior de Economia e Gestão, Lisbon.

[5] CHAPPELL, D., and D.A. PEEL (1979), "On the Political Theory of the Business Cycle", *Economics Letters*, **2**, 327-332.

[6] COHEN, Daniel, and Phillipe MICHEL (1988), "How Should Control Theory Be Used to Calculate a Time-Consistent Government Policy?", *The Review of Economic Studies*, **55**, 263-274.

[7] ELLIS, Christopher J., and Mark A. THOMA (1991), "Partisan Effects in Economies with Variable Electoral Terms", *Journal of Money, Credit, and Banking*, **23**, **No. 4**, November, 728-741.

[8] EUROPEAN COMMISSION (1997), "Economic Policy in EMU: Part A", *Economic Papers* **No. 124**.

[9] GINSBURGH, Victor, and Philippe MICHEL (1983), "Random Timing of Elections and the Political Business Cycle", *Public Choice*, **40**, 155-164.

[10] HALL, S.G., and S.G.B. HENRY (1988), **Macroeconomic Modelling**, North-Holland, Amsterdam.

[11] KAYSER, Mark A. (1998), "Convergence? EMU, Trade and the Strategic Timing of Elections", paper presented at the UC Berkeley conference 'EMU: Getting the Starting Game Right', mimeo, UCLA.

[12] KING, Mervyn (1998), "The Political Economy of European Monetary Union", *EIB Lecture Series*, September, Department of Economics, European University Institute, Florence.

[13] LÄCHLER, Ulrich (1982), "On Political Business Cycles with Endogenous Election Dates", *Journal of Public Economics*, **17**, 111-117.

[14] LÉONARD, Daniel, and Ngo Van LONG (1992), **Optimal Control Theory and Static Optimization in Economics**, Cambridge University Press, Cambridge.

[15] LEVINE, Paul, and David CURRIE (1987), "The Design of Feedback Rules in Linear Stochastic Rational Expectations Models", *Journal of Economic Dynamics and Control*, **11**, 1-28.

[16] MARKINK, A.J., and F. Van der PLOEG (1989), "Dynamic Policy Simulation of Linear Models with Rational Expectations of Future Events: A Computer Package", *Discussion Paper* **No. 8906**, CentER for Economic Research, Tilburg University.

[17] MILLER, Marcus, and Mark SALMON (1985a), "Policy Coordination and Dynamic Games", in **International Economic Policy Coordination**, edited by Willem H. Buiter and Richard C. Marston, Cambridge University Press, Cambridge, 184-227.

[18] MILLER, Marcus, and Mark SALMON (1985b), "Dynamic Games and the Time Inconsistency of Optimal Policy in Open Economies", *The Economic Journal* (Supplement), **85**, 124-137.

[19] MILLER, Marcus, and Mark SALMON (1990), "When Does Coordination Pay?", *Journal of Economic Dynamics and Control*, **14**, 553-569.

[20] MILLER, Marcus, Mark SALMON and Alan SUTHERLAND (1991), "Time Consistency, Discounting and the Returns to Co-operation", in **International Economic Policy Co-ordination**, edited by Carlo Carraro, Didier Laussel, Mark Salmon and Antoine Soubeyram, Basil Blackwell, Oxford.

[21] MINFORD, Patrick, Kent MATTHEWS and Satwant MARWAHA (1979), "Terminal Conditions as a Means of Ensuring Unique Solutions for Rational Expectations Models with Forward Expectations", *Economics Letters*, **4**, **No. 2**, 117-120.

[22] SAPIR, André, and Khalid SEKKAT (1999), "Optimum Electoral Areas: Should Europe adopt a single election day?", *European Economic Review,* **43**, **No. 8**, August, 1595-1619.

[23] TAKAYAMA, Akira (1994), **Analytical Methods in Economics**, Harvester Wheatsheaf, New York.