Universidade de Évora - Instituto de Investigação e Formação Avançada

Programa de Doutoramento em Matemática

Tese de Doutoramento

Generalized repunit numerical semigroups

Maria Isabel de Magalhães Colaço

Orientador(es) | Manuel Baptista Branco
Ignacio Ojeda Martinez de Castilha

Évora 2025

Universidade de Évora - Instituto de Investigação e Formação Avançada

Programa de Doutoramento em Matemática

Tese de Doutoramento

Generalized repunit numerical semigroups

Maria Isabel de Magalhães Colaço

Orientador(es) | Manuel Baptista Branco
Ignacio Ojeda Martinez de Castilha

Évora 2025

A tese de doutoramento foi objeto de apreciação e discussão pública pelo seguinte júri nomeado pelo Diretor do Instituto de Investigação e Formação Avançada:

Presidente	|	Carlos Correia Ramos (Universidade de Évora)
Vogais	|	Alberto Vigneron-Tenorio (Universidad de Cadiz) Ignacio Ojeda Martinez de Castilha (Universidad de Extremadura) (Orientador) Ilda Carla Mendes Inácio (Universidade da Beira Interior) Juan Ignacio Garcia-Garcia (Universidad de Cadiz) Pedro Macias Marques (Universidade de Évora)

Évora 2025

Abstract

This thesis is devoted to the study of a new class of numerical semigroups, which we call generalized repunit numerical semigroups. Firstly, we describe minimal systems of binomial generators for the defining ideals of all corresponding monomial curves (over a fixed field). Then, we describe the Apéry sets, relative to the multiplicity, and solve other classical problems of Numerical Semigroup Theory like the Frobenius problem or the computation of the genus, by means of closed formulas for the entire class. Generalized repunit numerical semigroups are Wilf and have other interesting properties such as being homogeneous and having Cohen-Macaulay tangent cone. Furthermore, from the structure of the toric ideal of a generalized repunit numerical semigroup, the complete Betti sequence of its coordinate ring is derived. Finally, we explicitly describe a minimal graded free resolution of this ring, for the grading given by the semigroup.

Keywords: Numerical semigroup; Frobenius problem; Monomial curve; Determinantal ideal; Minimal free resolution.

Resumo

Semigrupos numéricos repunit generalizados

Esta tese é dedicada ao estudo de uma nova classe de semigrupos, que designamos por semigrupos numéricos repunit generalizados. Em primeiro lugar, descrevemos sistemas minimais de geradores binomiais para os ideais definidores de todas as correspondentes curvas monomiais (sobre um corpo fixado). Em seguida, calculamos os seus conjuntos de Apéry, relativos à multiplicidade, e resolvemos outros problemas notáveis da Teoria dos Semigrupos Numéricos, como o problema de Frobenius ou o cálculo do género, por meio de fórmulas fechadas para a totalidade da classe. Os semigrupos numéricos repunit generalizados são Wilf e possuem outras propriedades interessantes, como serem homogéneos e possuirem cone tangente de Cohen-Macaulay. Além disso, a partir da estrutura do ideal tórico de um semigrupo numérico repunit generalizado, a sequência completa de Betti do seu anel de semigrupo é retirada. Finalmente, descrevemos explicitamente uma resolução livre minimal graduada deste anel, para a graduação dada pelo semigrupo.

Palavras-chave: Semigrupo numérico; Problema de Frobenius; Curva monomial; Ideal determinantal; Resolução livre minimal.

Contents

Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A numerical semigroup S is a subset of $\mathbb{N}$ containing zero such that S is closed under addition and $\mathbb{N} \setminus S$ is finite. The cardinality of $\mathbb{N} \setminus S$ is called the genus of S, here denoted $\mathrm{g}(S)$, and the greatest integer not in S is the Frobenius number of S, denoted $\mathrm{F}(S)$.

Given an element $s \in S\setminus\{0\}$, the Apéry set of S, relative to s, denoted $\mathrm{Ap}(S, s)$, so named after [**2**], is the subset $\{w_1, \ldots, w_s\}$ of S, $w_1 = 0 < \cdots < w_s$, such that, for $1 < i \leq s$, w_i is the least integer in S having s-residue distinct from those in $\{w_1, \ldots, w_{i-1}\}$. The set $\mathrm{Ap}(S, s)$ is thus a complete system of residues modulo s and, by adding the element s to it, we get a system of generators of S. Therefore, numerical semigroups are finitely generated (commutative) monoids. Furthermore, for a given numerical semigroup S, there exists a unique set $\{a_1, \ldots, a_n\} \subset \mathbb{N}$ such that $S = \mathbb{N}a_1 + \cdots + \mathbb{N}a_n$ and no proper subset of $\{a_1, \ldots, a_n\}$ generates S. In this case, the set $\{a_1, \ldots, a_n\}$ is the minimal system of generators of S, its cardinality is called the embedding dimension of S and $\min\{a_1, \ldots, a_n\}$ is called the multiplicity of S. The finiteness of the genus implies that $\gcd(a_1, \ldots, a_n) = 1$. In fact, one has that the necessary and sufficient condition for a subset $\mathcal{A}$ of $\mathbb{N}$ to generate a numerical semigroup is $\gcd(\mathcal{A}) = 1$.

The set $\mathrm{Ap}(S, s)$ retains much of the combinatorics of S, giving more information than an arbitrary system of generators of S. It can be used, for instance, to solve the membership problem in S, or to compute the Frobenius number and the genus of S, by means of the so called Selmer's formulas. The computation of other numerical invariants of S, like the pseudo-Frobenius numbers of S, introduced by Fröberg et al. in [**21**], or the number of these elements, called the type of S, rely on the combinatorics of $\mathrm{Ap}(S, s)$.

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a subset of $\mathbb{N}$ minimally generating a numerical semigroup. The Frobenius number and the genus of the numerical semigroup $S = \mathbb{N}a_1 + \cdots + \mathbb{N}a_n$ have natural interpretations in the study of nonnegative integer solutions of Diophantine equations of the form $a_1x_1 + \cdots + a_nx_n = b$, where $b \in \mathbb{N}$. Concretely, the Frobenius number of S is exactly the last integer b for which there are no nonnegative integer solutions of the equation and the cardinality of the set of those b for which no such solutions exist is the genus. In fact, since any nontrivial submonoid of $\mathbb{N}$ is isomorphic to a numerical semigroup, one has that the study of numerical semigroups is equivalent to the study of nonnegative integer solutions of non homogeneous Diophantine equations with positive coefficients and one can find in the early literature concerning numerical semigroups many examples where this classical problem is treated (see for example [**7**], [**8**], [**17**], [**18**] and [**29**]).

Frobenius proposed in his lectures the problem of finding a formula for $F(S)$ in terms of its minimal system $\{a_1, \ldots, a_n\}$ of S, a problem that is in the origins of numerical semigroup theory and that is since then known as the Frobenius problem. The formulas for the Frobenius number and genus of a numerical semigroup minimally generated by $\{a_1, a_2\}$ go back to the $19th$ century, (see for instance [**46**]), given by two certain (linear related) quadratic symmetric polynomials in the indeterminates a_1, a_2. In [**16**], F. Curtis proved that for numerical semigroups of embedding dimension 3 we cannot expect formulas for the Frobenius number of a certain type; particularly no polynomials formulas in terms of $\{a_1, a_2, a_3\}$ exist. Despiste the complexity of the Frobenius problem, it has been solved for certain classes of numerical semigroups, by means of closed formulas for the entire class.

In 1965, Rédei proved, in [**39**], that finitely generated commutative monoids are finitely presented, that is, a finitely generated commutative monoid S can be given by a set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of generators of S together with a finite set of relations among these generators, encoded by a set $\rho \subset \mathbb{N}^n \times \mathbb{N}^n$, referred to as a presentation of S by $\mathcal{A}$, which is a generating set for the congruence given by $\mathcal{A}$. For numerical semigroups, or more generally for finitely generated submonoids of $\mathbb{N}^d$, d a positive integer, which are the monoids of special interest in this thesis, here called positive affine semigroups, one has also that any two minimal presentations, with respect to set inclusion, have the same cardinality.

Computing a minimal presentation of S from $\mathcal{A}$ is not easy in general. The interested reader may refer to [**42**, Section 8.3] or [**11**] for a detailed development of this topic.

Let $\Bbbk$ be a field. It is well known that the computation of presentations of a finitely generated submonoid S of $\mathbb{Z}^d$, d a positive integer, can be achieved through the general theory of toric rings. Particularly, if S is the numerical semigroup generated by $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\Bbbk[S] = \Bbbk[t^{a_1}, \ldots, t^{a_n}] \subset \Bbbk[t]$ is its numerical semigroup ring then, the kernel of the $\Bbbk-$algebra surjective homomorphism

$$\Bbbk[x_1, \ldots, x_n] \to \Bbbk[S];\ x_i \mapsto t^{a_i},\ i = 1, \ldots, n,$$

denoted $I_{\mathcal{A}}$, is an homogeneous ideal for the grading given in $\Bbbk[x_1, \ldots, x_n]$ by $\mathcal{A}$. One has that $I_{\mathcal{A}}$ is a prime binomial ideal of the polynomial ring $\Bbbk[x_1, \ldots, x_n]$, and (minimal) presentations of S translate into (minimal) systems of binomial generators of $I_{\mathcal{A}}$, by a correspondence which is known as the Herzog correspondence after the celebrated paper [**26**]. In particular, the existence of a finite presentation of S may also be seen as a consequence of Hilbert's basis theorem.

If S is the numerical semigroup generated by $\mathcal{A} = \{a_1, \ldots, a_n\}$, then the toric ideal $I_{\mathcal{A}}$ is the defining ideal of the affine curve parametrized by $t \mapsto (t^{a_1}, \ldots, t^{a_n})$, $t \in \Bbbk$, which is called a monomial curve, and $\Bbbk[S]$ is (up to isomorphism) its coordinate ring. The theory of (non-trivial) numerical semigroups is essentially the theory of these (non-normal and hence singular) irreducible curves and common invariants of a numerical semigroup S, like the embedding dimension, multiplicity, genus (also called the degree of singularity in this setting) or the conductor are so named due to this bridge from Commutative Algebra and Algebraic Geometry to Numerical Semigroup Theory. It is also known that the Cohen-Macaulay type of $\Bbbk[S]$, that is, the rank of the last module in a minimal free resolution of this ring, coincides with the type of S, in the sense of Fröberg et al.

Several methods have been developed for computing minimal systems of binomial generators of $I_{\mathcal{A}}$ (see, for instance, [**6**] and [**36**]). We note that, for monomials curves in 4-dimensional affine space, we have the remarkable phenomena that the so-called arithmetic binomial ranks (see [**30**]) can be arbitrary large, firstly provided by the examples of Bresinsky and Hoa. In fact, this is true for embedding dimensions larger than three, as proved by Bresinsky in [**9**].

The study of the defining ideal of monomials curves is a long-established research topic since J. Herzog, in [**26**], characterized the minimal systems of binomial generators of (all) the monomial curves in affine three-dimensional space. The elegance of Herzog's result for the three-dimensional case contrasts with the fact that no explicit description is known for the general case. Particular advances are just known for low-dimensional cases (see, e.g., [**10**] or more recently in [**31**] and the references therein) or for special families of monomial curves like the one corresponding to the numerical semigroups that will be presented in this thesis; due to its proximity to our work, we highlight the article by D.P. Patil [**37**].

A positive affine semigroup S, not even a numerical semigroup, need not have a unique minimal presentation. If S has a unique minimal presentation then, S is said uniquely presented (see for instance [**22**], where the uniqueness of the minimal presentation is discussed from the semigroup point of view).

Despite not being the aim of this thesis, we emphasize that the study of the defining ideal of monomials curves have its own interest for applications to other areas such as linear programming (see, e.g., [**48**]), coding theory (see, e.g., [**33**] or algebraic statistics, where the minimal systems of binomial generators are called Markov bases and the uniqueness property has special consideration (see [**1**]).

The study of free resolutions of numerical semigroup rings has also gained a considerable interesest for researchers since the classification of monomial curves in affine three-dimensional space was achieved by Herzog. In this embedding dimension, the structure of the minimal resolution is clear: for symmetric numerical semigroups the structure is Koszul and non-symmetric numerical semigroups have determinantal structure. For embedding dimension $n > 3$, we have that the type of monomials curves is unbounded, a result firstly provided by Backelin for $n = 4$. For some particular classes of numerical semigroup rings of embedding dimension 4, the whole resolution has been constructed (see, for instance, [**44**]). Results with no restriction on the embedding dimension also exist, P. Gimenez at al explicitly construct the graded minimal free resolutions for monomial curves defined by arithmetic sequences (see [**24**]).

The reader interested in delving deeper into Numerical Semigroups can consult [**42**] or [**3**], for Ideals and Gröbner Bases in general [**15**], and for Binomial Ideals and Semigroup Algebras [**35**, Chapter 7] or [**13**, Chapter 6].

This thesis is organized as follows: the first chapter of this thesis provides the preliminaries on numerical semigroups and their semigroup algebras, needed for a full comprehension of the subsequent chapters 2, 3 and 4, where the detailed results on the family of generalized repunit numerical semigroups are given.

A repunit number is an integer whose representation in a base b consists of copies of the single digit 1. Throughout this thesis, we will write $r_b(\ell)$ for the $\ell-$th repunit number in base b, that is,

$$r_b(\ell) = \sum_{j=0}^{\ell-1} b^j$$

and, for completeness, we set $r_b(0) = 0$.

Given two positive integers $n > 1$ and a, consider the submonoid of generated by $\{a_1, a_2, \ldots\}$, where

$$a_i = r_b(n) + a r_b(i-1), \text{ for every } i \geq 1.$$

Clearly, the condition $\gcd(a_1, a) = 1$ is necessary and sufficient for such a monoid to be a numerical semigroup (see Chapter 3 of this thesis for a detailed proof of this fact). In this case, it is called a generalized repunit numerical semigroup as it generalizes the repunit numerical semigroups introduced by D. Torrão et al. (see [**40, 41**]) and it is denoted $S_a(b, n)$. Furthermore, as proved in Chapter 3, $S_a(b, n)$ is minimally generated by $\{a_1, \ldots, a_n\}$, so it is of embedding dimension n.

In what follows, $\mathcal{A} = \{a_1, \ldots, a_n\}$ denotes the minimal system of the generalized repunit numerical semigroup $S_a(b, n)$.

In Chapter 2, by using Gröbner basis techniques, we prove that $I_{\mathcal{A}}$ is minimally generated by the 2×2 minors of the matrix

$$X := \begin{pmatrix} x_1^b & \dots & x_{n-1}^b & x_n^b \\ x_2 & \dots & x_n & x_1^{a+1} \end{pmatrix}.$$

This is one of the main results of this chapter. As an immediate consequence, we have that generalized numerical semigroups are homogeneous in the sense of [**27**]. In fact, as we point out in Chapter 4, they are of homogeneous type.

In Chapter 2, we also obtain that the $2 \times 2-$minors of X form a minimal Gröbner basis with respect to a family of $\mathcal{A}$-graded reverse lexicographical term orders on $\Bbbk[x_1, \dots, x_n]$ and conclude that for $n > 3$, the ideal $I_{\mathcal{A}}$ has a unique minimal system of generators if and only if $a < b - 1$.

We note that the over-mentioned results are also valid for $b = 1$. In this case, $a_i = n + (i-1)\,a,\ i \geq 1$, is an arithmetic sequence that generates a MED semigroup, provided that $\gcd(n, a) = 1$.

From a classical point of view, where much investigation on numerical semigroups is centered on characterizing families of semigroups via the properties they fulfill, we can say that generalized repunit numerical semigroups, in which we also include the case $b = 1$, can be characterized as those semigroups having determinantal toric ideals according to the above description. In fact, as we will see in further results, much of the properties of this class of numerical semigroups rely on that structure. Following this line, see [**47**], where the relation of $I_{\mathcal{A}}$ having determinantal structure and the behavior of Pseudo-Frobenius numbers of S is analyzed.

In Chapter 3, by using the minimal system of binomial generators of $I_{\mathcal{A}}$ described in Chapter 2 and partially thanks to a result of Gastinger, pointed out in [**23**], we explicitly compute the Apéry set of $S_a(b, n)$, that is, the Apéry set of $S_a(b, n)$ with respect to the multiplicity a_1, a result that highlighs the internal structure of generalized repunit numerical semigroups as the Apéry set of $S_a(b, i)$ can be obtained from the Apéry set of $S_a(b, i-1)$, for every $i \geq 3$, for instance.

Then, we solve the Frobenius Problem and compute the genus of a generalized repunit numerical semigroup, by means of closed formulas in terms of n, b and a. These formulas are the main results of this chapter.

Finally, in Chapter 3, we compute the whole set of pseudo-Frobenius numbers of $S_a(b,n)$ obtaining that its cardinality is $n-1$. As an immediate consequence of this result we have that generalized repunit numerical semigroups gives another family of numerical semigroups satisfying Wilf conjecture, one of most notable open problems of Numerical Semigroup Theory, enunciated in [**49**], declaring the (multiplicative) inverse of the embedding dimension of a numerical semigroup S as the greatest lower bound for the probability of finding an element of S up to $\mathrm{F}(S)$. Although there are some families of numerical semigroups for which this conjecture is known to be true, the general case remains unsolved. A very good source for the state of the art of this problem is [**19**].

Denote simply by S the generalized repunit numerical semigroup $S_a(b,n)$. In Chapter 4, the ring $\Bbbk[S]$ is referred to as a generalized repunit $\Bbbk$-algebra.

Since $I_{\mathcal{A}}$ is a determinantal ideal, the generalized repunit $\Bbbk-$algebra $\Bbbk[S]$ can be (minimally) resolved by the Eagon-Northcott complex introduced in [**20**]. The main result of Chapter 4 is the explicit description of a minimal $S-$graded free of resolution of $\Bbbk[S]$, basically in terms of $\mathcal{A}$.

CHAPTER 1

Preliminaries and theoretical overview

In this chapter, we will present some basic definitions and known results that are necessary for a better understanding of the meaning and coherence of the articles compiled in this thesis. Some more specific definitions and known results will be presented in the body of this thesis.

The proofs of the results in the first section can be consulted in [**42**]. In the section 2 we mainly used [**34**], while in the section 3 we followed the preliminary section of [**28**].

1. Numerical semigroups and their most notable elements

From now on and, along the whole thesis, $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ the set of nonnegative integers.

A **binary operation** on a non-empty set S is a map $+ : S \times S \to S$. The image of $(x, y) \in S\times S$ is usually denoted by $x+y$. A **semigroup** is a non-empty set, S, equipped with a binary operation that verifies the associative law, that is, $(x+y)+z = x+(y+z)$, for all $x, y, z \in S$. Usually we also omit the binary operation $+$ when referring to a semigroup and write S as $(S, +)$ instead. A **monoid** is a semigroup with an **neutral element**, that is, there exists an element $e \in S$ satisfying $x + e = e + x = x$ for all $x \in S$. The neutral element, if it exists, is unique.

A **submonoid** of a monoid $(S, +)$ is a subset of S that is a monoid for the binary operation $+$. Equivalently, a submonoid of a monoid $(S, +)$ is a subset of S that contains the neutral element and is closed under the binary operation $+$. It is clear that S and $\{e\}$ are submonoids of S, called the trivial submonoids of S. Moreover, one can easily see that the intersection of any family of submonoids of S is also a submonoid of S.

Given a subset $\mathcal{A}$ of a monoid S, the **monoid generated** by $\mathcal{A}$, denoted by $\langle\mathcal{A}\rangle$, is the least (with respect to set inclusion) submonoid of S containing A, which turns out to be the intersection of all submonoids of S containing $\mathcal{A}$. Note that, by definition, if

$\mathcal{A} = \varnothing$ then $\langle \mathcal{A} \rangle = \{e\}$, the proper trivial submonoid of S. If $\mathcal{A}$ is a non-empty set, we say that $\mathcal{A}$ is a system of generators of S if $\langle \mathcal{A} \rangle = S$. In this case, we also say that S is **generated by** $\mathcal{A}$. Furthermore, we have that $\mathcal{A}$ is a **minimal system of generators** of S if no proper subset of $\mathcal{A}$ generates S. A monoid S is a **finitely generated monoid** if there exists a system of generators of S with finitely many elements.

Given two monoids $(S_1, +_1)$ and $(S_2, +_2)$, a map $f : S_1 \to S_2$ is a **monoid homomorphism** if $f(x +_1 y) = f(x) +_2 f(y)$ for all $x, y \in S_1$ and $f(e_1) = e_2$, where e_i is the neutral element of S_i, $i = 1, 2$. We say that f is a monoid isomorphism if it is bijective.

A **commutative monoid** is a monoid $(S, +)$ such that the binary operation $+$ verifies the commutative law, that is, $x + y = y + x$, for all $x, y \in S$. Given a non-empty subset $\mathcal{A}$ of a commutative monoid S, it is clear that

$$\langle \mathcal{A} \rangle = \left\{ \sum_{i=1}^{n} u_i\, a_i \mid n \in \mathbb{N} \setminus \{0\}\,,\ a_i \in \mathcal{A},\ u_i \in \mathbb{N} \text{ and } i \in \{1, \ldots, n\} \right\},$$

where, for each $a \in S$ and $u \in \mathbb{N}$, we define $0a = e$ and $ua = (u-1)a + a$.

A finitely generated commutative monoid $(S, +)$ is **cancellative** if $x + z = y + z$ implies $x = y$, for every x, y and $z \in S$; equivalently, if it is isomorphic to a submonoid of a finitely generated commutative group. Moreover, if the group is torsion-free, then S is said to be **torsion-free**, too.

All the monoids considered in this thesis are finitely generated submonoids of $\mathbb{Z}^d$, for some integer $d \geq 1$. Therefore, they are finitely generated, commutative, cancellative and torsion free. A monoid that is isomorphic to a finitely generated submonoid of $\mathbb{Z}^d$, for some integer $d \geq 1$, is usually called an **affine semigroup**.

Our main objects of study are numerical semigroups.

Definition 1. A **numerical semigroup** S is a submonoid of $\mathbb{N}$, with usual sum of natural numbers, such that $\mathbb{N} \setminus S$ is finite.

A numerical semigroup other than $\mathbb{N}$ is called a non-trivial numerical semigroup.

An alternative way to define a numerical semigroup is given by the following result.

Proposition 2. *Let S be a submonoid of* $(\mathbb{N}, +)$*. Then S is numerical semigroup if and only if* $\gcd(S) = 1$.

PROOF. See, e.g., [**42**, Lema 2.1]. □

Numerical semigroups classify the infinite submonoids of $\mathbb{N}$.

PROPOSITION 3. *Every non-trivial submonoid of* $(\mathbb{N}, +)$ *is isomorphic to a numerical semigroup.*

PROOF. See, e.g., [**42**, Proposition 2.2]. □

Given two subsets A and B of commutative monoid S, we write $A+B$ for the subset $\{x+y \mid x \in A, y \in B\}$ of S. Given a submonoid S of $\mathbb{N}^d$ and $S^* = S \setminus \{\mathbf{0}\}$ we have that the set $S^* + S^*$ is the subset of S of those elements which are the sum of two non-zero elements in S.

LEMMA 4. *Let* S *is a submonoid of* $\mathbb{N}^d$*, with the coordinate-wise sum. Then* $S^* \setminus (S^* + S^*)$ *is system of generators of* S*. Furthermore, every system of generators of* S *contains* $S^* \setminus (S^* + S^*)$.

PROOF. See [**42**, Lemma 2.3] and the comment below it. □

Note that, by lemma 4, we have that every submonoid S of $(\mathbb{N}^d, +)$ has a unique minimal system of generators, denoted $\mathrm{msg}(S)$. That is,

$$\mathrm{msg}(S) = S^* \setminus (S^* + S^*).$$

Let us see that if $d = 1$, then $\mathrm{msg}(S)$ is finite. But first, we introduce what is probably the most versatile tool in numerical semigroup theory.

DEFINITION 5. Let S be a numerical semigroup. The **Apéry set of S with respect to an element** $s \in S^*$ (so named in honor of [**2**]) is

$$\mathrm{Ap}(S, s) := \{w \in S \mid w - s \notin S\}.$$

Apéry sets can be computed efficiently, as shown for example in [**32**] (see also Proposition 23).

LEMMA 6. *Let* S *be a numerical semigroup and* $s \in S^*$*. Then* $\mathrm{Ap}(S, s)$ *is a complete system of residues modulo* a*. Concretely,* $\mathrm{Ap}(S, s) = \{0 = w(0), w(1), \ldots, w(s-1)\}$*, where* $w(i)$ *is the least element in* S^* *congruent with* i *modulo* s*, for all* $i \in \{0, \ldots, s-1\}$.

PROOF. See, e.g., [**42**, Lemma 2.4]. □

REMARK 7. Knowing the set $\mathrm{Ap}(S,s)$ allows us to solve the membership problem in S. From the previous result, an integer n belongs to S if and only if $n \geq w(n \bmod s)$, where $n \bmod s$ stands for the least nonnegative residue of n, modulo s.

The next result follows directly from the characterization of an Apéry set given in Lemma 6.

LEMMA 8. *Let S be a numerical semigroup and $s \in S^*$. Then for each $n \in S$ there exists a unique $(k,w) \in \mathbb{N} \times \mathrm{Ap}(S,s)$ such that $n = ks + w$.*

PROOF. See, e.g., [**42**, Lemma 2.6]. □

By Lemma 8 we conclude that $S = \langle \mathrm{Ap}(S,s) \cup \{s\} \rangle$. So together with Lemma 4 we have the following result.

THEOREM 9. *Every numerical semigroup admits a unique minimal system of generators. Moreover, this minimal system of generators is finite.*

The cardinality of the unique minimal system of generators of a numerical semigroup $S, \mathrm{msg}(S)$, is called the **embedding dimension** of S, denoted by $\mathrm{e}(S)$.

Obviously, the (unique) minimal system of generators of a finitely generated submonoid of $\mathbb{Z}^d$ is finite. So, for affine semigroups, the term embedding dimension also applies analogously.

REMARK 10. The finiteness of the cardinality of the minimal system of generators of a numerical semigroup cannot be extended to the submonoids of $\mathbb{Z}^d$. For example, the submonoid of $\mathbb{N}^2 \subset \mathbb{Z}^2$ (minimally) generated by $\{(1,n) \mid n \in \mathbb{N}\} \cup \{(n,1) \mid n \in \mathbb{N}\}$ is not finitely generated.

Despite the above, using Proposition 3 and Theorem 9, we get the next result.

COROLLARY 11. *Every submonoid of $(\mathbb{N},+)$ has a unique minimal system of generators, which in addition is finite.*

Let us now introduce some other notable elements of the theory of numerical semigroups.

Let S be a numerical semigroup. The smallest element of S^* is called the **multiplicity** of S and is denoted by $\mathrm{m}(S)$. It is clear that $\mathrm{m}(S)$ is not in $S^* + S^*$.

REMARK 12. Let S be a numerical semigroup and denote by $\mathrm{Ap}(S)$ the Apéry set of S, relative to its multiplicity $\mathrm{m}(S)$. Clearly, $\mathrm{msg}(S) \setminus \{\mathrm{m}(S)\}$ is a subset of $\mathrm{Ap}(S) \setminus \{0\}$, and since $\mathrm{Ap}(S)$ has cardinality $\mathrm{m}(S)$, we have that $\mathrm{m}(S)$ is an upper bound for the embedding dimension of S, that is, $\mathrm{e}(S) \leq \mathrm{m}(S)$.

Considering Proposition 2, it makes sense to talk about the largest element not in S. This element is usually known as the **Frobenius number** of S and is denoted here by $\mathrm{F}(S)$. The Frobenius problem is the search for closed formulas for the Frobenius number of a numerical semigroup in terms of its minimal system of generators (see [**38**] for more details).

The elements in $\mathbb{N}\setminus S$ are called the **gaps** of S, and the cardinality of the set of gaps of S is called the **genus** of S, denoted by $\mathrm{g}(S)$, which is sometimes referred to as the gender or degree of singularity of S.

EXAMPLE 13. Let $n > 1$ be an integer. Consider the subsemigroup of $(\mathbb{N}, +)$ given by $S(n) = \{m \in \mathbb{N} \mid m \geq n\}$. We have that $S = S(n) \cup \{0\}$ is a numerical semigroup. Clearly, $\mathrm{m}(S) = n$, $\mathrm{g}(S) = n - 1$ and $\mathrm{F}(S) = n - 1 = \mathrm{m}(S) - 1$.

The numerical semigroups shown in the example above are classical. In the literature they are commonly called **half-line** semigroups (or superficial numerical semigroups).

A **maximal embedding numerical semigroup** or MED numerical semigroup for short, is a numerical semigroup S such that $\mathrm{e}(S) = \mathrm{m}(S)$ (see remark 12). The half-line $S = S(n) \cup \{0\}$ is MED, since $\mathrm{msg}(S) = \{n, \ldots, 2n - 1\}$. More generally, we have,

EXAMPLE 14. Let $n > 1$ be an integer and a a positive integer relatively prime with n. The submonoid of $\mathbb{N}$ (minimally) generated by $\mathcal{A} = \{a_1, \ldots, a_n\}$, where (a_i) is the arithmetic sequence given by $a_i = n + (i - 1)a$, $i \geq 1$, is MED.

Selmer's formulas ([**45**]), summarized in the next result, relate Apéry sets to the Frobenius number and the genus of S.

THEOREM 15. *Let S be a numerical semigroup and let $s \in S^*$. Then:*

(1) $\mathrm{F}(S) = \max(\mathrm{Ap}(S,s)) - s$;
(2) $\mathrm{g}(S) = \frac{1}{s}\left(\sum_{w \in \mathrm{Ap}(S,s)} w\right) - \frac{s-1}{2}$.

PROOF. See, e.g. [**42**, Proposition 2.12]. ☐

Let S be a numerical semigroup. Following the terminology introduced in [**43**], we say that the an integer x is a **pseudo-Frobenius number** of S if $x \in \mathbb{Z} \setminus S$ and $x + s \in S$ for all $s \in S^*$. We denote by $\mathrm{PF}(S)$ the set of pseudo-Frobenius numbers of S and its cardinality is the **type** of S, denoted by $\mathrm{t}(S)$. Clearly, $\mathrm{F}(S) \in \mathrm{PF}(S)$.

The next result gives us a characterization of the set of pseudo-Frobenius numbers in terms of the Apéry sets.

PROPOSITION 16. *Let S be a numerical semigroup and let x be an element in S^*. Then*

$$\mathrm{PF}(S) = \{w - x \mid w \in \mathrm{maximals}_{\leq_S}(\mathrm{Ap}(S,x))\},$$

where $\leq_S$ is the partial order on $\mathbb{Z}$ such that $a \leq_S b$ if and only if $b - a \in S$.

PROOF. See, e.g. [**42**, Proposition 2.20]. ☐

The partial order shown in the result above is explored again in Proposition 25, with a little more generality.

From Proposition 16 and by an argument much similar to the one made in remark 12, we obtain an upper bound for the type of a numerical semigroup, we have that $\mathrm{t}(S) \leq \mathrm{m}(S) - 1$. Notice MED numerical semigroups have maximal type, that is, if S is MED then, $\mathrm{m}(S) = \mathrm{t}(S) + 1$, since, in this case, $\mathrm{Ap}(S) \setminus \{0\}$ consist of $\mathrm{m}(S) - 1$ incomparable elements with respect to $\leq_S$. In fact, the condition $\mathrm{m}(S) = \mathrm{t}(S) + 1$, caracterizes MED semigroups (see, for instance, Proposition 11 in [**3**]).

The element $\mathrm{c}(S) = \mathrm{F}(S) + 1$, called the **conductor** of S, is the smallest element $c \in S$ with the property that $x \geq c \Rightarrow x \in S$, for x any nonnegative integer. A **non-gap** of S is an element of S in $\{x \in \mathbb{N} \mid 0 \leq x \leq \mathrm{F}(S)\}$, that is, an element of $\{s \in S \mid s < \mathrm{F}(S)\}$. The cardinality of the set of non-gaps is denoted $\mathrm{n}(S)$. It is clear that $\mathrm{n}(S) + \mathrm{g}(S) = \mathrm{c}(S)$. In particular, $\mathrm{c}(S)$ is an upper bound for both the cardinalities of $g(S)$ and $\mathrm{n}(S)$.

In [**49**] Wilf conjectured that $\mathrm{c}(S) \leq \mathrm{e}(S)\,\mathrm{n}(S)$, for any numerical semigroup S. This question is still largely open and is one of the most important problems in numerical semigroup theory. The numerical semigroups that satisfy Wilf's condition are said to be Wilf. By [**21**, Theorem 20], a numerical semigroup S satisfying $\mathrm{t}(S) \leq \mathrm{e}(S) - 1$ is Wilf. Although the Wilf conjecture is known to be true for some families of numerical semigroups, the general case remains unsolved. A very good source for the state of the art of this problem is [**19**].

2. Presentations and minimal systems of binomial generators

Definition 17. Let S be a commutative monoid. A **congruence** $\sim$ on S is an equivalence relation on S that is additively closed, that is, $\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$ for $\mathbf{a}, \mathbf{b}$ and $\mathbf{c} \in S$.

Let $\phi : S \to S'$ be a monoid homomorphism. The **kernel of** ϕ is defined as

$$\ker(\phi) := \big\{(\mathbf{a}, \mathbf{b}) \in S \times S \mid \phi(\mathbf{a}) = \phi(\mathbf{b})\big\}.$$

In this case, the relation on S determined by $\ker(\phi) \subset S \times S$ is a congruence.

If S is the commutative monoid finitely generated by $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, then the monoid homomorphism determined by $\mathcal{A}$

$$\deg_{\mathcal{A}} : \mathbb{N}^n \longrightarrow S;\ \mathbf{e}_i \longmapsto \mathbf{a}_i,\ i = 1, \ldots, n, \tag{1}$$

where $\mathbf{e}_i$ denotes the element in $\mathbb{N}^n$ whose $i-$th coordinate is 1 with all other coordinates 0, is surjective and gives a **presentation of** S, namely,

$$S = \mathbb{N}^n / \sim_{\mathcal{A}},$$

where $\sim_{\mathcal{A}} = \ker(\deg_{\mathcal{A}})$. In the literature, this $\deg_{\mathcal{A}}$ is called the **factorization map associated to** $\mathcal{A}$ and accordingly, the fiber $\deg_{\mathcal{A}}^{-1}(\mathbf{a})$ is called the **set of factorizations** of $\mathbf{a} \in S$ with respect to $\mathcal{A}$.

Let $\Bbbk$ be a field. Given a commutative and finitely generated monoid S, the **semigroup algebra of** S is the direct sum

$$\Bbbk[S] := \bigoplus_{\mathbf{a} \in S} \mathrm{Span}_{\Bbbk}\{\chi^{\mathbf{a}}\}$$

with multiplication $\chi^{\mathbf{a}} \cdot \chi^{\mathbf{b}} = \chi^{\mathbf{a}+\mathbf{b}}$. If $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a generating set of S, then the factorization map $\deg_{\mathcal{A}}$ induces a map of semigroup algebras

$$\widehat{\deg_{\mathcal{A}}} : \Bbbk[x_1, \ldots, x_n] \to \Bbbk[S] \; ; \quad \mathbf{x}^{\mathbf{u}} \mapsto \chi^{\deg(\mathbf{u})}. \tag{2}$$

Observe that $\Bbbk[x_1, \ldots, x_n] = \Bbbk[\mathbb{N}^n]$.

Theorem 18. *With the above notation, if*

$$I_{\mathcal{A}} := \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \;\mid\; \deg_{\mathcal{A}}(\mathbf{u}) = \deg_{\mathcal{A}}(\mathbf{v}) \rangle \subseteq \Bbbk[x_1, \ldots, x_n], \tag{3}$$

then $\ker(\widehat{\deg_{\mathcal{A}}}) = I_{\mathcal{A}}$, *so that* $\Bbbk[S] \cong \Bbbk[x_1, \ldots, x_n]/I_{\mathcal{A}}$. *Moreover,* $I_{\mathcal{A}}$ *is spanned as a* $\Bbbk-$*vector space by* $\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \deg_{\mathcal{A}}(\mathbf{u}) = \deg_{\mathcal{A}}(\mathbf{v})\}$.

Proof. See, e.g. [**34**, Theorem 13]. □

As a consequence of the above theorem and the noetherianity of $\Bbbk[x_1, \ldots, x_n]$, it follows that every finitely generated commutative monoid is finitely presented (see e.g. [**34**, Theorem 11] for more details).

A **minimal presentation** of a commutative and finitely generated monoid S is a presentation of S which is minimal for the inclusion relation, that is, a presentation of S that does not properly contain a presentation of S. Observe that, by Theorem 18, minimal presentations of S are in one-to-one correspondence with minimal systems of binomial generators of $I_{\mathcal{A}}$, for given $\mathcal{A}$.

In general, S does not have unique presentation (uniquely presented finitely generated commutative monoids are studied in [**22**]). Despite of this fact, we have the following result.

In the next section we will delve into finitely generated positive commutative monoids.

Proposition 19. *Let S be a finitely generated submonoid of $\mathbb{Z}^d$. If S is positive, i.e., $S \cap (-S) = \{0\}$, then all minimal presentations of S have the same cardinality.*

Proof. See, e.g. [**42**, Corollary 8.13] or [**11**, Section 1]. □

From now on, S denotes a finitely generated submonoid of $\mathbb{Z}^d$. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ be the minimal system of generators of S, that is, $\mathcal{A} = \operatorname{msg}(S)$. In this case, $\deg_{\mathcal{A}}$

is naturally extended to a group homomorphism

$$\mathbb{Z}^n \longrightarrow \mathbb{Z}^d;\ \mathbf{e}_i \longmapsto \mathbf{a}_i,\ i = 1, \ldots, n,$$

so that, $\ker(\deg_{\mathcal{A}})$ can be seen as a subgroup (sublattice) of $\mathbb{Z}^n$. Therefore, the **dimension of** S, denoted $\dim(S)$, is well-defined as $n - \operatorname{rank}(\ker(\deg_{\mathcal{A}}))$.

PROPOSITION 20. *With the above notation, the dimension of* S *is equal to the Krull dimension of* $\Bbbk[S]$.

PROOF. See, e.g., [**35**, Proposition 7.5]. □

Observe that numerical semigroups are (isomorphic to) the submonoids of $\mathbb{N}$ of dimension one (Proposition 3). In fact, in this case, $\Bbbk[S]$ is the coordinate ring of a monomial curve in the affine space $A^n_{\Bbbk}$, where n is the cardinality of the minimal system of generators of S.

Computing a minimal presentation of S from $\mathcal{A}$ is not easy in general. The interested reader may refer to [**42**, Section 8.3] or [**11**] for a detailed development of this topic. In this section, we only briefly detail the methods used in the following chapter which essentially consist of determining whether a given subset of relations of $\ker(\deg_{\mathcal{A}}))$ (equivalently, a subset of binomials of $I_{\mathcal{A}}$) generates the entire presentation (equivalently, generates $I_{\mathcal{A}}$).

Given $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbf{Z}^n$, we write $\mathbf{u}^+$ for the element of $\mathbf{Z}^n$ whose $i-$th coordinate is u_i if $u_i \geq 0$ and 0 otherwise, and we write $\mathbf{u}^-$ for $\mathbf{u}^+ - \mathbf{u}$. Also, for a given vetor $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbf{N}^n$, the monomial $x_1^{u_1} \ldots x_n^{u_n} \in \Bbbk[x_1, \ldots, x_n]$ is denoted $\mathbf{x}^{\mathbf{u}}$.

PROPOSITION 21. *Let* $\mathcal{B}$ *be a system of generators of* $\ker(\deg_{\mathcal{A}}) \subseteq \mathbb{Z}^n$*, let* $I_{\mathcal{B}}$ *be the ideal of* $\Bbbk[x_1, \ldots, x_n]$ *generated by* $\{\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \mid \mathbf{u} \in \mathcal{B}\}$ *and let* J *be a subideal of* $I_{\mathcal{A}}$ *containing* $I_{\mathcal{B}}$*. Then,* $J = I_{\mathcal{A}}$ *if and only if* x_i *is a non-zerodivisor of* $\Bbbk[x_1, \ldots, x_n]/J$*, for every* $i \in \{1, \ldots, n\}$.

PROOF. See, e.g., [**34**, Proposition 38]. □

Recall that, for an arbitrary ideal I of $\Bbbk[x_1, \ldots, x_n]$ and $f \in \Bbbk[x_1, \ldots, x_n]$, the **quotient of** I **by** f is defined as

$$(I : f^\infty) := \{g \in \Bbbk[x_1, \ldots, x_n] \mid gf^s \in I,\ \text{for some } s \in \mathbb{N}\}.$$

Observe that f is a non-zerodivisor in $\mathbb{k}[x_1, \ldots, x_n]/I$ if and only if $(I : f^\infty) = I$. Therefore, by the following theorem due to Bigatti et al. ([**6**]), given an ideal J containing $I_\mathcal{B}$ one can indeed check the criterion given by Proposition 21.

Let $\prec_i$ be the term order on $\mathbb{k}[x_1, \ldots, x_n]$ defined by the following matrix

$$M := \left(\begin{array}{ccc|c|ccc} a_1 & \ldots & a_i & a_{i+1} & a_{i+2} & \ldots & a_n \\ \hline 0 & & -1 & 0 & 0 & \ldots & 0 \\ & \iddots & & \vdots & \vdots & & \vdots \\ -1 & & 0 & 0 & 0 & \ldots & 0 \\ \hline 0 & \ldots & 0 & 0 & 0 & & -1 \\ \vdots & & \vdots & \vdots & & \iddots & \\ 0 & & 0 & 0 & -1 & & 0 \end{array}\right).$$

We observe that $\prec_i$ is the $\mathcal{A}-$graded reverse lexicographical term order on $\mathbb{k}[x_1, \ldots, x_n]$ induced by $x_i \prec_i x_{i-1} \prec_i \ldots \prec_i x_1 \prec_i x_n \prec_i \ldots \prec_i x_{i+1}$; in particular, x_i is the smallest variable for $\prec_i$.

Theorem 22. *Let J be a subideal of $I_\mathcal{A}$. If $\{x_i^{\alpha_1} f_1, \ldots, x_i^{\alpha_r} f_r\}$ is the Gröbner basis of J with respect to $\prec_i$, then $\{f_1, \ldots, f_r\}$ is the Gröbner basis of $(J : x_i^\infty)$ with respect to $\prec_i$.*

Proof. See [**6**, Theorem 3.1]. □

Alternatively, if S is a numerical semigroup, the following result has as a consequence another effective way to implement Proposition 21 originally due to Gastinger ([**23**]).

Proposition 23. *With the above notation, the map*

$$\{\mathbf{x}^\mathbf{u} \in \mathbb{k}[x_1, \ldots, x_n] \mid \mathbf{x}^\mathbf{u} \notin \mathrm{in}_{\prec_i}(I_\mathcal{A} + \langle x_i \rangle)\} \longrightarrow \mathrm{Ap}(S, a_i); \quad \mathbf{x}^\mathbf{u} \longmapsto \deg_\mathcal{A}(\mathbf{u}),$$

is a bijection for every $i \in \{1, \ldots, n\}$.

Proof. See, e.g. [**28**, Proposition 3.1]. □

Now, since the $\{\mathbf{x}^\mathbf{u} \in \mathbb{k}[x_1, \ldots, x_n] \mid \mathbf{x}^\mathbf{u} \notin \mathrm{in}_{\prec_i}(J + \langle x_i \rangle)\}$ is basis of $\mathbb{k}[x_1, \ldots, x_n]/(J + \langle x_i \rangle)$ as $\mathbb{k}-$vector space, the next result is an immediate consequence of Proposition 23, COROLLARY 24 (Gastinger, 1990). *Let $J \subseteq I_{\mathcal{A}}$ be a subideal. The necessary and sufficient condition for $J = I_{\mathcal{A}}$ is that* $\dim_{\Bbbk}(\Bbbk[x_1,\ldots,x_n]/(J+\langle x_i\rangle)) = a_i$ *for some (any, in fact) $i \in \{1,\ldots,n\}$.*

3. Minimal free resolutions of semigroup algebras

Let S be the submonoid of $\mathbb{Z}^d$ generated by $\mathcal{A} = \{\mathbf{a}_1,\ldots,\mathbf{a}_n\}$. By Theorem 18, we have that $I_{\mathcal{A}} = \ker(\widehat{\deg_{\mathcal{A}}})$ is spanned as a $\Bbbk-$vector space by the set of binomials

$$\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u},\mathbf{v} \in \mathbb{N}^n \text{ with } \deg_{\mathcal{A}}(\mathbf{u}) = \deg_{\mathcal{A}}(\mathbf{v})\}. \tag{4}$$

Observe that $\Bbbk[x_1,\ldots,x_n]$ is $S-$graded via $\deg(x_i) = \mathbf{a}_i,\ i = 1,\ldots,n$. This grading is known as the $\mathcal{A}-$**grading** on $\Bbbk[x_1,\ldots,x_n]$. The semigroup algebra $\Bbbk[S] = \oplus_{\mathbf{a}\in S}\operatorname{Span}_{\Bbbk}\{\chi^{\mathbf{a}}\}$ also has a natural $S-$grading. Under these gradings, the map of semigroup algebras $\widehat{\deg_{\mathcal{A}}}$ is a graded map. Hence, the ideal $I_{\mathcal{A}} = \ker(\widehat{\deg_{\mathcal{A}}})$ is $S-$homogeneous.

The following result is [**34**, Proposition 29].

PROPOSITION 25. *With the above notation, if $\mathbf{a}_1,\ldots,\mathbf{a}_n$ are non-zero, then following are equivalent:*

(1) *The fibers of map* $\deg_{\mathcal{A}}(-)$ *are finite.*
(2) $\deg_{\mathcal{A}}^{-1}(\mathbf{0}) = \{(0,\ldots,0)\}$.
(3) $S \cap (-S) = \{0\}$, *that is to say,* $\mathbf{a} \in S$ *and* $-\mathbf{a} \in S \Rightarrow \mathbf{a} = \mathbf{0}$.
(4) *The relation* $\mathbf{a}' \preceq \mathbf{a} \Longleftrightarrow \mathbf{a}' - \mathbf{a} \in S$ *is a partial order on S.*

As mentioned in remark [**34**, Remark 30], if the conditions of Proposition 25 hold, the monoid S generated by $\mathcal{A}$ is said to be **positive**. When S is positive, $\mathfrak{m} = \langle x_1,\ldots,x_n\rangle$ is the only $S-$homogeneous maximal ideal in $\Bbbk[x_1,\ldots,x_n]$. Recall that a graded ideal $\mathfrak{m}$ in a graded ring R is a **graded maximal ideal** or ***maximal ideal** if the only graded ideal properly containing $\mathfrak{m}$ is R itself. Graded rings with a unique graded maximal ideal are known as **graded local rings** or ***local rings**. Many results valid for local rings are also valid for graded local rings, starting with Nakayama's Lemma. In particular, the $S-$graded minimal free resolution of any finitely generated $\Bbbk[x_1,\ldots,x_n]-$-module is well defined as explained below (see also [**13**, Section 1.5] and [**12**]).

In what follows, S is a finitely generated submonoid of $\mathbb{Z}^d$ such that $S \cap (-S) = \{0\}$, called a positive affine semigroup. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be the minimal generating set of S, that is, $\mathcal{A} = \mathrm{msg}(S)$. Without loss of generality, we assume $d = \dim(S)$.

The ring $\Bbbk[x_1, \ldots, x_n]$ has a natural $\mathcal{S}-$graded structure given by assigning degree $\mathbf{a}_i$ to $x_i,\ i = 1, \ldots, n$; indeed,

$$\Bbbk[x_1, \ldots, x_n] = \bigoplus_{\mathbf{a} \in \mathcal{S}} \Bbbk[x_1, \ldots, x_n]_{\mathbf{a}},$$

where $\Bbbk[x_1, \ldots, x_n]_{\mathbf{a}}$ denotes the $\Bbbk-$vector space generated by the monomials $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_n^{u_n}$ such that $\sum_{i=1}^{e} u_i \mathbf{a}_i = \mathbf{a}$, and $\Bbbk[x_1, \ldots, x_n]_{\mathbf{a}} \cdot \Bbbk[x_1, \ldots, x_n]_{\mathbf{a}'} = \Bbbk[x_1, \ldots, x_n]_{\mathbf{a}+\mathbf{a}'}$. The surjective $\mathcal{S}-$graded ring homomorphism

$$\widehat{deg_{\mathcal{A}}} : \Bbbk[x_1, \ldots, x_n] \longrightarrow \Bbbk[\mathcal{S}]; x_i \mapsto \mathbf{t}^{\mathbf{a}_i}$$

endows $\Bbbk[\mathcal{S}]$ with a structure of $\mathcal{S}-$graded $\Bbbk[x_1, \ldots, x_n]-$module. For simplicity, in what follows, we write $\varphi_0 = \widehat{deg_{\mathcal{A}}}$.

By Theorem 18, the kernel of φ_0, denoted $I_{\mathcal{A}}$, is a binomial ideal, the **toric ideal of** $\mathcal{S}$; such that $\Bbbk[\mathcal{S}] \cong \Bbbk[x_1, \ldots, x_n]/I_{\mathcal{A}}$. Thus, minimal generating systems of $I_{\mathcal{A}}$ gives rise to minimal representations of $\Bbbk[\mathcal{S}]$ as $\Bbbk[x_1, \ldots, x_n]-$module. Indeed, if $M_{\mathcal{A}} := \{f_1, \ldots, f_{\beta_1}\}$ is a minimal system of generators of $I_{\mathcal{A}}$, then

$$\Bbbk[x_1, \ldots, x_n]^{\beta_1} \xrightarrow{\varphi_1} \Bbbk[x_1, \ldots, x_n] \xrightarrow{\varphi_0} \Bbbk[\mathcal{S}] \to 0$$

is an exact sequence, where φ_1 is the homomorphism of $\Bbbk[x_1, \ldots, x_n]-$modules whose matrix with respect to the corresponding standard bases is $(f_1, \ldots, f_{\beta_1})$. Since $I_{\mathcal{A}}$ is $\mathcal{S}-$homogeneous (equivalently, a binomial ideal, see e.g. [**34**, Theorem 1]), then φ_1 is also $\mathcal{S}-$graded, that is, homogeneous of degree 0 after an appropriate degree shifting.

Now, if $\ker \varphi_1 \neq 0$, we can consider a minimal system of $\mathcal{S}-$graded generators of $\ker \varphi_1$, proceed as above defining a $\mathcal{S}-$graded homomorphism of $\Bbbk[\mathbf{x}]-$modules φ_2 and so on. By the Hilbert Syzygy Theorem, this process cannot continue indefinitely, giving rise to the **$\mathcal{S}-$graded minimal free resolution of** $\Bbbk[\mathcal{S}]$:

$$0 \to \Bbbk[x_1, \ldots, x_n]^{\beta_p} \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_2} \Bbbk[x_1, \ldots, x_n]^{\beta_1} \xrightarrow{\varphi_1} \Bbbk[x_1, \ldots, x_n] \xrightarrow{\varphi_0} \Bbbk[\mathcal{S}] \to 0.$$

For $\mathbf{b} \in \mathcal{S}$, we write $\beta_{i,\mathbf{b}}$ for the number of minimal generators of $\ker \varphi_i$ of $\mathcal{S}-$degree $\mathbf{b}$. Of course, $\beta_{i,\mathbf{b}}$ may be 0. Here it is convenient to recall that

$$\beta_{i,\mathbf{b}} = \dim_{\Bbbk} \operatorname{Tor}_i^{\Bbbk[x_1,\ldots,x_n]}(\Bbbk, \Bbbk[\mathcal{S}])_{\mathbf{b}}$$

(see, e.g. [**35**, Lemma 1.32]) is an invariant of $\Bbbk[\mathcal{S}]$ for every $i > 0$ and $\mathbf{b} \in \mathcal{S}$. The integer number $\beta_{i,\mathbf{b}}$ is called the $i-$**th Betti number of** $\Bbbk[\mathcal{S}]$ **in degree b** and $\beta_i = \sum_{\mathbf{b} \in \mathcal{S}} \beta_{i,\mathbf{b}}$ is called the $i-$**th (total) Betti number of** $\Bbbk[\mathcal{S}]$. Clearly, $\Bbbk[x_1, \ldots, x_n]^{\beta_i} = \bigoplus_{\mathbf{b} \in \mathcal{S}} \Bbbk[x_1, \ldots, x_n]^{\beta_{i,\mathbf{b}}}$ and

$$\varphi_i : \bigoplus_{\mathbf{b} \in \mathcal{S}} (\Bbbk[x_1, \ldots, x_n](-\mathbf{b}))^{\beta_{i,\mathbf{b}}} \longrightarrow \bigoplus_{\mathbf{b} \in \mathcal{S}} (\Bbbk[x_1, \ldots, x_n](-\mathbf{b}))^{\beta_{i-1,\mathbf{b}}}$$

is homogeneous of degree 0, for every $i = 1, \ldots, p$.

Notice that there are finitely many non-zero Betti numbers. The elements $\mathbf{b} \in \mathcal{S}$ such that $\beta_{1,\mathbf{b}} \neq 0$ are called in literature Betti elements and the set of Betti elements of $\mathcal{S}$ is usually denoted by $\operatorname{Betti}(\mathcal{S})$ (see, [**22**] for more details).

The maximum i such that $\beta_i \neq 0$ is called the **projective dimension of** $\Bbbk[\mathcal{S}]$, denoted $\operatorname{pd}_{\Bbbk[\mathbf{x}]}(\Bbbk[\mathcal{S}])$. By the Auslander–Buchsbaum formula (see, e.g. [**13**, Theorem 1.3.3]), one has

$$\operatorname{depth}(\Bbbk[\mathcal{S}]) = n - \operatorname{pd}_{\Bbbk[\mathbf{x}]}(\Bbbk[\mathcal{S}]). \tag{5}$$

Recall that when $\operatorname{depth}(\Bbbk[\mathcal{S}]) = d$ (equivalently, $\operatorname{pd}_{\Bbbk[\mathbf{x}]}(\Bbbk[\mathcal{S}]) = \operatorname{codim}(\Bbbk[\mathcal{S}]) = n - d$), then $\Bbbk[\mathcal{S}]$ is Cohen-Macaulay (much more information can be found at [**13**]). We extend this terminology to $\mathcal{S}$, by saying that $\mathcal{S}$ is **Cohen-Macaulay** when $\Bbbk[\mathcal{S}]$ is. In this case, if $p = \operatorname{pd}_{\Bbbk[\mathbf{x}]}(\Bbbk[\mathcal{S}])$, then β_p is called the **type of** $\Bbbk[S]$.

PROPOSITION 26. *If S is a numerical semigroup and $p = \operatorname{pd}_{\Bbbk[\mathbf{x}]}(\Bbbk[\mathcal{S}])$, then*

$$\operatorname{PF}(S) = \{\mathbf{b} - \sum_{i=1}^{n} \mathbf{a}_i \mid \beta_{p,\mathbf{b}} \neq 0\}.$$

Moreover, if $\beta_{p,\mathbf{b}} \neq 0$, then $\beta_{p,\mathbf{b}} = 1$; in particular, the type of S is equal to the type of $\Bbbk[S]$

PROOF. See, e.g. [**32**, Corollary 17]. □

CHAPTER 2

Minimal systems of binomial generators for the ideals of certain monomial curves

In number theory, a repunit is a number whose representation in a base b consists of copies of the single digit 1. In binary, these are known as Mersenne numbers. The term repunit, which stands for repeated unit, was coined by Albert H. Beiler in [**5**]. Given $\ell \in \mathbb{N}\setminus\{0\}$, we set $r_b(\ell)$ for the repunit number of length ℓ in base b, that is $r_b(\ell) = \sum_{j=0}^{\ell-1} b^j$, with the convention $r_b(0) = 0$.

Let n and a be two positive integers, with $n > 1$, and consider the sequence defined by

$$a_i := r_b(n) + a\, r_b(i-1), i \geq 1.$$

The main result of this chapter is the explicit determination of a minimal system of binomial generators of the defining ideal of the monomial curve associated to

$$\mathcal{A} = \{a_1, \ldots, a_n\},$$

provided that $\gcd(\mathcal{A}) = \gcd(a_1, a) = 1$. Concretely, we prove that $I_{\mathcal{A}}$ is minimally generated by the 2×2 minors of the matrix

$$X := \begin{pmatrix} x_1^b & \ldots & x_{n-1}^b & x_n^b \\ x_2 & \ldots & x_n & x_1^{a+1} \end{pmatrix}.$$

It is worth to say that the submonoids of $\mathbb{N}$ generated by $\mathcal{A}$, with $\gcd(\mathcal{A}) = 1$, are studied in detail in Chapter 3. These numerical semigroups, which are in fact minimally generated by $\mathcal{A}$, are called generalized repunit numerical semigroups by us, as they generalize the numerical semigroups introduced by D. Torrão et al. (obtained in case $a = b^n$). Therefore, the above result gives in particular minimal presentations of repunit numerical semigroups, an original result not considered by Torrão's et al, when studying these semigroups.

In this chapter, we also characterize the uniquely presented generalized repunit numerical semigroups. The cases $n = 2$ or $n = 3$ are well known; clearly, $I_{\mathcal{A}}$ is generic in these cases. For $n > 3$, we prove that the ideal $I_{\mathcal{A}}$ has a unique minimal system of generators if and only if and $a < b - 1$. In particular, for $n > 3$, repunit numerical semigroups are never uniquely presented. Finally, notice that our results applies in case $b = 1$, that is, for MED arithmetic numerical semigroups.

mathematics

MDPI

Article

Minimal Systems of Binomial Generators for the Ideals of Certain Monomial Curves

Manuel B. Branco [1], **Isabel Colaço** [2] and **Ignacio Ojeda** [3,*]

[1] Departamento de Matemáticas, Universidade de Évora, 7000-671 Évora, Portugal; mbb@uevora.pt
[2] Departamento de Matemática e Ciências Físicas, Instituto Politécnico de Beja, 7800-295 Beja, Portugal; isabel.colaco@ipbeja.pt
[3] Departamento de Matemáticas, Universidad de Extremadura, 06071 Badajoz, Spain
* Correspondence: ojedamc@unex.es

Abstract: Let a, b and $n > 1$ be three positive integers such that a and $\sum_{j=0}^{n-1} b^j$ are relatively prime. In this paper, we prove that the toric ideal I associated to the submonoid of $\mathbb{N}$ generated by $\{\sum_{j=0}^{n-1} b^j\} \cup \{\sum_{j=0}^{n-1} b^j + a \sum_{j=0}^{i-2} b^j \mid i = 2, \ldots, n\}$ is determinantal. Moreover, we prove that for $n > 3$, the ideal I has a unique minimal system of generators if and only if $a < b - 1$.

Keywords: binomial ideal; semigroup ideal; minimal system of generators; determinantal ideal; Gröbner basis; indispensability

MSC: primary: 13P10, 20M14; secondary: 52B20

check for updates

Citation: Branco, M.B.; Colaço I.; Ojeda, I. Minimal Systems of Binomial Generators for the Ideals of Certain Monomial Curves. *Mathematics* **2021**, *9*, 3204. https://doi.org/10.3390/math9243204

Academic Editor: Philippe Gimenez, Ignacio García Marco and Eduardo Sáenz De Cabezón

Received: 8 November 2021
Accepted: 9 December 2021
Published: 11 December 2021

1. Introduction

Let $\mathbb{k}$ be a field and let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a set of positive integers. It is well known that the kernel of the $\mathbb{k}-$algebra homomorphism

$$\varphi_{\mathcal{A}} : \mathbb{k}[x_1, \ldots, x_n] \to \mathbb{k}[t^{a_1}, \ldots, t^{a_n}];\ x_i \mapsto t^{a_i},\ i = 1, \ldots, n, \tag{1}$$

where $x_1, \ldots, x_n$ and t are indeterminates, is a binomial ideal (see [1], or [2] for a more recent reference). Clearly, $\ker(\varphi_{\mathcal{A}})$ is the defining ideal of a monomial curve.

Let b be a positive integer and set $r_b(\ell)$ for the $\ell-$th repunit number in base b, that is,

$$r_b(\ell) = \sum_{j=0}^{\ell-1} b^j.$$

By convention, $r_b(0) = 0$.

The main result in this paper is the explicit determination of a minimal system of binomial generators of $I := \ker \varphi_{\mathcal{A}}$ for

$$\mathcal{A} = \{a_i := r_b(n) + a\, r_b(i-1) \mid i = 1, \ldots, n\},$$

where a and $n > 1$ are positive integers. We prove that I is minimally generated by the 2×2 minors of the matrix

$$X := \begin{pmatrix} x_1^b & \ldots & x_{n-1}^b & x_n^b \\ x_2 & \ldots & x_n & x_1^{a+1} \end{pmatrix}, \tag{2}$$

provided that $\gcd(a_1, \ldots, a_n) = \gcd(a, r_b(n))$ is equal to 1. In this case, as an immediate consequence, we have that the so-called binomial arithmetical rank of I (see, e.g., [3]) is equal to $\binom{n}{2}$.

Furthermore, we obtain that the $2 \times 2-$minors of X form a minimal Gröbner basis with respect to a family of $\mathcal{A}$-graded reverse lexicographical term orders on $\Bbbk[x_1, \ldots, x_n]$ (Theorem 1) and, applying ([4], Corollary 14), we conclude that for $n > 3$, the ideal I has a unique minimal system of generators if and only if and $a < b - 1$ (Corollary 2).

The submonoids of $\mathbb{N}$ generated by $\mathcal{A}$ are studied in detail in [5] as a generalization of the numerical semigroups introduced by D. Torrão et al. (see [6,7]); in this context, Corollary 2 provides a minimal presentation of the submonoid of $\mathbb{N}$ generated by $a_1, \ldots, a_n$, providing an original result not considered in Torrão's PhD thesis.

To achieve our main result (Theorem 1), we first compute the ideal J of the projective monomial curve defined by the kernel of the $\Bbbk-$algebra homomorphism

$$\Bbbk[x_1, \ldots, x_n] \to \Bbbk[t^{r_b(0)}\, s, \ldots, t^{r_b(n-1)}\, s]; \quad x_i \mapsto t^{r_b(i-1)}\, s,\ i = 1, \ldots, n, \tag{3}$$

where s is also an indeterminate. This intermediate result (Proposition 1) has its own interest, as it exhibits another family of semigroup ideals that are determinantal and have unique minimal system of binomial generators (Corollary 1).

Throughout the paper, we keep the notation established in this introduction. Moreover, as the case $n = 2$ is trivial and the case $n = 3$ is well known for any a_1, a_2 and a_3 (see [1]), we suppose that $n > 3$ whenever necessary.

The explicit description of minimal systems of binomial generators of monomial curves, and in a broader context of toric ideals, is a long-established research topic since J. Herzog, in his celebrated paper [1], characterized the minimal systems of binomial generators of (all) the monomial curves in affine three-dimensional space. The elegance of Herzog's result for the three-dimensional case contrasts with the fact that no explicit description is known for the general case. Particular advances are just known for low-dimensional cases (see, e.g., [8] or more recently in [9] and the references therein) or for special families of monomial curves as presented in this paper; due to its proximity to the present work, we highlight the article by D.P. Patil [10] as one among many others.

We finally emphasize that, despite of not being the aim this paper, the study of the defining ideal of monomials curves have its own interest for applications to other areas such as linear programming (see, e.g., [11]), coding theory (see, e.g., [12] or algebraic statistics, where the minimal systems of bionomial generators are called Markov bases and the uniqueness property has special consideration (see [13]).

2. Preliminaries

Let a, b and n be three positive integers such that $n > 3$. Consider the sequence of positive integers $(a_i)_{i \geq 1}$ such that

$$a_i := r_b(n) + a\, r_b(i-1),$$

for every $i \geq 1$.

In this section, we present several lemmas that reflect the arithmetic structure of the sequence $(a_i)_{i \geq 1}$. In addition, we present the family of term orders that will be used throughout the paper.

Lemma 1. *The following equality holds:* $a_{n+k} = a_k + a\, b^{k-1}\, a_1$, *for all* $k \geq 1$. *In particular,* $a_{n+1} = (1+a)\, r_b(n)$.

Proof. It suffices to observe that $r_b(n+k-1) = r_b(k-1) + b^{k-1}\, r_b(n)$, for all $k \geq 1$, and, consequently, that $a_{n+k} = r_b(n) + +a\, r_b(n+k-1) = a_1 + a\, r_b(n+k-1) = a_1 + a\, (r_b(k-1) + b^{k-1} r_b(n)) = a_k + a\, b^{k-1}\, a_1$, for all $k \geq 1$. Finally, as $a_1 = r_b(n)$, the last statement is straightforward □

Notice that, by Lemma 1, the set $\mathcal{A} = \{a_1, \ldots, a_n\}$ is a system of generators of the submonoid of $\mathbb{N}$ generated by the sequence $(a_i)_{i \geq 1}$.

Lemma 2. *For each pair of positive integers j and k, it holds that*

$$b\,a_j + a_{j+k} = b\,a_{j+k-1} + a_{j+1}.$$

Proof. As $a_{j+k} = a_1 + a\,r_b(j+k-1) = a_1 + a\,(r_b(j-1) + b^{j-1}\,r_b(k)) = a_j + a\,b^{j-1}\,r_b(k)$, we conclude that

$$\begin{aligned} b\,a_j + a_{j+k} &= b\,a_j + a_j + a\,b^{j-1}\,r_b(k) = \\ &= b\,a_j + a_j + a\,b^{j-1}\,(b\,r_b(k-1)+1) = \\ &= b\,(a_j + a\,b^{j-1}\,r_b(k-1)) + a_j + a\,b^{j-1} = \\ &= b\,a_{j+k-1} + (a_1 + a\,r_b(j-1)) + a\,b^{j-1} = \\ &= b\,a_{j+k-1} + a_{j+1}, \end{aligned}$$

as claimed. □

Let $\prec_i$ be the term order on $\Bbbk[x_1,\ldots,x_n]$ defined by the following matrix

$$M := \left(\begin{array}{ccc|c|ccc} a_1 & \ldots & a_i & a_{i+1} & a_{i+2} & \ldots & a_n \\ \hline 0 & & -1 & 0 & 0 & \ldots & 0 \\ & \iddots & & \vdots & \vdots & & \vdots \\ -1 & & 0 & 0 & 0 & \ldots & 0 \\ \hline 0 & \ldots & 0 & 0 & 0 & & -1 \\ \vdots & & \vdots & \vdots & & \iddots & \\ 0 & & 0 & 0 & -1 & & 0 \end{array}\right).$$

We observe that $\prec_i$ is the $\mathcal{A}$-graded reverse lexicographical term order on $\Bbbk[x_1,\ldots,x_n]$ induced by $x_i \prec_i x_{i-1} \prec_i \ldots \prec_i x_1 \prec_i x_n \prec_i \ldots \prec_i x_{i+1}$; in particular, x_i is the smallest variable for $\prec_i$.

Lemma 3. *If $j \in \{1,\ldots,n-2\}$ and $k \in \{j+1,\ldots,n-1\}$, then*

$$x_j^b x_{k+1} \prec_i x_{j+1} x_k^b$$

if and only if $i \leq j$ or $k+1 \leq i$.

Proof. By Lemma 2, $b\,a_j + a_{k+1} = a_{j+1} + b\,a_k$, so we just need to decide what the variable x_j, x_{j+1}, x_k or x_{k+1} is cheapest for the order defined by the last $n-1$ rows of M. As $j < j+1 \leq k < k+1$, according to the definition of $\prec_i$, the variable x_{k+1} is cheaper than the other three when $j \leq i$ or $k+1 \leq i$; thus, $x_j^b x_{k+1} \prec_i x_{j+1} x_k^b$ in these cases. Conversely, if $j+1 \leq i \leq k$, then either x_k or x_{j+1} is cheaper than the others if $k = i$ or $k \neq i$, respectively. Therefore $x_j^b x_{k+1} \succ_i x_{j+1} x_k^b$ when $j+1 \leq i \leq k$, and we are done. □

3. Gröbner Bases and Minimal Generators for *J*

We keep the notation of the Introduction and Section 2.

Let $I_2(Y)$ be the ideal of $\Bbbk[x_1,\ldots,x_n]$ generated by the $2\times 2-$minors of

$$Y := \begin{pmatrix} x_1^b & x_2^b & \ldots & x_{n-1}^b \\ x_2 & x_3 & \ldots & x_n \end{pmatrix}.$$

Let $\mathcal{G}_1^{(i)}, \mathcal{G}_2^{(i)}$ and $\mathcal{G}_3^{(i)}$ be defined as follows:

$$\mathcal{G}_1^{(i)} = \left\{ \underline{x_{j+1} x_k^b} - x_j^b x_{k+1} \;\middle|\; j \in \{i,\ldots,n-2\},\, k \in \{j+1,\ldots,n-1\} \right\},$$

$$\mathcal{G}_2^{(i)} = \left\{ \underline{x_{j+1}x_k^b} - x_j^b x_{k+1} \mid j \in \{1,\ldots,i-2\},\, k \in \{j+1,\ldots,i-1\} \right\},$$
$$\mathcal{G}_3^{(i)} = \left\{ \underline{x_j^b x_{k+1}} - x_{j+1}x_k^b \mid j \in \{1,\ldots,i-1\},\, k \in \{i,\ldots,n-1\} \right\}$$

and let $\mathcal{G}_Y^{(i)}$ be equal to $\mathcal{G}_1^{(i)} \cup \mathcal{G}_2^{(i)} \cup \mathcal{G}_3^{(i)}$.

Notice that, by Lemma 3, the underlined monomials are the leading terms with respect to $\prec_i$ of the corresponding binomials.

Proposition 1. *With the above notation, the set $\mathcal{G}_Y^{(i)}$ is the reduced Gröbner basis of $I_2(Y)$ with respect to $\prec_i$. In particular, the cardinality of $\mathcal{G}_Y^{(i)}$ is $\binom{n-1}{2}$.*

Proof. First, let us see that $\mathcal{G}_Y^{(i)}$ is a Gröbner basis. By the Buchberger's Criterion (see, e.g., [14], Theorem 3.3), it suffices to verify that each S-pair of elements in $\mathcal{G}_Y^{(i)}$ can be reduced to zero by $\mathcal{G}_Y^{(i)}$ using the division algorithm. To do this, we distinguish several cases:

- Let $f \in \mathcal{G}_1^{(i)}$, that is to say, $f = x_{j+1}x_k^b - x_j^b x_{k+1}$, for some $j \in \{i,\ldots,n-2\}$ and $k \in \{j+1,\ldots,n-1\}$.
  - Let $g = x_{l+1}x_m^b - x_l^b x_{m+1} \in \mathcal{G}_1^{(i)}$. If $\gcd(x_{j+1}x_k^b, x_{l+1}x_m^b) = 1$, then $S(f,g)$ reduces to zero with respect to $\{f,g\} \subset \mathcal{G}_Y^{(i)}$. Otherwise, $j = l, j+1 = m, k = l+1$ or $k = m$. If $\boxed{j = l}$ then $S(f,g) = x_m^b(-x_j^b x_{k+1}) - x_k^b(-x_j^b x_{m+1}) = x_j^b(x_k^b x_{m+1} - x_m^b x_{k+1})$ reduces to zero with respect to $\mathcal{G}_Y^{(i)}$. If $\boxed{j+1 = m}$, then
  $$S(f,g) = x_{l+1}x_{j+1}^{b-1}(-x_j^b x_{k+1}) - x_k^b(-x_l^b x_{j+2}).$$
  Now, as $i \le j < j+1 \le k < k+1$ and $i \le l < l+1 \le m = j+1 < j+2$, the leading term of $S(f,g)$ with respect to $\prec_i$ is $x_k^b x_l^b x_{j+2}$. Then $S(f,g) = x_l^b(\underline{x_k^b x_{j+2}} - x_{j+1}^b x_{k+1}) + x_{j+1}^{b-1}x_{k+1}(x_l^b x_{j+1} - x_{l+1}x_j^b)$ reduces to zero with respect to $\mathcal{G}_Y^{(i)}$. By symmetry, the case $\boxed{k = l+1}$ is completely similar to the latter one. Finally, if $\boxed{k = m}$, then $S(f,g) = -x_{l+1}(-x_j^b x_{k+1}) - x_{j+1}(-x_l^b x_{k+1}) = x_{k+1}(x_j^b x_{l+1} - x_{j+1}x_l^b)$ reduces to zero with respect to $\mathcal{G}_Y^{(i)}$.
  - Let $g = x_{l+1}x_m^b - x_l^b x_{m+1} \in \mathcal{G}_2^{(i)}$. If $\gcd(x_{j+1}x_k^b, x_{l+1}x_m^b) = 1$, then $S(f,g)$ reduces to zero with respect to $\{f,g\} \subset \mathcal{G}_Y^{(i)}$. Otherwise, $j = l, j+1 = m, k = l+1$ or $k = m$. First, we observe that the cases $j = l$ and $k = m$ produce the same S-polynomial as in the corresponding case for $g \in \mathcal{G}_1^{(i)}$; so, we just focus on the cases $j+1 = m$ and $k = l+1$. If $\boxed{j+1 = m}$, then $i \le j < j+1 \le k < k+1$ and $l < l+1 \le m = j+1 < j+2 = m-1 \le i$, therefore $i < j+1 = m < i$, a contradiction. Finally, if $\boxed{k = l+1}$, then $i \le j < j+1 \le k < k+1$ and $l < l+1 = k \le m < m+1 \le i$, so $i < k = l+1 < i$, a contradiction again.
  - Let $g = x_l^b x_{m+1} - x_{l+1}x_m^b \in \mathcal{G}_3^{(i)}$. If $\gcd(x_{j+1}x_k^b, x_l^b x_{m+1}) = 1$, then $S(f,g)$ reduces to zero with respect to $\{f,g\} \subset \mathcal{G}_Y^{(i)}$. Otherwise, $j+1 = l, j = m, k = l$ or $k = m+1$. If $\boxed{j+1 = l}$, then $i \le j < j+1 = l \le k < k+1$ and $l \le i-1$; so $i < j+1 = l \le i-1$, a contradiction. If $\boxed{j = m}$ (or $\boxed{k = l}$, respectively) then $S(f,g) = x_m^b(x_k^b x_{l+1} - x_l^b x_{k+1})$ (or $S(f,g) = x_{k+1}(x_{j+1}x_m^b - x_j^b x_{m+1})$, respectively) reduces to zero with respect to $\mathcal{G}_Y^{(i)}$. Finally, if $\boxed{k = m+1}$, then
  $$S(f,g) = x_l^b(-x_j^b x_{m+2}) - x_{j+1}x_{m+1}^{b-1}(-x_{l+1}x_m^b).$$

Now, as $i \leq j < j+1 \leq k = m+1 < k+1$, $l \leq i-1$ and $i \leq m$, then x_{l+1} or x_{m-1} is cheaper than the others for the order induced by the last $n-1$ rows of the matrix M, therefore leading term of $S(f,g)$ is $x_l^b x_j^b x_{k+1}$ and thus, $S(f,g) = -x_j^b(\underline{x_l^b x_{m+2}} - x_{l+1}x_{m+1}^b) - x_{l+1}x_{m+1}^{b-1}(x_j^b x_{m+1} - x_{j+1}x_m^b)$ reduces to zero with respect to $\mathcal{G}_Y^{(i)}$.

- Let $f \in \mathcal{G}_2^{(i)}$, that is to say, $f = x_{j+1}x_k^b - x_j^b x_{k+1}$, for some $j \in \{1,\dots,i-2\}$ and $k \in \{j+1,\dots,i-1\}$.
  - Let $g = x_{l+1}x_m^b - x_l^b x_{m+1} \in \mathcal{G}_2^{(i)}$. If $\gcd(x_{j+1}x_k^b, x_{l+1}x_m^b) = 1$, then $S(f,g)$ reduces to zero with respect to $\{f,g\} \subset \mathcal{G}_Y^{(i)}$. Otherwise, $j = l, j+1 = m, k = l+1$ or $k = m$. If $\boxed{j = l}$ (or $\boxed{k = m}$, respectively), then $S(f,g) = x_j^b(x_k^b x_{m+1} - x_{k+1}x_m^b)$ (or $S(f,g) = x_{k+1}(x_{l+1}x_j^b - x_l^b x_{j+1})$, respectively) reduces to zero with respect to $\mathcal{G}_Y^{(i)}$. If $\boxed{j+1 = m}$, then
  $$S(f,g) = x_{l+1}x_m^{b-1}(-x_{m-1}^b x_{k+1}) - x_k^b(-x_l^b x_{m+1})$$
  and, as $l+1 \leq m = j+1 \leq k \leq i-1$, the leading term of $S(f,g)$ is equal to $x_{l+1}x_m^{b-1}x_{m-1}^b x_{k+1}$. Thus, $S(f,g) = -x_m^{b-1}x_{k+1}(\underline{x_{l+1}x_{m-1}^b} - x_l^b x_m) + x_l^b(x_k^b x_{m+1} - x_m^b x_{k+1})$ reduces to zero with respect to $\mathcal{G}_Y^{(i)}$; observe that $l < l+1 \leq m$ implies that the leading term of $x_{l+1}x_{m-1}^b - x_l^b x_m$ is actually $x_{l+1}x_{m-1}^b$. Finally, by symmetry, the case $\boxed{k = l+1}$ is completely similar to the latter one.
  - Let $g = x_l^b x_{m+1} - x_{l+1}x_m^b \in \mathcal{G}_3^{(i)}$. If $\gcd(x_{j+1}x_k^b, x_l^b x_{m+1}) = 1$, then $S(f,g)$ reduces to zero with respect to $\{f,g\} \subset \mathcal{G}_Y^{(i)}$. Otherwise, $j+1 = l, j = m, k = l$ or $k = m+1$. If $\boxed{j+1 = l}$, then
  $$S(f,g) = x_l^{b-1}x_{m+1}(-x_{l-1}^b x_{k+1}) - x_k^b(-x_{l+1}x_m^b).$$
  Furthermore, as $l = j+1 \leq k \leq i-1$ and $l \leq i-1 < i \leq m < m+1$, we have that the leading term is $x_l^{b-1}x_{m+1}x_{l-1}^b x_{k+1}$ if $k = l$ and $x_k^b x_{l+1}x_m^b$ otherwise. In ther first case, $S(f,g) = x_{m+1}x_{k-1}^b + x_k x_m^b$ reduces to zero with respecto to $\mathcal{G}_Y^{(i)}$. In the second case, $S(f,g) = x_m^b(\underline{x_k^b x_{l+1}} - x_{k+1}x_l^b) + x_l^{b-1}x_{k+1}(x_m^b x_l - x_{m+1}x_{l-1}^b)$ reduces to zero with respect to $\mathcal{G}_Y^{(i)}$. If $\boxed{j = m}$ (or $\boxed{k = l}$, respectively) then $S(f,g) = x_m^b(x_k^b x_{l+1} - x_l^b x_{k+1})$ (or $S(f,g) = x_{k+1}(x_{j+1}x_m^b - x_j^b x_{m+1})$, respectively) reduces to zero with respect to $\mathcal{G}_Y^{(i)}$. Finally, if $\boxed{k = m+1}$, then $j+1 \leq k = m+1 \leq i-1, l \leq i-1$ and $i \leq m$; so, $m+1 < i \leq m$, a contradiction.
- Let $f \in \mathcal{G}_3^{(i)}$, that is to say, $f = x_j^b x_{k+1} - x_{j+1}x_k^b$, for some $j \in \{1,\dots,i-1\}$ and $k \in \{i,\dots,n-1\}$.
  - Let $g = x_l^b x_{m+1} - x_{l+1}x_m^b \in \mathcal{G}_3^{(i)}$. If $\gcd(x_j^b x_{k+1}, x_l^b x_{m+1}) = 1$, then $S(f,g)$ reduces to zero with respect to $\{f,g\} \subset \mathcal{G}_Y^{(i)}$. Otherwise, $j = l, j = m+1, k+1 = l$ or $k = m$. As $j \leq i-1 < i \leq k$ and $l \leq i-1 < i \leq m$, the cases $\boxed{j = m+1}$ and $\boxed{k+1 = m}$ cannot occur. If $\boxed{j = l}$ (or $\boxed{k = m}$, respectively), then $S(f,g) = x_{l+1}(x_{k+1}x_m^b - x_{m+1}x_k^b)$ (or $S(f,g) = x_m^b(x_l^b x_{j+1} - x_j^b x_{l+1})$, respectively) reduces to zero with respect $\mathcal{G}_Y^{(i)}$.

Once we know that $\mathcal{G}_Y^{(i)}$ is Gröbner basis, it is immediate to see that it is reduced since the leading term of $f \in \mathcal{G}_Y^{(i)}$ does not divide any other monomial that appears in a binomial of $\mathcal{G}_Y^{(i)} \setminus \{f\}$.

It remains to prove that $\mathcal{G}_Y^{(i)}$ generates $I_2(Y)$. Clearly, $\mathcal{G}_Y^{(i)}$ is contained in the set of $2 \times 2-$minors of Y. Moreover, as the cardinality of $\mathcal{G}_1^{(i)}, \mathcal{G}_2^{(i)}$ and $\mathcal{G}_3^{(i)}$ are

$$\Big((n-1)-i\Big) + \Big((n-1)-i-1\Big) + \ldots + 1 = \binom{n-i}{2},$$

$$\Big((i-1)-1\Big) + \Big((i-1)-2\Big) + \ldots + 1 = \binom{i-1}{2}$$

and

$$(i-1)(n-i),$$

respectively, we have that the cardinality of $\mathcal{G}_Y^{(i)}$ is equal to $\binom{n-1}{2}$ which is the number of $2 \times 2-$minors of Y. Therefore, $\mathcal{G}_Y^{(i)}$ generates $I_2(Y)$ and we are done. $\square$

Example 1. *We observe that the reduced Gröbner basis, $\mathcal{G}_Y^{(i)}$, of $I_2(Y)$ with respect to $\prec_i$ is not an universal Gröbner basis. For example, if $n = b = 5$ and $\prec$ is the term order defined by*

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

then one can check (using, for example, Singular [15]) that the reduced Gröbner basis of the ideal $I_2(Y)$ with respect to $\prec$ has eight generators; however, $\mathcal{G}_Y^{(i)}$ contains $\binom{5-1}{2} = 6$ binomials only.

Alternatively, one can see that $\mathcal{G}_Y^{(i)}$ is not an universal Gröbner basis of $I_2(Y)$ by using ([16], Theorem 4.1).

We now consider the $2 \times n-$integer matrix B whose $j-$th column is

$$\mathbf{a}_j := \begin{pmatrix} r_b(j-1) \\ 1 \end{pmatrix}, \quad j = 1, \ldots, n.$$

Remark 1. *Observe that $a_j = \big(a, r_b(n)\big) \cdot \mathbf{a}_j$, for every $j = 1, \ldots, n$.*

Notice that the semigroup ideal associated to $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is equal to J; indeed, J is the kernel of (3).

Corollary 1. *The ideal J is minimally generated by the $2 \times 2-$minors of Y. Moreover, J has a unique minimal system of binomial generators.*

Proof. Let $I_2(Y)$ the ideal generated by the $2 \times 2-$minors of Y. Since $b\,\mathbf{a}_j + \mathbf{a}_{k+1} = \mathbf{a}_{j+1} + b\,\mathbf{a}_k$ for every j and k, we have that $I_2(Y) \subseteq J$.

Conversely, let C be the $(n-2)\times n-$matrix

$$\begin{pmatrix} b & -1 & -b & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & b & -1 & -b & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & -1 & -b & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & b & -1 & -b & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & b & -(b+1) & 1 \end{pmatrix}$$

and let I_C be the ideal of $\Bbbk[x_1,\dots,x_n]$ generated by

$$\{\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \mid \mathbf{u} \text{ is a row of } C\},$$

where $\mathbf{u}_+$ and $\mathbf{u}_-$ denote the positive and negative parts of $\mathbf{u}$, respectively. Clearly, $I_C \subseteq I_2(Y)$.

Now, as the determinant of the submatrix of C consisting in the last $n-2$ columns is 1, the rows of C generates a rank $n-2$ subgroup G_C of $\mathbb{Z}^n$ such that $\mathbb{Z}^n/G_C$ is torsion free. Moreover, as $B\,C^\top = 0$, we conclude that the rows of C generate $\ker_{\mathbb{Z}}(B)$. Therefore, by ([14], Lemma 7.6),

$$J = I_C : \Big(\prod_j x_j\Big)^\infty \subseteq I_2(Y) : \Big(\prod_j x_j\Big)^\infty.$$

By Proposition 1 and ([17], Theorem 3.1), we have that $I_2(Y) : x_i^\infty = I_2(Y)$ for every $i = 1,\dots,n$. So, $I_2(Y) : (\prod_j x_j)^\infty = I_2(Y)$ and, consequently, $J \subseteq I_2(Y)$ as desired.

Finally, by Proposition 1, we conclude that the $2\times 2-$minors of Y form a minimal system generators of J and, ([4], Corollary 14), we conclude that J has a unique minimal system of binomial generators. □

We recall that semigroup ideals minimally generated by a Graver basis have unique minimal system of binomials generators (see ([4], Corollary 16)). As Graver bases are in particular universal Gröbner bases (see [18], Proposition 4.11), by Example 1, we can assure the minimal system of binomial generators of J is not a Graver basis.

4. Gröbner Basis and Minimal Generators for *I*

We maintain the notation of the Introduction and the previous Sections, and we set $\mathcal{G}_4^{(i)}$ to be equal to

$$\Big\{\underline{x_1^{a+1}x_l^b} - x_{l+1}x_n^b \mid l = 1,\dots,i-1\Big\} \bigcup \Big\{\underline{x_{l+1}x_n^b} - x_1^{a+1}x_l^b \mid l = i,\dots,n-1\Big\},$$

where the underlined monomials again highlight the leading terms with respect to $\prec_i$ of the corresponding binomials.

Let $I_2(X)$ be the ideal of $\Bbbk[x_1,\dots,x_n]$ generated by the $2\times 2-$minors of the matrix X defined in (2).

Theorem 1. *The set $\mathcal{G}^{(i)} = \mathcal{G}_Y^{(i)} \cup \mathcal{G}_4^{(i)}$ is a minimal Gröbner basis of $I_2(X)$ with respect to $\prec_i$. In particular, the cardinality of $\mathcal{G}^{(i)}$ is $\binom{n}{2}$.*

Proof. Proceeding as in the proof of Proposition 1, we first need to prove that $S(f,g)$ reduces to zero with respect to $\mathcal{G}^{(i)}$, for every $f,g \in \mathcal{G}^{(i)}$. However, as, by Proposition 1, $\mathcal{G}_Y^{(i)}$ is already a Gröbner basis with respect to $\prec_i$ and the leading terms with respect to $\prec_i$

of the binomials in $\mathcal{G}_4^{(i)}$ are relatively prime, it suffices to prove that $S(f,g)$ reduces to zero with respect to $\mathcal{G}^{(i)}$, for every $f \in \mathcal{G}_Y^{(i)}$ and $g \in \mathcal{G}_4^{(i)}$. To do this we distinguish three cases:

- $f \in \mathcal{G}_1^{(i)} = \{\underline{x_{j+1}x_k^b} - x_j^b x_{k+1} \mid j \in \{i,\dots,n-2\},\ k \in \{j+1,\dots,n-1\}\}$. If $j \neq l$ and $k \neq l+1$, then the leading terms of f and g are relatively prime and there is nothing to prove. Therefore, it suffices to consider the cases $j = l$ or $k = l+1$.
  - If $j = l$, then $l \geq i$; otherwise, the leading terms of f and g are relatively prime, and $S(f,g) = x_n^b(-x_l^b x_{k+1}) - x_k^b(-x_1^{a+1}x_l^b) = -x_l^b(x_{k+1}x_n - x_1^{a+1}x_l^b)$ reduces to zero with respect to $\mathcal{G}_4^{(i)}$.
  - If $k = l+1$ then $n-2 \geq k-1 = l \geq j \geq i$, otherwise the leading terms of f and g are relatively prime, and $S(f,g) = x_n^b(-x_j^b x_{l+2}) - x_{j+1}x_{l+1}^{b-1}(-x_1^{a+1}x_l^b) = \underline{-x_j^b x_n^b x_{l+2}} + x_1^{a+1}x_{j+1}x_{l+1}^{b-1}x_l^b$. Observe that the leading term of $S(f,g)$ is divisible by the leading term of $h := \underline{x_n^b x_{l+2}} - x_1^{a+1}x_{l+1}^b \in \mathcal{G}_4^{(i)}$. Therefore, $S(f,g) = x_j^b h - x_j^b x_1^{a+1}x_{l+1}^b + x_1^{a+1}x_{j+1}x_{l+1}^{b-1}x_l^b = x_j^b h - x_1^{a+1}x_{l+1}^{b-1}(x_j^b x_{l+1} - x_{j+1}x_l^b)$. Now, as $x_j^b x_{l+1} - x_{j+1}x_l^b \in \mathcal{G}_Y^{(i)}$, we are done.
- $f \in \mathcal{G}_2^{(i)} = \{\underline{x_{j+1}x_k^b} - x_j^b x_{k+1} \mid j \in \{1,\dots,i-2\},\ k \in \{j+1,\dots,i-1\}\}$. If $j+1 \neq l$ and $k \neq l$, then the leading terms of f and g are relatively prime and there is nothing to prove. So, it suffices to consider the cases $j = l-1$ or $k = l$.
  - If $j+1 = l$, then $1 \leq j = l-1 < k \leq i-1$, otherwise the leading terms of f and g are relatively prime, and $S(f,g) = x_1^{a+1}x_l^{b-1}(-x_{l-1}^b x_{k+1}) - x_k^b(-x_{l+1}x_n^b) = x_k^b x_{l+1}x_n^b - x_1^{a+1}x_{l-1}^b x_l^{b-1}x_{k+1}$. If $l = k$, then the S-polynomial $S(f,g) = x_k^b x_{k+1}x_n^b - x_1^{a+1}x_{k-1}^b x_k^{b-1}x_{k+1} = -x_k^{b-1}x_{k+1}(\underline{x_1^{a+1}x_{k-1}^b} - x_k x_n^b)$ reduces to zero with respect to $\mathcal{G}_4^{(i)}$; otherwise, the leading term of $S(f,g)$ is $x_k^b x_{l+1}x_n^b$ which is divisible by the leading term of $h := \underline{x_k^b x_{l+1}} - x_{k+1}x_l^b \in \mathcal{G}_4^{(i)}$. So, $S(f,g) = x_n^b h + x_n^b x_{k+1}x_l^b - x_1^{a+1}x_{l-1}^b x_l^{b-1}x_{k+1} = x_n^b h - x_l^{b-1}x_{k+1}(x_1^{a+1}x_{l-1}^b - x_n^b x_l)$. Now, since $x_1^{a+1}x_{l-1}^b - x_n^b x_l \in \mathcal{G}_4^{(i)}$, we are done.
  - If $k = l$, then $1 \leq j < k = l \leq i-1$, otherwise the leading terms of f and g are relatively prime, and $S(f,g) = x_1^{a+1}(-x_j^b x_{l+1}) - x_{j+1}(-x_{l+1}x_n^b) = -x_{l+1}(x_1^{a+1}x_j^b - x_{j+1}x_n^b)$. Now, since $x_1^{a+1}x_j^b - x_{j+1}x_n^b \in \mathcal{G}_4^{(i)}$, we are done.
- $f \in \mathcal{G}_3^{(i)} = \{\underline{x_j^b x_{k+1}} - x_{j+1}x_k^b \mid j \in \{1,\dots,i-1\},\ k \in \{i,\dots,n-1\}\}$. If $j \neq 1, j \neq l, k \neq l$ and $k \neq n-1$, then the leading terms of f and g are relatively prime and there is nothing to prove. Therefore, is suffices so consider the cases $j = 1, j = l, k = l$ or $k = n-1$.
  - If $j = 1$, then, in particular, $l < i$, otherwise the leading terms of f and g are relatively prime. Now, if $a+1 \geq b$, then $S(f,g) = \underline{x_1^{a+1-b}x_l^b(-x_2 x_k^b)} - x_{k+1}x_{l+1}x_n^b$ and its leading term is divisible by the leading term of $h := \underline{x_l^b x_2} - x_{l+1}x_1^b \in \mathcal{G}_2^{(i)} \cup \mathcal{G}_3^{(i)}$; then $S(f,g) = x_1^{a+1-b}x_k^b h - x_1^{a+1-b}x_k^b(-x_{l+1}x_1^b) - x_{k+1}x_{l+1}x_n^b = x_1^{a+1-b}x_k^b h - x_{l+1}(\underline{x_{k+1}x_n^b} - x_1^{a+1}x_k^b)$ which reduces to zero with respect to $\mathcal{G}_2^{(i)} \cup \mathcal{G}_3^{(i)} \cup \mathcal{G}_4^{(i)}$. Otherwise, if $a+1 < b$, then $S(f,g) = \underline{x_l^b(-x_2 x_k^b)} - x_1^{b-a-1}x_{k+1}(-x_{l+1}x_n^b)$ and its leading term is divisible by the leading term of $h := \underline{x_l^b x_2} - x_{l+1}x_1^b \in \mathcal{G}_2^{(i)} \cup \mathcal{G}_3^{(i)}$; then $S(f,g) = x_k^b h - x_k^b(-x_{l+1}x_1^b) - x_1^{b-a-1}x_{k+1}(-x_{l+1}x_n^b) = x_k^b h - x_1^{b-a-1}x_{l+1}(x_{k+1}x_n^b - x_1^{a+1}x_k^b)$ which reduces to zero with respect to $\mathcal{G}_2^{(i)} \cup \mathcal{G}_3^{(i)} \cup \mathcal{G}_4^{(i)}$, too.
  - If $j = l$, then $1 \leq l \leq i-1$; otherwise, the leading terms of f and g are relatively prime, and $S(f,g) = x_1^{a+1}(-x_{l+1}x_k^b) - x_{k+1}(-x_{l+1}x_n^b) = -x_{l+1}(x_{k+1}x_n^b - x_1^{a+1}x_k^b)$ which reduces to zero with respect to $\mathcal{G}_4^{(i)}$.

- If $k = l$, then $i \leq l \leq n-1$; otherwise, the leading terms of f and g are relatively prime, and $S(f,g) = x_n^b(-x_{j+1}x_l^b) - x_j^b(-x_1^{a+1}x_l^b) = x_l^b(\underline{x_1^{a+1}x_j^b} - x_{j+1}x_n^b)$ which reduces to zero with respect to $\mathcal{G}_4^{(i)}$.
- If $k = n-1$, then $1 \leq j < i \leq l < n$, otherwise the leading terms of f and g are relatively prime. In this case, $S(f,g) = x_{l+1}x_n^{b-1}(-x_{j+1}x_{n-1}^b) \underline{- x_j^b(-x_1^{a+1}x_l^b)}$ and, since the leading term of $S(f,g)$ is divisible by the leading term of $h := \underline{x_1^{a+1}x_j^b} - x_{j+1}x_n^b \in \mathcal{G}_4^{(i)}$, we have that $S(f,g) = x_l^b h - x_l^b(-x_{j+1}x_n^b) + x_{l+1}x_n^{b-1}(-x_{j+1}x_{n-1}^b)$ $= x_l^b h - x_{j+1}x_n^{b-1}(\underline{x_{l+1}x_{n-1}^b} - x_l^b x_n)$, and as $\underline{x_{l+1}x_{n-1}^b} - x_l^b x_n$ belongs to $\mathcal{G}_1^{(i)}$, we are done.

Now, as $S(f,g)$ reduces to zero with respect to $\mathcal{G}^{(i)}$ in all the three cases we conclude that $\mathcal{G}^{(i)}$ forms a Gröbner basis.

Once we know that $\mathcal{G}^{(i)}$ is a Gröbner basis, we observe that the leading terms of the binomials in $\mathcal{G}^{(i)}$ are not divisible by the leading term of any other binomial in $\mathcal{G}^{(i)}$ other than itself. That is to say, the Gröbner basis $\mathcal{G}^{(i)}$ is minimal.

Clearly, $\mathcal{G}^{(i)}$ is a subset of $2 \times 2-$minors of X. Moreover, its cardinality is equal to the cardinality of $\mathcal{G}_Y^{(i)}$, that is $\binom{n-1}{2}$, plus the cardinality, $n-1$, of $\mathcal{G}_4^{(i)}$. Therefore, $\mathcal{G}^{(i)}$ has cardinality equal to $\binom{n-1}{2} + (n-1) = \binom{n}{2}$ which is the number of $2 \times 2-$minors of X. Hence we conclude that $\mathcal{G}^{(i)}$ generates $I_2(X)$ and we are done. □

Example 2. *The minimal Gröbner basis, $\mathcal{G}^{(i)}$, of $I_2(X)$ with respect to $\prec_i$ is not reduced in general. For example, if $n = 4, a = 3$ and $b = 3$, then one can see (using, e.g., Singular [15]) that the binomial $x_4^4 - x_1x_2^4x_3^2$ belongs to the Gröbner basis of $I_2(X)$ with respect to $\prec_2$; however, $x_4^4 - x_1x_2^4x_3^2$ is not a minor of X.*

Corollary 2. *If $\gcd(a, r_b(n)) = 1$, then the ideal I is minimally generated by the $2 \times 2-$minors of X. In this case, if $n > 3$, then I has a unique minimal system of generators if and only if and $a < b-1$.*

Proof. By Theorem 1, to prove the first part of the statement it suffices to see that $I = I_2(X)$.

By Lemma 2, we have that $\varphi_{\mathcal{A}}(f) = 0$, for every $f \in \mathcal{G}^{(i)}$, where $\varphi_{\mathcal{A}}$ is the $\Bbbk-$algebra homomorphism define in (1). Therefore $I_2(X) \subseteq I$. Conversely, let L be the $(n-1) \times n-$matrix

$$\begin{pmatrix} b & -(b+1) & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & b & -(b+1) & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & b & -(b+1) & 1 \\ (a+1) & 0 & 0 & 0 & \ldots & 0 & b & -(b+1) \end{pmatrix}$$

and let I_L be the ideal of $\Bbbk[x_1, \ldots, x_n]$ generated by

$$\{\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \mid \mathbf{u} \text{ is a row of } L\}.$$

Clearly, $I_L \subseteq I_2(X)$.

On the one hand, a direct computation shows that the set of $(n-1) \times (n-1)-$minors of L is equal (up to sign of its elements) to $\{a_1, \ldots, a_n\}$ and therefore, by ([18] [Lemma 12.2),

$$I_L : \Big(\prod_{i=1}^{n} x_i\Big)^{\infty} = I$$

if and only if $\gcd(a_1, \ldots, a_n) = \gcd(a, r_b(n)) = 1$. On the other hand, by Theorem 1 and ([17], Theorem 3.1), we have that $I_2(X) : (x_i^\infty) = I_2(X)$ for every $i = 1, \ldots, n$, that is to say, $I_2(X) : \left(\prod_{i=1}^n x_i\right)^\infty = I_2(X)$. Putting this together we conclude that

$$I_2(X) = I_2(X) : \Big(\prod_{i=1}^n x_i\Big)^\infty \supseteq I_L : \Big(\prod_{i=1}^n x_i\Big)^\infty = I,$$

and thus $I = I_2(X)$ as claimed.

To prove the second part of the statement, we observe that, for every $i \neq n$, the non-leading term, $x_1^{a+1} x_l^b$, of the binomial $\underline{x_{l+1} x_n^b} - x_1^{a+1} x_l^b \in \mathcal{G}_4^{(i)}$ is divisible by the leading term of $\underline{x_1^b x_l} - x_2 x_{l-1}^b \in \mathcal{G}_3^{(i)}$, provided that $l \geq 3$ (otherwise, no such binomial in $\mathcal{G}_3^{(i)}$ exists), if and only $a + 1 \geq b$. Now, as these are the only divisibility relationships between the monomials of the binomials in $\mathcal{G}^{(i)}$, and $l \geq 3$ implicitly requires $n > 3$, we obtain that for $n > 3$, $\mathcal{G}^{(i)}$ is reduced for every $\prec_i$ if and only $a < b - 1$, and, by ([4], Corollary 14), we conclude that for $n > 3$, I has a unique minimal system of binomial generators if and only if and $a < b - 1$. □

Notice that the condition $\gcd(a, r_b(n)) = 1$ cannot be avoided.

Example 3. *Let $n = 4, a = 3$ and $b = 2$. In this case, $a_1 = r_b(4) = 15, a_2 = 18, a_3 = 24$ and $a_4 = 36$. Clearly, $\gcd(a_1, a_2, a_3, a_4) = \gcd(a, r_b(4)) = 3$. By direct computation, one can check that I is minimally generated by four binomials whereas $I_2(X)$ is minimally generated by six binomials. In particular, $I \neq I_2(X)$; in fact, one has that I is a minimal prime of $I_2(X)$.*

The following example shows the minimal system of generators of I for $n = 4$.

Example 4. *If $n = 4$, then the ideal $I \subset \Bbbk[x_1, x_2, x_3, x_4]$ is minimally generated by*

$$x_2^{b+1} - x_1^b x_3, x_1^b x_4 - x_2 x_3^b, x_3^{b+1} - x_2^b x_4$$

and

$$x_1^{a+b+1} - x_2 x_4^b, x_1^{a+1} x_2^b - x_3 x_4^b, x_4^{b+1} - x_1^{a+1} x_3^b$$

(recall that the first three binomials generates J). In [9], a complete classification of the monomial curves in $\mathbb{A}^4(\Bbbk)$ having a unique minimal system of generators is given. By ([9], Theorem 3.11), one has that I has a unique minimal system of generators if and only if $x_1^{a+1} x_3^b$ is not divisible by $x_1^b x_3$; equivalently $a < b - 1$ as we already knew by Corollary 2. Observe that the result on the uniqueness of the system of generators of I can be deduced from [19], too.

We end this paper by observing that, since both J and I are determinantal ideals by Corollaries 1 and 2, respectively, one can conveniently adapt ([20], Section 2.1) to compute the minimal free resolution of I and J using the Eagon–Northcott complex. In particular, one can prove that the $\Bbbk-$algebras $\Bbbk[x_1, \ldots, x_n]/J$ and $\Bbbk[x_1, \ldots, x_n]/I$ are Cohen–Macaulay of type $n - 2$ and $n - 1$, respectively (see ([20], Section 2.1 for further details)). The explicit computation of the minimal free resolution of $\Bbbk[x_1, \ldots, x_n]/J$ and $\Bbbk[x_1, \ldots, x_n]/I$ is a future work.

Author Contributions: All authors contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported in part by Grant MTM2017-84890-P funded by MCIN/FEDER-UE, by Grant PGC2018-096446-B-C21 funded by MCIN/AEI/10.13039/501100011033, by "ERDF A way of making Europe", and by Grant GR21055(FQM-024) funded by Junta de Extremadura (Spain)/FEDER funds.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors wish to thank the referees for their valuable comments and suggestions that have undoubtedly helped to increase the clarity of the presentation of this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Herzog, J. Generators and relations of abelian semigroups and semigroup rings. *Manuscripta Math.* **1970**, *3*, 175–193.
2. Matusevich, L.F.; Ojeda, I. Binomial ideals and congruences on $\mathbb{N}^n$. In *Singularities, Algebraic Geometry, Commutative Algebra, and Related Topics Festschrift for Antonio Campillo on the Occasion of His 65th Birthday*; Cassou-Noguès, P., Greuel, G.M., Macarro, L.N., Xambó-Descamps, S., Eds.; Springer: Berlin/Heidelberg, Germany, 2018; pp. 429–454.
3. Katsabekis, A. Arithmetical rank of binomial ideals. *Arch. Math.* **2017**, *109*, 323–334.
4. Ojeda, I; Vignerón-Tenorio, A. Indispensable binomials in semigroups ideals. *Proc. Amer. Math. Soc.* **2010**, *138*, 4205–4216.
5. Branco, M.B.; Colaço, I.; Ojeda, I. The Frobenius problem for generalized repunit numerical semigroups. arXiv:2112.01106 [math.AC]
6. Rosales, J.C.; Branco, M.B.; Torrão, D. The Frobenius problem for repunit numerical semigroups. *Ramanujan J.* **2016**, *40*, 323–334.
7. Rosales, J.C.; Branco, M.B.; Torrão, D. The Frobenius problem for Mersenne numerical semigroups. *Math. Z.* **2017**, *286*, 741–749.
8. Bresinsky, H. Binomial generating sets for monomial curves, with applications in $\mathbb{A}^4$. *Rend. Sem. Mat. Univ. Politec. Torino* **1988**, *46*, 353–370.
9. Katsabekis, A.; Ojeda, I. An indispensable classification of monomial curves in $\mathbb{A}^4(\Bbbk)$. *Pac. J. Math.* **2014**, *268*, 96–116.
10. Patil, D.P. Minimal sets of generators for the relation ideals of certain monomial curves. *Manuscripta Math.* **1993**, *80*, 239–248.
11. Villarreal, R.H. *Monomial Algebras*; Monographs and Textbooks in Pure and Applied Mathematics; Marcel Dekker: New York, NY, USA, 2001; Volume 238.
12. Martínez-Moro, E.; Munuera C.; Ruano, D. (Eds.) *Advances in Algebraic Geometry Codes*; Series on Coding Theory and Cryptology; World Scientific Publishing: Hackensack, USA 2008; Volume 5.
13. Aoki, S.; Hara, H.; Takemura, A. *Markov Bases in Algebraic Statistics*; Springer Series in Statistics; Springer: New York, NY, USA, 2012; Volume 199.
14. Miller, E.; Sturmfels, B. *Combinatorial Commutative Algebra*; Graduate Texts in Mathematics; Springer: New York, NY, USA 2005; Volume 227.
15. Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H. SINGULAR 4-2-1—A Computer Algebra System for Polynomial Computations. Available online: http://www.singular.uni-kl.de (accessed on 28 October 2021).
16. Boocher, A.; Robeva, E. Robust toric ideals. *J. Symb. Comput.* **2015**, *68*, 254–264.
17. Bigatti, A.M.; La Scala, R.; Robbiano, L. Computing toric ideals. *J. Symb. Comput.* **1999**, *27*, 351–365.
18. Sturmfels, B. *Gröbner Bases and Convex Polytopes*; AMS University Lecture Series; American Mathematical Society: Providence, RI, USA, 1996; Volume 8.
19. Katsabekis, A.; Thoma, A. Specializations of multigradings and the arithmetical rank of lattice ideals. *Comm. Algebra* **2010**, *38*, 1904–1918.
20. Gimenez, P.; Sengupta, I.; Srinivasan, H. Minimal free resolutions for certain affine monomial curves. *Contemp. Math.* **2011**, *555*, 87–95.

CHAPTER 3

The Frobenius problem for generalized repunit numerical semigroups

In this chapter, we delve into the combinatorics of generalized repunit numerical semigroups. Let $b > 1$ be a integer. We recall that $(r_b(\ell))_{\ell\in\mathbb{N}}$ is the sequence such that $r_b(0) = 0$, defined by the recurrence relation $r_b(\ell) = br_b(\ell - 1) + 1$, $\ell > 0$.

Given two positive integers $n > 1$ and a, we firstly consider the the submonoid of $\mathbb{N}$ generated by $\{a_1, a_2, \ldots\}$, such that

$$a_i = r_b(n) + ar_b(i - 1), \text{ for every } i \geq 1.$$

We prove that the condition $\gcd(a_1, a) = 1$ is necessary and sufficient for such a monoid to be a numerical semigroup. In this case, it is called a generalized repunit numerical semigroup by us, or a grepunit semigroup for short, denoted $S_a(b, n)$, and we prove that $S_a(b, n)$ is minimally generated by $\mathcal{A} = \{a_1, \ldots, a_n\}$.

All $S_a(b, n)$ share a particular combinatorial invariant, a certain subposet of $\mathbb{N}^{n-1}$, ordered by the standard product order, firstly introduced in the study of repunit numerical semigroups, denoted $R(b, n)$ as a set, from which we explicitly describe the Apéry set of $S_a(b, n)$, relative to its multiplicity $a_1 = r_b(n)$, denoted $\mathrm{Ap}(S_a(b, n))$. Notice that, by Proposition 22 given in Section 2 of Chapter 1, the set $R(b, n)$ can be read on the minimal system of $I_{\mathcal{A}}$, for each a.

Then, we get closed formulas for the Frobenius number and genus of grepunit semigroups as functions of a, b and n, which are the main results of these chapter. Furthermore, we prove that the maximals of $R(b, n)$ corresponds in a one-to-one manner to the maximals of $\mathrm{Ap}(S_a(b, n))$, ordered by the partial order given by $S_a(b, n)$, by the restriction of the factorization map $\deg_{\mathcal{A}}$ to $\{0\} \times R(b, n)$. In particular, we obtain that the type of $S_a(b, n)$ is $n - 1$ and, thus, since $S_a(b, n)$ has embedding dimension n, we have that grepunit semigroups are Wilf.

Notice we describe the whole set of Pseudo-Frobenius numbers of $S_a(b,n)$ as the set of terms of the arithmetic sequence starting with $ba_n - a_1$ and whose step is the common diference $ba_i - a_{i+1}$. As point out in [**47**], this step can be read on the matrix X, defined in Chapter 2, which gives the minors of the minimal system of $I_{\mathcal{A}}$. Also, notice the type of $S_a(b,n)$ is already clear from the determinantal structure of $I_{\mathcal{A}}$, as explained in the next and final chapter of this thesis.

The forthcoming paper is published in [Branco, M.B., Colaço, I. & Ojeda, I. The Frobenius Problem for Generalized Repunit Numerical Semigroups. Mediterr. J. Math. 20, 16 (2023)] `https://doi.org/10.1007/s00009-022-02233-w`.

Mediterr. J. Math. (2023) 20:16
https://doi.org/10.1007/s00009-022-02233-w

Mediterranean Journal of Mathematics

The Frobenius Problem for Generalized Repunit Numerical Semigroups

Manuel B. Branco, Isabel Colaço and Ignacio Ojeda

Abstract. In this paper, we introduce and study the numerical semigroups generated by $\{a_1, a_2, \ldots\} \subset \mathbb{N}$ such that a_1 is the repunit number in base $b > 1$ of length $n > 1$ and $a_i - a_{i-1} = a\, b^{i-2}$, for every $i \geq 2$, where a is a positive integer relatively prime with a_1. These numerical semigroups generalize the repunit numerical semigroups among many others. We show that they have interesting properties such as being homogeneous and Wilf. Moreover, we solve the Frobenius problem for this family, by giving a closed formula for the Frobenius number in terms of a, b and n, and compute other usual invariants such as the Apéry sets, the genus or the type.

Mathematics Subject Classification. Primary 13P10, 20M14; Secondary 52B20.

Keywords. Numerical semigroup, Apéry sets, Frobenius problem, genus, type, Wilf's conjecture.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A numerical semigroup S is a subset of $\mathbb{N}$ containing zero which is closed under addition of natural numbers and such that $\mathbb{N} \backslash S$ is finite. The cardinality of $\mathbb{N} \backslash S$ is called the genus of S, denoted $\mathrm{g}(S)$.

Numerical semigroups have a unique finite minimal system of generators, that is, given a numerical semigroup S there exists a unique set $\{a_1, \ldots, a_e\} \subset \mathbb{N}$ such that

$$S = \mathbb{N}a_1 + \cdots + \mathbb{N}a_e$$

and no proper subset of $\{a_1, \ldots, a_e\}$ generates S (see [10, Theorem 2.7]). In this case, the set $\{a_1, \ldots, a_e\}$ is the minimal system of generators of S,

This work was supported in part by Grant MTM2017-84890-P funded by MCIN/FEDER-UE, by Grant PGC2018-096446-B-C21 funded by MCIN/AEI/10.13039/501100011033, by "ERDF A way of making Europe", and by Grant GR21055 (FQM-024) funded by Junta de Extremadura (Spain)/FEDER funds.

Birkhäuser

Published online: 03 December 2022

its cardinality is called the embedding dimension of S, denoted $\mathrm{e}(S)$, and $\min\{a_1,\ldots,a_e\}$ is called the multiplicity of S, denoted $\mathrm{m}(S)$. Notice that the finiteness of the genus implies that $\gcd(a_1,\ldots,a_e)=1$. In fact, one has that the necessary and sufficient condition for a subset $\mathcal{A}$ of $\mathbb{N}$ to generate a numerical semigroup is $\gcd(\mathcal{A})=1$ (see, e.g., [10, Lemma 2.1]).

Let S be a numerical semigroup. Since $\mathbb{N}\setminus S$ is finite, there exists the greatest integer not in S which is called the Frobenius number of S, denoted $\mathrm{F}(S)$. The so-called Frobenius problem deals with finding a closed formula for $\mathrm{F}(S)$ in terms of the minimal systems of generators of S, if possible (see, e.g., [9]).

Let b be a positive integer greater than 1. Set $a_1=\sum_{j=0}^{n-1} b^j$ and consider the sequence $(a_i)_{i\geq 1}$ defined by the recurrence relation

$$a_i - a_{i-1} = a\, b^{i-2}, \text{ for every } i \geq 2,$$

where a and n are positive integers. In this paper, we study the numerical semigroups, $S_a(b,n)$, generated by $\{a_1,a_2,\ldots\}$, provided that $\gcd(a_1,a)=1$. This last condition is necessary and sufficient for $S_a(b,n)$ to be a numerical semigroup (see Proposition 4). In this case, we say that $S_a(b,n)$ is a generalized repunit numerical semigroup as it generalizes the repunit numerical semigroups studied in [11] (see Example 7).

Clearly, the generalized repunit numerical semigroup $S_a(b,n)$ has multiplicity a_1. Moreover, by Theorem 6, we have that $S_a(b,n)$ is minimally generated by $\{a_1,\ldots,a_n\}$; in particular, the embedding dimension of $S_a(b,n)$ is n.

The main results in this paper are Theorem 22, which provides the following formula for the Frobenius number of $S_a(b,n)$:

$$\mathrm{F}(S_a(b,n)) = \begin{cases} (n-1)\ (b^n-1-a) + a\left(\sum_{j=0}^{n-1} b^j\right) & \text{if } a < b^n-1; \\ b^n - 1 - a + a\left(\sum_{j=0}^{n-1} b^j\right) & \text{if } a > b^n - 1, \end{cases}$$

and Corollary 26, which gives the following formula for the genus of $S_a(b,n)$:

$$\mathrm{g}(S_a(b,n)) = \frac{(n-1)\, b^n + a\left(\sum_{j=1}^{n-1} b^j\right)}{2}.$$

To achieve these results, we take advantage of Selmer's formulas, summarized in Proposition 20. These formulas depend on the Apéry sets of $S_a(b,n)$. We explicitly compute the Apéry set of $S_a(b,n)$ with respect to a_1 (Theorem 15). This is a result that may seem technical; however, it reflects the internal structure of generalized repunit numerical semigroups. For instance, the Apéry set of $S_a(b,i)$ can be obtained from the Apéry set of $S_a(b,i-1)$, for every $i\geq 3$. This is Corollary 19 whose statement is a stronger version of [8, Theorem 3.3] partially thanks to the fact that generalized repunit numerical semigroups are homogeneous in the sense of [5] (Proposition 17).

The last section of the paper is devoted to the computation of the pseudo-Frobenius numbers of $S_a(b,n)$. Concretely, using our results in Sect. 3, we explicitly compute the whole set of pseudo-Frobenius numbers of $S_a(b,n)$ and we obtain that its cardinality is $n-1$ (Proposition 29). So, we prove that the type of $S_a(b,n)$ is equal to $n-1$ which implies that $S_a(b,n)$ is Wilf (see Sect. 5 for further details).

Generalized repunit numerical semigroups have other interesting properties. Without going further, using Gröbner basis techniques, in [1] it is proved that the toric ideal associated to $S_a(b,n)$ is determinantal and that the cardinal of any minimal presentation of $S_a(b,n)$ is $\binom{n}{2}$. Moreover, following [7], we proved that, for $n > 3$, generalized repunit numerical semigroups are uniquely presented, in the sense of [4], if and only if $a < b-1$.

Finally, we note that our results are also valid for the case $b = 1$. In this case, $a_i = n + (i-1)\,a,\ i \geq 1$, is an arithmetic sequence that generates a MED semigroup, provided that $\gcd(n,a) = 1$. These semigroups are widely known (see, e.g., [10, Section 3]); for this reason and for the sake of simplicity, we consider $b > 1$; so that a_1 is properly a repunit number.

2. Generalized Repunit Numerical Semigroups

Let $b > 1$ be a positive integer.

Definition 1. A repunit number in base b is an integer whose representation in base b contains only the digit 1.

We write $r_b(\ell)$ for the repunit number in base b of length ℓ, that is,

$$r_b(\ell) = \sum_{j=0}^{\ell-1} b^j = \frac{b^\ell - 1}{b-1}.$$

By convention, we assume $r_b(0) = 0$.

Example 2. The first six repunit numbers in base 2 are $1, 3, 7, 15, 31, 63\ldots$, whereas the first six repunit numbers in base 3, are $1, 4, 13, 40, 121, 364\ldots$. Observe that repunit numbers in base 2 are the Mersenne numbers.

Here and in what follows, a and n denote two positive integers.

Notation 3. Set $a_i := r_b(n) + a\, r_b(i-1),\ i \geq 1$. Observe that $a_1 = r_b(n)$ and $a_i - a_{i-1} = a\, b^{i-2}$, for every $i \geq 2$. We write $S_a(b,n)$ for the submonoid of $\mathbb{N}$ generated by $a_i,\ i \geq 1$.

If $n = 1$, then $a_1 = 1$ and therefore $S_a(b,n) = \mathbb{N}$. So, in the following we assume that $n > 1$.

Proposition 4. *$S_a(b,n)$ is a numerical semigroup if and only if* $\gcd(r_b(n),a) = 1$.

Proof. Let $d = \gcd(r_b(n),a)$. By definition, $S_a(b,n) \subseteq d\,\mathbb{N}$. Now, if $S_a(b,n)$ is a numerical semigroup, then $\mathbb{N}\backslash d\,\mathbb{N} \subset \mathbb{N}\backslash S_a(b,n)$ has finitely many elements, and hence $d = 1$. Conversely, if $\gcd(r_b(n),a) = 1$, then $\gcd(a_1,a_2) = \gcd(r_b(n), r_b(n)+a) = \gcd(r_b(n),a) = 1$. So, $a_1\mathbb{N} + a_2\mathbb{N}$ is a numerical semigroup containing $S_a(b,n)$. Therefore, $\mathbb{N}\backslash S_a(b,n) \subseteq \mathbb{N}\backslash(a_1\mathbb{N}+a_2\mathbb{N})$ has finitely many elements, that is to say, $S_a(b,n)$ is a numerical semigroup. □

Let us prove that if $\gcd(r_b(n), a) = 1$, then $\{a_1, \ldots, a_n\}$ is the minimal generating set of $S_a(b, n)$. We begin with a useful lemma.

Lemma 5. *The following equality holds: $a_{n+i} = a_i + a\, b^{i-1}\, a_1$, for all $i \geq 1$.*

Proof. It suffices to observe that $r_b(n+i-1) = r_b(i-1) + b^{i-1}\, r_b(n)$, for all $i \geq 1$, and, consequently, that $a_{n+i} = a_1 + a\, r_b(n+i-1) = a_1 + a\,(r_b(i-1) + b^{i-1} r_b(n)) = a_i + a\, b^{i-1}\, a_1$, for all $i \geq 1$. □

Observe that the previous lemma already implies that $\{a_1, \ldots, a_n\}$ generates $S_a(b, n)$. So, it remains to see that $\{a_1, \ldots, a_n\}$ is minimal for the inclusion. In this case, by [10, Theorem 2.7], $\{a_1, \ldots, a_n\}$ will be the (unique) minimal system of generators of $S_a(b, n)$.

Theorem 6. *If $S_a(b, n)$ is a numerical semigroup, then $\{a_1, \ldots, a_n\}$ is the minimal system of generators of $S_a(b, n)$. In particular, the embedding dimension of $S_a(b, n)$ is n.*

Proof. By Lemma 5, we have that $\{a_1, \ldots, a_n\}$ is a system of generators of $S_b(a, n)$. Now, since $a_1 < \cdots < a_n$, to see the minimality property, it suffices to prove that $a_i \notin \langle a_1, \ldots, a_{i-1} \rangle$ for every $i \in \{2, \ldots, n\}$. By the condition $\gcd(a_1, a_2) = \gcd(a_1, a) = 1$ this is true for $i = 2$. Also when $a = 1$, we have that $a_i - a_k = r_b(i-1) - r_b(k-1) = \sum_{j=k-1}^{i-2} b^j < a_1$, for every $k \leq i$, and consequently, $a_i \notin \langle a_1, \ldots, a_{i-1} \rangle$.

So, from now on we assume $a > 1$ and $i \in \{3, \ldots, n\}$. If $a_i \in \langle a_1, \ldots, a_{i-1} \rangle$, then there exist $u_1, \ldots, u_{i-1} \in \mathbb{N}$ such that $a_i = \sum_{j=1}^{i-1} u_j a_j$. Therefore, $a_i = a_1 + a\, r_b(i-1)$ is equal $\big(\sum_{j=1}^{i-1} u_j\big) a_1 + a \sum_{j=1}^{i-1} u_j r_b(j-1)$ and thus

$$\left(\sum_{j=1}^{i-1} u_j \right) a_1 \equiv a_1 \pmod{a}.$$

Now, since $S_a(b, n)$ is a numerical semigroup, by Proposition 4, we have $\gcd(a_1, a) = \gcd(r_b(n), a) = 1$, and we conclude that $\sum_{j=1}^{i-1} u_j \equiv 1 \pmod{a}$. If $\sum_{j=1}^{i-1} u_j = 1$, then there exists $k \in \{1, \ldots, i-1\}$ such that $u_k = 1$ and $u_j = 0$ for every $j \neq k$, that is to say, $a_i = a_k$ which is not possible because $k < i$. Thus, there exists a positive integer N such that $\sum_{j=1}^{i-1} u_j = 1 + N\, a$. Therefore, $a_i = (1 + N\, a)\, a_1 \geq (1 + a)\, a_1 = a_{n+1}$, where the last equality follows from Lemma 5. However, this inequality implies $i \geq n + 1$, in contradiction to our assumption. □

We emphasize that the hypothesis $S_a(b, n)$ is a numerical semigroup (equivalently, $\gcd(a_1, \ldots, a_n) = 1$) cannot be avoided for the minimality property of $\{a_1, \ldots, a_n\}$; for example, if $b = 2, a = 5$ and $n = 4$, we have that $a_1 = 15, a_2 = 20, a_3 = 30$ and $a_4 = 50$; clearly, a_1 and a_2 suffice to generate $S_a(b, n)$, in this case.

Here and throughout this section, we suppose $\gcd(r_b(n), a) = 1$ so that $S_a(b, n)$ is a numerical semigroup with multiplicity a_1 and, by Theorem 6, of embedding dimension n. We call these semigroups generalized repunit numerical semigroups or grepunit semigroups for short.

Example 7.

(i) If $a = 2^n, a_1 = 2^n - 1$ and $b = 2$, then $S_a(b, n)$ is a Mersenne numerical semigroup (see [12]).
(ii) If $a = b^n$, then $S_a(b, n)$ is a repunit numerical semigroup (see [11]).

The numerical semigroups in Example 7 are part of the larger family of those numerical semigroups which are closed respect to the action of affine maps. A numerical semigroup S is said to be closed respect to the action of an affine map if there exists $\alpha \in \mathbb{N}\backslash\{0\}$ and $\beta \in \mathbb{Z}$ such that $\alpha s + \beta \in S$, for every $s \in S\backslash\{0\}$.

Corollary 8. *$S_a(b, n)$ is closed by the action of the affine map $x \mapsto b\,x + a - (b^n - 1)$.*

Proof. We first, observe that

$$\begin{aligned} b\,a_j + a - (b^n - 1) &= b\,a_j + a - (b-1) r_b(n) \\ &= b\,r_b(n) - (b-1)\,r_b(n) + a\,b\,r_b(j-1) + a \\ &= r_b(n) + a\,b\,r_b(j-1) + a = r_b(n) + a\,r_b(j) \\ &= a_{j+1}, \end{aligned}$$

for every $j \geq 1$.

Now, since $\{a_1, a_2, \ldots, a_n\}$ generates $S_a(b, n)$, given $s \in S\backslash\{0\}$, there exist $u_i \in \mathbb{N},\ i = 1, \ldots, n$, with $u_j \neq 0$ for some j, such that $s = \sum_{i=1}^n u_i a_i$. Therefore,

$$\begin{aligned} b\,s + a - (b^n - 1) &= \sum_{i=1}^{n} (u_i b)\,a_i + a - (b^n - 1) \\ &= \sum_{\substack{i=1 \\ i \neq j}}^{n} (u_i b)\,a_i + (u_j b)\,a_j + a - (b^n - 1) \\ &= \sum_{\substack{i=1 \\ i \neq j}}^{n} (u_i b)\,a_i + \big((u_j - 1) b\big)\,a_j + b\,a_j + a - (b^n - 1) \\ &= \sum_{\substack{i=1 \\ i \neq j}}^{n} (u_i b)\,a_i + \big((u_j - 1) b\big)\,a_j + a_{j+1} \in S, \end{aligned}$$

as claimed. □

In [13], numerical semigroups which are closed respect to the action of the affine maps $x \mapsto \alpha x + \beta$, with $\alpha \in \mathbb{N}\backslash\{0\}$ and $\beta \in \mathbb{N}$, are studied. Therefore, by Corollary 8, the grepunit semigroup $S_a(b, n)$ belongs to the family studied in [13] if and only if $a - (b^n - 1) > 0$; equivalently, $a > b^n - 1$.

Remark 9. Grepunit semigroups could be seen also as shifted numerical monoids in the sense of [8]; since, by Theorem 6, $S_a(b, n)$ is minimally generated by

$$\{a_1, a_1 + a\,r_b(1), \ldots, a_1 + a\,r_b(n-1)\}.$$

Nevertheless, the hypothesis $r_b(n) = a_1 > a^2\, r_b(n-1)^2$ required in [8] only holds for grepunit semigroups when $n = 2$ and $a^2 < b + 1$. Indeed, $r_b(n) > a^2\, r_b(n-1)^2$ if and only if

$$a^2 < \frac{r_b(n)}{r_b(n-1)^2} = \frac{1 + b\, r_b(n-1)}{r_b(n-1)^2} = \frac{1}{r_b(n-1)^2} + \frac{b}{r_b(n-1)}.$$

Now, as the right hand side is either $1+b$, if $n = 2$, or less than $(1+2b)/(1+b)^2 < 1$, otherwise, we are done.

To finish this section, we make explicit a set of relations which characterizes $S_a(b, n)$.

Lemma 10. *For each pair of integers $i \geq 1$ and $j \geq 1$, it holds that $b\, a_i + a_{i+j} = b\, a_{i+j-1} + a_{i+1}$.*

Proof. Since $a_{i+j} = a_1 + a\, r_b(i + j - 1) = a_1 + a\,(r_b(i-1) + b^{i-1}\, r_b(j)) = a_i + a\, b^{i-1}\, r_b(j)$, for every j, we have that

$$\begin{aligned} b\, a_i + a_{i+j} &= b\, a_i + a_i + a\, b^{i-1}\, r_b(j) \\ &= b\, a_i + a_i + a\, b^{i-1}\,(b\, r_b(j-1) + 1) \\ &= b\,(a_i + a\, b^{i-1}\, r_b(j-1)) + a_i + a\, b^{i-1} \\ &= b\, a_{i+j-1} + (a_1 + a\, r_b(i-1)) + a\, b^{i-1} \\ &= b\, a_{i+j-1} + a_{i+1}, \end{aligned}$$

as claimed. □

Let us delve into what it is said in Lemma 10. Consider the subgroup $\mathcal{L}$ of $\mathbb{Z}^n$ generated by the rows of the $(n-1) \times n-$matrix

$$A = \begin{pmatrix} b & -(b+1) & 1 & 0 \dots 0 & 0 & 0 \\ 0 & b & -(b+1) & 1 \dots 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \dots b & -(b+1) & 1 \\ (a+1) & 0 & 0 & 0 \dots 0 & b & -(b+1) \end{pmatrix}.$$

We first observe that, by Lemma 5, all the equalities in Lemma 10 can be written in the form

$$\mathbf{v}\, A \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = 0$$

for some $\mathbf{v} \in \mathbb{Z}^{n-1}$. Moreover, taking into account that, by Lemma 5 again, $a_{n+1} = (a+1)\, a_1$, a direct computation shows that the maximal minors of A are equal to $-a_1, a_2, -a_3, \dots, (-1)^n a_n$. So, $\mathbb{Z}^n/\mathcal{L}$ is a group of rank $n-1$ which is torsion free if and only if $\gcd(a_1, \dots, a_n) = 1$; equivalently, $S_a(b, n)$ is a numerical semigroup by Proposition 4. Thus, in this case, we have that the semigroup homomorphism

$$S_a(b, n) \longrightarrow \mathbb{N}^n/\mathcal{L}; \quad s = \sum_{i=1}^{n} u_i a_i \longmapsto (u_1, \dots, u_n) + \mathcal{L},$$

is an isomorphism; that is to say, $S_a(b,n)$ is the finitely generated commutative monoid corresponding to the congruence $\sim$ on $\mathbb{N}^n$ defined by $\mathbf{u} \sim \mathbf{v} \iff \mathbf{u} - \mathbf{v} \in \mathcal{L}$.

Now, as a straightforward consequence of the results in [6, Section 7.1] we conclude the following.

Corollary 11. *Let* $\daleth$ *be a field. The semigroup ideal of* $\daleth[x_1, \ldots, x_n]$ *associated to* $S_a(b,n)$ *is equal to the ideal* $I_{\mathcal{L}}$ *generated by*

$$\{x_1^{u_1} \cdots x_n^{u_n} - x_1^{v_1} \cdots x_n^{v_n} \;\mid\; u_i - v_i \in \mathcal{L},\; i = 1, \ldots, n\}. \tag{1}$$

Observe that accordingly to the definition of $\mathcal{L}$ the binomials in $I_{\mathcal{L}}$ not involving x_1 are homogeneous.

3. Apéry Sets of Grepunit Semigroups

The main aim of this section is to determine the Apéry set of a grepunit semigroup with respect to its multiplicity. Let us start by recalling what Apéry sets of a numerical semigroup are.

Definition 12. Let S be a numerical semigroup. The Apéry set of S with respect to $s \in S$, denoted $\mathrm{Ap}(S,s)$, is defined as

$$\mathrm{Ap}(S,s) = \{\omega \in S \mid \omega - s \notin S\}.$$

For the sake of simplicity, we write $\mathrm{Ap}(S)$ for the Apéry set of S with respect to its multiplicity, that is, $\mathrm{Ap}(S) = \mathrm{Ap}(S, \mathrm{m}(S))$.

Let a, b and n be three positive integers such that $b > 1, n > 1$ and $\gcd(r_b(n), a) = 1$. As mentioned above, the main objective of this section is to compute $\mathrm{Ap}(S_a(b,n))$. To this end, we first introduce the sets $R(b,i)$.

Definition 13. Let $i \geq 2$ be an integer and define $R(b,i)$ to be the subset of $\mathbb{N}^{i-1}$ whose elements $(u_2, \ldots, u_i)$ satisfy

(a) $0 \leq u_j \leq b$, for every $j = 2, \ldots, i$;
(b) if $u_j = b$, then $u_k = 0$ for every $k < j$.

Observe that

$$R(b,i) = (R(b,i-1) \times \{0, \ldots, b-1\}) \cup \{(0, \ldots, 0, b)\} \subset \mathbb{N}^{i-1}, \tag{2}$$

for every $i \geq 3$.

Lemma 14. *The cardinality of* $R(b,i)$ *is equal to* $r_b(i)$*, for every* $i \geq 2$.

Proof. We proceed by induction on i. If $i = 2$, then $R(b,i) = \{0, 1, \ldots, b\}$ and $r_b(i) = b + 1$. Suppose that $i > 2$ and that the result is true for $i - 1$. By (2), the cardinality of $R(b,i)$ is equal to b times the cardinality of $R(b,i-1)$ plus one. Since, by induction hypothesis, the cardinality of $R(b,i-1)$ is equal to $r_b(i-1)$ and $r_b(i) = b\, r_b(i-1) + 1$, we are done. □

Recall that, by Proposition 4, we have that $S_a(b,n)$ is a grepunit semigroup if and only if $\gcd(r_b(n),a) = 1$. In this case, $S_a(b,n)$ is minimally generated by

$$a_1 := r_b(n), a_2 := r_b(n) + a\, r_b(1), \ldots, a_n := r_b(n) + a\, r_b(n-1),$$

by Theorem 6.

Theorem 15. *With the above notation, we have that*

$$\mathrm{Ap}(S_a(b,n)) = \left\{ \sum_{i=2}^{n} u_i a_i \mid (u_2,\ldots,u_n) \in R(b,n) \right\}.$$

Proof. For the sake of simplicity of notation, we write S for $S_a(b,n)$.

As $\mathrm{Ap}(S) \subset \mathbb{N}$, its elements are naturally ordered: $0 = \omega_1 < \ldots < \omega_{a_1}$. So, we can proceed by induction on the index j of ω_j. If $j = 1$, then $\omega_j = 0$; so, by taking $(0,\ldots,0) \in R(b,n)$ we are done. Suppose now that $j > 1$ and that the result is true for every $j' < j$. Let k be the smallest index such that $\omega_j - a_k \in S$. Clearly, $(\omega_j - a_k) - a_1 \notin S$; otherwise $\omega_j - a_1 = \big((\omega_j - a_k) - a_1\big) + a_k \in S$, in contradiction with the fact that $\omega_j \in \mathrm{Ap}(S)$. Therefore $\omega_j - a_k \in \mathrm{Ap}(S)$ and, by induction hypothesis, there exists $(u_2,\ldots,u_n) \in R(b,n)$ such that $\omega_j - a_k = \sum_{i=2}^{n} u_i a_i$. Thus,

$$\omega_j = \sum_{i=2}^{k-1} u_i a_i + (u_k+1)a_k + \sum_{i=k+1}^{n} u_i a_i = (u_k+1)a_k + \sum_{i=l+1}^{n} u_i a_i,$$

where the second equality follows from the minimality of k. Let us see that $(0,\ldots,0,u_k+1,u_{k+1},\ldots,u_n)$ lies in $R(b,n)$. If $u_k + 1 \le b$ and $u_i < b,\ i \in \{k+1,\ldots,n\}$, we are done. So, we distinguish two cases:

- If $u_k + 1 > b$, then $u_k + 1 = b + 1$. In this case, $(u_k+1)a_k = (b+1)\,a_k = b\,a_k + a_k = b\,a_{k-1} + a_{k+1}$, where the last equality follows from Lemma 10.
- If $u_i = b$, for some $i \in \{k+1,\ldots,n\}$, then $u_k = 0$ and so $(u_k+1)a_k + u_i a_i = a_k + b a_i = b\,a_{k-1} + a_{i+1}$, where the last equality follows from Lemma 10 again.

In both cases, we obtain that $\omega_j - a_{k-1} \in S$ which contradicts the minimality of k. Hence, none of these two cases can occur. □

In [5], the notion of homogeneous numerical semigroups is introduced. Recall that if S is a numerical semigroup minimally generated by $\{a_1,\ldots,a_n\}$, then the set of lengths of $s \in S$ is defined as

$$\mathsf{L}_S(s) := \left\{ \sum_{j=1}^{n} u_j \;\middle|\; s = \sum_{j=1}^{n} u_j\, a_j,\ u_j \ge 0 \right\}.$$

Definition 16. A numerical semigroup is said to be homogeneous if $\mathsf{L}_S(s)$ is a singleton for each $s \in \mathrm{Ap}(S)$.

Proposition 17. *The numerical semigroup $S_a(b,n)$ is homogeneous.*

Proof. By [5, Proposition 3.9] and Corollary 11, it suffices to observe that one of the terms of each non-homogeneous element in (1) for the standard grading on ⅂$[x_1, \ldots, x_n]$ is divisible by x_1. □

Let us see now that, for $i \in \{3, \ldots, n\}$, the Apéry set $\mathrm{Ap}(S_a(b, i))$ can be constructed from the set $R(b, i-1)$. But, first we need a further piece of notation.

Recall that, by Proposition 4, we have that $S_a(b, i)$ is a grepunit semigroup if and only if $\gcd(r_b(i), a) = 1$. In this case, $S_a(b, i)$ is minimally generated by

$$a_1^{(i)} := r_b(i), a_2^{(i)} := a_1^{(i)} + a\, r_b(1), \ldots, a_i^{(i)} := a_1^{(i)} + a\, r_b(i-1),$$

by Theorem 6.

Corollary 18. *Let* $i \in \{3, \ldots, n\}$. *If* $\gcd(r_b(i), a) = 1$, *then* $\omega \in \mathrm{Ap}(S_a(b, i))$ *if and only if* $\omega = b\, a_i^{(i)}$ *or there exist* $(u_2, \ldots u_{i-1}) \in R(b, i-1)$ *and* $u_i \in \{0, \ldots, b-1\}$ *such that*

$$\omega = \sum_{j=2}^{i-1} u_j a_j^{(i-1)} + b^{i-1} \left(\sum_{j=2}^{i-1} u_j \right) + u_i\, a_i^{(i)}. \tag{3}$$

Proof. We observe that $a_j^{(i)} = a_j^{(i-1)} + r_b(i) - r_b(i-1) = a_j^{(i-1)} + b^{i-1},\ j = 0, \ldots, i-1$. So, (3) becomes

$$\omega = \sum_{j=1}^{i-1} u_j a_j^{(i)} + u_i\, a_i^{(i)}$$

and, taking into account (2), our claim readily follows from Theorem 15. □

Notice that, by Theorem 15, an immediate consequence of Corollary 18 is that $\mathrm{Ap}(S_a(b, i))$ can be constructed from $\mathrm{Ap}(S_a(b, i-1))$ provided that both $S_a(b, i)$ and $S_a(b, i-1)$ are numerical semigroups; indeed, the first and second summands in the right hand side of (3) correspond to the elements of $\mathrm{Ap}(S_a(b, i-1))$ and their lengths, respectively. Recall that, by Proposition 17, all elements in $\mathrm{Ap}(S_a(b, i-1))$ have the same length.

If S is a homogeneous numerical semigroup, we write $\mathsf{m}_S(s)$ for the length of $s \in \mathrm{Ap}(S)$. The notation $\mathsf{m}_S(s)$ is usually reserved for the minimal length $s \in S$; clearly, no ambiguity occurs in our case.

The following result is a straightforward consequence of Corollary 18.

Corollary 19. *Let* $i \in \{3, \ldots, n\}$. *If* $\gcd(r_b(i), a) = \gcd(r_b(i-1), a) = 1$, *then* $\omega \in \mathrm{Ap}(S_a(b, i))$ *if and only if* $\omega = b\, a_i^{(i)}$ *or there exist* $\omega' \in \mathrm{Ap}(S_a(b, i-1))$ *and* $u_i \in \{0, \ldots, b-1\}$ *such that*

$$\omega = \omega' + b^{i-1}\, \mathsf{m}_{S_a(b,i-1)}(\omega') + u_i\, a_i^{(i)}. \tag{4}$$

Corollary 19 maintains a great similarity with [8, Theorem 3.3]; however, the techniques used are very different and, more importantly, we do not require the hypothesis $r_b(n) > a^2\, r_b(n-1)^2$ (see Remark 9).

4. The Frobenius Problem

Let a, b and n be three positive integers such that $b{>}1$, $n{>}1$ and $\gcd(r_b(n), a) = 1$.

In this section, we address the Frobenius problem for $S_a(b,n)$. More precisely, we will give a formula for $\mathrm{F}(S_a(b,n))$ in terms of a, b and n. To do this, we take advantage of the following result due to Selmer (see, e.g., [10, Proposition 2.21]).

Proposition 20. *Let S be a numerical semigroup and let $s \in S\backslash\{0\}$. Then,*

(a) $\mathrm{F}(S) = \max \mathrm{Ap}(S,s) - s$;
(b) $\mathrm{g}(S) = \frac{1}{s}\left(\sum_{\omega\in \mathrm{Ap}(S,s)} \omega\right) - \frac{s-1}{2}$.

Before giving our formula for the Frobenius number, we show an interesting result which will be used below and later in the last section. As in the previous section, we write $\mathrm{Ap}(S)$ for $\mathrm{Ap}(S, \mathrm{m}(S))$.

Lemma 21. *If $\alpha_i := a_i + \sum_{j=i}^{n}(b-1)\, a_j,\ i = 2,\ldots,n$, then the following holds:*

(a) $\alpha_i \in \mathrm{Ap}(S_a(b,n))$, *for every* $i = 2,\ldots,n$
(b) *If* $a < b^n - 1$, *then* $\alpha_2 > \ldots > \alpha_n$, *and if* $a > b^n - 1$, *then* $\alpha_2 < \ldots < \alpha_n$.
(c) *For each* $\omega \in \mathrm{Ap}(S_a(b,n))$, *there exists* $i \in \{2,\ldots,n\}$ *such that* $\omega \le \alpha_i$.

Proof. Part (a) is nothing but a particular case of Theorem 15. To prove (b), it suffices to observe that

$$\alpha_i - \alpha_{i+1} = b\, a_i - a_{i+1} = b^n - 1 - a,\ i = 2,\ldots,n-1,$$

and note that $b^n - 1 \neq a$ because $\gcd(r_b(n), a) = 1$ by hypothesis. Finally, to prove (c) we can take advantage of Theorem 15 which state that for each $\omega \in \mathrm{Ap}(S_a(b,n))$, there exist $(u_2,\ldots,u_n) \in R(b,n)$ such that $\omega = \sum_{j=2}^{n} u_j\, a_j$. Clearly, by the definition of $R(b,n)$, if u_i is the leftmost nonzero entry in $(u_2,\ldots,u_n) \in R(b,n)$, then

$$\sum_{j=2}^{n} u_j\, a_j \le a_i + \sum_{j=i}^{n}(b-1)\, a_j = \alpha_i$$

and we are done. □

Theorem 22. *The Frobenius number of $S_a(b,n)$ is equal to*

(a) $(n-1)\,(b^n - 1 - a) + a\, a_1$, *if* $a < b^n - 1$;
(b) $b^n - 1 - a + a\, a_1$, *if* $a > b^n - 1$.

Proof. By Lemma 21, we have that $\max \mathrm{Ap}(S_a(b,n))$ is equal to either $a_2 + \sum_{j=2}^{n}(b-1)\, a_j$, if $a < b^n - 1$, or $b\, a_n$, if $a > b^n - 1$. So, we distinguish two cases:

(a) If $a < b^n - 1$, then $\max \mathrm{Ap}(S_a(b,n)) = \alpha_2$. So, by Selmer's formula (Proposition 20), we obtain that

$$\begin{aligned}\mathrm{F}(S_a(b,n)) &= \left(a_2 + \sum_{j=2}^{n}(b-1)\,a_j\right) - a_1 = \sum_{j=2}^{n}(b-1)\,a_j + a\\ &= \sum_{j=2}^{n}(b-1)\left(a_1 + a\sum_{k=0}^{j-2} b^j\right) + a\\ &= (n-1)(b-1)a_1 + a\sum_{j=2}^{n}(b^{j-1}-1) + a\\ &= (n-1)(b-1)a_1 + a\,a_1 - (n-1)a\\ &= (n-1)\left(b^n - 1 - a\right) + a\,a_1.\end{aligned}$$

(b) If $a > b^n - 1$, then $\max \mathrm{Ap}(S_a(b,n)) = \alpha_n$. So, by Selmer's formula we conclude that

$$\begin{aligned}\mathrm{F}(S_a(b,n)) &= b\,a_n - a_1 = b\,(a_1 + a\,r_b(n-1)) - a_1\\ &= (b-1)a_1 + a\,b,\, r_b(n-1) = b^n - 1 + a\,(a_1 - 1)\\ &= b^n - 1 - a + a\,a_1.\end{aligned}$$

□

Observe that condition $a > b^n - 1$ corresponds to the case considered in [13] (see the comment after Corollary 8).

5. The Genus of Grepunit Semigroups

In this section, we use Selmer's formulas (Proposition 20) to compute the genus of grepunit semigroups in terms of a, b and n.

The following results state some useful properties of the sets $R(b,i)$.

Lemma 23. *Let $b > 1$ be an integer. For each $i \geq 2$, the following holds:*

$$\sum_{(u_2,\ldots u_i)\in R(b,i)} \left(\sum_{j=2}^{i} u_j\right) = \sum_{j=2}^{i} \frac{b^i + b^{i-(j-1)}}{2}.$$

Proof. We proceed by induction on i. If $i = 2$, then $R(b,i) = \{0, 1, \ldots, b\}$ and our claim readily follows. Suppose that $i > 2$ and that the result is true for $i - 1$. Now, since by Lemma 14 the cardinality of $R(b, i-1)$ is equal to $r_b(i-1)$, by (2), we have that

$$\begin{aligned}&\sum_{(u_2,\ldots u_i)\in R(b,i)} \left(\sum_{j=2}^{i} u_j\right)\\ &\quad= \sum_{(u_2,\ldots u_{i-1})\in R(b,i-1)} b\left(\sum_{j=2}^{i-1} u_j\right) + r_b(i-1)\left(\sum_{k=0}^{b-1} k\right) + b.\end{aligned}$$

So, by induction hypothesis, we obtain that

$$\sum_{(u_2,\dots u_i)\in R(b,i)}\left(\sum_{j=2}^{i} u_j\right) = b\sum_{j=2}^{i-1}\frac{b^{i-1}+b^{(i-1)-(j-1)}}{2} + r_b(i-1)\left(\sum_{k=0}^{b-1} k\right) + b$$
$$= \sum_{j=2}^{i-1}\frac{b^{i}+b^{i-(j-1)}}{2} + \frac{b^{i-1}-1}{b-1}\,\frac{b(b-1)}{2} + b$$
$$= \sum_{j=2}^{i-1}\frac{b^{i}+b^{i-(j-1)}}{2} + \frac{b^{i}+b}{2} = \sum_{j=2}^{i}\frac{b^{i}+b^{i-(j-1)}}{2},$$

as claimed. □

As in previous sections, let a, b and n be three positive integers such that $b > 1$ and $n > 1$. Given $i \in \{2,\dots,n\}$, we write

$$a_1^{(i)} := r_b(i), a_2^{(i)} := a_1^{(i)} + a\, r_b(1), \dots, a_i^{(i)} := a_1^{(i)} + a\, r_b(i-1).$$

Proposition 24. *Let $i \in \{2,\dots,n\}$. Then,*

$$\sum_{(u_2,\dots u_i)\in R(b,i)}\left(\sum_{j=2}^{i} u_j a_j^{(i)}\right) = \sum_{j=2}^{i}\frac{b^{i}+b^{i-(j-1)}}{2}\, a_j^{(i)}.$$

Proof. We proceed by induction on i. If $i = 2$, then $R(b,i) = \{0,\dots,b\}$ and

$$\sum_{u_2\in\{0,\dots,b\}}\left(u_2\, a_2^{(2)}\right) = \left(\sum_{u_2\in\{0,\dots,b\}} u_2\right) a_2^{(2)} = \frac{b\,(b+1)}{2}\, a_2^{(2)}.$$

Suppose now $i > 2$ and that the result is true for $i-1$. Since $a_j^{(i)} = a_j^{(i-1)} + b^{i-1},\ j = 1,\dots,i-1$, by (2), we have that

$$\sum_{(u_2,\dots u_i)\in R(b,i)}\left(\sum_{j=2}^{i} u_j a_j^{(i)}\right) = \sum_{(u_2,\dots,u_{i-1})\in R(b,i-1)} b\left(\sum_{j=2}^{i-1} u_j a_j^{(i-1)}\right) +$$
$$+ b^{i-1}\left(\sum_{(u_2,\dots,u_{i-1})\in R(b,i-1)}\left(\sum_{j=2}^{i-1} u_j\right)\right) +$$
$$+ r_b(i-1)\sum_{k=0}^{b-1} k\, a_i^{(i)} + b\, a_i^{(i)}.$$

By induction hypothesis, the first summand of the right hand side is equal to

$$b\left(\sum_{j=2}^{i-1}\frac{b^{i-1}+b^{i-1-(j-1)}}{2}\, a_j^{(i-1)}\right) = \sum_{j=2}^{i-1}\frac{b^{i}+b^{i-(j-1)}}{2}\, a_j^{(i-1)},$$

and, by Lemma 23, the second summand of the right hand side is equal to

$$b^{i-1}\left(\sum_{j=2}^{i-1}\frac{b^i+b^{i-(j-1)}}{2}\right)=\sum_{j=2}^{i-1}\frac{b^i+b^{i-(j-1)}}{2}\,b^{i-1}$$
$$=\sum_{j=2}^{i-1}\frac{b^i+b^{i-(j-1)}}{2}\,\left(a_j^{(i)}-a_j^{(i-1)}\right).$$

Therefore,

$$\sum_{(u_2,\dots u_i)\in R(b,i)}\left(\sum_{j=2}^{i}u_j a_j^{(i)}\right)$$
$$=\sum_{j=2}^{i-1}\frac{b^i+b^{i-(j-1)}}{2}\,a_j^{(i)}++r_b(i-1)\sum_{k=0}^{b-1}k\,a_i^{(i)}+b\,a_i^{(i)}$$
$$=\sum_{j=2}^{i-1}\frac{b^i+b^{i-(j-1)}}{2}\,a_j^{(i)}+\left(\sum_{k=0}^{i-2}b^k\right)\frac{(b-1)b}{2}\,a_i^{(i)}+b\,a_i^{(i)}$$
$$=\sum_{j=2}^{i-1}\frac{b^i+b^{i-(j-1)}}{2}\,a_j^{(i)}+\frac{b^i+b}{2}\,a_i^{(i)}=\sum_{j=2}^{i}\frac{b^i+b^{i-(j-1)}}{2}\,a_j^{(i)},$$

as claimed. □

The next result follows immediately from Theorem 15 and Proposition 24.

Corollary 25. *Let $i\in\{2,\dots,n\}$. Then*

$$\sum_{\omega\in \mathrm{Ap}(S_a(b,i))}\omega=\sum_{j=2}^{i}\frac{b^i+b^{i-(j-1)}}{2}\,a_j^{(i)}.$$

Now, as a direct consequence of Corollary 25 and Selmer's formula (Proposition 20), we obtain the following formula for the genus of $S_a(b,n)$, provided that $\gcd((r_b(n),a)=1$.

$$\mathrm{g}(S_a(b,n))=\frac{\sum_{j=2}^{n}\frac{b^n+b^{n-(j-1)}}{2}\,a_j}{a_1}-\frac{a_1-1}{2}$$
$$=\frac{1}{a_1}\sum_{j=2}^{n}\frac{b^n+b^{n-(j-1)}}{2}\,a_j-\frac{a_1-1}{2} \tag{5}$$

where $a_j=a_j^{(n)},\ j=1,\dots,n$, as usual.

Corollary 26. *The genus of $S_a(b,n)$ is equal to*

$$\mathrm{g}(S_a(b,n))=\frac{(n-1)\,b^n+(a_1-1)\,a}{2}.$$

Proof. Since $a_j = a_1 + a\, r_b(j-1)$, by Eq. (5), we have that

$$\begin{aligned} \mathrm{g}(S_a(b,n)) &= \frac{1}{2\,a_1}\sum_{j=2}^{n}(b^n + b^{n-(j-1)})\,(a_1 + a\, r_b(j-1)) - \frac{a_1-1}{2} \\ &= \frac{1}{2}\sum_{j=2}^{n}(b^n + b^{n-(j-1)}) + \frac{a}{2a_1}\sum_{j=2}^{n}(b^n + b^{n-(j-1)})\, r_b(j-1) - \frac{a_1-1}{2} \\ &= \frac{(n-1)\,b^n}{2} + \frac{a_1-1}{2} + \frac{a}{2a_1}\sum_{j=2}^{n}(b^n + b^{n-(j-1)})\, r_b(j-1) - \frac{a_1-1}{2} \\ &= \frac{(n-1)\,b^n}{2} + \frac{a}{2a_1}\sum_{j=2}^{n}(b^n + b^{n-(j-1)})\, r_b(j-1). \end{aligned}$$

Now, taking into account that $r_b(j-1) = \frac{b^{j-1}-1}{b-1}$, we obtain that

$$\begin{aligned} \mathrm{g}(S_a(b,n)) &= \frac{(n-1)\,b^n}{2} + \frac{a}{2a_1(b-1)}\sum_{j=2}^{n}(b^n + b^{n-(j-1)})\,(b^{j-1}-1) \\ &= \frac{(n-1)\,b^n}{2} + \frac{a}{2a_1(b-1)}\sum_{j=1}^{n-1}(b^{n+j} - b^{n-j}) \\ &= \frac{(n-1)\,b^n}{2} + \frac{a}{2a_1(b-1)}\Big((b^n-1)\sum_{j=1}^{n-1} b^j\Big) \\ &= \frac{(n-1)\,b^n + (a_1-1)\,a}{2}, \end{aligned}$$

as claimed. □

Example 27. If $a = b = 3$ and $n = 4$, then the grepunit semigroup $S = S_a(b,n)$ is minimally generated by $a_1 = 40, a_2 = 43, a_3 = 52$ and $a_4 = 79$. By Corollary 25, we have that $\sum_{\omega\in \mathrm{Ap}(S)}\omega = 54\,a_2 + 45\,a_3 + 42\,a_4 = 7980$. So, by (5), we conclude that

$$\mathrm{g}(S) = \frac{7980}{40} - \frac{39}{2} = 180.$$

Note that, by Corollary 26, we can get $\mathrm{g}(S)$ without computing $\sum_{\omega\in \mathrm{Ap}(S)}\omega$.

6. The Type of Grepunit Semigroups. Wilf's Conjecture

Let S be a numerical semigroup and let $\mathrm{n}(S)$ be the cardinality of $\{s \in S \mid s < \mathrm{F}(S)\}$. Clearly, $\mathrm{g}(s) + \mathrm{n}(S) = \mathrm{F}(S) + 1$. In [14], H.S. Wilf conjectured that

$$\mathrm{F}(S) \le \mathrm{e}(S)\ \mathrm{n}(S) - 1, \tag{6}$$

where $\mathrm{e}(S)$ denotes the embedding dimension of S.

Although there are many families of numerical semigroups for which this conjecture is known to be true, the general case remains unsolved. The numerical semigroups that satisfy Wilf's conjecture are said to be Wilf (see for example the survey [2]).

In this section, we will prove that the grepunit semigroups are Wilf. Since $\mathrm{n}(S) = \mathrm{F}(S) - \mathrm{g}(s) + 1$, we can take advantage of our explicit formulas for the Frobenius number (Theorem 22) and for the genus (Corollary 26) to check that (6) holds. However, we will follow a different approach: we will prove that the grepunit semigroups are Wilf as an immediate consequence of the computation of their pseudo-Frobenius numbers.

Recall that an integer x is a pseudo-Frobenius number of a numerical semigroup S, if $x \notin S$ and $x + S \in S$, for all $s \in S \backslash \{0\}$. We denote by $\mathrm{PF}(S)$ the set of pseudo-Frobenius numbers of S. The cardinality of $\mathrm{PF}(S)$ is called the type of S and it is denoted $\mathrm{t}(S)$.

Given a numerical semigroup S, we write $\preceq_S$ for the partial order on $\mathbb{Z}$ such that $y \preceq_S x$ if and only if $x - y \in S$.

The following result is [10, Proposition 2.20].

Proposition 28. *Let S be a numerical semigroup. If s is a nonzero element of S, then*

$$\mathrm{PF}(S) = \left\{ \omega - s \ \mid \ \omega \in \mathrm{Maximals}_{\preceq_S} \mathrm{Ap}\left(S, s\right) \right\}.$$

As in the previous sections, let a, b and n be three positive integers such that $b > 1, n > 1$ and $\gcd(r_b(n), a) = 1$.

Proposition 29. *The set of maximal elements of $\mathrm{Ap}\left(S_a(b,n)\right)$ with respect to $\preceq_{S_a(b,n)}$ is equal to*

$$\left\{ a_i + (b-1)\sum_{j=i}^{n} a_j \ \mid \ i = 2, \ldots, n \right\}.$$

Proof. Set $S := S_a(b,n)$ and $\alpha_i := a_i + \sum_{j=i}^{n}(b-1)\, a_j,\ i = 2, \ldots, n$. From the proof of Lemma 21(c), we have that, for each $\omega \in \mathrm{Ap}(S)$, there exists α_i such that $\alpha_i - \omega \in S$. Therefore, $\mathrm{Maximals}_{\preceq_S} \mathrm{Ap}(S) \subseteq \{\alpha_2, \ldots, \alpha_n\}$. Now, since by Lemma 21, $\alpha_2 > \ldots > \alpha_n$, if $a < b^n - 1$, and $\alpha_2 < \ldots < \alpha_n$, if $a > b^n - 1$; in order to prove the reverse inclusion, we distinguish two cases:

- If $a < b^n - 1$, then $\alpha_j - \alpha_i = (i-j)(b^n - (a+1)) = \alpha_{n-(i-j)} - \alpha_n$, for every $i > j$. Now, if there exist $i > j$ such that $\alpha_i \preceq_S \alpha_j$, then $\alpha_{n-(i-j)} - \alpha_n \in S$. So, $\alpha_{n-(i-j)} = \alpha_n + \sum_{l=2}^{n} u_l a_l$ for some $u_l \in \mathbb{N},\ l = 2, \ldots, n$, not all zero. If $k \in \{2, \ldots, n\}$ is such that $u_k \neq 0$, then $\alpha_{n-(i-j)} = \alpha_n + a_k + s$, where $s = \sum_{l=2}^{n} u_l a_l - a_k \in S$. Thus, since $\alpha_n = b a_n$ and $b a_n + a_k = b a_{k-1} + (a+1) a_1$, we conclude that $\alpha_{n-(i-j)} - a_1 = b a_{k-1} + a\, a_1 + s \in S$, which is not possible because $\alpha_{n-(i-j)} \in \mathrm{Ap}(S)$.
- If $a > b^n - 1$, then $\alpha_j - \alpha_i = (j-i)((a+1) - b^n) = \alpha_{j-i+2} - \alpha_2$, for every $i < j$. Now, if there exist $i > j$ such that $\alpha_i \preceq_S \alpha_j$, then $\alpha_{j-i+2} - \alpha_2 \in S$. So, $\alpha_{j-i+2} = \alpha_2 + \sum_{l=2}^{n} u_l a_l = a_2 + \sum_{l=2}^{n}(u_l + b - 1) a_l$, for some $u_l \in \mathbb{N},\ l = 1, \ldots, n$, not all zero. If $k \in \{2, \ldots, n\}$ is such that $u_k \neq 0$, then $\alpha_{n-(i-j)} = a_2 + b\, a_k + s$, where $s = \sum_{l=2}^{n}(u_l + b - 1) a_l - b\, a_k \in S$. Thus, since $a_2 + b\, a_k = b a_1 + a_{k+1}$, if $k < n$, and $a_2 + b\, a_k = (b + a + 1)\, a_1$, if $k = n$, we conclude that $\alpha_{j-i+2} - a_1 \in S$, a contradiction again.

Finally, since we have shown that $\alpha_i - \alpha_j \notin S_a(b,n)$, for every $i,j \in \{2,\ldots,n\}$, we conclude that $\alpha_i \in \mathrm{Maximals}_{\preceq_S}(\mathrm{Ap}(S))$, for every $i = 2,\ldots,n$, as desired. □

Corollary 30. *The set of pseudo-Frobenius numbers of $S_a(b,n)$ is equal to*

$$\{(n-i+1)\,(b^n-1-a)+a\,a_1 \ \mid\ i=2,\ldots,n\}\,.$$

Consequently, the type of $S_a(b,n)$ is $n-1$.

Proof. By Propositions 28 and 29, $\mathrm{PF}(S_a(b,n)) = \{a_i + (b-1)\sum_{j=i}^{n} a_j - a_1 \ \mid\ i=2,\ldots,n\}$. Now, taking into account that

$$\begin{aligned} a_i + (b-1)\sum_{j=i}^{n} a_j - a_1 &= a\,r_b(i-1) + (b-1)(n-i+1)a_1 \\ &\quad + (b-1)\,a\sum_{j=i}^{n} r_b(j-1) \\ &= a\,r_b(i-1) + (n-i+1)(b^n-1) + a\sum_{j=i}^{n}(b^{j-1}-1) \\ &= a\,r_b(i-1) + (n-i+1)(b^n-1-a) \\ &\quad + a\,(a_1 - r_b(i-1)) \\ &= (n-i+1)(b^n-1-a) + a\,a_1, \end{aligned}$$

for every $i \in \{2,\ldots,n\}$, and we are done. □

Finally, since, by [3, Theorem 20], $\mathrm{F}(S) \le (\mathrm{t}(S)+1)\,\mathrm{n}(S) - 1$, for every numerical semigroup S, and, by Corollary 30, $\mathrm{t}(S_a(b,n)) + 1 = \mathrm{e}(S_a(b,n))$, we conclude that generalized repunit numerical semigroups are Wilf.

Acknowledgements

The authors wish to thank the referees for their valuable comments and suggestions that have undoubtedly helped to increase the clarity of the presentation of this paper.

Funding. Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data Availability Statement. All data generated or analyzed during this study are included in this published article.

References

[1] Branco, M.B., Colaço, I., Ojeda, I.: Minimal binomial systems of generators for the ideals of certain monomial curves. Mathematics **9**(24), 3204 (2021)

[2] Delgado, M.: Conjecture of Wilf: A Survey. Numerical Semigroups, Springer INdAM Ser., vol. 40, pp. 39–62. Springer, Cham (2020)

[3] Fröberg, R., Gottlieb, C., Häggkvist, R.: On numerical semigroups. Semigroup Forum **35**, 63–83 (1987)

[4] García-Sánchez, P.A., Ojeda, I.: Uniquely presented finitely generated commutative monoids. Pac. J. Math. **248**(1), 91–105 (2010)

[5] Jafari, R., Zarzuela Armengou, S.: Homogeneous numerical semigroups. Semigroup Forum **97**, 278–306 (2018)

[6] Miller, E., Sturmfels, B.: Combinatorial Commutative Algebra. Graduate Texts in Mathematics, vol. 227. Springer, New York (2005)

[7] Ojeda, I., Vigneron-Tenorio, A.: Indispensable binomials in semigroups ideals. Proc. Am. Math. Soc. **138**, 4205–4216 (2010)

[8] O'Neill, C., Pelayo, R.: Apéry sets of shifted numerical monoids. Adv. Appl. Math. **97**, 27–35 (2018)

[9] Ramírez Alfonsín, J.L.: The Diophantine Frobenius Problem, Oxford Lecture Series in Mathematics and its Applications (2005)

[10] Rosales, J.C., García-Sánchez, P.A.: Numerical Semigroups. Developments in Mathematics, vol. 20. Springer, New York (2009)

[11] Rosales, J.C., Branco, M.B., Torrão, D.: The Frobenius problem for repunit numerical semigroups. Ramanujan J. **40**, 323–334 (2016)

[12] Rosales, J.C., Branco, M.B., Torrão, D.: The Frobenius problem for Mersenne numerical semigroups. Math. Z. **286**, 741–749 (2017)

[13] Ugolini, S.: On numerical semigroups closed with respect to the action of affine maps. Publ. Math. Debrecen **90**, 149–167 (2017)

[14] Wilf, H.S.: A circle-of-lights algorithm for the "money-changing problem". Am. Math. Mon. **85**(7), 562–565 (1978)

Manuel B. Branco
Departamento de Matemáticas
Universidade de Évora
7000-671 Évora
Portugal
e-mail: mbb@uevora.pt

Isabel Colaço
Departamento de Matemática e Ciências Físicas
Instituto Politécnico de Beja
7800-295 Beja
Portugal
e-mail: `isabel.colaco@ipbeja.pt`

Ignacio Ojeda
Departamento de Matemáticas
Universidad de Extremadura
06071 Badajoz
Spain
e-mail: `ojedamc@unex.es`

Received: December 2, 2021.

Revised: June 2, 2022.

Accepted: November 25, 2022.

CHAPTER 4

Minimal free resolutions of generalized repunit algebras

In this chapter, the generalized repunit numerical semigroup $S_a(b, n)$ will be simply denoted by S, so $\mathcal{A} = \{a_1, \ldots, a_n\}$ is the minimal system of generators of S, where $a_i = r_b(n) + a r_b(i-1)$, for $i = 1, \ldots, n$, and the positive integer a satisfies $\gcd(a_1, a) = 1$.

Let $\mathbb{k}$ be a field. The $\mathbb{k}$-algebra $\mathbb{k}[S]$ is referred to be a generalized repunit algebra in this chapter.

As proved in Chapter 2, we have,

$$I_{\mathcal{A}} = I_2 \begin{pmatrix} x_1^b & \ldots & x_{n-1}^b & x_n^b \\ x_2 & \ldots & x_n & x_1^{a+1} \end{pmatrix},$$

where $I_2(X)$ stands for the ideal generated by the 2×2 minors of the matrix X.

Since $I_{\mathcal{A}}$ is a determinantal ideal, the generalized repunit algebra $\mathbb{k}[S]$ can be resolved by the Eagon-Northcott complex of the matrix X. As an immediate consequence of this result, we have that generalized repunit numerical semigroups of a same embedding dimension n have the same Betti numbers, which are $\beta_0 = 1$ and $\beta_j = j\binom{n}{j+1}$, for $0 < j \leq n-1$. Moreover, since S is homogeneous and its tangent cone is Cohen-Macaulay, we have that S is of homogeneous type, that is, the Betti sequences of S and of its tangent cone coincide. The main result is the explicit description of a minimal $S-$graded free of resolution of $\mathbb{k}[S]$, in terms of $\mathcal{A}$. Notice all these results are also valid in case $b = 1$, as proved by P. Gimenez at al. in [**25**].

The following paper is a preprint available at `https://arxiv.org/abs/2312.06013` distributed under CC BY 4.0: Creative Commons Attribution. It is published in [Colaço, I., & Ojeda, I. (2024). Minimal free resolution of generalized repunit algebras. Communications in Algebra, 1–8)] `https://doi.org/10.1080/00927872.2024.2394968`.

MINIMAL FREE RESOLUTION OF GENERALIZED REPUNIT ALGEBRAS

ISABEL COLAÇO AND IGNACIO OJEDA

Abstract. Let $\Bbbk$ be an arbitrary field and let $b > 1, n > 1$ and a be three positive integers. In this paper we explicitly describe a minimal $S-$graded free resolution of the semigroup algebra $\Bbbk[S]$ when S is a generalized repunit numerical semigroup, that is, when S is the submonoid of $\mathbb{N}$ generated by $\{a_1, a_2, \ldots, a_n\}$ where $a_1 = \sum_{j=0}^{n-1} b^j$ and $a_i - a_{i-1} = a\, b^{i-2},\ i = 2, \ldots, n$, with $\gcd(a, a_1) = 1$.

1. Introduction

Let $\Bbbk[\mathbf{x}] = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over an arbitrary field $\Bbbk$, let S be the numerical semigroup with minimal system of generators $A = \{a_1, \ldots, a_n\} \subset \mathbb{N}$ (see [13] for details on numerical semigroups) and let $\Bbbk[S] := \bigoplus_{a\in S} \Bbbk\chi^a$ be the semigroup $\Bbbk-$algebra of S.

Considering the ring $\Bbbk[\mathbf{x}]$ graded by S via $\deg(x_i) = a_i,\ i = 1, \ldots, n$, we have that the kernel of the $\Bbbk-$algebra homomorphism

$$\varphi_A : \Bbbk[\mathbf{x}] \longrightarrow \Bbbk[S], x_i \mapsto \chi^{a_i}$$

determines a presentation of $\Bbbk[S]$ as $S-$graded $\Bbbk[\mathbf{x}]-$module. Indeed, the so-called *toric ideal* $I_A := \ker(\varphi_A)$ is known (see, e.g. [15, Lemma 4.1]) to be generated by

$$\left\{ \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \ \mid\ \sum_{i=1}^{n} u_i a_i = \sum_{i=1}^{n} v_i a_i,\ \mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{N}^n \right\},$$

where $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_n^{u_n}$. In particular, it is homogeneous for the grading determined by S.

So, if $\{f_i := \mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i} \ \mid\ i = 1, \ldots, \beta_1\}$ is a minimal generating system of I_A and φ_0 denotes the corresponding canonical projection, then

$$\Bbbk[\mathbf{x}]^{\beta_1} \overset{\varphi_1 := (f_1, \ldots, f_{\beta_1})}{\longrightarrow} \Bbbk[\mathbf{x}] \overset{\varphi_0}{\longrightarrow} \Bbbk[\mathbf{x}]/I_A \cong \Bbbk[S] \to 0$$

2020 *Mathematics Subject Classification.* Primary: 16W50, 13D02 secondary: 20M14.

Key words and phrases. Graded rings and modules; syzygies, resolutions, complexes and commutative rings; commutative semigroups.

The authors are partially supported by Proyecto de Excelencia de la Junta de Andalucía (ProyExcel_00868) and by Proyecto de investigación del Plan Propio - UCA 2022-2023 (PR2022-011). The second author is partially supported by grant PID2022-138906NB-C21 funded by MICIU/AEI/10.13039/501100011033 and by ERDF/EU', and by research group FQM024 funded by Junta de Extremadura (Spain)/FEDER funds.

is exact and $S-$graded by suitable degree shiftings of the leftmost free module. Now, one can compute a minimal system of generators of the kernel of φ_1, say $\{\mathbf{f}_{12}, \ldots, \mathbf{f}_{\beta_2 2}\} \subset \Bbbk[\mathbf{x}]^{\beta_1}$, so that the sequence

$$\Bbbk[\mathbf{x}]^{\beta_2} \stackrel{\varphi_2 := (\mathbf{f}_{12}|\ldots|\mathbf{f}_{\beta_2 2})}{\longrightarrow} \Bbbk[\mathbf{x}]^{\beta_1} \stackrel{\varphi_1}{\longrightarrow} \Bbbk[\mathbf{x}] \stackrel{\varphi_0}{\longrightarrow} \Bbbk[\mathbf{x}]/I_A \cong \Bbbk[S] \to 0$$

is exact and, after the appropriate degree shifts, $S-$graded. So, by repeating this process as many times as necessary until reaching $\ker \varphi_p = 0$, which is guaranteed by the Hilbert syzygy theorem (see, e.g., [5, Theorem 1.13]), we obtain a *minimal $S-$graded free resolution of* $\Bbbk[S]$. The minimal free resolution is unique up to isomorphism (see [5, Section 20.1]). The $\beta_i,\ i = 1, \ldots, p$, are called *Betti numbers of* $\Bbbk[S]$ (see Remark 5 for more details).

Computing a minimal free resolution of $\Bbbk[S]$ is possible using Groebner bases techniques. Other related tasks are to characterize the minimal free resolution $S-$graded in terms of the combinatorics within S (see, for example, [3, 12]) or, for special cases of S, to describe explicitly a minimal $S-$graded free resolution of S in terms of S basically (see e.g. [9]). This article is about the latter.

Let b and n be two integers greater than one and let S be the submonoid of $\mathbb{N}$ generated by $\{a_1, a_2, \ldots\} \subset \mathbb{N}$, where

$$a_1 = \sum_{j=0}^{n-1} b^j \ \text{ and } \ a_i - a_{i-1} = a\, b^{i-2},\ i \geq 2,$$

for $a \in \mathbb{Z}_+$ relatively prime with a_1. In [2], it is proved that S is a numerical semigroup whose minimal generating system is $A := \{a_1, \ldots, a_n\}$. These numerical semigroups are called *generalized repunit numerical semigroups* (see [1, 2]) as they generalize the repunit numerical semigroups introduced in [14].

The aim of this paper is to explicitly describe a minimal $S-$graded free of resolution of $\Bbbk[S]$ when S is a generalized repunit numerical semigroup. In what follows, we consider S to be a generalized repunit numerical semigroup and refer $\Bbbk[S]$ as a generalized repunit $\Bbbk-$algebra.

We notice that if $b = 1$, then S is generated by an arithmetic sequence. In this case, the $S-$graded free of resolution of $\Bbbk[S]$ is fully described by P. Gimenez et al. in [9]. The minimal free resolution of numerical semigroups generated by arithmetic sequences has its own interest as, for instance, the Betti numbers of $\Bbbk[S]$ and the coordinate ring of its tangent cone ring coincide. We emphasize that, by [1, Corollary 2] and [7, Theorem 3.12], generalized repunit $\Bbbk-$algebras also have this property.

Finally, we emphasize that in [6, 16] similar techniques are applied to families closely related to ours. In particular, in [6, Section 4] the authors use Eagon-Northcott complexes to compute the Pseudo-Frobenius numbers of numerical semigroups associated to certain determinantal ideals. These ideas are brilliantly generalized in [16, Section 2.1].

2. The minimal free resolution

Let $b > 1, n > 1$ and $a > 1$ be three fixed integer numbers such that a and $a_1 = \sum_{j=0}^{n-1} b^j$ are relatively prime. With the same notation as in the introduction, let S be the generalized repunit numerical semigroup generated by $A = \{a_1, \ldots, a_n\}$.

In [1] it is proved that I_A is minimally generated by $2 \times 2-$minors of the matrix

$$X := (x_{ij}) = \begin{pmatrix} x_1^b & \cdots & x_{n-1}^b & x_n^b \\ x_2 & \cdots & x_n & x_1^{a+1} \end{pmatrix}. \tag{1}$$

Therefore, since I_A is a determinantal ideal, the generalized repunit $\mathbb{k}-$algebra $\mathbb{k}[S]$ can be resolved by the Eagon-Northcott complex introduced in [4] and described below.

Let y_1, y_2 be two indeterminates and let M_j be the $\mathbb{k}[\mathbf{x}]-$submodule of $\mathbb{k}[\mathbf{x}][y_1, y_2]$ generated by the monomials in y_1 and y_2 of degree j. Define

$$\mathbb{k}[\mathbf{x}]_j^X := \bigwedge^{j+1} \mathbb{k}[\mathbf{x}]^n \otimes_{\mathbb{k}[\mathbf{x}]} M_{j-1},\ j = 1, \ldots, n-1,$$

where $\bigwedge^{j+1} \mathbb{k}[\mathbf{x}]^n$ is the degree $j+1$ component of the exterior algebra of the free $\mathbb{k}[x]-$module $\mathbb{k}[x]^n$. Thus, if $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the usual basis of $\mathbb{k}[\mathbf{x}]^n$; that is, the basis of $\mathbb{k}[\mathbf{x}]^n$ such that $\mathbf{e}_i$ has a one in place i and zeros elsewhere, for each $i \in \{1, \ldots, n\}$, then the $\mathbb{k}[\mathbf{x}]-$module $\bigwedge^{j+1} \mathbb{k}[\mathbf{x}]^n$ is generated by $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_{j+1}}$, for each $1 \leq i_1 < \cdots < i_{j+1} \leq n$, for each $j \in \{1, \ldots, n-1\}$.

Now, since the codimension of I_A is $n-1$, because I_A defines an irreducible monomial curve in the $n-$dimensional affine space over $\mathbb{k}$, by [4, Theorem 2], we conclude that

$$0 \to \mathbb{k}[\mathbf{x}]_{n-1}^X \xrightarrow{d_{n-1}} \mathbb{k}[\mathbf{x}]_{n-2}^X \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} \mathbb{k}[\mathbf{x}]_1^X \xrightarrow{d_1} \mathbb{k}[\mathbf{x}] \longrightarrow \mathbb{k}[\mathbf{x}]/I_A \cong \mathbb{k}[S] \to 0$$

is a minimal free resolution of $\mathbb{k}[S]$, with

$$d_1(\mathbf{e}_i \wedge \mathbf{e}_j \otimes 1) = \begin{vmatrix} x_{1i} & x_{1j} \\ x_{2i} & x_{2j} \end{vmatrix}, \text{ for every } 1 \leq i < j \leq n, \tag{2}$$

and

$$d_j(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_{j+1}} \otimes y_1^{u_1} y_2^{u_2}) = \sum_{k \overset{*}{=} 1}^{2} \sum_{l=1}^{j+1} (-1)^{l+1} x_{k i_l} \mathbf{e}_{i_1} \wedge \cdots \wedge \widehat{\mathbf{e}_{i_l}} \wedge \cdots \wedge \mathbf{e}_{i_{j+1}} \otimes y_1^{u_1} y_2^{u_2} y_k^{-1}, \tag{3}$$

for every $1 \leq i_1 < \cdots < i_{j+1} \leq n,\ u_1, u_2 \in \mathbb{N}$ such that $u_1 + u_2 = j-1$ and $j \in \{2, \ldots, n-1\}$, where the asterisk means that we only sum over those k for which $u_k > 0$ and $\widehat{\mathbf{e}_{i_l}}$ means omitting $\mathbf{e}_{i_l}$.

Lemma 1. *For each $j \in \{1, \ldots, n-1\}$, the $\mathbb{k}[\mathbf{x}]-$module $\mathbb{k}[\mathbf{x}]_j^X$ is isomorphic to $\mathbb{k}[\mathbf{x}]^{j\binom{n}{j+1}}$.*

Proof. Since $\bigwedge^{j+1} \mathbb{k}[\mathbf{x}]^n$ and M_{j-1} are isomorphic as $\mathbb{k}[\mathbf{x}]-$modules to $\mathbb{k}[\mathbf{x}]^{\binom{n}{j+1}}$ and $\mathbb{k}[\mathbf{x}]^j$, respectively, for each $j \in \{1, \ldots, n-1\}$, we have that

$$\mathbb{k}[\mathbf{x}]_j^X = \bigwedge^{j+1} \mathbb{k}[\mathbf{x}]^n \otimes_{\mathbb{k}[\mathbf{x}]} M_{j-1} \cong \mathbb{k}[\mathbf{x}]^{\binom{n}{j+1}} \otimes_{\mathbb{k}[\mathbf{x}]} \mathbb{k}[\mathbf{x}]^j \cong \mathbb{k}[\mathbf{x}]^{j\binom{n}{j+1}},$$

for each $j = 1, \ldots, n-1$. □

So, applying the previous lemma, we have the following.

Proposition 2. *Let ϕ_0 be the identity map of $\Bbbk[\mathbf{x}]$ and, for each $j \in \{1, \ldots, n-1\}$, fix a $\Bbbk[\mathbf{x}]-$module isomorphism $\phi_j : \Bbbk[\mathbf{x}]_j^X \to \Bbbk[\mathbf{x}]^{j\binom{n}{j+1}}$. If $\beta_j = j\binom{n}{j+1}$ and $\delta_j = \phi_{j-1} \circ d_j \circ \phi_j^{-1}$, $j = 1, \ldots, n-1$, then*

$$0 \to \Bbbk[\mathbf{x}]^{\beta_{n-1}} \xrightarrow{\delta_{n-1}} \Bbbk[\mathbf{x}]^{\beta_{n-2}} \xrightarrow{\delta_{n-2}} \cdots \xrightarrow{\delta_2} \Bbbk[\mathbf{x}]^{\beta_1} \xrightarrow{\delta_1} \Bbbk[\mathbf{x}] \longrightarrow \Bbbk[\mathbf{x}]/I_A \cong \Bbbk[S] \to 0 \tag{4}$$

is a minimal free resolution of $\Bbbk[S]$.

For a clearer understanding of Proposition 2, we provide, as an illustrative example, the well-known minimal free resolution of $\Bbbk[S]$ for $n = 3$ (see, for example, [11, Theorem 2.3] or, in broader generality, the Hilbert-Burch theorem [5, Theorem 20.15]).

Example 3. Let S be the numerical semigroup generated by $a_1 = 1 + b + b^2, a_2 = 1 + b + b^2 + a$ and $a_3 = 1 + b + b^2 + a(1 + b)$. In this case, a minimal free resolution of $\Bbbk[S]$ is equal to

$$0 \to \Bbbk[\mathbf{x}]^2 \xrightarrow{\delta_2} \Bbbk[\mathbf{x}]^3 \xrightarrow{\delta_1} \Bbbk[\mathbf{x}] \longrightarrow \Bbbk[S] \to 0$$

where δ_2 and δ_1 are the $\Bbbk[\mathbf{x}]-$module homomorphisms whose matrices with respect to the corresponding usual bases are

$$A_2 = \begin{pmatrix} x_1^b & x_2 \\ x_2^b & x_3 \\ x_3^b & x_1^{a+1} \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} x_2^b x_1^{a+1} - x_3^{b+1} & -x_1^{a+b+1} + x_2 x_3^b & x_1^b x_3 - x_2^{b+1} \end{pmatrix},$$

respectively. Clearly, in this case, $\beta_1 = 3$ and $\beta_2 = 2$.

3. The minimal $S-$graded free resolution

The minimal free resolution (4), given in the previous section by using Eagon-Northcott, is not $S-$graded in general. The reason for this is that the maps δ_i, $i = 1, \ldots, n-1$, described in Proposition 2, are not necessarily $S-$homogeneous of degree 0.

To achieve a minimal free resolution $S-$graded of $\Bbbk[S]$, we must appropriately shift the free $\Bbbk[\mathbf{x}]-$modules that appear in $\Bbbk[\mathbf{x}]^{\beta_j}$, $j = 1, \ldots, n$, in such a way the maps δ_j, $j = 1, \ldots, n$, defined in Proposition 2 become S-homogeneous of degree 0. More precisely, we need to find positive integers $s_{j,k}$, $k = 1, \ldots, \beta_j$, $j = 1, \ldots, n-1$, such that the maps in the minimal free resolution

$$0 \to \bigoplus_{k=1}^{\beta_{n-1}} \Bbbk[\mathbf{x}](-s_{n-1,k}) \xrightarrow{\delta_{n-1}} \bigoplus_{k=1}^{\beta_{n-2}} \Bbbk[\mathbf{x}](-s_{n-2,k}) \xrightarrow{\delta_{n-2}} \cdots$$

$$\cdots \xrightarrow{\delta_2} \bigoplus_{k=1}^{\beta_1} \Bbbk[\mathbf{x}](-s_{1,k}) \xrightarrow{\delta_1} \Bbbk[\mathbf{x}] \longrightarrow \Bbbk[\mathbf{x}]/I_A \cong \Bbbk[S] \to 0$$

are $S-$homogeneous of degree 0. Recall that $\Bbbk[\mathbf{x}](-s)$ means that the basis elements of $\Bbbk[\mathbf{x}](-s)$ as $\Bbbk[x]-$module, say 1, have degree s. Thus, for example, $x_1 \in \Bbbk[\mathbf{x}](-s)$ has $S-$degree $a_1 + s$.

Example 4. By considering the degree-shift isomorphisms

$$\Bbbk[\mathbf{x}]^2 \cong \Bbbk[\mathbf{x}](-b\,a_1 - (b+1)a_3) \bigoplus \Bbbk[\mathbf{x}](-a_2 - (b+1)a_3)$$

and

$$\Bbbk[\mathbf{x}]^3 \cong \Bbbk[\mathbf{x}](-(b+1)a_3) \bigoplus \Bbbk[\mathbf{x}](-a_2 - b\,a_3) \bigoplus \Bbbk[\mathbf{x}](-(b+1)a_2)$$

in Example 3, we obtain a minimal $S-$graded free resolution of $\Bbbk[S]$ because these shifts make δ_2 and δ_1 $S-$homogeneous of degree 0.

Remark 5. Given $j \in \{1, \ldots, n-1\}$, we have that $\mathrm{Tor}_j^{\Bbbk[\mathbf{x}]}(\Bbbk, \Bbbk[S])_s \neq 0$ if and only if $s = s_{j,k}$ for some $1 \leq k \leq \beta_j$ (see, e.g. [10, Lemma 1.32]); in fact, the number of $s_{j,k}$'s that are equal to a given $s \in S$ is $\dim(\mathrm{Tor}_j^{\Bbbk[\mathbf{x}]}(\Bbbk, \Bbbk[S])_s)$. Summarizing, the integers $s_{j,k}$ are uniquely determined by $\Bbbk[S]$.

Our goal is to compute the integers $s_{i,k}$. To start, we introduce additional notation. From now on we will write $a_{n+1} = (a+1)a_1$ and $c = b^n - 1 - a$.

Lemma 6. *With the notation above,* $b\,a_i = c + a_{i+1},\ i = 1, \ldots, n$.

Proof. Clearly, $b\,a_1 = b\left(\sum_{j=0}^{n-1} b^j\right) = b^n - 1 + a_1 = b^n - 1 - a + a_2 = c + a_2$. Now, if $i \in \{2, \ldots, n\}$, then

$$\begin{aligned} b\,a_i &= b\,a_1 + b\,a\left(\sum_{j=0}^{i-2} b^j\right) = c + a_2 + a\left(\sum_{j=1}^{i-1} b^j\right) \\ &= c + a_1 + a + a\left(\sum_{j=1}^{i-1} b^j\right) = c + a_1 + a\left(\sum_{j=0}^{i-1} b^j\right) = c + a_{i+1}, \end{aligned}$$

and we are done. □

Proposition 7. *The maps in the exact sequence*

$$\bigoplus_{i=1}^{n-1}\left(\bigoplus_{j=0}^{i-1} \Bbbk[\mathbf{x}](-a_{n-i+1} - b\,a_{n-j})\right) \xrightarrow{\delta_1} \Bbbk[\mathbf{x}] \longrightarrow \Bbbk[\mathbf{x}]/I_A \cong \Bbbk[S] \to 0$$

are $S-$homogeneous of degree 0. *In particular, the set* $\left\{s_{1,k} \mid k = 1, \ldots, \beta_1 = \binom{n}{2}\right\}$ *is equal to* $\{c + a_{i_1} + a_{i_2} \mid 1 < i_1 < i_2 \leq n+1\}$.

Proof. In [1] it is proved that I_A is minimally generated by $2 \times 2-$minors of the matrix X defined in (1). Then first part follows straightforward from the definition of d_1 (see (2)). Now, the second part is an immediate consequence of Lemma 6. □

Recall that $x \in \mathbb{N} \setminus S$ is said to be a *pseudo-Frobenius element of* S if $x + s \in S$ for every $s \in S \setminus \{0\}$. This set is known to be finite and is denoted by $\mathrm{PS}(S)$ (see [13, Section 2.4] for more details).

Lemma 8. *The set* $\{s_{n-1,k} \mid k = 1, \ldots, \beta_{n-1} = n-1\}$ *is equal to*

$$\left\{k\,c + \sum_{i=2}^{n+1} a_i \mid k \in \{1, \ldots, \beta_{n-1} = n-1\}\right\}.$$

Proof. By [8, Corollary 17], we have that

$$\mathrm{PF}(S) = \left\{ s - \sum_{i=1}^{n} a_i \mid s \in \{s_{n-1,1}, \ldots, s_{n-1,n-1}\} \right\}.$$

Now since, by [2, Corollary 30], $\mathrm{PF}(S) = \{k\, c + a\, a_1 \mid k = 1, \ldots, n-1\}$, our claim follows. □

Proposition 9. *The maps in the exact sequence*

$$0 \to \bigoplus_{k=1}^{n-1} \Bbbk[\mathbf{x}](-s_{n-1,k}) \xrightarrow{\delta_{n-1}} \bigoplus_{k=2}^{n-1} \left(\bigoplus_{j=1}^{n} \Bbbk[\mathbf{x}](b\, a_j - s_{n-1,k}) \right)$$

are $S-$homogeneous of degree 0. Moreover, the set $\{s_{n-2,k} \mid k = 1, \ldots, \beta_{n-2} = (n-2)n\}$ is equal to

$$\left\{ k\, c + \sum_{\substack{i=2 \\ i \neq j+1}}^{n+1} a_i \mid k \in \{1, \ldots, n-2\} \text{ and } j \in \{1, \ldots, n\} \right\}.$$

Proof. For the first part, it suffices to observe the matrix of the map d_{n-1} defined in (3) with respect to the usual bases is

$$\begin{pmatrix} x_1^b & x_2 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{n-1}^b & x_n & 0 & \ldots & 0 & 0 \\ x_n^b & x_1^{a+1} & 0 & \ldots & 0 & 0 \\ 0 & x_1^b & x_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & x_{n-1}^b & x_n & \ldots & 0 & 0 \\ 0 & x_n^b & x_1^{a+1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & x_1^b & x_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & x_{n-1}^b & x_n \\ 0 & 0 & 0 & \ldots & x_n^b & x_1^{a+1} \end{pmatrix}.$$

For the second part, we first observe that, by Lemma 8, we can suppose $s_{n-1,k} = k\, c + \sum_{i=2}^{n+1} a_i,\ k = 2, \ldots, n-1$. Therefore, by Lemma 6, we have that

$$\begin{aligned} s_{n-1,k} - b\, a_j = k\, c + \sum_{i=2}^{n+1} a_i - b\, a_j &= k\, c + \sum_{i=2}^{n+1} a_i - c - a_{j+1} \\ &= (k-1)c + \sum_{i=2}^{n+1} a_i - a_{j+1} = (k-1)c + \sum_{\substack{i=2 \\ i \neq j+1}}^{n+1} a_i, \end{aligned}$$

for every $j \in \{1, \ldots, n\}$ and $k \in \{2, \ldots, n-1\}$. □

Now, we can finally state and prove the main theorem about the minimal $S-$graded free resolution of the generalized repunit $\Bbbk-$algebra $\Bbbk[S]$.

Theorem 10. *The set* $B_j := \{s_{j,k} \mid k = 1, \ldots, \beta_j = j\binom{n}{j+1}\}$ *is equal to*

$$B'_j := \left\{k\,c + a_{i_1} + \cdots + a_{i_{j+1}} \;\mid\; k \in \{1, \ldots, j\} \text{ and } 1 < i_1 < \cdots < i_{j+1} \leq n+1\right\},$$

for every $j \in \{1, \ldots, n-1\}$.

Proof. First, we note that, by Propositions 7 and 9, we already know that the result is true for $j = 1$ and $j = n-1$, respectively.

Let $j \in \{2, \ldots, n-2\}$. If we fix a bijection $\sigma_j : B_j \to B'_j$, then we have the isomorphism of free $\Bbbk[\mathbf{x}]-$modules

$$\bigoplus_{k=1}^{\beta_j} \Bbbk[\mathbf{x}](-s_{j,k}) \longrightarrow F_j := \bigoplus_{1<i_1<\cdots<i_{j+1}\leq n+1} \left(\bigoplus_{k=1}^{j} \Bbbk[\mathbf{x}](-k\,c - a_{i_1} - \cdots - a_{i_{j+1}})\right)$$

such that the element in the usual basis of left-hand module which has a 1 at place $\Bbbk[\mathbf{x}](-s_{j,k})$ and zeros elsewhere is sent to the element in the usual basis of F_j which has a 1 at place $\Bbbk[\mathbf{x}](-\sigma(s_{j,k}))$ and zeros elsewhere . So, if we prove that, for each $j \in \{2, \ldots, n-2\}$, there exist $\Bbbk[\mathbf{x}]-$module isomorphisms $\phi_j : \Bbbk[\mathbf{x}]^X_j \to F_j$ and $\phi_{j-1} : \Bbbk[\mathbf{x}]^X_{j-1} \to F_{j-1}$ such that

$$\delta_j := \phi_{j-1} \circ d_j \circ \phi_j^{-1} : F_j \to F_{j-1}$$

is $S-$homogeneous of degree 0, where $d_j : \Bbbk[\mathbf{x}]^X_j \to \Bbbk[\mathbf{x}]^X_{j-1}$ is the $\Bbbk[\mathbf{x}]-$module homomorphism defined in Section 2, we are done since the $s_{j,k}$ are uniquely defined (see Remark 5).

Let $j \in \{2, \ldots, n-2\}$ and let us define the $\Bbbk[\mathbf{x}]-$module isomorphism $\phi_j : \Bbbk[\mathbf{x}]^X_j \to F_j$ such that $\phi_j(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_{j+1}} \otimes y_1^{u_1} y_2^{u_2})$ is equal to the element that has a 1 in the coordinate corresponding to the direct summand $\Bbbk[\mathbf{x}](-(u_1+1)\,c - a_{i_1+1} - \cdots - a_{i_{j+1}+1})$ and zeros elsewhere (recall that $u_1, u_2 \in \mathbb{N}$ verifies $u_1 + u_2 = j-1$). Since, given $j \in \{2, \ldots, n-2\}$ and $l \in \{1, \ldots, j\}$, the $S-$degree of the $k\,i_j-$th entry of the matrix X defined in (1) is

$$\begin{cases} b\,a_{i_l} & \text{if } \; k = 1; \\ a_{i_l+1} & \text{if } \; k = 2, \end{cases}$$

we have that the $S-$degree of $\phi_{j-1}(x_{ki_l}\mathbf{e}_{i_1} \wedge \cdots \wedge \widehat{\mathbf{e}_{i_l}} \wedge \cdots \wedge \mathbf{e}_{i_{j+1}} \otimes y_1^{u_1} y_2^{u_2} y_k^{-1})$ is

$$\begin{cases} b\,a_{i_l} + u_1\,c + a_{i_1+1} + \cdots + a_{i_{j+1}+1} - a_{i_l+1} & \text{if } \; k = 1; \\ a_{i_l+1} + (u_1+1)\,c + a_{i_1+1} + \cdots + a_{i_{j+1}+1} - a_{i_l+1} & \text{if } \; k = 2, \end{cases}$$

In the first case, by Lemma 6, we have that

$$b\,a_{i_l} + u_1\,c + a_{i_1+1} + \cdots + a_{i_{j+1}+1} - a_{i_l+1} =$$
$$c + a_{i_l+1} + u_1 c + a_{i_1+1} + \cdots + a_{i_{j+1}+1} - a_{i_l+1} =$$
$$(u_1+1)c + a_{i_1+1} + \cdots + a_{i_{j+1}+1}.$$

and, in the second case, we have that

$$a_{i_l+1} + (u_1+1)c + a_{i_1+1} + \cdots + a_{i_{j+1}+1} - a_{i_l+1} =$$

$$(u_1+1)c + a_{i_1+1} + \cdots + a_{i_{j+1}+1}.$$

So, the $S-$degree $\phi_{j-1}(x_{ki_l}\mathbf{e}_{i_1}\wedge\cdots\wedge\widehat{\mathbf{e}_{i_l}}\wedge\cdots\wedge\mathbf{e}_{i_{j+1}}\otimes y_1^{u_1}y_2^{u_2}y_k^{-1})$ is equal to the $S-$degree of $\mathbf{e}_{i_1}\wedge\cdots\wedge\mathbf{e}_{i_{j+1}}\otimes y_1^{u_1}y_2^{u_2}$, for every $j\in\{2,\ldots,n-2\}$ and $l\in\{1,\ldots,j\}$. Therefore, since

$$d_j(\mathbf{e}_{i_1}\wedge\cdots\wedge\mathbf{e}_{i_{j+1}}\otimes y_1^{u_1}y_2^{u_2}) = \sum_{k\overset{*}{=}1}^{2}\sum_{l=1}^{j+1}(-1)^{l+1}x_{ki_l}\mathbf{e}_{i_1}\wedge\cdots\wedge\widehat{\mathbf{e}_{i_l}}\wedge\cdots\wedge\mathbf{e}_{i_{j+1}}\otimes y_1^{u_1}y_2^{u_2}y_k^{-1},$$

for every $j\in\{2,\ldots,n-1\}$, we conclude that, for our choice of the isomorphisms $\phi_j,\ j=2,\ldots,n-2$, the maps $\delta_j=\phi_{j-1}\circ d_j\circ\phi_j$ are $S-$homogeneous of degree zero, for every $j=2,\ldots,n-2$. □

COI Statement. The authors declared that they have no conflict of interest.

Acknowledgments. The authors wish to thank Manuel B. Branco for his valuable suggestions that improved the presentation of this article. We also thank Naoyuki Matsuoka for letting us know his interesting results on Pseudo-Frobenius numbers and semigroup ideals associated to certain numerical semigroups.

References

[1] Branco, M.B.; Colaço, I.; Ojeda, I. Minimal binomial systems of generators for the ideals of certain monomial curves. Mathematics **9**(24) (2021), 3204.

[2] Branco, M.B.; Colaço, I.; Ojeda, I. The Frobenius problem for generalized repunit numerical semigroups. Mediterr. J. Math. **20**, art. no. 16 (2023).

[3] Briales, E.;Campillo, A.;Marijuán, C.; Pisón P. Combinatorics of syzygies for semigroup algebra. Collect. Math. **49** (1998), 239–256.

[4] Eagon, J.A.; Northcott, D.G. Ideals defined by matrices and a certain complex associated with them. Proc. Roy. Soc. London Ser. A **269** (1962), 188–204.

[5] Eisenbud, D. *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150. Springer-Verlag, New York (1995).

[6] Goto, S.; Van Kien, D.; Matsuoka, N.; Le Truong, H. *Pseudo-Frobenius numbers versus defining ideals in numerical semigroup rings*, Journal of Algebra **508**, (2018), 1–15.

[7] Jafari, R., Zarzuela Armengou, S.: *Homogeneous numerical semigroups*, Semigroup Forum **97**, (2018), 278–306.

[8] Márquez-Campos, G.; Ojeda, I.; Tornero, J.M. On the computation of the Apéry set of numerical monoids and affine semigroups. Semigroup Forum **91**(1) (2015), 139–158.

[9] Gimenez, P.; Sengupta, I.; Srinivasan, H. Minimal graded free resolutions for monomial curves defined by arithmetic sequences. J. Algebra **388** (2013), 294–310.

[10] Miller, E.; Sturmfels. B. Combinatorial commutative algebra, Graduate Texts in Mathematics, 227, Springer-Verlag, New York, 2005

[11] Ojeda Martínez de Castilla, I.; Pisón Casares, P. On the hull resolution of an affine monomial curve. J. Pure Appl. Algebra **192**(1-3) (2004), 53–67.

[12] Ojeda, I.; Vigneron-Tenorio A. Simplicial complexes and minimal free resolution of monomial algebras. J. Pure Appl. Algebra **214**(6) (2010), 850–861.

[13] Rosales, J.C.; García Sánchez, P.A. *Numerical semigroups.* Developments in Mathematics, vol. 20, Springer, New York, (2009).

[14] Rosales, J.C.; Branco, M.B.; Torrão, D. The Frobenius problem for repunit numerical semigroups. Ramanujan J. **40** (2016), 323–334.

[15] Sturmfels, B. Gröbner bases and convex polytopes. Univ. Lecture Ser., 8 American Mathematical Society, Providence, RI, 1996.

[16] Van Kien, D.; Matsuoka, N. . *Numerical Semigroup Rings of Maximal Embedding Dimension with Determinantal Defining Ideals*. In: Barucci, V., Chapman, S., D'Anna, M., Fröberg, R. (eds) Numerical Semigroups . Springer INdAM Series, vol 40. Springer, Cham. (2020).

Departamento de Matemática e Ciências Físicas, Instituto Politécnico de Beja, 7800-295 Beja, Portugal
Email address: isabel.colaco@ipbeja.pt

Departamento de Matemáticas, Universidad de Extremadura, 06071 Badajoz, Spain
Email address: ojedamc@unex.es

Bibliography

[1] Aoki, S., Hara, H., Takemura, A., *Markov Bases in Algebraic Statistics*; Springer Series in Statistics; Springer; New York, 2012, ISBN 978-1461437192.

[2] Apéry, R., *Sur les branches superlinéaires des courbes algébriques*, CR Acad. Sci. Paris, **222** (1946), pp. 1198-1200.

[3] Assi, A.; D'Anna, M.; García-Sánchez, P. A., *Numerical semigroups and applications*, Springer Cham, Switzerland, 2020, ISBN 978-3030549435.

[4] Barucci, V., Dobbs, D.E., Fontana, M., *Maximality Properties in Numerical Semigroups and Applications to One-Dimensional Analytically Irreducible Local Domains*, Memoirs of the Amer. Math. Soc. 598, American Mathematical Society, Providence, 1997, ISBN 9780821805442.

[5] Beiler, A.H., *Recreations in the Theory of Numbers: The Queen of Mathematics Entertains*, Dover Publications, New York, 1964, ISBN 978-0486210964.

[6] Bigatti, A. M.; La Scala, R.; Robbiano, L., *Computing Toric Ideals*, Journal of Symbolic Computation, **27** (1999), 4, pp. 351–365, DOI 10.1006/jsco.1998.0256

[7] Brauer, A., *On a Problem of Partitions*, American Journal of Mathematics, **64** (1942), 1, pp. 299-312, DOI 10.2307/2371684

[8] Brauer, A., Shockley, J.E. *On a problem of Frobenius*, J. Reine Angew. Math. **211** (1962), pp. 215-220.

[9] Bresinsky, H., *On prime ideals with generic zero* $x_i = t^{n_i}$, Proceedings of the American Mathematical Society, **47** (1975), 2 pp. 329–333, DOI 10.2307/2039739

[10] Bresinsky, H., *Binomial generating sets for monomial curves, with applications in* $\mathbb{A}^4$, Rend. Sem. Mat. Univ. Politec. Torino, **46** (1988), pp. 353–370.

[11] Briales, E., Campillo, A., Marijuán, C., Pisón, P. *Minimal Systems of Generators for Ideals of Semigroups*, Journal of Pure and Applied Algebra, **124** (1998), pp. 7-30, DOI 10.1016/S0022-4049(96)00106-5.

[12] Briales, E.; Campillo, A., Marijuán, C., Pisón, P., *Combinatorics of syzygies for semigroup algebra*, Collectanea Mathematica, **49** (1998), pp. 239-256.

[13] Bruns, W., Herzog, H. J., *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics, **39** (1993), pp. 403.

[14] Colaço, I., & Ojeda, I., *Minimal free resolution of generalized repunit algebras*, Communications in Algebra, (2024), pp. 1-8, DOI 10.1080/00927872.2024.2394968

[15] Cox, D.A., Little, J., O'Shea, D. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Springer-Verlag, 2015, ISBN 978-0387356501.

[16] Curtis, F., *On formulas for the Frobenius number of a numerical semigroup*, Mathematica Scandinavica, **67** (1990), 2, pp. 190-192, http://www.jstor.org/stable/24492663.

[17] Davison, J. L., *On the Linear Diphantine Problem of Frobenius*, Journal of Number Theory, **48** (1994), 3, pp. 353-363, DOI 10.1006/jnth.1994.1071

[18] Djawadi, M., Hofmeister, G., *Linear Diophantine Problems*, Archiv der Mathematik, **66** (1996), pp. 19-29, DOI 10.1007/BF01323979

[19] Delgado, M., *On a question of Eliahou and a conjecture of Wilf*, Mathematische Zeitschrif, **288** (2018), pp. 595-627, DOI 10.1007/s00209-017-1902-3

[20] Eagon, J. A., Northcott, D. G., *Ideals Defined by Matrices and a Certain Complex Associated with Them*, Proceedings of the Royal Society of London. Series A, **269** (1962), pp. 188-204, http://www.jstor.org/stable/2992698

[21] Fröberg, R., Gottlieb, C., Häggkvist, R., *On Numerical Semigroups*, Semigroup Forum, **35** (1987), pp. 63-83, DOI 10.1007/BF02573091

[22] García-Sánchez, P. A.; Ojeda, I., *Uniquely Presented Finitely Generated Commutative Monoids*, Pacific Journal of Mathematics, **248** (2010), 1, pp. 91-105, DOI 10.2140/pjm.2010.248.91

[23] Gastinger, W., *Über die Verschwindungsideale monomialer Kurven*, PhD thesis, Univ. Regensburg, Landshut, 1989.

[24] Gimenez, P., Sengupta, I., Srinivasan, H., *Minimal Free Resolutions for Certain Affine Monomial Curves*, Comtemporary Mathematics, **555**, (2011), pp. 87-95, DOI 10.1090/conm/555/10991

[25] Gimenez, P.; Sengupta, I.; Srinivasan, H., *Minimal Graded Free Resolutions for Monomial Curves Defined by Arithmetic Sequences*, Journal of Algebra, **388** (2013), pp. 294-310, DOI 10.1016/j.jalgebra.2013.04.026

[26] Herzog, J., *Generators and Relations of Abelian Semigroups and Semigroup Rings*, Manuscripta Mathematica, **3** (1970), 3, pp. 175-193, 10.1007/BF01273309

[27] Jafari, R., Zarzuela Armengou, S., *Homogeneous Numerical Semigroups*, Semigroup Forum, **97** (2018), pp. 278-306, 2, DOI 10.1007/s00233-018-9941-6

[28] Jafari, R., Ojeda, I., *On the Depth of Simplicial Affine Semigroup Rings*, Collectanea Mathematica, (2024), DOI 10.1007/s13348-023-00424-6

[29] Johnson, S.M., *A Linear Diophantine Problem*, Canadian Journal of Mathematics, **12** (1960), pp. 390-398, DOI 10.4153/CJM-1960-033-6

[30] Katsabekis, A., *Arithmetical Rank of Binomial Ideals*, Archiv der Mathematik, **109** (2017), pp. 323-334, DOI 10.1007/s00013-017-1071-y

[31] Katsabekis, A., Ojeda, I., *An Indispensable Classification of Monomial Curves in* $\mathbb{A}^4(\Bbbk)$, Pacific Journal of Mathematics, **268** (2014), pp. 96-116, DOI 10.2140/pjm.2014.268.95

[32] Márquez-Campos, G., Ojeda, I., Tornero, J. M., *On the Computation of the Apéry Set of Numerical Monoids and Affine Semigroups*, Semigroup Forum, **91** (2015), pp. 139-158, DOI 10.1007/s00233-014-9631-y

[33] Martínez-Moro, E., Munuera, C., Ruano, D., *Advances in Algebraic Geometry Codes*, Series on Coding Theory and Cryptology, World Scientific Publishing: Hackensack, USA, 2008, DOI 10.1142/6767

[34] Matusevich, L. F., Ojeda, I., *Binomial Ideals and Congruences on* $\mathbb{N}^n$: Singularities, Algebraic Geometry, Commutative Algebra, and Related Topics, Springer International Publishing, pp. 429-454, 2018, DOI 10.1007/978-3-319-96827-8

[35] Miller, E., Sturmfels, B., *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics, 227, Springer, New York, 2005, ISBN 978-0387237077

[36] Ojeda, I., Vignerón-Tenorio, A., *Indispensable Binomials in Semigroups Ideals*, Proc. of the American Mathematical Society, **138** (2009), pp. 4205–4216, DOI 10.1090/S0002-9939-2010-10456-2

[37] Patil, D. P., *Minimal Sets of Generators for the Relation Ideals of Certain Monomial Curves*, Manuscripta Mathematica, **80** (1993), pp. 239-248, DOI 10.1007/BF03026549

[38] Ramirez Alfonsín, J. L., *The Diophantine Forbenius Problem*, Oxford University Press, London, 2005, ISBN 978-0198568209.

[39] Rédei, L., *The Theory of Finitely Generated Commutative Semigroups*, Pergamon, Oxford, 1965, ISBN 978-1483123523.

[40] Rosales, J. C., Branco, M. B., Torrão, D., *The Frobenius Problem for Repunit Numerical Semigroups*, The Ramanujan Journal, **40** (2016), 2, pp. 323-334, DOI 10.1007/s11139-015-9719-3

[41] Rosales, J. C., Branco, M. B., Torrão, D., *The Frobenius Problem for Mersenne Numerical Semigroups*, Mathematische Zeitschrift, **286** (2017), 1, pp. 741-749, DOI 10.1007/s00209-016-1781-z

[42] Rosales, J. C., García-Sánchez, P. A., *Numerical semigroups*, Developments in Mathematics, Vol. 20, Springer-Verlag, New York, 2009, ISBN 978-1441901606.

[43] Rosales, J. C., Branco, M. B., *Numerical Semigroups That Can Be Expressed As an Intersection of Symmetric Numerical Semigroups*, Journal of Pure and Applied Algebra, **171** (2002), 2, pp. 303-314, DOI 10.1016/S0022-4049(01)00128-1

[44] Stamate, D. I., *Betti Numbers for Numerical Semigroup Rings, in Multigraded Algebra and Applications*, Springer Proc. Math. Stat., **238** (2018), pp. 133–157, DOI 10.1007/978-3-319-90493-1

[45] Selmer, E. S., *On Linear Diophantine Problem of Frobenius*, Journal Für Die Reine Und Angewandte Mathematik, **293/294** (1977), pp. 1-17.

[46] Sylvester, J. J., *Problem 7382, Mathematical Questions, with their Solutions from the Educacional Times*, **41** (1884), pp. 21

[47] Van Kien, D., Matsuoka, N., *Numerical Semigroup Rings of Maximal Embedding Dimension with Determinantal Defining Ideals*, Numerical Semigroups: IMNS 2018, Springer, Cham, 2020, DOI 10.1007/978-3-030-40822-0

[48] Villarreal, R.H., *Monomial Algebras*; Monographs and Textbooks in Pure and Applied Mathematics 238, Marcel Dekker, Inc., New York, 2001.

[49] Wilf, H. S., *Circle-Of-Lights Algorithm for Money Changing Problem*, The American Mathematical Monthly, **85** (1978), 7, pp. 562-565, DOI 10.2307/2320864