Flexible Involutive Meadows

Emanuele Bottazzi
University of Pavia, Italy
emanuele.bottazzi@unipv.it

Bruno Dinis *
Departamento de Matemática, Universidade de Évora, Portugal
bruno.dinis@uevora.pt

Abstract

We investigate a notion of inverse for neutrices inspired by Van den Berg and Koudjeti's decomposition of a neutrix as the product of a real number and an idempotent neutrix. We end up with an algebraic structure that can be characterized axiomatically and generalizes involutive meadows. The latter are algebraic structures where the inverse for multiplication is a total operation. As it turns out, the structures satisfying the axioms of flexible involutive meadows are of interest beyond nonstandard analysis.

1 Introduction

Neutrices and external numbers (which can be seen as translations of neutrices over the hyperreal line) were introduced by Van den Berg and Koudjeti in [25] as models of uncertainties, in the context of nonstandard analysis, and further developed in [24, 27, 16, 19, 20]. Neutrices were named after and inspired by Van der Corput's groups of functions [28] in an attempt to give a mathematically rigorous formulation to the art of neglecting small quantities – *ars negligendi*.

One of the long-standing open questions in the theory of external numbers is the definition of a suitable notion of inverse of a neutrix. For zeroless external numbers,

The authors are grateful to Imme van den Berg for valuable comments on a preliminary version of this paper, to João Dias for pointing out a typo in one of the axioms, and to the reviewers for many comments and suggestions that improved the paper.

*Supported by FCT - Fundação para a Ciência e Tecnologia under the projects: UIDP/04561/2020 and 10.54499/UIDB/04674/2020, and by the research centers CMAFcIO – Centro de Matemática, Aplicações Fundamentais e Investigação Operacional and CIMA – Centro de Investigação em Matemática e Aplicações.

that is, external numbers that don't contain 0 and therefore cannot be reduced to a neutrix, there is a sort of inverse, defined from the Minkowski product between sets (see Definition 3.4 below), but this cannot work as a proper inverse since in many instances the result turns out to be the empty set.

A meadow (see Sections 2.1 and 2.2 below for further details) is a sort of commutative ring with a multiplicative identity element and a total multiplicative inverse operation. The theory of meadows allows for two main options: (*i*) involutive meadows which define $0^{-1} = 0$, resulting into an equational theory closer to that of the original structure [6], (*ii*) common meadows which define 0^{-1} as a new error term that propagates through calculations [5] (see also [7, 15, 8, 9]). For some recent developments, see [8, 9, 12, 13, 14].

One of the motivations for the study of structures where the inverse of zero is defined comes from equational theories [23, 26, 6]. For instance, Ono and Komori introduced such structures motivated from the algebraic study of equational theories and universal theories of fields, and free algebras over all fields, respectively. A long-standing result by Birkhoff states that algebraic structures with an equational axiomatization – namely, whose axioms only involve equality, besides the functions and constants of the structure itself – are closed under substructures. Algebraic structures where the inverse is defined only for nonzero elements are not equational, since they have to use inequalities or quantifiers in their definition of a multiplicative inverse. Instead, involutive meadows and common meadows which, as mentioned above, define the inverse of zero as zero or a new error term, respectively, admit equational axiomatizations.

Equational axiomatizations of meadows based on known algebraic structures, such as $\mathbb{Q}$ and $\mathbb{R}$, are also of interest to computer science. According to Bergstra and Tucker [6], such equational axiomatizations allow for simple term rewriting systems and are easier to automate in formal reasoning.

Another motivation for the study of meadows is a philosophical interest in the definition of an inverse of zero (see e.g. [4, Section 3]), if one wants to assign a meaning to expressions such as 0^{-1} or $1/0$ (Bergstra and Middleburg argue that, in principle, these two operations need to be distinguished [4]).

It turns out that external numbers are particularly suitable for expressing the kind of concepts involved in the definition of the inverse of zero. The key insight is that, being convex subgroups of the hyperreal numbers (i.e. the extension of the real number system which includes nonstandard elements such as infinitesimals), neutrices are "error" terms in the following sense:

- the sum of a neutrix with itself or with one of its elements is still the same neutrix;

- the product of a neutrix by an appreciable (not infinitesimal and not infinitely large) number is still the same neutrix;
- the product of a neutrix by an external number is a neutrix.

These properties are similar to those of 0, that is neutrices are idempotent for addition and absorbent for multiplication. Therefore neutrices can be seen as generalized zeroes and are suitable to build models of meadows.

The fact that one is using hyperreals (or other non-archimedean field extensions of the real numbers) is crucial, because the real numbers only have two convex subgroups: $\{0\}$ and $\mathbb{R}$, while in the context of the hyperreals there are countably infinitely many, e.g. the set of all infinitesimals – denoted $\oslash$ – and the set of all limited numbers – denoted $\pounds$ (see the examples after Definition 3.1). In turn, external numbers are of the form $a+A$, where a is an hyperreal number and A is a neutrix and can therefore be seen as translations of neutrices. According to the interpretation of neutrices as error terms or generalized zeroes, external numbers can be interpreted as expressing a quantity with a degree of uncertainty.

By introducing an alternative way to define the inverse of a neutrix, inspired by a result of Van den Berg and Koudjeti [25] stating that every neutrix can be decomposed as the product of an hyperreal number and an idempotent neutrix, we end up with an algebraic structure that can be characterized axiomatically and generalizes involutive meadows. Since the new class of structures involves error terms, we call it the class of *flexible involutive meadow*, in the spirit of [22].

In summary, the contributions of the paper are the following.

- We answer a question about the inverse of a neutrix.
- We connect the inverse of a neutrix to meadows, specifically involutive meadows.
- Inspired by the properties of the inverse of a neutrix, we propose a new algebraic structure that generalises involutive meadows in a simple way, called flexible involutive meadows, and we provide an axiomatization of such structures.
- We give some models of flexible involutive meadows.

We start in Section 2 by recalling the axioms of common meadows and of involutive meadows. We also recall some notions and results concerning external numbers in Section 3. In Section 4 we introduce flexible involutive meadows and prove that the external numbers are a flexible involutive meadow. We also derive some properties of flexible involutive meadows and relate them with varieties and von Neumann regular rings. Then, in Section 5, we present additional models for meadows relying on the external numbers. Some final remarks and open questions are mentioned in Section 6.

2 Preliminary notions

In this section we recall the axioms of common meadows and involutive meadows. We also provide some motivation for the study of structures where division by zero is possible.

2.1 Involutive meadows

The axioms of involutive meadows are listed in Figure 1 (see also [2, 4]).

(I_1)	$(x+y)+z = x+(y+z)$
(I_2)	$x+y = y+x$
(I_3)	$x+0 = x$
(I_4)	$x+(-x) = 0$
(I_5)	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
(I_6)	$x \cdot y = y \cdot x$
(I_7)	$1 \cdot x = x$
(I_8)	$x \cdot (y+z) = x \cdot y + x \cdot z$
(I_9)	$(x^{-1})^{-1} = x$
(I_{10})	$x \cdot (x \cdot x^{-1}) = x$

Figure 1: Axioms for involutive meadows

The term *involutive* refers to the fact that taking inverses is an involution, as postulated by axiom (I_9). With the exception of axiom (I_{10}), the remaining axioms are quite standard, as they postulate the existence of operations of addition + and multiplication $\cdot$ which are associative, commutative, admit a neutral element (denoted 0 and 1 respectively). Furthermore, there is an inverse for addition, and multiplication is distributive with respect to addition. Axioms (I_9) and (I_{10}) entail that $0^{-1} = 0$ (see [6, Theorem 2.2]). Axiom (I_{10}) replaces the more usual $x \cdot x^{-1} = 1$, which is false for $x = 0$ (otherwise, $0 = 0 \cdot 0 = 0 \cdot 0^{-1} = 1$). This hints at the fact that, in general, one should not define x^{-1} as the element satisfying $x \cdot x^{-1} = 1$. This also ties in with rejecting division as the "inverse" of multiplication, as discussed in [4].

2.2 Common meadows

The axioms of common meadows are listed in Figure 2 (see also [5]).

(M_1)	$(x + y) + z = x + (y + z)$
(M_2)	$x + y = y + x$
(M_3)	$x + 0 = x$
(M_4)	$x + (-x) = 0 \cdot x$
(M_5)	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
(M_6)	$x \cdot y = y \cdot x$
(M_7)	$1 \cdot x = x$
(M_8)	$x \cdot (y + z) = x \cdot y + x \cdot z$
(M_9)	$-(-x) = x$
(M_{10})	$x \cdot x^{-1} = 1 + 0 \cdot x^{-1}$
(M_{11})	$(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$
(M_{12})	$(1 + 0 \cdot x)^{-1} = 1 + 0 \cdot x$
(M_{13})	$0^{-1} = \mathbf{a}$
(M_{14})	$x + \mathbf{a} = \mathbf{a}$

Figure 2: Axioms for common meadows

As with involutive meadows, some of the axioms are quite standard (namely (M_1) – (M_3), (M_5) – (M_7), (M_8), (M_9), and (M_{11})), as they postulate the existence of operations of addition + and multiplication $\cdot$ which are associative, commutative and admit a neutral element (denoted 0 and 1 respectively). Note that, in involutive meadows, the equations of axioms (M_9) and (M_{11}) can be derived from the other axioms (as discussed in [6]). Furthermore, there is an inverse for addition, multiplication is distributive with respect to addition, and the inverse of the product of two elements is the product of the inverses.

Axiom (M_4) postulates the existence of a sort of additive inverse for every element

x but with the caveat that the result of operating an element with its inverse is not the neutral element 0 but $0 \cdot x$.

Axioms (M_{10}) and (M_{12}) concern further properties of the inverse for multiplication. The novelty, compared with more familiar settings, is that they have "error" terms in the form of the product of an element x, (respectively, its inverse x^{-1}) by 0.

Axiom (M_{13}) defines 0^{-1} as an "error" term $\mathbf{a}$ (some authors denote this error term by $\perp$) that does not belong to the initial structure. Due to the presence of this error term, the result of $x \cdot x^{-1}$ is defined as $1 + 0 \cdot x^{-1}$. If $x \neq 0$ and $x \neq \mathbf{a}$, then $0 \cdot x^{-1} = 0$ (see [5, Proposition 2.3.1]) and we recover the usual result that holds in a field. If $x = 0$ or $x = \mathbf{a}$, then the additional term $0 \cdot x^{-1}$ is equal to $\mathbf{a}$.

Axiom (M_{12}) has a similar motivation: if $x \neq \mathbf{a}$, then we recover that the inverse of 1 is 1. If $x = \mathbf{a}$, then we get that the inverse of $\mathbf{a}$ is $\mathbf{a}$ itself.

3 Hyperreal numbers and external numbers

Let us recall some definitions and results about neutrices and external numbers. We will use ${}^*\mathbb{R}$ to denote an elementary equivalent extension of the real number system that includes nonstandard elements – such as infinitesimals – [21], and $\mathbb{R}$ to denote the usual set of real numbers.[1]

A number x is *infinitesimal* if $|x| < r$ for every positive $r \in \mathbb{R}$ and it is *infinite* if $|x| > r$ for every $r \in \mathbb{R}$. We use the notation $x \simeq 0$ to say that x is infinitesimal. We will also write $x \simeq y$, and say that x is *infinitely close* to y, if $x - y \simeq 0$. A number is said to be *finite* if it is not infinite, and *appreciable* if it is neither infinitesimal nor infinite.

Crucially, the hyperreals admit nontrivial convex subgroups for addition (for instance: the set of infinitesimals $\oslash$, the set of finite numbers $\pounds$).

This property is shared by other non-archimedean field extensions of $\mathbb{R}$. We will discuss this matter further in Section 4.4.

3.1 External numbers

Definition 3.1 (Neutrices)**.** A *neutrix* is a convex subset of ${}^*\mathbb{R}$ that is a subgroup for addition.

Some simple examples of neutrices are:

- $\oslash = \{x \in {}^*\mathbb{R} : x \simeq 0\}$;

[1] Note that this notation differs from the usual presentations of external numbers, according to which $\mathbb{R}$ already contains nonstandard elements.

- $\pounds = \{x \in {}^*\mathbb{R} : x \text{ is finite}\}$;
- if $\varepsilon \simeq 0$, $\varepsilon\pounds = \{x \in {}^*\mathbb{R} : \frac{x}{\varepsilon} \text{ is finite}\}$.

Definition 3.2 (External numbers)**.** An *external number* α is the sum of an hyperreal number a and a neutrix A in the following sense:

$$\alpha = a + A = \{a + r : r \in A\}.$$

An external number $\alpha = a + A$ that is not reduced to a neutrix (equivalently, such that $0 \notin \alpha$) is said to be *zeroless*.

The sum and product of external numbers is introduced in the following definition. We refer to [25, 16, 20] for their properties.

Definition 3.3. For $a, b \in {}^*\mathbb{R}$ and $A, B \subseteq {}^*\mathbb{R}$ (not necessarily neutrices), we define the Minkowski sum and product

$$\begin{aligned} (a + A) + (b + B) &= (a + b) + (A + B) \\ (a + A) \cdot (b + B) &= ab + aB + bA + AB, \end{aligned}$$

where

$$\begin{aligned} A + B &= \{x + y : x \in A \wedge y \in B\} \\ aB &= \{ay : y \in B\} \\ AB &= \{xy : x \in A \wedge y \in B\}. \end{aligned}$$

It is also possible to define a notion of division between subsets of hyperreal numbers, even if they contain 0.

Definition 3.4. For $A, B \subseteq {}^*\mathbb{R}$ (not necessarily neutrices), we define

$$A \cdot B^{-1} = \{x : xB \subseteq A\}.$$

Usually, for neutrices, $A \cdot B^{-1}$ is written as $\frac{A}{B}$. Here we chose the inverse notation since we are investigating structures related to meadows, whose axioms are commonly stated in terms of the inverse operation. For further discussion on the use of these operations we refer to [4].

Notice that Definition 3.4 doesn't allow us to obtain a *proper* inverse of a neutrix. In fact, if $A = \{1\}$ and B is a neutrix, $A \cdot B^{-1}$ is empty, since for no x we have $0 \cdot x \in A$. This example motivated us to look for alternative definitions of inverses of a neutrix.

Let us finish this section by recalling some results on external numbers which will be useful later on.

If $x = a + A$ is zeroless, then $a^{-1} \cdot A \subseteq A$, and therefore $a^{-1} \cdot A + A = A$. In fact, $a^{-1} \cdot A \subseteq \oslash$.

The Taylor formula for $(a + A)^{-1}$ allows to obtain the following expression for the inverse of zeroless external numbers.

Proposition 3.5 ([25, p. 151],[20, Theorem 1.4.2]). *Let $x = a + A$ be a zeroless external number. Then*

$$x^{-1} = (a + A)^{-1} = a^{-1} + a^{-2} \cdot A.$$

We conclude with a list of basic algebraic properties of external numbers. Properties (3) and (4) are a consequence of the fact that, for a neutrix A, $A+A = A-A = A$. Property (7) replaces the usual distributivity formula, taking into account how error terms propagate (for further details on the distributivity formula for external numbers, see [16, Section 5]).

Proposition 3.6. *Let x, y, z be external numbers such that $x = a + A$. Then*

1. $x + (y + z) = (x + y) + z$;
2. $x + y = y + x$;
3. $x + A = x$;
4. $x + (-x) = A$;
5. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
6. $x \cdot y = y \cdot x$;
7. $x \cdot y + x \cdot z = x \cdot (y + z) + A \cdot y + A \cdot z$;
8. $(x^{-1})^{-1} = x$.

4 Flexible involutive meadows

A neutrix I is said to be *idempotent* if $I \cdot I = I$. As showed by Van den Berg and Koudjeti in [25] (see also [17]) every neutrix is a multiple of an idempotent neutrix.

Theorem 4.1 ([25, Theorem 7.4.2]). *Let N be a neutrix. Then, there exists an hyperreal number r and a unique idempotent neutrix I such that $N = r \cdot I$.*

We use the previous result to define inverses for neutrices.

Definition 4.2. Let $x = a + A$, where $A = r \cdot I$, for some hyperreal number r and idempotent neutrix I. We define the *inverse* of x, denoted x^{-1}, as follows:

$$x^{-1} = \begin{cases} a^{-1} + a^{-2} \cdot A & \text{if } a \neq 0 \\ r^{-1} \cdot I & \text{otherwise.} \end{cases}$$

The idempotent neutrices I can be seen as a generalized zeroes, since they share with 0 the properties $I + I = I$ and $I \cdot I = I$. Also, $I^{-1} = I$, similarly to $0^{-1} = 0$ in involutive meadows.

In the decomposition $N = r \cdot I$ of Theorem 4.1, the idempotent neutrix I is uniquely determined, but the number r is not. Nevertheless, the inverse given in Definition 4.2 is uniquely defined, as a consequence of the next proposition.

Proposition 4.3. *If $r, s \in {}^*\mathbb{R}$ and $r \neq s$ satisfy $N = r \cdot I = s \cdot I$, then also $r^{-1} \cdot I = s^{-1} \cdot I$.*

Proof. We may assume, without loss of generality that $0 < r < s$, which implies that $s^{-1} < r^{-1}$. Suppose towards a contradiction that $r^{-1} \cdot I \neq s^{-1} \cdot I$. By our assumptions over r and s, this implies $s^{-1} \cdot I \subsetneq r^{-1} \cdot I$. Then, there exists some $i \in I$ such that $i \cdot r^{-1} \notin s^{-1} \cdot I$. If we multiply by r, we obtain

$$i \notin s^{-1} \cdot (r \cdot I) = s^{-1} \cdot (s \cdot I) = I,$$

which contradicts the assumption that $i \in I$. Hence $r^{-1} \cdot I = s^{-1} \cdot I$. □

We show that the external numbers equipped with the inverse defined in Definition 4.2 satisfy the axioms given in Figure 3, where $N(x)$ denotes the neutrix part of the external number x. As such, one can also think of $N(x)$ as an error term, or a generalized zero, such that every x decomposes uniquely as $x = r + N(x)$ with $N(r) = 0$. We call any structure satisfying the axioms in Figure 3 a *flexible involutive meadow.*

(FI_1)	$(x+y)+z = x+(y+z)$
(FI_2)	$x+y = y+x$
(FI_3)	$x+N(x) = x$
(FI_4)	$x+(-x) = N(x)$
(FI_5)	$(x\cdot y)\cdot z = x\cdot(y\cdot z)$
(FI_6)	$x\cdot y = y\cdot x$
(FI_7)	$(1+N(x)\cdot x^{-1})\cdot x = x$
(FI_8)	$x\cdot y+x\cdot z = x\cdot(y+z)+N(x)\cdot y+N(x)\cdot z$
(FI_9)	$(x^{-1})^{-1} = x$
(FI_{10})	$x\cdot(x\cdot x^{-1}) = x$

Figure 3: Axioms for flexible involutive meadows

The axioms of flexible involutive meadows generalize the axioms of involutive meadows given in Figure 1 by replacing 0 with generalized zeroes $N(x)$, 1 with generalized ones (i.e. 1 plus an error term of the form $N(x)$), and distributivity by a generalized form of distributivity which holds for the external numbers. Notice that, in the context of the external numbers, the generalized zeroes take the form of neutrices.

Axiom (FI_7) is the flexible counterpart to (I_7), but replaces 1 with $1+N(x)\cdot x^{-1}$, and not simply with $1+N(x)$. In the setting of external numbers, this is necessary because if $N(x)\supseteq \pounds$, then $1+N(x)=N(x)$ would be a generalized zero. Axiom (FI_8) is a generalized distributivity axiom. In the setting of external numbers, the term $N(x)\cdot y+N(x)\cdot z$ is a neutrix, so the error term in (FI_8) is once again a neutrix. In fact, if one interprets $N(x)$ as being 0 for all x, then one recovers the axioms for involutive meadows. The proof is straightforward.

Lemma 4.4. *Let M be an involutive meadow. Then, if one defines $N(x)=0$ for all $x\in M$, the resulting structure is a flexible involutive meadow.*

In order to prove that the external numbers satisfy the axioms for flexible involutive meadows, we will use the following properties of the inverse of a neutrix.

Lemma 4.5. *Let $x=N(x)=r\cdot I$, with $r\in {}^*\mathbb{R}$ and I an idempotent neutrix. Then:*

1. $(1+x\cdot x^{-1})\cdot x = x$;

2. $(x^{-1})^{-1} = x$*;*

3. $x \cdot (x \cdot x^{-1}) = x$*.*

Proof. 1. We have that $x \cdot x^{-1} = N(x) \cdot x^{-1} = I$. Hence

$$(1 + x \cdot x^{-1}) \cdot x = (1 + I) \cdot (r \cdot I) = r \cdot I + r \cdot I^2 = r \cdot I = x.$$

2. We have

$$(x^{-1})^{-1} = (r^{-1}I)^{-1} = (r^{-1})^{-1}I = rI = x.$$

3. We have

$$x \cdot (x \cdot x^{-1}) = rI \cdot (rI \cdot r^{-1}I) = rI \cdot I = rI = x.$$ □

As proved below, the external numbers satisfy an additional property related to the following *Inverse Law* of involutive meadows:

$$x \neq 0 \Rightarrow x \cdot x^{-1} = 1$$

Involutive meadows that satisfy the Inverse Law are called *cancellation meadows* and are of particular interest. In fact, in [1] it is proved that every involutive meadow is a subdirect product of cancellation meadows.

In the setting of flexible involutive meadows, the inverse law is more suitably expressed by its flexible counterpart:

$$x \neq N(x) \Rightarrow x \cdot x^{-1} = 1 + e, \tag{4.1}$$

where e is a generalized zero (in the sense that $e + e = e$) such that $1 + e$ is not a generalized zero.

Moreover, by part 3 and part 4 of Proposition 3.6, the external numbers satisfy the following properties that generalize the properties of arithmetical meadows [4]:

$$\begin{array}{ll} (\mathrm{A}_1) & x + (-x) = N(x) \\ (\mathrm{A}_2) & x + N(x) = x. \end{array}$$

Notice that (A_2) is axiom (FI_3) of flexible involutive meadows.

Theorem 4.6. *The external numbers with the usual addition and multiplication and with the inverse introduced in Definition 4.2 satisfy the axioms for flexible involutive meadows plus the Flexible Inverse Law given by* (4.1) *and the properties* (A_1) *and* (A_2)*.*

Proof. By Proposition 3.6, in order to show that the external numbers are a flexible involutive meadow we only need to verify (FI_7) and (FI_{10}). If x is a neutrix, both axioms hold due to Lemma 4.5. Assume that $x = a + A$ is zeroless. Then, using the algebraic properties of the external numbers one derives

$$\begin{aligned}(1 + N(x) \cdot x^{-1}) \cdot x &= \left(1 + A\left(a^{-1} + a^{-2} \cdot A\right)\right)(a + A)\\ &= \left(1 + \left(a^{-1} \cdot A + a^{-2} \cdot A^2\right)\right)(a + A)\\ &= a + A + a^{-1} \cdot A^2 + A + a^{-1} \cdot A^3\\ &= a + A = x.\end{aligned}$$

Hence (FI_7) holds. As regarding (FI_{10}) one has

$$x(x \cdot x^{-1}) = (a + A)\left(1 + a^{-1} \cdot A\right) = a + A + A + a^{-1} \cdot A^2 = a + A = x.$$

Hence (FI_{10}) also holds and therefore the external numbers are a flexible involutive meadow.

We now show the Flexible Inverse Law. Let $x = a + A$ be a zeroless external number. Then

$$x \cdot x^{-1} = 1 + a^{-1} \cdot A + a^{-1} \cdot A + a^{-2} \cdot A^2 = 1 + a^{-1} \cdot A.$$

Since x is zeroless, $a^{-1} \cdot A \subseteq \oslash$, so the Flexible Inverse Law is satisfied. □

Corollary 4.7. *The axioms for flexible involutive meadows are consistent.*

4.1 Some properties of flexible involutive meadows

We now prove some basic properties of flexible involutive meadows. We start by showing an additive cancellation law and that $N(\cdot)$ is idempotent for addition.

Proposition 4.8. *Let M be a flexible involutive meadow and let $x, y, z \in M$. Then*

1. *$x + y = x + z$ if and only if $N(x) + y = N(x) + z$;*
2. *$N(x) + N(x) = N(x)$.*

Proof. 1. Suppose firstly that $x + y = x + z$. Then

$$N(x) + y = -x + x + y = -x + x + z = N(x) + z.$$

Suppose secondly that $N(x) + y = N(x) + z$. Then

$$x + y = x + N(x) + y = x + N(x) + z = x + z.$$

2. This follows from applying part (1) to axiom (FI_3). □

In order to prove other basic properties that allow one to operate with the $N(\cdot)$ function and with additive inverses one needs to assume the following two extra axioms

$$
\begin{array}{ll}
(N_1) & N(x+y) = N(x) \vee N(x+y) = N(y); \\
(N_2) & N(-x) = N(x).
\end{array}
$$

Proposition 4.9. *Let M be a flexible involutive meadow satisfying also* (N_1) *and* (N_2), *and let $x, y, z \in M$. Then*

1. $N(x+y) = N(x) + N(y)$;
2. $N(N(x)) = N(x)$;
3. *If* $x = N(y)$, *then* $x = N(x)$;
4. $-(-x) = x$;
5. $-(x+y) = -x - y$;
6. $N(x) = -N(x)$.

Proof. 1. One has

$$x + y + N(x) + N(y) = x + N(x) + y + N(y) = x + y.$$

Then by part (1) of Proposition 4.8

$$N(x+y) + N(x) + N(y) = N(x+y). \tag{4.2}$$

By (N_1) one has $N(x+y) = N(x)$ or $N(x+y) = N(y)$. Suppose that $N(x+y) = N(x)$. Then by (4.2) and Proposition 4.8,

$$N(x+y) = N(x) + N(x) + N(y) = N(x) + N(y).$$

If $N(x+y) = N(y)$ the proof is analogous.

2. Using Proposition 4.8 and part 1 we have

$$N(N(x)) = N(x-x) = N(x) + N(-x) = N(x) + N(x) = N(x).$$

3. Using part 2 we derive that $N(x) = N(N(y)) = N(y) = x$.

4. We have that
$$N(-(-x)) = N(-x) = N(x) = -x + x.$$
Hence
$$-(-x) = -(-x) + N(-(-x)) = -(-x) - x + x = N(-x) + x = N(x) + x = x.$$

5. By part 1
$$-(x+y) + x + y = N(x+y) = N(x) + N(y) = -x + x - y + y = -x - y + x + y.$$
Then by Proposition 4.8
$$-(x+y) + N(x+y) = -x - y + N(x+y).$$
Again using part 1 one obtains
$$-(x+y) + N(-(x+y)) = -x - y + N(-x) + N(-y) = -x + N(-x) - y + N(-y).$$
Hence $-(x+y) = -x - y$.

6. By part 4 we have
$$N(x) = -x + x = -x - (-x) = -(x - x) = -N(x).$$ □

4.2 Flexible involutive meadows are varieties

In [1], Bergstra and Bethke studied the relations between involutive meadows and varieties. One of their results is that involutive meadows are varieties. We prove that flexible involutive meadows are also varieties.

Let us start by recalling the definition of varieties in this context, following [10].

Definition 4.10. If $\mathcal{F}$ is a signature, then an *algebra* A of type $\mathcal{F}$ is defined as an ordered pair (A, F), where A is a nonempty set and F is a family of finitary operations on A in the language of $\mathcal{F}$ such that, for each n-ary function symbol f in $\mathcal{F}$, there is an n-ary operation f^A on A.

Definition 4.11. A nonempty class K of algebras of the same signature is called a *variety* if it is closed under subalgebras, homomorphic images, and direct products.

A result by Birkhoff entails that K is a variety if and only if it can be axiomatized by identities.

Definition 4.12. Let Σ be a set of identities over the signature $\mathcal{F}$; and define $M(\Sigma)$ to be the class of algebras A satisfying Σ. A class K of algebras is an *equational class* if there is a set of identities Σ such that $K = M(\Sigma)$. In this case we say that K is defined, or axiomatized, by Σ.

Theorem 4.13. *K is a variety if and only if it is an equational class.*

The class K of flexible involutive meadows is axiomatized by the identities in Figure 3 over the signature $\Sigma = \{+, \cdot, -, ^{-1}, 0, 1, N(\cdot)\}$, where N is a unary function that, when interpreted with the external numbers, corresponds to the neutrix part of a number x. As a consequence of Birkhoff's theorem, we have the following result.

Corollary 4.14. *Flexible involutive meadows are varieties.*

4.3 Flexible involutive meadows and commutative von Neumann regular rings

In the investigation of meadows, the relation with commutative von Neumann regular rings with a multiplicative identity element seems to be of particular interest [3, 4].

We recall that a semigroup $(S, \cdot)$ is said to be *Von Neumann regular* if

$$\forall x \in S\, \exists y \in S\, (x \cdot x \cdot y = x) .$$

A commutative von Neumann regular ring with a multiplicative identity is a Von Neumann regular commutative semigroup for both addition and multiplication.

Flexible involutive meadows are also commutative von Neumann regular rings with a multiplicative identity element.

Proposition 4.15. *Let M be a flexible involutive meadow. Then M is a Von Neumann regular commutative semigroup for both addition and multiplication.*

Proof. This is a simple consequence of associativity together with axioms (FI_4) and (FI_{10}). □

In [3, Lemma 2.11] it was shown that commutative von Neumann regular rings can be expanded in a unique way to an involutive meadow. Since involutive meadows are also flexible involutive meadows, von Neumann regular rings can be expanded to flexible involutive meadows. The expansion to flexible involutive meadows might not be unique, though, due to the presence of different error terms.

Further research on the connection between commutative von Neumann regular rings and flexible involutive meadows goes beyond the scope of this paper.

4.4 Solids are flexible involutive meadows

We finish this section by showing that instead of working with the external numbers, one can use a purely algebraic approach by working with a structure called a *solid*. A solid is a generalization of the notion of field in which there are generalized neutral elements for both addition and multiplication (e and u, respectively), and generalized inverses (s and d). For a full list of the solid axioms and for the definitions of the functions e, u, s and d we refer to the appendix in [18] or [19].

The following proposition compiles some results from [19, Propositions 2.12, 4.8 and Theorem 2.16] and [16, Proposition 4.12], which we use to show that solids are flexible involutive meadows.

Proposition 4.16. *Let S be a solid and let $x, y, z \in S$.*

1. *If $x = e(x)$, then $e(xy) = e(x)y$;*
2. *If $x \neq e(x)$, then $u(x)e(x) = xe(u(x)) = e(x)$;*
3. *$x(z + e(y)) = xz + xe(y)$;*
4. *If $x \neq e(x)$, then $d(d(x)) = x$.*

Theorem 4.17. *Every solid is a flexible involutive meadow.*

Proof. Let S be a solid. Most of the axioms of flexible involutive meadows are also axioms of solids, by considering $N(x) = e(x)$, $-x = s(x)$, $x^{-1} = d(x)$, and $1 = u$. The only non-obvious cases are the cases of axioms (FI_7), (FI_9) and (FI_{10}).

For axiom (FI_7), if $x \neq e(x)$, using the solid axioms and Proposition 4.16 we obtain

$$(1 + N(x) \cdot x^{-1}) \cdot x = x + e(x)d(x)x = x + e(x)u(x) = x + e(x) = x.$$

If $x = e(x)$, the result follows from Lemma 4.5(1).

Axiom (FI_9) follows from Lemma 4.5(2), if $x = e(x)$ and from Proposition 4.16(4).

Finally, axiom (FI_{10}) follows easily from the solid axioms if $x \neq e(x)$ and from Lemma 4.5(3) if $x = e(x)$. □

As it turns out, and as mentioned above, one is not forced to work in a nonstandard setting. Indeed, any non-archimedean ordered field yields a model of flexible involutive meadows.

Let $\mathbb{F}$ be a non-archimedean ordered field. Let $\mathcal{C}$ be the set of all convex subgroups for addition of $\mathbb{F}$ and Q be the set of all cosets with respect to the elements of $\mathcal{C}$. In [18] the elements of $\mathcal{C}$ were called *magnitudes* and Q was called the *quotient class* of $\mathbb{F}$ with respect to $\mathcal{C}$. In the same paper it was also shown that the quotient class of a non-archimedean field is a solid. Hence we have following corollary.

Corollary 4.18. *The quotient class of a non-archimedean ordered field is a flexible involutive meadow.*

5 Further models for meadows using external numbers

In the introduction we claimed that the external numbers are particularly suitable for expressing the kind of concepts involved in the definition of the inverse of zero. In order to support that claim, we explore further models for meadows inspired by the external numbers. We start by building a model for flexible involutive meadows over a finite field $\mathbb{F}$ and proceed by constructing a model for common meadows over $^*\mathbb{R}$.

5.1 Finite models of flexible involutive meadows

In this subsection we show that any finite field can be extended to a finite model of a flexible involutive meadow.

Recall that a finite field is isomorphic to $\mathbb{F}_{p^m}$, with p a prime number and m a positive integer. As a consequence, without loss of generality we assume that elements of the finite field, which we will simply denote $\mathbb{F}$ from here on, are of the form $a \mod p^m$ with $a \in \mathbb{N}$, so we can identify elements of $\mathbb{F}$ with natural numbers between 0 and $p^m - 1$.

Definition 5.1. Let $(\mathbb{F}, +, \cdot)$ be a finite field. Without loss of generality, we may think that the elements of $\mathbb{F}$ are natural numbers between 0 and $|\mathbb{F}| - 1$. We define $(\widehat{\mathbb{F}}, \oplus, \odot)$ as follows.

- For every $a \in \mathbb{F}$, we define the external number $\widehat{a} = a + \oslash$ and $\widehat{\mathbb{F}} = \{\widehat{a} : a \in \mathbb{F}\}$.
- For every nonzero $a \in \mathbb{F}$, we set $(\widehat{a})^{-1} = \widehat{a^{-1}}$.
- $(\widehat{0})^{-1} = \widehat{0}$ (this definition is motivated by, and indeed coincides with the one in Definition 4.2).
- The sum $\oplus$ and product $\odot$ over $\widehat{\mathbb{F}}$ are defined as:

$$\widehat{a} \oplus \widehat{b} = \widehat{a + b}$$

and

$$\widehat{a} \odot \widehat{b} = \widehat{ab}$$

where sums and product on the right hand side are the sums and product in $\mathbb{F}$.

Theorem 5.2. *Let $\mathbb{F}$ be a finite field. Then $\widehat{\mathbb{F}}$ satisfies the axioms for flexible involutive meadows.*

Proof. If $x \neq \widehat{0}$, the axioms are satisfied since $\mathbb{F}$ is a field and the operations in $\widehat{\mathbb{F}}$ are compatible with the ones in $\mathbb{F}$.

If $x = \widehat{0}$, axioms $(I_1) - (I_8)$ follow from the fact that $\mathbb{F}$ is a field.

As for axiom (I_9), if $x = \widehat{0}$, then $\widehat{0}^{\,-1} = \widehat{0}$, so that $(\widehat{0}^{\,-1})^{-1} = \widehat{0}^{\,-1} = \widehat{0}$.

Finally, for axiom (I_{10}), if $x = \widehat{0}$, then $\widehat{0} \cdot \widehat{0}^{\,-1} = 0$. □

Remark 5.3. Similar models for involutive meadows can be obtained without recurring to infinitesimals, as one could define the alternative model $\tilde{\mathbb{F}}$ by requiring the existence of an element $E \notin \mathbb{F}$ and defining, for each $a \in \mathbb{F}$ the element $\tilde{a} = a + E$ and the set $\tilde{\mathbb{F}} := \{\tilde{a} : a \in \mathbb{F}\}$ with the operations

$$\tilde{a} \oplus \tilde{b} = (a + E) + (b + E) = (a + b) + E,$$

(note that, in particular $E \oplus E = (0 + E) + (0 + E) = (0 + 0) + E = 0 + E = E$),

$$\tilde{a} \odot \tilde{b} = (a + E) \cdot (b + E) = (a \cdot b) + E$$

and

$$\tilde{a}^{-1} = a^{-1} + E$$

and, finally,

$$\tilde{0}^{-1} = 0 + E = \tilde{0}.$$

5.2 A model for common meadows based on $\mathbb{R}$

In this section we introduce a model $\widehat{\mathbb{R}}$ for the axioms of common meadows given in Figure 2. In our model, we consider the elements of $\mathbb{R}$ plus an error term in the form of a neutrix. In order to make things concrete, we choose to use the neutrix $\oslash$ but, in principle, any neutrix included in $\oslash$ can be used.

Elements of $\mathbb{R}$ will be represented by external numbers of the form $r + \oslash$ with $r \in \mathbb{R}$, while ${}^*\mathbb{R}$ will act as an inverse of the neutrix $\oslash$, which is the representative of 0. We can interpret $\oslash^{-1} = {}^*\mathbb{R}$ as the smallest neutrix collecting all the inverses of the elements of $\oslash$. The fact that the inverse of 0 has, in a sense, the maximum possible uncertainty is in good accord with both the intuition that division by 0 introduces an error term, and to the common practice of having the inverse of 0 not being a member of the original field [5].

The inverse of $^*\mathbb{R}$ is again $^*\mathbb{R}$. This choice can be justified in two ways. We can again interpret $^*\mathbb{R}^{-1}$ as the smallest neutrix collecting all the inverses of the elements of $^*\mathbb{R}$ or, alternatively, since $\oslash \subset {}^*\mathbb{R}$, the inverse of $^*\mathbb{R}$ should also be maximal.

Definition 5.4. We define the set $\widehat{\mathbb{R}}$ as follows:

- For every $r \in \mathbb{R}$, we set $\widehat{r} = r + \oslash \in \widehat{\mathbb{R}}$.
- $^*\mathbb{R} \in \widehat{\mathbb{R}}$.
- For every nonzero $r \in \mathbb{R}$, we set $\widehat{r}^{-1}$, with the quotient introduced in Definition 3.4 (see also Proposition 3.5).
- We define $(\widehat{0})^{-1} = {}^*\mathbb{R}$ and $^*\mathbb{R}^{-1} = {}^*\mathbb{R}$ (so $^*\mathbb{R}$ acts like the error term **a** in the definition of common meadow).
- The sum and product over $\widehat{\mathbb{R}}$ are the Minkowski operations introduced in Definition 3.3.

In the definition of real numbers as limits of Cauchy sequences, real numbers can be seen as being determined up to “infinitesimals”, which have an interpretation in terms of sequences converging to 0. In the model of common meadows introduced in the previous definition, this idea is expressed by the representation of r as $\hat{r} = r + \oslash$.

An immediate consequence of the previous definition is that for every $x \in \widehat{\mathbb{R}}$ we have

$$x + (\widehat{0})^{-1} = (\widehat{0})^{-1} + x = x \cdot (\widehat{0})^{-1} = (\widehat{0})^{-1} \cdot x = {}^*\mathbb{R}. \tag{5.1}$$

In the next lemma, we establish that the operations in $\widehat{\mathbb{R}}$ are compatible with those in $\mathbb{R}$.

Lemma 5.5. *For every $r, s \in \mathbb{R}$,*

- $\widehat{r+s} = \widehat{r} + \widehat{s}$*;*
- $\widehat{r \cdot s} = \widehat{r} \cdot \widehat{s}$*.*

Moreover, for every nonzero $r \in \mathbb{R}$, $\widehat{r}^{-1} = \widehat{r^{-1}}$.

Proof. The first two properties are a consequence of the following equalities

$$\widehat{r+s} = r + s + \oslash = (r + \oslash) + (s + \oslash) = \widehat{r} + \widehat{s}.$$

and, taking into account that r and s are real numbers,

$$\widehat{r \cdot s} = r \cdot s + \oslash = (r + \oslash) \cdot (s + \oslash) = \widehat{r} \cdot \widehat{s}.$$

As for the inverse, if $r \neq 0$, $\widehat{r^{-1}} = \frac{1}{r} + \oslash$, whereas, by Proposition 3.5,

$$\widehat{r}^{-1} = (r + \oslash)^{-1} = r^{-1} + r^{-2} \cdot \oslash = \frac{1}{r} + \oslash. \qquad \square$$

Corollary 5.6. *For every $r \in \mathbb{R}$,*

1. *if $r \neq 0$, then $\widehat{r} \cdot \widehat{r}^{-1} = \widehat{1}$;*
2. *if $r \neq 0$, then $\widehat{0} \cdot \widehat{r}^{-1} = \widehat{0}$;*
3. *$\widehat{1} = \widehat{1} + \widehat{0} \cdot \widehat{r}$;*
4. *$\widehat{r} + {}^*\mathbb{R} = {}^*\mathbb{R} + \widehat{r} = {}^*\mathbb{R}$ and $\widehat{r} \cdot {}^*\mathbb{R} = {}^*\mathbb{R} \cdot \widehat{r} = {}^*\mathbb{R}$;*
5. *${}^*\mathbb{R} + {}^*\mathbb{R} = {}^*\mathbb{R} - {}^*\mathbb{R} = {}^*\mathbb{R} \cdot {}^*\mathbb{R} = {}^*\mathbb{R}$.*

Equalities (1)–(3) in Corollary 5.6 can be obtained from the corresponding equalities for real numbers by repeated use of Lemma 5.5.

Theorem 5.7. *$\widehat{\mathbb{R}}$ is a model of axioms* (M_1)–(M_{14}) *of common meadows.*

Proof. We start by showing that axiom (M_1) is satisfied. If x, y and z are different from ${}^*\mathbb{R}$, then this is a consequence of Lemma 5.5. If at least one of x, y and z is equal to ${}^*\mathbb{R}$, then both sides of the equality evaluate to ${}^*\mathbb{R}$ by Corollary 5.6. The proof follows similar steps for axioms $(\mathrm{M}_2) - (\mathrm{M}_9)$. Note that, for axiom (M_8), we use also the fact that we have only one order of magnitude besides ${}^*\mathbb{R}$.

Let us show that axiom (M_{10}) is satisfied. If $x = \widehat{0}$:

$$\widehat{0} \cdot (\widehat{0})^{-1} = {}^*\mathbb{R} = \widehat{1} + \widehat{0} \cdot {}^*\mathbb{R}.$$

If $x = {}^*\mathbb{R}$: as a consequence of the definition of ${}^*\mathbb{R}^{-1}$, we have

$${}^*\mathbb{R} \cdot {}^*\mathbb{R}^{-1} = {}^*\mathbb{R} = \widehat{1} + \widehat{0} \cdot {}^*\mathbb{R} = \widehat{1} + \widehat{0} \cdot {}^*\mathbb{R}^{-1}.$$

We now turn to axiom (M_{11}). If x, y are not equal to 0 nor to ${}^*\mathbb{R}$, the axiom holds as a consequence of Lemma 5.5. Otherwise, due to the definition of the inverse, both sides are equal to ${}^*\mathbb{R}$.

For axiom (M_{12}), if $x = \widehat{r}$ for some $r \in \mathbb{R}$, the axiom holds as a consequence of Corollary 5.6 (3). If $x = {}^*\mathbb{R}$, then

$$(\widehat{1} + \widehat{0} \cdot {}^*\mathbb{R})^{-1} = (1 + {}^*\mathbb{R})^{-1} = {}^*\mathbb{R}^{-1} = {}^*\mathbb{R} = \widehat{1} + {}^*\mathbb{R} = \widehat{1} + \widehat{0} \cdot {}^*\mathbb{R}.$$

Axiom (M_{13}) is satisfied as a consequence of the definition of $(\widehat{0})^{-1}$. Finally, axiom (M_{14}) is satisfied as a consequence of (5.1). □

6 Final remarks and open questions

In this paper we have introduced the notion of flexible involutive meadow, by means of an equational axiomatization, and constructed some models based on the external numbers and non-archimedean fields. We have also shown a model for common meadows based on the real numbers and of involutive meadows based on finite fields. We would like to point out that, with similar techniques, one could also obtain meadows based on rational numbers.

The model for common meadows developed in Section 5.2 suggests that it is possible to study a flexible version of common meadows, in the spirit of what has been done in Section 4 for involutive meadows. In order to do so, it is possible to adapt the axioms by replacing 0 with $N(x)$, where $N(x)$ is an error term analogous to that of flexible involutive meadows.

In the context of external numbers, where $N(x)$ is the neutrix part of x, and for zeroless x, one has the inclusion $N(x) \subseteq x \cdot \oslash$. This grounds the interpretation of the flexible counterparts of axioms (M_4), (M_{10}) and (M_{12}). Moreover, as in the model discussed in Section 5.2, the element $\mathbf{a}$ can be taken as ${}^*\mathbb{R}$.

Section 4.4 unveils a connection between the two apparently very different algebraic structures of solids and flexible involutive meadows. This connection is in line with other works connecting algebraic structures related with meadows and structures arising in the context of nonstandard analysis (see [11, 15]). We believe that this line of research is worth exploring in future work.

To conclude, we mention some other possible directions of future work.

We would like to study related variants of meadows as well as their algebraic properties. For example, the study of *flexible* cancellation meadows, i.e. meadows in which the multiplicative cancellation axiom

$$x \neq 0 \wedge x \cdot y = x \cdot z \Rightarrow y = z$$

or its flexible counterpart (where we substitute 0 by an error term e) holds; or *flexible* arithmetical meadows in the sense of [4]; or *flexible* meadows of rational numbers (see e.g. [6]). Are flexible arithmetical meadows, i.e. flexible meadows satisfying

(A_1) and (A_2) (necessarily) connected with nonstandard models of arithmetic? As for the flexible meadows of rational numbers, are they a minimal algebra? If so, that might provide a connection with data types.

References

[1] Jan A. Bergstra and Inge Bethke. Subvarieties of the variety of meadows. *Scientific Annals of Computer Science*, 27(1):1–18, 2017.

[2] Jan A. Bergstra, Inge Bethke, and Alban Ponse. Equations for formally real meadows. *Journal of Applied Logic*, 13(2, Part B):1–23, 2015.

[3] Jan A. Bergstra, Yoram Hirshfeld, and John V. Tucker. Meadows and the equational specification of division. *Theoretical Computer Science*, 410(12):1261–1271, 2009.

[4] Jan A. Bergstra and Cornelis A. Middelburg. Inversive meadows and divisive meadows. *Journal of Applied Logic*, 9(3):203–220, 2011.

[5] Jan A. Bergstra and Alban Ponse. Division by zero in common meadows. In Rocco De Nicola and Rolf Hennicker, editors, *Software, Services, and Systems: Essays Dedicated to Martin Wirsing on the Occasion of His Retirement from the Chair of Programming and Software Engineering*, pages 46–61. Springer International Publishing, Cham, 2015.

[6] Jan A. Bergstra and John V. Tucker. The rational numbers as an abstract data type. *J. ACM*, 54(2):7–es, apr 2007.

[7] Jan A. Bergstra and John V. Tucker. On the axioms of common meadows: Fracterm calculus, flattening and incompleteness. *The Computer Journal*, 66(7):1565–1572, 2023.

[8] Jan A. Bergstra and John V. Tucker. Fracterm calculus for signed common meadows. *Transmathematica*, Jun. 2024.

[9] Jan A. Bergstra and John V. Tucker. Rings with common division, common meadows and their conditional equational theories, 2024. arXiv: 2405.01733.

[10] Stanley Burris and Hanamantagouda P. Sankappanavar. *A Course in Universal Algebra.* 2012.

[11] Ulderico Dardano, Bruno Dinis, and Giuseppina Terzo. Assemblies as semigroups. *Examples and Counterexamples*, 5, 2024.

[12] João Dias and Bruno Dinis. Towards an enumeration of finite common meadows. *International Journal of Algebra and Computation*, 34(6): 837–855, 2024.

[13] João Dias and Bruno Dinis. Artinian meadows, 2024. arXiv: 2407.07793.

[14] João Dias and Bruno Dinis. Flasque meadows, 2024. arXiv: 2407.08375.

[15] João Dias and Bruno Dinis. Strolling through common meadows. *Communications in Algebra*, 52(12): 5015–5042, 2024.

[16] Bruno Dinis and Imme van den Berg. Algebraic properties of external numbers. *J. Log. Anal.*, 3:9:1–30., 2011.

[17] Bruno Dinis and Imme van den Berg. Axiomatics for the external numbers of nonstandard analysis. *J. Logic & Analysis*, 9(7):1–47, 2017.

[18] Bruno Dinis and Imme van den Berg. On the quotient class of non-archimedean fields. *Indagationes Mathematicae*, 28(4):784–795, 2017.

[19] Bruno Dinis and Imme van den Berg. Characterization of distributivity in a solid. *Indagationes Mathematicae*, 29, Issue 2:580–600, 2018.

[20] Bruno Dinis and Imme van den Berg. *Neutrices and external numbers: a flexible number system.* Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2019. With a foreword by Claude Lobry.

[21] Robert Goldblatt. *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis.* Graduate Texts in Mathematics, 188. Springer, New York, 1998.

[22] Júlia Justino and Imme van den Berg. Cramer's rule applied to flexible systems of linear equations. *Electron. J. Linear Algebra*, 24(Special issue for the 2011 Directions in Matrix Theory Conference):126–152, 2012/13.

[23] Yuichi Komori. Free algebras over all fields and pseudo-fields. In *Report 10, Faculty of Science, Shizuoka University*, pages 9 – 15, 1975.

[24] Fouad Koudjeti. *Elements of External Calculus with an application to Mathematical Finance, Ph.D. thesis.* Labyrinth publications. Capelle a/d IJssel, The Netherlands, 1995.

[25] Fouad Koudjeti and Imme van den Berg. Neutrices, external numbers, and external calculus. In F. and M. Diener, editors, *Nonstandard analysis in practice*, Universitext, pages 145–170. Springer, Berlin, 1995.

[26] Hiroakira Ono. Equational theories and universal theories of fields. *Journal of the Mathematical Society of Japan*, 35(2):289 – 306, 1983.

[27] Imme van den Berg. A decomposition theorem for neutrices. *Annals of Pure and Applied Logic*, 161:7:851 – 865, 2010.

[28] Johannes G. van der Corput. Introduction to the neutrix calculus. *J. Analyse Math.*, 7:281–399, 1959/1960.

Received 14 February 2024