RENORMALIZATION OF TRANSLATED CONE EXCHANGE TRANSFORMATIONS

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Abstract. In this paper, we investigate a class of non-invertible piecewise isometries on the upper half-plane known as Translated Cone Exchanges. We provide a geometric construction for the first return map to the middle cone for a broad class of parameters, then we show a recurrence in the first return map tied to the diophantine properties of the parameters, and subsequently prove the infinite renormalizability of the first return map for these parameters.

1. Introduction

Piecewise isometries (PWIs) a class of maps that can be generally described as a "cutting-and-shuffling" action of a metric space, specifically a partitioning of the phase space into at most countably many convex pieces called *atoms*, which are each moved according to an isometry. The phase space of these maps can be partitioned into two (or three) subsets based on the dynamics – a polygon or disc packing of periodic islands known as the *regular set*, and its complement, the set of points whose orbit either lands on, or accumulates on, the discontinuity set. Some authors choose to further distinguish those points in the pre-images of the discontinuity and those points which accumulate on it. The most well-known and well-understood examples of such maps are the *interval exchange transformations (IETs)*, which arise as return maps to cross-sections of some measured foliations [49] and also as generalisations of circle rotations [50, 51, 52] and their encoding spaces generalise sturmian shifts [53]. Furthermore, interval exchanges which aren't irrational rotations are known to be almost always weakly mixing [54] but never strongly mixing [55]. Piecewise isometries in general, however, are not as well-known and as a subset of this class, interval exchanges are in many ways exceptional, due in part to being one-dimensional, as well as the invariance of Lebesgue measure.

In the more general setting, although the inherent lack of hyperbolicity restricts the variety of possible behaviours, for example it is known that all piecewise isometries have zero topological entropy [56], piecewise isometries are still capable of quite complex behaviour; many examples show the presence of unbounded periodicity and an underlying renormalizability which structures the dynamics near the discontinuities [57, 58, 59, 60, 61, 62, 63]; numerical evidence suggests the existence of invariant curves in the exceptional set which seem fractal-like and form barriers to ergodicity [61, 62, 64, 65]; there are conjectured conditions for piecewise isometries to have sensitive dependence on initial conditions [66].

Renormalization in theoretical physics and nonlinear dynamical systems has a long-standing history, see for example [67, 68, 69, 70, 71, 72, 73], driven by the problem of understanding phenomena that occur at many spatial and temporal scales, particularly near phase transitions, periodic points, or in the case of piecewise isometries, the discontinuity set.

In this paper, we investigate the renormalizability of a class of piecewise isometries called Translated Cone Exchanges on the closure of the upper half-plane $\overline{\mathbb{H}}$. In particular, we use a geometric construction to describe the action of a first return map to a subset containing the origin, and show that this map displays renormalizable behaviour locally to the origin in accordance with diophantine approximation of one of its parameters.

This paper is organized as follows. In Section 2, we introduce the family of maps we will investigate, namely, Translated Cone Exchange transformations. In Section 3 we will develop some tools that will be useful in the next section. Section 4 contains the main result of this paper, concerning renormalization around) for our class of maps. Finally, in Section 5 we present an example for fixed values of the parameters.

2. Translated Cone Exchange transformations

Let $\mathbb{H} \subset \mathbb{C}$ denote the upper half plane, and let $\overline{\mathbb{H}}$ be its closure in $\mathbb{C}$, that is

$$\overline{\mathbb{H}} = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}.$$

A *Translated Cone Exchange transformation (TCE)* is a PWI $(\mathcal{P}, F)$ defined on the closed upper half plane $\overline{\mathbb{H}}$. Let $\mathbb{B}$ be the set

$$\mathbb{B} = \left\{\alpha \in (0,\pi)^{d+2} : \|\alpha\|_{\ell_1} = \pi\right\},$$

and let $\mathbb{A}$ be a subset of $\mathbb{B}$ defined by

$$\mathbb{A} = \{\alpha = (\alpha_0, ..., \alpha_{d+1}) \in \mathbb{B} : \alpha_0 = \alpha_{d+1}\}.$$

Typically, when $\alpha \in \mathbb{A}$, we denote $\beta = \alpha_0 = \alpha_{d+1}$. Next, for some $\alpha = (\alpha_0, ..., \alpha_{d+1}) \in \mathbb{B}$, partition the interval $[0, \pi]$ by subintervals

$$W_j = \begin{cases} [0, \alpha_0) & \text{if } j = 0 \\ [\alpha_0, \alpha_0 + \alpha_1] & \text{if } j = 1 \\ \left(\sum\limits_{k=0}^{j-1} \alpha_k, \sum_{k=0}^{j} \alpha_k\right] & \text{if } j \in \{2, ..., d+1\} \end{cases}.$$

We then define the partition $\mathcal{P}$ as

$$\mathcal{P} = \{P_j : j \in \{0, ..., d+1\}\},$$

where

$$P_1 = \{0\} \cup \{z \in \mathbb{H} : \mathrm{Arg}\, z \in W_1\}$$

and

$$P_j = \{z \in \overline{\mathbb{H}} : \mathrm{Arg}\, z \in W_j\} \text{ for } j \neq 1.$$

The mapping F is defined as a composition

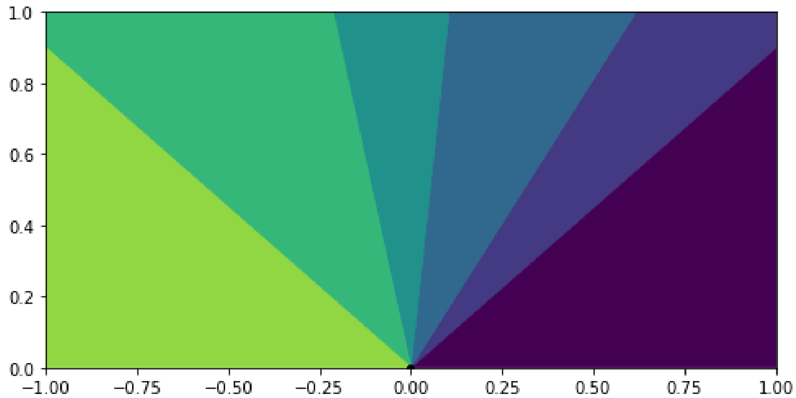

FIGURE 1. An example of a partition of the closed upper half plane $\overline{\mathbb{H}}$ into 6 cones.

$$F(z) = G \circ E(z), \tag{2.1}$$

where E is a permutation of the cones $P_1, ..., P_d$, and G is a piecewise horizontal translation. Formally, let $\mathrm{Sym}(d)$ denote the group of permutations on the set $\{1, ..., d\}$, $\tau \in \mathrm{Sym}(d)$, and let

$$\theta_j(\alpha, \tau) = \sum_{\tau(k)<\tau(j)} \alpha_k - \sum_{k<j} \alpha_k.$$

When α and τ are unambiguous, we may refer to $\theta_j(\alpha, \tau)$ simply as θ_j. The map $E : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ is then defined as

$$E(z) = \begin{cases} z & \text{if } z \in P_0 \cup P_{d+1} \\ ze^{i\theta_j} & \text{if } z \in P_j, j \in \{1, ..., d\} \end{cases}.$$

Note that E is invertible Lebesgue-almost everywhere in $\overline{\mathbb{H}}$. We define the *middle cone* P_c of F as

$$P_c = \bigcup_{j=1}^{d} P_j = \mathbb{H} \setminus (P_0 \cup P_{d+1}).$$

The map $G : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ is defined as

$$G(z) = \begin{cases} z - \rho & \text{if } z \in P_0, \\ z - \eta & \text{if } z \in P_c, \\ z + \lambda & \text{if } z \in P_{d+1}, \end{cases} \tag{2.2}$$

where $\rho, \lambda \in (0, \infty)$ are rationally independent and $0 < \eta < \lambda$.

Simulations of the orbits of points under some TCEs appears to reveal self-similarity as one zooms in towards the real line, such as in figure 2. One way to investigate

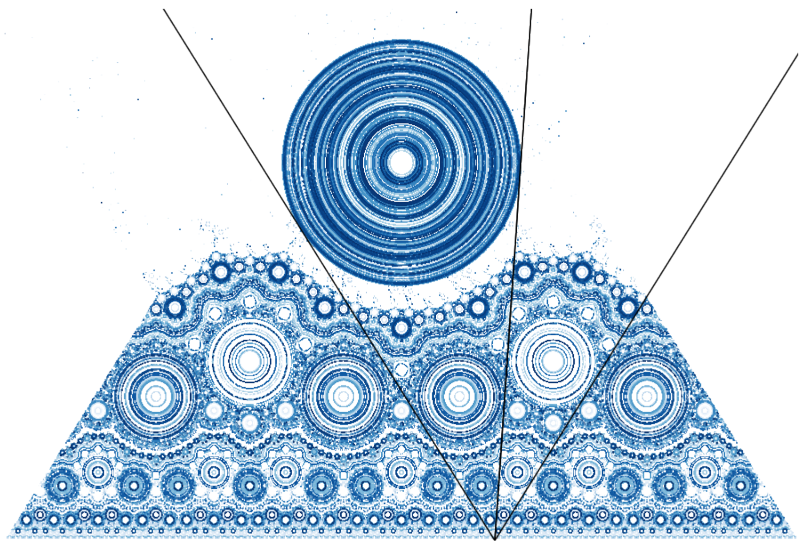

FIGURE 2. A plot of the first 10000 elements of the forward orbits of 200 randomly chosen points under a TCE with parameters $\alpha = (0.5, \pi - 2.5), \tau = (1\ 0), \lambda = \Phi$, and $\eta = \Phi^2$. Each orbit is given a (non-unique) colour to illustrate the trajectories. Note the appearance of renormalizable behaviour as features such as patterns of periodic disks appear to repeat closer to the real line, but scaled down.

this behaviour is by applying renormalization techniques. In particular, let $h : P_c \to \mathbb{N} \setminus \{0\}$ denote the *first return time* of $z \in P_c$ to P_c under F, that is

$$h(z) = \inf \{n > 0 : F^n(z) \in P_c\}. \tag{2.3}$$

The *first return map* $R : P_c \to P_c$ of F to P_c is then defined as

$$R(z) = F^{h(z)}(z). \tag{2.4}$$

Observe that for all $z \in P_c$, $R(z) = F^{h(z)}(z) = G^{h(z)} \circ E(z)$, since E is the identity outside of P_c.

3. Tools

In this section we prove some preliminary results that will serve as tools for more detailed investigation of the renormalization of TCEs.

Let $A : \overline{\mathbb{H}} \to \{-1, 0, 1\}$ denote the *address* of a point in $\overline{\mathbb{H}}$, defined by

$$A(z) = \begin{cases} 1 & \text{if } z \in P_0 \\ 0 & \text{if } z \in P_c \\ -1 & \text{if } z \in P_{d+1} \end{cases}$$

and define the *itinerary* of a point $z \in \mathbb{H}$ as the sequence $\iota(z) = (\iota_n(z))_{n \in \mathbb{N}}$, where $\iota_n(z) = A(F^n(z))$.

We bgin by proving a simple lemma regarding the dynamics of the point 0 (as it is far easier to understand than that of an arbitrary point $z \in P_c$).

Lemma 3.1. *Let* $\alpha \in \mathbb{B}$, $\tau \in \mathrm{Sym}(d)$, $\lambda, \rho \in \mathbb{R}$ *such that* $\lambda/\rho \notin \mathbb{Q}$, *and* $0 < \eta < \lambda$. *If* $z \in P_c$, *then for all* $1 \leq j \leq h(z)$,

$$F^j(z) = E(z) + F^j(0).$$

Proof. Suppose not, for a contradiction. Then there is some n with $0 \leq n \leq h(z)$ such that $F^n(z) \neq E(z) + F^n(0)$, and without loss of generality assume that n is the smallest such integer. Clearly $n > 1$, so for all $0 \leq j \leq n-1$, $F^j(z) - E(z) = F^j(0)$, and therefore $\iota_j(z) = \iota_j(0)$ for all $0 \leq j \leq n-2$, but $\iota_{n-1}(z) \neq \iota_{n-1}(0)$. Since $n \leq h(z)$, we cannot have $F^j(z) \in P_c$ for all $1 \leq j \leq n-1$. Hence for addresses of the $(n-1)^{\text{th}}$ iterates of z and 0 to disagree, one of two cases must occur:

1. $F^{n-1}(z) \in P_{d+1}$ and $F^{n-1}(0) \in P_c \cup P_0$; or

2. $F^{n-1}(z) \in P_0$ and $F^{n-1}(0) \in P_c \cup P_{d+1}$.

Since the orbit of 0 is restricted to $\mathbb{R}$ and since $E(0) = 0$, the second parts of each case become $G^{n-1}(0) \geq 0$ and $G^{n-1}(0) \leq 0$, respectively.

Similarly to the proof of Lemma **??**, suppose $E(z) = z'$, and let $\varepsilon_1 = \mathrm{Im}(z')\cot(\alpha_{d+1})$ and $\varepsilon_2 = \mathrm{Im}(z')\cot(\alpha_0)$. Then $z' \in P_c$ if and only if $-\varepsilon_1 \leq \mathrm{Re}(z') \leq \varepsilon_2$. Note that since G is a horizontal translation, $F^{n-1}(z) = G^{n-1}(E(z)) = G^{n-1}(z') \in P_{d+1}$ if and only if $\mathrm{Re}(G^{n-1}(z')) < -\varepsilon_1$. Similarly $G^{n-1}(z') \in P_0$ if and only if $\mathrm{Re}(G^{n-1}(z')) > \varepsilon_2$. The two above cases above can thus be reformulated as:

1. $\mathrm{Re}(G^{n-1}(z')) < -\varepsilon_1$ and $G^{n-1}(0) \geq 0$; or

2. $\mathrm{Re}(G^{n-1}(z')) > \varepsilon_2$ and $G^{n-1}(0) \leq 0$.

In case 1, we get $0 \leq G^{n-1}(0) = G^{n-1}(z') - z' = \mathrm{Re}(G^{n-1}(z')) - \mathrm{Re}(z')$. Hence $\mathrm{Re}(G^{n-1}(z)) \geq \mathrm{Re}(z')$ and thus

$$-\varepsilon_1 \leq \mathrm{Re}(z') \leq \mathrm{Re}(G^{n-1}(z')) < -\varepsilon_1,$$

which is a contradiction. Case 2 leads to a similar contradiction. Therefore there is no such n.

□

Recall from the theory of continued fractions that the n^{th} convergent to an irrational real number $\lambda = [a_0; a_1, a_2, ...]$ is a fraction $p_n/q_n = [a_0; a_1, ..., a_n]$, where p_n, q_n are coprime integers and $q_n > 0$. The numbers p_n, q_n can be generated by the recursive relations:

$$\begin{aligned} p_0 &= a_0, & q_0 &= 1, \\ p_1 &= a_1 a_0 + 1, & q_1 &= a_1, \\ p_n &= a_n p_{n-1} + p_{n-2}, & q_n &= a_n q_{n-1} + q_{n-2}. \end{aligned} \tag{3.1}$$

Furthermore, the convergents to λ satisfy the property that for all positive integers $q < q_{n+1}$ and all $p \in \mathbb{Z}$,

$$|q_n\lambda - p_n| \leq |q\lambda - p|,$$

with equality only when $(p,q) = (p_n, q_n)$.

3.1. **Continued Fractions.** Let $g : [0,1] \to [0,1]$ denote the Gauss map, given by

$$g(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

In particular, if $\lambda = [0; \lambda_1, \lambda_2, \lambda_3, ...] \in [0,1]$, then

$$g(\lambda) = \frac{1}{\lambda} - \lambda_1 = [0; \lambda_2, \lambda_3, ...].$$

Let $\lambda = [0; \lambda_1, \lambda_2, ...] \in [0, \infty) \setminus \mathbb{Q}$. To start, let

$$\mathcal{N}_\lambda = \{(m,n) \in \mathbb{N}^2 : 0 \leq n \leq \lambda_{m+1}\},$$

and define an indexing function $w_\lambda : \mathcal{N}_\lambda \to \mathbb{N}$ by

$$w_\lambda(m,n) = \begin{cases} n & \text{if } m = 0, \\ \lambda_1 + ... + \lambda_m + n & \text{if } m > 0. \end{cases}$$

Note that w_λ is surjective and if we define the subset $\mathcal{N}_\lambda^< \subset \mathcal{N}$ to be

$$\mathcal{N}_\lambda^< = \{(m,n) \in \mathbb{N}^2 : 0 \leq n < \lambda_{m+1}\},$$

then $w_\lambda \restriction_{\mathcal{N}_\lambda^<}$ is a bijection. Furthermore,

$$w_\lambda(m, \lambda_{m+1}) = w_\lambda(m+1, 0).$$

From now on, we denote the j^{th} coefficient of the continued fraction expansion of $g^m(\lambda)$ by $g^m(\lambda)_j$. The next proposition gives us a nice property of w_λ.

Proposition 3.2. *Let $j, m, n \in \mathbb{N}$. Then $(m+j, n) \in \mathcal{N}_\lambda$ if and only if $(j,n) \in \mathcal{N}_{g^m(\lambda)}$. In particular,*

$$w_\lambda(m+j, n) = \lambda_1 + ... + \lambda_m + w_{g^m(\lambda)}(j,n).$$

Proof. We have that $(m+j, n) \in \mathcal{N}_\lambda$ is equivalent to $0 \leq n \leq \lambda_{m+j+1}$. We also have that $\lambda_{m+j+1} = g^m(\lambda)_{j+1}$, so $0 \leq n \leq g^m(\lambda)_{j+1}$. This is equivalent to $(j,n) \in \mathcal{N}_{g^m(\lambda)}$.

If $m = j = 0$, then the second part of our lemma is clearly true.

Assume $j = 0$ and $m > 0$. Then

$$w_\lambda(m+j, n) = w_\lambda(m,n) = \lambda_1 + ... + \lambda_m + n = \lambda_1 + ... + \lambda_m + w_{g^m(\lambda)}(0,n),$$

where the final equality is true since $(m,n) \in \mathcal{N}_\lambda$ is equivalent to $(0,n) \in \mathcal{N}_{g^m(\lambda)}$.

Finally, suppose $m, j > 0$. Then

$$\begin{aligned} w_\lambda(m+j, n) &= \lambda_1 + ... + \lambda_m + \lambda_{m+1} + ... + \lambda_{m+j} + n \\ &= \lambda_1 + ... + \lambda_m + g^m(\lambda)_1 + ... + g^m(\lambda)_j + n. \end{aligned}$$

Note that since $(m+j, n) \in \mathcal{N}_\lambda$ is equivalent to $(j,n) \in \mathcal{N}_{g^m(\lambda)}$, we have

$$w_\lambda(m+j, n) = \lambda_1 + ... + \lambda_m + w_{g^m(\lambda)}(j,n).$$

□

We set the *semiconvergents* to λ as the fractions

$$\frac{P_{w_\lambda(m,n)}(\lambda)}{Q_{w_\lambda(m,n)}(\lambda)} = \begin{cases} [0;\lambda_1,...,\lambda_m] \text{ if } n=0, \\ \\ [0;\lambda_1,...,\lambda_m,n] \text{ if } n\neq 0. \end{cases}$$

Note that by a standard result in the theory of continued fractions, we have that

$$\frac{P_{w_\lambda(m,n)}(\lambda)}{Q_{w_\lambda(m,n)}(\lambda)} = \begin{cases} \dfrac{np_m(\lambda)+p_{m-1}(\lambda)}{nq_m(\lambda)+q_{m-1}(\lambda)} & \text{if } n\neq 0, \\ \\ \dfrac{p_m(\lambda)}{q_m(\lambda)} & \text{if } n=0, \end{cases}$$

so that the numerators and denominators coincide. We define the *signed errors of the semiconvergents of* λ as

$$\Delta_{w_\lambda(m,n)}(\lambda) = Q_{w_\lambda(m,n)}(\lambda)\lambda - P_{w_\lambda(m,n)}(\lambda). \tag{3.2}$$

Intuitively, the sequence $(\Delta_k)_{k\in\mathbb{N}}$ gives us information on how close rational approximations to λ get as the size of the denominator increases.

By expanding the definitions of $P_{w_\lambda(m,n)}$ and $Q_{w_\lambda(m,n)}$, we see

$$\begin{aligned} \Delta_{w_\lambda(m,n)}(\lambda) &= n(q_m\lambda - p_m) + q_{m-1}\lambda - p_{m-1} \\ &= n\Delta_{w_\lambda(m,0)}(\lambda) + \Delta_{w_\lambda(m-1,0)}(\lambda), \end{aligned} \tag{3.3}$$

for $(m,n)\in\mathcal{N}_\lambda$ with $m\geq 1$. Moreover, by expanding the recurrence relation for p_m and q_m and rearranging terms, we have the additional property

$$\begin{aligned} \Delta_{w_\lambda(m,0)} &= q_m\lambda - p_m \\ &= \lambda_m(q_{m-1}\lambda - p_{m-1}) + q_{m-2}\lambda - p_{m-2} \\ &= \lambda_m\Delta_{w_\lambda(m-1,0)}(\lambda) + \Delta_{w_\lambda(m-2,0)}(\lambda), \end{aligned}$$

for $m\in\mathbb{N}, m\geq 2$. Indeed, using these two recurrence properties, we can verify that $\Delta_{w_\lambda(m,0)}(\lambda) = \Delta_{w_\lambda(m-1,\lambda_m)}(\lambda)$.

From now on, we shall write $\Delta_{w(m,n)}(x) = \Delta_{w_x(m,n)}(x)$ to simplify our notation. A result by Bates et al. [43] presents an interesting connection between iterates of the Gauss map and consecutive errors in the approximation of λ by its convergents.

Lemma 3.3 (Theorem 10 of [43])**.** *Let* $\lambda = [0;\lambda_1,\lambda_2,...]\in[0,1)\setminus\mathbb{Q}$. *For all* $m\in\mathbb{N}$,

$$g^m(\lambda) = \frac{q_m\lambda - p_m}{p_{m-1} - q_{m-1}\lambda}. \tag{3.4}$$

Equation (3.4) can be equivalently formulated as

$$g^m(\lambda) = -\frac{\Delta_{w(m,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)}. \tag{3.5}$$

Lemma 3.4. *Let $\lambda = [0; \lambda_1, \lambda_2, ...] \in [0, 1) \setminus \mathbb{Q}$. For all $(m, n) \in \mathcal{N}_\lambda$ with $m \geq 1$,*

$$\Delta_{w(0,n)}(g^m(\lambda)) = -\frac{\Delta_{w(m,n)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)}. \tag{3.6}$$

Proof. For $n = 0$, $\Delta_{w(0,0)}(g^m(\lambda)) = g^m(\lambda)$, so (3.6) holds.

Next, observe that

$$\Delta_{w(j,0)}(g^m(\lambda)) = g^m(\lambda)_j \Delta_{w(j-1,0)}(g^m(\lambda)) + \Delta_{w(j-2,0)}(g^m(\lambda)).$$

Since $g^m(\lambda)_j = \lambda_{m+j}$, this becomes

$$\Delta_{w(j,0)}(g^m(\lambda)) = \lambda_{m+j} \Delta_{w(j-1,0)}(g^m(\lambda)) + \Delta_{w(j-2,0)}(g^m(\lambda)).$$

We thus see that

$$\Delta_{w(0,n)}(g^m(\lambda)) = n(q_0 g^m(\lambda) - p_0) + q_{-1}\lambda - p_{-1},$$

and by recalling p_{-1}, q_{-1}, p_0, and q_0 from (3.1), we get

$$\Delta_{w(0,n)}(g^m(\lambda)) = n g^m(\lambda) - 1.$$

Using (3.5), we can substitute $g^m(\lambda)$ and rearrange terms to get

$$\begin{aligned}\Delta_{w(0,n)}(g^m(\lambda)) &= -n\left(\frac{\Delta_{w(m,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)} + 1\right) \\ &= -\frac{n\Delta_{w(m,0)}(\lambda) + \Delta_{w(m-1,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)}.\end{aligned}$$

Finally, from this and (3.3) we get (3.6). □

Corollary 3.5. *For all $(m + j, n) \in \mathcal{N}_\lambda$, where $m + j \in \mathbb{N}$, $m \geq 1$, we have*

$$\Delta_{w(0,n)}(g^{m+j}(\lambda)) = -\frac{\Delta_{w(m,n)}(g^j(\lambda))}{\Delta_{w(m-1,0)}(g^j(\lambda))} \tag{3.7}$$

Proof. This follows from Lemma 3.4 with $g^j(\lambda)$ instead of λ. □

Our next Lemma is an important tool for determining scaling properties of these errors.

Lemma 3.6. *Let $\lambda = [0; \lambda_1, \lambda_2, ...] \in [0, 1) \setminus \mathbb{Q}$. For all $(m + j, n) \in \mathcal{N}_\lambda$ such that $m, j \in \mathbb{N}$ and $m \geq 1$,*

$$\Delta_{w(j,n)}(g^m(\lambda)) = -\frac{\Delta_{w(m+j,n)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)}. \tag{3.8}$$

Proof. Let us prove first that (3.8) holds for $n = 0$. By multiplying and dividing by $\Delta_{w(m+k-1,0)}$ for all $0 \leq k \leq j$, we get

$$\frac{\Delta_{w(m+j,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)} = \prod_{k=0}^{j} \frac{\Delta_{w(m+k,0)}(\lambda)}{\Delta_{w(m+k-1,0)}(\lambda)}$$

Then, using (3.5), we get

$$\frac{\Delta_{w(m+j,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)} = \prod_{k=0}^{j} -\Delta_{w(0,0)}(g^{m+k}(\lambda)).$$

Rearranging this last expression and using (3.7), we get

$$\frac{\Delta_{w(m+j,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)} = (-1)^{j+1}\Delta_{w(0,0)}(g^m(\lambda)) \prod_{k=1}^{j} -\frac{\Delta_{w(k,0)}(g^m(\lambda))}{\Delta_{w(k-1,0)}(g^m(\lambda))}$$

We then simplify the product by cancelling terms in the numerator and denominator to get

$$\begin{aligned}\frac{\Delta_{w(m+j,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)} &= (-1)^{j+1}\Delta_{w(0,0)}(g^m(\lambda)) \left((-1)^j \frac{\Delta_{w(j,0)}(g^m(\lambda))}{\Delta_{w(0,0)}(g^m(\lambda))}\right) \\ &= -\Delta_{w(j,0)}(g^m(\lambda)).\end{aligned}$$

Finally, for general $(m+j,n) \in \mathcal{N}_\lambda$, $m, j \in \mathbb{N}$, $m \geq 1$, we have

$$\frac{\Delta_{w(m+j,n)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)} = \frac{\Delta_{w(m+j,n)}(\lambda)}{\Delta_{w(m+j-1,0)}(\lambda)} \frac{\Delta_{w(m+j-1,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)}$$

Using (3.6) and (3.8) for $n = 0$, we get

$$\frac{\Delta_{w(m+j,n)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)} = \Delta_{w(0,n)}(g^{m+j}(\lambda))\Delta_{w(j-1,0)}(g^m(\lambda)),$$

and then using (3.7) gives us

$$\begin{aligned}\frac{\Delta_{w(m+j,n)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)} &= -\frac{\Delta_{w(j,n)}(g^m(\lambda))}{\Delta_{w(j-1,0)}(g^m(\lambda))}\Delta_{w(j-1,0)}(g^m(\lambda)) \\ &= -\Delta_{w(j,n)}(g^m(\lambda)),\end{aligned}$$

as required.

□

With these properties in mind, we will now define the rhombi which will partition a neighbourhood of the middle cone P_c. Recall that $\mathcal{N}_\lambda^<$ denotes the subset of $\mathcal{N}_\lambda$ defined by

$$\mathcal{N}_\lambda^< = \{(m,n) \in \mathbb{N}^2 : 0 \leq n < \lambda_{m+1}\}.$$

For $(m,n) \in \mathcal{N}_\lambda^<$, let $S'_{w_\lambda(m,n)}(\lambda)$ be the set defined by

$$S'_{w_\lambda(m,n)}(\lambda) = \begin{cases} (P_0 - \Delta_{w(m,0)}(\lambda)) \cap (P_c - \Delta_{w(m,n+1)}(\lambda)) \\ \cap P_c \cap (P_{d+1} - (n\Delta_{w(m,0)}(\lambda) + \Delta_{w(m-1,0)}(\lambda)) & \text{if } m \text{ is even} \\ (P_0 - (n\Delta_{w(m,0)}(\lambda) + \Delta_{w(m-1,0)}(\lambda)) \cap P_c \\ \cap (P_c - \Delta_{w(m,n+1)}(\lambda)) \cap (P_{d+1} - \Delta_{w(m,0)}(\lambda)) & \text{if } m \text{ is odd} \end{cases}. \tag{3.9}$$

From now on we shall write $S'_{w(m,n)}(x) = S'_{w_x(m,n)}(x)$ for brevity. By observing that $\Delta_{w(m,n)}(\lambda) > 0$ when m is even and $\Delta_{w(m,n)}(\lambda) < 0$ when m is odd, we can clearly see that $S_{w(m,n)}(\lambda) \neq \emptyset$ for all $(m,n) \in \mathcal{N}_\lambda^<$. Additionally, since every point in $S'_{w(m,n)}(\lambda)$ has positive imaginary part, the boundary of $S'_{w(m,n)}(\lambda)$ consists of segments of the non-horizontal boundary lines of P_0, P_c, and P_{d+1}, and all of these lines either have angle α_0 or α_{d+1}. Thus, $S'_{w(m,n)}(\lambda)$ is a quadrilateral, and its opposing sides must be parallel, so it is a parallelogram.

Since opposite edges of $S'_{w(m,n)}(\lambda)$ are parallel, the sidelengths of $S'_{w(m,n)}(\lambda)$ are uniquely determined by the perpendicular distances between opposing edges. In the case that m is even, these are in turn uniquely determined by the distances between the vertices of the pairs of cones $P_0 - \Delta_{w(m+1,0)}(\lambda)$ and P_c, and $P_c - \Delta_{w(m,n+1)}(\lambda)$ and $P_{d+1} - \Delta_{w(m,n)}(\lambda)$. In the case that m is odd, the perpendicular distances are determined by the distances between the vertices of pairs of cones $P_0 - \Delta_{w(m,n)}(\lambda)$ and $P_c - \Delta_{w(m,n+1)}(\lambda)$, and P_c and $P_{d+1} - \Delta_{w(m+1,0)}(\lambda)$. Since $\Delta_{w(m,n+1)}(\lambda) - \Delta_{w(m,n)}(\lambda) = \Delta_{w(m+1,0)}(\lambda)$, we know that these distances are equal. Therefore, the sidelengths of $S'_{w(m,n)}(\lambda)$ are all equal, so it is a rhombus for all $\lambda \in [0,1) \setminus \mathbb{Q}$ and all $(m,n) \in \mathcal{N}_\lambda^<$.

An interesting property of these rhombi can be found by an application of Lemma 3.6.

Theorem 3.7. *Let $\lambda \in [0,1) \setminus \mathbb{Q}$. For all $(m+j,n) \in \mathcal{N}_\lambda^<$ such that $m, j \in \mathbb{N}$ and $m \geq 1$ is even,*

$$\frac{1}{-\Delta_{w(m-1,0)}(\lambda)} S'_{w(m+j,n)}(\lambda) = S'_{w(j,n)}(g^m(\lambda)).$$

Proof. Firstly, note that since m is even, $\Delta_{w(m-1,0)}(\lambda) < 0$. Hence $-1/\Delta_{w(m-1,0)}(\lambda) > 0$. Hence,

$$\frac{1}{-\Delta_{w(m-1,0)}(\lambda)}(P_k + x) = P_k - \frac{x}{\Delta_{w(m-1,0)}(\lambda)},$$

for all $k \in \{1, ..., d\}$. Thus, we have

$$\begin{aligned} &\frac{1}{-\Delta_{w(m-1,0)}(\lambda)} S'_{w(m+j,n)}(\lambda) \\ &= \begin{cases} \left(P_0 + \dfrac{\Delta_{w(m+j,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)}\right) \cap \left(P_c + \dfrac{\Delta_{w(m+j,n+1)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)}\right) & \\ \cap P_c \cap \left(P_{d+1} + \dfrac{n\Delta_{w(m+j,0)}(\lambda) + \Delta_{w(m+j-1,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)}\right), & \text{if } m+j \text{ is even,} \\ \left(P_0 + \dfrac{n\Delta_{w(m+j,0)}(\lambda) + \Delta_{w(m+j-1,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)}\right) \cap P_c & \\ \cap \left(P_c + \dfrac{\Delta_{w(m+j,n+1)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)}\right) \cap \left(P_{d+1} + \dfrac{\Delta_{w(m+j,0)}(\lambda)}{\Delta_{w(m-1,0)}(\lambda)}\right), & \text{if } m+j \text{ is odd.} \end{cases} \end{aligned}$$

Using (3.8) many times give us

$$\frac{1}{-\Delta_{w(m-1,0)}(\lambda)} S'_{w(m+j,n)}(\lambda)$$

$$= \begin{cases} (P_0 - \Delta_{w(j,0)}(g^m(\lambda))) \cap (P_c - \Delta_{w(j,n+1)}(g^m(\lambda))) \\ \cap P_c \cap (P_{d+1} - (n\Delta_{w(j,0)}(g^m(\lambda)) + \Delta_{w(j-1,0)}(g^m(\lambda)))), & \text{if } j \text{ is even,} \\ (P_0 - (n\Delta_{w(j,0)}(g^m(\lambda)) + \Delta_{w(j-1,0)}(g^m(\lambda)))) \cap P_c \\ \cap (P_c - \Delta_{w(j,n+1)}(g^m(\lambda))) \cap (P_{d+1} - \Delta_{w(j,0)}(g^m(\lambda))), & \text{if } j \text{ is odd.} \end{cases}$$

Finally, comparing this with (3.9) gets

$$\frac{1}{-\Delta_{w(m-1,0)}(\lambda)} S'_{w(m+j,n)}(\lambda) = S'_{w(j,n)}(g^m(\lambda)).$$

□

This theorem seems to suggest the possibility of infinite renormalizability of the first return maps to P_c for a whole class of TCEs. At the very least, if indeed the first return map of a TCE to P_c is an isometry on $S'_{w(m,n)}(\lambda)$ for $(m,n) \in \mathcal{N}_\lambda^<$ with $m \geq m_0$ and some $m_0 \in \mathbb{N}$, then at the very least the partition matches with its potential renormalization.

3.2. **Conjugacies within the class of TCEs.** Let $\alpha = (\alpha_0, ..., \alpha_{d+1}) \in \mathbb{B}^{d+2}$, $\tau \in \mathrm{Sym}(d)$, and $\lambda, \rho, \eta \in \mathbb{R}$, $\lambda, \rho > 0$, $-\rho < \eta < \lambda$, and let F be the TCE with these parameters.

For $c > 0$, let

$$F_c(z) = G_c \circ E(z), \tag{3.10}$$

denote the TCE, where

$$G_c(z) = \begin{cases} z + \lambda/c, & \text{if } z \in P_{d+1}, \\ z - \eta/c, & \text{if } z \in P_c, \\ z - \rho/c, & \text{if } z \in P_0. \end{cases}$$

Define $S_c : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ as uniform scaling by c about the origin

$$S_c(z) = cz. \tag{3.11}$$

Proposition 3.8. *Let F be as in* (2.1), *F_c as in* (3.10), *and S_c as in* (3.11). *Then*

$$F_c = S_c^{-1} \circ F \circ S_c. \tag{3.12}$$

Proof. Firstly, observe that for all $j \in \{0, ..., d+1\}$,

$$cP_j = P_j,$$

from which we can deduce that

$$cz \in P_j \text{ if and only if } z \in P_j. \tag{3.13}$$

Let $z \in \overline{\mathbb{H}}$. From (3.11), we get

$$S_c^{-1} \circ F \circ S_c(z) = \frac{1}{c} F(cz),$$

and by expanding F as in (2.1), we have

$$S_c^{-1} \circ F \circ S_c(z) = \begin{cases} \frac{1}{c}(cz + \lambda) & \text{if } cz \in P_{d+1} \\ \frac{1}{c}(e^{i\theta_j}(cz) - \eta) & \text{if } cz \in P_j, j \in \{1, ..., d\} \\ \frac{1}{c}(cz - \rho) & \text{if } cz \in P_0 \end{cases} .$$

Recalling (3.13), distributing the multiplication by $1/c$, and comparing with (3.10), we get

$$\begin{aligned} S_c^{-1} \circ F \circ S_c(z) &= \begin{cases} z + \lambda/c & \text{if } z \in P_{d+1} \\ e^{i\theta_j} z - \eta/c & \text{if } z \in P_j, j \in \{1, ..., d\} \\ z - \rho/c & \text{if } z \in P_0 \end{cases} \\ &= F_c(z). \end{aligned}$$

□

A consequence of Proposition 3.8 becomes clear when we apply (3.12) when $c = \rho$. In this case, we see that the TCE F is topologically conjugate to the TCE F_ρ with parameters α, τ, λ/ρ, 1, and η/ρ. Therefore, when investigating the dynamics of TCEs (at least from a topological perspective), we can normalize ρ to 1 without losing any information.

Let $\overline{\alpha}$ and $\overline{\tau}$ denote the reverse of α and τ, respectively, in the sense that

$$\overline{\alpha}_j = (\alpha_{d+1}, \alpha_d, ..., \alpha_0), \tag{3.14}$$

and

$$\overline{\tau} = \begin{pmatrix} \overline{\tau}_0 \\ \overline{\tau}_1 \end{pmatrix} = \begin{pmatrix} d & d-1 & ... & 1 \\ \tau(d) & \tau(d-1) & ... & \tau(1) \end{pmatrix}. \tag{3.15}$$

Let $\mathcal{P} = \{P_k : k \in \{0, ..., d+1\}\}$ denote the partition of $\overline{\mathbb{H}}$ into $d+2$ cones according to α, and let $\overline{\mathcal{P}} = \{\overline{P}_k : k \in \{0, ..., d+1\}\}$ denote a similar partition of $\overline{\mathbb{H}}$ according to $\overline{\alpha}$. Define the TCE

$$\overline{F} = \overline{G} \circ \overline{E}, \tag{3.16}$$

where

$$\overline{E}(z) = \begin{cases} z, & \text{if } z \in P_0 \cup P_{d+1}, \\ e^{i\theta_j(\overline{\alpha}, \overline{\tau})} z, & \text{if } z \in P_j, j \in \{1, ..., d\}, \end{cases}$$

and

$$\overline{G}(z) = \begin{cases} z+\rho, & \text{if } z \in P_{d+1}, \\ z+\eta, & \text{if } z \in P_c, \\ z-\lambda, & \text{if } z \in P_0. \end{cases}$$

Define $\mathcal{R} : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ by

$$\mathcal{R}(z) = \mathcal{R}(x+iy) = -x+iy = -\overline{z}, \tag{3.17}$$

that is, the reflection of $z = x+iy$ about the imaginary axis, where $x, y \in \mathbb{R}$, $y \geq 0$. Note that $\mathcal{R}$ is an involution, i.e. $\mathcal{R}^{-1} = \mathcal{R}$. Furthermore, $\mathcal{R}$ is clearly additive.

Proposition 3.9. *Let F be as in* (2.1), *$\overline{F}$ as in* (3.16), *and $\mathcal{R}$ as in* (3.17). *Then*

$$\overline{F} = \mathcal{R} \circ F \circ \mathcal{R}. \tag{3.18}$$

Proof. Observe that by (3.14) and (3.15), we get

$$\begin{aligned} \theta_j(\overline{\alpha}, \overline{\tau}) &= \sum_{\overline{\tau}_1(k)<\overline{\tau}_1(j)} \overline{\alpha}_k - \sum_{\overline{\tau}_0(k)<\overline{\tau}_0(j)} \overline{\alpha}_k \\ &= \sum_{\tau(d+1-k)>\tau(d+1-j)} \alpha_{d+1-k} - \sum_{d+1-k>d+1-j} \alpha_{d+1-k}. \end{aligned}$$

By relabelling k to $d+1-k$, we get

$$\theta_j(\overline{\alpha}, \overline{\tau}) = \sum_{\tau(k)>\tau(d+1-j)} \alpha_k - \sum_{k>d+1-j} \alpha_k.$$

Recall that

$$\pi = \sum_{k=0}^{d+1} \alpha_k.$$

This implies that

$$\begin{aligned} \theta_j(\overline{\alpha}, \overline{\tau}) &= \left(\pi - \alpha_0 - \alpha_{d+1} - \sum_{\tau(k)\leq\tau(d+1-j)} \alpha_k\right) - \left(\pi - \alpha_0 - \alpha_{d+1} - \sum_{k\leq d+1-j} \alpha_k\right) \\ &= -\left(\sum_{\tau(k)\leq\tau(d+1-j)} \alpha_k - \sum_{k\leq d+1-j} \alpha_k\right). \end{aligned}$$

Note that since τ is a permutation of a finite set, $\tau(x) = \tau(y)$ if and only if $x = y$. Thus, we have

$$\theta_j(\overline{\alpha}, \overline{\tau}) = -\left(\sum_{\tau(k)<\tau(d+1-j)} \alpha_k - \sum_{k<d+1-j} \alpha_k\right) = -\theta_{d+1-j}(\alpha, \tau). \tag{3.19}$$

Additionally, note that for all $j \in \{0, ..., d+1\}$,

$$\mathcal{R}(P_j) = \overline{P}_{d+1-j}.$$

Thus,

$$\mathcal{R}(z) \in P_j \text{ if and only if } z \in \overline{P}_{d+1-j}. \tag{3.20}$$

By expanding the definitions of F as in (2.1), we get

$$\mathcal{R} \circ F \circ \mathcal{R}(z) = \begin{cases} \mathcal{R}(\mathcal{R}(z) + \lambda), & \text{if } \mathcal{R}(z) \in P_{d+1}, \\ \mathcal{R}(e^{i\theta_j(\alpha,\tau)}\mathcal{R}(z) - \eta), & \text{if } \mathcal{R}(z) \in P_j, j \in \{1, ..., d\}, \\ \mathcal{R}(\mathcal{R}(z) - \rho), & \text{if } \mathcal{R}(z) \in P_0. \end{cases}$$

Using (3.20), and the definition of $\mathcal{R}$ as in (3.17), we get

$$\mathcal{R} \circ F \circ \mathcal{R}(z) = \begin{cases} \mathcal{R}(-\overline{z} + \lambda), & \text{if } z \in \overline{P}_{d+1} \\ \mathcal{R}(-e^{i\theta_j(\alpha,\tau)}\overline{z} - \eta), & \text{if } z \in \overline{P}_{d+1-j}, j \in \{1, ..., d\}, \\ \mathcal{R}(-\overline{z} - \rho), & \text{if } z \in \overline{P}_0. \end{cases}$$

Using the additivity of $\mathcal{R}$, as well as the fact that $\mathcal{R}(x) = -x$ when $x \in \mathbb{R}$, we have that

$$\begin{aligned} \mathcal{R} \circ F \circ \mathcal{R}(z) &= \begin{cases} \mathcal{R}(-\overline{z} + \lambda), & \text{if } z \in \overline{P}_{d+1}, \\ \mathcal{R}(-e^{i\theta_j(\alpha,\tau)}\overline{z} - \eta), & \text{if } z \in \overline{P}_{d+1-j}, j \in \{1, ..., d\}, \\ \mathcal{R}(-\overline{z} - \rho), & \text{if } z \in \overline{P}_0, \end{cases} \\ &= \begin{cases} z - \lambda, & \text{if } z \in \overline{P}_{d+1} \\ e^{-i\theta_j(\alpha,\tau)}z + \eta, & \text{if } z \in \overline{P}_{d+1-j}, j \in \{1, ..., d\} \\ z + \rho, & \text{if } z \in \overline{P}_0 \end{cases}. \end{aligned}$$

Observe that

$$\mathcal{R} \circ F \circ \mathcal{R}(z) = e^{-i\theta_j}z + \eta \text{ when } z \in \overline{P}_{d+1-j},$$

is equivalent to

$$\mathcal{R} \circ F \circ \mathcal{R}(z) = e^{-i\theta_{d+1-j}}z + \eta \text{ when } z \in \overline{P}_j.$$

Thus,

$$\mathcal{R} \circ F \circ \mathcal{R}(z) = \begin{cases} z + \rho, & \text{if } z \in \overline{P}_0, \\ e^{-i\theta_{d+1-j}(\alpha,\tau)}z + \eta, & \text{if } z \in \overline{P}_j, j \in \{1, ..., d\}, \\ z - \lambda, & \text{if } z \in \overline{P}_{d+1}. \end{cases}$$

Finally, using (3.19) and comparing with the definition of $\overline{F}$ as in (3.16), this becomes

$$\mathcal{R} \circ F \circ \mathcal{R}(z) = \begin{cases} z + \rho, & \text{if } z \in \overline{P}_0, \\ e^{i\theta_j(\overline{\alpha},\overline{\tau})} z + \eta, & \text{if } z \in \overline{P}_j, j \in \{1, ..., d\}, \\ z - \lambda, & \text{if } z \in \overline{P}_{d+1}, \end{cases}$$
$$= \overline{F}(z).$$

□

4. Renormalization around zero

In this section we investigate renormalizability of TCES around the origin for a broad range of values of λ and η.

Let $\lambda \in [0, 1) \setminus \mathbb{Q}$ and $\eta \in \mathbb{R}$ such that $0 < \eta = p - q\lambda < \lambda$ for some $p, q \in \mathbb{N}$. Note that for the following we equip $\mathcal{N}_\lambda^<$ with an ordering $<'$ which is the lexicographical order on $\mathbb{N} \times \mathbb{N}$ restricted to $\mathcal{N}_\lambda^<$. Recall that the lexicographical order $<_{\mathbb{N}\times\mathbb{N}}$ on $\mathbb{N}\times\mathbb{N}$ (with respect to the canonical ordering $<$ on $\mathbb{N}$) is defined by

$$(x, y) <_{\mathbb{N}\times\mathbb{N}} (w, z) \text{ if and only if } \begin{cases} x < w, \text{ or} \\ x = w \text{ and } y < z. \end{cases}$$

Let (m_0, n_0) be the smallest element of $\mathcal{N}_\lambda^<$, with respect to $<'$, such that

$$P_{w(m_0,n_0)} \geq p \text{ and } Q_{w(m_0,n_0)} \geq q. \tag{4.1}$$

Let $k_0 = w(m_0, n_0)$, and we define the following family of sets

$$\mathcal{S}(\lambda, \eta) = \{S_k(\lambda, \eta) : k \in \mathbb{N}\},$$

where

$$S_k(\lambda, \eta) = S'_{k_0+k}(\lambda). \tag{4.2}$$

For brevity, we will omit the arguments from $\mathcal{S}$, S_k, and S'_k when λ and η are unambiguous.

Note that for all $(m, n) \in \mathcal{N}_\lambda^<$,

$$\Delta_{w(m,n)} = Q_{w(m,n)}\lambda - P_{w(m,n)} = -\eta + (Q_{w(m,n)} - q)\lambda - (P_{w(m,n)} - p).$$

We define a sequence $(h_{w(m,n)})_{(m,n)\in\mathcal{N}_\lambda^<}$ of positive integers by

$$h_{w(m,n)} = (Q_{w(m,n)} - q) + (P_{w(m,n)} - p) + 1. \tag{4.3}$$

We can establish some recurrence relations for the sequence $h_{w(m,n)}$ using those of $P_{w(m,n)}$ and $Q_{w(m,n)}$.

Proposition 4.1. *Let $(m, n) \in \mathcal{N}_\lambda^<$. Then*

$$h_{w(m,n+1)} = (n + 1)h_{w(m,0)} + h_{w(m-1,0)} + (n + 1)(p + q - 1).$$

Moreover, if $(m, n) \geq' (m_0, n_0)$, then $h_{w(m,n)} > 0$.

Proof. Recall that $Q_{w(m,n+1)} = (n+1)Q_{w(m,0)} + Q_{w(m-1,0)}$ and $P_{w(m,n+1)} = (n+1)P_{w(m,0)} + P_{w(m-1,0)}$. Therefore,

$$\begin{aligned} h_{w(m,n+1)} &= (Q_{w(m,n+1)} - q) + (P_{w(m,n+1)} - p) + 1 \\ &= ((n+1)Q_{w(m,0)} + Q_{w(m-1,0)} - q) + ((n+1)P_{w(m,0)} + P_{w(m-1,0)} - p) + 1 \\ &= (n+1)(Q_{w(m,0)} + P_{w(m,0)}) + \big((Q_{w(m-1,0)} - q) + (P_{w(m-1,0)} - p) + 1\big) . \end{aligned}$$

Using (4.3), we get

$$\begin{aligned} h_{w(m,n+1)} &= (n+1)((Q_{w(m,0} - q) + (P_{w(m,0)} - p) + 1 + (p+q-1)) + h_{w(m-1,0)} \\ &= (n+1)h_{w(m,0)} + h_{w(m-1,0)} + (n+1)(p+q-1). \end{aligned}$$

Also, note that (m_0, n_0) is the smallest pair in $\mathcal{N}_\lambda^<$ such that $Q_{w(m,n)} \geq q$ and $P_{w(m,n)} \geq p$. Thus, it is simple to deduce that $h_{w(m,n)} > 0$ if $(m,n) \geq' (m_0, n_0)$. ☐

We won't attempt to find recursive relations for all iterates of F at 0, but we will at least calculate the orbit of 0 at iterates given by the sequence $(h_{w(m,n)})$.

Lemma 4.2. *Let $(m,n) \in \mathcal{N}_\lambda^<$ such that $(m,n) \geq' (m_0, n_0)$. Then*

$$F^{h_{w(m,n)}}(0) = \Delta_{w(m,n)}.$$

Proof. Suppose, for a contradiction, that $F^{h_{w(m,n)}}(0) \neq \Delta_{w(m,n)}$. Let $(a_t)_{t\in\mathbb{N}}$ and $(b_t)_{n\in\mathbb{N}}$ denote the sequences defined by

$$F^t(0) = -\eta + b_t\lambda - a_t. \tag{4.4}$$

Note that since $\eta = p - q\lambda$, we have

$$F^t(0) = q\lambda - p + b_t\lambda - a_t = (b_t + q)\lambda - (a_t + p),$$

which is never equal to 0 since λ is irrational and $b_t + q$ and $a_t + p$ are both positive. Observe that $(a_t)_t$ and $(b_t)_t$ are both non-decreasing and obey the following rule:

$$(a_{t+1}, b_{t+1}) = \begin{cases} (a_t, b_t + 1) & \text{if } F^t(0) < 0 \\ (a_t + 1, b_t) & \text{if } F^t(0) > 0 \end{cases}.$$

Given that $a_0 = a_1 = b_0 = b_1 = 0$, we can deduce that the sequences $(a_t)_t$ and $(b_t)_t$ achieve every non-negative integer value, that is, for any $N \in \mathbb{N}$, there is some $t \in N$ such that $a_t = N$, and similarly there is some $t' \in \mathbb{N}$ such that $b_{t'} = N$. Moreover, a simple inductive argument shows that for all integers $t \geq 1$,

$$a_t + b_t + 1 = t.$$

Next, observe that $F^{h_{w(m,n)}}(0) \neq \Delta_{w(m,n)}$ is equivalent to the statement that $a_{h_{w(m,n)}} \neq P_{w(m,n)}$ and $b_{h_{w(m,n)}} \neq Q_{w(m,n)}$. However, notice that

$$(Q_{w(m,n)} - q) + (P_{w(m,n)} - p) + 1 = h_{w(m,n)} = a_{h_{w(m,n)}} + b_{h_{w(m,n)}} + 1.$$

This implies one of two cases.

(1) $a_{h_{w(m,n)}} < P_{w(m,n)} - p$ and $b_{h_{w(m,n)}} > Q_{w(m,n)} - q$; or
(2) $a_{h_{w(m,n)}} > P_{w(m,n)} - p$ and $b_{h_{w(m,n)}} < Q_{w(m,n)} - q$.

Suppose case (1). Since $(b_t)_t$ is non-decreasing and takes every non-negative integer value, we know that there is some non-negative integer $t^* < h_{w(m,n)}$ such that $b_{t^*} = Q_{w(m,n)} - q$. Thus,

$$F^{t^*}(0) = -\eta + (Q_{w(m,n)} - q)\lambda - a_{t^*}.$$

Note that $a_{t^*} < a_t < P_{w(m,n)} - p$. Now, suppose $\Delta_{w(m,n)} > 0$. Then for any $0 \le j \le P_{w(m,n)} - p - a_{t^*}$, we have

$$F^{t^*}(0)-j = -\eta+(Q_{w(m,n)}-q)\lambda-(a_{t^*}+j) > (Q_{w(m,n)}-q)\lambda-(P_{w(m,n)}-p) = \Delta_{w(m,n)} > 0.$$

Thus, $F^{t^*+j}(0) = F^{t^*}(0) - j$ for $0 \le j \le P_{w(m,n)} - p - a_{t^*}$. In particular,

$$F^{t^*+P_{w(m,n)}-p-a_{t^*}}(0) = -\eta + (Q_{w(m,n)} - q)\lambda - (P_{w(m,n)} - p).$$

But then

$$\begin{aligned} t^* + P_{w(m,n)} - p - a_{t^*} &= (Q_{w(m,n)} - q) + (P_{w(m,n)} - p) + 1 \\ &= h_{w(m,n)}. \end{aligned}$$

And this implies that

$$\begin{aligned} F^{h_{w(m,n)}}(0) &= F^{t^*+P_{w(m,n)}-p-a_{t^*}}(0) \\ &= -\eta + (Q_{w(m,n)} - q)\lambda - (P_{w(m,n)} - p) \\ &= \Delta_{w(m,n)}. \end{aligned}$$

But this contradicts our assumption that $F^{h_{w(m,n)}}(0) \neq \Delta_{w(m,n)}$.

Now suppose that $\Delta_{w(m,n)} < 0$. Let $0 \le j < P_{w(m,n)} - p - a_{t^*}$. Then either $F^{t^*}(0) - j > 0$ or $\Delta_{w(m,n)} < F^{t^*}(0) - j < 0$. But $\Delta_{w(m,n)} < F^{t^*}(0) - j < 0$ implies

$$|(Q_{w(m,n)}\lambda - (a_{t^*} + j + p)| < |\Delta_{w(m,n)}| = |Q_{w(m,n)}\lambda - P_{w(m,n)}|,$$

and $a_{t^*} + j + p < P_{w(m,n)}$. This contradicts the 'best approximate' property of the semiconvergent $P_{w(m,n)}/Q_{w(m,n)}$. Thus $F^{t^*}(0) - j > 0$ for all $0 \le j < P_{w(m,n)} - p - a_{t^*}$. Thus, by a similar argument to before, we reach the contradiction that $F^{h_{w(m,n)}}(0) = \Delta_{w(m,n)}$.

In case (2), $a_{h_{w(m,n)}} > P_{w(m,n)} - p$ and $b_{h_{w(m,n)}} < Q_{w(m,n)} - q$. We can reach a similar contradiction as above, by using a similar argument where the roles of $(a_t)_t$ and $(b_t)_t$ are interchanged.

This exhausts all cases, so our assumption that $F^{h_{w(m,n)}}(0) \neq \Delta_{w(m,n)}$ must be false. □

Lemma 4.3. *Let $m \in \mathbb{N}$ such that $(m, 0) \geq' (m_0, n_0)$. Then, for all $1 \le t < h_{w(m+1,0)}$,*

$$|F^t(0)| \geq |\Delta_{w(m,0)}|.$$

Proof. recall that $F^t(0) = -\eta + b_t\lambda - a_t$, for some $a_t, b_t \in \mathbb{N}$. Since $(a_t)_t$ and $(b_t)_t$ are non-decreasing and $t < h_{w(m+1,0)}$, we have that $b_t \le q_{m+1} - q$ and $a_t \le p_{m+1} - p$, not both equal. Hence either $b_t + q < q_{m+1}$ or $a_t + p < p_{m+1}$ and $b_t + q \le q_{m+1}$. In either case, by the best approximate property of convergents

$$|F^t(0)| = |(b_t + q)\lambda - (a_t + p)| \ge |q_m\lambda - p_m| = |\Delta_{w(m,0)}|.$$

□

Lemma 4.4. *Let* $(m, n) \in \mathcal{N}_\lambda^<$. *Then for all* $1 \le t < h_{w(m,n+1)}$,

$$F^t(0) \ge \Delta_{w(m,0)} \text{ or } F^t(0) \le n\Delta_{w(m,0)} + \Delta_{w(m-1,0)} \text{ if } m \text{ is even,}$$
$$F^t(0) \le \Delta_{w(m,0)} \text{ or } F^t(0) \ge n\Delta_{w(m,0)} + \Delta_{w(m-1,0)} \text{ if } m \text{ is odd.}$$

Proof. From (**??**), recall that if m is even and $n > 0$, then $P_{w(m,n)}/Q_{w(m,n)} > \lambda$ is a best approximate from above, which implies

$$b\lambda - a \le Q_{w(m,n)}\lambda - P_{w(m,n)} < 0, \tag{4.5}$$

for all $a, b \in \mathbb{Z}$ with $0 < b < Q_{w(m,n+1)}$ such that $P_{w(m,n)}/Q_{w(m,n)} \ne a/b > \lambda$.

Let $(a_t)_{t\in\mathbb{N}}$ and $(b_t)_{t\in\mathbb{N}}$ be the sequences described by (4.4). Suppose that $1 \le t \le h_{w(m,n)}$ with $F^t(0) < 0$. Then

$$b_t \le Q_{w(m,n+1)} - q \text{ and } a_t \le P_{w(m,n+1)} - p,$$

since $(a_t)_t$ and $(b_t)_t$ are non-decreasing. Thus,

$$b_t + q \le Q_{w(m,n+1)} \text{ and } a_t + p \le P_{w(m,n+1)}, \tag{4.6}$$

not both equal. Therefore, from (4.4) we know that

$$F^t(0) = (b_t + q)\lambda - (a_t + p),$$

and by (4.5) with (4.6), we have

$$F^t(0) \le \begin{cases} Q_{w(m,n)}\lambda - P_{w(m,n)}, & \text{if } n \ne 0, \\ q_{m-1}\lambda - p_{m-1}, & \text{if } n = 0. \end{cases}$$

Recalling the definition of $\Delta_{w(m,n)}$ as in (3.2), we get

$$F^t(0) \le \begin{cases} \Delta_{w(m,n)}, & \text{if } n \ne 0, \\ \Delta_{w(m-1,0)}, & \text{if } n = 0. \end{cases}$$

Finally, by using (3.3), we have

$$F^t(0) \le n\Delta_{w(m,0)} + \Delta_{w(m-1,0)}.$$

If $F^t(0) > 0$, then by Lemma 4, we have

$$F^t(0) \ge \Delta_{w(m,0)}.$$

From (**??**), recall that if m is off and $n > 0$, then $P_{w(m,n)}/Q_{w(m,n)} < \lambda$ is a best approximate from below, that is

$$b\lambda - a \geq Q_{w(m,n)}\lambda - P_{w(m,n)} > 0,$$

for all $a, b \in \mathbb{Z}$ with $0 < b < Q_{w(m,n+1)}$ such that $P_{w(m,n)}/Q_{w(m,n)} \neq a/b < \lambda$. Thus, in the case that m is odd and $F^t(0) > 0$, then

$$b_t + q \leq Q_{w(m,n+1)} \text{ and } a_t + p \leq P_{w(m,n+1)},$$

not both equal. Thus, similarly to the above case where m is even, we have

$$\begin{aligned} F^t(0) &= (b_t + q)\lambda - (a_t + p) \\ &\geq \begin{cases} Q_{w(m,n)}\lambda - P_{w(m,n)} & \text{if } n \neq 0 \\ q_{m-1}\lambda - p_{m-1} & \text{if } n = 0 \end{cases} \\ &= n\Delta_{w(m,0)} + \Delta_{w(m-1,0)}. \end{aligned}$$

On the other hand, if $F^t(0) < 0$, then by Lemma 4,

$$F^t(0) \leq \Delta_{w(m,0)}.$$

□

In order to prove the next theorem, we will distinguish between the following two cases and we will prove them separately. We will first prove that if

$$z \in E^{-1}(S_{w(m,n)}),$$

then

$$h(z) = h_{w(m,n+1)}.$$

and then we will prove that if

$$h(z) = h_{w(m,n+1)},$$

then

$$F^{h(z)}(z) = E(z) + \Delta_{w(m,n+1)}.$$

Theorem 4.5. *Let $\alpha \in \mathbb{B}$, $\tau \in \mathrm{Sym}(d)$, $\lambda \in [0,1) \setminus \mathbb{Q}$, and $0 < \eta = p - q\lambda < \lambda$ for some $p, q \in \mathbb{N}$. Let $R(z)$ be as in* (2.4)*, $h(z)$ as in* (2.3)*, $S'_{w(m,n)}$ as in* (3.9)*, and $h_{w(m,n)}$ as in* (4.3)*. For all $(m,n) \in \mathcal{N}_\lambda^<$ with $(m,n) \geq' (m_0, n_0)$, $h(E^{-1}(S'_{w(m,n)}))$ exists and is equal to $h_{w(m,n+1)}$. Moreover, let $z \in E^{-1}(S'_{w(m,n)})$. Then*

$$R(z) = E(z) + \Delta_{w(m,n+1)}. \tag{4.7}$$

Proof. Let $z \in P_c$.

Assume $z \in E^{-1}(S'_{w(m,n)})$. Observe that $E(z) + F^{h_{w(m,n+1)}}(0) \in P_c$, so $h(z) \leq h_{w(m,n+1)}$. By Lemma 3.1, we know that

$$F^{h(z)}(z) = E(z) + F^{h(z)}(0).$$

Suppose, for a contradiction, that $h(z) < h_{w(m,n+1)}$. We will prove the contradiction for even and odd m separately, starting with the case that m is even. By Lemma 4.4, we know that either $F^{h(z)}(0) \geq \Delta_{w(m,0)}$ or $F^{h(z)}(0) \leq n\Delta_{w(m,0)} + \Delta_{w(m-1,0)}$. Since m is even, recall that

$$S'_{w(m,n)} = (P_0 - \Delta_{w(m,0)}) \cap (P_c - \Delta_{w(m,n+1)}) \cap P_c \cap (P_{d+1} - (n\Delta_{w(m,0)} + \Delta_{w(m-1,0)})).$$

Observe that

$$\begin{aligned} F^{h(z)}(z) &\in S'_{w(m,n)} + F^{h(z)}(0) \\ &\subset (P_0 + (F^{h(z)}(0) - \Delta_{w(m,0)})) \cap (P_{d+1} + (F^{h(z)}(0) - (n\Delta_{w(m,0)} + \Delta_{w(m-1,0)}))). \end{aligned}$$

Therefore, if $F^{h(z)}(0) \geq \Delta_{w(m,0)}$, then

$$F^{h(z)}(z) \in P_0 + (F^{h(z)}(0) - \Delta_{w(m,0)}) \subset P_0.$$

But by the definition of $h(z)$ as in (2.3), we have

$$F^{h(z)}(z) = R(z) \in P_c,$$

a contradiction. Similarly, if $F^{h(z)}(0) \leq n\Delta_{w(m,0)} + \Delta_{w(m-1,0)}$, then

$$F^{h(z)}(z) \in P_{d+1} + (F^{h(z)}(0) - (n\Delta_{w(m,0)} + \Delta_{w(m-1,0)})) \subset P_{d+1},$$

which also contradicts $F^{h(z)}(z) \in P_c$.

Now suppose m is odd. Then

$$S'_{w(m,n)} = (P_0 - (n\Delta_{w(m,0)} + \Delta_{w(m-1,0)})) \cap P_c \cap (P_c - \Delta_{w(m,n+1)})) \cap (P_{d+1} - \Delta_{w(m,0)}).$$

By Lemma 4.4, we know that either

$$F^{h(z)}(0) \geq n\Delta_{w(m,0)} + \Delta_{w(m-1,0)} \text{ or } F^{h(z)}(0) \leq \Delta_{w(m,n)}.$$

Clearly, either of these cases give similar contradictions as before. Therefore our assumption that $h(z) < h_{w(m,n+1)}$ must be false, so in fact $h(z) = h_{w(m,n+1)}$.

($\star\star$). Suppose $h(z) = h_{w(m,n+1)}$. By Lemma 3.1,

$$F^{h(z)}(z) = E(z) + F^{h(z)}(0),$$

and by Lemma 4.2 we know that

$$F^{h_{w(m,n+1)}}(0) = \Delta_{w(m,n+1)}.$$

Combining these two with our assumption gives us that if $h(z) = h_{w(m,n+1)}$, then $F^{h(z)}(z) = E(z) + \Delta_{w(m,n+1)}$. □

With this Theorem, as well as Theorem 3.7, we can prove the existence of a renormalization scheme around the point 0. First, we will find the definition of the first return map R on the rest of P_c.

Lemma 4.6. *Let* S_k *be as in* (4.2)*, and* $(m_0, n_0) \in \mathcal{N}_\lambda^<$ *as in* (4.1)*. Let*

$$U = \{0\} \cup \bigcup_{k=0}^{\infty} S_k.$$

Then

$$U = \begin{cases} (P_0 - \Delta_{w(m_0,0)}) \cap P_c \cap (P_{d+1} - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)}), & \text{if } m_0 \text{ is even,} \\ (P_0 - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)}) \cap P_c \cap (P_{d+1} - \Delta_{w(m_0,0)}), & \text{if } m_0 \text{ is odd,} \end{cases}$$

and U *is convex.*

Proof. We will first prove the equality

$$U = \begin{cases} (P_0 - \Delta_{w(m_0,0)}) \cap P_c \cap (P_{d+1} - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)})), & \text{if } m_0 \text{ is even,} \\ (P_0 - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)})) \cap P_c \cap (P_{d+1} - \Delta_{w(m_0,0)}), & \text{if } m_0 \text{ is odd.} \end{cases}.$$

Observe that for all $k \in \mathbb{N}$,

$$S_k \subset P_c.$$

Additionally, if $(m, n) \in \mathcal{N}_\lambda^<$ such that $w(m, n) = k + k_0$, then

$$S_k \subset \begin{cases} \bigcup_{m=m_0}^{\infty} (P_0 - \Delta_{w(m,0)}) \cup \bigcup_{(m,n)\geq(m_0+1,0)} (P_0 - (n\Delta_{w(m,0)} + \Delta_{w(m-1,0)})), & \text{if } m_0 \text{ is even,} \\ \bigcup_{m=m_0+1}^{\infty} (P_0 - \Delta_{w(m,0)}) \cup \bigcup_{(m,n)\geq(m_0,n_0)} (P_0 - (n\Delta_{w(m,0)} + \Delta_{w(m-1,0)})), & \text{if } m_0 \text{ is odd.} \end{cases}$$

Note that $P_0 - (n\Delta_{w(m+1,0)} + \Delta_{w(m,0)}) \subset P_0 - \Delta_{w(m,0)}$ for all $(m + 1, n) \in \mathcal{N}_\lambda^<$. Thus, we have

$$S_k \subset \begin{cases} P_0 - \Delta_{w(m_0,0)}, & \text{if } m_0 \text{ is even,} \\ P_0 - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)}), & \text{if } m_0 \text{ is odd.} \end{cases}.$$

With a similar argument, we can show that

$$S_k \subset \begin{cases} P_{d+1} - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)}), & \text{if } m_0 \text{ is even,} \\ P_{d+1} - \Delta_{w(m_0,0)}, & \text{if } m_0 \text{ is odd.} \end{cases},$$

Altogether, we deduce that

$$U \subseteq \begin{cases} (P_0 - \Delta_{w(m_0,0)}) \cap P_c \cap (P_{d+1} - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)})), & \text{if } m_0 \text{ is even,} \\ (P_0 - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)})) \cap P_c \cap (P_{d+1} - \Delta_{w(m_0,0)}), & \text{if } m_0 \text{ is odd.} \end{cases}.$$

Suppose m_0 is even, and let

$$z \in (P_0 - \Delta_{w(m_0,0)}) \cap P_c \cap (P_{d+1} - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)})),$$

with $z \neq 0$. Then there is some $(m, n) \in \mathcal{N}_\lambda^<$ with m is odd and $(m, n) \geq' (m_0, n_0)$ such that

$$z \in P_0 - (n\Delta_{w(m,0)} + \Delta_{w(m-1,0)}),$$

and there is some $(m', n') \in \mathcal{N}_\lambda^<$ with m' even and $(m', n') \geq' (m_0, n_0)$ such that

$$z \in P_{d+1} - (n\Delta_{w(m,0)} + \Delta_{w(m-1,0)}).$$

Suppose that (m, n) and (m', n') are the largest such pairs, which is well-defined since $z \neq 0$. Since m is odd and m' is even, we either have $m < m'$ or $m' < m$. Suppose $m < m'$. Then

$$z \in (P_0 - (n\Delta_{w(m,0)} + \Delta_{w(m-1,0)})) \cap P_c \cap (P_{d+1} - \Delta_{w(m,0)}).$$

Since (m, n) is the largest pair such that $z \in P_0 - (n\Delta_{w(m,0)} + \Delta_{w(m-1,0)})$, and noting that $(n + 1)\Delta_{w(m,0)} + \Delta_{w(m-1,0)} = \Delta_{w(m,n+1)}$ since $n + 1 > 0$, we have that

$$z \in (P_c - \Delta_{w(m,n+1)}) \text{ or } z \in (P_{d+1} - \Delta_{w(m,n+1)}).$$

However, $P_{d+1} - \Delta_{w(m,n+1)} \subset P_{d+1}$ since m is odd and so $\Delta_{w(m,n+1)} > 0$. Importantly,

$$P_c \cap (P_{d+1} - \Delta_{w(m,n+1)}) = \emptyset,$$

and thus $z \in P_c - \Delta_{w(m,n+1)}$. Therefore,

$$z \in (P_0 - (n\Delta_{w(m,0)} + \Delta_{w(m-1,0)})) \cap (P_c - \Delta_{w(m,n+1)}) \cap P_c \cap (P_{d+1} - \Delta_{w(m,0)}) = S'_{w(m,n)},$$

so clearly $z \in U$. In the case that $m' < m$ we can use a similar argument to prove that

$$z \in (P_0 - \Delta_{w(m',0)}) \cap P_c \cap (P_c - \Delta_{w(m',n'+1)}) \cap (P_{d+1} - (n'\Delta_{w(m',0)} + \Delta_{w(m'-1,0)})) = S'_{w(m',n')},$$

so that $z \in U$. If m_0 is odd and

$$z \in (P_0 - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)})) \cap P_c \cap (P_{d+1} - \Delta_{w(m_0,0)}),$$

with $z \neq 0$, then we can use a similar argument to prove that $z \in U$. Hence, we have

$$U = \begin{cases} (P_0 - \Delta_{w(m_0,0)}) \cap P_c \cap (P_{d+1} - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)})) & \text{if } m_0 \text{ is even} \\ (P_0 - (n_0\Delta_{w(m_0,0)} + \Delta_{w(m_0-1,0)})) \cap P_c \cap (P_{d+1} - \Delta_{w(m_0,0)}) & \text{if } m_0 \text{ is odd} \end{cases}.$$

To show that U is convex, one must note that the cones P_0, P_c, and P_{d+1} and all their translates are convex sets, and that the intersection of convex sets is also convex. □

Recall that for any given $0 < \eta = p - q\lambda < \lambda$, (m_0, n_0) is the minimum element of $\mathcal{N}_\lambda^<$ such that $P_{w(m_0,n_0)} \geq p$ and $Q_{w(m_0,n_0)} \geq q$. Observe that for any $(m, n) \in \mathcal{N}_\lambda^<$, if $0 < \eta < p - q\lambda$ is chosen such that $(m_0, n_0) \leq' (m, n)$, then

$$S_{w(m,n)-w(m_0,n_0)} = S'_{w(m,n)},$$

is such that $R(z) = E(z) + \Delta_{w(m,n)}$ for all $z \in S_{w(m,n)-w(m_0,n_0)}$. Define $U_k(\lambda, \eta) = \bigcup_{j=k}^{\infty} S_j$ for $k \geq w(m_0, n_0)$ (omitting the arguments where unambiguous), and let R' be the first return map to P_c of the TCE defined by

$$F'(z) = G' \circ E(z),$$

where

$$G'(z) = \begin{cases} z + \lambda, & \text{if } z \in P_{d+1}, \\ z - \eta', & \text{if } z \in P_c, \\ z - 1, & \text{if } z \in P_0. \end{cases}$$

α, τ, λ, and η', where $0 < \eta' = p' - q'\lambda < \lambda$ is such that $p' \leq p$ and $q' \leq q$. If (m_0', n_0') the the minimal element of $\mathcal{N}_\lambda^<$ such that $P_{w(m_0',n_0')} \geq p'$ and $Q_{w(m_0',n_0')} \geq q'$, then $(m_0', n_0') \leq' (m_0, n_0)$. Furthermore,

$$R \restriction_{U_{w(m_0,n_0)}} = R' \restriction_{U_{w(m_0,n_0)}} .$$

Let $R' : P_c \to P_c$ be the first return map of F' to P_c, where F' is the TCE defined by

$$F'(z) = G' \circ E(z),$$

where

$$G'(z) = \begin{cases} z + g^2(\lambda), & \text{if } z \in P_{d+1}, \\ z - \eta', & \text{if } z \in P_c, \\ z - 1, & \text{if } z \in P_0. \end{cases}$$

We now proof our main result on this paper.

Theorem 4.7. *Let $\alpha \in \mathbb{B}$, $\tau \in \mathrm{Sym}(d)$, $\lambda \in [0,1) \setminus \mathbb{Q}$, $0 < \eta = p - q\lambda < \lambda$.*
Let R be as in (2.4), and let

$$(m_0, n_0) = \min\{(m,n) \in \mathcal{N}_\lambda^< : P_{w(m,n)}(\lambda) \geq p \text{ and } Q_{w(m,n)}(\lambda) \geq q\}$$

Let $0 < \eta' = p' - q'g^2(\lambda) < g^2(\lambda)$ such that the pair (m_0', n_0') defined by

$$(m_0', n_0') = \min\{(m,n) \in \mathcal{N}_{g^2(\lambda)}^< : P_{w(m,n)}(g^2(\lambda)) \geq p' \text{ and } Q_{w(m,n)}(g^2(\lambda)) \geq q'\},$$

satisfies $(m_0' + 2, n_0') \leq' (m_0 + 2, 0)$ (within the set $\mathcal{N}_\lambda^<$.)
Then for all $z \in U_{w(m_0,0)}(g^2(\lambda), \eta')$, the following holds:

$$R'(z) = \frac{1}{-\Delta_{w(1,0)}(\lambda)} R((-\Delta_{w(1,0)}(\lambda))z).$$

Proof. Recall that by Theorem 3.7, we have

$$\frac{1}{-\Delta_{w(1,0)}(\lambda)} S'_{w(j+2,n)}(\lambda) = S'_{w(j,n)}(g^2(\lambda)),$$

for all $(j+2, n) \in \mathcal{N}_\lambda^<$ with $(j+2,n) \geq' (m_0+2, 0)$, i.e. $j \geq m_0$. Note that by Proposition 3.2,

$$(m_0' + 2, n_0') \leq' (m_0 + 2, 0) \text{ in } \mathcal{N}_\lambda^<,$$

is equivalent to the statement that

$$(m_0', n_0') \leq' (m_0, 0) \text{ in } \mathcal{N}^{<}_{g^2(\lambda)}.$$

Hence,

$$\frac{1}{-\Delta_{w(1,0)}(\lambda)} U_{w(m_0+2,0)}(\lambda) = U_{w(m_0,0)}(g^2(\lambda)).$$

Let $z \in S_k(\lambda, \eta) = S'_{w(m+2,n)}(\lambda)$ for some $k = w_\lambda(m+2,n) - w_\lambda(m_0+2,0)$. Also note that since $cP_j = P_j$ for all $c > 0$ and E consists of rotations about 0, we have

$$E(cz) = cE(z), \tag{4.8}$$

for $c > 0$. Therefore $z \in S'_{w(m,n)}(g^2(\lambda, \eta'))$, and by expanding R as in (4.7) we get

$$\frac{1}{-\Delta_{w(1,0)}(\lambda)} R\left((-\Delta_{w(1,0)}(\lambda))z\right) = \frac{1}{-\Delta_{w(1,0)}(\lambda)}\left(E((-\Delta_{w(1,0)}(\lambda))z) + \Delta_{w(m+2,n+1)}(\lambda)\right).$$

Using 4.8, we get

$$\begin{aligned}\frac{1}{-\Delta_{w(1,0)}(\lambda)} R\left((-\Delta_{w(1,0)}(\lambda))z\right) &= \frac{1}{-\Delta_{w(1,0)}(\lambda)}\left((-\Delta_{w(1,0)}(\lambda))E(z) + \Delta_{w(m+2,n+1)}(\lambda)\right) \\ &= E(z) + \left(-\frac{\Delta_{w(m+2,n+1)}(\lambda)}{\Delta_{w(1,0)}(\lambda)}\right).\end{aligned}$$

Now, using (3.8) and comparing with the formula for R' as in (4.7), we see

$$\begin{aligned}\frac{1}{-\Delta_{w(1,0)}(\lambda)} R\left((-\Delta_{w(1,0)}(\lambda))z\right) &= E(z) + \Delta_{w(m,n+1)}(g^2(\lambda)) \\ &= R'(z).\end{aligned}$$

□

The immediate consequence of Theorem 4.7 is the infinite renormalizability towards 0 of TCEs for which $\lambda \in [0,1) \setminus \mathbb{Q}$ and $0 < \eta = p - q\lambda < \lambda$.

5. Return maps

As an example, we will set $\alpha \in \mathbb{B}$, $\lambda = \Phi$, $\rho = 1$, and $\eta = \Phi^2$, where

$$\Phi = \frac{\sqrt{5}-1}{2}. \tag{5.1}$$

In this case the the space $\mathcal{N}^{<}_{\lambda} = \mathbb{N} \times \{0\} \cong \mathbb{N}$ since $\lambda = [0; \bar{1}]$ and so $\lambda_k = 1$ for all $k \in \mathbb{N}$. Thus, $w(m,n) = w(m) = m$. Also due to $\lambda_k = 1$ for each $k \in \mathbb{N}$, the semiconvergents P_m/Q_m simply coincide with the convergents p_m/q_m, and the convergents are in this case defined by

$$\begin{aligned} p_0 &= 0, & q_0 &= 1, \\ p_1 &= 1, & q_1 &= 1, \\ p_n &= p_{n-1} + p_{n-2}, & q_n &= q_{n-1} + q_{n-2}. \end{aligned}$$

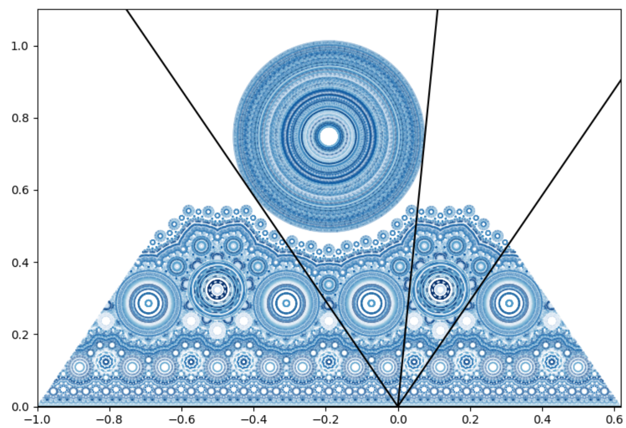

FIGURE 3. A plot of 1500 iterates of 500 uniformly chosen points within the box $[-1, \lambda] \times [0, 1.1]$ under the TCE with parameters $\alpha = (\pi/2 - 0.6, 0.5, 0.7, \pi - 0.6)$, $\tau = (1\,0)$, $\lambda = \Phi$, $\rho = 1$, and $\eta = \Phi^2$. The first 400 points of each orbit are omitted to remove transients.

It is thus clear that $p_n = F_n$ and $q_n = F_{n+1}$, where $(F_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence with $F_0 = 0$ and $F_1 = 1$.

The first return times $(h_m)_{m\in\mathbb{N}}$ in the case of $\lambda = \Phi$ are given by

$$h_m = (q_m - 1) + (p_m - 1) + 1 = F_{m+1} + F_m - 1 = F_{m+2} - 1$$

for all $n \in \mathbb{N}$, where F_n is the n-th Fibonacci number, with initial values $F_0 = 0$, $F_1 = 1$. Observe that $h_n = h_{n-1} + h_{n-2} + 1$ for all $n \geq 2$, and $h_0 = 2$, $h_1 = 4$. Now note that $\eta = \Phi^2 = 1 - \Phi = 1 - \lambda$, and thus we have

$$m_0 = \min\{m \in \mathbb{N} : p_m \geq 1, q_m \geq 1\} = 1.$$

The errors of the convergents are given by

$$\Delta_m(\Phi) = q_m\lambda - p_m = F_{m+1}\Phi - F_m. \tag{5.2}$$

Proposition 5.1. *We have*

$$\Delta_m(\Phi) = -(-\Phi)^{m+1}$$

Proof. Observe that

$$\begin{aligned}
\Delta_m(\Phi) &= F_{m+1}\Phi - F_m \\
&= F_m\Phi + F_{m-1}\Phi - F_m \\
&= F_{m-1}\Phi - (1 - \Phi)F_m \\
&= -\Phi(F_m\Phi - F_{m-1}) \\
&= -\Phi\Delta_{m-1}(\Phi).
\end{aligned}$$

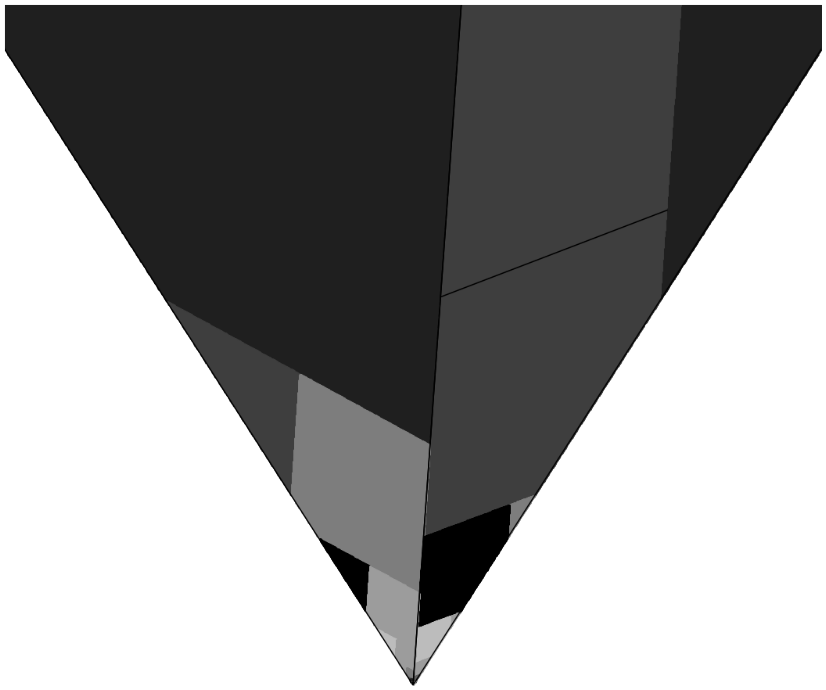

FIGURE 4. A partition of P_c in the case where $\alpha = (1, 0.5, \pi - 2.5, 1)$, $\tau = (1\,0)$, $\lambda = \Phi$, $\rho = 1$, and $\eta = \Phi^2$. A cascading pattern towards the origin can be seen, but its geometric structure becomes clearer after we apply the cone exchange E.

Recall that $p_0 = 0$ and $q_0 = 1$. Then a simple inductive argument shows us that

$$\Delta_m(\Phi) = (-\Phi)^m \Delta_0(\Phi) = (-\Phi)^m (q_0 \Phi - p_0) = -(-\Phi)^{m+1}$$

□

Noting that the recurrence relations for p_m and q_m give us $p_{-1} = 1$ and $q_{-1} = 0$ and so we can set $\Delta_{-1} = -1 = -(-\Phi)^0$, which remains consistent with (5.2). With this proposition in mind, we can define the partition $\mathcal{S} = \{S_m : m \in \mathbb{N}\}$ by

$$\begin{aligned} S_m &= \begin{cases} (P_0 - \Delta_m) \cap P_c \cap (P_c - \Delta_{m+1}) \cap (P_{d+1} - \Delta_{m-1}), & \text{if } m \text{ is even,} \\ (P_0 - \Delta_{m-1}) \cap (P_c - \Delta_{m+1}) \cap P_c \cap (P_{d+1} - \Delta_m), & \text{if } m \text{ is odd.} \end{cases} \\ &= \begin{cases} (P_0 + (-\Phi)^{m+1}) \cap P_c \cap (P_c + (-\Phi)^{m+2}) \cap (P_{d+1} + (-\Phi)^m), & \text{if } m \text{ is even,} \\ (P_0 + (-\Phi)^m) \cap (P_c + (-\Phi)^{m+2}) \cap P_c \cap (P_{d+1} + (-\Phi)^{m+1}), & \text{if } m \text{ is odd.} \end{cases} \end{aligned}$$

These are rhombi, as can be seen in figure 5. It is also clear to see that for all $m \in \mathbb{N}$.

$$S_{m+2} = \Phi^2 S_m. \tag{5.3}$$

The ribbon in figure 5, including the similarly coloured rhombus adjacent to it, is the set

$$X = P_c \cap (P_c - (\lambda - \eta)) \cap (P_{d+1} + \eta) = P_c \cap (P_c - \Phi^3) \cap (P_{d+1} + \Phi^2), \tag{5.4}$$

and the cone in the same figure is the set

$$Y = P_c \cap (P_c + \eta) = P_c \cap (P_c + \Phi^2). \tag{5.5}$$

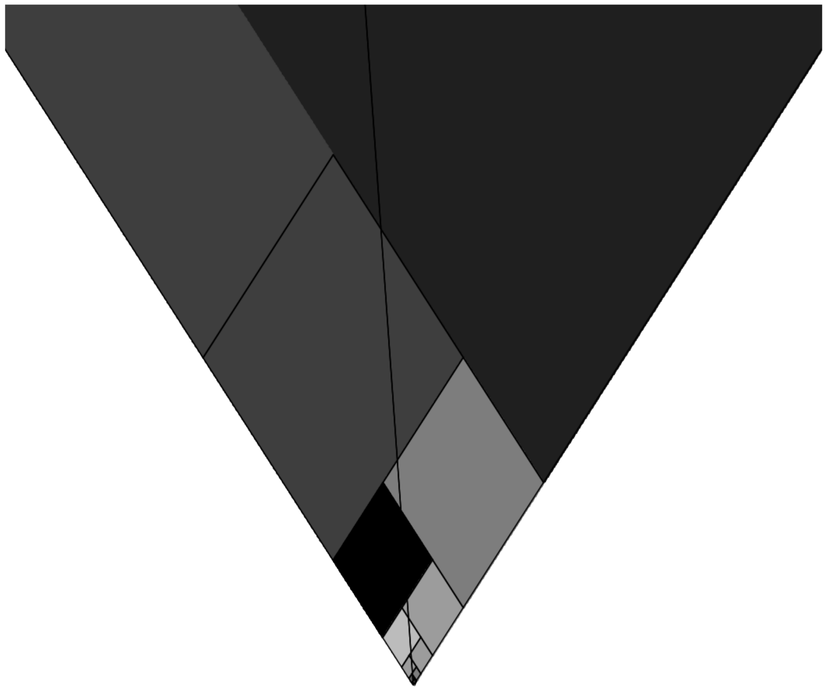

FIGURE 5. The same partition as in 4, but after an application of E, which reveals an alternating pattern of rhombi.

We are interested in the pre-image of these sets under E. In particular, define the partition

$$\mathcal{P}' = \left\{E^{-1}(S) \cap P_j : j \in \{1, ..., d\}, S \in \{Y, X, S_2, S_3, ..., \}\right\}.$$

This partition can be seen in figure 4. As a consequence of the next theorem, $(\mathcal{P}', R)$ is a PWI with a countably infinite number of atoms. We also define a separate family of sets

$$\mathcal{Q} = \left\{Q_{n,j} = E^{-1}(S_n) \cap P_j : n \in \mathbb{N}, j \in \{1, ..., d\}\right\},$$

which includes only the rhombi, and thus forms a partition of only a subset of the middle cone. Note that since $\lambda_m = 1$ for all $m \in \mathbb{N}$ and $m_0 = 1$, $m^* = 0$. We see that the set U from Lemma **??**is given by

$$\begin{aligned} U &= \{0\} \cup \bigcup_{m=0}^{\infty} S_m \\ &= (P_0 - \Delta_0) \cap P_c \cap (P_{d+1} - \Delta_{-1}) \\ &= (P_0 - \Phi) \cap P_c \cap (P_{d+1} + 1). \end{aligned}$$

Also observe that by removing S_0 and S_1 we get

$$U_2 = \{0\} \cup \bigcup_{m=2}^{\infty} S_m = (P_0 - \Phi^3) \cap P_c \cap (P_{d+1} + \Phi^2).$$

From this, we notice that

$$U_2 \cup X = P_c \cap (P_{d+1} + \Phi^2) \cap \left((P_c - \Phi^3) \cup (P_0 - \Phi^3)\right),$$

but since $P_{d+1} - \Phi^3 \subset P_{d+1}$, we know that $P_c \cap (P_{d+1} - \Phi^3) = \emptyset$, so

$$\begin{aligned} U_2 \cup X &= P_c \cap (P_{d+1} + \Phi^2) \cap \big((P_{d+1} - \Phi^3) \cup (P_c - \Phi^3) \cup (P_0 - \Phi^3)\big) \\ &= P_c \cap (P_{d+1} + \Phi^2) \cap \overline{\mathbb{H}} \\ &= P_c \cap (P_{d+1} + \Phi^2). \end{aligned}$$

Therefore, we can see that

$$U_2 \cup X \cup Y = P_c \cap \left((P_c + \Phi^2) \cup (P_{d+1} + \Phi^2)\right),$$

and a similar argument tells us that $P_c \cap (P_0 + \Phi^2) = \emptyset$ and so finally we get,

$$U_2 \cup X \cup Y = P_c \cap \left((P_0 + \Phi^2) \cup (P_c + \Phi^2) \cup (P_{d+1} + \Phi^2)\right) = P_c \cap \overline{\mathbb{H}} = P_c.$$

Therefore, $\mathcal{P}'$ is a partition of P_c up to a set of Lebesgue measure 0, namely it is a partition of $P_c \setminus \{0\}$.

Note that for all $(m, n) \in \mathcal{N}_\Phi^<$, $w(m, 1) = w(m + 1, 0)$, and so by recalling that $w(m, 0) = w(m) = m$, the condition that $w(m, 1) \geq w(m_0, 0)$ is equivalent to the condition that

$$w(m + 1, 0) \geq w(m_0, 0),$$

which is itself equivalent to

$$m \geq m_0 - 1 = 0.$$

With this in mind, Theorem **??** tells us that for all $m \in \mathbb{N}$, if $z \in E^{-1}(S_m)$, then

$$h(z) = h_{w(m,1)} = h_{w(m+1,0)} = h_{m+1} = F_{m+3} - 1,$$

and

$$R(z) = E(z) + \Delta_{m+1} = E(z) - (-\Phi)^{m+2},$$

Observe that $\lambda = \Phi$ is a special case of irrational number within $[0, 1]$ in the sense that it is a fixed point of the Gauss map g. Thus, $g^2(\Phi) = \Phi$ and thus we can choose $\eta' = \eta = 1 - \Phi = \Phi^2$ and Theorem 4.7 tells us that the first return map R exhibits exact self-similarity within U. In particular, for all $z \in U_{w(0,0)} = U_0 = U$, we have the following conjugacy

$$R(z) = \frac{1}{-\Delta_1} R((-\Delta_1)z) = \frac{1}{\Phi^2} R(\Phi^2 z).$$

One consequence of this is that if there exists a periodic point $z \in S_m$ and the period of z is k, then z is a periodic point of R with period k/h_{m+1}. The self-similarity shows that for all $n \in \mathbb{Z}$ such that $2n \geq -m$, $\Phi^{2n} z$ is a periodic point of R, thus also a periodic point of F whose period is an integer multiple of h_{m+2n+1}. In particular, there is a sequence $(z_n)_{n \in \mathbb{N}}$ given by

$$z_n = \Phi^{2n-m} z, \tag{5.6}$$

so that for all $n \in \mathbb{N}$, $z_n \in S_{2n+\tilde{m}}$ and the period of z_n is an integer multiple of $h_{2n+\tilde{m}+1}$, where $\{0, 1\} \ni \tilde{m} \cong m \pmod 2$.

Given a map $f : X \to X$, let $\mathcal{O}_f^+(x)$ denote the forward orbit of $x \in X$ under f, that is

$$\mathcal{O}_f^+(x) = \{f^n(x) : n \in \mathbb{N}\}.$$

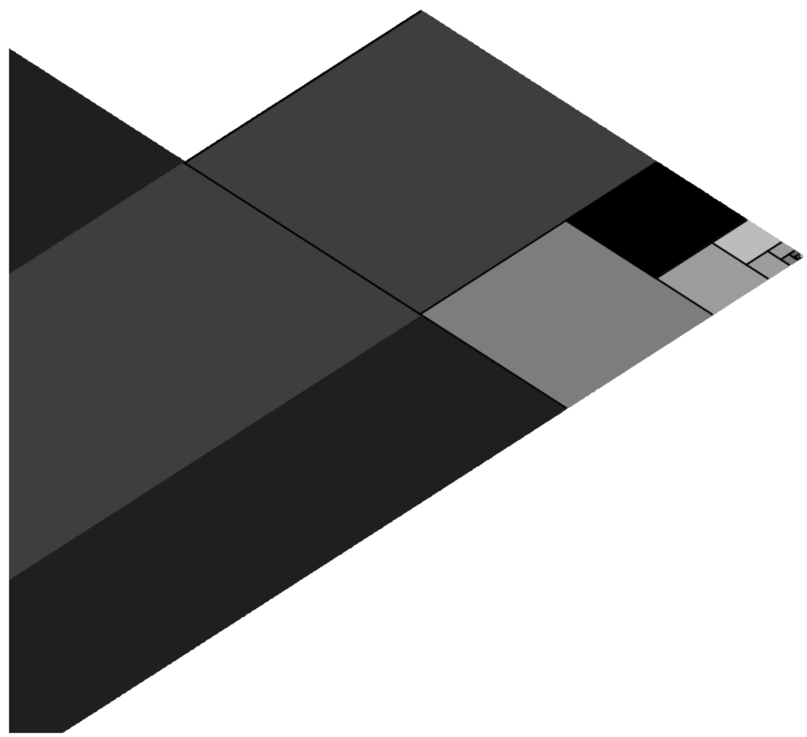

FIGURE 6. The same partition of P_c as in figure 4, after an application of R, which has shifted the rhombi alternately.

Proposition 5.2. *Suppose there exists a periodic point $z \in S_m$ for some $m \in \mathbb{N}$, and let $(z_n)_{n\in\mathbb{N}}$ be the sequence of periodic points given by* (5.6)*. Then the sequence $\left(\mathcal{O}_F^+(z_n)\right)_{n\in\mathbb{N}}$ of periodic orbits accumulates on the interval $[-1, \Phi]$.*

Proof. Let $n \in \mathbb{N}$. Note that

$$\{F^j(z_n) : 1 \leq j \leq h(z)\} \subset \mathcal{O}_F^+(z_n). \tag{5.7}$$

Lemma 3.1 tells us that for all $1 \leq j \leq h(z_n)$,

$$F^j(z_n) = E(z_n) + F^j(0).$$

Therefore,

$$|F^j(z_n) - F^j(0)| = |E(z_n)| = |z_n|. \tag{5.8}$$

Let $H \in \mathbb{N}$. Then there exists an $N \in \mathbb{N}$ such that $h(z_n) \geq H$ for all $n \geq N$, and thus (5.8) holds for all $1 \leq j \leq H$. Now let $\varepsilon > 0$ be small. Then there exists an $N' \in \mathbb{N}$ such that for all integers $n \geq N'$ such that

$$|z_{n'}| < \varepsilon. \tag{5.9}$$

Set $N^* = \max\{N, N'\}$. Then for all integers $n \geq N^*$, both (5.8) holds for all $1 \leq j \leq H$ and (5.9) holds. Hence, for all $n \geq N^*$ we have

$$|F^j(z) - F^j(0)| < \varepsilon,$$

for all $1 \leq j \leq H$. Since H and ε are independent and arbitrary, we conclude that the sequence $\left(\mathcal{O}_F^+(z_n)\right)_{n\in\mathbb{N}}$ accumulates on the set $\mathcal{O}_F^+(0)$.

By Proposition **??**, F is a 2-IET everywhere on the interval $[-1, \lambda]$ except on the preimages of 0, since $F(0) = -\eta = 1 - \lambda$, contrary to $F(x) = x + \lambda$ for $x \in [-1, 0)$ and $F(x) = x - 1$ for $x \in (0, \lambda)$.

Therefore F is conjugate to an irrational rotation almost everywhere, since $\lambda = \Phi$ is irrational. In particular, since λ is irrational and $\eta = 1 - \lambda$, we know that by for example Lemma 4 that $F^j(0) = F^{j-1}(-\eta)$ is bounded away from 0 for all integers $j > 0$, so $F^j(0) \neq 0$ for all $j > 0$.

Hence, the orbit of $F(0) = -\eta$ under F is also the orbit under an irrational rotation, and thus the orbit of 0 is dense in the interval $[-1, \lambda]$, i.e.

$$\overline{\mathcal{O}_F^+(0)} = [-1, \lambda].$$

Therefore, the sequence $\left(\mathcal{O}_F^+(z_n)\right)_{n\in\mathbb{N}}$ accumulates on the interval $[-1, \lambda]$. □

References

[1] Adler, R., Kitchens, B., Tresser, C. (2001). *Dynamics of non-ergodic piecewise affine maps of the torus.* Ergodic Theory and Dynamical Systems **21**, 959-999.

[2] Adler, R., Kitchens, B., Martens, M., Tresser, C., Wu, C.W., (2003). *The mathematics of halftoning.* IBM J. Res. Develop., **47**, 5-15.

[3] Alevy, I., Kenyon, R., Yi, R., (2018). *A family of minimal and renormalizable rectangle exchange maps.* arXiv:1803.06369

[4] Ashwin, P., Chambers, W., Petrov, G., (1997). *Lossless digital filter overflow oscillations; approximation of invariant fractals.* Internat. J. Bifurcation Appl. Eng. 7 , no. 11, 2603–2610.

[5] Ashwin, P. , Goetz, A. (2006). *Polygonal invariant curves for a planar piecewise isometry.* Trans. Amer. Math. Soc. **358** no. 1, 373-390.

[6] Ashwin, P., Goetz, A. (2010). *Cone exchange transformations and boundedness of orbits.* Cambridge University Press, 30(5), 1311-1330.

[7] Ashwin, P., Goetz, A., Peres, P. , Rodrigues, A. (2018). *Embeddings of Interval Exchange Transformations in Planar Piecewise Isometries.* arXiv:1805.00245.

[8] Avila, A., Forni, G. (2007) *Weak mixing for interval exchange transformations and translation flows*, Ann. of Math. (2) , 165(2), 637-664.

[9] Cornfeld, I. P. , Fomin, S. V., Sinai, Ya G. (1982) *Ergodic Theory* Grundlehren der Mathematisches Wissenschaften, 245, Springer-Verlag.

[10] Davies, A. C., (1995). *Nonlinear oscillations and chaos from digital filters overflow*, Phil. Trans. Roy. Soc. A, 353, 85-99.

[11] Deane, Jonathan H. B. (2006). *Piecewise isometries: applications in engineering.* Meccanica, **41**, no. 3, 241-252.

[12] Goetz, A. (2000). *Dynamics of piecewise isometries.* Illinois journal of mathematics, **44**, 465 – 478, (2000).

[13] Goetz, A., Dynamics of piecewise isometries. Thesis (Ph.D.) University of Illinois at Chicago. 1996.

[14] Hooper, W. Patrick (2013) *Renormalization of polygon exchange maps arising from corner percolation.* Inventiones mathematicae, 191(2), 255-320.

[15] Keane, M. S., (1975) *Interval exchange transformations*, Math. Z. 141, 25-31.

[16] Kocarev, Lj. , Wu, C. W., Chua, L. O., (1996). *IEEE Trans. Circuits Systems II*, **43** , no. 3, 234246.

[17] A. Ya. Khinchin, *Continued Fractions.* Translated from the third Russian edition (Moscow, 1961) by Scripta Technica. University of Chicago Press, Chicago, 1964. xii.

[18] Lowenstein, John H., Vivaldi, Franco, *Renormalization of a one-parameter family of piecewise isometries* arXiv:1406.6910

[19] Scott, A. J., (2003). *Hamiltonian mappings and circle packing phase spaces: Numerical investigations.* Phys. D, 181, 45-52.

[20] Schwartz, R., (2007). *Unbounded orbits for outer billiards.* J. Modern Dynam. **3**, 371 − 424.

[21] Scott, A. J., Holmes, C.A., Milburn, G. (2001). *Hamiltonian mappings and circle packing phase spaces.* Phys. D, 155, 34-50.

[22] Sturman, R., Meier, S., Ottino, J., Wiggins, S. (2008). *Linked twist map formalism in two and three dimensions applied to mixing in tumbled granular flows.* Journal of Fluid Mechanics, 602, 129-174.

[23] Avila, A. (2015). *Global theory of one-frequency Schrdinger operators.* Acta Math., **215** (1), 1–54.

[24] Buzzi, J. (2001). *Piecewise isometries have zero topological entropy.* Ergodic Theory and Dynamical Systems, **21**, 1371–1377.

[25] Peter Ashwin and Xin-Chu Fu (2001). *Tangencies in invariant disc packings for certain planar piecewise isometries are rare.* Dynamical Systems, **16**, 333–345.

[26] Poggiaspalla, G. (2006). *Self-similarity in piecewise isometric systems.* Dynamical Systems, **21**, 147–189.

[27] Adler, Roy and Kitchens, Bruce and Tresser, Charles (2001). *Dynamics of non-ergodic piecewise affine maps of the torus.* Ergodic Theory and Dynamical Systems, **21**, 959–999.

[28] Akiyama, Shigeki and Brunotte, Horst and Pethö, Attila and Steiner, Wolfgang (2008). *Periodicity of Certain Piecewise Affine Planar Maps.* Tsukuba Journal of Mathematics, **32**, 197–251.

[29] Boshernitzan, M.D. and Carroll, C.R. (1997). *An extension of Lagranges theorem to interval exchange transformations over quadratic fields.* Journal dAnalyse Mathematique, **72**, 21–44.

[30] Goetz, Arek and Poggiaspalla, Guillaume (2004). *Rotations by* $\pi/7$. Nonlinearity, **17**, 1787–1802.

[31] Kouptsov, K. L. and Lowenstein J. H. and Vivaldi, F. (2002). *Quadratic rational rotations of the torus and dual lattice maps.* Nonlinearity, **15**, 1795–1842.

[32] Lowenstein J. H. and Kouptsov, K. L. and Vivaldi, F. (2003). *Recursive tiling and geometry of piecewise rotations by* $\pi/7$. Nonlinearity, **17**, 371–395.

[33] Goetz, Arkadiusz (1996). *Dynamics of piecewise isometries.* University of Illinois at Chicago.

[34] Ashwin, Peter and Goetz, Arek (2005). *Invariant Curves and Explosion of Periodic Islands in Systems of Piecewise Rotations.* SIAM Journal on Applied Dynamical Systems, **4**, 437–458.

[35] Ashwin, Peter and Goetz, Arek and Peres, Pedro and Rodrigues, Ana (2020). *Embeddings of interval exchange transformations into planar piecewise isometries.* Ergodic Theory and Dynamical Systems, **40**, 1153–1179.

[36] Ashwin, P. and Fu, X. C. (2002). *On the geometry of orientation-preserving planar piecewise isometries.* Journal of Nonlinear Science, **12**, 207–240.

[37] Tsujii, Masato (1999). *Absolutely continuous invariant measures for expanding piecewise linear maps.* Hokkaido University Preprint Series in Mathematics, **442**, 1–28.

[38] Mauldin, R. Daniel and Williams, Stanley C. (1988). *Hausdorff dimension in graph directed constructions.* Transactions of the American Mathematical Society, **309**, 811–829.

[39] Peres, P., Rodrigues, A. (2018). *On the dynamics of Translated Cone Exchange Transformations.* arXiv.1809.05496

[40] Goetz, Arek (1998). *Dynamics of a piecewise rotation.* Discrete & Continuous Dynamical Systems, **4**, 593.

[41] Kahng, Byungik (2009). *Redefining chaos: Devaney-chaos for piecewise continuous dynamical systems.* International Journal of Mathematical Models and Methods in Applied Sciences, **3**, 317–326.

[42] Chen, Zhan-he and Fu, Xin-chu and Yu, Rong-zhong (2009). *Dynamical complexity of planar piecewise isometries.* Inproceedings: 2009 International Workshop on Chaos-Fractals Theories and Applications, 11–15.

[43] Bates, B. and Bunder, M. and Tognetti, K. (2005). *Continued Fractions and The Gauss Map.* Academia Paedagogica Nyiregyhaziensis. Acta Mathematica, **21**, 113–125.

[44] Avila, A.; Lyubich, M. (2011). *The full renormalization horseshoe for unimodal maps of higher degree: exponential contraction along hybrid classes.* Publications Mathmatiques de l'IHES, **114**, pp. 171–223.

[45] McMullen, C.T. (1998). *Self-similarity of Siegel disks and Hausdorff dimension of Julia sets.* Acta Math. **180(2)**: 247–292.

[46] Sinai, Ya., Khanin, K.M. (1989). *Smoothness of conjugacies of diffeomorphisms of the circle with rotations.* Russian Mathematical Surveys, **44**, 57–82.

[47] Sullivan, D. (1992). *Bounds, quadratic differentials and renormalization conjectures.* AMS Centennial Publications, II. Mathematics into the 21st Century. (1988 Centennial Symposium, Aug. 8–12), American Mathematical Society, Providence, R. I., 417–466.

[48] Tresser, T., Coullet, P. (1978). *Iterations d'endomorphismes et groupe de renormalisation.* Journal de Physique, Colloque C5, supplment au n 8, tome 39. C. R. Acad. Sci. Paris Ser. A **287** (1978), no. 7, C5–25.

[49] Masur, Howard, (1982). *Interval Exchange Transformations and Measured Foliations*, Ann. of Math. 115, no. 1, 169200.

[50] Boshernitzan, M.D., Carroll, (1997). *An extension of Lagrange's theorem to interval exchange transformations over quadratic fields.* C.R. J. Anal. Math. 72: 21.

[51] Zorich, A., (1996). *Finite Gauss measure on the space of interval exchange transformation. Lyapunov exponents*, Ann. Inst. Fourier, Grenoble, 46, 325-370.

[52] Luca Marchese, (2011). *The Khinchin Theorem for interval-exchange transformations*, Journal of Modern Dynamics 5, no. 1, 123-183.

[53] Sbastien Ferenczi, Luca Q. Zamboni, (2008). *Languages of k-interval exchange transformations*, Bulletin of the London Mathematical Society 40, no. 4, 705714.

[54] Avila, Artur, and Giovanni Forni, (2007). *Weak Mixing for Interval Exchange Transformations and Translation Flows*, Ann. of Math. 165, no. 2, 63764.

[55] Katok, A. B. (1980). *Interval exchange transformations and some special flows are not mixing*, Israel J. Math. 35, 301-310.

[56] Buzzi, Jérôme (2001). *Piecewise isometries have zero topological entropy.* Ergodic Theory and Dynamical Systems, **21**, 1371–1377.

[57] Goetz, Arek, (2000). *A self-similar example of a piecewise isometric attractor*, Dynamical Systems: From Crystal to Chaos, 248-258.

[58] Goetz, Arek and Poggiaspalla, Guillaume (2004). *Rotations by* $\pi/7$. Nonlinearity, **17**, 1787–1802.

[59] Adler, Roy and Kitchens, Bruce and Tresser, Charles (2001). *Dynamics of non-ergodic piecewise affine maps of the torus.* Ergodic Theory and Dynamical Systems, **21**, 959–999.

[60] Lowenstein J. H. and Kouptsov, K. L. and Vivaldi, F. (2003). *Recursive tiling and geometry of piecewise rotations by* $\pi/7$. Nonlinearity, **17**, 371–395.

[61] Ashwin, Peter and Goetz, Arek (2005). *Invariant Curves and Explosion of Periodic Islands in Systems of Piecewise Rotations.* SIAM Journal on Applied Dynamical Systems, **4**, 437–458.

[62] Ashwin, P. , Goetz, A. (2006). *Polygonal invariant curves for a planar piecewise isometry.* Trans. Amer. Math. Soc. **358** no. 1, 373-390.

[63] Poggiaspalla, G. (2006). *Self-similarity in piecewise isometric systems.* Dynamical Systems, **21**, 147–189.

[64] Ashwin, Peter and Goetz, Arek and Peres, Pedro and Rodrigues, Ana (2020). *Embeddings of interval exchange transformations into planar piecewise isometries*. Ergodic Theory and Dynamical Systems, **40**, 1153–1179.

[65] Lynn, Thomas F. and Ottino, Julio M. and Umbanhowar, Paul B. and Lueptow, Richard M. (2020). *Identifying invariant ergodic subsets and barriers to mixing by cutting and shuffling: Study in a birotated hemisphere*, 101, no. 1.

[66] Kahng, Byungik, (2009). *Singularities of two-dimensional invertible piecewise isometric dynamics*, Chaos: An Interdisciplinary Journal of Nonlinear Science, 19, no. 2.

[67] Wilson, Kenneth G. (1983). *The renormalization group and critical phenomena*, Reviews of Modern Physics, 55, no. 3, 583–600.

[68] P. Coullet, C. Tresser (1978). *Itrations d'endomorphismes et groupe de renormalisation*, Journal de Physique Colloques, 39, no. C5, 25–28.

[69] Feigenbaum, M.J. (1978). *Quantitative universality for a class of nonlinear transformations*, Journal of Statistical Physics, 19, 2552.

[70] Feigenbaum, M.J. (1979). *The universal metric properties of nonlinear transformations*, Journal of Statistical Physics, 21, 669706.

[71] McMullen, Curtis T. (1994), *Complex dynamics and renormalization*, Princeton University Press, no. 135.

[72] Rauzy, G. (1979). *changes dintervalles et transformations induites*, Acta Arithmetica, 34, no. 4, 315328.

[73] Veech, W. A., (1982). *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. 115, 201-242.

[74] Peres, P. (2019). *Renormalization in Piecewise Isometries*, University of Exeter.

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