An Analytic Approach to Capillary Pressure in Tree-Shaped Networks

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Abstract: This study analyzes the pressure in capillary tree-shaped networks, based on the thermodynamics and the geometric description of the network. The presence of equilibrium and non-equilibrium effects on pressure is accounted. In particular, the presence of equilibrium effects is shown to be essential for describing the pressure in the network. An approach to the definition of equilibrium capillary pressure is presented in terms of interfacial energies of the phases but also in terms of the geometry characteristics of the tree-shaped network. Among others, the equilibrium capillary pressure is shown to depend on number of branching levels and number of daughter branches. Finally, a criterion is established to evaluate the importance of equilibrium and non-equilibrium effects on the capillary pressure.

INTRODUCTION

The study of capillary-driven flows is a subject of great interest due to its widespread applications in oil recovery, hydrology, catalysis, membranes, among others [1]. The dynamic invasion of fluid into a capillary tube was first studied analytically by [2] and [3]. The so-called Lucas-Washburn equation describes the time penetration of liquid into a capillary. While this approach holds promise for describing the long-time capillary flow, it neglects the influence of the forces that prevail in the initial stage of the flow. Other authors [4,5] studied the thermodynamic and kinetic effects on capillary penetration in systems of limited size. In addition, Ichikawa and Satoda [6] characterized the interface dynamics of capillary flow. The capillary rise dominated by inertial forces was investigated by [7], and Zhmud *et al.* [8] presented an overview of the solutions for the different time regimes. Other authors, Siebold *et al.* [9] and Chan *et al.* [10] have performed capillary experiments to study the effect of the dynamic contact angle and to determine the factors that influence the significance of gravity, respectively. Raiskinmaki *et al.* [11] included the effect of entrance pressure loss on the capillary driven flow. An approach that accounts for non-equilibrium effects on capillary pressure was proposed by [12]. Others [13] developed a model for the capillary rise in porous media which incorporates dynamic saturation gradients in the media. Based on experiments, Cheng *et al.* [14] showed a connection between the capillary pressure and the fluid interfacial area in porous media. Other studies [15] extended the equilibrium capillary pressure based on the first law of thermodynamics to a fractal porous media. The dynamic capillary rise due to hydrostatic effects was studied by [16].

Research on tree-shaped capillary networks, though dating back one century [17], is still a topic of great interest due to its widespread applications in physics, biology, geology and engineering. The simulation of fluid flow through these structures requires an accurately knowledge of the pressure distribution of a fluid through the structure. Studies performed in porous media point out the dependence of capillary pressure on the rate change of moisture inside the media, suggesting that departures from equilibrium are dependent on the movement of fluid interfaces (see for example, Beliaeve and Hassanizadeh [12], Deinert *et al.* [15] and Fan *et al.* [18]). Therefore, capillary pressure is an essential quantity that deserves further analysis.

Here we focus on tree-shaped networks. The interest in tree-shaped flow networks is spreading through different fields. Trees for cooling in electronics were studied by [19,20]. Applications for single-phase flow and two-phase flow have been proposed by [19,21,22]. Concepts for tree-shaped heat and mass exchangers were proposed by Coppens *et al.* [23] and Tondeur *et al.* [24].

This paper combines the thermodynamics and geometric characteristics of the system to describe capillary pressure in tree-shaped networks. Based on the approach developed, the non-equilibrium and equilibrium effects on the capillary pressure are identified. Besides, the equilibrium capillary pressure is defined in terms of readily measurable parameters. The influence of geometric parameters of the tree-shaped network is studied to obtain a better understanding of this quantity.

TREE-SHAPED NETWORKS:

Geometric and Operational Parameters

The geometric parameters of the tree-shaped network are defined in Fig. **1**. This network has N branches of ducts, from level 0 to level n. Each duct branches into m daughter branches at the next level. The ducts are round capillary tubes of different radii (R_i) and lengths (L_i; i=0,1,…,n). The radii and lengths of the ducts are sized relative to one another, in accordance with [17,25,26]:

$$\frac{R_i}{R_{i+1}} = a_r \tag{1}$$

$$\frac{L_i}{L_{i+1}} = b_l \tag{2}$$

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where a_r and b_l are scale factors independent of i. The relationship between the size of the first duct (level 0) and the size of the ducts at level i is given by

$$\frac{R_i}{R_0} = a_r^{\,i} \tag{3}$$

$$\frac{L_i}{L_0} = b_l^{\,i} \tag{4}$$

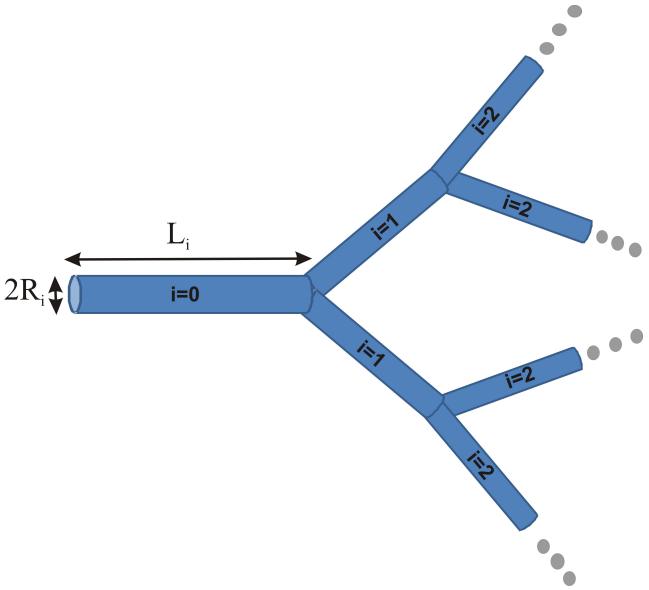

Fig. (1). Tree-shaped capillary network (N=3; m=2).

One basic feature of tree-shaped networks is that pairing, or bifurcation of ducts (dichotomy), is an optimized feature of the flow architecture (see for example [17,25]). Therefore, m takes a value of 2. If the flow is laminar and fully developed, the minimization of flow resistance yields the scale factor $a_r=2^{-1/3}$, which in physiology is known as Hess-Murray law [26]. Moreover, for bifurcations (m=2) the scale factors b_l and a_r range from 2^{-1} to $2^{-1/3}$, respectively. If the flow is turbulent, a_r and b_l are $2^{-3/7}$ and $2^{-1/7}$, respectively [25]. In both flows it seems that is a geometric ratio of R/L^3 which is preserved.

The total number of branches is

$$N = \sum_{i=0}^{n} m^i = \frac{1-m^q}{1-m} \tag{5}$$

with q = n+1 and the volume of the whole network is

$$V_s = \sum_{i=0}^{n} m^i \alpha_{vs} R_i^2 L_i \tag{6}$$

where α_{vs} is a shape factor (i.e., π). Substituting Eqs. (3) and (4) into (6) we obtain

$$V_s = \alpha_{vs} R_0^2 L_0 \sum_{i=0}^{n} m^i a_r^{2i} b_l^i = \\ = \alpha_{vs} R_0^2 L_0 \frac{1-(m a_r^2 b_l)^q}{1-m a_r^2 b_l} \tag{7}$$

In a similar procedure, the surface area is

$$A_s = \sum_{i=0}^{n} m^i \alpha_{as} R_i L_i = \alpha_{as} R_0 L_0 \frac{1-(m a_r b_l)^q}{1-m a_r b_l} \tag{8}$$

Here m is the number of branches, n is the final branching level and α_{as} is a shape factor (*e.g.*, π). Equations (7) and (8) show the volume and the surface area of the tree-shaped network in terms of the geometric characteristics of the first duct, number of branches and scale factors.

CAPILLARY PRESSURE:

Thermodynamic Model

We assume a wetting fluid entering in a capillary tube under a pressure $p_{w,}$ and that this fluid distributes itself by displacing a non-wetting fluid at pressure p_n. The first law of thermodynamics, for the system under study, can be mathematically expressed by [27]

$$\dot{U} = F\dot{r} + \sum_j \zeta_{w,j} \dot{A}_{w,j} - p_w \dot{V}_w - p_n \dot{V}_n + T\dot{S} \tag{9}$$

where U is the internal energy of the fluid, T is the fluid temperature, V_w is the volume of wetting fluid, V_n is the volume of non-wetting fluid, A is the interfacial area of the fluid, S is the entropy of the fluid, ζ is the interfacial energy of the wetting fluid in contact with the *i* non-wetting phase and the first right-hand term denotes the power done by force F to move the solid-fluid interfaces.

Consider a slow, capillary-driven flow under isothermal conditions. Under these conditions, it is reasonable to consider that the internal energy and the entropy of the fluid are constant [15]. Therefore, as $\dot{V}_w + \dot{V}_n = 0$, Eq. (9) can be written as

$$P_c \dot{V}_w = (p_n - p_w)\dot{V}_w = -F\dot{r} - \sum_j \zeta_{w,j} \dot{A}_{w,j} \tag{10}$$

In summary, Eq. (10) shows how the capillary pressure P_c results of the motion of the solid-fluid interfaces (non-equilibrium effect), as well as of the interfacial energy of the wetting fluid in contact with the non-wetting phases and the fluid interfacial area (equilibrium effect).

According to [15,18] the first right-hand term of Eq. (10) is related to the contact line velocity and the volume of wetting fluid. Therefore, Eq. (10) can be rearranged to yield

$$P_c = -\mu \frac{\partial}{\partial t}\left(\frac{V_w}{V}\right) - \sum_j \zeta_{w,j} \left(\frac{\partial A_{w,j}}{\partial V}\right)_{U,S,T} \tag{11}$$

where μ is a coefficient that accounts for the movement of fluid-solid contact lines and V_W/V is the moisture content. Therefore, the first right-hand term accounts for the non-equilibrium effects (*i.e.,* the change of fluid-solid contact lines and the change of moisture content in time), and the second right-hand term accounts for the change of interfacial areas with fluid volume (equilibrium effects)

$$P_0 = -\sum_j \zeta_{w,j} \left(\frac{\partial A_{w,j}}{\partial V}\right)_{U,S,T} \tag{12}$$

Equation (12) represents the so-called equilibrium capillary pressure [15,18] and shows that this pressure depends both on the interfacial energy of the wetting fluid and on the variation in fluid interfacial areas with respect to fluid volume. Therefore, it is also a function of network's saturation. These results are in agreement with several studies (see for example [12,15,18,28]), which recognize that the interfaces play a key role in describing the multiphase fluid flow.

EQUILIBRIUM FLUID PRESSURE IN A TREE-SHAPED NETWORK

Consider a tree-shaped network as shown as in Fig. **1**. In order to obtain the equilibrium capillary pressure, we must provide the variation in fluid interfacial area with fluid (network) volume. Therefore, a description of the geometry of the network is required.

Consider a two-phase flow where a liquid (*e.g.*, water, wetting phase) displaces air (non-wetting phase) in a tree-shaped network. To implement Equation (12) we need to determine $\partial A/\partial V$ and this can be obtained from the geometry of the tree-shaped network. In a similar procedure to that described above, the interfacial area with corresponding pore volume for a situation where a liquid displaces air are expressed as

$$A_a = \sum_{i=0}^{n} m^i \alpha_{aa} R_i^2 = \alpha_{aa} R_0^2 \frac{1-(ma_r^2)^q}{1-ma_r^2} \tag{13}$$

$$V_a = \sum_{i=0}^{n} m^i \alpha_{va} R_i^3 = \alpha_{va} R_i^3 \frac{1-(ma_r^3)^q}{1-ma_r^3} \tag{14}$$

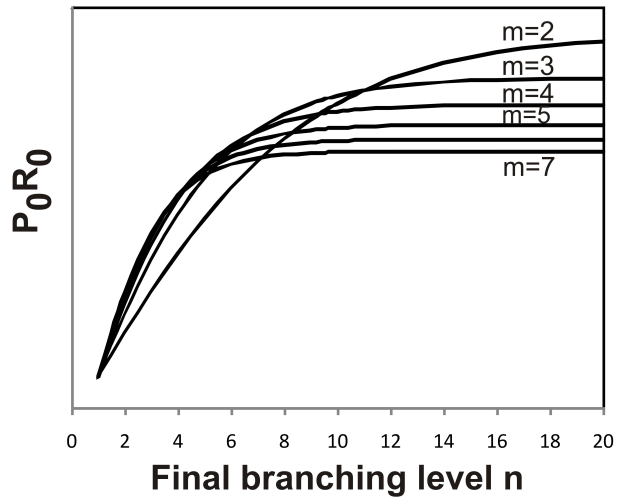

Fig. (2). Dependence of P_0R_0 on branching level n for different number of daughter branches m.

Here the subscript a means air, and α_{aa} and α_{va} are shape factors (*e.g.*, 4π and $4/3\pi$ for a spherical interface, respectively). Therefore, according to Equations (7) and (8) and Equations (13) and (14), the derivatives $\partial A/\partial V$ are given by

$$\frac{\partial A_{w,s}}{\partial V} = \frac{\partial A_{w,s}}{\partial R_0}\left(\frac{\partial V}{\partial R_0}\right)^{-1} = \frac{\chi_s}{R_0}\left[\frac{1-(ma_r b_l)^q}{1-ma_r b_l}\right]\left[\frac{1-ma_r^2 b_l}{1-(ma_r^2 b_l)^q}\right] \tag{15}$$

$$\frac{\partial A_{w,a}}{\partial V_a} = \frac{\partial A_{w,a}}{\partial R_0}\left(\frac{\partial V_a}{\partial R_0}\right)^{-1} = \frac{\chi_a}{R_0}\left[\frac{1-(ma_r^2)^q}{1-ma_r^2}\right]\left[\frac{1-ma_r^3}{1-(ma_r^3)^q}\right] \tag{16}$$

with $\chi_s=(1/2)\alpha_{as}/\alpha_{vs}$ and $\chi_a=(2/3)\alpha_{aa}/\alpha_{va}$. Substituting Eqs. (15) and (16) into Eq. (12) yields

$$P_0 = -\frac{\left(\eta_{w,s}+\eta_{w,a}\right)}{R_0} \tag{17}$$

with

$$\eta_{w,s} = \chi_s \zeta_{w,s}\left[\frac{1-(ma_r b_l)^q}{1-ma_r b_l}\right]\left[\frac{1-ma_r^2 b_l}{1-(ma_r^2 b_l)^q}\right] \tag{18}$$

$$\eta_{w,a} = \chi_a \zeta_{w,a}\left[\frac{1-(ma_r^2)^q}{1-ma_r^2}\right]\left[\frac{1-ma_r^3}{1-(ma_r^3)^q}\right] \tag{19}$$

Equation (17) presents the equilibrium capillary pressure in terms of the radius of the first duct (R_0), scale factors (a_r, b_l), branches characteristics (m, n), shape factors (χ_a, χ_s) and interfacial energies ($\zeta_{w,a}$, $\zeta_{w,s}$) of the respective phase. Equation (17) is also dependent of the network's saturation. For a saturated network

$$P_{0,sat} = \frac{\chi_s \zeta_{w,s}}{R_0}\left[\frac{1-(ma_r b_l)^q}{1-ma_r b_l}\right]\left[\frac{1-ma_r^2 b_l}{1-(ma_r^2 b_l)^q}\right] \tag{20}$$

Based on this formulation, we were able to research some aspects concerning the parameters that affect the equilibrium capillary pressure. The results are plotted in Figs. **2** to **4**.

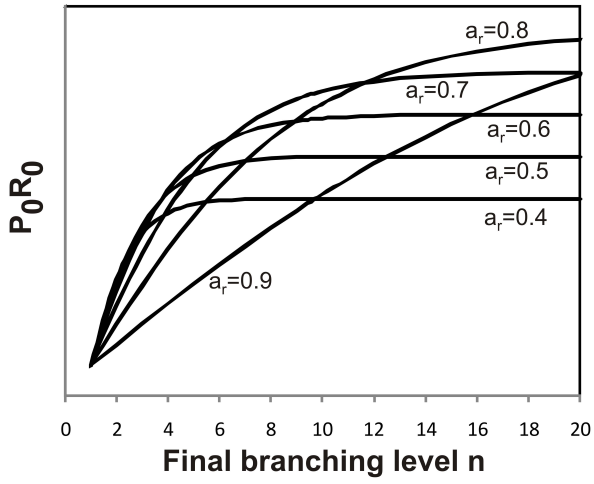

Fig. (3). Dependence of P_0R_0 on branching level n for different scale factors a_r (m=2; $b_l = 2^{-1}$).

Fig. (**2**) shows the effect of the branching level and the number of daughter branches on equilibrium capillary pressure. According to this figure, we come to the conclusion that the P_0R_0 is strongly dependent on n and m. Besides, for $n \leq 6$ the pairing, or bifurcation of ducts (m=2) generates the lowest equilibrium capillary pressure. On the other hand, for $n \geq 12$ a network with bifurcating ducts has the highest equilibrium capillary pressure if compared with higher number of daughter branches. For each m there is a certain value of n for which increasing the level of branching do not affect the equilibrium capillary pressure. This is reached first (i.e., at lower branching levels n) for higher number of daughter branches.

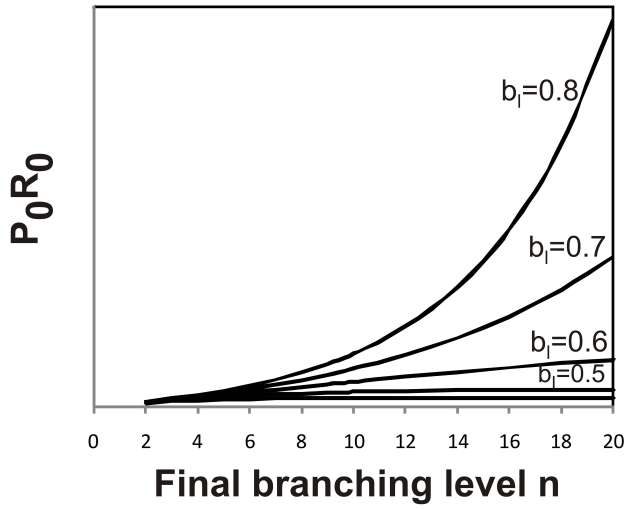

Fig. (4). Dependence of P_0R_0 on branching level n for different scale factors b_l (m=2; $a_r=2^{-1/3}$).

The effects of scale factor a_r and b_l are presented in Figs. (**3**) and (**4**). Fig. (**3**) reveals that when the scale factor a_r increases, the maximum equilibrium capillary pressure occurs at higher branching levels n. Besides, for each a_r there is a certain value of n for which increasing the level of branching do not affect the equilibrium capillary pressure. This value of n is reached first for lower values of a_r. Fig. (**4**) indicates that the equilibrium capillary pressure becomes higher at high scale factor b_l, which is an expected result since an higher scale factor provides a longer network.

At this stage, it is also possible to establish a criterion to evaluate the importance of equilibrium and non-equilibrium effects on the capillary pressure. According to Eqs. (11) and (17), the equilibrium pressure is dominant when

$$\frac{1}{R_0} > \frac{\mu}{\eta_{w,a} + \eta_{w,s}} \frac{\partial}{\partial t}\left(\frac{V_w}{V}\right) \tag{21}$$

The equilibrium capillary pressure persists in time as the main driving potential, as long as the geometry of the network obeys this criterion (Eq. 21).

CONCLUDING REMARKS

Capillary fluid flow has been the focus of many studies because of its importance in nature and in different complex flow structures. This study focused on the understanding of capillary pressure in a tree-shaped network. An approach is presented by combining the thermodynamics and the geometric description of the tree-shaped network. The non-equilibrium effects of the capillary, arising from forces acting on the fluid-solid contact line, are compared with the equilibrium effects determined by the change in interfacial areas with fluid volume. Finally, an approach is presented whereby equilibrium capillary pressure is shown to be related with the geometry of the network.

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NOMENCLATURE

A	=	Interfacial area of the fluid
a_r	=	Scale factors for radius
b_l	=	Scale factors for length
F	=	Force to move the solid-fluid interfaces
L	=	Duct length
m	=	Number of branches
N	=	Branches of ducts
n	=	Final branching level
p_n	=	Pressure of non-wetting fluid
p_w	=	Pressure of wetting fluid
R_0	=	Duct radius of the first duct (level 0)
R	=	Duct radius
S	=	Fluid entropy
T	=	Fluid temperature
U	=	Fluid internal energy
V_n	=	Volume of non-wetting fluid
V_w	=	Volume of wetting fluid
V_W/V	=	Moisture content

Greek symbols

ζ	=	Interfacial energy of the wetting fluid in contact with the non-wetting phase
α_a	=	Shape factor for area
α_v	=	Shape factor for volume
μ	=	Coefficient that accounts for the movement of fluid-solid contact lines

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