Non-negative solutions of systems of ODEs with coupled boundary conditions

Gennaro Infante a,⇑, Feliz M. Minhós b, Paolamaria Pietramala a

a Dipartimento di Matematica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy
b Department of Mathematics, University of Évora, Rua Romão Ramalho 59, 7000-671 Évora, Portugal

A B S T R A C T

We provide a new existence theory of multiple positive solutions valid for a wide class of systems of boundary value problems that possess a coupling in the boundary conditions. Our conditions are fairly general and cover a large number of situations. The theory is illustrated in details in an example. The approach relies on classical fixed point index.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

The problem of the existence of positive solutions for systems of local and nonlocal boundary value problems (BVPs) has received an increased attention by researchers, see for example the papers of Agarwal et al. [1–3], Ahmad and Graef [4], Ahmad and Nieto [5], Henderson et al. [14], Lan and Lin [23], Precup [28,29], Yang and Kong [36], Yang and Zhang [37] and references therein. Between systems of BVPs of particular interest are those where the boundary conditions (BCs) are coupled. Systems with coupled BCs can be applied to Lotka–Volterra models, reaction-diffusion phenomena and interaction problems, see for example the works of Amann [7], Leung [24] and Mehmeti and Nicaise [27]. A recent paper in this line of research is the one by Asif and Khan [8], who study the four-point coupled system

\[ \begin{align*}
  u''(t) + f_1(t, u(t), v(t)) &= 0, & t &\in (0, 1), \\
  v''(t) + f_2(t, u(t), v(t)) &= 0, & t &\in (0, 1), \\
  u(0) &= 0, & u(1) &= \delta_{12} v(\eta_{12}), \\
  v(0) &= 0, & v(1) &= \delta_{22} u(\eta_{22}).
\end{align*} \] (1)

The authors prove, via the well-known Guo–Krasnosel'skiĭ theorem on cone compression-expansion, the existence of one positive solution of the system (1) by means of an associated auxiliary system of Hammerstein integral equations, namely...
An integral representation of the type (2) is also used in the papers [9,38]. Yuan et al. in [38] study, by means of a nonlocal alternative of Leray–Schauder type and the Guo–Krasnosel’skiı ˘ fixed-point theorem, the existence of one or two positive solutions for a semi-positone system of fractional differential equations subject to four-point BCs. In [9] Cui and Sun study, via fixed point index theory, the existence of one positive solution for the system

\begin{align}
  u''(t) + f_1(t, u(t), v(t)) &= 0, & t &\in (0, 1), \\
  v''(t) + f_2(t, u(t), v(t)) &= 0, & t &\in (0, 1), \\
  u(0) &= 0, & u(1) &= \beta_{12}[v], \\
  v(0) &= 0, & v(1) &= \beta_{22}[u],
\end{align}

where \( \beta_{ij}[\cdot] \) are linear functionals defined via (positive) Stieltjes measures.

We mention that Stieltjes integrals are also used in the framework of nonlocal coupled BCs in the paper of Kang and Wei [18] (where the Leggett and Williams fixed point theorem is used) and in two recent papers of Goodrich [11,12] (who uses the Guo–Krasnosel’skiı ˘ fixed-point theorem); one interesting feature of [11,12] is the possibility to use signed measures, in the line of the paper by Webb and Infante [32].

Here we provide a new approach for a wide class of systems of BVPs that possess, along the coupling in the nonlinearities of the differential equations, also a coupling in the BCs and we prove, under suitable conditions, existence of multiple non-negative solutions. Our idea is to give an existence theory valid for systems of perturbed Hammerstein integral equations of the type

\begin{align}
  u(t) &= \gamma_1(t)\beta_{11}[u] + \gamma_1(t)\beta_{12}[v] + \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s))\,ds, \\
  v(t) &= \gamma_2(t)\beta_{21}[v] + \gamma_2(t)\beta_{22}[u] + \int_0^1 k_2(t, s)g_2(s)f_2(s, u(s), v(s))\,ds,
\end{align}

where \( \gamma_i \) are continuous functions and \( \beta_{ij}[\cdot] \) are linear functionals defined via Stieltjes measures. A system of perturbed Hammerstein integral equations similar to (4) is investigated by Infante and Pietramala in [16], with the intent of dealing with BVPs with nonlinear BCs, allowing a coupling in the nonlinearities \( f_1 \) and \( f_2 \) but not in the BCs. The methodology of [16] relies on an extensions of the results of [32] to the context of systems.

We illustrate our theory with an example of a system of second and fourth order ordinary differential equations

\begin{align}
  u''(t) + g_1(t)f_1(t, u(t), v(t)) &= 0, & t &\in (0, 1), \\
  v''''(t) + g_2(t)f_2(t, u(t), v(t)) &= 0, & t &\in (0, 1),
\end{align}

subject to the nonlocal boundary conditions

\begin{align}
  u(0) &= \beta_{11}[u], & u(1) &= \beta_{12}[v], \\
  v(0) &= \beta_{21}[v], & v(1) &= 0, & v''(0) &= 0, & v''''(1) + \beta_{22}[u] &= 0.
\end{align}

The system of ordinary differential equations (5), with local BCs, can be used as a model for the stationary states of a one-dimensional bridge, with a coupling between the cable and the roadbed. Here the cable is seen as vibrating string and the roadbed as a vibrating beam, see for example the papers of Lazer and McKenna [20], Lü et al. [25], Sun [30] and the doctoral thesis of Matas [26]. The boundary conditions (6) involve functionals of the form

\[ \beta_{ij}[w] = \int_0^1 w(s)\,dB_{ij}(s), \]

and include, as special cases, \( m \)-point and integral conditions, when

\[ \beta_{ij}[w] = \sum_{j=1}^m \delta_j w(\eta_j) \quad \text{and} \quad \beta_{ij}[w] = \int_0^1 \delta_j(s)w(s)\,ds. \]

For previous work on Riemann–Stieltjes integral BCs we refer the reader, for example, to the papers of Karakostas and Tsamatos [19] and Webb [31,33]. We point out that nonlocal conditions have a physical interpretation; for example the coupled condition

\[ u(0) = u(1) = v(1) = v''(0) = v(0) = 0, \quad v''(1) + \delta u(\eta) = 0, \]

models a feedback control mechanism, where the bending moment in the right end of the beam is related to the displacement registered in a point \( \eta \) of the string.
We prove our results by means of the classical fixed point index theory (see for example the review of Amann [6] and the book of Guo and Lakshmikantham [13]) and also make use of ideas from the papers [16,17,32].

2. Positive solutions for systems of integral equations

In order to utilize the classical fixed point index theory to find positive solutions of the system of integral equations

\[
\begin{align*}
    u(t) &= \gamma_{11}(t)\beta_{11}[u] + \gamma_{12}(t)\beta_{12}[v] + \int_0^1 k_1(t,s)g_1(s)f_1(s,u(s),v(s))\,ds, \\
    v(t) &= \gamma_{21}(t)\beta_{21}[v] + \gamma_{22}(t)\beta_{22}[u] + \int_0^1 k_2(t,s)g_2(s)f_2(s,u(s),v(s))\,ds,
\end{align*}
\]

(7)

we make the following hypotheses on the terms that occur in (7):

- For every \(i = 1, 2, f_i : [0, 1] \times [0, \infty) \to [0, \infty)\) satisfies Carathéodory conditions, that is, \(f_i(\cdot, u, v)\) is measurable for each fixed \((u, v)\) and \(f_i(\cdot, \cdot)\) is continuous for almost every (a.e.) \(t \in [0, 1]\) and for each \(r > 0\) there exists \(\phi_r \in L^\infty[0, 1]\) such that

\[
f_i(t, u, v) \leq \phi_r(t) \quad \text{for } u, v \in [0, r] \text{ and a.e. } t \in [0, 1].
\]

- For every \(i = 1, 2, k_i : [0, 1] \to [0, \infty)\) is measurable, and for every \(\tau \in [0, 1]\) we have

\[
\lim_{|t-s| \to 0} \left| k_i(t, s) - k_i(\tau, s) \right| = 0 \quad \text{for a.e. } s \in [0, 1].
\]

- For every \(i = 1, 2\), there exist a subinterval \([a_i, b_i] \subseteq [0, 1]\), a function \(\Phi_i \in L^\infty[0, 1]\), and a constant \(c_i \in (0, 1]\), such that

\[
k_i(t, s) < \Phi_i(s) \quad \text{for } t \in [0, 1] \text{ and a.e. } s \in [0, 1],
\]

\[
k_i(t, s) > c_i\Phi_i(s) \quad \text{for } t \in [a_i, b_i] \text{ and a.e. } s \in [0, 1].
\]

- For every \(i, j = 1, 2, g_i \Phi_i \in L^1[0, 1]\), \(g_i \geq 0\) a.e., and \(\int_0^1 \Phi_i(s)g_i(s)\,ds > 0\).

- For every \(i, j = 1, 2, \beta_{ij}[\cdot] \) is a linear functional given by

\[
\beta_{ij}[w] = \int_0^1 w(s)\,dB_i(s),
\]

involving Riemann–Stieltjes integrals; \(B_i\) is of bounded variation and \(dB_i\) is a positive measure.

- For every \(i, j = 1, 2, \gamma_{ij} \in C[0, 1]\), \(\gamma_{ij}(t) > 0\) for every \(t \in [0, 1]\), \(\beta_i[\gamma_{ij}] < 1\) and there exists \(c_i \in (0, 1]\) such that

\[
\gamma_{ij}(t) > c_i\|\gamma_{ij}\|_{\infty} \quad \text{for every } t \in [a_i, b_i],
\]

where \(\|w\|_{\infty} := \max\{|w(t)|, t \in [0, 1]\}\).

We work in the space \(C[0, 1] \times C[0, 1]\) endowed with the norm

\[
\|(u, v)\| := \max\{\|u\|_{\infty}, \|v\|_{\infty}\}.
\]

Let

\[
\tilde{K}_i := \{w \in C[0, 1] : w(t) \geq 0 \quad \text{for } t \in [0, 1] \text{ and } \min_{t \in [a_i, b_i]} w(t) \geq \tilde{c}_i\|w\|_{\infty}\},
\]

where \(\tilde{c}_i = \min(c_i, c_{i1}, c_{i2})\), and consider the cone \(K\) in \(C[0, 1] \times C[0, 1]\) defined by

\[
K := \{(u, v) \in \tilde{K}_1 \times \tilde{K}_2\}.
\]

For a positive solution of the system (7) we mean a solution \((u, v) \in K\) of (7) such that \(\|(u, v)\| > 0\).

Under our assumptions, we show that the integral operator

\[
T(u, v)(t) = \begin{pmatrix}
    T_1(u, v)(t) \\
    T_2(u, v)(t)
\end{pmatrix} := \begin{pmatrix}
    \gamma_{11}(t)\beta_{11}[u] + \gamma_{12}(t)\beta_{12}[v] + F_1(u, v)(t) \\
    \gamma_{21}(t)\beta_{21}[v] + \gamma_{22}(t)\beta_{22}[u] + F_2(u, v)(t)
\end{pmatrix},
\]

(8)

where

\[
F_i(u, v)(t) := \int_0^1 k_i(t, s)g_i(s)f_i(s, u(s), v(s))\,ds,
\]

leaves the cone \(K\) invariant and is compact.

**Lemma 1.** The operator (8) maps \(K\) into \(K\) and is compact.
Proof. Take \((u, v) \in K\) such that \(\|(u, v)\| < r\). Then we have, for \(t \in [0, 1]\),

\[
T_1(u, v)(t) = \gamma_{11}(t)\beta_{11}[u] + \gamma_{12}(t)\beta_{12}[v] + \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s))\, ds
\]

therefore

\[
\|T_1(u, v)\| \leq \|\gamma_{11}\|_{\infty}\beta_{11}[u] + \|\gamma_{12}\|_{\infty}\beta_{12}[v] + \int_0^1 \Phi_1(s)g_1(s)f_1(s, u(s), v(s))\, ds.
\]

Then we obtain

\[
\min_{t \in [0, 1]} T_1(u, v)(t) \geq c_{11}\|\gamma_{11}\|_{\infty}\beta_{11}[u] + c_{12}\|\gamma_{12}\|_{\infty}\beta_{12}[v] + c_1 \int_0^1 \Phi_1(s)g_1(s)f_1(s, u(s), v(s))\, ds \geq \tilde{c}_1\|T_1(u, v)\|.
\]

Hence we have \(T_1(u, v) \in \tilde{K}_1\). In a similar manner we proceed for \(T_2(u, v)\).

Moreover, the map \(T_0\) is compact since the components \(T_i\) are sum of two compact maps: the compactness of \(F_i\) is well-known and, since \(\gamma_{1i}\) and \(\gamma_{2i}\) are continuous, the perturbation \(\gamma_{1i}(t)\beta_{1i}[u] + \gamma_{2i}(t)\beta_{2i}[v]\) maps bounded sets into bounded subsets of a finite dimensional space.

We use the following (relative) open bounded sets in \(K\):

\[
K_\rho = \{(u, v) \in K : \|(u, v)\| < \rho\},
\]

and

\[
V_\rho = \{(u, v) \in K : \min_{t \in [0, 1]} u(t) < \rho \text{ and } \min_{t \in [0, 1]} v(t) < \rho\}.
\]

The set \(V_\rho\) (in the context of systems) was introduced by Infante and Pietramala [15] and is equal to the set called \(\Omega_\rho^{c}\) by Franco, Infante and O’Regan [10]. \(\Omega_\rho^{c}\) is an extension to the case of systems of a set given by Lan [22]. The advantage of the notation \(V_\rho\) is that sheds light on the fact (see also the paper by Infante and Webb [17]) that choosing \(c\) as large as possible provides a weaker condition to be satisfied by the functions \(f_i\) in Lemmas 3 and 4. Note that \(K_\rho \subset V_\rho \subset K_{\rho r}\), where \(c = \min(c_1, c_2)\). We denote by \(\partial K_\rho\) and \(\partial V_\rho\), the boundary of \(K_\rho\) and \(V_\rho\), relative to \(K\).

In the next Lemma we make use of the notation

\[
\mathcal{X}_ij(s) := \int_0^1 k_i(t, s)dB_j(t), \quad i, j = 1, 2,
\]

and we prove that the index is 1 on \(K_\rho\).

Lemma 2. Assume that

\[
(1) \text{ there exists } \rho > 0 \text{ such that for every } i = 1, 2
\]

\[
\|\gamma_{1i}\|_{\infty}\beta_{1i}[\gamma_{12}]_{\infty}\beta_{12}[1] + \|\gamma_{12}\|_{\infty}\beta_{12}[1] + \int_0^1 \left(\frac{1}{m_i} + \frac{1}{m_i} \right) \int_0^1 \mathcal{X}_{ii}(s)g_i(s)\, ds < 1,
\]

where

\[
f_{i, \rho} = \sup \left\{ \frac{f_i(t, u, v)}{\rho} : (t, u, v) \in [0, 1] \times [0, \rho] \times [0, \rho], \frac{1}{m_i} \right\}
\]

and \(m_i = \sup_{t \in [0, 1]} \int_0^1 k_i(t, s)g_i(s)\, ds\).

Then the fixed point index, \(i_k(T, K_\rho)\), is equal to 1.

Proof. We show that \(\mu(u, v) \neq T(u, v)\) for every \((u, v) \in \partial K_\rho\) and for every \(\mu \geq 1\); this ensures that the index is 1 on \(K_\rho\). In fact, if this does not happen, there exist \(\mu \geq 1\) and \((u, v) \in \partial K_\rho\) such that \(\mu(u, v) = T(u, v)\). Assume, without loss of generality, that \(\|u\| = \rho\) and \(\|v\| = \rho\). Then

\[
\mu u(t) = \gamma_{11}(t)\beta_{11}[u] + \gamma_{12}(t)\beta_{12}[v] + F_1(u, v)(t)
\]

and therefore, since \(v(t) \leq \rho\), for all \(t \in [0, 1]\),

\[
\mu u(t) \leq \gamma_{11}(t)\beta_{11}[u] + \gamma_{12}(t)\beta_{12}[\rho] + F_1(u, v)(t) = \gamma_{11}(t)\beta_{11}[u] + \rho \gamma_{12}(t)\beta_{12}[1] + F_1(u, v)(t).
\]

Applying \(\beta_{11}\) to both sides of (10) gives

\[
\mu \beta_{11}[u] \leq \beta_{11}[\gamma_{11}]\beta_{11}[u] + \rho \beta_{11}[\gamma_{12}]\beta_{12}[1] + \beta_{11}[F_1(u, v)]
\]

Thus we have

\[
(\mu - \beta_{11}[\gamma_{11}])\beta_{11}[u] \leq \rho \beta_{11}[\gamma_{12}]\beta_{12}[1] + \beta_{11}[F_1(u, v)].
\]
Taking the supremum of \( \frac{\beta_{11}[u]}{\mu - \beta_{11}[\gamma_{12}]} \) gives
\[
\beta_{11}[u] = \frac{\beta_{11}[\gamma_{12}][\beta_{12}[1]}{\mu - \beta_{11}[\gamma_{12}]} + \beta_{11}[F_1(u, v)].
\]

Substituting into (10) gives
\[
\mu u(t) \leq \gamma_{11}(t) \left( \frac{\beta_{11}[\gamma_{12}][\beta_{12}[1]}{\mu - \beta_{11}[\gamma_{12}]} + \beta_{11}[F_1(u, v)] \right) + \rho \gamma_{12}(t)\beta_{12}[1] + F_1(u, v)(t)
\]
\[
= \frac{\beta_{11}[\gamma_{12}][\beta_{12}[1]}{\mu - \beta_{11}[\gamma_{12}]} + \gamma_{11}(t) \int_0^1 \mathcal{K}_1(s)g_1(s, u(s), v(s)) ds + \rho \gamma_{12}(t)\beta_{12}[1] + F_1(u, v)(t).
\]

Since \( \mu > 1 \), one has \( \frac{\beta_{11}[\gamma_{12}]}{\mu - \beta_{11}[\gamma_{12}]} \leq \frac{1}{\mu - \beta_{11}[\gamma_{12}]} \) and therefore
\[
\mu u(t) \leq \frac{\beta_{11}[\gamma_{12}][\beta_{12}[1]}{1 - \beta_{11}[\gamma_{12}]} + \gamma_{11}(t) \int_0^1 \mathcal{K}_1(s)g_1(s, u(s), v(s)) ds + \rho \gamma_{12}[1] \beta_{12}[1] + F_1(u, v)(t).
\]

Taking the supremum of \( t \) on \([0, 1]\) gives
\[
\mu \rho \leq \rho \|\gamma_{11}\|_\infty \beta_{11}[\gamma_{12}][\beta_{12}[1]} + \rho \|\gamma_{12}\|_\infty \beta_{12}[1] + \rho f_1(t, u, v)(t)
\]
\[
\int_0^1 \mathcal{K}_1(s)g_1(s, u(s), v(s)) ds + \frac{1}{M_r}.
\]

Using the hypothesis (9) we obtain \( \mu \rho < \rho \). This contradicts the fact that \( \mu > 1 \) and proves the result. \( \square \)

**Lemma 3.** Assume that

\((p^0_\rho)\) there exist \( \rho > 0 \) such that for every \( i = 1, 2 \)
\[
f_i(p, \rho; c) \left( \frac{\|\gamma_{11}\|_\infty}{1 - \beta_{11}[\gamma_{12}]} \right) \int_a^b K_1(s)g_i(s) ds + \frac{1}{M_i} > 1,
\]

where
\[
f_1(p, \rho; c) = \inf \left\{ \frac{f_1(t, u, v)}{\rho} : (t, u, v) \in \left[ a_1, b_1 \right] \times [0, \rho/c] \times \left[ 0, \rho/c \right] \right\},
\]
\[
f_2(p, \rho; c) = \inf \left\{ \frac{f_2(t, u, v)}{\rho} : (t, u, v) \in \left[ a_2, b_2 \right] \times [0, \rho/c] \times [0, \rho/c] \right\}
\]
and
\[
\frac{1}{M_i} = \inf \int_a^b k_i(t, s)g_i(s) ds.
\]

Then \( i_e(T, V_\rho) = 0 \).

**Proof.** Let \( e(t) = 1 \) for \( t \in [0, 1] \). Then \((e, e) \in K\). We prove that
\[
(u, v) \neq T(u, v) + \mu(e, e) \quad \text{for} \quad (u, v) \in \partial V_\rho \quad \text{and} \quad \mu \geq 0.
\]

In fact, if this does not happen, there exist \((u, v) \in \partial V_\rho \) and \( \mu \geq 0 \) such that \((u, v) = T(u, v) + \mu(e, e)\). Without loss of generality, we can assume that for all \( t \in [a_1, b_1] \) we have
\[
\rho \leq u(t) \leq \rho/c, \quad \min u(t) = \rho \quad \text{and} \quad 0 \leq v(t) \leq \rho/c.
\]

Then, for \( t \in [a_1, b_1] \), we obtain
\[
\begin{align*}
u(t) &= \gamma_{11}(t)\beta_{11}[u] + \gamma_{12}(t)\beta_{12}[v] + \int_0^1 k_1(t, s)g_1(s, u(s), v(s)) ds + \mu e(t) \\
\text{and therefore} & \quad u(t) \geq \gamma_{11}(t)\beta_{11}[u] + F_1(u, v)(t) + \mu e(t).
\end{align*}
\]

Applying \( \beta_{11} \) to both sides of (12) gives
\[
\beta_{11}[u] \geq \beta_{11}[\gamma_{12}][\beta_{12}[1] + \beta_{11}[F_1(u, v)] + \mu \beta_{11}[e].
\]

This can be written in the form
\[
(1 - \beta_{11}[\gamma_{12}])[\beta_{11}[u] \geq \beta_{11}[F_1(u, v)] + \mu \beta_{11}[e].
\]
that is
\begin{equation}
\beta_{11}[u] \geq \frac{\beta_{11}[F_1(u, v) \gamma_{11}(t)]}{(1 - \beta_{11}[\gamma_{11}])} + \frac{\mu \beta_{11}[e]}{(1 - \beta_{11}[\gamma_{11}])}.
\end{equation}

Thus, (12) becomes
\begin{equation}
u(t) \geq \frac{\gamma(t)}{(1 - \beta_{11}[\gamma_{11}])} \times \int_{a_0}^{b_1} \mathcal{K}(s)g(s)f(s, u(s), v(s))ds + \frac{\mu \gamma(t)}{(1 - \beta_{11}[\gamma_{11}])} + \int_{a_0}^{b_1} k(t, s)g(s)f(s, u(s), v(s))ds + \mu.
\end{equation}

Then we have, for \( t \in [a_1, b_1] \),
\begin{equation}
u(t) \geq \frac{c_{11}[\gamma_{11}]}{(1 - \beta_{11}[\gamma_{11}])} \times \int_{a_0}^{b_1} \mathcal{K}(s)g(s)f(s, u(s), v(s))ds + \frac{\mu c_{11}[\gamma_{11}]}{(1 - \beta_{11}[\gamma_{11}])} + \int_{a_0}^{b_1} k(t, s)g(s)f(s, u(s), v(s))ds + \mu
\end{equation}

Taking the minimum over \([a_1, b_1]\) gives
\begin{equation}
\min_{t \in [a_1, b_1]} \nu(t) \geq \rho f_{1,(p,c)} \left( \frac{c_{11}[\gamma_{11}]}{(1 - \beta_{11}[\gamma_{11}])} \times \int_{a_0}^{b_1} \mathcal{K}(s)g(s)ds + \frac{1}{M_1} \right) + \mu
\end{equation}

Using the hypothesis (11) we obtain \( \rho = \min_{t \in [a_1, b_1]} \nu(t) > \rho + \mu \), a contradiction. \( \Box \)

The following Lemma also shows that the index is 0 on \( V_0 \); the idea here is similar to the one in Lemma 4 of [16]: this time we have to control the growth of just one nonlinearity, \( f_k \), at the cost of having to deal with a larger domain. For other results on the existence of solutions with different growth on the nonlinearities see the works [28,29] and the paper by Yang [35].

**Lemma 4.** Assume that
\begin{equation}
(f_k) \quad \text{there exist } \rho > 0 \text{ such that for some } i = 1, 2
\end{equation}
\begin{equation}
f_{i,(p,c)} \left( \frac{c_{11}[\gamma_{11}]}{(1 - \beta_{11}[\gamma_{11}])} \times \int_{a_0}^{b_1} \mathcal{K}(s)g(s)ds + \frac{1}{M_1} \right) > 1,
\end{equation}

where
\begin{equation}
f_{i,(p,c)} = \inf \left\{ \frac{f_i(t, u, v)}{\rho} : (t, u, v) \in [a_1, b_1] \times [0, \rho/c] \times [0, \rho/c] \right\}.
\end{equation}

Then \( i_k(T, V_0) = 0 \).

**Proof.** Suppose that the condition (13) holds for \( i = 1 \). Let \( e(t) \equiv 1 \) for \( t \in [0, 1] \). Then \( (e, e) \in K \). We prove that
\begin{equation}
(u, v) \neq T(u, v) + \mu(e, e) \quad \text{for } (u, v) \in \partial V_0 \text{ and } \mu > 0.
\end{equation}

In fact, if this does not happen, there exist \((u, v) \in \partial V_0 \text{ and } \mu > 0 \text{ such that } (u, v) = T(u, v) + \mu(e, e)\). So, for all \( t \in [a_1, b_1] \), \min u(t) \leq \rho \text{ and for } t \in [a_2, b_2], \min v(t) \leq \rho \text{ We have, for } t \in [0, 1],
\begin{equation}
u(t) \geq \frac{\gamma(t)}{(1 - \beta_{11}[\gamma_{11}])} \times \int_{a_0}^{b_1} \mathcal{K}(s)g(s)f(s, u(s), v(s))ds + \frac{1}{M_1} + \mu.
\end{equation}

and, as in the proof of Lemma 3,
\begin{equation}
u(t) \geq \frac{\gamma(t)}{(1 - \beta_{11}[\gamma_{11}])} \times \int_{a_0}^{b_1} \mathcal{K}(s)g(s)f(s, u(s), v(s))ds + \frac{1}{M_1} + \mu
\end{equation}

Then we have
\begin{equation}
\min_{t \in [a_1, b_1]} \nu(t) \geq \rho f_{1,(p,c)} \left( \frac{c_{11}[\gamma_{11}]}{(1 - \beta_{11}[\gamma_{11}])} \times \int_{a_0}^{b_1} \mathcal{K}(s)g(s)ds + \frac{1}{M_1} \right) + \mu
\end{equation}
Using the hypothesis (13) we obtain \( \min_{t \in [a1, b1]} u(t) > \rho + \mu \geq \rho \), a contradiction. \[ \square \]

The above Lemmas can be combined to prove the following Theorem, here we deal with the existence of at least one, two or three solutions. We stress that, by expanding the lists in conditions (S5), (S6) below, it is possible to state results for four or more positive solutions, see for example the paper by Lan [21] for the type of results that might be stated. We omit the proof which follows from the properties of fixed point index.

**Theorem 5.** The system (7) has at least one positive solution in \( K \) if either of the following conditions hold.

(S1) There exist \( \rho_1, \rho_2 \in (0, \infty) \) with \( \rho_1/c < \rho_2 \) such that \((l^0_1), (l^0_2)\) hold.

(S2) There exist \( \rho_1, \rho_2 \in (0, \infty) \) with \( \rho_1 < \rho_2 \) such that \((l^1_1), (l^1_2)\) hold.

The system (7) has at least two positive solutions in \( K \) if one of the following conditions hold:

(S3) There exist \( \rho_1, \rho_2, \rho_3 \in (0, \infty) \) with \( \rho_1/c < \rho_2 < \rho_3 \) such that \((l^0_1), (l^0_2), (l^0_3)\) hold.

(S4) There exist \( \rho_1, \rho_2, \rho_3 \in (0, \infty) \) with \( \rho_1 < \rho_2 \) and \( \rho_2/c < \rho_3 \) such that \((l^1_1), (l^1_2), (l^1_3)\) hold.

The system (7) has at least three positive solutions in \( K \) if one of the following conditions hold:

(S5) There exist \( \rho_1, \rho_2, \rho_3, \rho_4 \in (0, \infty) \) with \( \rho_1/c < \rho_2 < \rho_3 < \rho_4 \) such that \((l^0_1), (l^0_2), (l^0_3), (l^0_4)\) hold.

(S6) There exist \( \rho_1, \rho_2, \rho_3, \rho_4 \in (0, \infty) \) with \( \rho_1 < \rho_2 \) and \( \rho_2/c < \rho_3 < \rho_4 \) such that \((l^1_1), (l^1_2), (l^1_3), (l^1_4)\) hold.

**Remark 1.** Note that, if the nonlinearities \( f_1 \) and \( f_2 \) have some extra positivity properties, a solution \((u, v)\) achieved by means of the above Theorem has further positivity properties. For example, if the condition (S1) holds and moreover we assume that \( f_1(t, 0, v) > 0 \) in \([a1, b1] \times (0) \times [0, \rho_2] \) and \( f_2(t, u, 0) > 0 \) in \([a2, b2] \times [0, \rho_2] \times (0) \), the solution \((u, v)\) of the system (7) is such that \( \|u\|_{\infty} \) and \( \|v\|_{\infty} \) are strictly positive. The Remark 2 of [16] should read accordingly.

3. An application to coupled systems of BVPs

We study the existence of positive solutions for the system of second order differential equations

\[
\begin{align*}
\dot{u}(t) + g_1(t)f_1(t, u(t), v(t)) &= 0, \quad t \in (0, 1), \\
\dot{v}(t) + g_2(t)f_2(t, u(t), v(t)) &= 0, \quad t \in (0, 1),
\end{align*}
\]

(14)

with the nonlocal boundary conditions

\[
\begin{align*}
&u(0) = \beta_{11}[u], \quad u(1) = \beta_{12}[v], \\
&v(0) = \beta_{21}[v], \quad v(1) = 0, \quad \nu''(0) = 0, \quad \nu''(1) + \beta_{22}[u] = 0.
\end{align*}
\]

(15)

We rewrite this differential system in the integral form

\[
\begin{align*}
u(t) &= (1 - t)\beta_{21}[v] + \frac{1}{6} t(1 - t^2)\beta_{22}[u] + \int_0^1 k_2(t, s)g_2(s, u(s), v(s))\,ds, \\
\dot{u}(t) &= (1 - t)\beta_{11}[u] + tf_1(t) + \int_0^1 k_1(t, s)g_1(s, u(s), v(s))\,ds,
\end{align*}
\]

where

\[
k_1(t, s) = \begin{cases} s(1 - t), & s \leq t, \\
(1 - s), & s > t,
\end{cases}
\quad \text{and} \quad
k_2(t, s) = \begin{cases} \frac{1}{4}s(1 - t)(2s - t^2), & s \leq t, \\
\frac{1}{3}t(1 - s)(2s - t^2), & s > t,
\end{cases}
\]

are non-negative continuous functions on \([0, 1] \times [0, 1]\).

The intervals \([a1, b1]\) and \([a2, b2]\) may be chosen arbitrarily in \((0, 1)\). It is easy to check that

\[
k_1(t, s) \leq s(1 - s) := \Phi_1(s), \quad \min_{t \in [1, b1]} k_1(t, s) \geq c_1 s(1 - s),
\]

where \(c_1 = \min\{1 - b_1, a_1\}\). Furthermore, see the paper by Webb et al. [34], we have that

\[
k_2(t, s) \leq \Phi_2(s) := \begin{cases} \frac{2}{3}s(1 - s^2), & 0 \leq s \leq \frac{1}{2}, \\
\frac{2}{3}s(2 - s^2), & \frac{1}{2} < s < 1,
\end{cases}
\]

and

\[
k_2(t, s) \geq c_2(t)\Phi_2(s),
\]

where
\[ c_2(t) = \begin{cases} 
\frac{1}{2}t(t - t^2), & \text{for } t \in [0, 1/2], \\
\frac{1}{2}t(t - 1)(2 - t), & \text{for } t \in (1/2, 1], 
\end{cases} \]

so that

\[ c_2 = \min_{t \in [a_2, b_2]} c_2(t) > 0. \]

The existence of multiple solutions of the system (14)–(15) follows from Theorem 5.

In the next example we illustrate the constants that occur in our theory.

**Example 1.** Consider the system

\[ \begin{align*}
&u' + (1/8)(u^3 + t^3v^3) + 2 = 0, \quad t \in (0, 1), \\
&v'' = \sqrt{u} + 13v^2, \quad t \in (0, 1), \\
&u(0) = (1/2)u(1/4), \quad u(1) = (1/3)v(3/4), \\
&v(0) = (1/4)v(1/3), \quad v(1) = v'(0) = 0, \quad v''(1) + (1/5)u(2/3) = 0.
\end{align*} \]

In this case the nonlocal conditions are given by the functionals \( \beta_1[w] = \delta_1w(\eta_1) \).

The choice \( [a_1, b_1] = [a_2, b_2] = [1/4, 3/4] \) gives

\[ c_1 = 1/4, \quad c_2 = 45\sqrt{3}/128, \quad c_{11} = c_{12} = c_{21} = 1/4, \quad c_{22} = 45\sqrt{3}/128. \]

\[ m_1 = 8, \quad M_1 = 16, \quad m_2 = 384/5, \quad M_2 = 768/5. \]

We have that \( \beta_{11}[\eta_{11}] = 3/8 \) and \( \beta_{21}[\eta_{21}] = 1/6. \)

Since \( \mathcal{K}_{11}(s) = \alpha_{11}(\eta_{11}, s) \) we obtain

\[ \int_0^1 \mathcal{K}_{11}(s) \, ds = 3/64 \quad \text{and} \quad \int_{1/4}^{3/4} \mathcal{K}_{11}(s) \, ds = 1/32, \]

\[ \int_0^1 \mathcal{K}_{21}(s) \, ds = 11/3888 \quad \text{and} \quad \int_{1/4}^{3/4} \mathcal{K}_{21}(s) \, ds = 3985/1990656. \]

Then, for \( \rho_1 = 1/8, \rho_2 = 1 \) and \( \rho_3 = 11 \), we have (the constants that follow have been rounded to 2 decimal places unless exact)

\[ \inf_{f} \left\{ f_1(t, u, v) : (t, u, v) \in [1/4, 3/4] \times [0, 1/2] \times [0, 1/2] \right\} = f_1(1/4, 0, 0) > 13.34\rho_1, \]

\[ \sup_{f} \left\{ f_1(t, u, v) : (t, u, v) \in [0, 1] \times [0, 1] \times [0, 1] \right\} = f_1(1, 1, 1) < 3\rho_2, \]

\[ \sup_{f_2} \left\{ f_2(t, u, v) : (t, u, v) \in [0, 1] \times [0, 1] \times [0, 1] \right\} = f_2(1, 1, 1) < 59.95\rho_2, \]

\[ \inf_{f} \left\{ f_2(t, u, v) : (t, u, v) \in [1/4, 3/4] \times [11, 44] \times [0, 44] \right\} = f_2(1/4, 11, 0) > 13.34\rho_3, \]

\[ \inf_{f_2} \left\{ f_2(t, u, v) : (t, u, v) \in [1/4, 3/4] \times [44, 11] \times [0, 11] \right\} = f_2(1/4, 0, 11) > 140.63\rho_3, \]

that is the conditions \((l_{\rho_1}^0)\ast (l_{\rho_2}^0)\) and \((l_{\rho_3}^0)\ast (l_{\rho_3}^0)\) are satisfied; therefore the system (16) has at least two positive solutions in \( K \).

**References**


