CRAMER'S RULE APPLIED TO FLEXIBLE SYSTEMS OF LINEAR EQUATIONS*

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Abstract. Systems of linear equations, called flexible systems, with coefficients having uncertainties of type $o(.)$ or $O(.)$ are studied. In some cases an exact solution may not exist but a general theorem that guarantees the existence of an admissible solution, in terms of inclusion, is presented. This admissible solution is produced by Cramer's Rule; depending on the size of the uncertainties appearing in the matrix of coefficients and in the constant term vector some adaptations may be needed.

Key words. Cramer's Rule, External Numbers, Nonstandard Analysis.

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1. Introduction. The aim of this work is to find conditions that guarantee the existence of an admissible solution, in terms of inclusion, for systems of linear equations which have entries that are not exact: the matrix of coefficients and/or the constant term vector of the system have coefficients with uncertainties of type $o(.)$ or $O(.)$. Uncertainties of this kind can be seen as groups of functions and they have been generalized by Van der Corput [1] in a theory of neglecting where these uncertainties are called neutrices. We use an alternative approach to Van der Corput's program within nonstandard analysis where neutrices will now be convex *external* subsets of the nonstandard real number system which are groups for addition; an example is given by the external set of all infinitesimals.

The kind of systems under consideration will be called *flexible systems of linear equations*. We will show that admissible solutions of a non-singular non-homogeneous flexible system of linear equations are given by Cramer's Rule, with some restrictions induced by the size of the uncertainties of the system. For a review of Cramer's Rule we refer to [9] and [4].

This article has the following structure. In Section 2 we recall the notions of neutrix and external number and their operations. In Section 3 we define flexible

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systems of linear equations and introduce the notions of admissible and exact solutions. In Section 4 we present conditions upon the size of the uncertainties appearing in a flexible system of linear equations that guarantee that an admissible solution is produced by Cramer's Rule. We also investigate appropriate adaptations under weaker conditions. We then present the Main Theorem and give some examples that illustrate it. In Section 5 we present the proof of the Main Theorem. In Section 6 we present some applications of the Main Theorem. We start by showing that an admissible solution of a reduced flexible system of 2 by 2 linear equations given by Cramer's Rule is always an admissible solution produced by Gauss-Jordan elimination. Then we show that the admissible solution is in fact the exact solution of the system.

To indicate strict set identity we will use the symbol "$=$". The symbol "$\subseteq$" represents inclusion. Strict inclusion is denoted by "$\subset$".

2. Neutrices and External numbers. The setting of this article is the axiomatic nonstandard analysis IST as presented by Nelson in [8]. A recent introduction to IST is contained in [3]. We use freely external sets where we follow the approach HST as indicated in [5]; this is an extension of an essential part of IST. For a thorough introduction to external numbers with proofs we refer to [6] and [7].

We recall that within IST the nonstandard numbers are already present in the standard set $\mathbb{R}$. *Infinitesimal numbers* (or *infinitesimals*) are real numbers that are smaller, in absolute value, than any positive standard real number. *Infinitely large numbers* are reciprocals of infinitesimals, i.e. real numbers larger than any standard real number. *Limited numbers* are real numbers which are not infinitely large and *appreciable numbers* are limited numbers which are not infinitesimals. The external set of all infinitesimal numbers is denoted by $\oslash$, the external set of all limited numbers is denoted by $\pounds$, the external set of all positive appreciable numbers is denoted by @ and the external set of all positive infinitely large numbers by $\not\infty$.

A *neutrix* is an additive convex subgroup of $\mathbb{R}$. Except for $\{0\}$ and $\mathbb{R}$, all neutrices are external sets. The most common neutrices are $\oslash$ and $\pounds$. All other neutrices contain $\pounds$ or are contained in $\oslash$. Examples of neutrices contained in $\oslash$ are $\varepsilon\pounds$, $\varepsilon\oslash$ and $\pounds\varepsilon^{\not\infty}$, numbers smaller than any standard power of ε, where ε is a positive infinitesimal. Examples of neutrices that contain $\pounds$ are $\omega\pounds$, $\omega\oslash$ and $\omega^2\pounds$, where ω is an infinitely large number. The external class of all neutrices is denoted by $\mathcal{N}$. Neutrices are totally ordered by inclusion. Addition and multiplication on $\mathcal{N}$ are defined by the Minkowski operations as it follows:

$$A + B = \{a + b \mid (a, b) \in A \times B\}$$

and

$$AB = \{ab \mid (a, b) \in A \times B\},$$

for $A, B \in \mathcal{N}$.

The sum of two neutrices is the largest one for inclusion.

PROPOSITION 2.1. *If* $A, B \in \mathcal{N}$, *then* $A + B = \max(A, B)$.

Neutrices are invariant under multiplication by appreciable numbers.

PROPOSITION 2.2. *If* $A \in \mathcal{N}$, *then* $@A = A$.

An *external number* is the algebraic sum of a real number and a neutrix. If $a \in \mathbb{R}$ and $A \in \mathcal{N}$, then $\alpha \equiv a + A \in \mathbb{E}$ and A is called the *neutrix part of* α, being denoted as $N(\alpha)$; $N(\alpha)$ is unique but a is not because for all $c \in \alpha$, $\alpha = c + N(\alpha)$. We then say that c is a *representative of* α. Clearly, neutrices are external numbers such that the representative may be chosen equal to 0. All classical real numbers are external numbers with the neutrix part equal to $\{0\}$. The external class of all external numbers is denoted by $\mathbb{E}$. An external number α is called *zeroless*, if $0 \notin \alpha$. Let $\alpha = a + A$ be zeroless. Then its *relative uncertainty* $R(\alpha)$ is defined by the neutrix A/a. Notice that $A/a = A/\alpha$, hence $R(\alpha)$ is independent of the choice of a; also $R(\alpha) \subseteq \oslash$ (see Lemmas 5.1 and 5.2). Let $\alpha = a + A$ and $\beta = b + B$ be two external numbers. Then either α and β are disjoint or one contains the other. Addition, subtraction, multiplication and division of α with β are given by Minkowski operations. One shows that

$$\begin{aligned}
\alpha + \beta &= a + b + \max(A, B);\\
\alpha - \beta &= a - b + \max(A, B);\\
\alpha\beta &= ab + \max(aB, bA, AB)\\
&= ab + \max(aB, bA) \text{ if } \alpha \text{ or } \beta \text{ is zeroless;}\\
\frac{\alpha}{\beta} &= \frac{a}{b} + \frac{1}{b^2}\max(aB, bA) = \frac{\alpha\beta}{b^2}, \text{ with } \beta \text{ zeroless.}
\end{aligned}$$

The relation $\alpha \leqslant \beta$ if and only if $]-\infty, \alpha] \subseteq]-\infty, \beta]$ is a relation of total order compatible with addition and multiplication. In practice, calculations with external numbers tend to be rather straightforward as it will be illustrated by the following examples.

Let ε be a positive infinitesimal. Then

$$(6 + \oslash) + (-2 + \varepsilon£) = (6 - 2) + (\oslash + \varepsilon£) = 4 + \oslash;$$

$$\begin{aligned}
(6 + \oslash)(-2 + \varepsilon£) &= 6(-2) + (-2)\oslash + 6\varepsilon£ + \oslash\varepsilon£\\
&= -12 + \oslash + \varepsilon£ + \varepsilon\oslash = -12 + \oslash;
\end{aligned}$$

$$\begin{aligned}\frac{6+\oslash}{-2+\varepsilon\pounds} &= \frac{6}{-2}\cdot\frac{1+\oslash/6}{1+\varepsilon\pounds/2} = (-3)\,\frac{1+\oslash}{1+\varepsilon\pounds}\\ &= (-3)\,(1+\oslash)(1+\varepsilon\pounds) = -3+\oslash.\end{aligned}$$

However, multiplication of external numbers is not fully distributive, for instance

$$\oslash\varepsilon = \oslash(1+\varepsilon-1) \subset \oslash(1+\varepsilon) - \oslash\cdot 1 = \oslash + \oslash = \oslash.$$

Yet distributivity can be entirely characterized [2]. Let $\alpha = a+A$, β and γ be external numbers, where $a \in \mathbb{R}$ and A is a neutrix. Important cases where distributivity is verified are

$$a(\beta+\gamma) = a\beta + a\gamma \tag{2.1}$$

and

$$(a+A)\beta = a\beta + A\beta. \tag{2.2}$$

Also subdistributivity always holds, this means that $\alpha(\beta+\gamma) \subseteq \alpha\beta+\alpha\gamma$; the property follows from the well-kown property of subdistributivity of interval calculus.

Definition 2.3. Let A be a neutrix and α be an external number. We say that α is an *absorber* of A if $\alpha A \subset A$.

Example 2.4. According to Proposition 2.2, appreciable numbers are not absorbers. So an absorber must be an infinitesimal. Let ε be a positive infinitesimal. Then ε is an absorber of $\oslash$ because $\varepsilon\oslash \subset \oslash$. However, not necessarily all infinitesimals are absorbers of a given neutrix, for instance $\varepsilon\pounds\varepsilon^{-\not\infty} = \pounds\varepsilon^{-\not\infty}$.

3. Flexible systems of linear equations. In this section we introduce some notations and define the flexible systems and some related notions.

Notation 3.1. Let $m, n \in \mathbb{N}$ be standard. For $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$, let $\alpha_{ij} = a_{ij} + A_{ij}$, with $a_{ij} \in \mathbb{R}$ and $A_{ij} \in \mathcal{N}$. We denote

1. $\mathcal{A} = [\alpha_{ij}]$, an $m \times n$ matrix
2. $\overline{\alpha} = \max\limits_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} |\alpha_{ij}|$
3. $\overline{a} = \max\limits_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} |a_{ij}|$
4. $\overline{A} = \max\limits_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} A_{ij}$
5. $\underline{A} = \min\limits_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} A_{ij}$.

In particular, for a column vector $\mathcal{B} = [\beta_i]$, with $\beta_i = b_i + B_i \in \mathbb{E}$ for $1 \leqslant i \leqslant n$, we denote $\overline{\beta} = \max\limits_{1\leqslant i\leqslant n} |\beta_i|$, $\overline{b} = \max\limits_{1\leqslant i\leqslant n} |b_i|$, $\overline{B} = \max\limits_{1\leqslant i\leqslant n} B_i$ and $\underline{B} = \min\limits_{1\leqslant i\leqslant n} B_i$.

We observe that not all equations with external numbers can be solved in terms of equalities. For instance, no external number, or even set of external numbers, satisfies the equation $\oslash\xi = \pounds$ since one should have $\xi \subseteq \pounds$ and $\oslash\pounds = \oslash \subset \pounds$. So we will study inclusions instead of equalities.

Definition 3.2. Let $m, n \in \mathbb{N}$ be standard and $\alpha_{ij} = a_{ij} + A_{ij}, \beta_i = b_i + B_i$, $\xi_j = x_j + X_j \in \mathbb{E}$ for $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$. We call

$$\begin{cases} \alpha_{11}\xi_1 + & \dots & +\alpha_{1j}\xi_j + & \dots & +\alpha_{1n}\xi_n & \subseteq \beta_1 \\ \vdots & & \vdots & & \vdots & \vdots \\ \alpha_{m1}\xi_1 + & \dots & +\alpha_{mj}\xi_j + & \dots & +\alpha_{mn}\xi_n & \subseteq \beta_m \end{cases}$$

a *flexible system of linear equations.*

Definition 3.3. Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$, and let $\mathcal{B} = [\beta_i]$ be a column vector, with $\beta_i = b_i + B_i \in \mathbb{E}$ for all $i, j \in \{1, ..., n\}$.

1. $\mathcal{A}$ is called a *non-singular matrix* if $\Delta = \det \mathcal{A}$ is zeroless.
2. $\mathcal{B}$ is called an *upper zeroless vector* if $\overline{\beta}$ is zeroless.

Definition 3.4. Let $n \in \mathbb{N}$ be standard and $\alpha_{ij} = a_{ij} + A_{ij}, \beta_i = b_i + B_i$, $\xi_j = x_j + X_j \in \mathbb{E}$ for all $i, j \in \{1, ..., n\}$. Consider the square flexible system of linear equations

$$\begin{cases} \alpha_{11}\xi_1 + & \dots & +\alpha_{1j}\xi_j + & \dots & +\alpha_{1n}\xi_n & \subseteq \beta_1 \\ \vdots & & \vdots & & \vdots & \vdots \\ \alpha_{n1}\xi_1 + & \dots & +\alpha_{nj}\xi_j + & \dots & +\alpha_{nn}\xi_n & \subseteq \beta_n \end{cases}, \tag{3.1}$$

with matrix representation given by $\mathcal{AX} \subseteq \mathcal{B}$. If $\mathcal{A}$ is a non-singular matrix, the system is called *non-singular.* If $\mathcal{B}$ is an upper zeroless vector, the system is called *non-homogeneous.* Moreover, if 1 is a representative of $\overline{\alpha}$, $\mathcal{A}$ is called a *reduced matrix* and we speak about a *reduced system.* If external numbers $\xi_1, ..., \xi_n$ can actually be found to satisfy (3.1), the column vector $(\xi_1, ..., \xi_n)^T$ is called an *admissible solution* of $\mathcal{AX} \subseteq \mathcal{B}$. A solution $\xi = (\xi_1, ..., \xi_n)^T$ of the system (6.2) is *maximal* if no (external) set $\eta \supset \xi$ satisfies this flexible system. If $\xi_1, ..., \xi_n$ satisfy the system (3.1) with equalities, the column vector $(\xi_1, ..., \xi_n)^T$ is called the *exact solution* of $\mathcal{AX} \subseteq \mathcal{B}$.

4. Existence of admissible solutions. Not all non-singular non-homogeneous flexible systems of linear equations can be resolved by Cramer's Rule. We need to control the uncertainties of the system in order to guarantee that Cramer's Rule produces a valid solution and, if necessary, to make some adaptations. The matrix $\mathcal{A}$ of coefficients has to be more precise, in a sense, than the constant term vector $\mathcal{B}$. The general theorem presented in this section shows that, under certain conditions upon the size of the uncertainties appearing in a non-singular non-homogeneous flexible system of linear equations, it is possible to guarantee the existence of admissible solutions by Cramer's Rule. Even when not all of those conditions are satisfied it is still possible, in some cases, to obtain an admissible solution given by adapting Cramer's Rule, where we neglect some uncertainties of the system.

In this section we will simply call a non-singular non-homogeneous flexible system of linear equations *flexible system* and a reduced non-singular non-homogeneous flexible system of linear equations *reduced flexible system*.

We start by defining the kind of precision needed in order to control the uncertainties appearing in a flexible system.

Definition 4.1. Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a non-singular matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$, and $\mathcal{B} = [\beta_i]_{n\times 1}$ be an upper zeroless vector, with $\beta_i = b_i + B_i \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$.

We define the *relative uncertainty of* $\mathcal{A}$ by

$$R(\mathcal{A}) = \overline{A}\,\overline{\alpha}^{n-1} \diagup \Delta.$$

We define the *relative precision of* $\mathcal{B}$ by

$$P(\mathcal{B}) = \underline{B} \diagup \overline{\beta}.$$

Remark 4.2. If $\mathcal{A} = [\alpha]$, with $\alpha = a + A$ zeroless, the relative uncertainty of $\mathcal{A}$ reduces to A/a, the relative uncertainty of the external number $\det \mathcal{A} = \alpha$. In general $R(\mathcal{A})$ gives an upper bound of the relative uncertainty of $\det \mathcal{A}$. Note that if $\overline{\alpha} \subseteq @$ we simply have $R(\mathcal{A}) = \overline{A} \diagup \Delta$.

Notation 4.3. Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$, and $\mathcal{B} = [\beta_i]$ be a column vector, with $\beta_i = b_i + B_i \in \mathbb{E}$, for $1 \leqslant i, j \leqslant n$. We denote

$$\mathcal{M}_j = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1(j-1)} & \beta_1 & \alpha_{1(j+1)} & \cdots & \alpha_{11} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{n(j-1)} & \beta_n & \alpha_{n(j+1)} & \cdots & \alpha_{nn} \end{bmatrix}$$

$$\mathcal{M}_j(b) = \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1(j-1)} & b_1 & \alpha_{1(j+1)} & \dots & \alpha_{11} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n1} & \dots & \alpha_{n(j-1)} & b_n & \alpha_{n(j+1)} & \dots & \alpha_{nn} \end{bmatrix}$$

$$\mathcal{M}_j(a,b) = \begin{bmatrix} a_{11} & \dots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \dots & a_{11} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{n(j-1)} & b_n & a_{n(j+1)} & \dots & a_{nn} \end{bmatrix}.$$

THEOREM 4.4. (Main Theorem) *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be a non-singular matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ and $\Delta = \det \mathcal{A} = d + D$, and let $\mathcal{B} = [\beta_i]$ be an upper zeroless vector, with $\beta_i = b_i + B_i \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$. Consider the flexible system $\mathcal{AX} \subseteq \mathcal{B}$ where $\mathcal{X} = [\xi_i]$, with $\xi_i = x_i + X_i \in \mathbb{E}$ for all $i \in \{1, ..., n\}$.*

1. *If $R(\mathcal{A}) \subseteq P(\mathcal{B})$, then*

$$\mathcal{X} = \begin{bmatrix} \frac{\det \mathcal{M}_1(b)}{d} \\ \vdots \\ \frac{\det \mathcal{M}_n(b)}{d} \end{bmatrix}$$

is an admissible solution of $\mathcal{AX} \subseteq \mathcal{B}$.

2. *If $R(\mathcal{A}) \subseteq P(\mathcal{B})$ and Δ is not an absorber of $\underline{B}$, then*

$$\mathcal{X} = \begin{bmatrix} \frac{\det \mathcal{M}_1(b)}{\Delta} \\ \vdots \\ \frac{\det \mathcal{M}_n(b)}{\Delta} \end{bmatrix}$$

is an admissible solution of $\mathcal{AX} \subseteq \mathcal{B}$.

3. *If $R(\mathcal{A}) \subseteq P(\mathcal{B})$, Δ is not an absorber of $\underline{B}$ and $\underline{B} = \overline{B}$, then*

$$\mathcal{X} = \begin{bmatrix} \frac{\det \mathcal{M}_1}{\Delta} \\ \vdots \\ \frac{\det \mathcal{M}_n}{\Delta} \end{bmatrix}$$

is an admissible and maximal solution of $\mathcal{AX} \subseteq \mathcal{B}$.

We will call $\left(\frac{\det \mathcal{M}_1}{\Delta}, ..., \frac{\det \mathcal{M}_n}{\Delta}\right)^T$ the *Cramer-solution* of the flexible system (3.1).

So Part 3 of Theorem 4.4 states conditions guaranteeing that the Cramer-solution maximally satisfies (3.1).

Under the weaker conditions of Part 2, one is forced to substitute the constant term vector $\mathcal{B}$ by a representative, the uncertainties occurring in $\mathcal{B}$ possibly being too large. If only the condition on the relative precision $R(\mathcal{A}) \subseteq P(\mathcal{B})$ is known to hold, also the determinant Δ must be substituted by a representative. The condition that Δ should not be so small as to be an absorber of $\underline{B}$ may be seen, in a sense, as a generalization of the usual condition on non-singularity of determinant of the matrix of coefficients, i.e. that this determinant should be non-zero.

We show now some examples which illustrate the role of the conditions presented in Theorem 4.4.

The first two examples show that not all flexible systems can be resolved by Cramer's Rule and also illustrate the importance of the condition on precision in a flexible system.

Example 4.5. Let ε be a positive infinitesimal. Consider the following non-homogeneous flexible system of linear equations

$$\begin{cases} (3+\varepsilon\oslash)\,x + (-1+\oslash)\,y = 1+\varepsilon\pounds \\ (2+\varepsilon\pounds)\,x + (1+\varepsilon\oslash)\,y = \varepsilon\pounds. \end{cases}$$

A real part of this system is given by $\begin{cases} 3x-y=1 \\ 2x+y=0 \end{cases}$ which has the exact solution $\begin{cases} x=\frac{1}{5} \\ y=-\frac{2}{5} \end{cases}$.

We have $\Delta = \begin{vmatrix} 3+\varepsilon\oslash & -1+\oslash \\ 2+\varepsilon\pounds & 1+\varepsilon\oslash \end{vmatrix} = 5+\oslash$, which is zeroless. So the initial system is non-singular. When we apply Cramer's Rule, we get

$$x = \frac{\begin{vmatrix} 1+\varepsilon\pounds & -1+\oslash \\ \varepsilon\pounds & 1+\varepsilon\oslash \end{vmatrix}}{\Delta} = \frac{1+\varepsilon\pounds}{5+\oslash} = \frac{1}{5}+\oslash$$

$$y = \frac{\begin{vmatrix} 3+\varepsilon\oslash & 1+\varepsilon\pounds \\ 2+\varepsilon\pounds & \varepsilon\pounds \end{vmatrix}}{\Delta} = \frac{-2+\varepsilon\pounds}{5+\oslash} = -\frac{2}{5}+\oslash.$$

However, this is not a valid solution because

$$\begin{aligned}(3+\varepsilon\oslash)\,x + (-1+\oslash)\,y &= (3+\varepsilon\oslash)\left(\frac{1}{5}+\oslash\right) + (-1+\oslash) - \frac{2}{5}+\oslash \\ &= 1+\oslash \supset 1+\varepsilon\pounds\end{aligned}$$

and

$$(2+\varepsilon\pounds)\,x + (1+\varepsilon\oslash)\,y = (2+\varepsilon\pounds)\left(\frac{1}{5}+\oslash\right) + (1+\varepsilon\oslash)\left(-\frac{2}{5}+\oslash\right) = \oslash \supset \varepsilon\pounds.$$

In fact, using representatives, it is easy to show that this system does not have solutions at all.
We have $R(\mathcal{A}) = \overline{A}\overline{\alpha} \diagup \Delta = \frac{3\oslash}{5+\oslash} = \oslash$ and $P(\mathcal{B}) = \underline{B} \diagup \overline{\beta} = \frac{\varepsilon \pounds}{1+\varepsilon \pounds} = \varepsilon \pounds$. So $R(\mathcal{A}) \nsubseteq P(\mathcal{B})$ and Theorem 4.4 cannot be applied, although Δ is not an absorber of $\underline{B}$, since $\Delta \underline{B} = \varepsilon \pounds = \underline{B}$, and $\underline{B} = \overline{B} = \varepsilon \pounds$.

EXAMPLE 4.6. Let ε be a positive infinitesimal. Consider the following flexible system:

$$\begin{cases} 3x + (-1+\varepsilon\oslash)\, y = 1 + \varepsilon \pounds \\ 2x + y = \varepsilon \pounds. \end{cases}$$

Its matrix representation is given by $\mathcal{A}\mathcal{X} = \mathcal{B}$, where

$$\mathcal{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathcal{A} = \begin{bmatrix} 3 & -1+\varepsilon\oslash \\ 2 & 1 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 1+\varepsilon \pounds \\ \varepsilon \pounds \end{bmatrix}.$$

We have $\overline{A} = \varepsilon\oslash$, $\underline{B} = \varepsilon \pounds$ and $\Delta = \det \mathcal{A} = \begin{vmatrix} 3 & -1+\varepsilon\oslash \\ 2 & 1 \end{vmatrix} = 5 + \varepsilon\oslash$ zeroless. Also (i) $R(\mathcal{A}) = \varepsilon\oslash \subseteq \varepsilon \pounds = P(\mathcal{B})$, (ii) Δ is not an absorber of $\underline{B}$ since $\Delta \underline{B} = \varepsilon \pounds = \underline{B}$ and (iii) $\underline{B} = \varepsilon \pounds = \overline{B}$. Hence all the conditions of Part 3 of Theorem 4.4 are satisfied. By applying Cramer's Rule we get

$$x = \frac{\begin{vmatrix} 1+\varepsilon \pounds & -1+\varepsilon\oslash \\ \varepsilon \pounds & 1 \end{vmatrix}}{\Delta} = \frac{1+\varepsilon \pounds}{5+\varepsilon\oslash} = \frac{1}{5} + \varepsilon \pounds$$

$$y = \frac{\begin{vmatrix} 3 & 1+\varepsilon \pounds \\ 2 & \varepsilon \pounds \end{vmatrix}}{\Delta} = \frac{-2+\varepsilon \pounds}{5+\varepsilon\oslash} = -\frac{2}{5} + \varepsilon \pounds.$$

When testing the validity of this solution, we have indeed that

$$3x + (-1+\varepsilon\oslash)\, y = 3\left(\frac{1}{5} + \varepsilon \pounds\right) + (-1+\varepsilon\oslash)\left(-\frac{2}{5} + \varepsilon \pounds\right) = 1 + \varepsilon \pounds$$

and

$$2x + y = 2\left(\frac{1}{5} + \varepsilon \pounds\right) + \left(-\frac{2}{5} + \varepsilon \pounds\right) = \varepsilon \pounds.$$

Notice that this system has the same real part as the previous system, to which Cramer's Rule could not be applied.

The following example also satisfies the conditions of Part 3 of Theorem 4.4, which guarantee the validity of the solution produced by Cramer's Rule.

Example 4.7. Let ε be a positive infinitesimal. Consider the following flexible system

$$\begin{cases} \left(1+\varepsilon^2\oslash\right)x+y+\left(1+\varepsilon^3\pounds\right)z=\frac{1}{\varepsilon}+\varepsilon\oslash \\ \left(2+\varepsilon^3\pounds\right)x+\left(-1+\varepsilon^2\oslash\right)y-z=\varepsilon\oslash \\ \left(\varepsilon+\varepsilon^3\oslash\right)x+y+\left(2+\varepsilon^2\oslash\right)z=1+\varepsilon\oslash. \end{cases}$$

Given its matrix representation $\mathcal{A}\mathcal{X}=\mathcal{B}$, one has that

$$\Delta=\begin{vmatrix} 1+\varepsilon^2\oslash & 1 & 1+\varepsilon^3\pounds \\ 2+\varepsilon^3\pounds & -1+\varepsilon^2\oslash & -1 \\ \varepsilon+\varepsilon^3\oslash & 1 & 2+\varepsilon^2\oslash \end{vmatrix}=-3+\varepsilon^2\oslash \text{ is zeroless,}$$

$R(\mathcal{A})=\overline{A}\overline{\alpha}^2/\Delta=\frac{4\varepsilon^2\oslash}{-3+\varepsilon^2\oslash}=\varepsilon^2\oslash$ and $P(\mathcal{B})=\underline{B}/\overline{\beta}=\frac{\varepsilon\oslash}{\frac{1}{\varepsilon}+\varepsilon\oslash}=\varepsilon^2\oslash$. So (i) $R(\mathcal{A})\subseteq P(\mathcal{B})$, (ii) Δ is not an absorber of $\underline{B}$ since $\Delta\underline{B}=\varepsilon\oslash=\underline{B}$ and (iii) $\underline{B}=\overline{B}=\varepsilon\oslash$. When we apply Cramer's Rule, we get

$$x=\frac{\begin{vmatrix} \frac{1}{\varepsilon}+\varepsilon\pounds & 1 & 1+\varepsilon^3\pounds \\ \varepsilon\pounds & -1+\varepsilon^2\oslash & -1 \\ 1+\varepsilon\pounds & 1 & 2+\varepsilon^2\oslash \end{vmatrix}}{\Delta}=\frac{-\frac{1}{\varepsilon}+\varepsilon\oslash}{-3+\varepsilon^2\oslash}=\frac{1}{3\varepsilon}+\varepsilon\oslash$$

$$y=\frac{\begin{vmatrix} 1+\varepsilon^2\oslash & \frac{1}{\varepsilon}+\varepsilon\pounds & 1+\varepsilon^3\pounds \\ 2+\varepsilon^3\pounds & \varepsilon\pounds & -1 \\ \varepsilon+\varepsilon^3\oslash & 1+\varepsilon\pounds & 2+\varepsilon^2\oslash \end{vmatrix}}{\Delta}=\frac{2-\frac{4}{\varepsilon}+\varepsilon\oslash}{-3+\varepsilon^2\oslash}=\frac{4}{3\varepsilon}-\frac{2}{3}+\varepsilon\oslash$$

$$z=\frac{\begin{vmatrix} 1+\varepsilon^2\oslash & 1 & \frac{1}{\varepsilon}+\varepsilon\pounds \\ 2+\varepsilon^3\pounds & -1+\varepsilon^2\oslash & \varepsilon\pounds \\ \varepsilon+\varepsilon^3\oslash & 1 & 1+\varepsilon\pounds \end{vmatrix}}{\Delta}=\frac{\frac{2}{\varepsilon}-2+\varepsilon\oslash}{-3+\varepsilon^2\oslash}=-\frac{2}{3\varepsilon}+\frac{2}{3}+\varepsilon\oslash.$$

When testing the validity, we find that $(x,y,z)^T$ satisfies the equations. Indeed

$$\begin{aligned}&\left(1+\varepsilon^2\oslash\right)x+y+\left(1+\varepsilon^3\pounds\right)z\\ &=\left(1+\varepsilon^2\oslash\right)\left(\frac{1}{3\varepsilon}+\varepsilon\oslash\right)+\left(\frac{4}{3\varepsilon}-\frac{2}{3}+\varepsilon\oslash\right)+\left(1+\varepsilon^3\pounds\right)\left(-\frac{2}{3\varepsilon}+\frac{2}{3}+\varepsilon\oslash\right)=\frac{1}{\varepsilon}+\varepsilon\oslash\end{aligned}$$

$$\begin{aligned}&\left(2+\varepsilon^3\pounds\right)x+\left(-1+\varepsilon^2\oslash\right)y-z\\ &=\left(2+\varepsilon^3\pounds\right)\left(\frac{1}{3\varepsilon}+\varepsilon\oslash\right)+\left(-1+\varepsilon^2\oslash\right)\left(\frac{4}{3\varepsilon}-\frac{2}{3}+\varepsilon\oslash\right)-\left(-\frac{2}{3\varepsilon}+\frac{2}{3}+\varepsilon\oslash\right)=\varepsilon\oslash\end{aligned}$$

$$\begin{aligned}&\left(\varepsilon+\varepsilon^3\oslash\right)x+y+\left(2+\varepsilon^2\oslash\right)z\\ &=\left(\varepsilon+\varepsilon^3\oslash\right)\left(\frac{1}{3\varepsilon}+\varepsilon\oslash\right)+\left(\frac{4}{3\varepsilon}-\frac{2}{3}+\varepsilon\oslash\right)+\left(2+\varepsilon^2\oslash\right)\left(-\frac{2}{3\varepsilon}+\frac{2}{3}+\varepsilon\oslash\right)=1+\varepsilon\oslash.\end{aligned}$$

The next example refers to Part 2 of Theorem 4.4.

Example 4.8. Let ε be a positive infinitesimal. Consider the following flexible system:

$$\begin{cases} 3x + (-1+\varepsilon\oslash)\, y = 1+\oslash \\ 2x + y = \varepsilon\pounds. \end{cases}$$

Its matrix representation is given by $\mathcal{A}\mathcal{X} = \mathcal{B}$, with

$$\mathcal{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathcal{A} = \begin{bmatrix} 3 & -1+\varepsilon\oslash \\ 2 & 1 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 1+\oslash \\ \varepsilon\pounds \end{bmatrix}.$$

We have $\overline{A} = \varepsilon\oslash$ and $\underline{B} = \varepsilon\pounds$. The determinant $\Delta = \det\mathcal{A} = \begin{vmatrix} 3 & -1+\varepsilon\oslash \\ 2 & 1 \end{vmatrix} = 5+\varepsilon\oslash$ is zeroless. One has $R(\mathcal{A}) = \varepsilon\oslash \subseteq \varepsilon\pounds = P(\mathcal{B})$ and Δ is not an absorber of $\underline{B}$. However $\underline{B} = \varepsilon\pounds \neq \oslash = \overline{B}$. So this system satisfies only the conditions of Part 2 of Theorem 4.4. Cramer's Rule yields

$$x = \frac{\begin{vmatrix} 1+\oslash & -1+\varepsilon\oslash \\ \varepsilon\pounds & 1 \end{vmatrix}}{\Delta} = \frac{1+\oslash}{5+\varepsilon\oslash} = \frac{1}{5}+\oslash$$
$$y = \frac{\begin{vmatrix} 3 & 1+\oslash \\ 2 & \varepsilon\pounds \end{vmatrix}}{\Delta} = \frac{-2+\oslash}{5+\varepsilon\oslash} = -\frac{2}{5}+\oslash.$$

This is not a valid solution. Indeed

$$2x + y = \frac{2}{5}+\oslash+\left(-\frac{2}{5}+\oslash\right) = \oslash \supset \varepsilon\pounds.$$

If we ignore the uncertainties of the constant term vector in $\det\mathcal{M}_1$ and $\det\mathcal{M}_2$, by Part 2 of Theorem 4.4, Cramer's Rule produces an admissible solution:

$$x = \frac{\begin{vmatrix} 1 & -1+\varepsilon\oslash \\ 0 & 1 \end{vmatrix}}{\Delta} = \frac{1}{5+\varepsilon\oslash} = \frac{1}{5}+\varepsilon\oslash$$
$$y = \frac{\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix}}{\Delta} = \frac{-2}{5+\varepsilon\oslash} = -\frac{2}{5}+\varepsilon\oslash.$$

When testing the validity of this solution, we have indeed that

$$3x + (-1+\varepsilon\oslash)\, y = \frac{3}{5}+\varepsilon\oslash+\frac{2}{5}+\varepsilon\oslash = 1+\varepsilon\oslash \subseteq 1+\oslash$$

and

$$2x + y = \frac{2}{5} + \varepsilon \oslash - \frac{2}{5} + \varepsilon\oslash = \varepsilon\oslash \subseteq \varepsilon\pounds.$$

In the last example we may apply only Part 1 of Theorem 4.4.

Example 4.9. Let ε be a positive infinitesimal. Consider the following flexible system:

$$\begin{cases} 3x + \left(-1 + \varepsilon^2\oslash\right) y = 1 + \oslash \\ 2\varepsilon x + \varepsilon y = \varepsilon\pounds. \end{cases}$$

Here the matrix representation is given by $\mathcal{A}\mathcal{X} = \mathcal{B}$, with

$$\mathcal{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathcal{A} = \begin{bmatrix} 3 & -1+\varepsilon^2\oslash \\ 2\varepsilon & \varepsilon \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 1+\oslash \\ \varepsilon\pounds \end{bmatrix}.$$

We have $\overline{A} = \varepsilon^2\oslash$ and $\underline{B} = \varepsilon\pounds$. The determinant $\Delta = \det \mathcal{A} = \begin{vmatrix} 3 & -1+\varepsilon^2\oslash \\ 2\varepsilon & \varepsilon \end{vmatrix} = 5\varepsilon + \varepsilon^3\oslash$ is infinitesimal, yet zeroless. It holds that $R(\mathcal{A}) = \varepsilon\oslash \subseteq \varepsilon\pounds = P(\mathcal{B})$ but Δ is an absorber of $\underline{B}$ because $\Delta\underline{B} = \varepsilon^2\pounds \subset \varepsilon\pounds = \underline{B}$. So this system satisfies the condition of Part 1 of Theorem 4.4. By applying Cramer's Rule we get

$$x = \frac{\begin{vmatrix} 1+\oslash & -1+\varepsilon^2\oslash \\ \varepsilon\pounds & \varepsilon \end{vmatrix}}{\Delta} = \frac{\varepsilon\pounds}{5\varepsilon + \varepsilon^3\oslash} = \pounds$$

$$y = \frac{\begin{vmatrix} 3 & 1+\oslash \\ 2\varepsilon & \varepsilon\pounds \end{vmatrix}}{\Delta} = \frac{\varepsilon\pounds}{5\varepsilon + \varepsilon^3\oslash} = \pounds.$$

These results are clearly not valid, because

$$3x + \left(-1+\varepsilon^2\oslash\right) y = 3\pounds + \left(-1+\varepsilon^2\oslash\right)\pounds = \pounds \supset 1 + \oslash.$$

Observe that the results produced by Cramer's Rule are not even zeroless though the determinant is zeroless and the constant term vector is upper zeroless.
If we ignore the uncertainties of the constant term vector and the uncertainty of Δ, by the application of Part 1 of Theorem 4.4, the solution produced by Cramer's Rule is now admissible. One has

$$x = \frac{\begin{vmatrix} 1 & -1+\varepsilon^2\oslash \\ 0 & \varepsilon \end{vmatrix}}{d} = \frac{\varepsilon}{5\varepsilon} = \frac{1}{5}$$

$$y = \frac{\begin{vmatrix} 3 & 1 \\ 2\varepsilon & 0 \end{vmatrix}}{d} = -\frac{2\varepsilon}{5\varepsilon} = -\frac{2}{5}.$$

When testing the validity of this solution, we have indeed that

$$3x+\left(-1+\varepsilon^2\oslash\right)y=\frac{3}{5}-\frac{2}{5}\left(-1+\varepsilon^2\oslash\right)=1+\varepsilon^2\oslash\subset 1+\oslash$$

and

$$2\varepsilon x+\varepsilon y=\frac{2\varepsilon}{5}-\frac{2\varepsilon}{5}=0\subset\varepsilon\pounds.$$

5. Proof of Theorem 4.4. We present now some preliminary results and some Lemmas that will be used in the proof of Theorem 4.4.

We start by recalling some simple results about calculation properties of external numbers.

LEMMA 5.1. *Let $\alpha = a + A$ be a zeroless external number. Then its relative uncertainty $R(\alpha) = A/a$ satisfies*

$$\frac{A}{a}\subseteq\oslash.$$

Proof. Since $\alpha = a + A$ is zeroless, one has $0 \notin \alpha$ and so $|a| > A$. Hence $\frac{A}{a} < 1$ and so $\frac{A}{a} \subseteq \oslash$ because there is no neutrix strictly included in $\pounds$ and which strictly contains $\oslash$. □

LEMMA 5.2. *Let A be a neutrix and $\beta = b + B$ be a zeroless external number. Then $\frac{A}{\beta} = \frac{A}{b}$ and $A\beta = Ab$.*

Proof. Since $B \subseteq b\oslash$ by Lemma 5.1, $AB \subseteq \oslash bA \subseteq bA$. Hence $\frac{A}{\beta} = \frac{0+A}{b+B} = \frac{bA}{b^2} = \frac{A}{b}$ and $A\beta = (0 + A)(b + B) = \max(bA, AB) = Ab$. □

LEMMA 5.3. *Let $a \in \mathbb{R}$, $A \in \mathcal{N}$ and $n \in \mathbb{N}$ be standard. If $|a| > A$, then*

$$N\left((a+A)^n\right)=a^{n-1}A.$$

Proof. Since $|a| > A$, we have $(a + A)^2 = (a + A)(a + A) = a^2 + aA$. So $(a + A)^3 = (a + A)(a + A)^2 = (a + A)\left(a^2 + aA\right) = a^3 + a^2A$. Using external induction, we conclude that

$$(a+A)^n=a^n+a^{n-1}A.$$

Hence $N\left((a+A)^n\right) = a^{n-1}A$. □

Below some useful upper bounds with respect to matrices and determinants will be derived.

REMARK 5.4. Let $\mathcal{A} = [\alpha_{ij}]$ be a reduced non-singular matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$ and $\Delta = \det \mathcal{A}$. Since Δ is zeroless, one has $\overline{\alpha} \subseteq 1 + \oslash$ by Lemma 5.1. Consequently $A_{ij} \subseteq \oslash$ for all $i, j \in \{1, ..., n\}$, hence $\overline{A} \subseteq \oslash$.

LEMMA 5.5. *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be a reduced non-singular matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$ and $\Delta = \det \mathcal{A} = d + D$. Then*

$$D = N(\Delta) \subseteq \overline{A}.$$

Proof. Let S_n denote the set of all permutations of the set $\{1, 2, ..., n\}$ and $\sigma = (p_1, ..., p_n) \in S_n$. Let $\gamma_\sigma = (a_{1p_1} + A_{1p_1}) \cdot ... \cdot (a_{np_n} + A_{np_n})$. Because $\overline{a} = 1$, by Remark 5.4, one has $|a_{kp_k}| \leqslant \overline{a} = 1$ and $A_{kp_k} \subseteq \overline{A} \subseteq \oslash$ for all $k \in \{1, ..., n\}$. So, by Lemma 5.3, $N(\gamma_\sigma) \subseteq N\left(\left(1 + \overline{A}\right)^n\right) = \overline{A}$.
Now,

$$\Delta = \begin{vmatrix} a_{11} + A_{11} & ... & a_{1n} + A_{1n} \\ \vdots & & \vdots \\ a_{n1} + A_{n1} & ... & a_{nn} + A_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \gamma_\sigma$$
$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(a_{1p_1} \cdot ... \cdot a_{np_n} + N(\gamma_\sigma)\right),$$

with $\operatorname{sgn}(\sigma) \in \{-1, 1\}$. Then

$$N(\Delta) = \sum_{\sigma \in S_n} N(\gamma_\sigma) \subseteq n!\overline{A} = \overline{A}. \qquad \square$$

LEMMA 5.6. *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]_{n \times n}$ be a reduced non-singular matrix with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ and $\mathcal{B} = [\beta_i]_{n \times 1}$ be an upper zeroless vector with $\beta_i = b_i + B_i \in \mathbb{E}$, for $1 \leqslant i, j \leqslant n$. Then, for all $j \in \{1, ..., n\}$*

(i) $\det \mathcal{M}_j < 2n!\overline{\beta}$.

(ii) $N(\det \mathcal{M}_j(b)) \subseteq \overline{b}.\overline{A}$ *and* $N(\det \mathcal{M}_j) \subseteq \overline{b}.\overline{A} + \overline{B}$.

Proof. Let S_n be the set of all permutations of $\{1, 2, ..., n\}$ and $\sigma = (p_1, ..., p_n)$ a permutation of S_n. We have $\overline{\beta}$ zeroless and, for $1 \leqslant j \leqslant n$,

$$\mathcal{M}_j = \begin{bmatrix} \alpha_{11} & ... & \alpha_{1(j-1)} & \beta_1 & \alpha_{1(j+1)} & ... & \alpha_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n1} & ... & \alpha_{n(j-1)} & \beta_n & \alpha_{n(j+1)} & ... & \alpha_{nn} \end{bmatrix}.$$

Let $\gamma_\sigma = \alpha_{1p_1} \cdot \ldots \cdot \alpha_{(j-1)p_{j-1}} \alpha_{(j+1)p_{j+1}} \cdot \ldots \cdot \alpha_{np_n}$ and $i (= i_\sigma)$ be such that $\operatorname{sgn}(\sigma)\, \gamma_\sigma \beta_i$ is one of the terms of $\det \mathcal{M}_j$. Because $\overline{a} = 1$, by Remark 5.4, it holds that $\overline{\alpha} \subseteq 1 + \oslash$ and $\overline{A} \subseteq \oslash$. So $|\gamma_\sigma| \leqslant \overline{\alpha}^{n-1} \leqslant 1 + \oslash$.

(i) One has

$$\det \mathcal{M}_j = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \gamma_\sigma \beta_i \leqslant \sum_{\sigma \in S_n} |\gamma_\sigma \beta_i| \leqslant n!\, (1 + \oslash)\, \overline{\beta} < 2n! \overline{\beta}.$$

(ii) By Lemma 5.3, $N(\gamma_\sigma) \subseteq N\left(\left(1 + \overline{A}\right)^{n-1}\right) = \overline{A}$. Then, for $1 \leqslant j \leqslant n$

$$\begin{aligned} N(\det \mathcal{M}_j(b)) &= N\left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \gamma_\sigma b_i\right) = \sum_{\sigma \in S_n} N(\gamma_\sigma b_i) \\ &= \sum_{\sigma \in S_n} b_i N(\gamma_\sigma) \subseteq n! \overline{b}.\overline{A} = \overline{b}.\overline{A}. \end{aligned}$$

So, for $1 \leqslant j \leqslant n$

$$\begin{aligned} N(\det \mathcal{M}_j) &= N\left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \gamma_\sigma \beta_i\right) = \sum_{\sigma \in S_n} N(\gamma_\sigma \beta_i) \\ &= \sum_{\sigma \in S_n} \gamma_\sigma N(\beta_i) + \beta_i N(\gamma_\sigma) \subseteq \sum_{\sigma \in S_n} |\gamma_\sigma|\, \overline{B} + b_i N(\gamma_\sigma) \\ &\subseteq n!\left(\overline{B} + \overline{b}.\overline{A}\right) = \overline{B} + \overline{b}.\overline{A}. \qquad \square \end{aligned}$$

Lemma 5.7. *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be a reduced non-singular matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ and $\Delta = \det \mathcal{A} = d + D$, and let $\mathcal{B} = [\beta_i]$ be an upper zeroless vector, with $\beta_i = b_i + B_i \in \mathbb{E}$, for $1 \leqslant i, j \leqslant n$. Consider the reduced flexible system $\mathcal{A}\mathcal{X} \subseteq \mathcal{B}$. Assume that $\mathcal{X} = [\xi_j]$, with $\xi_j = x_j + X_j \in \mathbb{E}$ for all $j \in \{1, ..., n\}$, is an admissable solution, and $R(\mathcal{A}) \subseteq P(\mathcal{B})$. Then*

1. *$\overline{A}\overline{x} \subseteq \left(\overline{A} / \Delta\right) \overline{\beta} \subseteq \underline{B}$, with $\overline{x} = \max_{1 \leqslant j \leqslant n} |x_j|$.*
2. *If $N(\xi_j) \subseteq \underline{B}$ for all $j \in \{1, ..., n\}$, for all $i \in \{1, ..., n\}$ one has*

$$N\left(\sum_{j=1}^{n} \alpha_{ij} \xi_j\right) \subseteq N(\beta_i).$$

Proof. 1. Because $\mathcal{A}$ is a non-singular matrix, Δ is zeroless. So $d \neq 0$. Moreover, since $\mathcal{A}$ is a reduced matrix, $\overline{a} = 1$ and so $R(\mathcal{A}) = \overline{A} / \Delta$.

By Cramer's Rule $\begin{bmatrix} \frac{\det \mathcal{M}_1(a,b)}{d} \\ \vdots \\ \frac{\det \mathcal{M}_n(a,b)}{d} \end{bmatrix}$ is the only solution of the classical linear system $\mathcal{P}\mathcal{Y} = \mathcal{C}$, where $\mathcal{P} = [a_{ij}]_{n\times n}$ is a real matrix and $\mathcal{Y} = [x_i]_{n\times 1}$ and $\mathcal{C} = [b_i]_{n\times 1}$ are real column vectors, with $i, j \in \{1, ..., n\}$.

So $\overline{x} = \left|\frac{\det \mathcal{M}_k(a,b)}{d}\right|$ for some $k \in \{1, ..., n\}$. By Part (i) of Lemma 5.6 we have in particular that $\det \mathcal{M}_k(a,b) < 2n!\overline{b} \leqslant 2n!\overline{\beta}$. Then using Lemma 5.2

$$\begin{aligned} \overline{A}\overline{x} &= \overline{A}\frac{\det \mathcal{M}_k(a,b)}{d} \subseteq \frac{\overline{A}}{d}2n!\overline{\beta} = \frac{\overline{A}}{\Delta}\overline{\beta} \\ &= R(\mathcal{A})\overline{\beta} \subseteq P(\mathcal{B})\overline{\beta} = \left(\underline{B}\diagup\overline{\beta}\right)\overline{\beta} \subseteq \underline{B}. \end{aligned}$$

Hence $\overline{A}\overline{x} \subseteq \left(\overline{A}\diagup\Delta\right)\overline{\beta} \subseteq \underline{B}$.

2. Suppose that $N(\xi_j) \subseteq \underline{B}$ for all $j \in \{1, ..., n\}$. Then, using Lemma 5.2 and Part 1, one has for all $i \in \{1, ..., n\}$

$$\begin{aligned} N\left(\sum_{j=1}^{n}\alpha_{ij}\xi_j\right) &= \sum_{j=1}^{n} N\left(\alpha_{ij}\xi_j\right) = \sum_{j=1}^{n}\left(\alpha_{ij}N\left(\xi_j\right) + \xi_j N\left(\alpha_{ij}\right)\right) \\ &= \sum_{j=1}^{n}\left(a_{ij}N\left(\xi_j\right) + x_j N\left(\alpha_{ij}\right)\right) \subseteq \sum_{j=1}^{n}\left(\overline{a}\underline{B} + \overline{x}\overline{A}\right) \\ &= n\left(\underline{B} + \overline{x}\overline{A}\right) \subseteq \underline{B} + \underline{B} = \underline{B} \subseteq N\left(\beta_i\right). \end{aligned}$$

Hence $N\left(\sum\limits_{j=1}^{n}\alpha_{ij}\xi_j\right) \subseteq N(\beta_i)$, for all $i \in \{1, ..., n\}$. $\square$

We are now able to present the proof of the Theorem *4.4,* starting with the case of reduced flexible systems.

Proof of Theorem 4.4. We assume first that $\overline{a} = 1$. Because $\mathcal{A}$ is a non-singular matrix, $\Delta = \det \mathcal{A} = d + D$ is zeroless. So $d \neq 0$ and $\frac{1}{\Delta} = \frac{1}{d+D} = \frac{1}{d} + \frac{D}{d^2}$. Hence, by Lemma 5.2

$$N\left(\frac{1}{\Delta}\right) = \frac{D}{d^2} = \frac{D}{\Delta^2}. \tag{5.1}$$

For all $i, j \in \{1, ..., n\}$, let $x = [x_j]$ be a solution of the system $\sum_{j=1}^{n} a_{ij}x_j = b_i$. Then by distributivity regarding multiplication by real numbers [2] and Part 1 of

Lemma 5.7

$$\begin{aligned}\alpha_{i1}x_1+...+\alpha_{in}x_n &= (a_{i1}+A_{i1})\,x_1+...+(a_{in}+A_{in})\,x_n\\ &= (a_{i1}x_1+...+a_{in}x_n)+(A_{i1}x_1+...+A_{in}x_n)\\ &\subseteq b_i+\overline{A}\overline{x}\subseteq b_i+\underline{B}\subseteq b_i+B_i=\beta_i.\end{aligned}$$

To complete the proof consider now the neutricial part of the system $\mathcal{A}\mathcal{X}\subseteq\mathcal{B}$.

1. By Part (ii) of Lemma 5.6, Lemma 5.2 and Part 1 of Lemma 5.7, for all $j\in\{1,...,n\}$

$$N\left(\frac{\det\mathcal{M}_j\,(b)}{d}\right)=\frac{1}{d}N\left(\det\mathcal{M}_j\,(b)\right)\subseteq\frac{\overline{b}.\overline{A}}{d}=\left(\overline{A}\diagup\Delta\right)\overline{\beta}\subseteq\underline{B}. \tag{5.2}$$

So $N\left(\xi_j\right)=N\left(\frac{\det\mathcal{M}_j(b)}{d}\right)\subseteq\underline{B}$ for all $j\in\{1,...,n\}$. Hence $\mathcal{X}=\left[\frac{\det\mathcal{M}_j(b)}{d}\right]_{1\leqslant j\leqslant n}$ is a solution of $\mathcal{A}\mathcal{X}\subseteq\mathcal{B}$ by Part 2 of Lemma 5.7.

2. Suppose that Δ is not an absorber of $\underline{B}$. So $\underline{B}\subseteq\Delta\underline{B}$ and we have

$$\underline{B}\diagup\Delta\subseteq\underline{B}. \tag{5.3}$$

Then using Lemma 5.2 and formula (5.1), for all $j\in\{1,...,n\}$

$$\begin{aligned}N\left(\xi_j\right)=N\left(\frac{\det\mathcal{M}_j\,(b)}{\Delta}\right)&=\frac{1}{\Delta}N\left(\det\mathcal{M}_j\,(b)\right)+\det\mathcal{M}_j\,(b)\cdot N\left(\frac{1}{\Delta}\right)\\ &=\frac{1}{d}N\left(\det\mathcal{M}_j\,(b)\right)+\det\mathcal{M}_j\,(b)\cdot\frac{D}{\Delta^2}\\ &=N\left(\frac{\det\mathcal{M}_j\,(b)}{d}\right)+\frac{\det\mathcal{M}_j\,(b)}{\Delta}\cdot\frac{D}{\Delta}.\end{aligned}$$

Using formula (5.2), Part (i) of Lemma 5.6 and Lemma 5.5 one derives

$$N\left(\frac{\det\mathcal{M}_j\,(b)}{d}\right)+\frac{\det\mathcal{M}_j\,(b)}{\Delta}\frac{D}{\Delta}\subseteq\underline{B}+\frac{2n!\overline{\beta}}{\Delta}\frac{\overline{A}}{\Delta}=\underline{B}+\frac{\left(\overline{A}\diagup\Delta\right)\overline{\beta}}{\Delta}.$$

Moreover, by Part 1 of Lemma 5.7 and formula (5.3)

$$\frac{\left(\overline{A}\diagup\Delta\right)\overline{\beta}}{\Delta}\subseteq\underline{B}\diagup\Delta\subseteq\underline{B}. \tag{5.4}$$

Hence for all $j\in\{1,...,n\}$

$$N\left(\xi_j\right)\subseteq\underline{B}+\underline{B}=\underline{B}.$$

Therefore Part 2 of Lemma 5.7 implies that $\mathcal{X}=\left[\frac{\det\mathcal{M}_j(b)}{\Delta}\right]_{1\leqslant j\leqslant n}$ is a solution of $\mathcal{A}\mathcal{X}\subseteq\mathcal{B}$.

3. Suppose now that Δ is not an absorber of $\underline{B}$ and that $\underline{B}=\overline{B}$. Then using Lemma 5.6 and formula (5.1), for all $j\in\{1,...,n\}$

$$\begin{aligned} N\left(\xi_j\right) &= N\left(\frac{\det\mathcal{M}_j}{\Delta}\right)=\frac{1}{\Delta}N\left(\det\mathcal{M}_j\right)+\det\mathcal{M}_j\cdot N\left(\frac{1}{\Delta}\right)\\ &\subseteq \frac{1}{\Delta}\left(\overline{b}.\overline{A}+\overline{B}\right)+2n!\overline{\beta}N\left(\frac{1}{\Delta}\right)=\frac{1}{\Delta}\left(\overline{b}.\overline{A}+\underline{B}\right)+\overline{\beta}\frac{D}{\Delta^2}. \end{aligned}$$

By Lemmas 5.2 and 5.5 and formula (5.3)

$$\frac{1}{\Delta}\left(\overline{b}.\overline{A}+\underline{B}\right)+\overline{\beta}\frac{D}{\Delta^2}\subseteq\overline{\beta}\left(\overline{A}/\Delta\right)+\underline{B}/\Delta+\frac{\overline{\beta}}{\Delta}\left(\overline{A}/\Delta\right)\subseteq\left(\overline{A}/\Delta\right)\overline{\beta}+\underline{B}+\frac{1}{\Delta}\left(\overline{A}/\Delta\right)\overline{\beta}.$$

It follows from Part 1 of Lemma 5.7 and formula (5.4) that $\left(\overline{A}/\Delta\right)\overline{\beta}\subseteq\underline{B}$ and $\frac{1}{\Delta}\left(\overline{A}/\Delta\right)\overline{\beta}\subseteq\underline{B}$. So

$$N\left(\xi_j\right)\subseteq\underline{B}. \tag{5.5}$$

Hence $\mathcal{X}=\left[\frac{\det\mathcal{M}_j}{\Delta}\right]_{1\leqslant j\leqslant n}$ is a solution of $\mathcal{A}\mathcal{X}\subseteq\mathcal{B}$ by Part 2 of Lemma 5.7.

As for the general case, let $\overline{a}$ be arbitrary. Because $\mathcal{A}=[\alpha_{ij}]$ is a non-singular matrix, $\Delta=\det\mathcal{A}$ is zeroless. So $d\neq 0$ and $\overline{a}\neq 0$. Consider the $n\times n$ matrix $\mathcal{A}'=[\alpha_{ij}/\overline{a}]\equiv[c_{ij}+C_{ij}]$ and the column vector $\mathcal{B}'=[\beta_i/\overline{a}]$. Then $\mathcal{A}'$ is a non-singular matrix and $\mathcal{B}'$ is an upper zeroless vector, with $\overline{c}=\max\limits_{1\leqslant i,j\leqslant n}|c_{ij}|=1$. So $\mathcal{A}'\mathcal{X}\subseteq\mathcal{B}'$ is a reduced flexible system with the same solutions as the system $\mathcal{A}\mathcal{X}\subseteq\mathcal{B}$. One has

$$R\left(\mathcal{A}'\right)=\frac{\left(\overline{A}/\overline{a}\right)\overline{c}^{n-1}}{\Delta/\overline{a}^n}=\overline{A}\overline{a}^{n-1}/\Delta=R\left(\mathcal{A}\right)\subseteq P\left(\mathcal{B}\right)=\left(\underline{B}/\Delta\right)\left(\overline{a}/\overline{a}\right)=P\left(\mathcal{B}'\right).$$

Hence $\mathcal{X}=\left[\frac{\det\mathcal{M}_j/\overline{a}^n}{\Delta/\overline{a}^n}\right]_{1\leqslant j\leqslant n}=\left[\frac{\det\mathcal{M}_j}{\Delta}\right]_{1\leqslant j\leqslant n}$ satisfies the equation $\mathcal{A}'\mathcal{X}\subseteq\mathcal{B}'$. Then $\mathcal{X}$ satisfies also the equation $\mathcal{A}\mathcal{X}\subseteq\mathcal{B}$.

Finally we prove that $\mathcal{X}$ is maximal. Indeed, let $\xi_1,...,\xi_n$ be such that $\left(\xi_1,...,\xi_n\right)^T$ satisfies (6.2), and $x_j\in\xi_j$ for $1\leqslant j\leqslant n$. Then for every choice of representatives $a_{ij}\in\alpha_{ij}$ with $1\leqslant i,j\leqslant n$ there exist $b_1\in\beta_1$,..., $b_n\in\beta_n$ such that

$$\left\{\begin{array}{llll} a_{11}x_1+ & ... & +a_{1n}x_n & =b_1\\ \vdots & & \vdots & \vdots\\ a_{n1}x_1+ & ... & +a_{nn}x_n & =b_n \end{array}\right. .$$

Put

$$d=\det\begin{bmatrix} a_{11} & ... & a_{1n}\\ \vdots & & \vdots\\ a_{n1} & ... & a_{nn} \end{bmatrix}.$$

Then $x_j = \frac{\mathcal{M}_j(a,b)}{d} \in \frac{\det \mathcal{M}_j}{\Delta}$ for $1 \leqslant j \leqslant n$. Hence $\xi_j \subseteq \frac{\det \mathcal{M}_j}{\Delta}$ for $1 \leqslant j \leqslant n$ and so $\mathcal{X}$ is maximal. □

6. On Gauss-Jordan elimination. Theorem 4.4 yields closed form formulae for column vectors of external numbers satisfying the flexible system (3.1) by inclusion. In this section we study their relation with solutions obtained by Gauss-Jordan elimination, which are of more practical interest. This will be done by direct verification in the case of a reduced non-singular non-homogeneous flexible system of 2 by 2 linear equations. The verifications in the general case need some additional lemmas and will be the subject of a second article.

The solution of reduced flexible systems by the operations of Gauss-Jordan elimination corresponds to multiplication by certain matrices. Sum and product of matrices will be defined pointwise.

Indeed, let $\mathcal{A} = [\alpha_{ij}]_{m\times n}$, $\mathcal{B} = \left[\beta_{ij}\right]_{m\times n}$ and $\mathcal{C} = \left[\gamma_{jk}\right]_{n\times p}$, where $m, n, p \in \mathbb{N}$, $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n, 1 \leqslant k \leqslant p$ and $\alpha_{ij}, \beta_{ij}, \gamma_{jk}$ are all external numbers. Then

$$\mathcal{A} + \mathcal{B} = \left[\alpha_{ij} + \beta_{ij}\right]_{m\times n}$$

and

$$\mathcal{A}\mathcal{C} = \left[\sum_{1\leqslant j\leqslant n} \alpha_{ij}\gamma_{jk}\right]_{m\times p}.$$

One difficulty to overcome is the fact that multiplication of matrices with external numbers is not fully distributive and associative. These are consequences of the fact that multiplication of external numbers is not fully distributive. For an example, let $A \supset \{0\}$ be a neutrix. Then

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right)\begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \equiv [0]$$

and

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \neq [0].$$

Still, monotony for inclusion is preserved in the following way: Let $\gamma_{ij} \in \mathbb{E}$ for $1 \leqslant i, j \leqslant 2$ and let $U, V, X, Y \in \mathcal{N}$ with $U \subseteq X$ and $V \subseteq Y$. Then

$$\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} \subseteq \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}. \tag{6.1}$$

Indeed

$$\begin{aligned}\begin{bmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{bmatrix}\begin{bmatrix}U\\ V\end{bmatrix} &= \begin{bmatrix}\gamma_{11}U+\gamma_{12}V\\ \gamma_{21}U+\gamma_{22}V\end{bmatrix}\\ &\subseteq \begin{bmatrix}\gamma_{11}X+\gamma_{12}Y\\ \gamma_{21}X+\gamma_{22}Y\end{bmatrix} = \begin{bmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{bmatrix}\begin{bmatrix}X\\ Y\end{bmatrix}.\end{aligned}$$

We use the property of subdistributivity of interval calculus in the next proposition on matrix calculation with differences. We consider the general case, for the proof is straightforward.

Proposition 6.1. *Let $n \in \mathbb{N}$ be standard and let $\alpha_{ij}, \beta_i, \xi_j \in \mathbb{E}$ for all $i, j \in \{1, ..., n\}$. Assume*

$$\begin{bmatrix}\alpha_{11} & \cdots & \alpha_{1n}\\ \vdots & & \vdots\\ \alpha_{n1} & \cdots & \alpha_{nn}\end{bmatrix}\begin{bmatrix}\xi_1\\ \vdots\\ \xi_n\end{bmatrix} \subseteq \begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}.$$

Let $B_i = N(\beta_i)$ for all $i \in \{1, ..., n\}$. Let $x_i, y_i \in \xi_i$ and $u_i = x_i - y_i$ for $1 \leqslant i \leqslant n$. Then the column vector $(u_1, ..., u_n)^T$ satisfies

$$\begin{bmatrix}\alpha_{11} & \cdots & \alpha_{1n}\\ \vdots & & \vdots\\ \alpha_{n1} & \cdots & \alpha_{nn}\end{bmatrix}\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} \subseteq \begin{bmatrix}B_1\\ \vdots\\ B_n\end{bmatrix}.$$

Proof. It follows from subdistributivity that for $1 \leqslant i \leqslant n$

$$\begin{aligned}\alpha_{i1}u_1 + ... + \alpha_{in}u_n &= \alpha_{i1}(x_1 - y_1) + ... + \alpha_{in}(x_n - y_n)\\ &\subseteq \alpha_{i1}x_1 - \alpha_{i1}y_1 + ... + \alpha_{in}x_n - \alpha_{in}y_n\\ &= \alpha_{i1}x_1 + ... + \alpha_{in}x_n - (\alpha_{i1}y_1 + ... + \alpha_{in}y_n)\\ &\subseteq \beta_i - \beta_i = B_i. \qquad \square\end{aligned}$$

For the solution of reduced flexible systems by the operations of Gauss-Jordan elimination we will consider matrices with real entries. Then, taking profit of (2.1), distributivity holds to a large extent, which leads to some convenient simplifications. Below we will maintain the notations of Notation 3.1.

Definition 6.2. Let $\alpha_{12}, \alpha_{21}, \alpha_{22}, \beta_1, \beta_2, \xi_1, \xi_2 \in \mathbb{E}$. Let $a_{12} \in \alpha_{12}, a_{21} \in \alpha_{21}$ and $a_{22} \in \alpha_{22}$. Consider the reduced non-singular non-homogeneous flexible system of linear equations

$$\left\{\begin{array}{ccccc}(1 + A_{11})\xi_1 & + & \alpha_{12}\xi_2 & \subseteq \beta_1\\ \alpha_{21}\xi_1 & + & \alpha_{22}\xi_2 & \subseteq \beta_2.\end{array}\right. \tag{6.2}$$

Let $d = a_{22} - a_{21}a_{12}$, then $d \neq 0$. We define matrices $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ by

$$\mathcal{G}_1 = \begin{bmatrix} 1 & 0 \\ -a_{21} & 1 \end{bmatrix}, \mathcal{G}_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{d} \end{bmatrix}, \mathcal{G}_3 = \begin{bmatrix} 1 & -a_{12} \\ 0 & 1 \end{bmatrix}.$$

We write $\mathcal{G}[.]$ to indicate the repeated multiplication of matrices $\mathcal{G}_3(\mathcal{G}_2(\mathcal{G}_1 \cdot [.]))$.

Observe that, with $\mathcal{A} = \begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the matrix $\mathcal{G}_1$ corresponds to the subtraction of a_{21} times the first row of the second row of $\mathcal{A}$, the matrix $\mathcal{G}_2$ divides the second row of $\mathcal{G}_1\mathcal{A}$ by d and the matrix $\mathcal{G}_3$ subtracts the second row a_{12} times of the first row of $\mathcal{G}_2(\mathcal{G}_1\mathcal{A})$. These are the appropriate Gauss-Jordan elimination operations for the matrix $\mathcal{A}$, indeed $\mathcal{G}\mathcal{A} = I_2$ with $\mathcal{G}_3(\mathcal{G}_2 \cdot \mathcal{G}_1) = \frac{1}{d}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & 1 \end{bmatrix}$.

Definition 6.3. Let $(x, y) \in \mathbb{R}^2$. We call $(x, y)^T$ a *Gauss-solution* of (6.2) if for all choices of representatives of $\alpha_{12}, \alpha_{21}, \alpha_{22}$ and corresponding matrices one has

$$\mathcal{G}\begin{bmatrix} 1 + A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \subseteq \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

We will asume that $N(\beta_1) = N(\beta_2) \equiv B$. In case Δ is not an absorber of B and $\overline{A}/\Delta \subseteq B/\overline{\beta}$, every element of the solution given by Cramer's Rule is a Gauss-solution and vice-versa. This will be shown in the remaining part of this section. We start with some useful properties of multiplication of matrices.

Because the matrices $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ contain only real numbers, by (2.2) distributivity holds with respect to expressions of the form $a + A$, with $a \in \mathbb{R}$ and $A \in \mathcal{N}$. Hence

$$\mathcal{G}\begin{bmatrix} 1 + A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \mathcal{G}\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{6.3}$$

Lemma 6.4. *Consider the reduced non-singular non-homogeneous flexible system (6.2). Assume that Δ is not an absorber of B. Let $a_{12} \in \alpha_{12}, a_{21} \in \alpha_{21}$ and $a_{22} \in \alpha_{22}$. Then*

1. $B = B\Delta = B/\Delta$.
2. $\mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix}$.
3. *If $\overline{A}/\Delta \subseteq B/\overline{\beta}$ one has*

$$\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \subseteq \begin{bmatrix} B/\overline{\beta} & B/\overline{\beta} \\ B/\overline{\beta} & B/\overline{\beta} \end{bmatrix}$$

and

$$\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} \subseteq \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix}.$$

Proof. *1.* Because (6.2) is a reduced non-singular flexible system, $0 < |\Delta| \leqslant 2 + \oslash \leqslant 3$. Moreover, Δ is not an absorber of B. So

$$B \subseteq \Delta B \subseteq 3B = B.$$

Hence $B = B\Delta$. Moreover $B / \Delta = (B\Delta)/\Delta = B(\Delta/\Delta) = B$, since $\Delta/\Delta \subseteq 1 + \oslash$.

2. Firstly, since $|a_{21}| \leqslant 1$, one has

$$\mathcal{G}_1\begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -a_{21} & 1 \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} B \\ a_{21}B + B \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix}.$$

Secondly, by Part 1,

$$\mathcal{G}_2\begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{d} \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} B \\ \frac{B}{d} \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix}.$$

Thirdly, since $|a_{12}| \leqslant 1$,

$$\mathcal{G}_3\begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} 1 & -a_{12} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} B + a_{12}B \\ B \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix}.$$

Hence

$$\mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} = \mathcal{G}_3\left(\mathcal{G}_2\left(\mathcal{G}_1 \cdot \begin{bmatrix} B \\ B \end{bmatrix}\right)\right) = \begin{bmatrix} B \\ B \end{bmatrix}.$$

3. If $\overline{A} / \Delta \subseteq B / \overline{\beta}$, by Part 1 one has $\overline{A} \subseteq B / \overline{\beta}$. Then, because for all $i, j \in \{1, 2\}$, $A_{ij} \subseteq \overline{A} \subseteq B / \overline{\beta}$, using formula (6.1) and Part 2, one obtains, whenever b is a representative of $\overline{\beta}$

$$\begin{aligned} \mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} &\subseteq \mathcal{G}\begin{bmatrix} B/\overline{\beta} & B/\overline{\beta} \\ B/\overline{\beta} & B/\overline{\beta} \end{bmatrix} = \mathcal{G}\begin{bmatrix} B/b & B/b \\ B/b & B/b \end{bmatrix} \\ &= \frac{1}{b}\mathcal{G}\begin{bmatrix} B & B \\ B & B \end{bmatrix} = \frac{1}{b}\begin{bmatrix} B & B \\ B & B \end{bmatrix} = \begin{bmatrix} B/\overline{\beta} & B/\overline{\beta} \\ B/\overline{\beta} & B/\overline{\beta} \end{bmatrix}. \end{aligned}$$

Moreover, also using Lemma 5.1

$$\begin{aligned} \mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} &\subseteq \mathcal{G}\begin{bmatrix} B/\overline{\beta} & B/\overline{\beta} \\ B/\overline{\beta} & B/\overline{\beta} \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} \\ &\subseteq \mathcal{G}\begin{bmatrix} \oslash & \oslash \\ \oslash & \oslash \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} \subseteq \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix}. \end{aligned}$$

□

We also need a property on the order of magnitude of the entries of a matrix with respect to its determinant.

LEMMA 6.5. *Let* $\mathcal{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$ *be the matrix of coefficients of the reduced non-singular flexible system (6.2) and* $\Delta = \det \mathcal{A}$. *Then* $|\alpha_{12}| > \oslash\Delta$ *or* $|\alpha_{22}| > \oslash\Delta$.

Proof. One has $\Delta = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$, with $|\alpha_{ij}| \leqslant 1 + \oslash$ for all $i, j \in \{1, 2\}$. Suppose that $\alpha_{12} \subseteq \oslash\Delta$ and $\alpha_{22} \subseteq \oslash\Delta$. Then $\alpha_{11}\alpha_{22} \subseteq (1+\oslash) \oslash \Delta = \oslash\Delta$ and $\alpha_{12}\alpha_{21} \subseteq \oslash (1+\oslash) \Delta = \oslash\Delta$. So $\Delta \subseteq \oslash\Delta$, which is absurd because Δ is zeroless. Hence $|\alpha_{12}| > \oslash\Delta$ or $|\alpha_{22}| > \oslash\Delta$. □

The next two propositions yield a lower bound on the uncertainty of Cramer-solutions and an upper bound on the uncertainty of Gauss-solutions.

PROPOSITION 6.6. *Consider the reduced non-singular non-homogeneous flexible system of linear equations* (6.2). *Assume that* Δ *is not an absorber of* B *and that* $\overline{A}/\Delta \subseteq B/\overline{\beta}$. *Then*

$$N\left(\frac{\det \mathcal{M}_1}{\Delta}\right) = N\left(\frac{\det \mathcal{M}_2}{\Delta}\right) = B.$$

Proof. By formula (5.5), $N\left(\frac{\det \mathcal{M}_1}{\Delta}\right) \subseteq B$ and $N\left(\frac{\det \mathcal{M}_2}{\Delta}\right) \subseteq B$. On the other hand one has

$$\begin{aligned} a_{22}B + a_{12}B &\subseteq (a_{22}B + b_1A_{22} + BA_{22}) + (a_{12}B + b_2A_{12} + BA_{12}) \\ &= N\left(\det \begin{bmatrix} b_1 + B & a_{12} + A_{12} \\ b_2 + B & a_{22} + A_{22} \end{bmatrix}\right) = N\left(\det \mathcal{M}_1\right). \end{aligned}$$

By Lemma 6.5, $|\alpha_{12}| > \oslash\Delta$ or $|\alpha_{21}| > \oslash\Delta$. So $a_{22} = c_1 d$, with $|c_1| > \oslash$, or $a_{12} = c_2 d$, with $|c_2| > \oslash$. Using Part 1 of Lemma 6.4, we find $a_{22}B = c_1 dB = c_1 B \supseteq B$ or $a_{12}B = c_2 dB = c_2 B \supseteq B$. Therefore $B \subseteq a_{22}B + a_{12}B \subseteq N(\det \mathcal{M}_1)$. Hence

$$\frac{B}{\Delta} \subseteq \frac{N\left(\det \mathcal{M}_1\right)}{\Delta} \subseteq N\left(\frac{\det \mathcal{M}_1}{\Delta}\right).$$

Again by Part 1 of Lemma 6.4 one has $B = \frac{B}{\Delta}$. So $B \subseteq N\left(\frac{\det \mathcal{M}_1}{\Delta}\right)$ and we conclude that $N\left(\frac{\det \mathcal{M}_1}{\Delta}\right) = B$.

The proof is the same for $N\left(\frac{\det \mathcal{M}_2}{\Delta}\right) = B$. □

PROPOSITION 6.7. *Consider the reduced non-singular non-homogeneous flexible system of linear equations* (6.2). *Assume that* Δ *is not an absorber of* B *and that*

$\overline{A}/\triangle \subseteq B/\overline{\beta}$. Let $x_{1,}, x_{2,} y_1, y_2 \in \mathbb{R}$ such that $(x_1, x_2)^T$ and $(y_1, y_2)^T$ are Gauss-solutions of (6.2)*. Let $u_1 = x_1 - y_1$ and $u_2 = x_2 - y_2$. Then $u_1 \in B$ and $u_2 \in B$.*

Proof. Let $a_{12} \in \alpha_{12}, a_{21} \in \alpha_{21}$ and $a_{22} \in \alpha_{22}$. Then

$$\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \subseteq \begin{bmatrix} B \\ B \end{bmatrix}, \tag{6.4}$$

for, using Part 2 of Lemma 6.4,

$$\begin{aligned}
&\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&\subseteq \mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
&\subseteq \mathcal{G}\begin{bmatrix} b_1+B \\ b_2+B \end{bmatrix} - \mathcal{G}\begin{bmatrix} b_1+B \\ b_2+B \end{bmatrix} \\
&= \mathcal{G}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} - \mathcal{G}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} \\
&= \begin{bmatrix} B \\ B \end{bmatrix} - \begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix}.
\end{aligned}$$

Also

$$\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \subseteq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \oslash & \oslash \\ \oslash & \oslash \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{6.5}$$

Indeed, by distributivity, Part 3 of Lemma 6.4 and Lemma 5.1

$$\begin{aligned}
\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \mathcal{G}\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&\subseteq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} B/\overline{\beta} & B/\overline{\beta} \\ B/\overline{\beta} & B/\overline{\beta} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&\subseteq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \oslash & \oslash \\ \oslash & \oslash \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\end{aligned}$$

Assume $(u_1, u_2) \in \mathbb{R}^2$ such that $(u_1, u_2)^T$ satisfies

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \oslash & \oslash \\ \oslash & \oslash \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \subseteq \begin{bmatrix} B \\ B \end{bmatrix}. \tag{6.6}$$

Then

$$\begin{cases} u_1 + \oslash u_1 + \oslash u_2 \subseteq B \\ u_2 + \oslash u_1 + \oslash u_2 \subseteq B. \end{cases} \tag{6.7}$$

Suppose first that $\max(|u_1|, |u_2|) = |u_1|$. So $u_1 + \oslash u_1 + \oslash u_2 = u_1 + \oslash u_1 = (1+\oslash) u_1$. If $u_1 \notin B$, also $u_1/2 \notin B$. Hence $|u_1 + \oslash u_1 + \oslash u_2| > |u_1|/2 \notin B$, which contradicts the first equation of system (6.7). Therefore $u_1 \in B$ and also $u_2 \in B$. The case that $\max(|u_1|, |u_2|) = |u_2|$ is analogous. Hence all solutions $(u_1, u_2)^T$ of (6.6) satisfy $u_1 \in B$ and $u_2 \in B$. By (6.5) all solutions of (6.4) satisfy (6.6). Hence all solutions of (6.4) satisfy $u_1 \in B$ and $u_2 \in B$. □

By Part 3 of Theorem 4.4, if $\triangle$ is not an absorber of B and $\overline{A}/\Delta \subseteq B/\overline{\beta}$, a Cramer-solution of the system (6.2) is an admissible solution. We show now that under these conditions any element of this solution is a Gauss-solution.

THEOREM 6.8. *Assume that $\triangle$ is not an absorber of B and that $\overline{A}/\Delta \subseteq B/\overline{\beta}$. Let $(x,y)^T \in \left(\frac{\det \mathcal{M}_1}{\Delta}, \frac{\det \mathcal{M}_2}{\Delta}\right)^T$. Then $(x,y)^T$ is a Gauss-solution of* (6.2).

Proof. Let $a_{12} \in \alpha_{12}, a_{21} \in \alpha_{21}$ and $a_{22} \in \alpha_{22}$. Choose $b_1 \in \beta_1$ and $b_2 \in \beta_2$ and let $b = \max(|b_1|, |b_2|)$. Put $d_1 = b_1 a_{22} - b_2 a_{12}$, $d_2 = b_2 - b_1 a_{21}$ and $d = a_{22} - a_{12}a_{21}$. One has $|d_1| \leqslant 3b$ and $|d_2| \leqslant 3b$.

We assume first that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{d_1}{d} \\ \frac{d_2}{d} \end{bmatrix}$. Then

$$\mathcal{G}\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{d_1}{d} \\ \frac{d_2}{d} \end{bmatrix} = \mathcal{G}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Now we prove that

$$\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \subseteq \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix}.$$

Indeed, using Parts 3 and 1 of Lemma 6.4, one obtains that

$$\begin{aligned}\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &\subseteq \begin{bmatrix} B/b & B/b \\ B/b & B/b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{B}{b}x + \frac{B}{b}y \\ \frac{B}{b}x + \frac{B}{b}y \end{bmatrix} \\ &= \begin{bmatrix} \frac{B}{b}\frac{d_1}{d} + \frac{B}{b}\frac{d_2}{d} \\ \frac{B}{b}\frac{d_1}{d} + \frac{B}{b}\frac{d_2}{d} \end{bmatrix} \subseteq \begin{bmatrix} \frac{B}{b}\frac{b}{d} + \frac{B}{b}\frac{b}{d} \\ \frac{B}{b}\frac{b}{d} + \frac{B}{b}\frac{b}{d} \end{bmatrix} \\ &= \begin{bmatrix} \frac{B}{\Delta} \\ \frac{B}{\Delta} \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix} = \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix}.\end{aligned}$$

Then it follows by distributivity that

$$\begin{aligned}\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \mathcal{G}\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &\subseteq \mathcal{G}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} = \mathcal{G}\begin{bmatrix} b_1 + B \\ b_2 + B \end{bmatrix} = \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.\end{aligned}$$

Hence $(x,y)^T$ is a Gauss-solution of (6.2).

Finally, let $\begin{bmatrix} x' \\ y' \end{bmatrix} \in \begin{bmatrix} \frac{\det \mathcal{M}_1}{\triangle} \\ \frac{\det \mathcal{M}_2}{\triangle} \end{bmatrix}$ be arbitrary. By Proposition 6.6 one has $N\left(\frac{\det \mathcal{M}_1}{\Delta}\right) = N\left(\frac{\det \mathcal{M}_2}{\Delta}\right) = B$. So $\begin{bmatrix} x' \\ y' \end{bmatrix} \in \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix}$. Then by distributivity and Lemma 6.4

$$\begin{aligned}
&\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} \\
&\subseteq \mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} \\
&\subseteq \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \mathcal{G}\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} + \mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} \\
&\subseteq \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} \\
&= \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} = \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.
\end{aligned}$$

Hence $(x',y')^T$ is also a Gauss-solution of (6.2). □

Next theorem is a converse to Theorem 6.8. Under the usual conditions, a Gauss-solution must be an element of the Cramer-solution.

THEOREM 6.9. *Assume that $\triangle$ is not an absorber of B and that $\overline{A}\diagup\Delta \subseteq B\diagup\overline{\beta}$. Let $(x,y)^T$ be a Gauss-solution of* (6.2)*. Then $(x,y)^T$ satisfies* (6.2)*, in fact $(x,y)^T \in \left(\frac{\det \mathcal{M}_1}{\triangle}, \frac{\det \mathcal{M}_2}{\triangle}\right)^T$.*

Proof. Let $a_{12} \in \alpha_{12}, a_{21} \in \alpha_{21}$ and $a_{22} \in \alpha_{22}$. Choose $b_1 \in \beta_1$ and $b_2 \in \beta_2$ and let $b = \max(|b_1|, |b_2|)$. Put $d_1 = b_1 a_{22} - b_2 a_{12}$, $d_2 = b_2 - b_1 a_{21}$ and $d = a_{22} - a_{12}a_{21}$. It follows from Theorem 6.8 that $(x,y)^T = \left(\frac{d_1}{d}, \frac{d_2}{d}\right)^T$ is a Gauss-solution, and it clearly satisfies (6.2). Let $(x',y')^T$ be an arbitrary Gauss-solution of (6.2). By Propositions 6.7 and 6.6 it holds that $x' \in \frac{d_1}{d} + B = \frac{\det \mathcal{M}_1}{\triangle}$ and $y' \in \frac{d_2}{d} + B = \frac{\det \mathcal{M}_2}{\triangle}$. Then it follows from Part 3 of Theorem 4.4 that $(x,y)^T$ satisfies (6.2). □

THEOREM 6.10. *Assume that $\triangle$ is not an absorber of B and that $\overline{A}\diagup\Delta \subseteq B\diagup\overline{\beta}$. Then the Cramer-solution of the reduced flexible system* (6.2) *equals the external set of all Gauss-solutions.*

Proof. By Theorem 6.8 and 6.9 it holds that $\left(\frac{\det \mathcal{M}_1}{\Delta}, \frac{\det \mathcal{M}_2}{\Delta}\right)^T$ is equal to the external set of all Gauss-solutions. □

This final theorem implies that the external set of all Gauss-solutions, being equal to the Cramer-solution, by Part 3 of Theorem 4.4, also constitutes an admissible and maximal solution of the reduced flexible system (6.2).

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