JOHN WILEY & SONS, LTD., THE ATRIUM, SOUTHERN GATE, CHICHESTER P019 8SQ, UK

*** PROOF OF YOUR ARTICLE ATTACHED, PLEASE READ CAREFULLY ***

After receipt of your corrections your article will be published initially within the online version of the journal.

PLEASE NOTE THAT THE PROMPT RETURN OF YOUR PROOF CORRECTIONS WILL ENSURE THAT THERE ARE NO UNNECESSARY DELAYS IN THE PUBLICATION OF YOUR ARTICLE

☐ **READ PROOFS CAREFULLY**

ONCE PUBLISHED ONLINE OR IN PRINT IT IS NOT POSSIBLE TO MAKE ANY FURTHER CORRECTIONS TO YOUR ARTICLE

§ This will be your only chance to correct your proof
§ Please note that the volume and page numbers shown on the proofs are for position only

☐ **ANSWER ALL QUERIES ON PROOFS** (Queries are attached as the last page of your proof.)

§ List all corrections and send back via e-mail to the production contact as detailed in the covering e-mail, or mark all corrections directly on the proofs and send the scanned copy via e-mail. Please do not send corrections by fax or post

☐ **CHECK FIGURES AND TABLES CAREFULLY**

§ Check sizes, numbering, and orientation of figures
§ All images in the PDF are downsampled (reduced to lower resolution and file size) to facilitate Internet delivery. These images will appear at higher resolution and sharpness in the printed article
§ Review figure legends to ensure that they are complete
§ Check all tables. Review layout, titles, and footnotes

☐ **COMPLETE COPYRIGHT TRANSFER AGREEMENT (CTA) if you have not already signed one**

§ Please send a scanned signed copy with your proofs by e-mail. **Your article cannot be published unless we have received the signed CTA**

☐ **OFFPRINTS**

§ 25 complimentary offprints of your article will be dispatched on publication. Please ensure that the correspondence address on your proofs is correct for dispatch of the offprints. If your delivery address has changed, please inform the production contact for the journal – details in the covering e-mail. Please allow six weeks for delivery.

Additional reprint and journal issue purchases

§ Should you wish to purchase a minimum of 100 copies of your article, please visit http://www3.interscience.wiley.com/aboutus/contact_reprint_sales.html
§ To acquire the PDF file of your article or to purchase reprints in smaller quantities, please visit http://www3.interscience.wiley.com/aboutus/ppv-articleselect.html. Restrictions apply to the use of reprints and PDF files – if you have a specific query, please contact permreq@wiley.co.uk. Corresponding authors are invited to inform their co-authors of the reprint options available
§ To purchase a copy of the issue in which your article appears, please contact cs-journals@wiley.co.uk upon publication, quoting the article and volume/issue details
§ Please note that regardless of the form in which they are acquired, reprints should not be resold, nor further disseminated in electronic or print form, nor deployed in part or in whole in any marketing, promotional or educational contexts without authorization from Wiley. Permissions requests should be directed to mailto: permreq@wiley.co.uk

NME 2686
pp: 1–29 (col.fig.: 16,17)
PROD. TYPE: COM
ED: Harish
PAGN: Babu – SCAN:

INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng (2009)
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nme.2686

Smooth finite strain plasticity with non-local pressure support

P. M. A. Areias[1,*,†,‡] and T. Rabczuk[2,‡] Q1

[1] *ICIST—Instituto de Engenharia de Estruturas, Território e Construção, IST—Instituto Superior Técnico, Universidade Técnica de Lisboa, Portugal* Q2
[2] *Mechanical Engineering Department, University of Canterbury, Christchurch, New Zealand*

SUMMARY

The aim of this work is to introduce an alternative framework to solve problems of finite strain elasto-plasticity including anisotropy and kinematic hardening coupled with any isotropic hyperelastic law. After deriving the constitutive equations and inequalities without any of the customary simplifications, we arrive at a new general elasto-plastic system. We integrate the elasto-plastic algebraico-differential system and replace the loading–unloading condition by a Chen–Mangasarian smooth function to obtain a non-linear system solved by a trust region method. Despite being non-standard, this approach is advantageous, since quadratic convergence is always obtained by the non-linear solver and very large steps can be used with negligible effect in the results. Discretized equilibrium is, in contrast with traditional approaches, smooth and well behaved. In addition, since no *return mapping* algorithm is used, there is no need to use a predictor. The work follows our previous studies of element technology and highly non-linear visco-elasticity. From a general framework, with exact linearization, systematic particularization is made to prototype constitutive models shown as examples. Our element with non-local pressure support is used. Examples illustrating the generality of the method are presented with excellent results.

Received 1 February 2008; Revised 30 January 2009; Accepted 25 May 2009

KEY WORDS: finite strain; plasticity; complementarity; mixed methods

1. INTRODUCTION

Association of the return-mapping technique [1–3] and well-founded mixed formulations [4, 5] led to a standardization of elasto-plastic modeling with finite elements (see the treatise by Belytschko and co-workers [6]). However, the return-mapping algorithm still poses challenges to systematization: the predictor in the presence of damage may give a false indication and there is an implied inequality for the plastic multiplier. Another problem for implicit return mapping occurs when the

*Correspondence to: P. M. A. Areias, ICIST—Instituto de Engenharia de Estruturas, Território e Construção, IST—Instituto Superior Técnico, Universidade Técnica de Lisboa, Portugal.
†E-mail: pareias@civil.ist.utl.pt
‡Assistant Professor.

hardening law depends on the plastic strain rate and temperature, such as in the Johnson–Cook model and certain Nemat-Nasser models [7]: the predictor can also give a false indication, since typically the plastic strain rate may raise the yield stress and produce spurious results. Empirical integration methods have been devised for each case (see, e.g. [8]).

Besides the problems with the predictor, the convergence radius is often not satisfactory (Crisfield and Norris [9] show dense clouds of points due to step halving). Here, we introduce a new finite strain elasto-plastic algorithm able to include, in the same underlying framework, kinematic hardening, anisotropy, damage, etc. This differs from our recent application to non-linear viscoelasticity [10], where the constitutive system was smooth.

Few, if any, attempts have been made to integrate the general finite strain elasto-plastic constitutive system. Typically, strong assumptions and simplifications are made, which substantially reduce the true complexity of the problem (small elastic strains and/or isotropic plasticity and coaxiality are often assumed). For the continuum case, a notable exception has been the work of Nemat-Nasser [3], which showed the complete system in continuum form. However, to our knowledge, no attempt to implicitly integrate it has been made.

As for the finite element part, recent experiments with the inf–sup (IS) test (e.g. [5]) led to a simple element with non-local pressure support (see also [10]).

Concerning certain simplified (small strain or coaxial) constitutive models, solutions based on the dual formulation are now strongly established both with predictor/corrector [1] and Shur-based methodologies [11]. We propose here an alternative that consists of smoothing the loading/unloading condition (also called complementarity condition, see Han and Reddy [12, p. 60, Equation 3.37] and solving monolithically the resulting system. That system can therefore be solved by classical Newton-based root finders. Accuracy of the solution depends on the smoothing parameter, which measures the distance to the origin of the complementarity graph. Mathematical foundations of this method were established by Chen and Mangasarian [13].

Numerous works have shown the advantages of ordinary differential equation (ODE) integration and correct calculation of the derivatives in loading for elasto-plastic problems. This is often called 'consistent linearization' and was introduced for *smooth* constitutive calculations context by Hughes and Taylor [14] and for non-smooth problems by Simo and co-workers (see the monograph [2]). The linearization consists of an application of the chain rule and, in the finite strain case, use of Lie group theory.

For elastoplasticity, however, the overall problem remains non-smooth [12, 15, 16]. For a sequence of global iterations, a given quadrature point can have successive loading/unloading or reloading/unloading states. Because derivatives are not continuous, erratic behavior is often observed.

The primal version of the FEM in small strains has a relatively straightforward weak form with a differential inclusion. For finite strains, because the elastic part of the deformation gradient depends implicitly on the stress, this simplicity cannot be retained. The dual form is advantageous from the implementation point of view. Numerous papers have dealt with J_2 [17–20] plasticity in finite strains including kinematic hardening. Cost-effective algorithms are then adopted for von-Mises plasticity, based on radial-return technique (parallel trial elastic strain and final deviatoric stress [1]) that reduces the constitutive solution to one algebraic equation. In that case, for finite strains the additional condition of coaxiality of the strain measures and the Kirchhoff stress is either verified or imposed. This started with the paper by Weber and Anand [21]. Semi-implicit methods, which freeze the flow vector in the solution, hence retaining the attractiveness of the Key and Krieg approach for more complex cases, have been disseminated by Moran and co-workers

DOI: 10.1002/nme

(see, e.g. [22]). Other yield functions require the direct use of Lee's decomposition [23] and monolithic integration. This has been done for a similar case by Hartmann *et al.* [24] and it was applied to the von-Mises yield criterion. The lack of smoothness of the problem is still not tackled consistently for this case. We first enumerate our requirements to clarify the options:

- Use of isotropic hyperelastic law: $\mathbf{T}_t \equiv \mathbf{T}_t(\mathbf{V}^{\mathrm{e}})$, where $\mathbf{V}^{\mathrm{e}}$ is the left *elastic* stretch tensor and $\mathbf{T}_t$ is the Kirchhoff stress measure (see [25, p.142] for the isotropy limitation).
- Unique framework for viscoelasticity, viscoplasticity and elastoplasticity with no restrictions in the form either of the flow law or the yield function.
- Use of any *kinematic* hardening model (including multi-surface models) as an additional equations to the system.
- Element-independence: specific properties of the elements, such as mixed or hybrid techniques should not be used to simplify the constitutive calculations.

For moderate elastic strains (often the case for metals), simplified methods are often used, such as the 'rotated configuration' by Areias and Belytschko [26]. With the previous work [10], topics covered are:

- Quantified evaluation of absence of locking and spurious modes in the nearly incompressible regime.
- Integration of the constitutive ODE and incompressibility preservation.
- Objectivity and monotonicity of the back-stress treatment.
- Smoothing the loading/unloading condition or use of a non-smooth solver.

The first two themes were treated in our previous work. In the essence, the behavior of an element with constraints introduced by the material it represents is indicated by the inspection of the IS value with mesh refinement. We evaluated this behavior [10] and it confirms, for the specific conditions shown here, that the stability and convergence are satisfied.

2. FORMULATION OF THE COUPLED EQUILIBRIUM/CONSTITUTIVE PROBLEM

2.1. *Governing equations*

A given open set $\Omega_0 \subset \mathbb{R}^3$ is the reference configuration of a given body: each point X is associated by a bijective map to its position in that configuration: $X \rightarrow \mathbf{X} \in \Omega_0$. See Figure 1 for a clarification of this notation. In the absence of discontinuities of maps defined in Ω_0, a unique deformation map $\varphi(\mathbf{X}) \in \mathscr{H}^1(\Omega_0)^{n_{\mathrm{sd}}}$ exists such that any position besides the reference one is determined $\mathbf{x} = \varphi(\mathbf{X})$ with a the difference being $\mathbf{u} = \mathbf{x} - \mathbf{X}$. We use the standard notation n_{sd} as the number of space dimensions. The deformation gradient is obtained as $\mathbf{F} = \nabla_0 \varphi(\mathbf{X})$, where ∇_0 represents the gradient with respect to $\mathbf{X}$. The Jacobian of the deformation map is given by $J = \det \mathbf{F}$ and represents the local volume ratio.

The deformation gradient is decomposed into elastic (e) and plastic (p) parts, using Lee's decomposition [23]: $\mathbf{F} = \mathbf{F}^{\mathrm{e}}\mathbf{F}^{\mathrm{p}}$ where $\mathbf{F}^{\mathrm{e}}$ includes the lattice rotation, in the sense of Nemat-Nasser (see [3, p. 250]), but with a redefinition of $\mathbf{F}^{\mathrm{e}}$

$$\mathbf{F} = \mathbf{V}^{\mathrm{e}}\mathbf{Q}\mathbf{U}^{\mathrm{p}} \tag{1}$$

DOI: 10.1002/nme

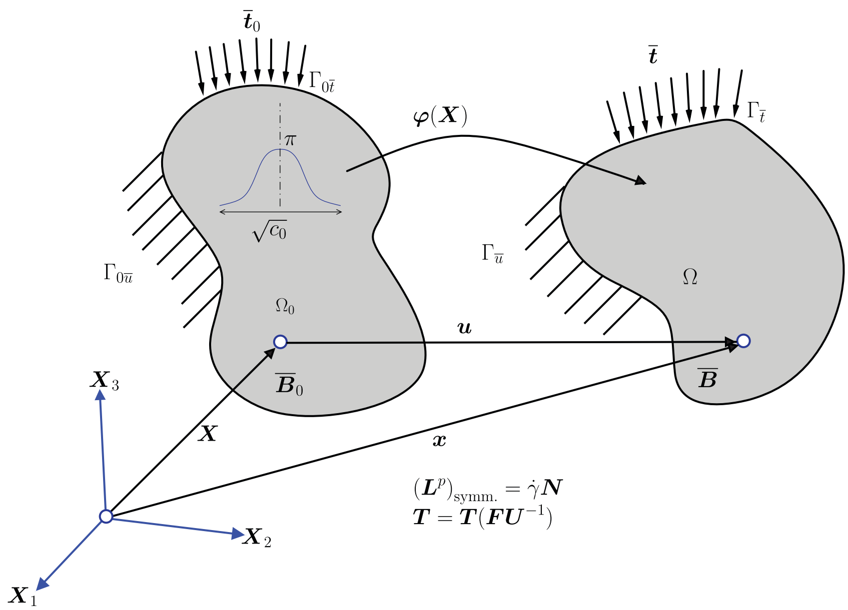


Figure 1. Problem description: finite strain elastoplasticity with non-local pressure.

Color Online, B&W in Print

with $\mathbf{V}^{\mathrm{e}}$ being the left elastic stretch tensor, $\mathbf{Q}$ the orthogonal rotation tensor (such that $\mathbf{F}^{\mathrm{e}}=\mathbf{V}^{\mathrm{e}}\mathbf{Q}$, a variant of [3] valid for isotropic elastic laws) and $\mathbf{U}^{\mathrm{p}}=\mathbf{F}^{\mathrm{p}}$ the right plastic stretch tensor. Since there is no danger of mixing $\mathbf{V}^{\mathrm{e}}$ and $\mathbf{U}^{\mathrm{p}}$ with their total counterparts we drop the superscripts from this point on. From the elastic part we extract the elastic left Cauchy–Green tensor $\mathbf{V}^2=\mathbf{F}^{\mathrm{e}}\mathbf{F}^{\mathrm{eT}}$ and from the plastic part the right Cauchy–Green tensor $\mathbf{C}=\mathbf{U}^2$ is obtained (we also omit the superscripts e and p for $\mathbf{V}$ and $\mathbf{C}$, respectively). The first Piola–Kirchhoff stress $\mathbf{P}$ is related to the body forces $\mathbf{B}_0(\mathbf{X})\in L^2(\Omega_0)$ by the equilibrium equation. Cauchy stresses (required for the elasto-plastic model) are obtained as $\boldsymbol{\sigma}=(1/J)\mathbf{P}\mathbf{F}^{\mathrm{T}}$. Body forces are assumed to be defined in the reference configuration.

The outer boundary of Ω_0 is partitioned into two sets: the Neumann set, $\Gamma_{0\bar{t}}$ where some stress components are known and the Dirichlet set, $\Gamma_{0\bar{u}}$ where some components of displacement are known. Also used is the Kirchhoff stress, $\mathbf{T}_t=J\boldsymbol{\sigma}$ that depends on $\mathbf{V}$, which indeed restricts the elastic law to be isotropic [25]. Since for metals, elastomers and other materials often $J\cong 1$ then we can write $\boldsymbol{\sigma}\cong\mathbf{T}_t$.

After introducing a yield function, ϕ, we can calculate the flow vector $\mathbf{N}$, which is the gradient with respect to its tensorial argument. This is convenient to remove one term in the linearization operation. When writing the back stresses $\mathbf{B}$, it is assumed that these are Kirchhoff back stresses.

The strong form of the governing equations is first shown. The system consists of equilibrium equations, essential and natural boundary conditions and the constitutive laws for both stress, plastic rate and rate of back stresses. In addition, there is the loading/unloading condition (also known as the complementarity condition [12]), a switch between purely hyperelastic and hyperelastic/plastic behavior. After regularization of the governing equations, we introduce a smooth version of the

DOI: 10.1002/nme

complementarity condition and perform a semi-discretization of the partial differential equation (PDE) part of the equations followed by a backward time-stepping method.

For isotropic elasticity and symmetric flow vector, the problem can be written as:
Find $\mathbf{u}(X,t)$, $\mathbf{U}(X,t)$, $\dot{\gamma}(\mathbf{X},t)$ and $\mathbf{B}(X,t)$ such that

$$\nabla_0 \cdot \mathbf{P}^{\mathrm{T}} + \mathbf{B}_0 = \mathbf{0} \quad \text{in } \Omega_0 \tag{2}$$

$$\mathbf{u} = \overline{\mathbf{u}} \quad \text{on } \Gamma_{\overline{u}} \tag{3}$$

$$\mathbf{t} = \bar{\mathbf{t}} \quad \text{on } \Gamma_{\bar{t}} \tag{4}$$

$$\mathbf{T} = \mathbf{T}(\mathbf{F}\mathbf{U}^{-1}) \tag{5}$$

$$\mathbf{P} = \pi \mathbf{F}^{-\mathrm{T}} + \mathbf{T}\mathbf{F}^{-\mathrm{T}} \tag{6}$$

$$\mathbf{N} = \mathbf{N}(\mathbf{T}, \mathbf{B}, \gamma) \tag{7}$$

$$\dot{\mathbf{U}} = \underset{\dot{\mathbf{U}}_t}{\arg}[(\mathbf{F}\mathbf{U}^{-1}\dot{\mathbf{U}}_t\mathbf{F}^{-1})_{\text{symm.}} - \dot{\gamma}\mathbf{N} = \mathbf{0}] \tag{8}$$

$$\overset{\circ}{\mathbf{B}} = \dot{\gamma}\mathbf{g}(\mathbf{T}, \mathbf{B}, \gamma) \tag{9}$$

$$\phi(\mathbf{T}, \mathbf{B}, \gamma) \leqslant 0 \tag{10}$$

$$\dot{\gamma} \geqslant 0 \tag{11}$$

$$\phi(\mathbf{T}, \mathbf{B}, \gamma)\dot{\gamma} = 0 \tag{12}$$

where $(\bullet)_{\text{symm.}}$ indicates the symmetric part of $\bullet$.

The system consists of a second-order PDE (2) with boundary conditions (3)–(4), three algebraic equations (5)–(7), two first-order ODEs (8)–(9) and the complementarity condition (10)–(12). The latter can be replaced by a first-order non-smooth ODE

$$c_d\dot{\gamma} - [c_d\dot{\gamma} + \phi(\mathbf{T}, \mathbf{B}, \gamma)]_+ = 0 \tag{13}$$

where $[x]_+ = \max(0, x)$ for $x \in \mathbb{R}$ and $c_d \in \mathbb{R}^+$ is a dimensional parameter ensuring dimensional consistency. In (8) $\dot{\mathbf{U}}_t$ is unknown.

Frequently, authors fail to recognize the intricate form of the flow law (8) and provide ill-explained explicit approximations to it. However, authoritative works in the subject clearly advertise this fact (see, e.g. Equation 4.9.27 in [3] and the derivations in [27]). Note that a closed-form solution of (8) exists and is used here, perhaps for the first time. Using Voigt notation (identified by a subscript v) we can write Equation (8) as

$$\dot{\mathbf{U}}_v = \dot{\gamma}\mathbf{\Phi}^{-1}\mathbf{N}_v \tag{14}$$

with $\mathbf{\Phi}^{-1}$ being calculated by Mathematica [28] with the AceGen add-on.

In Equation (9) we use the Lie derivative with respect to the elastic velocity gradient. As stated by Johansson *et al.* [20] the material derivative is not objective, the fact that being overlooked by many authors, even in recent papers.

The constitutive pressure, $\widetilde{\pi}$, is completely defined given J, by means of a constitutive equation [29]. In contrast, the equilibrium pressure, π, which is used in the equilibrium system, is

DOI: 10.1002/nme

obtained indirectly from $\widetilde{\pi}$ by a inhomogeneous Helmholtz equation which is added to the global system. The constitutive pressure, $\widetilde{\pi}$, is obtained using a convex bulk strain energy density and reads

$$\widetilde{\pi} \equiv g(J) = \kappa[J^2 - J + \ln(J)] \tag{15}$$

and the equilibrium pressure, π, is obtained from the solution of the following inhomogeneous Helmholtz equation [30]:

$$\pi - c_0 \nabla_0^2 \pi - \widetilde{\pi} = 0 \tag{16}$$

where ∇_0^2 is the Laplace operator with respect to the material coordinates. The parameter c_0 controls the non-locality of the pressure field. After imposing a zero flux in the boundaries, $\nabla_0 \pi \cdot \mathbf{N}_0 = 0$ for $\mathbf{X} \in \Gamma_0$ (this was introduced by Lasry and Belytschko [31]) we can write a weak form of (16)

$$\int_{\Omega_0} [\pi^{\Delta}(\pi - \widetilde{\pi}) + c_0 \nabla_0 \pi^{\Delta} \cdot \nabla_0 \pi] \, \mathrm{d}V_0 = 0 \tag{17}$$

for all admissible variations $\pi^{\Delta} \in [\mathscr{H}^1(\Omega_0)^1]$ with $[\mathscr{H}^1(\Omega_0)^{n_{\mathrm{sd}}}]$ denoting the Sobolev space of square-integrable functions with weak derivatives up to order one with range in $\mathbb{R}^{n_{\mathrm{sd}}}$. The stabilizing effect of (17) in the solution is illustrated in the diagram of Figure 2. In this diagram, we show the effect of $\sqrt{c_0}$ and the distribution of $\widetilde{\pi}$ in the response π. We can observe that c_0 has a strong effect in the width and height of the equilibrium pressure and that spikes in pressure are filtered.

2.2. *Smoothing of the complementarity condition*

Owing to the presence of the plus function in (13), the constitutive system is non-smooth. Although specific solvers have been developed to solve this type of problems, (e.g. [32]) smoothing methods have also been very successful (the paper by Areias and Rabczuk [33] shows an example). By using the Chen and Mangasarian [13] smoothing method, we can use a smooth root finder. The 'plus' function $[x]_+$ is replaced by the smooth ramp function $S(x)$: $[x]_+ \cong S(x)$.

This function is given by

$$S(x) = x + \frac{1}{\beta} \ln(1 + \mathrm{e}^{-\beta x}) \tag{18}$$

where β is a parameter controlling the accuracy of reproduction of the original function. The parameter β is obtained as a fraction of the initial yield stress σ_{y0} as

$$\beta = \frac{0.693147}{\mathrm{tol}\sigma_{y0}} \tag{19}$$

where tol is a new constitutive property. This value of β is obtained by solving:

$$\min_x \left\{ x^2 + \left[\frac{\ln(1 - \mathrm{e}^{-\beta x})}{\beta} \right]^2 \right\} \tag{20}$$

DOI: 10.1002/nme

NME 2686

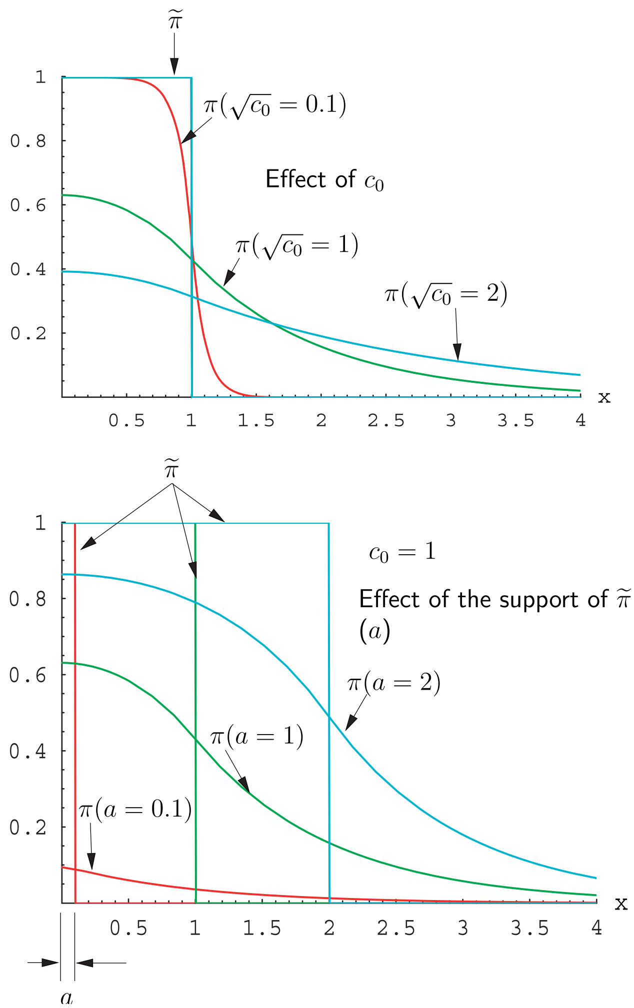


Figure 2. Effect of c_0 and the spatial distribution of $\widetilde{\pi}$. We use the conditions $\pi'(x=0)=0$ and $\lim_{x\to\infty}\pi'=0$. Smearing of $\widetilde{\pi}$ occurs and this allows an artificial support for the pressure. The equation $\pi - c_0\pi'' - \widetilde{\pi} = 0$ is solved in $x \in]0, +\infty[$ and $\widetilde{\pi} = 1$ for $x \leqslant a$.

Color Online, B&W in Print

After performing the necessary substitutions, the graph of $c_d\dot{\gamma} - S[c_d\dot{\gamma} + \phi] = 0$ for normalized ϕ is shown in Figure 3 for several values of tol. We can see that tol corresponds to the maximum difference in the yield function near the tip of the complementarity condition, i.e. $\dot{\gamma}\phi = 0$. It converges to the exact result as tol is decreased, since

$$\lim_{\beta\to+\infty} [S(x) - [x]_+] = 0 \quad \forall x \in \mathbb{R} \tag{21}$$

can be proved.

DOI: 10.1002/nme

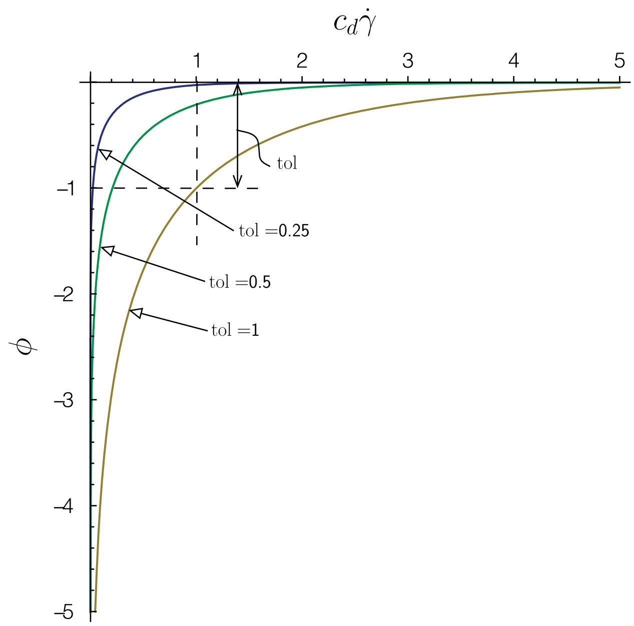


Figure 3. The smoothed complementarity condition for tol $\in\{0.25, 0.5, 1\}$.

Color Online, B&W in Print

2.3. *Time integration, constitutive solution and linearization*

Time integration provides constitutive quantities at time step t given all quantities in $t_0 < t$ and $\mathbf{F}$ at time step t; $\mathbf{F}$ is provided by the equilibrium iteration. To calculate the remaining constitutive quantities, we introduce a general integration procedure, prone to exact linearization of most particular elasto-plastic laws. Specific laws will be casted into this framework.

To simplify the notation we introduce the following auxiliary quantities:

$$\mathsf{Z} = (\mathbf{U}^{-1})_v \tag{22}$$

$$v\mathsf{Z}_0 = (\mathbf{U}_0^{-1})_v \tag{23}$$

with the subscript v being the Voigt form of the corresponding argument, which must be a symmetric tensor. In general, using the Voigt matrix $\mathbf{v}$,

$$\mathbf{v} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \tag{24}$$

we obtain $\mathsf{Z}_{v_{ij}} = [\mathbf{U}^{-1}]_{ij}$, $i, j = 1, 2, 3$; the matrix form of Z being denoted as $\mathbf{Z}$. The notation $(\bullet)' = \partial\bullet/\partial\Delta\gamma$ is also used for conciseness.

To clarify the notation, blackboard style is used for the Voigt form of full minor-symmetric fourth-order tensors, calligraphic style is used for the Voigt form of fourth-order tensors with

DOI: 10.1002/nme

minor symmetry in the first two indices only; sans–serif notation is reserved for the Voigt form of second-order tensors.

Using this notation, the smoothed complementarity condition is integrated implicitly as

$$\mu\Delta\gamma - S\{\mu\Delta\gamma + \phi[\mathsf{T}(\mathsf{Z}), \mathsf{B}, \Delta\gamma]\} = 0 \tag{25}$$

and the integrated flow law is given by

$$\underbrace{[(\mathbf{F}\mathbf{Z}\mathbf{Z}_0^{-1}\mathbf{F}^{-1})_{\text{symm.}}]_v}_{\mathbb{P}\mathsf{Z}} = \{\mathsf{I} - \Delta\gamma\mathsf{N}[\mathsf{T}(\mathsf{Z}), \mathsf{B}, \Delta\gamma]\} \tag{26}$$

with $\mathsf{I} = \{1\ 1\ 1\ 0\ 0\ 0\}^{\mathrm{T}}$. The fourth-order symmetric tensor $\mathbb{P}$ is calculated using Mathematica 6.0 [28] as the derivative of the left-hand side with respect to Z.

Back stresses are integrated using the approximation $\Delta t\mathbf{L} \cong (\mathbf{F}\mathbf{F}_0^{-1} - \mathbf{I})$ as

$$\mathsf{B} - \mathsf{B}_0 = \mathbb{B}\mathsf{B} + \Delta\gamma\mathsf{g}[\mathsf{T}(\mathsf{Z}), \mathsf{B}, \Delta\gamma] \tag{27}$$

with $\mathbb{B}$ being also calculated as a derivative of the corresponding Truesdell rate with respect to B.

Linearization of Equations (25)–(27) is required both for the constitutive solution and the application of the equilibrium Newton–Raphson method. When solving the constitutive system, unknowns are $\Delta\gamma$, Z and B. For the equilibrium Newton–Raphson method, the derivative of T with respect to $\mathbf{F}$ is required:

$$\mathscr{E} = \frac{\partial \mathsf{T}}{\partial \mathbf{F}} \tag{28}$$

The non-linear constitutive system consists of finding the roots of the following equations:

$$R_1 = \mu\Delta\gamma - S\{\mu\Delta\gamma + \phi[\mathsf{T}(\mathsf{Z}), \mathsf{B}, \Delta\gamma]\} \tag{29}$$

$$\mathsf{R}_2 = \mathbb{P}\mathsf{Z} - \mathsf{I} + \Delta\gamma\mathsf{N}[\mathsf{T}(\mathsf{Z}), \mathsf{B}, \Delta\gamma] \tag{30}$$

$$\mathsf{R}_3 = \mathsf{B} - \mathsf{B}_0 - \mathbb{B}\mathsf{B} - \Delta\gamma\mathsf{g}[\mathsf{T}(\mathsf{Z}), \mathsf{B}, \Delta\gamma] \tag{31}$$

with zero left-hand sides.

The Jacobian of this system is concisely written as

$$\mathbf{J} = \begin{bmatrix} J_{11} & \mathsf{J}_{12}^{\mathrm{T}} & \mathsf{J}_{13}^{\mathrm{T}} \\ \mathsf{J}_{21} & \mathbb{J}_{22} & \mathbb{J}_{23} \\ \mathsf{J}_{31} & \mathbb{J}_{32} & \mathbb{J}_{33} \end{bmatrix} \tag{32}$$

where

$$J_{11} = \mu - S'(\mu + \phi') \tag{33}$$

$$\mathsf{J}_{12} = -S'\mathsf{N}_2^{\mathrm{T}}\mathbb{M} \tag{34}$$

$$\mathsf{J}_{13} = S'\mathsf{N}_2 \tag{35}$$

$$\mathsf{J}_{21} = \mathsf{N} \tag{36}$$

DOI: 10.1002/nme

$$\mathbb{J}_{22} = \mathbb{P} + \Delta\gamma \mathbb{U}\mathbb{M} \tag{37}$$

$$\mathbb{J}_{23} = -\Delta\gamma \mathbb{U} \tag{38}$$

$$\mathsf{J}_{31} = -\mathsf{g} - \Delta\gamma \mathsf{g}' \tag{39}$$

$$\mathbb{J}_{32} = -\Delta\gamma \mathbb{G}\mathbb{M} \tag{40}$$

$$\mathbb{J}_{33} = \mathbb{I} - \mathbb{B} - \Delta\gamma \mathbb{H} \tag{41}$$

and

$$\mathsf{N}_2 = \frac{\partial \phi}{\partial \mathsf{T}} \tag{42}$$

$$\mathbb{M} = \frac{\partial \mathsf{T}}{\partial \mathsf{Z}} = \left(\frac{\partial \mathbf{T}}{\partial \mathbf{F}^{\mathrm{e^T}}} \mathbf{F} \right)_v \tag{43}$$

$$\mathbb{U} = \frac{\partial \mathsf{N}}{\partial \mathsf{T}} \tag{44}$$

$$\mathbb{G} = \frac{\partial \mathsf{g}}{\partial \mathsf{T}} \tag{45}$$

$$\mathbb{H} = \frac{\partial \mathsf{g}}{\partial \mathsf{B}} \tag{46}$$

Note that $\mathsf{N} \neq \mathsf{N}_2$ since, in the latter, the Voigt form of $\mathbf{T}$ is used. We can write

$$\mathsf{N}_2 = \mathbb{I}_6 \mathsf{N} \tag{47}$$

where $\mathbb{I}_6$ is obtained from the 6×6 identity matrix by replacing 1 by 2 in the $(4,4)$, $(5,5)$ and $(6,6)$ components.

It can be observed that the deformation gradient is present in $\mathbb{P}$, $\mathbb{B}$ and T; the infinitesimal variation of T with $\mathbf{F}$ is simply given as a differential form as

$$\mathrm{d}\mathsf{T} = \mathscr{E}_{n+1} : \mathrm{d}\mathbf{F} = \mathbb{M}\,\mathrm{d}\mathsf{Z} + \mathscr{N} : \mathrm{d}\mathbf{F} \tag{48}$$

where

$$\mathscr{N} = \frac{\mathrm{d}\mathsf{T}}{\mathrm{d}\mathbf{F}^{\mathrm{e}}} \mathbf{Z} \tag{49}$$

The remaining non-trivial derivatives are

$$\mathscr{D} = \left[\frac{\mathrm{d}}{\mathrm{d}\mathbf{F}} (\mathbf{F}\mathbf{Z}\mathbf{Z}_0^{-1}\mathbf{F}^{-1})_{\mathrm{symm.}} \right]_v \tag{50}$$

$$\mathscr{B} = \frac{\mathrm{d}}{\mathrm{d}\mathbf{F}} (\mathbb{B}\mathsf{B}) \tag{51}$$

which we calculate using Mathematica 6.0 [28] with the AceGen add-on.

DOI: 10.1002/nme

Linearization follows directly from the solution of the modified linear system (whose coefficient matrix is still $\mathbf{J}$)

$$\mathbf{J}\begin{Bmatrix} \mathrm{d}\gamma \\ \mathrm{d}\mathbf{Z} \\ \mathrm{d}\mathbf{B} \end{Bmatrix} = -\begin{Bmatrix} \mathbf{J}_{1F} : \mathrm{d}\mathbf{F} \\ \mathscr{J}_{2F} : \mathrm{d}\mathbf{F} \\ \mathscr{J}_{3F} : \mathrm{d}\mathbf{F} \end{Bmatrix} \tag{52}$$

where:

$$\mathbf{J}_{1F} = -S'\mathbf{N}_2^{\mathrm{T}}\mathscr{N} \tag{53}$$

$$\mathscr{J}_{2F} = \mathscr{D} + \Delta\gamma\mathbb{U}\mathscr{N} \tag{54}$$

$$\mathscr{J}_{3F} = -\mathscr{B} - \Delta\gamma\mathbb{G}\mathscr{N} \tag{55}$$

Of course, only the second equation in (52) is relevant, which can be re-written as

$$\mathrm{d}\mathbf{Z} = \underbrace{-[(\mathbf{J}^{-1})_{21} \otimes \mathbf{J}_{1F} + (\mathbf{J}^{-1})_{22}\mathscr{J}_{2F} + (\mathbf{J}^{-1})_{23}\mathscr{J}_{3F}]}_{\mathscr{T}_{ZF}} : \mathrm{d}\mathbf{F} \tag{56}$$

resulting in the sum of an elastic and an elasto-plastic tangent:

$$\mathscr{E} = \mathscr{N} + \mathbb{M}\mathscr{T}_{ZF} \tag{57}$$

Accuracy of this approach can be assessed for the plane-stress case (imposed by zeroing the T_{33} stress by modifying F_{33}). Iso-error maps for $\mathrm{tol} = 1\times 10^{-2}$ and plane stress are shown in Figure 4. Note that these are finite strain error maps and, in addition to the constitutive integration error, kinematical approximations (such as the velocity gradient $\mathbf{L}$) also contribute for the error.

2.4. *Specific constitutive equations*

Prototype models are used for the testing purposes. Both von-Mises and Hill yield criteria are used, and both isotropic and combined isotropic/kinematic hardening laws are inspected. The yield function $\phi(\mathsf{T}, \mathsf{B}, \Delta\gamma)$ is decomposed as (Table I) **Q3**

$$\phi(\mathsf{T}, \mathsf{B}, \Delta\gamma) = y(\mathsf{T} - \mathsf{B}) - \sigma_y(\Delta\gamma) \tag{58}$$

where y is called the equivalent stress and σ_y is the hardening function. More complex models (see [37]) do not require large modifications.

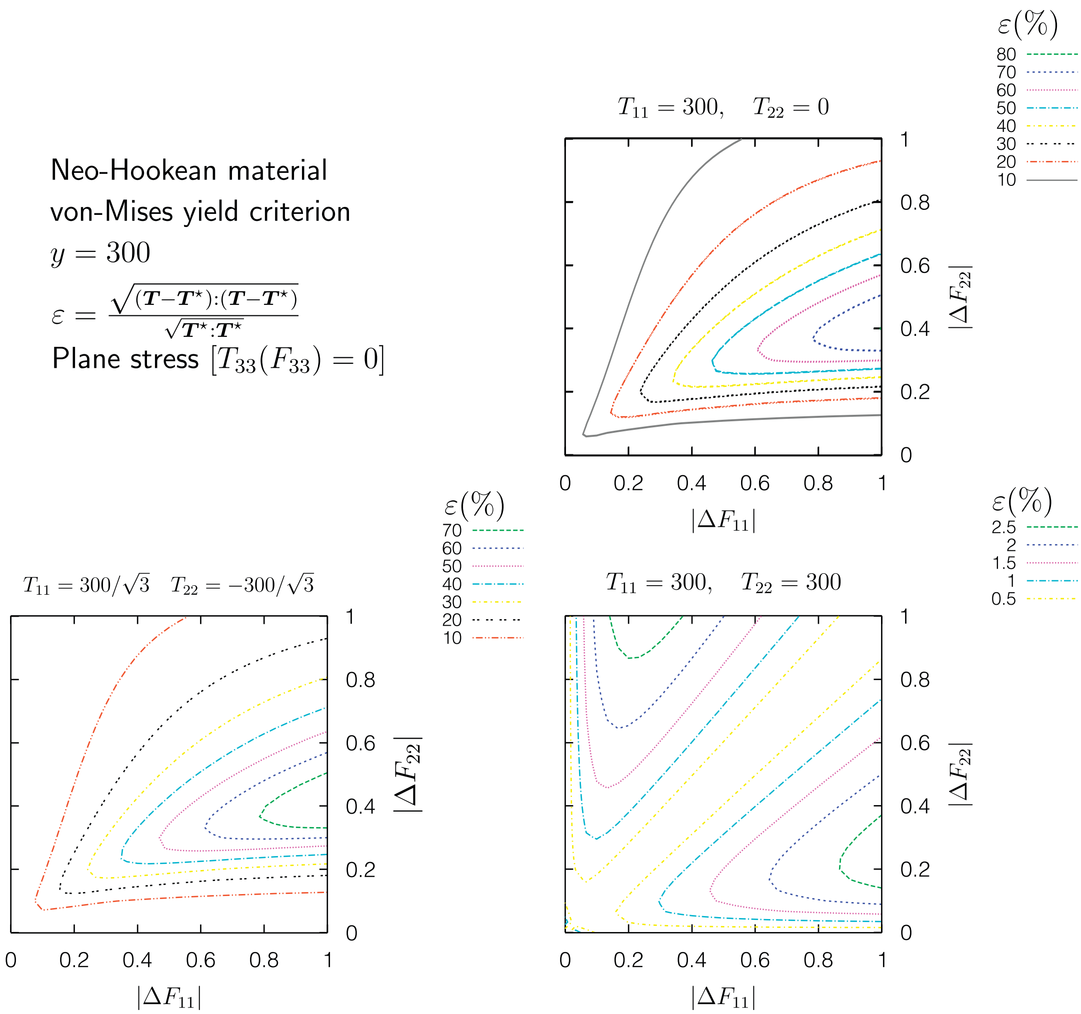
2.5. *Weak form—equilibrium and first variation*

The Galerkin method is used to obtain a weak form of equilibrium. This was performed before for the equilibrium pressure in (17). The test functions for the equilibrium equation are now vector fields, which we denote by $\mathbf{x}^{\Delta} \in [\mathscr{H}^1(\Omega_0)^3]$. After writing the weak form, a symmetric discretization is employed to obtain an algebraic system.

To simplify the notation, we introduce the notation $\mathbf{T} = \mathbf{T}_t - g(J)\mathbf{I}$, resulting in:

$$W^{\Delta} = \int_{\Omega_0} \{\mathbf{T} : \nabla\mathbf{x}^{\Delta} + c_0\nabla_0\pi \cdot \nabla_0\pi^{\Delta} - g(J)\pi^{\Delta} + \pi\mathbf{I} : \nabla\mathbf{x}^{\Delta} + \pi\pi^{\Delta} + \mathbf{B}_0 \cdot \mathbf{x}^{\Delta}\} = 0 \tag{59}$$

DOI: 10.1002/nme



Color Online, B&W in Print

Figure 4. Finite strain iso-error maps (tol $= 1\times10^{-2}$) for plane stress.

Table I. Back-stress laws in Voigt form.

Back-stress law	$g(\mathsf{T}, \mathsf{B}, \dot{\gamma})$
Isotropic	$\mathbf{0}$
Tsakmakis and Willuweit [34] version of Armstrong–Frederick [35]	$(c\mathsf{N} - b\mathsf{B})$
Burlet–Cailletaud [36]	$[c\mathsf{N} - b\frac{1}{\mathbf{N}:\mathbf{N}}(\mathbf{N}\otimes\mathbf{N}):\mathbf{B}]$

A Newton–Raphson-based solver is used, and therefore exact derivatives of all quantities in (59) are required. The time derivative of $\mathbf{T}$ is given by

$$\dot{\mathbf{T}} = \nabla\dot{\mathbf{x}}\mathbf{T} + \mathbf{T}\nabla\dot{\mathbf{x}}^{\mathrm{T}} + \mathbb{D} : \nabla\dot{\mathbf{x}} \tag{60}$$

where $\mathbb{D}$ can be calculated from $\mathscr{E}$ as

$$\mathbb{D}_{(qn)(kp)} = \underbrace{\mathscr{E}_{(qn)kj} F_{pj}}_{\mathbb{F}_{qnkp}} - T_{(qp)}\delta_{(nk)} - T_{(pn)}\delta_{(kq)} \tag{61}$$

where the parenthesis in the index notation point out the Voigt grouping.

This useful relation (61) between moduli is obtained by re-deriving the work of Truesdell and Noll ([38, pp. 131–133]). Despite being more direct than the often used push-back/linearize/push-forward procedure, the authors could not find this relation in the bibliography. The reader can note that no deviatoric projection is required (as in, e.g. [2]).

Time derivative of $\mathbf{T}:\nabla\mathbf{x}^{\Delta}$ is given by the well-known (cf. [39]) relation:

$$(\mathbf{T}:\dot{\nabla}\mathbf{x}^{\Delta}) = \nabla\mathbf{x}^{\Delta}:\mathbb{D}:\nabla\dot{\mathbf{x}} + (\nabla\dot{\mathbf{x}}^{\mathrm{T}}\nabla\mathbf{x}^{\Delta} + \nabla\dot{\mathbf{x}}^{\Delta}):\mathbf{T} \tag{62}$$

The linearized form of the equations is easily obtained with this relation, and the final partitioned result is shown

$$\dot{W}^{\Delta} = \int_{\Omega_0} \{\nabla\mathbf{x}^{\Delta}:\mathbb{D}:\nabla\dot{\mathbf{x}} + \mathbf{T}:(\nabla\dot{\mathbf{x}}^{\mathrm{T}}\nabla\mathbf{x}^{\Delta}) - \pi(\nabla\dot{\mathbf{x}}^{\mathrm{T}}:\nabla\mathbf{x}^{\Delta}) \tag{63}$$

$$-\pi^{\Delta} J g'(J)\mathbf{I}:\nabla\dot{\mathbf{x}} + \dot{\pi}\mathbf{I}:\nabla\mathbf{x}^{\Delta} + c_0\nabla_0\pi^{\Delta}\cdot\nabla_0\dot{\pi} + \pi^{\Delta}\dot{\pi}\}\,\mathrm{d}\Omega_0 \tag{64}$$

where, for the formulation in use, $\nabla\dot{\mathbf{x}}^{\Delta} = \mathbf{0}$.

2.6. *Discretization*

The discretization is performed for the nodal unknowns $\{\mathbf{x}_K, \pi_K\}$ as

$$\mathbf{x}_h = \sum_{K=1}^{4} N_K \mathbf{x}_K \tag{65}$$

$$\pi_h = \sum_{K=1}^{4} N_K \pi_K \tag{66}$$

where K represents a local node number.

A symmetric discretization is employed for the test functions:

$$\mathbf{x}_h^{\Delta} = \sum_{K=1}^{4} N_K \mathbf{x}_K^{\Delta} \tag{67}$$

$$\pi_h^{\Delta} = \sum_{K=1}^{4} N_K \pi_K^{\Delta} \tag{68}$$

The gradients are calculated using the shape function derivative tables $\mathbf{N}_n$ and $\mathbf{N}_{n0}$ from

$$\nabla\mathbf{x}_h^{\mathrm{T}} = \mathbf{N}_n\mathbf{x}_n \tag{69}$$

$$\nabla_0\pi_h = \mathbf{N}_{n0}\boldsymbol{\pi}_n \tag{70}$$

$$\nabla\mathbf{v}_h^{\mathrm{T}} = \mathbf{N}_n\mathbf{v}_n \tag{71}$$

$$\nabla_0\xi_h = \mathbf{N}_{n0}\boldsymbol{\xi}_n \tag{72}$$

DOI: 10.1002/nme

where

$$\mathbf{N}_n = \begin{bmatrix} \dfrac{\mathrm{d}N_1}{\mathrm{d}x_1} & \dfrac{\mathrm{d}N_2}{\mathrm{d}x_1} & \dfrac{\mathrm{d}N_3}{\mathrm{d}x_1} & \dfrac{\mathrm{d}N_4}{\mathrm{d}x_1} \\ \dfrac{\mathrm{d}N_1}{\mathrm{d}x_2} & \dfrac{\mathrm{d}N_2}{\mathrm{d}x_2} & \dfrac{\mathrm{d}N_3}{\mathrm{d}x_2} & \dfrac{\mathrm{d}N_4}{\mathrm{d}x_2} \\ \dfrac{\mathrm{d}N_1}{\mathrm{d}x_3} & \dfrac{\mathrm{d}N_2}{\mathrm{d}x_3} & \dfrac{\mathrm{d}N_3}{\mathrm{d}x_3} & \dfrac{\mathrm{d}N_4}{\mathrm{d}x_3} \end{bmatrix} \tag{73}$$

and

$$\mathbf{N}_{n0} = \begin{bmatrix} \dfrac{\mathrm{d}N_1}{\mathrm{d}X_1} & \dfrac{\mathrm{d}N_2}{\mathrm{d}X_1} & \dfrac{\mathrm{d}N_3}{\mathrm{d}X_1} & \dfrac{\mathrm{d}N_4}{\mathrm{d}X_1} \\ \dfrac{\mathrm{d}N_1}{\mathrm{d}X_2} & \dfrac{\mathrm{d}N_2}{\mathrm{d}X_2} & \dfrac{\mathrm{d}N_3}{\mathrm{d}X_2} & \dfrac{\mathrm{d}N_4}{\mathrm{d}X_2} \\ \dfrac{\mathrm{d}N_1}{\mathrm{d}X_3} & \dfrac{\mathrm{d}N_2}{\mathrm{d}X_3} & \dfrac{\mathrm{d}N_3}{\mathrm{d}X_3} & \dfrac{\mathrm{d}N_4}{\mathrm{d}X_3} \end{bmatrix} \tag{74}$$

Unknowns and nodal test parameters are grouped as:

$$\mathbf{x}_n = \begin{bmatrix} \mathbf{x}_1^{\mathrm{T}} \\ \mathbf{x}_2^{\mathrm{T}} \\ \mathbf{x}_3^{\mathrm{T}} \\ \mathbf{x}_4^{\mathrm{T}} \end{bmatrix} \tag{75}$$

$$\boldsymbol{\pi}_n = \begin{Bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{Bmatrix} \tag{76}$$

$$\mathbf{x}_n^{\Delta} = \begin{bmatrix} \mathbf{x}_1^{\Delta\mathrm{T}} \\ \mathbf{x}_2^{\Delta\mathrm{T}} \\ \mathbf{x}_3^{\Delta\mathrm{T}} \\ \mathbf{x}_4^{\Delta\mathrm{T}} \end{bmatrix} \tag{77}$$

$$\boldsymbol{\pi}_n^{\Delta} = \begin{Bmatrix} \pi_1^{\Delta} \\ \pi_2^{\Delta} \\ \pi_3^{\Delta} \\ \pi_4^{\Delta} \end{Bmatrix} \tag{78}$$

DOI: 10.1002/nme

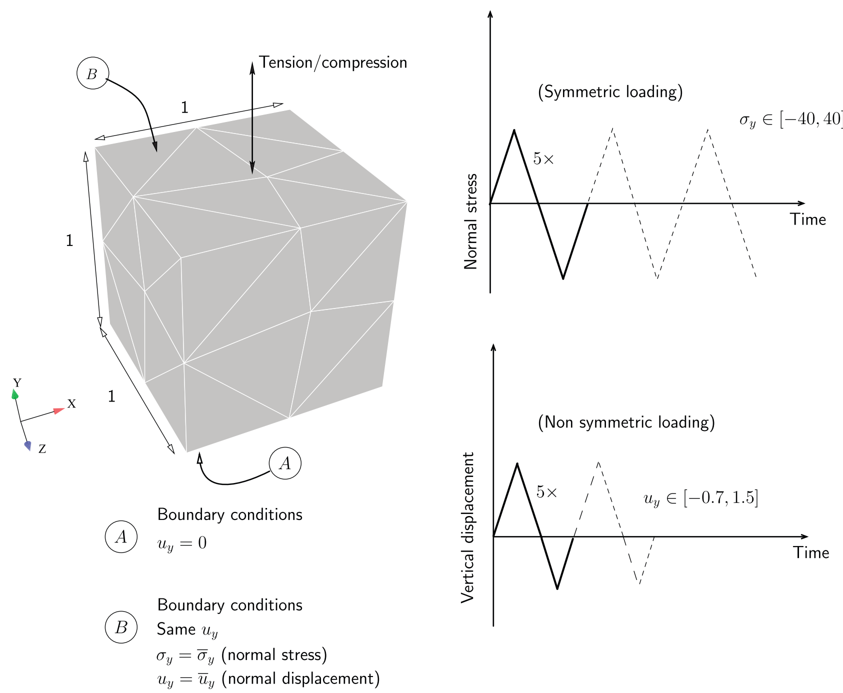


Figure 5. Problem description of a 1D test under strain and stress control (normal and shear stress). A comparison is made with results from Dettmer and Reese [19]. The von-Mises criterion is used with fixed $y = 0.35$, $c = 100$ and $b \in \{0, 1.35, 2.7\}$. Elastic properties for the neo–Hookean material are $\mu = 80$ and $\kappa = 320$. The non-local parameter is $c_0 = 0.2$. Consistent units are used.

Color Online, B&W in Print

The deformation gradient is calculated as

$$\mathbf{F}_h = \mathbf{x}_n^{\mathrm{T}} \mathbf{N}_{n0}^{\mathrm{T}} \tag{79}$$

from which the table of the updated shape function derivatives is obtained:

$$\mathbf{N}_n = \mathbf{F}_h^{-\mathrm{T}} \mathbf{N}_{n0} \tag{80}$$

A specific quadrature scheme is used. Terms involving the deviatoric Kirchhoff stress $\mathbf{T}$ use one Gauss point and the remaining terms use four Gauss points to correctly sample the pressure terms.

The element forces and stiffness matrices (note that N_{jK}^n are uniform for 4-node tetrahedra) are given by

$$f_{Ki}^x = \frac{V_0}{4} \left\{ 4 T_{ij} N_{jK}^n + \sum_{\mathsf{l}=1}^{4} [N_K(\xi_\mathsf{l}) B_{0i} + N_{iK}^n N_L(\xi_\mathsf{l}) \pi_L] \right\} \tag{81}$$

$$f_K^\pi = c_0 V_0 N_{iK}^0 N_{iL}^0 \pi_L + \frac{V_0}{4} \sum_{\mathsf{l}=1}^{4} (N_K(\xi_\mathsf{l}) \{ N_L(\xi_\mathsf{l}) \pi_L - g[J(\xi_\mathsf{l})] \}) \tag{82}$$

DOI: 10.1002/nme

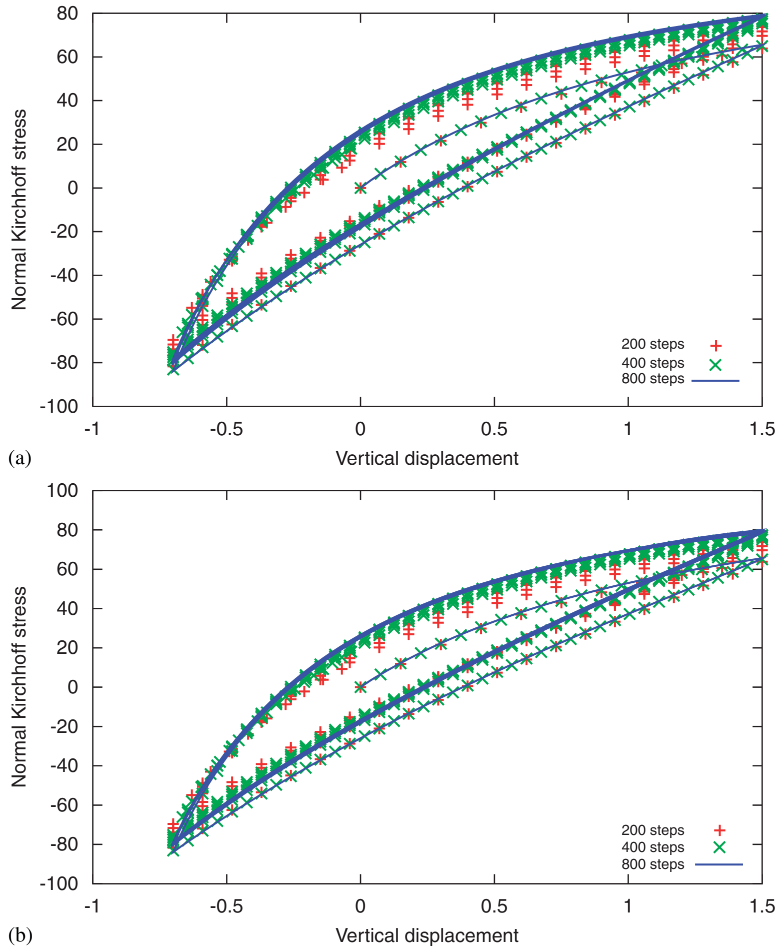


Figure 6. Effect of the number of steps for a five cycle loading with $\mathrm{tol} = 1 \times 10^{-4}$ and $b = 1.35$ (strain control): (a) Armstrong-Frederick and (b) Burlet-Cailletaud.

Color Online, B&W in Print

$$K^{x^\Delta x}_{KiLm} = \frac{V_0}{4}\left[4 N^n_{jK} N^n_{nL} \mathscr{G}_{(ij)mn} - N^n_{mK} N^n_{iL} \sum_{l=1}^{4} N_M(\xi_l)\pi_M\right] \tag{83}$$

$$K^{x^\Delta \pi}_{KiL} = \frac{V_0 N^n_{iK}}{4} \sum_{l=1}^{4} N_L(\xi_l) \tag{84}$$

$$K^{\pi^\Delta \pi}_{KL} = c_0 V_0 N^0_{iK} N^0_{iL} + \frac{V_0}{4} \sum_{l=1}^{4} N_K(\xi_l) N_L(\xi_l) \tag{85}$$

1

$$K^{\pi^\Delta x}_{KLi} = -\frac{V_0 N^n_{iL}}{4} \sum_{l=1}^{4} \{N_K(\xi_l) J(\xi_l) g'[J(\xi_l)]\} \tag{86}$$

DOI: 10.1002/nme

NME 2686

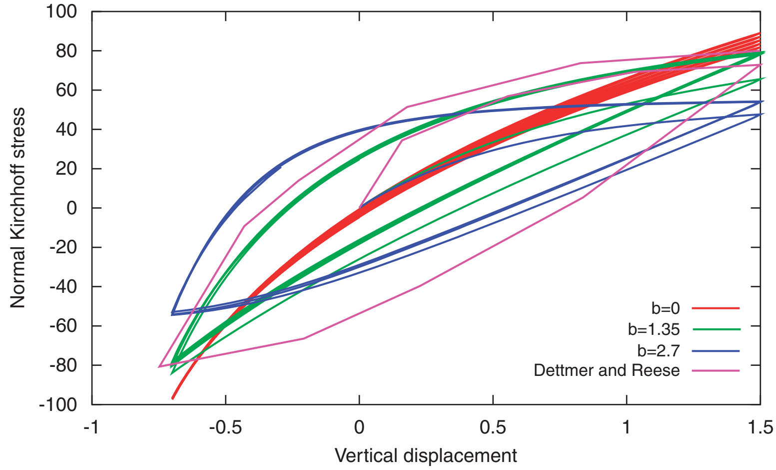


Figure 7. Strain control: Kirchhoff stress (Armstrong–Frederick and Burlet–Cailletaud are very close).

Color Online, B&W in Print

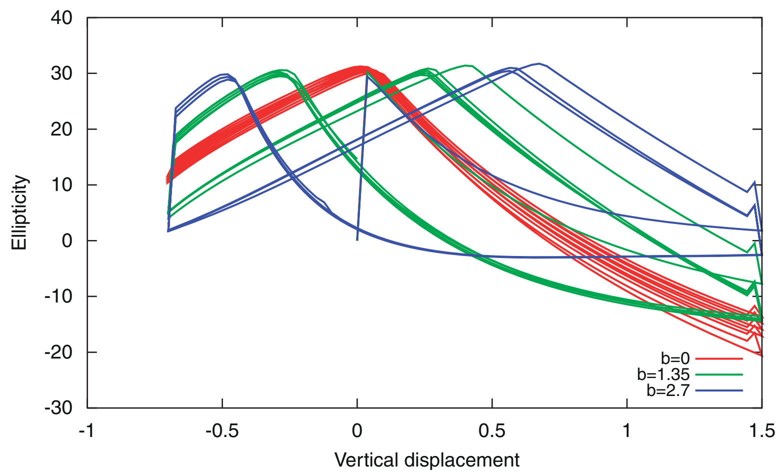


Figure 8. Strain control: ellipticity indicator (Armstrong–Frederick and Burlet–Cailletaud are very close).

Color Online, B&W in Print

where V_0 is the tetrahedron volume

$$V_0 = \frac{1}{6}\det\begin{bmatrix} X_{11}-X_{31} & X_{21}-X_{31} & X_{41}-X_{31} \\ X_{12}-X_{32} & X_{22}-X_{32} & X_{42}-X_{32} \\ X_{13}-X_{33} & X_{23}-X_{33} & X_{43}-X_{33} \end{bmatrix} \tag{87}$$

with the notation X_{Ki}; $K=1,\ldots,4, i=1,\ldots,3$. The often costly constitutive update is performed at one Gauss point only, whereas the less expensive pressure terms are evaluated at four Gauss points (summation with index I). This decoupling is first proposed here.

DOI: 10.1002/nme

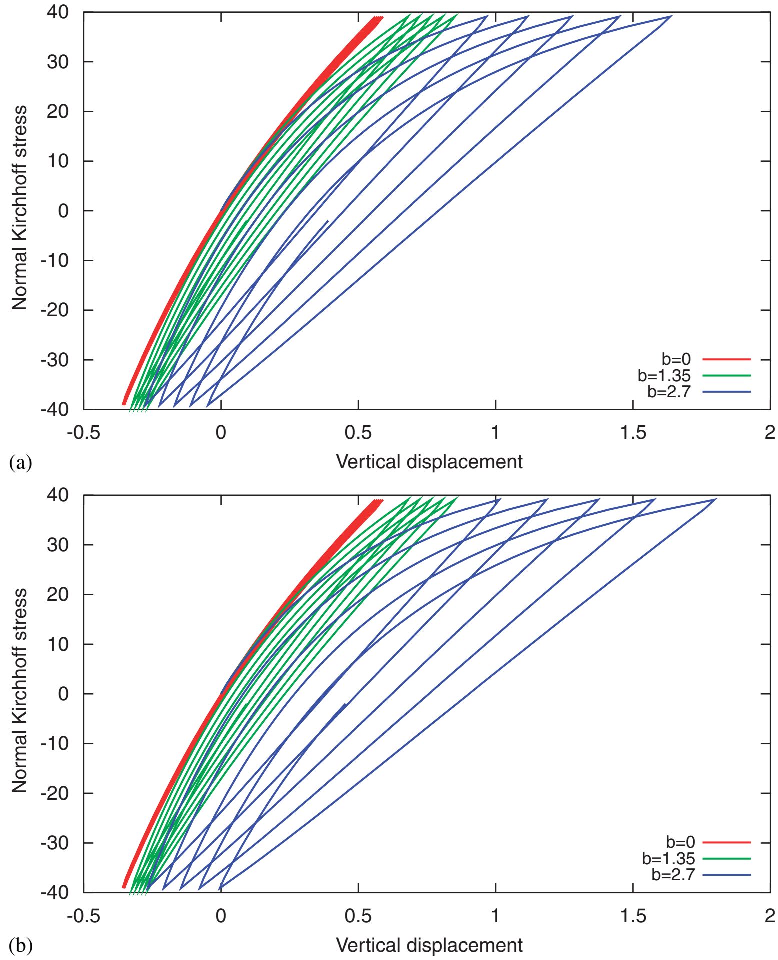


Figure 9. Stress control: Kirchhoff stress (both Armstrong–Frederick and Burlet–Cailletaud are shown): (a) Armstrong-Frederick and (b) Burlet-Cailletaud.

The reader can note that, in Equation (83), the modulus $\mathscr{G}$ was used, which is the spatial tangent modulus used for the strong ellipticity condition (see [38, Equations 45.6 and 45.14] where the notation B is used):

$$\mathscr{G}_{(ij)kl} = \mathbb{D}_{(ij)(kl)} + \delta_{(ik)} T_{(jl)} \tag{88}$$

The ellipticity indicator is obtained as follows:

$$e = \min_{\alpha_1,\alpha_2} \arg\min_{\lambda} \{\lambda \,|\, \det[n_i(\alpha_1,\alpha_2) n_k(\alpha_1,\alpha_2) \mathscr{G}_{(ij)kl} - \lambda \delta_{jl}] = 0\} \tag{89}$$

DOI: 10.1002/nme

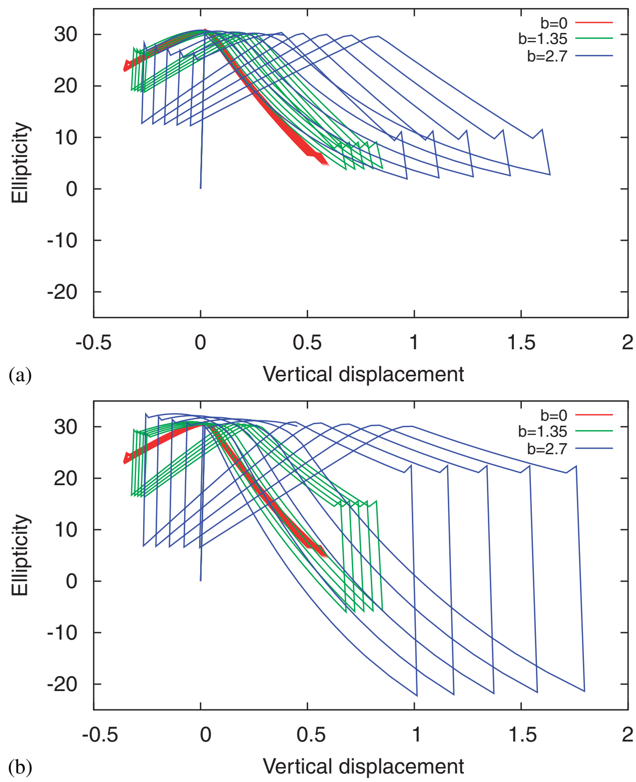


Figure 10. Stress control: ellipticity indicator (both Armstrong–Frederick and Burlet–Cailletaud are shown): (a) Armstrong-Frederick and (b) Burlet-Cailletaud.

with **n** being a unit vector, function of two angles α_1, α_2:

$$\mathbf{n} = \begin{Bmatrix} \sin\alpha_1 \cos\alpha_2 \\ \sin\alpha_1 \sin\alpha_2 \\ \cos\alpha_1 \end{Bmatrix} \tag{90}$$

Therefore, all quantities required for a constitutive *analysis* are readily available from the finite element implementation. Equation (89) involves a large number of eigenvalue evaluations and will be used in the next section.

2.7. *Armstrong–Frederick/Burlet–Cailletaud hardening*

With the purpose of also verifying the patch-test satisfaction, we use a cube of irregular elements as shown in Figure 5 where the relevant data are presented. The Armstrong–Frederick [35] and Burlet–Cailletaud [36] kinematic hardening models are tested. The relevant data are shown in Figure 5: both strain and stress control are used. For strain control, the effect of the number of time steps is shown in Figure 6. Large steps can be used, with no signs of convergence problems and with only slight result differences.

DOI: 10.1002/nme

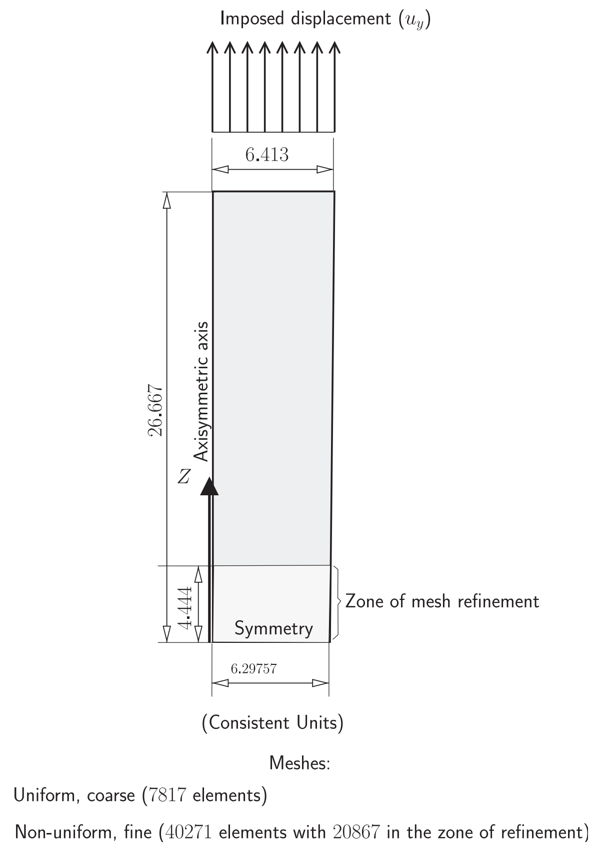


Figure 11. Geometry, boundary conditions and constitutive properties for the tension test. One-eighth of the specimen is meshed, since there are three symmetry planes.

For strain control, the evolution of the Kirchhoff stress is shown in Figure 7 and compared with the results of Dettmer and Reese [19] (their model A) . This comparison shows a difference, explained by the use of a different elastic model and kinematic part of the flow law. The ellipticity indicator (e) shows loss of ellipticity for large stretch values, as seen in Figure 8.

For stress control, results are shown in Figures 9 and 10. It is noticeable that, for the Burlet–Cailletaud model, strong ellipticity is lost in several occasions, as shown in the latter figure.

These losses are not unforeseen due to the presence of the stress tensor in the tangent modulus $\mathscr{G}$, the yield-limited tangent $\mathscr{E}$ and the shift in the yield surface origin. This means that a crack may occur when kinematic hardening is present even if the slope in the hardening law is positive.

3. TENSION TEST OF A TRUNCATED CONE: COMPARISON OF VON-MISES AND HILL CRITERIA

A truncated cone built out of ASTM A-533 steel is subject to an imposed displacement at its larger base. This geometry is used to induce necking and has been adopted in the past in finite element

DOI: 10.1002/nme

Table II. Tension test: constitutive properties (consistent units).

E	206.9
ν	0.29
$c0_{\text{small}}$	0.1
$c0_{\text{big}}$	2
$\text{tol}_{\text{small}}$	1×10^{-6}
tol_{big}	1×10^{-2}
σ_y	$0.45+(0.715-0.45)(1-\mathrm{e}^{-16.93\varepsilon_p})+0.12929\varepsilon_p$

simulations [17]. The test data were obtained by Norris *et al.* [40] who, besides having performed the experimental test, successfully used a 2D finite difference code to perform a simulation. For our test, we use the properties of their specimen $2499R$ with the hardening law fitted by Simo [17]. An extension to this validation test is made with an anisotropic variant.

In finite element simulations with the radial-return algorithm, convergence problems are known to occur after the limit point is reached (see [41, pp. 358–359], and in the past hybrid solution techniques (BFGS followed by Newton iterations) were employed to efficiently solve the problem. This indicates the inability of Newton method (where the exact derivative is used) to deal with a non-smooth problem.

The geometry, boundary conditions and material properties are summarized in Figure 11. Two yield criteria are used:

von-Mises:

$$y_{\text{VM}}=\frac{1}{\sqrt{2}}\sqrt{(T_{11}-T_{22})^2+(T_{22}-T_{33})^2+(T_{11}-T_{33})^2+3(T_{12}^2+T_{13}^2+T_{23}^2)} \tag{91}$$

and Hill (*with specific parameters*):

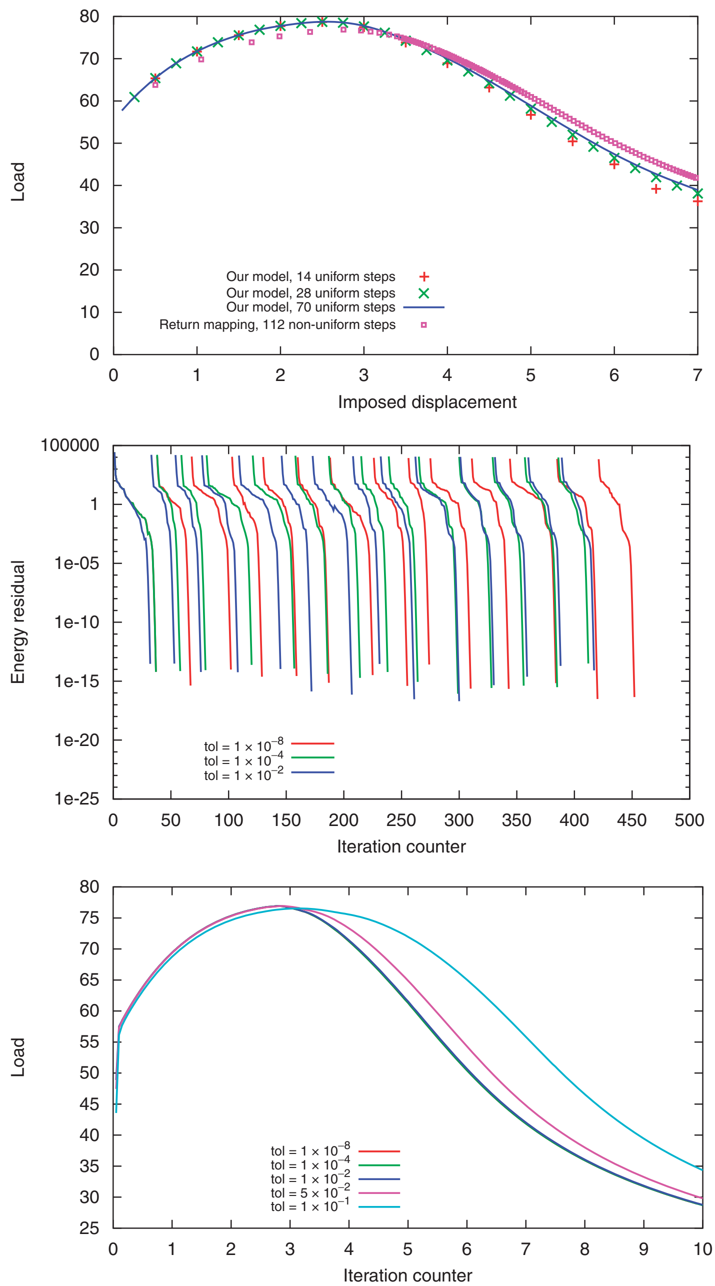
$$y_{\text{H}}=\tfrac{1}{2}\sqrt{4T_{11}^2-T_{11}T_{33}+T_{33}^2-7T_{11}T_{22}-T_{22}T_{33}+4T_{22}^2+6T_{12}^2+6T_{13}^2+6T_{23}^2} \tag{92}$$

The effects of both the tolerance tol and the parameter c_0 are inspected, as well as the mesh.(Table II) Two meshes are employed: a uniform mesh containing 7817 elements and a finer mesh with a localized refinement in the region indicated in Figure 11 containing 40 271 elements of which 20 867 are placed in the zone of mesh refinement (see Figure 11). For the purpose of stress convergence, a third mesh with 24 436 elements is also inspected. Only $\frac{1}{8}$th of the specimen geometry is meshed, since three planes of symmetry exist. The reason for this refinement is the same that led Norris *et al.* [40] to introduce points in that region during their finite difference simulation: large elements in the necking region tend to deform into high aspect ratio tetrahedra and resolution worsens.

It was also found in [40] that a teardrop region adjacent to the necking region is in the state of compression, forcing the outer annulus to withstand the tension (this is particularly acute at a certain distance from the specimen's center). We plot the longitudinal stress along the axisymmetric axis in Figure 13 for $u_y=7$. The shift to the left of the curve is apparent when the fine mesh is used. The reason for this is that the finer mesh captures more of the necking in terms of

DOI: 10.1002/nme

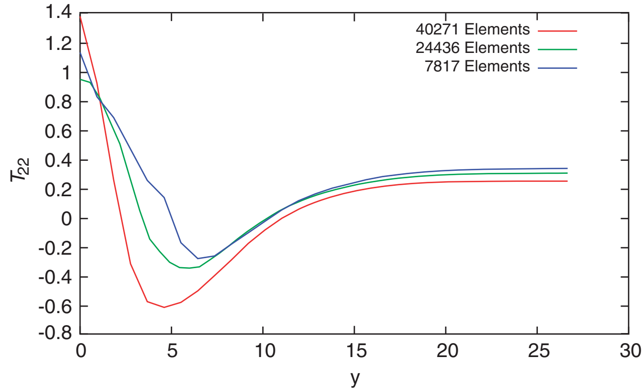
NME 2686



Color Online, B&W in Print

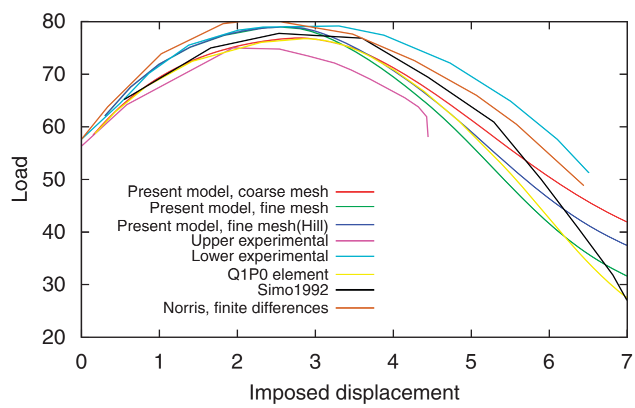
Figure 12. Load versus longitudinal displacement results for the present model compared with the return-mapping algorithm. 14, 28 and 70 uniform displacement steps are used with $\mathrm{tol} = 1\times 10^{-6}$ and $c_0 = 10$ consistent units (von-Mises law). Return mapping makes use of finite strain Simo's algorithm [17]. The energy residual and the effect of tol are also shown.

DOI: 10.1002/nme



Color Online, B&W in Print

Figure 13. Longitudinal stress along the axisymmetric axis for three mesh densities.



Color Online, B&W in Print

Figure 14. Load versus imposed displacement results for two meshes and several formulations. The curves are within the experimental envelope (the upper and lower experimental results from [40] were reproduced).

stress distribution, and hence elements further away from the high stress gradient region are not so stretched.

To inspect the robustness of our method, we fix each longitudinal displacement increment, $\Delta\overline{u}_y = \{0.5, 0.25, 0.1\}$ corresponding, respectively, to 14, 28 and 70 uniform displacement steps and observe the effect in the load/displacement results for the coarse mesh. With only 14 steps, the results used are sufficiently accurate. We tested Simo's [17] radial-return mapping in principal directions and show the results in Figure 12. It is clear that the proposed method allows very large steps in comparison with the return-mapping technique that makes 112 non-uniform steps. This results from step halving that occurs when, for a certain step, convergence fails. A large number of steps were also pointed out in [41] and Crisfield and Norris [9] for the 2D case. Very dense clouds of points due to step-cutting were also shown in the latter paper. Note that we use a trust region solution method which is typically more robust than the standard Newton method (the residual

DOI: 10.1002/nme

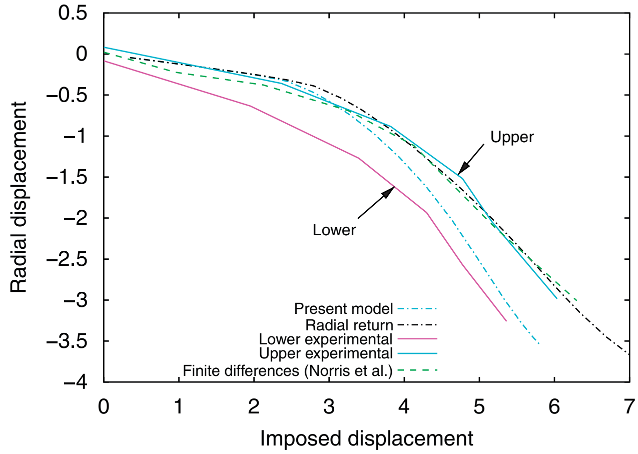


Figure 15. Tension test: necking displacement comparison.

norm is shown in Figure 12). We are not aware of other authors solving this problem with 14 uniform steps.

The value of tol has some effect on the convergence response, as the study in the same figure shows. This is despite the fact that the load-deflection results are nearly unchanged for reasonable values, as illustrated in Figure 12.

An acceptable load–displacement result is obtained with the coarse mesh. However, stress values and necking shape are better reproduced with finer meshes (Figure 13 shows the longitudinal stress for three mesh densities). In Figure 14, results are compared with the mixed Q_1–P_0 hexahedron, the numerical results by Simo [17] (with their finer mesh) and Norris *et al.* [40]. The experimental results from the same reference [40] are also shown.

Our results are within the experimental envelope obtained by Norris *et al.* [40] and more pronounced softening is obtained by the proposed method than with other methods. The hexahedron Q_1–P_0 used the radial-return algorithm (with the finer mesh) and produced results close to the ones by Simo [17]. Overall, there is some spreading in these results, since the methods are also different. The response of the Hill criterion is slightly stiffer, and this is expected.

The necking behavior is also important, since it explains the softening response (with strain hardening) and shows how good the element performance is. Numerical results are shown in Figure 15 and compared with the upper and lower experiments by Norris *et al.* [40]. With the exception of the radial-return results, both meshes are within the experimental envelope.

For the von-Mises criterion, the deformed mesh, the effective plastic strain and longitudinal stress contour plots are presented in Figure 16. We note that near the necking region, the outer ring is supporting the specimen while the specimen core is in compression, as observed by Norris *et al*. Results are smooth and relatively unaffected by the mesh aspect ratio in the necking region. A run was made with a more refined mesh but the results are indistinguishable.

Although the load-deflection results obtained using Hill criterion are close to the ones obtained with von-Mises criterion, the deformed mesh is obviously not (see Figure 17).

DOI: 10.1002/nme

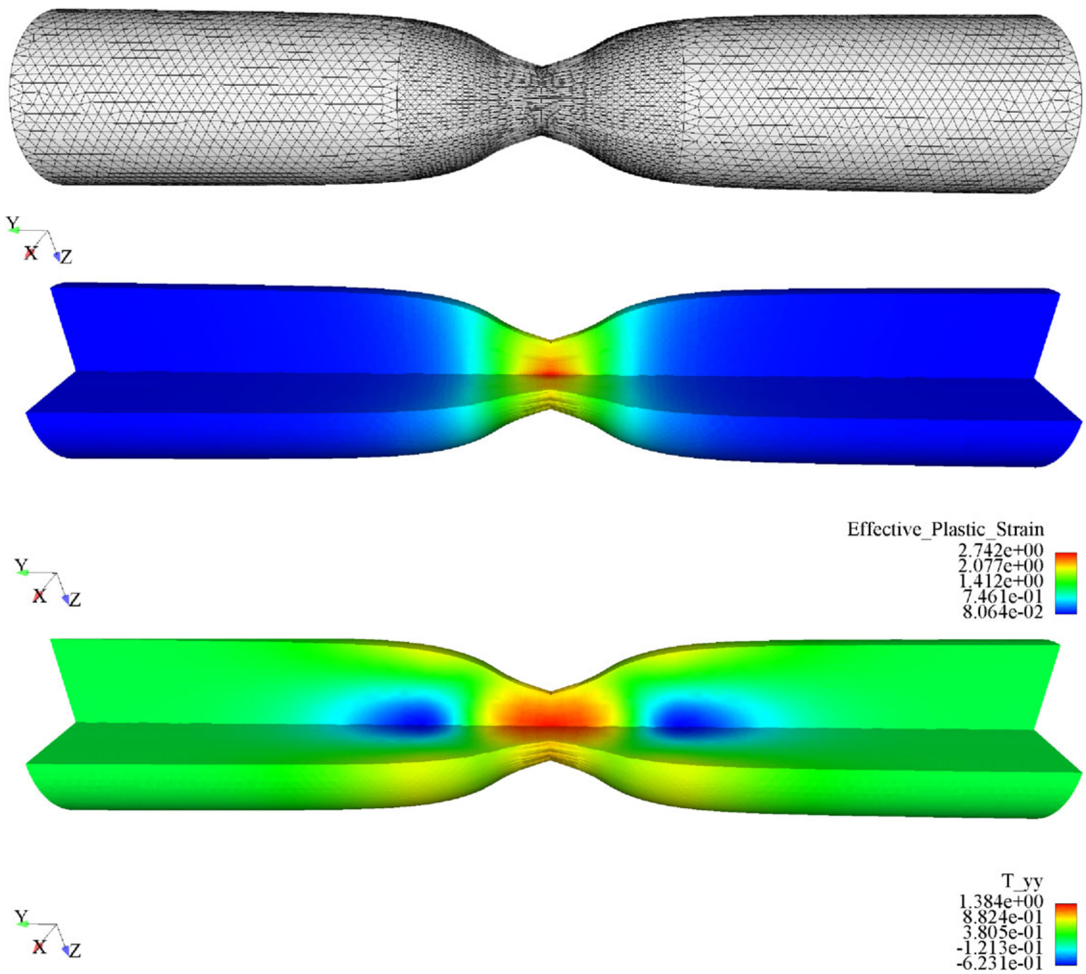


Figure 16. Deformed mesh ($u_y = 2\times 7$ consistent units), effective plastic strain contour plot and longitudinal stress for the fine mesh with $\mathrm{tol} = 1\times 10^{-6}$ and $c_0 = 2$ (von-Mises with neo–Hookean material).

The imposed displacement can reach very high values without any mesh distortion problems, see Figure 18 where the cross section for the Hill criterion is shown.

4. CONCLUSIONS

A general framework for finite strain plasticity with anisotropy and kinematic hardening was presented. Despite being restricted to elastic isotropy, the resulting model is much more general than the previous proposals. At the continuum level, Nemat-Nasser presented a closely related approach (see [3]). Smoothing by use of the Mangasarian functions replaced the exact complementarity condition, so that the return-mapping algorithm was avoided. Examples included kinematic

DOI: 10.1002/nme

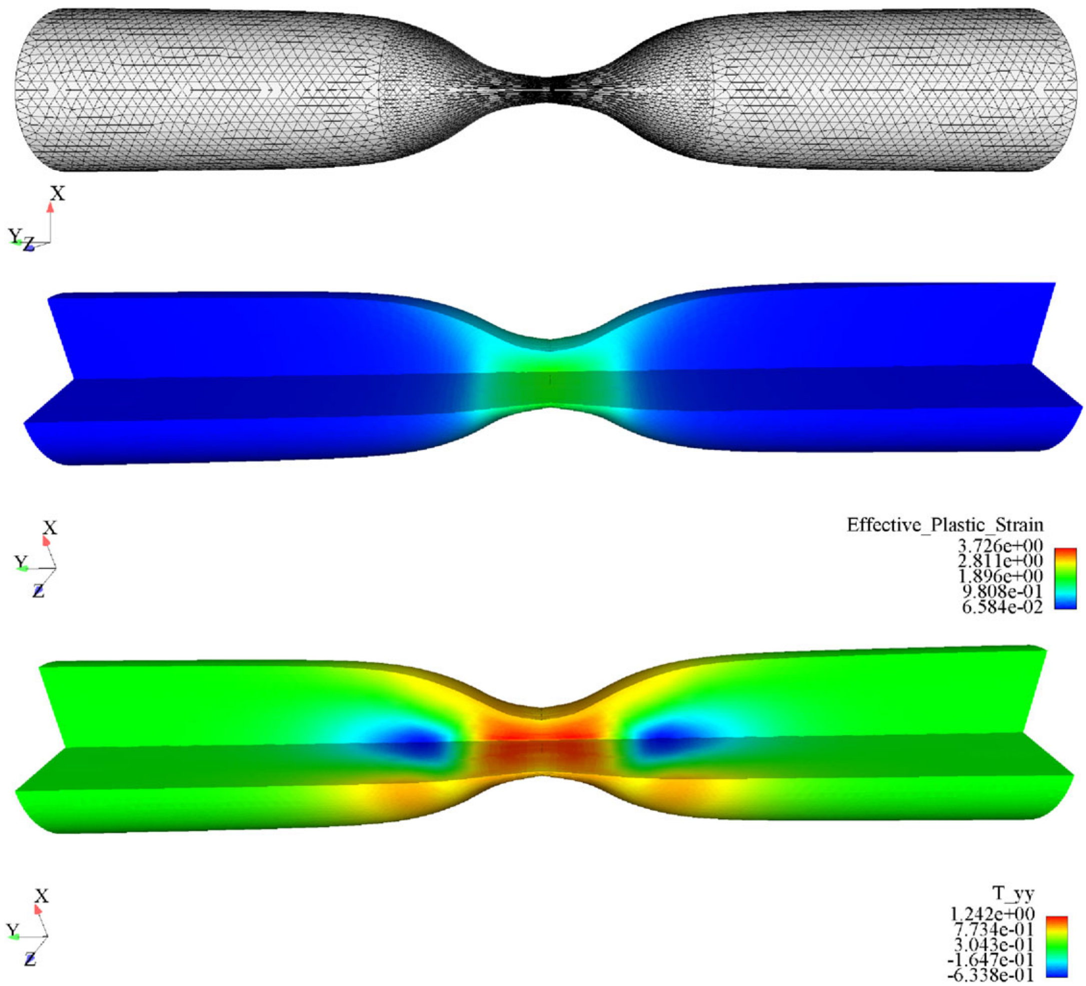


Figure 17. Deformed mesh ($u_y = 2\times 7$ consistent units), effective plastic strain contour plot and longitudinal stress for the fine mesh with $\mathrm{tol} = 1\times 10^{-6}$ and $c_0 = 2$ (Hill criterion with neo–Hookean material).

Color Online, Color in Print

hardening and plastic anisotropy; the ellipticity indicator was obtained for kinematic hardening, showing loss of strong ellipticity under strain control and, for the Burlet–Cailletaud model, under stress control.

The overall scheme is also computationally simpler than previous integration schemes. It was found that computational costs are higher than classical J_2 neo–Hookean-based or Hencky-based approaches, but arbitrary isotropic elastic laws, anisotropic flow laws, yield functions and hardening functions can be adopted. For example, to model feature–full viscoelasto-plastic elastomers (see [10]) and solid propellants this can be very important [42]. In addition, since no return mapping is required, nor a particular solution method for plasticity, we can include more complex behavior

DOI: 10.1002/nme

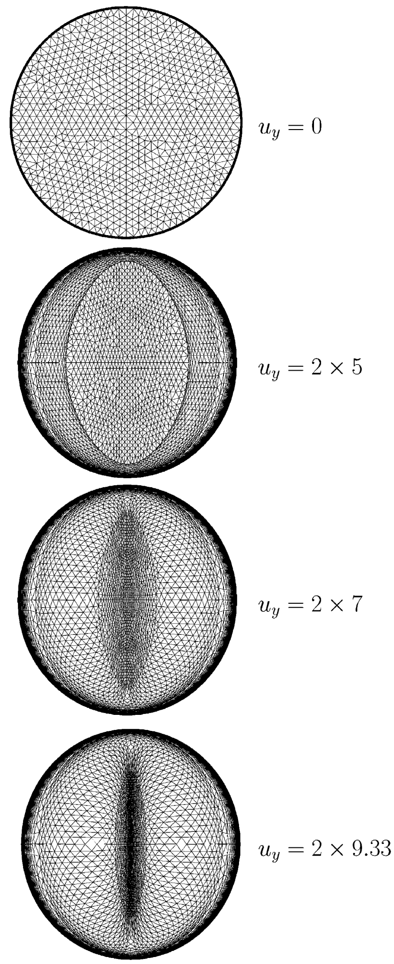


Figure 18. Cross-section evolution, Hill criterion.

without the effort usually required to derive closed-form quantities for specific return mappings in finite strain plasticity.

Mixed pressure–displacement elements with non-local pressure were employed. These were found to be very robust in previous works [10] and, besides being stable in the IS condition, the accuracy was found to be very good in numerical tests.

ACKNOWLEDGEMENTS

The first author gratefully acknowledges the advice and guidance of late Professor J. A. C. Martins concerning the complementarity condition.

REFERENCES

1. Key SW, Krieg RD. On the numerical implementation of inelastic time dependent and time independent, finite strain constitutive equations in structural mechanics. *Computer Methods in Applied Mechanics and Engineering* 1982; **33**:439–452.
2. Simo JC, Hughes TJR. *Computational Inelasticity* (corrected 2nd printing edn). Springer: Berlin, 2000.
3. Nemat-Nasser S. *Plasticity*: *A Treatise on Finite Deformation of Heterogeneous Inelastic Materials*. Cambridge Monographs on Mechanics. Cambridge University Press: Cambridge, 2004.
4. Brezzi F, Bathe K-J. A discourse on the stability conditions for mixed finite element formulations. *Computer Methods in Applied Mechanics and Engineering* 1990; **82**:27–57.
5. Brezzi F, Fortin M. *Mixed and Hybrid Finite Element Methods*. Springer: Berlin, 1991.
6. Belytschko T, Liu WK, Moran B. *Nonlinear Finite Elements for Continua and Structures*. Wiley: New York, 2000.
7. Nemat-Nasser S. Thermomechanical response of HSLA-65 steel plates: experiments and modeling. *Mechanics of Materials* 2005; **37**:379–405.
8. Peirce D, Shih CF, Needleman A. A tangent modulus method for rate dependent solids. *Computers and Structures* 1984; **18**:875–887.
9. Crisfield MA, Norris V. A stabilized large-strain elasto-plastic Q_1–P_0 method 1999; **46**:579–592. **Q4**
10. Areias PMA, Matouš K. Finite element formulation for modeling nonlinear viscoelastic elastomers. *Computer Methods in Applied Mechanics and Engineering* 2008; **197**:4702–4717.
11. Kulkarni DV, Tortorelli DA, Wallin M. A Newton–Schur alternative to the consistent tangent approach in computational plasticity. *Computer Methods in Applied Mechanics and Engineering* 2007; **196**:1169–1177.
12. Han W, Reddy BD. Plasticity: mathematical theory and numerical analysis. *Interdisciplinary Applied Mathematics. Mechanics and Materials*. Springer: Berlin, 1999.
13. Chen C, Mangasarian OL. Smoothing methods for convex inequalities and linear complementarity problems. *Mathematical Programming* 1995; **71**(1):51–69.
14. Hughes TJR, Taylor RL. Unconditionally stable algorithms for quasi-static elasto/viscoplastic finite element analysis. *Computers and Structures* 1978; **8**:169–173.
15. Reddy BD, Martin JB. Algorithms for the solution of internal variable problems in plasticity. *Computer Methods in Applied Mechanics and Engineering* 1991; **93**:253–273.
16. Maugin GA. *The Thermodynamics of Plasticity and Fracture*. Cambridge University Press: Cambridge, 1992.
17. Simo JC. Algorithms for static and dynamic multiplicative plasticity that preserves the classical return mapping schemes of the infinitesimal theory. *Computer Methods in Applied Mechanics and Engineering* 1992; **99**:61–112.
18. Reese S, Govindjee S. A theory of finite viscoelasticity and numerical aspects. *International Journal of Solids and Structures* 1998; **35**:3455–3482.
19. Dettmer W, Reese S. On the theoretical and numerical modelling of Armstrong–Frederick kinematic hardening in the finite strain regime. *Computer Methods in Applied Mechanics and Engineering* 2004; **193**:87–116.
20. Johansson G, Ekh M, Runesson K. Computational modeling of inelastic large ratcheting strains. *International Journal of Plasticity* 2005; **21**:955–980.
21. Weber G, Anand L. Finite deformation constitutive equations and a time integration procedure for isotropic, hyperelastic–viscoplastic solids. *Computer Methods in Applied Mechanics and Engineering* 1990; **79**:173–202.
22. Moran B, Ortiz M, Shih CF. Formulation of implicit finite element methods for multiplicative finite deformation plasticity. *International Journal for Numerical Methods in Engineering* 1990; **29**:483–514.
23. Lee EH. Elasto-plastic deformation at finite strains. *ASME Journal of Applied Mechanics* 1969; **36**:1–6.
24. Hartmann S, Luhrs G, Haupt P. An efficient stress algorithm with applications in viscoplasticity and plasticity. *International Journal for Numerical Methods in Engineering* 1997; **40**:991–1013.
25. Chadwick P. *Continuum Mechanics. Concise Theory and Problems* (2nd edn). Dover Publications: New York, 1999.
26. Areias PMA, Belytschko T. Analysis of finite strain anisotropic elastoplastic fracture in thin plates and shells. *Journal of Aerospace Engineering* (ASCE) 2006; **19**(4):259–270.
27. Lubarda VA. Duality in constitutive formulation of finite-strain elastoplasticity based on $f = f_e f_p$ and $f = f^p f^e$ decompositions. *International Journal of Plasticity* 1999; **15**:1277–1290.
28. Wolfram Research, Inc. *Mathematica*, Wolfram Research, Inc, Champaign, Illinois, version 6.0 edition, 2007.
29. Doll S, Schweizerhof K. On the development of volumetric strain energy functions. *Journal of Applied Mechanics* 2000; **67**(1):17–21.

30. Peerlings RHJ, de Borst R, Brekelmans WAM, de Vree JHP. Gradient enhanced damage for quasi-brittle materials. *International Journal for Numerical Methods in Engineering* 1996; **39**:3391–3403.
31. Lasry D, Belytschko T. Localization limiters in transient problems. *International Journal of Solids and Structures* 1988; **24**:581–597.
32. Pang J-S. Newton's method for b-differentiable equations. *Mathematics of Operations Research* 1990; **15**:311–341.
33. Areias PMA, Rabczuk T. Quasi-static crack propagation in plane and plate structures using set-valued traction-separation laws. *International Journal for Numerical Methods in Engineering* 2007. Q5
34. Tsakmakis C, Willuweit A. A comparative study of kinematic hardening at finite deformations. *International Journal of Non-Linear Mechanics* 2004.
35. Armstrong PJ, Frederick CO. A mathematical representation of the multiaxial Bauschinger effect. *General Electricity Generating Board RD/B/N731*, Berkeley Nuclear Laboratories, 1966.
36. Burlet H, Cailletaud G. Numerical techniques for cyclic plasticity at variable temperature. *Engineering Computations* 1986; **3**:143–153.
37. Chen X, Jiao R. Modified kinematic hardening rule for multiaxial ratcheting prediction. *International Journal of Plasticity* 2004; **20**:871–898.
38. Truesdell C, Noll W. *The Non-Linear Field Theories of Mechanics* (3rd edn). Springer: Berlin, 2004.
39. Bathe K-J. *Finite Element Procedures*. Prentice-Hall: Englewood Cliffs, NJ, 1996.
40. Norris Jr DM, Moran B, Scudder JK, Quiñones DF. A computer simulation of the tension test. *Journal of the Mechanics and Physics of Solids* 1978; **26**:1–19.
41. Zienkiewicz OC, Taylor RL. *The Finite Element Method* (5th edn), vol. 2. Butterworth, Heinemann: London, 2000.
42. Ozüpek S, Becker EB. Constitutive modeling of high-elongation solid propellants. *Journal of Engineering Materials and Technology* 1992; **114**:111–115.

DOI: 10.1002/nme

Author Queries Form

John Wiley

JOURNAL TITLE: **NME** **29/6/2009**

ARTICLE NO: 2686

Queries and / or remarks

Query No.	Details required	Author's response
Q1	Please check if the author affiliation link given is correct.	
Q2	There was a discrepancy in the spelling between publisher provided list (Pedro Miguel de Almeida) and ms (P.M.A Are accute i accute as) manuscript was followed Please check	
Q3	Please check the placement and citation of Tables 1 and 2.	
Q4	Please provide journal name for Ref.[9].	
Q5	Please provide DOI number if volume, page range are not yet available for Refs [33,34].	

COPYRIGHT TRANSFER AGREEMENT

Date: ____________________ Contributor name: ____

Contributor address: ____

Manuscript number (Editorial office only): ____

Re: Manuscript entitled ____

____ (the "Contribution")

for publication in ____ (the "Journal")

published by ____ ("Wiley-Blackwell").

Dear Contributor(s):
Thank you for submitting your Contribution for publication. In order to expedite the editing and publishing process and enable Wiley-Blackwell to disseminate your Contribution to the fullest extent, we need to have this Copyright Transfer Agreement signed and returned as directed in the Journal's instructions for authors as soon as possible. If the Contribution is not accepted for publication, or if the Contribution is subsequently rejected, this Agreement shall be null and void. **Publication cannot proceed without a signed copy of this Agreement.**

A. COPYRIGHT

1. The Contributor assigns to Wiley-Blackwell, during the full term of copyright and any extensions or renewals, all copyright in and to the Contribution, and all rights therein, including but not limited to the right to publish, republish, transmit, sell, distribute and otherwise use the Contribution in whole or in part in electronic and print editions of the Journal and in derivative works throughout the world, in all languages and in all media of expression now known or later developed, and to license or permit others to do so.

2. Reproduction, posting, transmission or other distribution or use of the final Contribution in whole or in part in any medium by the Contributor as permitted by this Agreement requires a citation to the Journal and an appropriate credit to Wiley-Blackwell as Publisher, and/or the Society if applicable, suitable in form and content as follows: (Title of Article, Author, Journal Title and Volume/Issue, Copyright © [year], copyright owner as specified in the Journal). Links to the final article on Wiley-Blackwell's website are encouraged where appropriate.

B. RETAINED RIGHTS

Notwithstanding the above, the Contributor or, if applicable, the Contributor's Employer, retains all proprietary rights other than copyright, such as patent rights, in any process, procedure or article of manufacture described in the Contribution.

C. PERMITTED USES BY CONTRIBUTOR

1. Submitted Version. Wiley-Blackwell licenses back the following rights to the Contributor in the version of the Contribution as originally submitted for publication:

a. After publication of the final article, the right to self-archive on the Contributor's personal website or in the Contributor's institution's/employer's institutional repository or archive. This right extends to both intranets and the Internet. The Contributor may not update the submission version or replace it with the published Contribution. The version posted must contain a legend as follows: This is the pre-peer reviewed version of the following article: FULL CITE, which has been published in final form at [Link to final article].

b. The right to transmit, print and share copies with colleagues.

2. Accepted Version. Re-use of the accepted and peer-reviewed (but not final) version of the Contribution shall be by separate agreement with Wiley-Blackwell. Wiley-Blackwell has agreements with certain funding agencies governing reuse of this version. The details of those relationships, and other offerings allowing open web use, are set forth at the following website: http://www.wiley.com/go/funderstatement. NIH grantees should check the box at the bottom of this document.

3. Final Published Version. Wiley-Blackwell hereby licenses back to the Contributor the following rights with respect to the final published version of the Contribution:

a. Copies for colleagues. The personal right of the Contributor only to send or transmit individual copies of the final published version in any format to colleagues upon their specific request provided no fee is charged, and further-provided that there is no systematic distribution of the Contribution, e.g. posting on a listserve, website or automated delivery.

b. Re-use in other publications. The right to re-use the final Contribution or parts thereof for any publication authored or edited by the Contributor (excluding journal articles) where such re-used material constitutes less than half of the total material in such publication. In such case, any modifications should be accurately noted.

c. Teaching duties. The right to include the Contribution in teaching or training duties at the Contributor's institution/place of employment including in course packs, e-reserves, presentation at professional conferences, in-house training, or distance learning. The Contribution may not be used in seminars outside of normal teaching obligations (e.g. commercial seminars). Electronic posting of the final published version in connection with teaching/training at the Contributor's institution/place of employment is permitted subject to the implementation of reasonable access control mechanisms, such as user name and password. Posting the final published version on the open Internet is not permitted.

d. Oral presentations. The right to make oral presentations based on the Contribution.

4. Article Abstracts, Figures, Tables, Data Sets, Artwork and Selected Text (up to 250 words).

a. Contributors may re-use unmodified abstracts for any non-commercial purpose. For on-line uses of the abstracts, Wiley-Blackwell encourages but does not require linking back to the final published versions.

b. Contributors may re-use figures, tables, data sets, artwork, and selected text up to 250 words from their Contributions, provided the following conditions are met:

(i) Full and accurate credit must be given to the Contribution.
(ii) Modifications to the figures, tables and data must be noted. Otherwise, no changes may be made.
(iii) The reuse may not be made for direct commercial purposes, or for financial consideration to the Contributor.
(iv) Nothing herein shall permit dual publication in violation of journal ethical practices.

D. CONTRIBUTIONS OWNED BY EMPLOYER

1. If the Contribution was written by the Contributor in the course of the Contributor's employment (as a "work-made-for-hire" in the course of employment), the Contribution is owned by the company/employer which must sign this Agreement (in addition to the Contributor's signature) in the space provided below. In such case, the company/employer hereby assigns to Wiley-Blackwell, during the full term of copyright, all copyright in and to the Contribution for the full term of copyright throughout the world as specified in paragraph A above.

2. In addition to the rights specified as retained in paragraph B above and the rights granted back to the Contributor pursuant to paragraph C above, Wiley-Blackwell hereby grants back, without charge, to such company/employer, its subsidiaries and divisions, the right to make copies of and distribute the final published Contribution internally in print format or electronically on the Company's internal network. Copies so used may not be resold or distributed externally. However the company/employer may include information and text from the Contribution as part of an information package included with software or other products offered for sale or license or included in patent applications. Posting of the final published Contribution by the institution on a public access website may only be done with Wiley-Blackwell's written permission, and payment of any applicable fee(s). Also, upon payment of Wiley-Blackwell's reprint fee, the institution may distribute print copies of the published Contribution externally.

E. GOVERNMENT CONTRACTS

In the case of a Contribution prepared under U.S. Government contract or grant, the U.S. Government may reproduce, without charge, all or portions of the Contribution and may authorize others to do so, for official U.S. Government purposes only, if the U.S. Government contract or grant so requires. (U.S. Government, U.K. Government, and other government employees: see notes at end)

F. COPYRIGHT NOTICE

The Contributor and the company/employer agree that any and all copies of the final published version of the Contribution or any part thereof distributed or posted by them in print or electronic format as permitted herein will include the notice of copyright as stipulated in the Journal and a full citation to the Journal as published by Wiley-Blackwell.

G. CONTRIBUTOR'S REPRESENTATIONS

The Contributor represents that the Contribution is the Contributor's original work, all individuals identified as Contributors actually contributed to the Contribution, and all individuals who contributed are included. If the Contribution was prepared jointly, the Contributor agrees to inform the co-Contributors of the terms of this Agreement and to obtain their signature to this Agreement or their written permission to sign on their behalf. The Contribution is submitted only to this Journal and has not been published before. (If excerpts from copyrighted works owned by third parties are included, the Contributor will obtain written permission from the copyright owners for all uses as set forth in Wiley-Blackwell's permissions form or in the Journal's Instructions for Contributors, and show credit to the sources in the Contribution.) The Contributor also warrants that the Contribution contains no libelous or unlawful statements, does not infringe upon the rights (including without limitation the copyright, patent or trademark rights) or the privacy of others, or contain material or instructions that might cause harm or injury.

CHECK ONE BOX:

☐ Contributor-owned work

ATTACH ADDITIONAL SIGNATURE PAGES AS NECESSARY

Contributor's signature ______________________ Date ____________

Type or print name and title ______________________

Co-contributor's signature ______________________ Date ____________

Type or print name and title ______________________

☐ Company/Institution-owned work (made-for-hire in the course of employment)

Company or Institution (Employer-for-Hire) ______________________ Date ____________

Authorized signature of Employer ______________________ Date ____________

☐ U.S. Government work

Note to U.S. Government Employees
A contribution prepared by a U.S. federal government employee as part of the employee's official duties, or which is an official U.S. Government publication, is called a "U.S. Government work," and is in the public domain in the United States. In such case, the employee may cross out Paragraph A.1 but must sign (in the Contributor's signature line) and return this Agreement. If the Contribution was not prepared as part of the employee's duties or is not an official U.S. Government publication, it is not a U.S. Government work.

☐ U.K. Government work (Crown Copyright)

Note to U.K. Government Employees
The rights in a Contribution prepared by an employee of a U.K. government department, agency or other Crown body as part of his/her official duties, or which is an official government publication, belong to the Crown. U.K. government authors should submit a signed declaration form together with this Agreement. The form can be obtained via http://www.opsi.gov.uk/advice/crown-copyright/copyright-guidance/publication-of-articles-written-by-ministers-and-civil-servants.htm

☐ Other Government work

Note to Non-U.S., Non-U.K. Government Employees
If your status as a government employee legally prevents you from signing this Agreement, please contact the editorial office.

☐ NIH Grantees

Note to NIH Grantees
Pursuant to NIH mandate, Wiley-Blackwell will post the accepted version of Contributions authored by NIH grant-holders to PubMed Central upon acceptance. This accepted version will be made publicly available 12 months after publication. For further information, see www.wiley.com/go/nihmandate.

WILEY AUTHOR DISCOUNT CARD

As a highly valued contributor to Wiley's publications, we would like to show our appreciation to you by offering a **unique 25% discount** off the published price of any of our books*.

To take advantage of this offer, all you need to do is apply for the **Wiley Author Discount Card** by completing the attached form and returning it to us at the following address:

The Database Group
John Wiley & Sons Ltd
The Atrium
Southern Gate
Chichester
West Sussex PO19 8SQ
UK

In the meantime, whenever you order books direct from us, simply quote promotional code **S001W** to take advantage of the 25% discount.

The newest and quickest way to order your books from us is via our new European website at:

http://www.wileyeurope.com

Key benefits to using the site and ordering online include:

- Real-time SECURE on-line ordering
- The most up-to-date search functionality to make browsing the catalogue easier
- Dedicated Author resource centre
- E-mail a friend
- Easy to use navigation
- Regular special offers
- Sign up for subject orientated e-mail alerts

So take advantage of this great offer, return your completed form today to receive your discount card.

Yours sincerely,

Verity Leaver
E-marketing and Database Manager

*TERMS AND CONDITIONS

This offer is exclusive to Wiley Authors, Editors, Contributors and Editorial Board Members in acquiring books (excluding encyclopaedias and major reference works) for their personal use. There must be no resale through any channel. The offer is subject to stock availability and cannot be applied retrospectively. This entitlement cannot be used in conjunction with any other special offer. Wiley reserves the right to amend the terms of the offer at any time.

REGISTRATION FORM
FOR 25% BOOK DISCOUNT CARD

To enjoy your special discount, tell us your areas of interest and you will receive relevant catalogues or leaflets from which to select your books. Please indicate your specific subject areas below.

Accounting [ ]
- Public [ ]
- Corporate [ ]

Chemistry [ ]
- Analytical [ ]
- Industrial/Safety [ ]
- Organic [ ]
- Inorganic [ ]
- Polymer [ ]
- Spectroscopy [ ]

Encyclopedia/Reference [ ]
- Business/Finance [ ]
- Life Sciences [ ]
- Medical Sciences [ ]
- Physical Sciences [ ]
- Technology [ ]

Earth & Environmental Science [ ]

Hospitality [ ]

Genetics [ ]
- Bioinformatics/Computational [ ]
  Biology [ ]
- Proteomics [ ]
- Genomics [ ]
- Gene Mapping [ ]
- Clinical Genetics [ ]

Medical Science [ ]
- Cardiovascular [ ]
- Diabetes [ ]
- Endocrinology [ ]
- Imaging [ ]
- Obstetrics/Gynaecology [ ]
- Oncology [ ]
- Pharmacology [ ]
- Psychiatry [ ]

Non-Profit [ ]

Architecture [ ]

Business/Management [ ]

Computer Science [ ]
- Database/Data Warehouse [ ]
- Internet Business [ ]
- Networking [ ]
- Programming/Software [ ]
  Development [ ]
- Object Technology [ ]

Engineering [ ]
- Civil [ ]
- Communications Technology [ ]
- Electronic [ ]
- Environmental [ ]
- Industrial [ ]
- Mechanical [ ]

Finance/Investing [ ]
- Economics [ ]
- Institutional [ ]
- Personal Finance [ ]

Life Science [ ]

Landscape Architecture [ ]

Mathematics/Statistics [ ]

Manufacturing [ ]

Material Science [ ]

Psychology [ ]
- Clinical [ ]
- Forensic [ ]
- Social & Personality [ ]
- Health & Sport [ ]
- Cognitive [ ]
- Organizational [ ]
- Developmental and Special Ed [ ]
- Child Welfare [ ]
- Self-Help [ ]

Physics/Physical Science [ ]

[ ] I confirm that I am a Wiley Author/Editor/Contributor/Editorial Board Member of the following publications:

SIGNATURE: ………

PLEASE COMPLETE THE FOLLOWING DETAILS IN BLOCK CAPITALS:

TITLE AND NAME: (e.g. Mr, Mrs, Dr) ………………………………………………………………………………………

JOB TITLE: ………

DEPARTMENT: ………..

COMPANY/INSTITUTION: ………………………………………………………………………………………………………

ADDRESS: …….

……

……

……

TOWN/CITY: ……

COUNTY/STATE: …….

COUNTRY: ………

POSTCODE/ZIP CODE: ………

DAYTIME TEL: ………

FAX: ……

E-MAIL: ……

YOUR PERSONAL DATA
We, John Wiley & Sons Ltd, will use the information you have provided to fulfil your request. In addition, we would like to:

1. Use your information to keep you informed by post, e-mail or telephone of titles and offers of interest to you and available from us or other Wiley Group companies worldwide, and may supply your details to members of the Wiley Group for this purpose.

   [ ] Please tick the box if you do not wish to receive this information

2. Share your information with other carefully selected companies so that they may contact you by post, fax or e-mail with details of titles and offers that may be of interest to you.

   [ ] Please tick the box if you do not wish to receive this information.

If, at any time, you wish to stop receiving information, please contact the Database Group (databasegroup@wiley.co.uk) at John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex PO19 8SQ, UK.

E-MAIL ALERTING SERVICE
We offer an information service on our product ranges via e-mail. If you do not wish to receive information and offers from John Wiley companies worldwide via e-mail, please tick the box [ ].

This offer is exclusive to Wiley Authors, Editors, Contributors and Editorial Board Members in acquiring books (excluding encyclopaedias and major reference works) for their personal use. There should be no resale through any channel. The offer is subject to stock availability and may not be applied retrospectively. This entitlement cannot be used in conjunction with any other special offer. Wiley reserves the right to vary the terms of the offer at any time.

Ref: S001W