**Universidade de Évora - Instituto de Investigação e Formação Avançada**

Programa de Doutoramento em Matemática

Tese de Doutoramento

# Existence, non existence and multiplicity of solutions for higher order boundary value problems

Gracino Francisco Rodrigues

Orientador(es) | Feliz Manuel Minhós

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Évora 2024

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# Existence, non existence and multiplicity of solutions for higher order boundary value problems

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## Abstract

This thesis deals with boundary value problems composed by coupled systems with different types of differential equations: with parameters, strongly nonlinear, of second and higher-order equations; with discontinuous nonlinearities, in regular and singular cases,...

This diversity can also be seen in various types of boundary conditions: Sturm-Liouville type boundary conditions, classical Dirichlet and two-point boundary conditions,

There are several results: sufficient conditions for the existence, non-existence and multiplicity of solutions (via Ambrosetti-Prodi alternative ); existence and location of a solution for impulsive problems, and numerical results related to a new three-dimensional non-Newtonian incompressible fluid model, where the viscosity and elasticity vary depending on the shear rate, this variation is of the power law type.

The most used tool is the method of lower and upper solutions, along with the properties of the Leray-Schauder topological degree and Schauder's fixed-point theorem.

**Keywords:** Coupled systems; Ambrosetti-Prodi alternative; Regular and singular phi-Laplacian equations; Generalized impulsive conditions; Generalized third-grade fluid.

# Existência, não existência e multiplicidade de soluções para problemas de valor de contorno de ordem superior

## Sumário

Esta tese trata de problemas de valores de contorno compostos por sistemas acoplados com diferentes tipos de equações diferenciais: com parâmetros, fortemente não lineares, de equações de segunda ordem e de ordem superior; com não linearidades descontínuas, em casos regulares e singulares,...

Esta diversidade também pode ser vista em vários tipos de condições de contorno: condições de contorno do tipo Sturm-Liouville, Dirichlet clássica e condições de contorno de dois pontos,

Os resultados são vários: condições suficientes para a existência, inexistência e multiplicidade de soluções (via alternativa Ambrosetti-Prodi); existência e localização de uma solução para problemas impulsivos,e resultados numéricos relacionados com um novo modelo tridimensional de fluido incompressível não-Newtoniano, onde a viscosidade e a elasticidade variam em função da taxa de cisalhamento, esta variação é do tipo lei de potência.

A ferramenta mais utilizada é o método das soluções inferiores e superiores, juntamente com as propriedades do grau topológico de Leray-Schauder e o teorema do ponto fixo de Schauder.

**Keywords:** Sistemas acoplados; Alternativa Ambrosetti-Prodi; Equações phi-Laplacianas regulares e singulares; Condições impulsivas generalizadas; Fluido Generalizado de ordem 3.

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Chapter 1

# Introduction

The type of solution, the existence, non-existence and/or multiplicity of solution, the qualitative structure of the solutions of non-linear Boundary Value Problems (BVP) depends both on the functional context of the non-linearities involved, and on the type of boundary conditions considered. Thus, the study of the solvency of systems of higher order nonlinear differential equations has considered these two variables: nonlinearity and boundary conditions.

## Motivation

The discussion of the existence and multiplicity of solutions as a function of parameters has been little considered in the literature on differential equations, and rarely applied to coupled systems of differential equations. The first part of this thesis aims to contribute to filling this gap. Coupled systems of second-order differential equations, where there is dependence between the various unknown variables, are frequently used in the modeling of phenomena in the physical, biological (see [2, 7, 100]), and social sciences, example in [48, 97]. They relate variables and parameters, with the aim of making predictions about the behavior of these problems.

In the context of differential equations, we have some phenomena that do not behave continuously; therefore, their characteristic is the abrupt change of state at certain moments in their development. Such changes result from disturbances whose duration is insignificant compared to the duration of the phenomenon as a whole. Thus, these disturbances can be understood as instantaneous, causing effects called impulses [58]. Impulsive differential equations can model many real phenomena with sudden and discontinuous jumps. These events can occur in many fields, such as population dynamics, control and optimization theory, chemistry, biology, and biotechnology, economics, pharmacokinetics, and other areas of physics and mechanical problems, as in [12, 41, 57].

To show the relevance of the three-dimensional model to be proposed, it is worth highlighting that in recent decades, the study of phenomena associated with fluids has attracted a large number of researchers from diverse areas such as mechanical engineering, biology, medicine, biomedical engineering, physics and mathematics.

## State of the art

The discussion of the existence and multiplicity of solutions as a function of parameters has been little considered in the literature on differential equations and rarely applied to coupled systems of differential equations. These types of equations were introduced in [5] and have since been studied by several authors in the context of different types of boundary value problems. As examples, we refer to [37, 109] for three- and two-point limit value problems; [86, 90] for Neumann-type boundary conditions; [32, 34, 65, 74, 104] for periodic problems; [82] for parametric problems with $(p, q)$-Laplacian equations; [30] with asymptotic conditions; [91] under conditions of coercion.

Coupled systems of second-order differential equations, where there is dependence between several unknown variables, have been studied in a huge variety of theoretical and applied situations, involving different types of boundary conditions, such as: [76, 67, 89, 98]. Furthermore, there are many real phenomena modeled by coupled systems, particularly in problems related to population dynamics, as in [3, 4, 9, 60]. For the classical approach to impulsive differential equations, we can refer, for example, to [58] for a general theory; [52] applying fixed point index; [61, 78, 85] for impulsive functional problems; [108] for a monotonous iterative technique to approximate the solution. The study of $\phi$-Laplacian impulsive problems can be seen, for example, in: [49] with periodic boundary conditions via a continuation theorem; [72, 73] for limited and unlimited intervals; [101] for fractional equations with $p$-Laplacian.

In the literature $\phi$ and $\psi$ are known as $\phi$, $\psi$-one-dimensional Laplacians, as they generalize the classical Laplacian and the more common $p$-Laplacian, and can be applied to many types of problems and model many real phenomena. As examples, we can mention: [102] proving the existence of a positive periodic solution for a $\varphi$-Laplacian Liénard equation with a singularity; [56] for multiple solutions of the $p$-Laplacian Dirichlet problem with discontinuities; [103] showing the existence of three positive solutions to the one-dimensional $p$-Laplacian problem; [28, 95] to obtain positive solutions to a superlinear $p$-Laplacian problem; [93] applying Krasnosel'skii's cone theory of expansion and compression to non-negative nonlinearities.

In recent years, some authors have studied the $\phi$-Laplacian singular homeomorphisms $\phi : (-r, r) \to \mathbb{R}$ with $0 < r < +\infty$. For example: [46, 47] for the $p$-Laplacian Operator; [10, 11, 84] obtaining results of existence and multiplicity; [14] proving the existence of heteroclinic solutions; [38] in the half-line along with the functional boundary conditions.

Based on the work of Lan et al. (see [59]) the equations relating to the $\phi$-Laplacians problems of one or higher-dimensional are a generalization of the $p$-Laplacians problems. It is known that $p$-Laplacian equations with one or more dimensions arise in the study of Newtonian fluids and non-Newtonian fluids, particularly in shear-thickening (or dilatant) fluids and shear-thinning (or pseudoplastic) fluids. Therefore, we intend to present numerical results relating with one specific shear-dependent viscoelastic third-grade fluid model to be proposed in this work. The work presented regarding fluids is based on the work of Caulk and Naghdi [26] and, also based, on the work of Carapau et al. [18, 19, 20, 17], but applied to a new three-dimensional model for a non-Newtonian fluid with shear-dependent viscoelasticity.

## Specific objectives

The objectives of this thesis are divided into three groups:

- For problems involving differential equations with parameters defined in limited domains, the aim is to obtain sufficient conditions to guarantee the existence, non-existence and multiplicity of solutions in 2nd order coupled systems of the Ambrosetti-Prodi type with values at two points.

- Define conditions sufficient to guarantee the existence and localization of solutions to impulsive problems with functional boundary conditions and generalized impulse functions and with finite impulse moments.

- Present numerical results to the unsteady volume flow rate, to the unsteady three-dimensional velocity field and including an analysis on perturbed flows, for a new three-dimensional non-Newtonian fluid model to be proposed in this work.

## Methodology

To obtain sufficient conditions to guarantee the existence, non-existence, and multiplicity of solutions in 2nd-order coupled systems, the arguments apply the method of lower and upper solutions. By defining a suitable auxiliary, homotopic, and truncated problem, it is possible to apply degree theory topological methods as a tool to prove the existence and nonexistence of a solution. In short, it is predicted that for parameter values such that there are lower and upper solutions, then there is also at least one solution, and this solution is located in an interval delimited by lower and upper solutions. The multiplicity discussion is carried out for particular cases of the boundary conditions considered at [69].

In the development of the mathematical tool to guarantee the existence and location of solutions in systems of coupled impulsive differential equations of second order, third order, and higher order, the first existing result is obtained from Schauder's fixed point theorem. The second also provides the location of a solution through the technique of lower and upper solutions.

Models associated with fluid flow generally involve coupled systems of time-dependent nonlinear partial differential (or ordinary) equations and algebraic equations. Its analytical and numerical treatment is complex and requires the use of sophisticated arguments. To overcome this computational problem, we use an alternative theory, the Cosserat theory associated with fluid dynamics (see [26, 42, 43, 44]), which allows us to approximate the three-dimensional model using simplified one-dimensional models. Based on this theoretical approach, we reduce the nonlinear three-dimensional equations governing the unsteady axisymmetric motion of a non-Newtonian incompressible fluid (or Newtonian fluid) to a one-dimensional system of ordinary (or partial) differential equations depending on time and a single spatial variable. From this new system, we obtain the unsteady equation for the mean pressure gradient and the wall shear stress, both depending on the unsteady volume flow rate, Womersley number, viscosity and viscoelastic parameters in a finite section of a straight tube with constant circular cross-section.

## Organization of the Thesis

In Chapter 2, we considered some boundary value problems composed of coupled systems of second-order differential equations with complete nonlinearities and general functional boundary conditions, verifying some monotonous assumptions. Chapter 3 deals with two particular cases of the problem presented in Chapter 2, systems of second-order differential equations with parameters: a coupled system with boundary conditions of the Sturm-Liouville type and a classical system with Dirichlet boundary conditions, and for each an Ambosetti-Prodi alternative is discussed. In the first we present sufficient conditions for the existence and non-existence of solutions and, in the second, for multiplicity.

The Chapter 4 presents sufficient conditions for the solvability of a second-order coupled system, composed of two differential equations involving different laplacians applied to fully discontinuous nonlinearities, two-point boundary conditions, and generalized impulsive effects. Applying the lower and upper solutions technique and Schauder's fixed point theorem, it is obtained an existence and localization theorem, based on local monotone properties on the nonlinearities and the impulsive functions.

In Chapter 5, a methodology is developed with sufficient conditions for the solubility of a third-order coupled system. It includes two differential equations involving different Laplacians, totally discontinuous nonlinearities, two-point boundary conditions, and generalized impulse conditions. We emphasize that this is the first time impulsive coupled systems with strongly nonlinear fully differential equations and generalized impulse effects are considered simultaneously. Furthermore, the singular case is applied to a special relativity model in classical electrodynamics.

As a consequence of the techniques developed in Chapters 4 and Chapter 5 and following constant advances in this work, Chapter 6 generalizes the solvability conditions and solution location for a coupled impulsive system of higher order, with the possibility of equations of different orders, with fully differential equations including different regular and singular Laplacians and generalized impulsive effects, dependent on variables and some derivatives. As in previous chapters, an application of our theory to the bending vibration of a single-span suspension bridge is presented.

The equations related to $\phi$-Laplacian problems of one dimension or higher are a generalization of $p$-Laplacian problems. It is known that $p$-Laplacian equations with one or more dimensions arise in the study of Newtonian fluids and non-Newtonian fluids, in particular in shear-thickening (or dilatant) fluids and shear-thinning (or pseudoplastic) fluids. In this sense, Chapter 7 aims to present numerical results to the unsteady volume flow rate, to the unsteady three-dimensional velocity field and including an analysis on perturbed flows, for a new three-dimensional non-Newtonian fluid model.

Chapter **2**

# Coupled systems with Ambrosetti-Prodi-type differential equations

This chapter, based on the paper [69], focuses on the search for sufficient conditions to require nonlinearities in order to be able to discuss, depending on the parameters, the existence of solutions for coupled systems of nonlinear second-order differential equations, of the type

$$\begin{cases} u''(x) + f(x, u(x), v(x), u'(x), v'(x)) = \mu\ m(x), \\ v''(x) + h(x, u(x), v(x), u'(x), v'(x)) = \lambda\ n(x), \end{cases} \tag{2.1}$$

for $x \in [0, 1]$, with $f, h : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ , $m, n : [0, 1] \to \mathbb{R}^+$ continuous functions and $\mu, \lambda$ are real parameter, along with boundary conditions

$$\begin{array}{rcl} a_1 u(0) - b_1 u'(0) & = & A_1, \\ c_1 u(1) + d_1\ u'(1) & = & B_1, \\ a_2 v(0) - b_2 v'(0) & = & A_2, \\ c_2 v(1) + d_2 v'(1) & = & B_2, \end{array} \tag{2.2}$$

where $a_i, b_i, c_i, d_i, A_i, B_i \in \mathbb{R}$, $i = 1, 2$, such that $a_i^2 + b_i > 0$, $c_i^2 + d_i > 0$, for $i$ fixed, and $b_i, d_i \geq 0$.

Coupled systems of second-order differential equations were studied by many authors, not only from a mathematical point of view, as in, for example, [76, 67, 89, 98], but also to model some real phenomena, as, for instance, Lokta-Volterra models, reaction diffusion processes, prey-predator systems, Sturm-Liouville problems, mathematical biology, chemical systems, as in [3, 4, 8, 9, 60]. Moreover, in the last few years, fractional calculus has had an increasing application to several real processes, as in [1, 51, 79].

Ambrosetti-Prodi-type problems, introduced in [5, 66], have been applied to several types of boundary conditions, as it can be seen in, for example: [37, 109], to separated two-point and three-point boundary value problems; [86, 90], for Neumann boundary conditions; [32, 34, 65, 74, 104] to periodic solutions; [82], for parametric problems with $(p, q)$-Laplace operator; [30], with asymptotic assumptions; [6], for fractional Laplacian; [81], with asymptotic sign-changed nonlinearities, among others.

Motivated by these works, we consider for the first time, as far as we know, a coupled second-order system composed of two Ambrosetti-Prodi-type differential equations together with two-point boundary conditions. The arguments are based on the lower and upper solutions method [32, 33, 36, 68, 91], which requires a new definition of lower and upper functions to overcome the couple variation on the nonlinearities (see Definition 2.1.1). A Nagumo condition (see, for example, [31, 53, 80]) plays an important role to control the first derivatives variation, and the theory of the topological degree (see, for instance, [39, 63]) is the main tool to prove the existence of solution for the parameters' values such that there are lower and upper solutions for (2.1), (2.2). Therefore, the main result is an existence and localization theorem, as it provides not only the existence of a solution, but also a range where the solution varies.

The chapter is organized as follows: Section 2.1 contains the functional framework, a definition of upper-lower solutions, the Nagumo conditions and “a priori" estimates of the first derivative of the unknown functions. In Section 2.2 we present an existence and location result and an example to show the applicability of the main theorem. Section 2.3 applies our main result to a stationary version of the model presented in [48, 97] for complex interactions in social media, the mechanisms and dynamics of information diffusion in online social networks.

## 2.1 Definitions and auxiliary results

In this section, some definitions and lemmas will be introduced for the subsequent analysis, we consider the following functional framework.

Let $X = C^1[0,1]$ be the usual Banach space equipped with the norm $\| \cdot \|_{C^1}$, defined by

$$\| w\|_{C^1} := \max\{\| w\|, \| w'\|\},$$

where

$$\|y_1\| := \max_{x\in[0,1]} |y_1(x)|,$$

and $X^2 = C^1[0,1] \times C^1[0,1]$ with the norm

$$\| (u,v)\|_{X^2} = \max\{\| u\|_{C^1}, \| v\|_{C^1}\}.$$

To apply the lower and upper solutions method, depending on the values of the parameters $\mu$ and $\lambda$, we introduce a new type of lower and upper definition:

**Definition 2.1.1.** *Let $a_i, b_i, c_i, d_i, A_i, B_i \in \mathbb{R}$, such that $a_i^2 + b_i > 0$, $c_i^2 + d_i > 0$ and $b_i, d_i \geq 0$, for $i = 1, 2$.*

*A pair of functions $(\gamma_1, \gamma_2) \in (C^2(]0,1[) \cap C^1([0,1]))^2$ is a lower solution of problem (2.1)-(2.2) if, for all $x \in [0,1]$,*

$$\begin{cases} \gamma_1''(x) + f(x,\gamma_1(x),\gamma_2(x),\gamma_1'(x),z_1) \geq \mu\ m(x), \forall z_1 \in \mathbb{R}, \\ \\ \gamma_2''(x) + h(x,\gamma_1(x),\gamma_2(x),y_1,\gamma_2'(x)) \geq \lambda\ n(x), \forall y_1 \in \mathbb{R}, \end{cases} \tag{2.3}$$

*and*

$$\begin{array}{rcl} a_1\gamma_1(0) - b_1\gamma_1'(0) & \leq & A_1, \\ c_1\gamma_1(1) + d_1\gamma_1'(1) & \leq & B_1, \\ a_2\gamma_2(0) - b_2\gamma_2'(0) & \leq & A_2, \\ c_2\gamma_2(1) + d_2\gamma_2'(1) & \leq & B_2. \end{array} \tag{2.4}$$

*A pair of functions is an upper solution of problem (2.1)-(2.2) if the reversed inequalities are verified.*

The so-called Nagumo condition establishes an “a priori" estimation for the first derivatives of the solutions of (2.1), provided that they satisfy adequate bounds, and is adapted here for coupled systems.

**Definition 2.1.2.** *Let $\gamma_i(x),\ \Gamma_i(x),\ i=1,2$, be continuous functions such that*

$$\gamma_1(x) \leq \Gamma_1(x),\ \gamma_2(x) \leq \Gamma_2(x),\ \forall x \in [0,1],$$

*and consider the sets*

$$S = \{(x, y_0, z_0, y_1, z_1) \in [0,1] \times \mathbb{R}^4 : \gamma_1(x) \leq y_0 \leq \Gamma_1(x),\ \gamma_2(x) \leq z_0 \leq \Gamma_2(x)\}. \tag{S}$$

*The continuous functions $f, h : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy a Nagumo-type condition relative to the intervals $[\gamma_1(x), \Gamma_1(x)]$ and $[\gamma_2(x), \Gamma_2(x)]$, for all $x \in [0,1]$, if there are $k_1,\ k_2$ , such that*

$$k_1 := \max\{\Gamma_1(1) - \gamma_1(0), \Gamma_1(0) - \gamma_1(1)\}, \tag{2.5}$$

$$k_2 := \max\{\Gamma_2(1) - \gamma_2(0), \Gamma_2(0) - \gamma_2(1)\}, \tag{2.6}$$

*and continuous positive functions $\varphi, \psi : [0, +\infty) \to (0, +\infty)$, verifying*

$$\int_{k_1}^{+\infty} \frac{ds}{\varphi(s)} = +\infty, \quad \int_{k_2}^{+\infty} \frac{ds}{\psi(s)} = +\infty. \tag{2.7}$$

*such that*

$$\begin{array}{lcl} |f(x, y_0, z_0, y_1, z_1)| & \leq & \varphi(|y_1|),\ \forall (x, y_0, z_0, y_1, z_1) \in S, \\ |h(x, y_0, z_0, y_1, z_1)| & \leq & \psi(|z_1|),\ \forall (x, y_0, z_0, y_1, z_1) \in S. \end{array} \tag{2.8}$$

The *priori* estimates for the first derivatives are given by the next lemma.

**Lemma 2.1.3.** *Suppose that the continuous functions $f, h : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy a Nagumo type condition relative to the intervals $[\gamma_1(x), \Gamma_1(x)]$ and $[\gamma_2(x), \Gamma_2(x)]$, for all $x \in [0,1]$.*

*Then for every solution $(u, v) \in (C^2[0,1])^2$ of (2.1) verifying*

$$\gamma_1(x) \leq u(x) \leq \Gamma_1(x) \quad \text{and} \quad \gamma_2(x) \leq v(x) \leq \Gamma_2(x),\ \forall x \in [0,1], \tag{2.9}$$

*there are $N_1, N_2 > 0$ (depending only on the parameters $\mu$ and $\lambda$ and the functions $m$, $n$, $\gamma_1,\ \Gamma_1,\ \gamma_2,\ \Gamma_2,\ \varphi$ and $\psi$), such that*

$$\|u'\| \leq N_1 \quad \text{and} \quad \|v'\| \leq N_2. \tag{2.10}$$

*Proof.* Let $(u(x), v(x))$ be a solution of (2.1) verifying (2.9). By the Mean Value Theorem, there are $x_0, x_1 \in [0,1]$ such that

$$u'(x_0) = u(1) - u(0) \quad \text{and} \quad v'(x_1) = v(1) - v(0). \tag{2.11}$$

If

$$|u'(x)| \leq k_1,\ \forall x \in [0,1],$$

then it is enough to define $N_1 := k_1$ and the proof is complete.

Moreover, the case $|u'(x)| > k_1,\ \forall x \in [0,1]$, is not possible. In fact, if $u'(x) > k_1,\ \forall x \in [0,1]$, we obtain, by (2.11), (2.9) and (2.5), the contradiction

$$u'(x_0) = u(1) - u(0) \leq \Gamma_1(1) - \gamma_1(0) \leq k_1.$$

If $u'(x) < -k_1,\ \forall x \in [0,1]$, the contradiction is similar.

Consider $N_i > k_i$, for each $i = 1, 2$, such that

$$\int_{k_1}^{N_1} \frac{ds}{\varphi(s) + |\mu|\|m\|} > 1, \quad \int_{k_2}^{N_2} \frac{ds}{\psi(s) + |\lambda|\|n\|} > 1, \tag{2.12}$$

and assume that there are $x_2, x_3 \in [0,1]$ with $x_2 < x_3$, such that

$$u'(x_2) \leq k_1 \quad \text{and} \quad u'(x_3) > k_1.$$

By continuity of $u'(x)$, there exists $x_4 \in [x_2, x_3]$ such that $u'(x_4) = k_1$.

By a convenient change of variable and using (2.1) and (2.8),

$$\begin{aligned}
\int_{u'(x_4)}^{u'(x_3)} \frac{ds}{\varphi(|s|) + |\mu|\|m\|} &= \int_{x_4}^{x_3} \frac{u''(x)}{\varphi(|u'(x)|) + |\mu|\|m\|} dx \\
&\leq \int_0^1 \frac{|u''(x)|}{\varphi(|u'(x)|) + |\mu|\|m\|} dx \\
&\leq \int_0^1 \frac{|\mu m(x) - f(x, u(x), v(x), u'(x), v'(x))|}{\varphi(|u'(x)|) + |\mu|\|m\|} dx, \\
&\leq \int_0^1 \frac{|\mu|\|m\| + |f(x, u(x), v(x), u'(x), v'(x))|}{\varphi(|u'(x)|) + |\mu|\|m\|} dx, \\
&\leq \int_0^1 \frac{|\mu|\|m\| + \varphi(|u'(x)|)}{\varphi(|u'(x)|) + |\mu|\|m\|} dx = 1,
\end{aligned}$$

and by (2.12)

$$\int_{u'(x_4)}^{u'(x_3)} \frac{ds}{\varphi(|s|) + |\mu|\|m\|} \leq 1 < \int_{k_1}^{N_1} \frac{ds}{\varphi(|s|) + |\mu|\|m\|}.$$

Therefore $u'(x_3) < N_1$, and as $x_3$ is taken arbitrarily, then $u'(x) < N_1$, for the values of $x$ whenever $u'(x) > k_1$.

The case for $x_2 > x_3$ follows similar arguments.

The other possible case where

$$u'(x_2) > -k_1 \quad \text{and} \quad u'(x_3) < -k_1,$$

can be proved by the previous techniques. Therefore $\|u'\| \leq N_1$.

By a similar method, it can be shown that $\|v'\| \leq N_2$. □

**Remark 2.1.4.** *From the previous demonstration, it follows that $N_1$ and $N_2$ can be considered independently of $\mu$ and $\lambda$, since $\mu$ and $\lambda$ belong to a bounded set.*

## 2.2 Main result

The first theorem is an existence and localization result:

**Theorem 2.2.1.** *Let* $f, h : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *be continuous functions. If there are lower and upper solutions of (2.1)-(2.2),* $(\gamma_1, \gamma_2)$ *and* $(\Gamma_1, \Gamma_2)$*, respectively, according Definition 2.1.1, such that*

$$\gamma_i(x) \leq \Gamma_i(x), \quad i = 1, 2, \ \forall x \in [0,1], \tag{2.13}$$

*and* $f$ *and* $h$ *verify Nagumo conditions as in Definition 2.1.2, relative to the intervals* $[\gamma_1(x), \Gamma_1(x)]$ *and* $[\gamma_2(x), \Gamma_2(x)]$*, for all* $x \in [0,1]$ *, with*

$$f(x, y_0, z_0, y_1, z_1) \text{ nondecreasing in } z_0, \tag{2.14}$$

*for* $x \in [0,1]$ *,* $\forall z_1 \in \mathbb{R}$*,*

$$\min\left\{\min_{x\in[0,1]} \gamma_1'(x), \min_{x\in[0,1]} \Gamma_1'(x)\right\} \leq y_1 \leq \max\left\{\max_{x\in[0,1]} \gamma_1'(x), \max_{x\in[0,1]} \Gamma_1'(x)\right\},$$

*and*

$$h(x, y_0, z_0, y_1, z_1) \text{ nondecreasing in } y_0,$$

*for* $x \in [0,1]$ *,* $\forall y_1 \in \mathbb{R}$*,*

$$\min\left\{\min_{x\in[0,1]} \gamma_2'(x), \min_{x\in[0,1]} \Gamma_2'(x)\right\} \leq z_1 \leq \max\left\{\max_{x\in[0,1]} \gamma_2'(x), \max_{x\in[0,1]} \Gamma_2'(x)\right\}.$$

*Then there is at least a pair* $(u(x), v(x)) \in (C^2[0,1])^2$ *solution of (2.1)-(2.2) and, moreover,*

$$\gamma_1(x) \leq u(x) \leq \Gamma_1(x), \quad \gamma_2(x) \leq v(x) \leq \Gamma_2(x), \quad \forall x \in [0,1]. \tag{2.15}$$

*Proof.* Define the functions $\alpha, \beta : [0,1] \times \mathbb{R} \to \mathbb{R}$ given by

$$\alpha(x, y_0) = \begin{cases} \Gamma_1(x) & \text{if} \quad y_0 > \Gamma_1(x) \\ y_0 & \text{if} \quad \gamma_1(x) \leq y_0 \leq \Gamma_1(x) \\ \gamma_1(x) & \text{if} \quad y_0 < \gamma_1(x) \end{cases} \tag{2.16}$$

and

$$\beta(x, z_0) = \begin{cases} \Gamma_2(x) & \text{if} \quad z_0 > \Gamma_2(x) \\ y_1 & \text{if} \quad \gamma_2(x) \leq z_0 \leq \Gamma_2(x) \\ \gamma_2(x) & \text{if} \quad z_0 < \gamma_2(x). \end{cases} \tag{2.17}$$

For $\theta, \vartheta \in [0,1]$, consider the truncated and perturbed auxiliary problem formed by the equations

$$\begin{cases} u''(x) + \theta \ f(x, \alpha(x, u(x)), \beta(x, v(x)), u'(x), v'(x)) = \\ \qquad\qquad u(x) + \theta \ [\mu \ m(x) - \alpha(x, u(x))], \\ \\ v''(x) + \vartheta \ h(x, \alpha(x, u(x)), \beta(x, v(x)), u'(x), v'(x)) = \\ \qquad\qquad v(x) + \vartheta \ [\lambda \ n(x) - \beta(x, v(x))], \end{cases} \tag{2.18}$$

for $x \in ]0,1[$, and the boundary conditions

$$\begin{aligned} u(0) &= \theta \ [A_1 - a_1\alpha(0, u(0)) + b_1 u'(0) + \alpha(0, u(0))], \\ u(1) &= \theta \ [B_1 - c_1\alpha(1, u(1)) - d_1 u'(1) + \alpha(1, u(1))], \\ v(0) &= \vartheta \ [A_2 - a_2\beta(0, v(0)) + b_2 v'(0) + \beta(0, v(0))], \\ v(1) &= \vartheta \ [B_2 - c_2\beta(1, v(1)) - d_2 v'(1) + \beta(1, v(1))] \end{aligned} \tag{2.19}$$

where $a_i, b_i, c_i, d_i, A_i, B_i \in \mathbb{R}$, $i = 1, 2$, such that $a_i^2 + b_i > 0$, $c_i^2 + d_i > 0$ and $b_i, d_i \geq 0$. Take $r_i > 0$, $i = 1, 2$, such that, $\forall x \in [0, 1]$,

$$\begin{aligned} -r_i < \gamma_i(x) \leq \Gamma_i(x) < r_i, \\ \mu\, m(x) - f(x, \gamma_1(x), \beta(x, v(x)), 0, v'(x)) - r_1 - \gamma_1(x) < 0, \\ \mu\, m(x) - f(x, \Gamma_1(x), \beta(x, v(x)), 0, v'(x)) + r_1 - \Gamma_1(x) > 0, \\ \lambda\, n(x) - h(x, \alpha(x, u(x)), \gamma_2(x), u'(x), 0) - r_2 - \gamma_2(x) < 0, \\ \lambda n(x) - h(x, \alpha(x, u(x)), \Gamma_2(x), u'(x), 0) + r_2 - \Gamma_2(x) > 0, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} |A_i - a_i\Gamma_i(0) + \Gamma_i(0)| < r_i, \quad |A_i - a_i\gamma_i(0) + \gamma_i(0)| < r_i, \\ |B_i - c_i\Gamma_i(1) + \Gamma_i(1)| < r_i, \quad |B_i - c_i\gamma_i(1) + \gamma_i(1)| < r_i. \end{aligned} \tag{2.21}$$

**Claim 1.** *Every solution* $(u(x), v(x))$ *of the problems (2.18) and (2.19) verifies*

$$|u(x)| < r_1 \text{ and } |v(x)| < r_2, \ \forall x \in [0, 1],$$

*independently of* $\theta, \vartheta \in [0, 1]$.

Assume, by contradiction, that there exist $\theta \in [0, 1]$, $(u(x), v(x))$ solution of (2.18) and (2.19) and $x \in [0, 1]$ such that $|u(x)| \geq r_1$.If $u(x) \geq r_1$, define

$$\max_{x \in [0,1]} u(x) := u(x_0).$$

For $x_0 \in ]0, 1[$ and $\theta \in ]0, 1]$, $u'(x_0) = 0$ and $u''(x_0) \leq 0$.By (2.20), we have the following contradiction

$$\begin{aligned} 0 &\geq u''(x_0) \\ &= u(x_0) + \theta\, [\mu\, m(x_0) - \alpha(x_0, u(x_0))] - \theta f(x_0, \alpha(x_0, u(x_0)), \beta(x_0, v(x_0)), u'(x_0), v'(x_0)) \\ &= u(x_0) + \theta\, [\mu\, m(x_0) - \Gamma_1(x_0)] - \theta f(x_0, \Gamma_1(x_0), \beta(x_0, v(x_0)), 0, v'(x_0)) \\ &\geq \theta\, [\mu\, m(x_0) - f(x_0, \Gamma_1(x_0), \beta(x_0, v(x_0)), 0, v'(x_0)) + u(x_0) - \Gamma_1(x_0)] \\ &\geq \theta\, [\mu\, m(x_0) - f(x_0, \Gamma_1(x_0), \beta(x_0, v(x_0)), 0, v'(x_0)) + r_1 - \Gamma_1(x_0)] > 0. \end{aligned}$$

If $\theta = 0$, the contradiction arises from

$$0 \geq u''(x_0) = u(x_0) \geq r_1 > 0.$$

If $x_0 = 0$, then

$$\max_{x \in [0,1]} u(x) := u(0),$$

and $u'(0^+) = u'(0) \leq 0$. By (2.19) and (2.21), we have

$$\begin{aligned} r_1 &\leq u(0) = \theta\, [A_1 - a_1\alpha(0, u(0)) + b_1 u'(0) + \alpha(0, u(0))] \\ &= \theta\, [A_1 - a_1\Gamma_1(0) + b_1 u'(0) + \Gamma_1(0)] \leq \theta\, [A_1 - a_1\Gamma_1(0) + \Gamma_1(0)] \\ &\leq |A_1 - a_1\Gamma_1(0) + \Gamma_1(0)| < r_1. \end{aligned}$$

If $x_0 = 1$ a contradiction is obtained analogously.

Then $u(x) < r_1$ for $x \in [0, 1]$ and regardless of $\theta$. The other possible case where

$$u(x) > -r_1, \text{ for } x \in [0, 1],$$

can be proved by similar techniques.

Therefore $|u(x)| < r_1$ for all $x \in [0,1]$, regardless of $\theta$ .

By a similar method, it can be proved that $|v(x)| < r_2$ for all $x \in [0,1]$ regardless of $\vartheta$.

**Claim 2.** For *every solution* $(u(x), v(x))$ *of the problems (2.18) and (2.19),*

$$|u'(x)| < N_1 \text{ and } |v'(x)| < N_2, \ \forall x \in [0,1],$$

*independently of* $\theta, \vartheta \in [0,1]$, with $N_1$ and $N_2$ *given by Lemma 2.1.3.*

Define the continuous functions $F_\theta, H_\vartheta \ : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$, by

$$F_\theta(x, y_0, z_0, y_1, z_1) := \theta \ f(x, \alpha(x, y_0), \beta(x, z_0), y_1, z_1) - y_0 + \theta \ \alpha(x, y_0),$$

and

$$H_\vartheta(x, y_0, z_0, y_1, z_1) := \vartheta \ h(x, \alpha(x, y_0), \beta(x, z_0), y_1, z_1) - z_0 + \vartheta \ \beta(x, z_0),$$

with $y_0 \in [-r_1, r_1]$ and $z_0 \in [-r_2, r_2]$.

As the functions $f, h : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy a Nagumo-type condition relative to the intervals $[\gamma_1(x), \Gamma_1(x)]$ and $[\gamma_2(x), \Gamma_2(x)]$, then

$$\begin{aligned} |F_\theta(x, y_0, z_0, y_1, z_1)| &\leq |f(x, \alpha(x, y_0), \beta(x, z_0), y_1, z_1)| + |y_0| + |\alpha(x, y_0)| \\ &\leq \varphi(|y_1|) + 2r_1, \end{aligned}$$

and

$$\begin{aligned} |H_\vartheta(x, y_0, z_0, y_1, z_1)| &\leq |h(x, \alpha(x, y_0), \beta(x, z_0), y_1, z_1)| + |z_0| + |\beta(x, z_0)| \\ &\leq \psi(|z_1|) + 2r_2. \end{aligned}$$

Therefore for continuous positive functions $\varphi_*, \psi_* : [0, +\infty) \to (0, +\infty)$, given by

$$\varphi_*(|y_1|) := \varphi(|y_1|) + 2r_1 \quad \text{and} \quad \psi_*(|z_1|) := \psi(|z_1|) + 2r_2,$$

then, clearly, $F_\theta$ and $H_\vartheta$ satisfy Nagumo conditions in the sets

$$E = \{(x, y_0, z_0, y_1, z_1) \in [0,1] \times \mathbb{R}^4 : |y_0| \leq r_1, \ \ |z_0| \leq r_2\},$$

and, by (2.7), we have

$$\int_{k_1}^{N_1} \frac{ds}{\varphi_*(s)} = \int_{k_1}^{N_1} \frac{ds}{\varphi(s) + 2r_1} > 1,$$

and

$$\int_{k_2}^{N_2} \frac{ds}{\psi_*(s)} = \int_{k_2}^{N_2} \frac{ds}{\psi(s) + 2r_2} > 1.$$

Therefore, by Lemma 2.1.3,

$$|u'(x)| < N_1 \text{ and } |v'(x)| < N_2, \ \forall x \in [0,1],$$

independently of $\theta, \vartheta \in [0,1]$.

**Claim 3.** *The problems (2.18) and (2.19), for $\theta = 1$ and $\vartheta = 1$, has at least one solution $(u, v)$.*

Define the operators

$$\mathcal{L} : (C^2([0,1]))^2 \to (C^2([0,1]))^2 \times \mathbb{R}^4,$$

given by

$$\mathcal{L}(u,v) = (u'' - u, v'' - v, u(0), u(1), v(0), v(1)),$$

and

$$\mathcal{N}_{(\theta,\vartheta)} : (C^1([0,1]))^2 \to (C^2([0,1]))^2 \times \mathbb{R}^4,$$

given by

$$\mathcal{N}_{(\theta,\vartheta)}(u,v) = (X, Y, A_\theta, B_\theta, A_\vartheta, B_\vartheta),$$

being

$$X := -\theta\ f(x, \alpha(x,u(x)), \beta(x,v(x)), u'(x), v'(x)) + \theta\ [\mu\ m(x) - \alpha(x,u(x))],$$

$$Y := -\vartheta\ h(x, \alpha(x,u(x)), \beta(x,v(x)), u'(x), v'(x)) + \vartheta\ [\lambda\ n(x) - \beta(x,v(x))],$$

$$\begin{aligned} A_\theta &:= \theta\ [A_1 - a_1\alpha(0,u(0)) + b_1 u'(0) + \alpha(0,u(0))], \\ B_\theta &:= \theta\ [B_1 - c_1\alpha(1,u(1)) - d_1 u'(1) + \alpha(1,u(1))], \\ A_\vartheta &:= \vartheta\ [A_2 - a_2\beta(0,v(0)) + b_2 v'(0) + \beta(0,v(0))], \end{aligned}$$

and

$$B_\vartheta := \vartheta\ [B_2 - c_2\beta(1,v(1)) - d_2 v'(1) + \beta(1,v(1))].$$

Since $\mathcal{L}$ is invertible, we define the completely continuous operator

$$\mathcal{T} : (C^2([0,1]))^2 \to (C^2([0,1]))^2,$$

given by

$$\mathcal{T}_{(\theta,\vartheta)}(u,v) = \mathcal{L}^{-1}\mathcal{N}_{(\theta,\vartheta)}(u,v).$$

For $M := \max\{r_1, r_2, N_1, N_2\}$ consider the set

$$\Omega = \{(u,v) \in C^2([0,1]) : \|(u,v)\| < M\}.$$

By Claims 1 and 2, for all $\theta, \vartheta \in [0,1]$, the degree $d(\mathcal{I} - \mathcal{T}_{(\theta,\vartheta)}, \Omega_1, 0)$ is well defined, and, by homotopy invariance,

$$d(\mathcal{I} - \mathcal{T}_{(0,0)}, \Omega, 0) = d(\mathcal{I} - \mathcal{T}_{(1,1)}, \Omega, 0).$$

As the equation $(u,v) = \mathcal{T}_{(0,0)}(u,v)$ admits only the null solution, then, by degree theory,

$$d(\mathcal{I} - \mathcal{T}_{(0,0)}, \Omega, 0) = \pm 1,$$

and, in particular, the equation $(u,v) = \mathcal{T}_{(1,1)}(u,v)$ has at least one solution.

That is, the problem is composed of the equations

$$\begin{cases} u''(x) + \ f(x, \alpha(x,u(x)), \beta(x,v(x)), u'(x), v'(x)) = \\ \qquad\qquad u(x) + \mu\ m(x) - \alpha(x,u(x)), \\ \\ v''(x) + \ h(x, \alpha(x,u(x)), \beta(x,v(x)), u'(x), v'(x)) = \\ \qquad\qquad v(x) + \lambda\ n(x) - \beta(x,v(x)), \end{cases} \tag{2.22}$$

and the boundary conditions

$$\begin{array}{rcl} u(0) &=& A_1 - a_1\alpha(0,u(0)) + b_1 u'(0) + \alpha(0,u(0)), \\ u(1) &=& B_1 - c_1\alpha(1,u(1)) - d_1 u'(1) + \alpha(1,u(1)), \\ v(0) &=& A_2 - a_2\beta(0,v(0)) + b_2 v'(0) + \beta(0,v(0)), \\ v(1) &=& B_2 - c_2\beta(1,v(1)) - d_2 v'(1) + \beta(1,v(1)) \end{array} \quad (2.23)$$

has at least one solution $(u_1(x), v_1(x))$ in $\Omega_1$.

**Claim 4.** *The functions $u_1(x)$ and $v_1(x)$ are a solution of the initial problem (2.1), (2.2).*

In fact, if the pair $(u_1(x), v_1(x)) \in (C^2[0,1])^2$ is a solution of (2.22) and (2.23), it will be also a solution of the initial problems (2.1) and (2.2), provided that

$$\gamma_1(x) \le u_1(x) \le \Gamma_1(x), \quad \gamma_2(x) \le v_1(x) \le \Gamma_2(x), \quad \forall x \in [0,1].$$

Suppose, by contradiction, that there exists $x \in [0,1]$ such that $u_1(x) > \Gamma_1(x)$, and define

$$\max_{x\in[0,1]} [u_1(x) - \Gamma_1(x)] := u_1(x_1) - \Gamma_1(x_1) > 0.$$

If $x_1 \in ]0,1[$ then

$$u_1'(x_1) = \Gamma_1'(x_1) \quad \text{and} \quad u_1''(x_1) \le \Gamma_1''(x_1),$$

and we have, by Definition 2.1.1, the following contradiction

$$\begin{array}{rcl} 0 &\ge& u_1''(x_1) - \Gamma_1''(x_1) \\ &=& -f(x_1, \alpha(x_1, u_1(x_1)), \beta(x_1, v_1(x_1)), u_1'(x_1), v_1'(x_1)) \\ && +u(x_1) + \mu\ m(x_1) - \alpha(x_1, u_1(x_1)) - \Gamma_1''(x_1) \\ &=& -f(x_1, \Gamma_1(x_1), \beta(x_1, v_1(x_1)), \Gamma_1'(x_1), v_1'(x_1)) + u(x_1) + \mu\ m(x_1) - \Gamma_1(x_1) - \Gamma_1''(x_1) \\ &\ge& -f(x_1, \Gamma_1(x_1), \Gamma_2(x_1), \Gamma_1'(x_1), v_1'(x_1)) + u(x_1) + \mu\ m(x_1) - \Gamma_1(x_1) - \Gamma_1''(x_1) \\ &>& -f(x_1, \Gamma_1(x_1), \Gamma_2(x_1), \Gamma_1'(x_1), v_1'(x_1)) + \mu\ m(x_1) - \Gamma_1''(x_1) \ge 0. \end{array}$$

For $x_1 = 0$,

$$\max_{x\in[0,1]} [u_1(x) - \Gamma_1(x)] := u_1(0) - \Gamma_1(0) > 0$$

and

$$u_1'(0^+) - \Gamma_1'(0^+) = u_1'(0) - \Gamma_1'(0) \le 0.$$

By the boundary conditions (2.23) and Definition 2.1.1, this contradiction is obtained

$$\begin{array}{rcl} \Gamma_1(0) < u_1(0) &=& A_1 - a_1\alpha(0,u(0)) + b_1 u'(0) + \alpha(0,u(0)) \\ &=& A_1 - a_1\Gamma_1(0) + b_1 u'(0) + \Gamma_1(0) \\ &\le& -b_1\Gamma_1'(0) + b_1 u'(0) + \Gamma_1(0) \\ &=& b_1[u'(0) - \Gamma_1'(0)] + \Gamma_1(0) \le \Gamma_1(0). \end{array}$$

So, $x_1 \neq 0$ and by a similar method it is shown that $x_1 \neq 1$. Therefore

$$u_1(x) \le \Gamma_1(x), \forall x \in [0,1].$$

Defining

$$\min_{x\in[0,1]}[u_1(x)-\gamma_1(x)] := u_1(x_2)-\gamma_1(x_2) < 0,$$

it can be proved that $u_1(x) \geq \gamma_1(x)$, for all $x \in [0,1]$, by an analogous process.

Using the same technique, it can be shown that

$$\gamma_2(x) \leq v_1(x) \leq \Gamma_2(x), \quad \forall x \in [0,1].$$

So, $(u_1(x), v_1(x))$ is a solution of (2.1)-(2.2), for the values of $\mu, \lambda \in \mathbb{R}$, such that there are lower and upper solutions according to Definition 2.1.1. □

**Example 2.2.2.** *Consider the system*

$$\begin{cases} u''(x) - \sin\left(\frac{\pi}{2}u(x)\right) + \dfrac{x}{2}v(x) - u'(x) + \dfrac{1}{v'^2(x)+1} = \mu \\ \\ v''(x) + e^{u(x)-1} - v(x) + \cos\left(\frac{\pi}{2}u'(x)\right) - v'(x) = \lambda, \end{cases} \tag{2.24}$$

*for $x \in [0,1]$, and $\mu, \lambda$ are real parameter, along with boundary conditions*

$$\begin{array}{rcl} u(0) & = & 0, \\ u(1)+u'(1) & = & 1, \\ v(0) & = & 0, \\ v(1)+v'(1) & = & 1. \end{array} \tag{2.25}$$

The functions $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2 : [0,1] \to \mathbb{R}$, given by

$$\begin{array}{rcl} \gamma_1(x) & = & -x, \ \gamma_2(x) = -x-1, \\ \Gamma_1(x) & = & x, \ \Gamma_2(x) = x+1, \end{array}$$

are, respectively, lower and upper solutions of (2.24) and (2.25) for $\mu$ and $\lambda$ such that

$$\mu \in [0,1] \quad \text{and} \quad \lambda \in \left[\frac{1}{e}-1, \frac{1}{e}+1\right].$$

The functions

$$f(x,y_0,z_0,y_1,z_1) = -\sin\left(\frac{\pi}{2}y_0\right) + \frac{x}{2}z_0 - y_1 + \frac{1}{z_1^2+1}$$

and

$$h(x,y_0,z_0,y_1,z_1) = e^{y_0-1} - z_0 + \cos\left(\frac{\pi}{2}y_1\right) - z_1$$

are continuous and satisfy the Nagumo conditions (2.7) and (2.8). By Theorem 2.2.1, the problems (2.24) and (2.25) has at least one solution $(u_1(x), v_1(x))$, which verifies

$$-x \leq u_1(x) \leq x \quad \text{and} \quad -x-1 \leq v_1(x) \leq x+1, \quad \forall x \in [0,1].$$

Figures 2.1 show the admissible region for solution $(u_1(x), v_1(x))$.

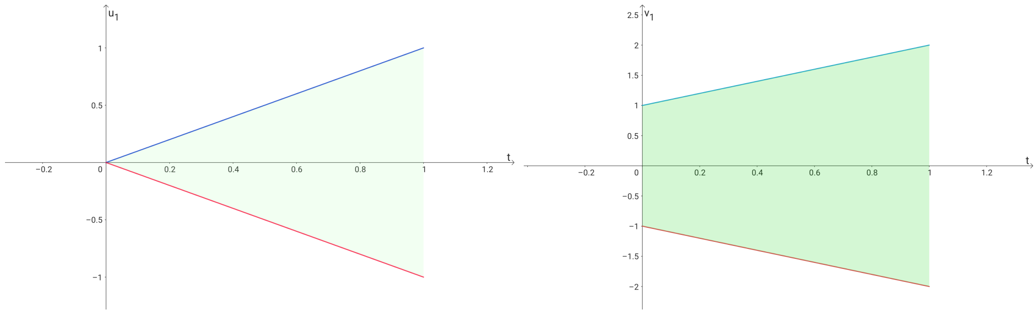



**Fig. 2.1:** At least one solution $(u_1(x), v_1(x))$. of problem (2.24) and (2.25) is located in the colored region, when $x \in [0, 1]$.

## 2.3 Application to the diffusion of information in social media

In [48, 97], the authors study the process of disseminating information in social media to obtain, mathematical predictability in news dissemination to increase the efficiency of the distribution of positive information and, at the same time, reduce information unwanted. In short, it is considered a model to describe the flow of information, based on two similar news sources, that spread with logistic growth independently together with an additional effect one over the other.

It is addressed that, for a given piece of information initiated by two specific users called *sources*, the density of influenced users in the network depends on the distance $x$ to any source. Based on this model we consider the following stationary system

$$\begin{cases} u''(x) + \dfrac{r_1}{d_1} u(x) \left(1 - \dfrac{u(x)}{K_1}\right) + \dfrac{\alpha_1}{d_1} u(x)v(x) = \mu\; m(x), \\ \\ v''(x) + \dfrac{r_2}{d_2} v(x) \left(1 - \dfrac{v(x)}{K_2}\right) + \dfrac{\alpha_2}{d_2} u(x)v(x) = \lambda\; n(x), \end{cases} \tag{2.26}$$

together with the boundary conditions

$$\begin{aligned} u(0) - b_1 u'(0) &= 1, \\ u'(1) &= 0, \\ v(0) - b_2 v'(0) &= 1, \\ v'(1) &= 0, \end{aligned} \tag{2.27}$$

where $b_i, d_i, r_i, K_i \in \mathbb{R}^+$ and $\alpha_i \geq 0$, $i = 1, 2$, having the following meaning:

- $u$ and $v$ are the density of information from the different sources
- $d_1, d_2$ represent the popularity of the two pieces of information;
- $r_1, r_2$ are the speed with which information spreads within groups of users with the same distance;
- $K_1, K_2$ represents the carrying capacity, which is the maximum possible density of influenced users;

- $\alpha_1$ measures the positive effect of news $v$ on $u$ and $\alpha_2$ measures the positive effect of news $u$ on $v$;
- $m, n : [0,1] \to \mathbb{R}^+$ are continuous functions, $\mu, \lambda$ real parameters, both relating with distance of each source.
- $b_1$, $b_2$ are related with the initial spread of each source.

Define

$$m \equiv \max_{x\in[0,1]} m(x) \quad \text{and} \quad n \equiv \max_{x\in[0,1]} n(x),$$

consider $K_1 = 1$ and $K_2 = 1$, in the sense that the maximum load capacity is 100%, and, as a numeric example, assume that

$$\begin{gathered} b_1 \in \left]0, \frac{1}{6}\right], \ b_2 \in \left]0, \frac{1}{2}\right], \\ \frac{2}{3} + \frac{11 r_1}{144 d_1} + \frac{11(4\pi^2-1)\alpha_1}{48\pi^2 d_1} < 1, \\ \frac{1}{8} + \frac{(4\pi^2-1) r_2}{16\pi^4 d_2} + \frac{11(4\pi^2-1)\alpha_2}{48\pi^2 d_2} < \frac{3}{2}. \end{gathered}$$

Then the functions

$$\begin{aligned} \gamma_1(x) &= \frac{3}{4}(x-1)^2, \ \gamma_2(x) = \frac{1}{2}(x-1)^2 \\ \Gamma_1(x) &= \frac{1}{12}x^4 - \frac{1}{6}x^2 + 1, \ \Gamma_2(x) = \frac{1}{4\pi^2}\cos^2\left(\frac{\pi}{2}x\right) + \frac{(4\pi^2-1)}{4\pi^2} \end{aligned}$$

are, respectively, lower and upper solutions of problems (2.26) and (2.27) for

$$\mu \in \left[\frac{1}{m}\left(\frac{2}{3} + \frac{11 r_1}{144 d_1} + \frac{11(4\pi^2-1)\alpha_1}{48\pi^2 d_1}\right), \frac{1}{m}\right] \tag{2.28}$$

and

$$\lambda \in \left[\frac{1}{n}\left(\frac{1}{8} + \frac{(4\pi^2-1) r_2}{16\pi^4 d_2} + \frac{11(4\pi^2-1)\alpha_2}{48\pi^2 d_2}\right), \frac{3}{2n}\right]. \tag{2.29}$$

Moreover, the problems (2.26) and (2.27) are particular cases of (2.1), (2.2), with

$$f(x, y_0, z_0, y_1, z_1) = \frac{r_1}{d_1} y_0 (1 - y_0) + \frac{\alpha_1}{d_1} y_0 z_0,$$

and

$$h(x, y_0, z_0, y_1, z_1) = \frac{r_2}{d_2} z_0 (1 - z_0) + \frac{\alpha_2}{d_2} y_0 z_0.$$

These functions verify the Nagumo conditions (2.7) and (2.8), relative to the intervals $y_0 \in [\gamma_1(x), \Gamma_1(x)]$ and $z_0 \in [\gamma_2(x), \Gamma_2(x)]$, as

$$\begin{aligned} |f(x, y_0, z_0, y_1, z_1)| &= \left|\frac{r_1}{d_1} y_0 (1 - y_0) + \frac{\alpha_1}{d_1} y_0 z_0\right| \\ &\leq \frac{r_1}{d_1} |y_0 (1 - y_0)| + \frac{\alpha_1}{d_1} |y_0 z_0| \\ &\leq \frac{1}{2d_1}\left(\frac{1}{2} r_1 + \alpha_1\right), \end{aligned}$$

$$\begin{aligned} |h(x, y_0, z_0, y_1, z_1)| &= \left| \frac{r_2}{d_2} z_0 \,(1 - z_0) + \frac{\alpha_2}{d_2} y_0 z_0 \right| \\ &\leq \frac{r_2}{d_2} \left| z_0 \,(1 - z_0) \right| + \frac{\alpha_2}{d_2} \left| y_0 z_0 \right| \\ &\leq \frac{1}{2d_2} \left( \frac{1}{2} r_2 + \alpha_2 \right). \end{aligned}$$

and, trivially,

$$\int^{+\infty} \frac{ds}{\frac{1}{2d_i}\left(\frac{1}{2} r_i + \alpha_i\right)} = +\infty, \text{ for } i = 1, 2.$$

So by Theorem 2.2.1 the problems (2.26) and (2.27) have at least one solution $(u(x), v(x))$, for the values of $\mu$ and $\lambda$ verifying (2.28) and (2.29), such that

$$\frac{3}{4}(x-1)^2 \leq u(x) \leq \frac{1}{12} x^4 - \frac{1}{6} x^2 + 1,$$

and

$$\frac{1}{2}(x-1)^2 \leq v(x) \leq \frac{1}{4\pi^2} \cos^2\left(\frac{\pi}{2} x\right) + \frac{(4\pi^2 - 1)}{4\pi^2},$$

for all $x \in [0, 1]$.

The Figures 2.2 show the regions where this solution lies.

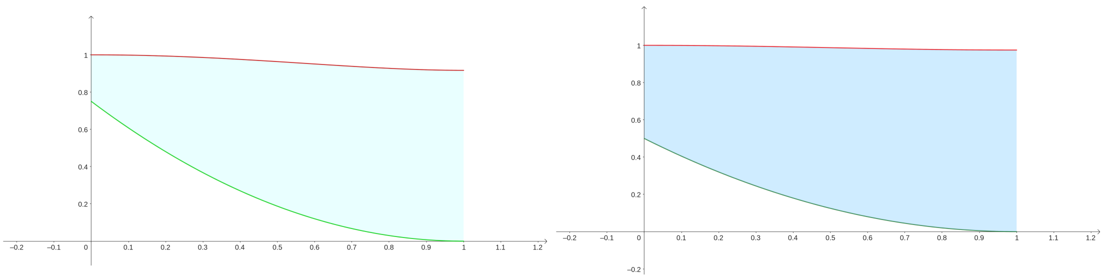


**Fig. 2.2:** At least one solution $(u(x), v(x))$. of problem (2.26) and (2.27) is located in the colored region, when $x \in [0, 1]$.

Chapter **3**

# Ambrosetti–Prodi alternative for systems of second-order differential equations

This chapter, based on the article [75], focuses on sufficient conditions to demand from nonlinearities in order to be able to discuss, depending on the parameters, the existence and non-existence of solutions for second-order systems of the type

$$\begin{cases} u_1''(t) + f(t, u_1(t), u_2(t), u_1'(t)) = \mu\, v_1(t) \\ u_2''(t) + g(t, u_1(t), u_2(t), u_2'(t)) = \lambda\, v_2(t), \end{cases} \tag{3.1}$$

for $t \in [0,1]$, with $f, g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$, $v_1, v_2 : [0,1] \to \mathbb{R}^+$ continuous functions and $\mu, \lambda$ are real parameter, along with boundary conditions

$$\begin{array}{rcl} a_i u_i(0) - b_i u_i'(0) & = & 0, \\ c_i u_i(1) + d_i u_i'(1) & = & 0 \end{array} \tag{3.2}$$

where $a_i, b_i, c_i, d_i \geq 0$, $i = 1, 2$, such that $a_i + b_i > 0$ and $c_i + d_i > 0$.

The multiplicity of solutions will be obtained for a particular case of (3.1), (3.2), that is,

$$\begin{cases} u_1''(t) + f(t, u_1(t), u_1'(t)) = \mu\, v_1(t) \\ u_2''(t) + g(t, u_2(t), u_2'(t)) = \lambda\, v_2(t), \end{cases} \tag{3.3}$$

for $t \in [0,1]$, with $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, with boundary conditions, this is

$$u_i(0) = u_i(1) = 0, \text{ for } i = 1, 2. \tag{3.4}$$

Although the above system is not coupled, we decided to present it, to show the readers how the multiplicity arguments can be used in the Ambrosetti-Prodi alternative. This technique, for proving the existence of multiple solutions, remains an open problem and will be the aim of future work.

Common to all of these problems is the discussion of the so-called Ambrosetti–Prodi alternative: there are some values $\xi_0$ and $\xi_1$, of a parameter $\xi$, such that the problem has no solution for $\xi < \xi_0$, at least one if $\xi = \xi_0$, or two solutions, for $\xi_0 < \xi < \xi_1$.

Recently, in [69], the authors presented a technique to discuss the existence of coupled systems of two Ambrosetti-Prodi-type second-order fully differential equations, where it is proved the existence of solutions for the values of the parameters for which there are lower and upper solutions for the system.

This paper extends, for the first time, as far as we know, the Ambrosetti–Prodi alternative to coupled systems of differential equations with two parameters: the existence and nonexistence of solutions is obtained for the problem (3.1), (3.2), and the multiplicity discussion for a particular case.

The method relies on the lower and upper solutions technique together with a Nagumo condition to estimate the values of the first derivatives. Leray-Schauder topological degree properties play a key role to obtain the multiplicity of solutions [39]. As it usual in this method, the results provide also a localization for such solutions in a strip bounded by lower and upper solutions. This feature is particularly useful in practice, as we can see, in the application of these theorems to study the population dynamics, namely to a Lotka-Volterra steady-state system with migration, in the last section.

The chapter is organized as it follows: Section 3.1 contains definitions and some auxiliary results, such as the *a priori* Nagumo estimation for the first derivatives and a previous result, used forward in the main results. The Section 3.2 and Sections 3.3 make the discussion on the two parameters for the existence and multiplicity of solutions, respectively. The Section 3.4 presents an application to study the interactions between two species under two scenarios: mutualism and neutralism.

## 3.1 Definitions and auxiliary results

In this section, some definitions, lemmas and theorem will be introduced for the subsequent analysis.

Let $X = C^1[0,1]$ be the usual Banach space equipped with the norm $\| \cdot \|_{C^1}$, defined by

$$\| x\|_{C^1} := \max\{\| x\|, \| x'\|\},$$

where

$$\|x\| := \max_{t\in[0,1]} |x(t)|,$$

and $X^2 = C^1[0,1] \times C^1[0,1]$ with the norm

$$\| (x,y)\|_{X^2} = \max\{\| x\|_{C^1}, \| y\|_{C^1}\} \tag{3.5}$$

The so-called Nagumo condition, introduced by [80], establishes an *a priori* estimation for the first derivative of the solution of the system (3.1), provided that it satisfies an adequate framework.

**Definition 3.1.1.** *Let $\alpha_i(t),\ \beta_i(t),\ i = 1,2,$ be continuous functions such that*

$$\alpha_i(t) \leq \beta_i(t), \text{ for all } t \in [0,1],$$

*and consider the set*

$$S = \{(t, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^3 : \alpha_1(t) \leq y_1 \leq \beta_1(t),\ \alpha_2(t) \leq y_2 \leq \beta_2(t)\}. \tag{S}$$

*A continuous function $h : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies a Nagumo-type condition in the set S, if, there is a continuous positive function $\varphi : [0,+\infty) \to (0,+\infty)$, verifying*

$$|h(t, y_1, y_2, y_3)| \leq \varphi(|y_3|), \tag{3.6}$$

*and such that*

$$\int_0^{+\infty} \frac{ds}{\varphi(s)} = +\infty. \tag{3.7}$$

The *a priori* estimate for the first derivatives is given by next lemma, following the arguments of [69].

**Lemma 3.1.2.** *Suppose that the continuous functions $f, g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfy a Nagumo type condition (3.6), (3.7) in $S$. Then for every solution $(u_1, u_2) \in (C^2[0,1])^2$ of (3.1) verifying*

$$\alpha_1(t) \leq u_1(t) \leq \beta_1(t) \quad \text{and} \quad \alpha_2(t) \leq u_2(t) \leq \beta_2(t), \ \forall t \in [0,1], \tag{3.8}$$

*there are $N_1 > 0$, $N_2 > 0$, such that*

$$\|u_1'\| \leq N_1 \quad \text{and} \quad \|u_2'\| \leq N_2. \tag{3.9}$$

**Remark 3.1.3.** *The constant $N_1$ depends only on the parameter $\mu$ and on the functions $v_1$, $\alpha_1$, and $\beta_1$. Analogously, $N_2$ depends only on $\lambda$, $v_2$, $\alpha_2$, and $\beta_2$.*
*However, if the parameters $\mu$ and $\lambda$, belong to bounded sets, $N_1$ and $N_2$ can be taken independently of $\mu$ and $\lambda$.*

To apply lower and upper solutions method, depending on the values of the parameters $\mu$ and $\lambda$, we take the followings coupled functions:

**Definition 3.1.4.** *Let $a_i, b_i, c_i, d_i \geq 0$, such that $a_i + b_i > 0$, $c_i + d_i > 0$, for $i = 1, 2$.*
*A pair of functions $(\gamma_1, \gamma_2) \in (C^2(]0,1[) \cap C^1([0,1]))^2$ is a lower solution of problem (3.1), (3.2) if, for all $t \in [0,1]$,*

$$\begin{cases} \gamma_1''(t) + f(t, \gamma_1(t), \gamma_2(t), \gamma_1'(t)) \geq \mu \ v_1(t), \\ \\ \gamma_2''(t) + g(t, \gamma_1(t), \gamma_2(t), \gamma_2'(t)) \geq \lambda \ v_2(t), \end{cases} \tag{3.10}$$

*and, for $i = 1, 2$,*

$$\begin{array}{rcl} a_i\gamma_i(0) - b_i\gamma_i'(0) & \leq & 0, \\ c_i\gamma_i(1) + d_i\gamma_i'(1) & \leq & 0. \end{array} \tag{3.11}$$

*A pair of functions $(\phi_1, \phi_2) \in (C^2(]0,1[) \cap C^1([0,1]))^2$ is an upper solution of problem (3.1), (3.2) if, for all $t \in [0,1]$,*

$$\begin{cases} \phi_1''(t) + f(t, \phi_1(t), \phi_2(t), \phi_1'(t)) \leq \mu \ v_1(t), \\ \\ \phi_2''(t) + g(t, \phi_1(t), \phi_2(t), \phi_2'(t)) \leq \lambda \ v_2(t), \end{cases} \tag{3.12}$$

*and, for $i = 1, 2$,*

$$\begin{array}{rcl} a_i\phi_i(0) - b_i\phi_i'(0) & \geq & 0, \\ c_i\phi_i(1) + d_i\phi_i'(1) & \geq & 0. \end{array} \tag{3.13}$$

The first theorem is an existence and localization result, which is a particular case of Theorem 3.1. of [69]. In short, it guarantees the existence of a solution for the values of $\mu$ and $\lambda$ such that there are lower and upper solutions of the problem (3.1), (3.2).

**Theorem 3.1.5.** *Let $f, g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous functions. If there are lower and upper solutions of (3.1)-(3.2), $(\gamma_1, \gamma_2)$ and $(\phi_1, \phi_2)$, respectively, according Definition 3.1.4, such that*

$$\gamma_i(x) \leq \phi_i(x), \quad i = 1, 2, \ \forall x \in [0,1], \tag{3.14}$$

*and $f$ and $g$ verify Nagumo conditions as in Definition 3.1.1, relative to the intervals $[\gamma_1(x), \phi_1(x)]$ and $[\gamma_2(x), \phi_2(x)]$, for all $x \in [0,1]$, with*

$$f(x, y_0, z_0, y_1) \text{ nondecreasing in } z_0, \tag{3.15}$$

*for $x \in [0,1]$,*

$$\min\left\{\min_{x\in[0,1]} \gamma_1'(x), \min_{x\in[0,1]} \phi_1'(x)\right\} \leq y_1 \leq \max\left\{\max_{x\in[0,1]} \gamma_1'(x), \max_{x\in[0,1]} \phi_1'(x)\right\},$$

*and*

$$g(x, y_0, z_0, z_1) \text{ nondecreasing in } y_0,$$

*for $x \in [0,1]$,*

$$\min\left\{\min_{x\in[0,1]} \gamma_2'(x), \min_{x\in[0,1]} \phi_2'(x)\right\} \leq z_1 \leq \max\left\{\max_{x\in[0,1]} \gamma_2'(x), \max_{x\in[0,1]} \phi_2'(x)\right\}.$$

*Then there is at least a pair $(u(x), v(x)) \in (C^2[0,1])^2$ solution of (3.1)-(3.2) and, moreover,*

$$\gamma_1(x) \leq u(x) \leq \phi_1(x), \quad \gamma_2(x) \leq v(x) \leq \phi_2(x), \quad \forall x \in [0,1]. \tag{3.16}$$

## 3.2 Existence and Non-Existence of Solution

A preliminary discussion of the values of parameters $\mu$ and $\lambda$ for which it is possible to guarantee the existence and non-existence of a solution for system (3.1) with boundary conditions (3.2), is given by next theorem:

**Theorem 3.2.1.** *Let $f, g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous functions that verify conditions Theorem 3.1.5. If there are $\mu_1, \lambda_1 \in \mathbb{R}$, $p > 0$ and $q > 0$ that satisfy*

$$\frac{f(t,0,0,0)}{v_1(t)} < \mu_1 < \frac{f(t, y_1, y_2, 0)}{v_1(t)}, \tag{3.17}$$

*for every $t \in [0,1]$, $y_1 \leq -p$ and $y_2 \in \mathbb{R}$,*

$$\frac{g(t,0,0,0)}{v_2(t)} < \lambda_1 < \frac{g(t, y_1, y_2, 0)}{v_2(t)}, \tag{3.18}$$

*for every $t \in [0,1]$, $y_1 \in \mathbb{R}$ and $y_2 \leq -q$, then there exist $\mu_0 < \mu_1$ and $\lambda_0 < \lambda_1$ (with possibility of $\mu_0 = -\infty$ and $\lambda_0 = -\infty$) such that:*

1. if $\mu < \mu_0$ or $\lambda < \lambda_0$, (3.1), (3.2) there is no solution;
2. if $\mu_0 < \mu \leq \mu_1$ and $\lambda_0 < \lambda \leq \lambda_1$, (3.1), (3.2) there is at least one solution.

*Proof.* **Claim 1**: *There exists $\bar{\mu} < \mu_1$ and $\lambda^* < \lambda_1$ such that (3.1), (3.2) has a solution for $\mu = \bar{\mu}$ and $\lambda = \lambda^*$.*

Defining

$$\bar{\mu} := \max_{t\in[0,1]} \left\{ \frac{f(t,0,0,0)}{v_1(t)} \right\} \text{ and } \lambda^* := \max_{t\in[0,1]} \left\{ \frac{g(t,0,0,0)}{v_2(t)} \right\},$$

there are $\bar{t}, t^* \in [0,1]$ such that

$$\frac{f(t,0,0,0)}{v_1(t)} \leq \bar{\mu} = \frac{f(\bar{t},0,0,0)}{v_1(\bar{t})} < \mu_1,$$

and

$$\frac{g(t,0,0,0)}{v_2(t)} \leq \lambda^* = \frac{g(t^*,0,0,0)}{v_2(t^*)} < \lambda_1,$$

for all $t \in [0,1]$.

Then the functions $\phi_1(t) \equiv 0$ and $\phi_2(t) \equiv 0$ are upper solutions of problem (3.1), (3.2) for $\mu = \bar{\mu}$ and $\lambda = \lambda^*$. On the other hand, $\gamma_1(t) = -p$ and $\gamma_2(t) = -q$ are lower solution of problem (3.1), (3.2) for $\mu = \bar{\mu}$ and $\lambda = \lambda^*$, since, by (3.17), and the boundary conditions

- $\gamma_1''(t) = 0 > \mu_1\ v_1(t) - f(t,-p,-q,0) > \bar{\mu}\ v_1(t) - f(t,-p,-q,0);$
- $\gamma_2''(t) = 0 > \lambda_1\ v_2(t) - g(t,-p,-q,0) > \lambda^*\ v_2(t) - g(t,-p,-q,0);$
- $-a_1 p \leq 0$ and $-c_1 p \leq 0;$
- $-a_2 q \leq 0$ and $-c_2 q \leq 0.$

As $f$ and $g$ satisfy Nagumo conditions on the set

$$S_1 = \{(t, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^3 : -p \leq y_1 \leq 0, \ \ -q \leq y_2 \leq 0\},$$

then, by Theorem 3.1.5, there exists at least one solution of problem (3.1), (3.2) for $\mu = \bar{\mu} < \mu_1$ and $\lambda = \lambda^* < \lambda_1$.

**Claim 2**: *If (3.1), (3.2) has a solution for $\mu = \sigma < \mu_1$ and $\lambda = \rho < \lambda_1$, then it has a solution for $\mu \in [\sigma, \mu_1]$ and $\lambda \in [\rho, \lambda_1]$.*

Let $(u_{1\sigma}(t), u_{2\rho}(t))$ be a solution of the problem (3.1), (3.2) for $\mu = \sigma < \mu_1$ and $\lambda = \rho < \lambda_1$, that is

$$\begin{cases} u_{1\sigma}''(t) + f(t, u_{1\sigma}(t), u_{2\rho}(t), u_{1\sigma}'(t)) = \sigma\ v_1(t) \\ \\ u_{2\rho}''(t) + g(t, u_{1\sigma}(t), u_{2\rho}(t), u_{2\rho}'(t)) = \rho\ v_2(t). \end{cases}$$

The pair of functions $(u_{1\sigma}(t), u_{2\rho}(t))$ is an upper solution of (3.1), (3.2), for the values of $\mu$ and $\lambda$ such that $\sigma \leq \mu \leq \mu_1$ and $\rho \leq \lambda \leq \lambda_1$, since

$$u_{1\sigma}''(t) = \sigma\ v_1(t) - f(t, u_{1\sigma}(t), u_{2\rho}(t), u_{1\sigma}'(t)) \leq \mu\ v_1(t) - f(t, u_{1\sigma}(t), u_{2\rho}(t), u_{1\sigma}'(t))$$

and

$$u_{2\rho}''(t) = \rho\ v_2(t) - g(t, u_{1\sigma}(t), u_{2\rho}(t), u_{2\rho}'(t)) \le \lambda\ v_2(t) - g(t, u_{1\sigma}(t), u_{2\rho}(t), u_{2\rho}'(t)).$$

while the boundary conditions are trivially checked.

For $p > 0$ and $q > 0$, defined in (3.23) and (3.24) consider $P > 0$ and $Q > 0$ large enough, such that

$$P \ge p,\ Q \ge q,\ u_{1\sigma}(0) \ge -P,\ u_{1\sigma}(1) \ge -P,\ u_{2\rho}(0) \ge -Q \text{ and } u_{2\rho}(1) \ge -Q. \quad (3.19)$$

Then $(-P, -Q)$ is lower solution of problem (3.1), (3.2) for $\mu \le \mu_1$ and $\lambda \le \lambda_1$, since, by (3.17) and the boundary conditions

$$\begin{aligned}
&\bullet\quad 0 > \mu_1\ v_1(t) - f(t, -P, -Q, 0) \ge \mu\ v_1(t) - f(t, -P, -Q, 0);\\
&\bullet\quad 0 > \lambda_1\ v_2(t) - g(t, -P, -Q, 0) \ge \lambda\ v_2(t) - g(t, -P, -Q, 0);\\
&\bullet\quad -a_1 P \le 0 \text{ and } -c_1 P \le 0;\\
&\bullet\quad -a_2 Q \le 0 \text{ and } -c_2 Q \le 0.
\end{aligned} \quad (3.20)$$

To apply Theorem 3.1.5, it remains to justify that

$$-P \le u_{1\sigma}(t) \text{ and } -Q \le u_{2\rho}(t), \quad \forall t \in [0,1].$$

Assume, by contradiction, that the first inequality is not verified. Then there exists $t \in [0,1]$ such that $u_{1\sigma}(t) < -P$ and define

$$\min_{t\in[0,1]} u_{1\sigma}(t) := u_{1\sigma}(t_0) < -P.$$

By (3.19), $u_{1\sigma}'(t_0) = 0$, $u_{1\sigma}''(t_0) \ge 0$ and, by (3.17), we obtain the contradiction

$$\begin{aligned}
0 &\le u_{1\sigma}''(t_0) = \sigma\ v_1(t_0) - f(t_0, u_{1\sigma}(t_0), u_{2\rho}(t_0), 0)\\
&\le \mu\ v_1(t_0) - f(t_0, u_{1\sigma}(t_0), u_{2\rho}(t_0), 0)\\
&\le \mu_1\ v_1(t_0) - f(t_0, u_{1\sigma}(t_0), u_{2\rho}(t_0), 0) < 0.
\end{aligned}$$

Then $-P \le u_{1\sigma}(t)$, for all $t \in [0,1]$.

Using a similar method, by (3.18) it can be shown that $-Q \le u_{2\rho}(t)$, for all $t \in [0,1]$.

Therefore, by Theorem 3.1.5, there exists at least one solution $(u_1(t), u_2(t))$ of the problem (3.1), (3.2) for the values of $\mu$ and $\lambda$ such that $\mu \in [\sigma, \mu_1]$ and $\lambda \in [\rho, \lambda_1]$.

**Claim 3**: *There exist $\mu_0$ and $\lambda_0$ such that:*

*for $\mu < \mu_0$ or $\lambda < \lambda_0$, (3.1), (3.2) has no solution;*

*for $\mu_0 < \mu \le \mu_1$ and $\lambda_0 < \lambda \le \lambda_1$, (3.1), (3.2) has at least one solution.*

Consider set

$$\mathcal{A} = \{(\mu, \lambda) \in \mathbb{R}^2 : (3.1),\ (3.2) \text{ has solution}\}. \quad (3.21)$$

with the order relation given by

$$(x, y) \leq (z, w) \Leftrightarrow x \leq z \wedge y \leq w.$$

The set $\mathcal{A}$ is not empty because, by Claim 1, $(\bar{\mu}, \lambda^*) \in \mathcal{A}$, and define

$$(\mu_0, \lambda_0) := \inf \mathcal{A}. \tag{3.22}$$

If (3.1), (3.2) has a solution for all $\mu < \mu_1$ and $\lambda < \lambda_1$, then $\mu_0 = -\infty$ and $\lambda_0 = -\infty$.

By Claim 1 and (3.22), $\mu_0 \leq \bar{\mu} < \mu_1$ and $\lambda_0 \leq \lambda^* < \lambda_1$. By Claim 2, (3.1), (3.2) has at least one solution for the values of $\mu$ and $\lambda$ such that $\mu_0 < \mu \leq \mu_1$ and $\lambda_0 < \lambda \leq \lambda_1$. □

Replacing, in conditions (3.17), (3.18), $f$, $g$, $y_1$ and $y_2$ by $-f$, $-g$, $-y_1$ and $-y_2$, respectively, we obtain a dual version of the previous theorem, whose proof follows the same type of arguments:

**Theorem 3.2.2.** *Let $f, g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous functions verifying the assumptions of Theorem 3.1.5.*
*If there are $\mu_1, \lambda_1 \in \mathbb{R}$, $p > 0$ and $q > 0$ that satisfy*

$$\frac{f(t,0,0,0)}{v_1(t)} > \mu_1 > \frac{f(t,y_1,y_2,0)}{v_1(t)}, \tag{3.23}$$

*for every $t \in [0,1]$, $y_1 \geq p$ and $y_2 \in \mathbb{R}$,*

$$\frac{g(t,0,0,0)}{v_2(t)} > \lambda_1 > \frac{g(t,y_1,y_2,0)}{v_2(t)}, \tag{3.24}$$

*for every $t \in [0,1]$, $y_1 \in \mathbb{R}$ and $y_2 \geq q$, then there exists $\mu_0 > \mu_1$ and $\lambda_0 > \lambda_1$ (with possibility $\mu_0 = +\infty$ and $\lambda_0 = +\infty$) such that:*
*1. if $\mu > \mu_0$ or $\lambda > \lambda_0$, (3.1), (3.2) has no solution;*
*2. if $\mu_0 > \mu \geq \mu_1$ and $\lambda_0 > \lambda \geq \lambda_1$, (3.1), (3.2) has at least one solution.*

## 3.3 Multiplicity of Solution

The multiplicity result is obtained for a particular case of problem (3.1), (3.2): a standard system where the differential equations are independent, that is, with $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$.

The arguments are based on the topological Leray-Schauder degree, along with strict lower and upper solutions, as in the next definition:

**Definition 3.3.1.** *i. A pair of functions $(\gamma_1, \gamma_2) \in (C^2(]0,1[) \cap C^1([0,1]))^2$ is a strict lower solution of problem (3.3), (3.4) if, for all $t \in [0,1]$,*

$$\begin{cases} \gamma_1''(t) + f(t, \gamma_1(t), \gamma_1'(t)) > \mu\ v_1(t), \\ \\ \gamma_2''(t) + g(t, \gamma_2(t), \gamma_2'(t)) > \lambda\ v_2(t), \end{cases} \tag{3.25}$$

*and*

$$\gamma_i(0) < 0,\ \ \gamma_i(1) < 0,\ \ \textit{for } i = 1, 2. \tag{3.26}$$

*ii. A pair of functions* $(\phi_1, \phi_2) \in (C^2(]0,1[) \cap C^1([0,1]))^2$ *is a strict upper solution of problem (3.3), (3.4) if, for all* $t \in [0,1]$,

$$\begin{cases} \phi_1''(t) + f(t, \phi_1(t), \phi_1'(t)) < \mu\ v_1(t), \\ \\ \phi_2''(t) + g(t, \phi_2(t), \phi_2'(t)) < \lambda\ v_2(t), \end{cases} \tag{3.27}$$

*and*

$$\phi_i(0) > 0,\ \ \phi_i(1) > 0,\ \ \textit{for } i = 1, 2. \tag{3.28}$$

For the functional framework, define the operators

$$\mathcal{L} : (C^2([0,1]))^2 \to (C([0,1]))^2 \times \mathbb{R}^4$$

given by

$$\mathcal{L}(u_1, u_2) = (u_1'', u_2'', u_1(0), u_1(1), u_2(0), u_2(1)) \tag{3.29}$$

and

$$\mathcal{N}_{(\mu,\lambda)} : (C^1([0,1]))^2 \to (C([0,1]))^2 \times \mathbb{R}^4$$

given by

$$\mathcal{N}_{(\mu,\lambda)}(u_1, u_2) = (X, Y, 0, 0, 0, 0),$$

being

$$X := -\theta\ f(t, \delta_1(t, u_1(t)), {u_1}'(t)) + u_1(t) + \theta\ [\mu\ v_1(t) - \delta_1(t, u_1(t))],$$

and

$$Y := -\vartheta\ g(t, \delta_2(t, u_2(t)), {u_2}'(t)) + u_2(t) + \vartheta\ [\lambda\ v_2(t) - \delta_2(t, u_2(t))].$$

Since $\mathcal{L}$ is invertible, we can define the completely continuous operator

$$\mathcal{T} : (C^2([0,1]))^2 \to (C([0,1]))^2$$

given by

$$\mathcal{T}_{(\theta,\vartheta)}(u_1, u_2) = \mathcal{L}^{-1}\mathcal{N}_{(\mu,\lambda)}(u_1, u_2).$$

Clearly, the operator $\mathcal{T}$ is compact, and the following lemma allows to evaluate the topological degree, $d(\mathcal{I} - \mathcal{T}, \Omega, (0,0))$:

**Lemma 3.3.2.** *Assume that there are strict lower and upper solutions of (3.3), (3.4),* $\gamma_i(t)$ *and* $\phi_i(t)$*, respectively, with*

$$\gamma_i(t) < \phi_i(t),\ \ i = 1, 2,\ \ \forall t \in [0,1],$$

*where the continuous functions* $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *verify the Nagumo conditions as in Definition 3.1.1, relative to the intervals* $[\gamma_1(t), \phi_1(t)]$ *and* $[\gamma_2(t), \phi_2(t)]$.
*Then, there is* $M > 0$ *such that, for*

$$\Omega = \left\{(u_1, u_2) \in (C^2([0,1]))^2 : \gamma_i(t) < u_i(t) < \phi_i(t), \|u_i'\| < M, i = 1, 2\right\}$$

*we have*

$$d(\mathcal{I} - \mathcal{T}_{(1,1)}, \Omega, (0,0)) = \pm 1.$$

**Remark 3.3.3.** *By Remark 3.1.3, it is possible to consider the same set $\Omega$ for all equations (3.3), regardless of $\mu$ and $\lambda$, provided that $\gamma_i(t)$ and $\phi_i(t)$ are strict lower and upper solutions of (3.3), (3.4) and $(\mu, \lambda)$ belongs to a bounded set.*

*Proof.* Consider the truncature functions $\delta_i : [0,1] \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$,

$$\delta_i(t, y_i) := \begin{cases} \phi_i(t) & \text{if} \quad y_i > \phi_i(t) \\ y_i & \text{if} \quad \gamma_i(t) \leq y_i \leq \phi_i(t) \\ \gamma_i(t) & \text{if} \quad y_i < \gamma_i(t). \end{cases} \tag{3.30}$$

For $\theta, \vartheta \in [0,1]$, consider the homotopic, truncated and perturbed problem composed by the system

$$\begin{cases} u_1''(t) + \theta \ f(t, \delta_1(t, u_1(t)), u_1'(t)) = u_1(t) + \theta \ [\mu \ v_1(t) - \delta_1(t, u_1(t))] \\ \\ u_2''(t) + \vartheta \ g(t, \delta_2(t, u_2(t)), u_2'(t)) = u_2(t) + \vartheta \ [\lambda \ v_2(t) - \delta_2(t, u_2(t))], \end{cases} \tag{3.31}$$

and the boundary conditions (3.4).

With these definitions, problem (3.31), (3.4) is equivalent to the operator equation

$$\mathcal{T}_{(\theta,\vartheta)}(u_1, u_2) = (u_1, u_2). \tag{3.32}$$

For $i = 1, 2$, take $R_i > 0$ such that, for every $t \in [0,1]$,

$$\begin{gathered} -R_i < \gamma_i(t) \leq \phi_i(t) < R_i, \\ \mu \ v_1(t) - f(t, \gamma_1(t), 0) - R_1 - \gamma_1(t) < 0, \\ \mu \ v_1(t) - f(t, \phi_1(t), 0) + R_1 - \phi_1(t) > 0, \\ \lambda \ v_2(t) - g(t, \gamma_2(t), 0) - R_2 - \gamma_2(t) < 0, \\ \lambda \ v_2(t) - g(t, \phi_2(t), 0) + R_2 - \phi_2(t) > 0. \end{gathered} \tag{3.33}$$

By Lemma 3.1.2, and applying the technique suggested in the proof of Theorem 3.1.5 (see [69], Theorem 3.1), adapted to strict lower and upper solutions $\gamma_i(t)$ and $\phi_i(t)$, there are positive real numbers $M_i$, $i = 1, 2$, such that

$$\|u_1'\| < M_1 \text{ and } \|u_2'\| < M_2,$$

independently of the parameters $\theta$ and $\vartheta$.

Defining

$$\Omega_1 = \{(u_1, u_2) \in (C^2([0,1]))^2 : \|u_i\| < R_i, \|u_i'\| < M_i, i = 1, 2\},$$

then every solution of (3.32) belongs to $\Omega_1$, for all $(\theta, \vartheta) \in [0,1]^2$, $(u_1, u_2) \notin \partial\Omega_1$ and, so, the degree $d(\mathcal{I} - \mathcal{T}_{(\theta,\vartheta)}, \Omega_1, (0,0))$ is well defined for every $(\theta, \vartheta) \in [0,1]^2$.

For $(\theta, \vartheta) = (0, 0)$, the equation $\mathcal{T}_{(0,0)}(u_1, u_2) = (u_1, u_2)$, that is, the homogeneous linear problems

$$\begin{cases} u_i''(t) - u_i(t) = 0 \\ u_i(0) = 0 \\ u_i(1) = 0, \ \text{for } i = 1, 2, \end{cases}$$

admits only the null solution, then, by degree theory, $d(\mathcal{I}-\mathcal{T}_{(0,0)},\Omega_1,(0,0))=\pm 1$, and by the homotopy invariance

$$\pm 1 = d(\mathcal{I}-\mathcal{T}_{(0,0)},\Omega_1,(0,0)) = d(\mathcal{I}-\mathcal{T}_{(1,1)},\Omega_1,(0,0)). \tag{3.34}$$

Therefore, problem (3.31), (3.4) has, at least, a solution $(\widetilde{u}_1,\widetilde{u}_2)$ for $(\theta,\vartheta)=(1,1)$.

Let us prove that $(\widetilde{u}_1,\widetilde{u}_2)\in\Omega$. Assume, by contradiction, that there is $t\in[0,1]$ such that $\gamma_1(t)\geq\widetilde{u}_1(t)$ and define

$$\max_{t\in[0,1]}(\gamma_1(t)-\widetilde{u}_1(t)) := \gamma_1(t_0)-\widetilde{u}_1(t_0)\geq 0.$$

By (3.4) and (3.26), $t_0\in]0,1[$, $\gamma_1'(t_0)=\widetilde{u}_1'(t_0)$, and $\gamma_1''(t_0)-\widetilde{u}_1''(t_0)\leq 0$. Therefore,by (3.15), we have the contradiction

$$\begin{aligned}\gamma_1''(t_0) &\leq \widetilde{u}_1''(t_0) = -f(t_0,\delta_1(t_0,\widetilde{u}_1(t_0)),\widetilde{u}_1{}'(t_0))+\widetilde{u}_1(t_0)+\mu v_1(t_0)-\delta_1(t_0,\widetilde{u}_1(t_0))\\ &= -f(t_0,\gamma_1(t_0),\gamma_1'(t_0))+\widetilde{u}_1(t_0)-\gamma_1(t_0)+\mu v_1(t_0)\\ &\leq -f(t_0,\gamma_1(t_0),\gamma_1'(t_0))+\mu v_1(t_0)<\gamma_1''(t_0).\end{aligned}$$

So, $\gamma_1(t)<\widetilde{u}_1(t)$, for all $t\in[0,1]$.

As the other inequalities can be obtained by similar arguments, we have

$$\gamma_1(t)<\widetilde{u}_1(t)<\phi_1(t),\ \gamma_2(t)<\widetilde{u}_2(t)<\phi_2(t),\ \forall t\in[0,1],$$

and, therefore, $(\widetilde{u}_1,\widetilde{u}_2)\in\Omega$ .

For

$$M := \max_{i=1,2}\left\{\|\gamma_i\|_{C^1},\|\phi_i\|_{C^1},M_i\right\}, \tag{3.35}$$

$\Omega\subset\Omega_1$, and, by (3.34) and the excision property of the degree,

$$\pm 1 = d(\mathcal{I}-\mathcal{T}_{(1,1)},\Omega_1,(0,0)) = d(\mathcal{I}-\mathcal{T}_{(1,1)},\Omega,(0,0)).$$

□

**Remark 3.3.4.** *Remark that, from (3.30), if $(u_1,u_2)\in\Omega$ is a solution of problem (3.31), (3.4), then is a solution of (3.3), (3.4), too.*

The multiplicity result requires extra assumptions on the nonlinearities:

**Theorem 3.3.5.** *Let $f,g:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be continuous functions such that there are $\mu_1,\lambda_1\in\mathbb{R}$, $p>0$ and $q>0$ that satisfy*

$$\frac{f(t,0,0)}{v_1(t)}<\mu_1<\frac{f(t,y_1,0)}{v_1(t)}, \tag{3.36}$$

*for every $t\in[0,1]$, $y_1\leq -p$,*

$$\frac{g(t,0,0)}{v_2(t)}<\lambda_1<\frac{g(t,y_1,0)}{v_2(t)}, \tag{3.37}$$

*for every* $t \in [0,1]$, $y_2 \leq -q$,

$$f(t, y_1, y_2) \text{ and } g(t, y_1, y_2) \text{ are nonincreasing on } y_1, \tag{3.38}$$

*for all* $(t, y_2) \in [0,1] \times \mathbb{R}$.
*Assume that there are* $k_i \in \mathbb{R}$, $i = 1, 2$, *with* $k_1 \geq -p$ *and* $k_2 \geq -q$, *such that, every solution* $(u_1(t), u_2(t))$ *of (3.3), (3.4), with* $\mu_0 < \mu_1$ *and* $\lambda_0 < \lambda_1$, *satisfies*

$$u_i(t) < k_i, \quad i = 1, 2, \quad \forall \in [0,1], \tag{3.39}$$

*and there exist* $m_i \in \mathbb{R}$, $i = 1, 2$, *such that*

$$f(t, y_1, y_2) \geq m_1 v_1(t) \tag{3.40}$$

*for* $(t, y_1, y_2) \in [0,1] \times [-p, k_1] \times \mathbb{R}$ *and*

$$g(t, y_1, y_2) \geq m_2 v_2(t) \tag{3.41}$$

*for* $(t, y_1, y_2) \in [0,1] \times [-q, k_2] \times \mathbb{R}$.
*Then numbers* $\mu_0$ *and* $\lambda_0$, *given by Theorem 3.2.1, are finite and:*
*1. if* $\mu < \mu_0$ *or* $\lambda < \lambda_0$, *(3.3), (3.4 ) there is no solution;*
*2. if* $\mu = \mu_0$ *and* $\lambda = \lambda_0$, *(3.3), (3.4 ) there is at least one solution;*
*3.if* $\mu_0 < \mu \leq \mu_1$ *and* $\lambda_0 < \lambda \leq \lambda_1$, *(3.3), (3.4) there are at least two solutions.*

*Proof.* **Claim 1**: *Every solution* $(u_1(t), u_2(t))$ *of the problem (3.3), (3.4), for* $(\mu, \lambda) \in ]\mu_0, \mu_1] \times ]\lambda_0, \lambda_1]$, *verifies*

$$-p < u_1(t) < k_1 \quad \text{and} \quad -q < u_2(t) < k_2, \quad \forall t \in [0,1].$$

By (3.39), it will suffice to prove that any solution $(u_1(t), u_2(t))$ of (3.3), (3.4), with $(\mu, \lambda) \in ]\mu_0, \mu_1] \times ]\lambda_0, \lambda_1]$, satisfies

$$-p < u_1(t) \quad \text{and} \quad -q < u_2(t), \quad \forall t \in [0,1].$$

Assume, by contradiction, that there is $\mu \in ]\mu_0, \mu_1]$, such that $u_1(t) \leq -p$, and define

$$\min_{t \in [0,1]} u_1(t) := u_1(t_1) \leq -p < 0.$$

By (3.4), $t_1 \in ]0,1[$, and, therefore,

$$u_1'(t_1) = 0, \quad u_1''(t) \geq 0.$$

By (3.36), the following contradiction holds

$$\begin{aligned} 0 &\leq u_1''(t_1) = \mu v_1(t_1) - f(t_1, u_1(t_1), u_1'(t_1)) \\ &\leq \mu_1 v_1(t_1) - f(t_1, u_1(t_1), 0) < 0. \end{aligned}$$

Therefore

$$-p < u_1(t) < k_1, \quad \forall t \in [0,1],$$

and, by (3.37) and similar arguments, it can be proved that

$$-q < u_2(t) < k_2, \quad \forall t \in [0,1].$$

**Claim 2**: *The numbers $\mu_0$ and $\lambda_0$ are finite.*

If, by contradiction, $\mu_0 = -\infty$ and $\lambda_0 = -\infty$, then, by Theorem 3.2.1, problem (3.3), (3.4) has a solution for any values of $\mu$ and $\lambda$ such that $\mu \leq \mu_1$ and $\lambda \leq \lambda_1$.

Let $(u_1(t), u_2(t))$ be a solution of (3.3), (3.4), for $\mu \leq \mu_1$ and $\lambda \leq \lambda_1$.

Then, by (3.41), we have

$$u_1''(t) = \mu v_1(t) - f(t, u_1(t), u_1'(t)) \leq \mu v_1(t) - m_1 v_1(t) = (\mu - m_1) v_1(t). \tag{3.42}$$

Define

$$v_{10} := \min_{t\in[0,1]} v_1(t) > 0,$$

and consider $\mu$ small enough, such that

$$m_1 - \mu > 0 \quad \text{and} \quad \frac{(m_1-\mu)v_{10}}{16} > k_1.$$

By (3.4), there is $t_2 \in ]0,1[$ such that $u_1'(t_2) = 0$.

For $t < t_2$ by (3.42),

$$u_1'(t) = -\int_t^{t_2} u_1''(\zeta)d\zeta \geq \int_t^{t_2} (m_1-\mu)v_1(\zeta)d\zeta \geq (m_1-\mu)(t_2-t)v_{10}.$$

For $t \geq t_2$

$$u_1'(t) = \int_{t_2}^{t} u_1''(\zeta)d\zeta \leq (\mu - m_1)(t-t_2)v_{10}.$$

Choose $I = [0, \frac{1}{4}]$, or $I = [\frac{3}{4}, 1]$, such that $|t_2 - t| \geq \frac{1}{4}$, for $t \in I$.

In the first case,

$$u_1'(t) \geq \frac{(m_1-\mu)v_{10}}{4}, \quad \forall t \in I,$$

and the following contradiction with (3.39) holds:

$$\begin{aligned} 0 &= \int_0^1 u_1'(t)dt = \int_0^{\frac{1}{4}} u_1'(t)dt + \int_{\frac{1}{4}}^1 u_1'(t)dt \\ &\geq \int_0^{\frac{1}{4}} \frac{(m_1-\mu)v_{10}}{4}dt - u_1\left(\frac{1}{4}\right) \\ &= \frac{(m_1-\mu)v_{10}}{16} - u_1\left(\frac{1}{4}\right) > k_1 - u_1\left(\frac{1}{4}\right). \end{aligned}$$

If $I = [\frac{3}{4}, 1]$, then

$$u_1'(t) \leq \frac{(\mu - m_1)v_{10}}{4}, \quad \forall t \in I,$$

and, following the same technique, an analogous contradiction is obtained. So $\mu_0$ is finite.

Analogously, it can be shown that $\lambda_0$ is finite.

**Claim 3**: *For* $(\mu, \lambda) \in ]\mu_0, \mu_1] \times ]\lambda_0, \lambda_1]$ *there is a second solution of (3.3), (3.4).*

As both $\mu_0$ and $\lambda_0$ are finite, by Theorem 3.2.1, there exist $\mu_{-1} < \mu_0$ or $\lambda_{-1} < \lambda_0$ such that (3.3), (3.4) has no solution for $\mu = \mu_{-1}$ or $\lambda = \lambda_{-1}$.

In the first case, by Lemma 3.1.2 and Remark 3.1.3, it is possible to consider $\rho > 0$, large enough, such that the estimation $\|u_i'\| < \rho$, $i = 1, 2$, holds for every solution $(u_1(t), u_2(t))$ of (3.3), (3.4), with $\mu \in [\mu_{-1}, \mu_1]$ or $\lambda \in [\lambda_{-1}, \lambda_1]$.

Consider

$$M_* := \max\{p, q, |k_i|, i = 1, 2\}, \tag{3.43}$$

and the set

$$\Omega_* = \{(u_1, u_2) \in (C^2([0,1]))^2 : \|u_i\| < M_*, \|u_i'\| < \rho, i = 1, 2\}. \tag{3.44}$$

Together with the linear operator $\mathcal{L}$, given by (3.29), define the nonlinear operator

$$\mathcal{N}^*_{(\mu,\lambda)} : (C^1([0,1]))^2 \to (C([0,1]))^2 \times \mathbb{R}^4$$

by

$$\mathcal{N}^*_{(\mu,\lambda)}(u_1, u_2) = \begin{pmatrix} \mu v_1(t) - f(t, u_1(t), u_1{}'(t)), \\ \lambda v_2(t) - g(t, u_2(t), u_2{}'(t)), \\ 0, 0, 0, 0 \end{pmatrix},$$

and the completely continuous operator

$$\mathcal{T}^* : (C^2([0,1]))^2 \to (C([0,1]))^2$$

given by

$$\mathcal{T}^*_{(\mu,\lambda)}(u_1, u_2) = \mathcal{L}^{-1}\mathcal{N}_{(\mu,\lambda)}(u_1, u_2).$$

By the definition of $\Omega_*$ and Claim 1, the degree $d(\mathcal{I} - \mathcal{T}^*_{(\mu,\lambda)}, \Omega^*, (0,0))$ is well defined for every $(\mu, \lambda) \in [\mu_{-1}, \mu_1] \times [\lambda_0, \lambda_{-1}]$, and, by degree theory,

$$d(\mathcal{I} - \mathcal{T}^*_{(\mu_{-1},\lambda_{-1})}, \Omega^*, (0,0)) = 0.$$

Therefore, for the homotopy $H : [0,1] \to \mathbb{R}^2$ on the parameters $(\mu, \lambda)$, given by

$$H(s) = ((1-s)\,\mu_{-1} + s\mu_1, (1-s)\,\lambda_{-1} + s\lambda_1),$$

it is clear that the degree $d(\mathcal{I} - \mathcal{T}^*_{H(s)}, \Omega^*, (0,0))$ is well defined for every $s \in [0,1]$, and $(\mu, \lambda) \in [\mu_{-1}, \mu_1] \times [\lambda_{-1}, \lambda_1]$.

By the invariance under homotopy,

$$0 = d(\mathcal{I} - \mathcal{T}^*_{(\mu_{-1},\lambda_{-1})}, \Omega^*, (0,0)) = d(\mathcal{I} - \mathcal{T}^*_{(\mu,\lambda)}, \Omega^*, (0,0)), \tag{3.45}$$

for $(\mu, \lambda) \in [\mu_{-1}, \mu_1] \times [\lambda_{-1}, \lambda_1]$.

Take $(\mu^*, \lambda^*) \in ]\mu_0, \mu_1] \times ]\lambda_0, \lambda_1] \subset [\mu_{-1}, \mu_1] \times [\lambda_{-1}, \lambda_1]$ and let $(u_1^*(t), u_2^*(t))$ be a solution of (3.3)-(3.4) with $(\mu, \lambda) = (\mu^*, \lambda^*)$, which exists by Theorem 3.2.1.

By Claim 1 and (3.43) it is possible to consider $\varepsilon_i > 0$, $i = 1, 2$, such that

$$|u_i^*(t) + \varepsilon_i| < M^*, \ i = 1, 2, \ \text{for } t\,[0, 1]\,. \tag{3.46}$$

For the functions given by

$$\widetilde{u}_1(t) := u_1^*(t) + \varepsilon_1, \ \widetilde{u}_2(t) := u_2^*(t) + \varepsilon_2,$$

the pair $(\widetilde{u}_1(t), \widetilde{u}_2(t))$ is a strict upper solution of (3.3), (3.4), for $\mu^* < \mu \leq \mu_1$ and $\lambda^* < \lambda \leq \lambda_1$, as we have:

$$\begin{aligned} \widetilde{u}_1''(t) &= u_1''^*(t) = \mu^* v_1(t) - f(t, u_1^*(t), u_1^{*\prime}(t)) \\ &< \mu v_1(t) - f(t, u_1^*(t), \widetilde{u}_1'(t)) \\ &\leq \mu v_1(t) - f(t, u_1^*(t) + \varepsilon_1, \widetilde{u}_1'(t)) \\ &= \mu v_1(t) - f(t, \widetilde{u}_1(t), \widetilde{u}_1'(t)). \end{aligned}$$

Analogously it can be proved that

$$\widetilde{u}_2''(t) = u_2''^*(t) < \lambda v_2(t) - g(t, \widetilde{u}_2(t), \widetilde{u}_2'(t)).$$

Moreover, the pair $(-p, -q)$ is a strict lower solution of (3.3), (3.4), for $\mu \leq \mu_1$ and $\lambda \leq \lambda_1$, as, by (3.36) and (3.37),

$$\begin{aligned} 0 &> \mu_1\, v_1(t) - f(t, -p, 0) \geq \mu\, v_1(t) - f(t, -p, 0), \\ 0 &> \lambda_1\, v_2(t) - g(t, -q, 0) \geq \lambda\, v_2(t) - g(t, -q, 0). \end{aligned}$$

By Claim 1,

$$-p < u_1^*(t) < u_1^*(t) + \varepsilon_1 = \widetilde{u}_1(t),$$

and

$$-q < u_2^*(t) < u_2^*(t) + \varepsilon_2 = \widetilde{u}_2(t), \quad \forall t \in [0, 1].$$

By Lemma 3.1.2 and Remark 3.3.3, there is $\rho_0 > 0$,independent of $\mu$ and $\lambda$, such that for the set

$$\Omega_\varepsilon = \left\{ \begin{array}{c} (u_1, u_2) \in (C^2([0, 1]))^2 : -p < u_1(t) < \widetilde{u}_1(t), \\ -q < u_2(t) < \widetilde{u}_2(t), \ \|u_i'\| < \rho_0, \ i = 1, 2 \end{array} \right\}$$

the degree

$$d(\mathcal{I} - \mathcal{T}^*_{(\mu,\lambda)}, \Omega_\varepsilon, (0, 0)) = \pm 1, \ \text{for } (\mu, \lambda) \in ]\mu_0, \mu_1] \times ]\lambda_0, \lambda_1]. \tag{3.47}$$

Assuming, in (3.44), $\rho > 0$ large enough such that, by (3.46), $\Omega_\varepsilon \subset \Omega^*$, then, by (3.45), (3.47) and the additivity property of the degree,

$$d(\mathcal{I} - \mathcal{T}^*_{(\mu,\lambda)}, \Omega^* - \overline{\Omega}_\varepsilon, (0, 0)) = \mp 1, \ \text{for } (\mu, \lambda) \in ]\mu_0, \mu_1] \times ]\lambda_0, \lambda_1].$$

Then, for $(\mu, \lambda) \in ]\mu_0, \mu_1] \times ]\lambda_0, \lambda_1]$, the problem (3.3), (3.4) has, at least two solutions: a solution in $\Omega_\varepsilon$ and other one in $\Omega^* - \overline{\Omega}_\varepsilon$, since $(\mu, \lambda)$ is arbitrary in $]\mu_0, \mu_1] \times ]\lambda_0, \lambda_1]$.

**Claim 4**: *For $(\mu, \lambda) = (\mu_0, \lambda_0)$ the problem (3.3)-(3.4) has, at least, a solution.*

Consider the sequence $(\mu_n, \lambda_n)$ such that $(\mu_n, \lambda_n) \in ]\mu_0, \mu_1] \times ]\lambda_0, \lambda_1]$, $\lim \mu_n = \mu_0$, and $\lim \lambda_n = \lambda_0$.

By Theorem 3.2.1, for each $(\mu_n, \lambda_n)$, the problem *(3.3)-(3.4)* has, at least, a solution $(u_{1n}(t), u_{2n}(t))$.

From the estimations given in Claim 1, and (3.43), $\|(u_{1n}, u_{2n})\| < M^*$, and by Lemma 3.1.2, there $\overline{\rho} > 0$ sufficiently large, such that

$$\|(u'_{1n}, u'_{2n})\| < \overline{\rho},$$

independently of $n$.Then the sequence $(u''_{1n}, u''_{2n})$ is bounded in $C([0,1])$, and,by the Arzèla-Ascoli theorem, there is a subsequence of $(u_{1n}(t), u_{2n}(t))$ that converges in $C^2([0,1])$ to a solution $(u_1(t), u_2(t))$ of (3.3)-(3.4) for $(\mu, \lambda) = (\mu_0, \lambda_0)$. □

A dual version of Theorem 3.3.5 can be given:

**Theorem 3.3.6.** *Let $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions such that there are $\mu_1, \lambda_1 \in \mathbb{R}$, $p > 0$ and $q > 0$ that satisfy*

$$\frac{f(t,0,0)}{v_1(t)} > \mu_1 > \frac{f(t,y_1,0)}{v_1(t)}, \tag{3.48}$$

*for every $t \in [0,1]$, $y_1 \geq p$,*

$$\frac{g(t,0,0)}{v_2(t)} > \lambda_1 > \frac{g(t,y_1,0)}{v_2(t)}, \tag{3.49}$$

*for every $t \in [0,1]$, $y_1 \geq q$,*

$$f(t,y_1,y_2) \text{ and } g(t,y_1,y_2) \text{ are nonincreasing on } y_1, \tag{3.50}$$

*for all $(t, y_2) \in [0,1] \times \mathbb{R}$.*
*Assume that there are $k_i \in \mathbb{R}$, $i = 1,2$, with $k_1 \leq p$ and $k_2 \leq q$, such that, every solution $(u_1(t), u_2(t))$ of (3.3), (3.4), with $\mu_0 > \mu_1$ and $\lambda_0 > \lambda_1$, satisfies*

$$u_i(t) > k_i, \quad i = 1,2, \quad \forall \in [0,1], \tag{3.51}$$

*and there exist $m_i \in \mathbb{R}$, $i = 1,2$, such that*

$$f(t,y_1,y_2) \leq m_1 v_1(t) \tag{3.52}$$

*for $(t, y_1, y_2) \in [0,1] \times [k_1, p] \times \mathbb{R}$ and*

$$g(t,y_1,y_2) \leq m_2 v_2(t) \tag{3.53}$$

*for $(t, y_1, y_2) \in [0,1] \times [k_2, q] \times \mathbb{R}$.*
*Then numbers $\mu_0$ and $\lambda_0$, given by Theorem 3.2.2, are finite and:*
*1. if $\mu > \mu_0$ or $\lambda > \lambda_0$, (3.3), (3.4 ) there is no solution;*
*2. if $\mu = \mu_0$ and $\lambda = \lambda_0$, (3.3), (3.4 ) there is at least one solution;*
*3. if $\mu_0 > \mu \geq \mu_1$ and $\lambda_0 > \lambda \geq \lambda_1$, (3.3), (3.4) there are at least two solutions.*

## 3.4 Application in a Lotka-Volterra Steady-state System with Migration

The Lotka-Volterra equations are often used to represent interactions between species. In their original version they describe prey-predator competition models. However, there are many other types of interaction occurring between species, we can cite mutualism and neutralism as examples. The study of population dynamics between two species can be considered the most elementary way to describe inter-specific and intra-specific interaction.

In [7], the importance of including spatial dependence in the Lotka-Volterra equations is shown, since the models depend only on time, assume that the spatial distributions of populations are homogeneous, but in most biological systems this assumption is not valid.

In this paper we present a steady-state model of interactive Lotka-Volterra of two species, adapted from the works [2, 100].

Consider the system of equations

$$\begin{cases} d_1 u_1''(x) + u_1(x)\left(\eta_1 - \delta_1 u_1(x) + \psi_1 u_2(x)\right) = \bar{\mu}\, v_1(x) \\ \\ d_2 u_2''(x) + u_2(x)\left(\eta_2 + \psi_2 u_1(x) - \delta_2 u_2(x)\right) = \bar{\lambda}\, v_2(x), \quad x \in [0,1], \end{cases} \tag{3.54}$$

with boundary conditions

$$\begin{array}{rcl} a_i u_i(0) - b_i u_i'(0) & = & 0 \\ u_i'(1) & = & 0, \text{ for } i = 1, 2. \end{array} \tag{3.55}$$

where $a_i, b_i, d_i > 0$ and $\eta_i, \delta_i, \psi_i \geq 0$, for $i = 1, 2$, having the following meaning:

- $u_1$ and $u_2$ are the population density;
- first term in each equation is responsible for dispersion with species-specific diffusion ($d_i$);
- the second term corresponds to the intrinsic growth of the species, with coefficients $\eta_i$ representing the growth rate of the species;
- $\delta_i$, the intra-specific competition coefficients;
- $\psi_i$, the inter-specific interaction coefficient;
- $v_1(x)$ and $v_2(x)$ can be defined as physical and geographic conditions of the domain region favoring, or not, the development of a species;
- the parameters $\bar{\mu}$ and $\bar{\lambda}$ are the weight of attraction or repulsion of the terms $v_1(x)$ and $v_2(x)$ for the respective populations.

### 3.4.1 Interaction by mutualism

Mutualism is an example of an interspecific ecological relationship that benefits all individuals involved in the interaction. In particular, the Lotka-Volterra model of mutualism is the case where the interaction coefficients $\psi_1$ and $\psi_2$ of the problem (3.54), (3.55) are positive.

Consider a numerical example of (3.54), (3.55), with $d_1 = 0.1$, $d_2 = 0.2$, $\eta_1 = 0.3$, $\eta_2 = 0.2$, $\delta_1 = 0.5$, $\delta_2 = 0.8$, $\psi_1 = 0.2$, $\psi_2 = 0.4$, $\frac{\bar{\mu}}{d_1} = \mu$, $\frac{\bar{\lambda}}{d_2} = \lambda$, $v_1(x) = \cos^2(x)$ and $v_2(x) = e^{-x}$.

So, we have the particular problem

$$\begin{cases} u_1''(x) + u_1(x)\,(3 - 5u_1(x) + 2u_2(x)) = \mu\ \cos^2(x) \\ \\ u_2''(x) + u_2(x)\,(1 + 2u_1(x) - 4u_2(x)) = \lambda\ e^{-x}, \quad x \in [0,1] \end{cases} \tag{3.56}$$

with boundary conditions

$$\begin{array}{rcl} u_i(0) - u_i'(0) & = & 0 \\ u_i'(1) & = & 0, \ \text{for } i = 1, 2. \end{array} \tag{3.57}$$

At $x = 0$ and $x = 1$ the boundary conditions of zero density can be interpreted as an inhospitable region, which the species cannot inhabit.

The assumptions of Theorem 3.2.2 are satisfied for every $x \in [0,1]$, and, by (3.23) and (3.24), it is possible to give some estimations on the parameters $\mu_1$ and $\lambda_1$ :

$$0 > \mu_1 > p\,(3 - 5p + 2q)\,,$$

and

$$0 > \lambda_1 > q\,(1 + 2p - 4q)\,,$$

for some $p$ and $q$ such that

$$\begin{cases} p\,(3 - 5p + 2q) < 0 \\ \\ q\,(1 + 2p - 4q) < 0. \end{cases} \tag{3.58}$$

Figure 3.1 shows the region of points $(p, q)$ where condition (3.58) holds.

By Definition 3.1.4 the functions

$$\begin{array}{rcl} (\gamma_1(x), \gamma_2(x)) & \equiv & (0,0) \\ (\phi_1(x), \phi_2(x)) & \equiv & (p,q) \end{array}$$

are, respectively, lower and upper solutions of problem (3.56), (3.57) for

$$\mu \in [p\,(3 - 5p + 2q)\,, 0] \quad \text{and} \quad \lambda \in [q\,(1 + 2p - 4q)\,, 0]\,. \tag{3.59}$$

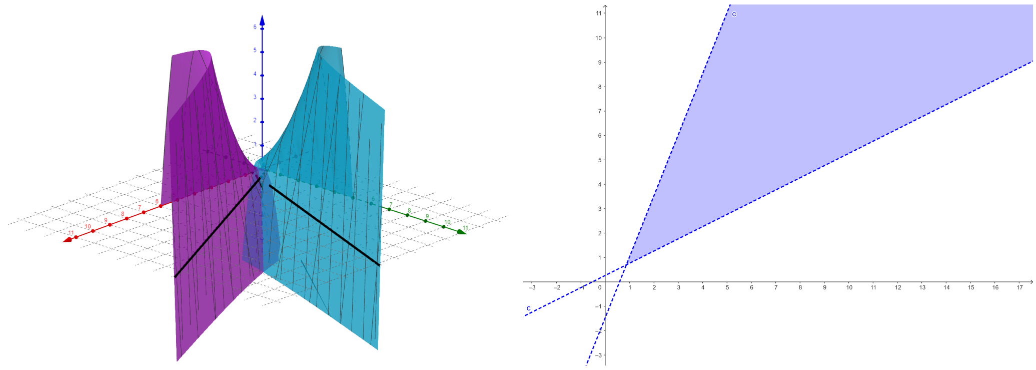


**Fig. 3.1:** Solution of system (3.58) and region of points $(p, q)$.

Moreover, the problem (3.56), (3.57) is a particular case of (3.1), (3.2), with

$$f(x, y_1, y_2, y_3) = y_1 \left(3 - 5y_1 + 2y_2\right),$$

and

$$g(x, y_1, y_2, y_3) = y_2 \left(1 + 2y_1 - 4y_2\right).$$

These functions verify the Nagumo conditions (3.6) and (3.7), relative to the intervals $y_1 \in [0, p]$ and $y_2 \in [0, q]$, as

$$\begin{aligned} |f(x, y_1, y_2, y_3)| &= |y_1 \left(3 - 5y_1 + 2y_2\right)| \\ &\leq |p \left(3 - 5p + 2q\right)| := \varphi_1(|y_3|), \end{aligned}$$

$$\begin{aligned} |g(x, y_1, y_2, y_3)| &= |y_2 \left(1 + 2y_1 - 4y_2\right)| \\ &\leq |q \left(1 + 2p - 4q\right)| := \varphi_2(|y_3|). \end{aligned}$$

and, trivially

$$\int_0^{+\infty} \frac{ds}{\varphi_i(s)} = +\infty, \text{ for } i = 1, 2.$$

So, by Theorem 3.1.5, for the values of $\mu$ and $\lambda$ verifying (3.59), the problem (3.56), (3.57) has at least one solution $(u_1(x), u_2(x))$, such that

$$0 \leq u_1(x) \leq p \quad \text{and} \quad 0 \leq u_2(x) \leq q$$

for all $x \in [0, 1]$.

Therefore, by Theorem 3.2.2 there are $\mu_0 > \mu_1$ and $\lambda_0 > \lambda_1$ such that problem (3.56) and (3.57) has no solution for $\mu > \mu_0 > 0$ or $\lambda > \lambda_0 > 0$, and has at least one solution for

$$0 > \mu \geq \mu_1 > p \left(3 - 5p + 2q\right) \quad \text{and} \quad 0 > \lambda \geq \lambda_1 > q \left(1 + 2p - 4q\right).$$

### 3.4.2 Interaction by neutralism

Neutralism is an ecological relationship in which there is no interspecific interaction and the two species evolve independently, i.e. when both interaction parameters are null.

Consider in (3.54) $\psi_1 = 0$ and $\psi_2 = 0$, and the numerical problem (3.56), (3.57), with the same values for the other parameters:

$$\begin{cases} u_1''(x) + u_1(x)\,(3 - 5u_1(x)) = \mu \cos^2(x) \\ \\ u_2''(x) + u_2(x)\,(1 - 4u_2(x)) = \lambda\, e^{-x}, \quad x \in [0,1] \end{cases} \tag{3.60}$$

with boundary conditions

$$u_i(0) \;=\; u_i(1) = 0, \text{ for } i = 1,2. \tag{3.61}$$

The assumptions (3.48) and (3.49) of Theorem 3.2.2 are satisfied for every $x \in [0,1]$, and the estimations of the parameters are given by

$$0 > \mu_1 > p\,(3 - 5p)\,, \text{ when } p > \frac{3}{5},$$

and

$$0 > \lambda_1 > q\,(1 - 4q)\,, \text{when } q > \frac{1}{4}.$$

Let $0 > \epsilon_1 > \frac{3}{5} - p$ and $0 > \epsilon_2 > \frac{1}{4} - q$ be real numbers, then the functions

$$\begin{aligned} (\gamma_1(x), \gamma_2(x)) &= (\epsilon_1, \epsilon_2) \\ (\phi_1(x), \phi_2(x)) &= (p, q) \end{aligned}$$

are, respectively, strict lower and upper solutions of problem (3.60), (3.61), according to Definition 3.3.1, for

$$\mu \in (p(3 - 5p), \epsilon_1(3 - 5\epsilon_1)) \quad \text{and} \quad \lambda \in (q(1 - 4q), \epsilon_2(1 - 4\epsilon_2))\,. \tag{3.62}$$

Moreover, the problem (3.60), (3.61) is a particular case of (3.3), (3.4), with

$$f(x, y_1, y_2) = y_1\,(3 - 5y_1)\,,$$

and

$$g(x, y_1, y_2) = y_1\,(1 - 4y_1)\,.$$

These functions verify the Nagumo conditions (3.6) and (3.7), relative to the intervals $y_1 \in [\epsilon_1, p]$ and $y_2 \in [\epsilon_2, q]$, as

$$\begin{aligned} |f(x, y_1, y_2)| &= |y_1\,(3 - 5y_1)| \\ &\leq |p\,(3 - 5p)| := \bar{\varphi}_1(|y_2|), \end{aligned}$$

$$\begin{aligned} |g(x, y_1, y_2)| &= |y_1\,(1 - 4y_1)| \\ &\leq |q\,(1 - 4q)| := \bar{\varphi}_2(|y_2|). \end{aligned}$$

and, trivially

$$\int_0^{+\infty} \frac{ds}{\bar{\varphi}_i(2)} = +\infty, \text{ for } i = 1,2.$$

So, by Theorem 3.1.5, for the values of $\mu$ and $\lambda$ verifying (3.62), the problem (3.60)-(3.61) has at least a solution $(u_1(x), u_2(x))$, such that

$$\epsilon_1 < u_1(x) < p \quad \text{and} \quad \epsilon_2 < u_2(x) < q, \tag{3.63}$$

for all $x \in [0, 1]$.

By Theorem 3.2.2 there are $\mu_0$ and $\lambda_0$, such that there is no solution if $\mu > \mu_0 = 0$ or $\lambda > \lambda_0 = 0$.

Chapter 4

# Second-order strongly nonlinear impulsive coupled systems

In this chapter, based on the paper [71], we consider the second order coupled system

$$\begin{cases} (\phi(u'(x)))' + f(x,u(x),u'(x),v(x),v'(x)) = 0, x \in M, \\ (\psi(v'(x)))' + g(x,u(x),u'(x),v(x),v'(x)) = 0, x \in N, \end{cases} \tag{4.1}$$

where $M = [a,b] \setminus \{x_1,...,x_m\}$ and $N = [a,b] \setminus \{\tau_1,...,\tau_n\}$, $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are increasing homeomorphisms such that $\phi(0) = \psi(0) = 0$ and $\phi(\mathbb{R}) = \psi(\mathbb{R}) = \mathbb{R}$, $f, g : [a,b] \times \mathbb{R}^4 \mapsto \mathbb{R}$, are $L^1$-Carathéodory functions, together with the boundary conditions

$$\begin{cases} u(a) = A, \ u'(b) = B, \\ v(a) = C, \ v'(b) = D, \ A, B, C, D \in \mathbb{R}. \end{cases} \tag{4.2}$$

The impulsive conditions are given by

$$\begin{array}{rcl} \Delta u(x_i) & = & I_i(x_i, u(x_i), u'(x_i), v(x_i)), \\ \Delta v(\tau_j) & = & J_j(\tau_j, u(\tau_j), v(\tau_j), v'(\tau_j)), \end{array} \tag{4.3}$$

being $\Delta u(x_i) = u(x_i^+) - u(x_i^-)$, $i = 1,2,...,m$, $\Delta v(\tau_j) = v(\tau_j^+) - v(\tau_j^-)$, $j = 1,2,...,n$, $I_i, J_j \in C([a,b] \times \mathbb{R}^3, \mathbb{R})$, and $x_k$ fixed points such that $a = x_0 < x_1 < x_2 < ... < x_m < x_{m+1} = b$, $a = \tau_0 < \tau_1 < \tau_2 < ... < \tau_n < \tau_{n+1} = b$.

Impulsive differential equations model many real processes in which the nonlinearities undergo periods of smooth behaviour followed by sudden discontinuous jumps in their values. These phenomena can be found in population dynamics, control and optimization theory, ecology, biology and biotechnology, economics, pharmacokinetics, and other physics and mechanics problems. For the classical approach to impulsive differential equations, we can refer, for example, [58], for a general theory; [52], applying fixed point index; [61, 78, 85], for functional impulsive problems; [108], for a monotone iterative technique for approximating the solution. The study of $\phi-$Laplacian impulsive problems can be seen, for instance in: [49], with periodic boundary conditions via a continuation theorem; [72, 73], for bounded and unbounded intervals; [101], for fractional equations with p-Laplacian.

Nonlinear coupled systems, where there are interactions between several unknown functions and their derivatives, have been studied in several works in recent years, such as: [94], applying Schauder's fixed point theorem; [35], for fractional differential equations at resonance via coincidence degree theory; [77], including different types of differential and integral equations.

Motivated by the above references, we apply, to the best of our knowledge, for the first time, the methods and techniques suggested in, for example, [15, 45], to an impulsive coupled system with fully differential equations including different Laplacians and generalized impulsive conditions, which jumps depend of both variables and some of its derivatives.

The chapter is organized as follows: Section 4.1 presents the definitions and some preliminary results such as a Nagumo-type condition and *a priori* bounds for both first derivatives. Section 4.2 contains the main result: an existence and localization theorem. In the Section 4.3, an example illustrates the potential applications of this theory.

## 4.1 Definitions and auxiliary results

This section will introduce some preliminary results for the subsequent analysis.

To establish the functional framework, define

$$y(x_k^{\pm}) := \lim_{x\to x_k^{\pm}} y(x),$$

and consider the sets of piecewise continuous functions:

$$PC_1[a,b] = \left\{ \begin{array}{c} u : u \in C([a,b],\mathbb{R}) \text{ continuous for } x \neq x_i, u(x_i) = u(x_i^-), \\ u(x_i^+) \text{ exists for } i = 1,2,3,...,m, \end{array} \right\}$$

$$PC_2[a,b] = \left\{ \begin{array}{c} v : v \in C([a,b],\mathbb{R}) \text{ continuous for } x \neq \tau_j, v(\tau_j) = v(\tau_j^-), \\ v(\tau_j^+) \text{ exists for } j = 1,2,3,...,n, \end{array} \right\}$$

and $PC_k^1[a,b] = \{y : y' \in PC_k[a,b],\ k = 1,2\}$.

Let $X_k := PC_k^1[a,b]$ be the usual Banach space equipped with the norm $\|\cdot\|_\infty$, defined by

$$\|y\|_{X_k} := \max\{\|y\|_\infty, \|y'\|_\infty\},$$

where

$$\|y\|_\infty := \sup_{a\le x\le b} |y(x)|$$

and $X^2 = PC_1^1[a,b] \times PC_2^1[a,b]$ with the norm

$$\|(u,v)\|_{X^2} = \max\{\|u\|_{X_1}, \|v\|_{X_2}\}.$$

For $(u,v)$ solution of problem (4.1)-(4.3), one must consider $(u(x),v(x)) \in X^2$, satisfying (4.1), the boundary conditions (4.2) and the impulse effects (4.3).

**Definition 4.1.1.** *A function $h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ is $L^1$-Carathéodory if*

*i. for each $(y_0, y_1, z_0, z_1) \in \mathbb{R}^4$, $x \mapsto h(x, y_0, y_1, z_0, z_1)$ is measurable on $[a,b]$;*

*ii. for almost every $x \in [a,b]$, $(y_0, y_1, z_0, z_1) \mapsto h(x, y_0, y_1, z_0, z_1)$ is continuous on $\mathbb{R}^4$;*

*iii. for each $L > 0$, there is a positive function $\rho_L \in L^1[a,b]$ such that, for a.e. $x \in [a,b]$, and $(y_0, y_1, z_0, z_1) \in \mathbb{R}^4$ with*

$$\max\{|y_0|, |y_1|, |z_0|, |z_1|\} < L, \tag{4.4}$$

*we have*

$$|h(x, y_0, y_1, z_0, z_1)| \le \rho_L(x). \tag{4.5}$$

The next lemma gives the unique solution for a suitable homogeneous problem related to (4.1)-(4.3).

**Lemma 4.1.2.** *Let $p, q : [a,b] \longrightarrow \mathbb{R}$ be $L^1$-Carathéodory functions. The problem composed by the differential system*

$$\begin{cases} (\phi(u'(x)))' + p(x) = 0 \\ (\psi(v'(x)))' + q(x) = 0 \end{cases} \tag{4.6}$$

*and conditions (4.2), (4.3), has a unique solution given by*

$$u(x) = A + \sum_{i\,:\,x>x_i} I_i(x_i, u(x_i), u'(x_i), v(x_i)) + \int_a^x \phi^{-1}\left(\phi(B) + \int_s^b p(\xi)d\xi\right) ds$$

*and*

$$v(x) = C + \sum_{j\,:\,x>\tau_j} J_j(\tau_j, u(\tau_j), v(\tau_j), v'(\tau_j)) + \int_a^x \psi^{-1}\left(\psi(D) + \int_s^b q(\xi)d\xi\right) ds.$$

*Proof.* Integrating the first equation of (4.6), for $x \in (x_n, b]$, we have, by (4.2),

$$u'(x) = \phi^{-1}\left(\phi(B) + \int_x^b p(s)ds\right),$$

and, by a new integration from $a$ to $x$,

$$u(x) = A + \int_a^x \phi^{-1}\left(\phi(B) + \int_s^b p(\xi)d\xi\right) ds.$$

For $x \in [a, x_1]$,

$$u(x) = A + \int_a^x \phi^{-1}\left(\phi(B) + \int_s^b p(\xi)d\xi\right) ds,$$

and by (4.3),

$$u(x) = A + \int_a^x \phi^{-1}\left(\phi(B) + \int_s^b p(\xi)d\xi\right) ds + I_1(x_1, u(x_1), u'(x_1), v(x_1), v'(x_1)).$$

Therefore, by induction, it can be proved that the solution of the first equation of the problem (4.6), (4.2), (4.3), for $x \in [a,b]$, is given by

$$u(x) = A + \sum_{i\,:\,x>x_i} I_i(x_i, u(x_i), u'(x_i), v(x_i), v'(x_i)) + \int_a^x \phi^{-1}\left(\phi(B) + \int_s^b p(\xi)d\xi\right) ds.$$

Likewise, for the second equation, we have

$$v(x) = C + \sum_{j\,:\,x>\tau_j} J_j(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j)) + \int_a^x \psi^{-1}\left(\psi(D) + \int_s^b q(\xi)d\xi\right) ds.$$

□

The Nagumo condition is an important tool for controlling the first derivatives:

**Definition 4.1.3.** *Let* $\gamma_k(x),\ \Gamma_k(x),\ k=1,2,$ *be piecewise continuous functions such that*

$$\gamma_1(x)\leq\Gamma_1(x),\ \gamma_2(x)\leq\Gamma_2(x),\ a.e.\ in\ [a,b],$$

*and consider the set*

$$S=\{(x,y_0,y_1,z_0,z_1)\in[a,b]\times\mathbb{R}^4:\gamma_1(x)\leq y_0\leq\Gamma_1(x),\ \gamma_2(x)\leq z_0\leq\Gamma_2(x)\}. \tag{S}$$

*The* $L^1-$*Carathéodory functions* $f,g:[a,b]\times\mathbb{R}^4\to\mathbb{R}$ *satisfy a Nagumo-type condition, if, there are* $\mu_k>0,\ k=1,2,$ *with*

$$\begin{aligned}\mu_1: &= \max_{i=0,1,2,\dots,m}\left\{\left|\frac{\Gamma_1(x_{i+1})-\gamma_1(x_i)}{x_{i+1}-x_i}\right|,\left|\frac{\gamma_1(x_{i+1})-\Gamma_1(x_i)}{x_{i+1}-x_i}\right|\right\},\\ \mu_2: &= \max_{j=0,1,2,\dots,n}\left\{\left|\frac{\Gamma_2(\tau_{j+1})-\gamma_2(\tau_j)}{\tau_{j+1}-\tau_j}\right|,\left|\frac{\gamma_2(\tau_{j+1})-\Gamma_2(\tau_j)}{\tau_{j+1}-\tau_j}\right|\right\},\end{aligned} \tag{4.7}$$

*and continuous positive functions* $\varphi_k:[0,+\infty)\to(0,+\infty),\ k=1,2,$ *verifying*

$$\int_{\phi(\mu_1)}^{\phi(+\infty)}\frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)}ds=+\infty,\ \int_{\psi(\mu_2)}^{\psi(+\infty)}\frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)}ds=+\infty, \tag{4.8}$$

*such that*

$$\begin{aligned}|f(x,y_0,y_1,z_0,z_1)| &\leq \varphi_1(|y_1|),\ \forall(x,y_0,y_1,z_0,z_1)\in S,\\ |g(x,y_0,y_1,z_0,z_1)| &\leq \varphi_2(|z_1|),\ \forall(x,y_0,y_1,z_0,z_1)\in S.\end{aligned} \tag{4.9}$$

This growth condition allows *a priori* estimations on the first derivatives.

**Lemma 4.1.4.** *Consider* $\gamma_k,\Gamma_k\in PC^1[a,b],\ k=1,2,$ *such that*

$$\gamma_k(x)\leq\Gamma_k(x),\ a.e.\ x\in[a,b],$$

*and let* $f,g:[a,b]\times\mathbb{R}^4\to\mathbb{R}$ *be* $L^1-$*Carathéodory functions satisfying a Nagumo-type condition, according to Definition 4.1.3. Then, there exist* $N_k>0,\ k=1,2,$ *such that for every solution* $(u,v)$ *of ( 4.1) on* $S$ *satisfies*

$$\|u'\|_\infty<N_1\ and\ \|v'\|_\infty<N_2.$$

**Remark 4.1.5.** *Note that* $N_1$ *depends only on* $\gamma_1,\Gamma_1$ *and* $\varphi_1$, *and* $N_2$ *on* $\gamma_2,\Gamma_2$ *and* $\varphi_2$.

*Proof.* Let $(u(x),v(x))\in S$ be a solution of (4.1).

By the Mean Value Theorem, there are $\bar{x}\in(x_i,x_{i+1})$ and $\tilde{x}\in(\tau_j,\tau_{j+1})$ such that

$$u'(\bar{x})=\frac{u(x_{i+1})-u(x_i^+)}{x_{i+1}-x_i}\quad\text{and}\quad v'(\tilde{x})=\frac{v(\tau_{j+1})-v(\tau_j^+)}{\tau_{j+1}-\tau_j}. \tag{4.10}$$

If $|u'(x)| \leq \mu_1, \ \forall x \in [a,b]$, then it is enough to define $N_1 := \mu_1$ and the proof is complete.

More the case $|u'(t)| > \mu_1, \ \forall x \in [a,b]$, with $\mu_1$ defined in (4.7), is not possible. In fact, if $u'(t) > \mu_1, \ \forall x \in [a,b]$, we obtain, by (4.10) and (4.7), the contradiction

$$u'(\bar{x}) = \frac{u(x_{i+1}) - u(x_i^+)}{x_{i+1} - x_i} \leq \frac{\Gamma_1(x_{i+1}) - \gamma_1(x_i)}{x_{i+1} - x_i} \leq \mu_1.$$

If $u'(x) < -\mu_1, \ \forall x \in [a,b]$, the contradiction is similar.

We assume that there are $\breve{x}, x^* \in (x_i, x_{i+1})$ with $\breve{x} < x^*$, such that

$$u'(\breve{x}) \leq \mu_1 \quad \text{and} \quad u'(x^*) > \mu_1.$$

By continuity of $u'(x)$, there exists $\hat{x} \in [\breve{x}, x^*]$ such that $u'(\hat{x}) = \mu_1$.

Consider $N_k > \mu_k$, $k = 1, 2$, such that

$$\int_{\phi(\mu_1)}^{\phi(N_1)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds > \mu_1(b-a), \quad \int_{\psi(\mu_2)}^{\psi(N_2)} \frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)} ds > \mu_2(b-a), \tag{4.11}$$

Making a convenient change of variable and using (4.8),

$$\begin{aligned}
\int_{\phi(u'(\hat{x}))}^{\phi(u'(x^*))} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds &= \int_{\hat{x}}^{x^*} \frac{|\phi^{-1}(\phi(u'(x)))|}{\varphi_1(|\phi^{-1}(\phi(u'(x)))|)} (\phi(u'(x)))' dx \\
&\leq \int_a^b \frac{|u'(x)|}{\varphi_1(|u'(x)|)} (\phi(u'(x)))' dx \\
&\leq \int_a^b \frac{|u'(x)| \cdot |-f(x, u(x), u'(x), v(x), v'(x))|}{\varphi_1(|u'(x)|)} dx, \\
&\leq \int_a^b \frac{|u'(x)| \cdot |\varphi_1(|u'(x)|)|}{\varphi(|u'(t)|} dx, \\
&\leq \int_a^b |u'(x)| dx = |u(b) - u(a)| \leq \mu_1(b-a),
\end{aligned}$$

and by (4.11)

$$\int_{\phi(u'(\hat{x}))}^{\phi(u'(x^*))} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds \leq \mu_1(b-a) < \int_{\phi(\mu_1)}^{\phi(N_1)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds.$$

Therefore $u'(x^*) < N_1$, and as $x^*$ is taken arbitrarily, then $u'(x) < N_1$, for the values of $x$ whenever $u'(x) > \mu_1$.

The case for $\breve{x} > x^*$ follows similar arguments.

The other possible case where

$$u'(\breve{x}) > -\mu_1 \quad \text{and} \quad u'(x^*) < -\mu_1.$$

can be proved by the previous techniques. Therefore $\|u'\| \leq N_1$.

By a similar method, it can be shown that $\|v'\| \leq N_2$. □

Lower and upper functions will be defined as follows:

**Definition 4.1.6.** *The pair of functions* $(\alpha_1(x), \alpha_2(x)) \in X^2$ *with* $(\phi(\alpha_1'(x)), \psi(\alpha_2'(x))) \in (AC[a,b])^2$, *is a lower solution of problem (4.1)-(4.3), if*

$$\begin{cases} (\phi(\alpha_1'(x)))' + f(x, \alpha_1(x), \alpha_1'(x), \alpha_2(x), w) \geq 0, \text{ for } w \in \mathbb{R} \\ (\psi(\alpha_2'(x)))' + g(x, \alpha_1(x), z, \alpha_2(x), \alpha_2'(x)) \geq 0, \text{ for } z \in \mathbb{R}, \\ \alpha_1(a) \leq A, \ \alpha_1'(b) \leq B, \\ \alpha_2(a) \leq C, \ \alpha_2'(b) \leq D, \\ \Delta\alpha_1(x_i) > I_i(x_i, \alpha_1(x_i), \alpha_1'(x_i), \alpha_2(x_i)), \\ \Delta\alpha_2(\tau_j) > J_j(\tau_j, \alpha_1(\tau_j), \alpha_2(\tau_j), \alpha_2'(\tau_j)). \end{cases} \tag{4.12}$$

*A pair of functions* $(\beta_1(x), \beta_2(x)) \in X^2$ *such that* $(\phi(\beta_1'(x)), \psi(\beta_2'(x))) \in (AC[a,b])^2$ *is an upper solution of problem (4.1)-(4.3) if the opposite inequalities hold.*

The arguments forward will require the following lemma, of [99]:

**Lemma 4.1.7.** *For* $v, w \in C(I)$ *such that* $v(x) \leq w(x)$, *for every* $x \in I$, *define*

$$q(x,u) = max\{v, min\{u, w\}\}.$$

*Then, for each* $u \in C^1(I)$ *the next two properties hold:*

*(a)* $\frac{d}{dx}q(x,u(x))$ *exists for a.e.* $x \in I$.

*(b) If* $u, u_m \in C^1(I)$ *and* $u_m \to u$ *in* $C^1(I)$ *then*

$$\frac{d}{dx}q(x,u_m(x)) \to \frac{d}{dx}q(x,u(x)) \text{ for a.e. } x \in I.$$

The Schauder's fixed point theorem, will be the key existence tool:

**Theorem 4.1.8.** *([106]) Let* $Y$ *be a nonempty, closed, bounded and convex subset of a Banach space* $X$, *and suppose that* $P : Y \to Y$ *is a compact operator. Then* $P$ *has at least one fixed point in* $Y$.

## 4.2 Existence and localization theorem

Consider the following assumptions:

**(H1)** $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are increasing homeomorphisms such that $\phi(0) = \psi(0) = 0$ and $\phi(\mathbb{R}) = \psi(\mathbb{R}) = \mathbb{R}$, and

$$\left|\phi^{-1}(w)\right| \leq \phi^{-1}(|w|), \text{ and } \left|\psi^{-1}(w)\right| \leq \psi^{-1}(|w|).$$

**(H2)** $f, g : [a,b] \times \mathbb{R}^4 \mapsto \mathbb{R}$, are $L^1$-Carathéodory such that

$$f(x, y_0, \alpha_1'(x), \alpha_2(x), z_1) \le f(x, y_0, y_1, z_0, z_1) \le f(x, y_0, \beta_1'(x), \beta_2(x), z_1),$$

for $\alpha_1'(x) \le y_1 \le \beta_1'(x)$, $\alpha_2(x) \le z_0 \le \beta_2(x)$, and $(x, y_0, z_1) \in [a,b] \times \mathbb{R}^2$, and

$$g(x, \alpha_1(x), y_1, z_0, \alpha_2'(x)) \le g(x, y_0, y_1, z_0, z_1) \le g(x, \beta_1(x), y_1, z_0, \beta_2'(x)),$$

for $\alpha_1(x) \le y_0 \le \beta_1(x)$, $\alpha_2'(x) \le z_1 \le \beta_2'(x)$, and $(x, y_1, z_0) \in [a,b] \times \mathbb{R}^2$.

**(H3)** $I_i, J_j \in C([a,b] \times \mathbb{R}^3, \mathbb{R})$, verify

$$I_i(x_i, y_0, y_1, \alpha_2(x_i)) \ge I_i(x_i, y_0, y_1, z_0) \ge I_i(x_i, y_0, y_1, \beta_2(x_i)),$$

for $i = 1, 2, ..., m$, $\alpha_2(x) \le z_0 \le \beta_2(x)$, $(y_0, y_1) \in \mathbb{R}^2$ and

$$J_j(\tau_j, \alpha_1(\tau_j), z_0, z_1) \ge J_j(\tau_j, u(\tau_j), z_0, z_1) \ge J_j(\tau_j, \beta_1(\tau_j), z_0, z_1).$$

for $j = 1, 2, ..., n$, $\alpha_1(x) \le y_0 \le \beta_1(x)$, $(z_0, z_1) \in \mathbb{R}^2$.

The main result is an existence and localization theorem, that is, not only it guarantees the existence of at least a solution but provides also a strip where this solution is localized.

**Theorem 4.2.1.** *Let $A, B, C, D \in \mathbb{R}$, and homeomorphisms $\phi$ and $\psi$ verifying (H1). Assume that there are lower and upper solutions of (4.1)-(4.3), $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$, respectively, such that*

$$\alpha_k(x) \le \beta_k(x), \quad k = 1, 2, \ \forall x \in [a,b],$$

*the $L^1-$Carathéodory functions $f, g : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy Nagumo conditions as in Definition 4.1.3, in the set*

$$S_* = \{(x, y_0, y_1, z_0, z_1) \in [a,b] \times \mathbb{R}^4 : \alpha_1(x) \le y_0 \le \beta_1(x),\ \alpha_2(x) \le z_0 \le \beta_2(x)\},$$

*and (H2) and (H3) hold.*
*Then, there is at least a pair $(u(x), v(x)) \in X^2$ solution of (4.1)-(4.3) and, moreover,*

$$\alpha_1(x) \le u(x) \le \beta_1(x), \quad \alpha_2(x) \le v(x) \le \beta_2(x), \quad \forall x \in [a,b], \tag{4.13}$$

*and*

$$\|u'\| \le N_1 \quad \text{and} \quad \|v'\| \le N_2,$$

*with $N_1$ and $N_2$ given by Lemma 4.1.4.*

*Proof.* Define the functions $\delta_k : [a,b] \times \mathbb{R} \to \mathbb{R}$, for $k = 1, 2$, given by

$$\delta_k(x, w) = \begin{cases} \beta_k(x) & \text{if} \quad w > \beta_k(x) \\ w & \text{if} \quad \alpha_k(x) \le w \le \beta_k(x) \\ \alpha_k(x) & \text{if} \quad w < \alpha_k(x). \end{cases} \tag{4.14}$$

Consider the following modified coupled system composed by the truncated and perturbed differential equations

$$\begin{cases} (\phi(u'(x)))' + f(x, \delta_1(x, u(x)), u'(x), \delta_2(x, v(x)), v'(x)) \\ \qquad + \dfrac{\delta_1(x, u(x)) - u(x)}{1 + |\delta_1(x, u(x)) - u(x)|} = 0, \\ (\psi(v'(x)))' + g(x, \delta_1(x, u(x)), u'(x), \delta_2(x, v(x)), v'(x)) \\ \qquad + \dfrac{\delta_2(x, v(x)) - v(x)}{1 + |\delta_2(x, v(x)) - v(x)|} = 0, \end{cases} \tag{4.15}$$

with the truncated impulsive conditions

$$\begin{aligned} \Delta u(x_i) &= I_i(x_i, \delta_1(x_i, u(x_i)), \tfrac{\partial \delta_1}{\partial x}(x_i, u(x_i)), \delta_2(x_i, v(x_i))), \\ \Delta v(\tau_j) &= J_j(\tau_j, \delta_1(\tau_j, u(\tau_j)), \delta_2(\tau_j, v(\tau_j)), \tfrac{\partial \delta_2}{\partial x}(\tau_j, v(\tau_j))), \end{aligned} \tag{4.16}$$

for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$, and the boundary conditions (4.2).

The proof will follow several steps for clearness.

**Step1:** *In the set*

$$S^* = \{(x, y_0, y_1, z_0, z_1) \in [a, b] \times \mathbb{R}^4 : \alpha_1(x) \le y_0 \le \beta_1(x),\ \alpha_2(x) \le z_0 \le \beta_2(x)\}, \tag{S*}$$

*every solution* $(u(x), v(x))$ *of problem (4.15), (4.2) and (4.16) satisfies*

$$\|u'\| \le N_1 \quad \text{and} \quad \|v'\| \le N_2. \tag{4.17}$$

The functions $F$ e $G$, defined in $S^*$, by

$$\begin{aligned} F(x, u(x), u'(x), v(x), v'(x)) \quad : \quad &= f(x, \delta_1(x, u(x)), u'(x), \delta_2(x, v(x)), v'(x)) \\ &+ \frac{\delta_1(x, u(x)) - u(x)}{1 + |\delta_1(x, u(x)) - u(x)|}, \end{aligned}$$

and

$$\begin{aligned} G(x, u(x), u'(x), v(x), v'(x)) \quad : \quad &= g(x, \delta_1(x, u(x)), u'(x), \delta_2(x, v(x)), v'(x)) \\ &+ \frac{\delta_2(x, v(x)) - u(x)}{1 + |\delta_2(x, v(x)) - u(x)|} \end{aligned}$$

satisfy the Nagumo type conditions, as in Definition 4.1.3, in $S^*$, with

$$|F(x, y_0, y_1, z_0, z_1)| \le \varphi_1(|y_1|) + 1$$

and

$$|G(x, y_0, y_1, z_0, z_1)| \le \varphi_2(|y_1|) + 1.$$

By Lemma 4.1.4, we have the *a priori* estimates (4.17).

**Step2:** *The problem (4.15), (4.2) and (4.16) has at least a solution* $(u^*(x), v^*(x)) \in X^2$.

Define $L > 0$ and $M > 0$, such that

$$L \geq \max \left\{ B, D, \|\alpha_k\|_{X_k}, \|\beta_k\|_{X_k}, N_k, \mu_k, k = 1, 2 \right\} \tag{4.18}$$

and

$$M > \max \left\{ \begin{array}{c} \sum_{i\,:\,x>x_i} \left| I_i(x_i, \delta_1(x_i, u(x_i)), \frac{\partial \delta_1}{\partial x}(x_i, u(x_i)), \delta_2(x_i, v(x_i)) \right|, \\ \\ \sum_{j\,:\,x>\tau_j} \left| J_j(\tau_j, \delta_1(\tau_j, u(\tau_j)), \delta_2(\tau_j, v(\tau_j)), \frac{\partial \delta_2}{\partial x}(\tau_j, v(\tau_j))) \right| \end{array} \right\}. \tag{4.19}$$

Since $F$ and $G$ are $L^1-$Carathéodory functions, there exist nonnegative function $\rho_{kL}(x) \in L^1([a,b])$, $k = 1, 2$, such that, for $\|(u,v)\|_{X^2} < L$, we have

$$\begin{aligned} \left|F(x, \delta_1(x, u(x)), u'(x), \delta_2(x, v(x)), v'(x))\right| &\leq \rho_{1L}(x), \\ \left|G(x, \delta_1(x, u(x)), u'(x), \delta_2(x, v(x)), v'(x))\right| &\leq \rho_{2L}(x), \quad a.e. x \in [a,b]. \end{aligned} \tag{4.20}$$

Defining the operators $T_1 : X^2 \to X_1\, T_2 : X^2 \to X_2$, and $T : X^2 \to X^2$ given by

$$T(u,v) = (T_1(u,v), T_2(u,v)), \tag{4.21}$$

with

$$\begin{aligned} (T_1(u,v))(x) &= A + \sum_{i\,:\,x>x_i} I_i(x_i, \delta_1(x_i, u(x_i)), \frac{\partial \delta_1}{\partial x}(x_i, u(x_i)), \delta_2(x_i, v(x_i))) \\ &\quad + \int_a^x \phi^{-1}\left(\phi(B) + \int_s^b F(\xi, \delta_1(\xi, u(\xi)), u'(\xi), \delta_2(\xi, v(\xi)), v'(\xi)) d\xi\right) ds \end{aligned}$$

and

$$\begin{aligned} (T_2(u,v))(x) &= C + \sum_{j\,:\,x>\tau_j} J_j(\tau_j, \delta_1(\tau_j, u(\tau_j)), \delta_2(\tau_j, v(\tau_j)), \frac{\partial \delta_2}{\partial x}(\tau_j, v(\tau_j)))) \\ &\quad + \int_a^x \psi^{-1}\left(\psi(D) + \int_s^b G(\xi, \delta_1(\xi, u(\xi)), u'(\xi), \delta_2(\xi, v(\xi)), v'(\xi)) d\xi\right) ds. \end{aligned}$$

**Claim 1**: *$T$ is well defined, continuous and uniformly bounded.*

By the Lebesgue dominated convergence Theorem, (4.13), (4.20), (4.19), and (4.17),

$$\begin{aligned} |(T_1(u,v))(x)| &\leq |A| + \sum_{i\,:\,x>x_i} \left| I_i(x_i, \delta_1(x_i, u(x_i)), \frac{\partial \delta_1}{\partial x}(x_i, u(x_i)), \delta_2(x_i, v(x_i))) \right| \\ &\quad + \int_a^x \left| \phi^{-1}\left(\phi(B) + \int_s^b F(\xi, \delta_1(\xi, u(\xi)), u'(\xi), \delta_2(\xi, v(\xi)), v'(\xi)) d\xi\right) ds \right| \\ &\leq |A| + \sum_{i\,:\,x>x_i} \left| I_i(x_i, \delta_1(x_i, u(x_i)), \frac{\partial \delta_1}{\partial x}(x_i, u(x_i)), \delta_2(x_i, v(x_i))) \right| \\ &\quad + \int_a^x \phi^{-1}\left(\left| \phi(B) + \int_s^b F(\xi, \delta_1(\xi, u(\xi)), u'(\xi), \delta_2(\xi, v(\xi)), v'(\xi)) d\xi \right|\right) ds \\ &\leq |A| + M + \int_a^b \phi^{-1}\left(|\phi(B)| + \int_s^b \rho_{1L}(\xi) d\xi\right) ds < +\infty, \end{aligned}$$

$$\begin{aligned}
\left|(T_1(u,v))'(x)\right| &= \left|\phi^{-1}\left(\phi(B)+\int_x^b F(\xi,\delta_1(\xi,u(\xi)),u'(\xi),\delta_2(\xi,v(\xi)),v'(\xi))d\xi\right)\right| \\
&\leq \phi^{-1}\left(|\phi(B)|+\int_a^b \rho_{1L}(\xi)d\xi\right) < +\infty.
\end{aligned}$$

Therefore $(T_1(u,v))(x) \in X_1$. The proof that $(T_2(u,v))(x) \in X_2$ is similar, and, so, therefore $T$ is well defined in $X^2$.

Moreover, define $\mathcal{B} \subseteq X^2$ by

$$\mathcal{B} \subseteq \{(u,v) \in X^2 : \|(u,v)\|_{X^2} \leq L\},$$

from the above, it clear that $T\mathcal{B}$ is uniformly bounded.

**Claim 2**: *$T$ is equicontinuous, that is, $T_1\mathcal{B}$ is equicontinuous on each interval $]x_i,x_{i+1}]$, for $i=0,1,...,m$, with $x_0=a$ and $x_{m+1}=b$, and $T_2\mathcal{B}$ is equicontinuous on each interval $]\tau_j,\tau_{j+1}]$, for $j=0,1,...,n$, with $\tau_0=a$ and $\tau_{n+1}=b$.*

Consider $\mathcal{I} \subseteq ]x_i,x_{i+1}]$ and $\tilde{x},x^* \in \mathcal{I}$ such that, without loss of generality, $\tilde{x} \leq x^*$. For $(u,v) \in \mathcal{B}$, we have

$$\begin{aligned}
&\lim_{\tilde{x}\to x^*} |(T_1(u,v))(x^*)-(T_1(u,v))(\tilde{x})| \leq \\
&\lim_{\tilde{x}\to x^*}\left|\sum_{a<x_i<x^*} I_i(x_i,\delta_1(x_i,u(x_i)),\frac{\partial\delta_1}{\partial x}(x_i,u(x_i)),\delta_2(x_i,v(x_i)))\right. \\
&\left.-\sum_{a<x_i<\tilde{x}} I_i(x_i,\delta_1(x_i,u(x_i)),\frac{\partial\delta_1}{\partial x}(x_i,u(x_i)),\delta_2(x_i,v(x_i)))\right| \\
&+\lim_{\tilde{x}\to x^*}\left|\int_{x_i}^{x^*}\phi^{-1}\left(\phi(B)+\int_s^b F(\xi,\delta_1(\xi,u(\xi)),u'(\xi),\delta_2(\xi,v(\xi)),v'(\xi))d\xi\right)ds\right. \\
&\left.-\int_{x_i}^{\tilde{x}}\phi^{-1}\left(\phi(B)+\int_s^b F(\xi,\delta_1(\xi,u(\xi)),u'(\xi),\delta_2(\xi,v(\xi)),v'(\xi))d\xi\right)ds\right| \\
\leq\ &\lim_{\tilde{x}\to x^*}\left|\int_{\tilde{x}}^{x^*}\phi^{-1}\left(|\phi(B)|+\int_s^b \rho_{1L}(\xi)d\xi\right)ds\right| = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\tilde{x}\to x^*}\left|(T_1(u,v))'(x^*)-(T_1(u,v))'(\tilde{x})\right| = \\
&\lim_{\tilde{x}\to x^*}\left|\phi^{-1}\left(\phi(B)+\int_{x^*}^b F(\xi,\delta_1(\xi,u(\xi)),u'(\xi),\delta_2(\xi,v(\xi)),v'(\xi))d\xi\right)\right. \\
&\left.-\phi^{-1}\left(\phi(B)+\int_{\tilde{x}}^b F(\xi,\delta_1(\xi,u(\xi)),u'(\xi),\delta_2(\xi,v(\xi)),v'(\xi))d\xi\right)\right| = 0.
\end{aligned}$$

Therefore, $T_1\mathcal{B}$ is equicontinuous on $X_1$.

Similarly, we can show that $T_2\mathcal{B}$ is equicontinuous on $X_2$, too. Thus, $T\mathcal{B}$ is equicontinuous on $X^2$.

**Claim 3**: $T\mathcal{B} : X^2 \to X^2$ *is equiconvergent at* $x = x_i$ and $x = \tau_j$.

First, let us prove the equiconvergence at $x = x_i^+$, for $i = 1, 2, ...m$. The proof of equiconvergence at $\tau = \tau_j^+$, for $j = 1, 2, ...n$, is analogous.

So, it follows

$$\begin{aligned} & \left| \left(T_1\left(u,v\right)\right)\left(x_i\right) - \lim_{x\to x_i^+} \left(T_1\left(u,v\right)\right)\left(x\right) \right| \leq \\ & \left| \sum_{a<x_k<x_i} I_k(x_k, \delta_1(x_k, u(x_k)), \frac{\partial \delta_1}{\partial x}(x_k, u(x_k)), \delta_2(x_k, v(x_k))) \right. \\ & \left. - \lim_{x\to x_i^+} \sum_{a<x_k<x} I_k(x_k, u(x_k), \frac{\partial \delta_1}{\partial x}(x_k, u(x_k)), \delta_2(x_k, v(x_k))) \right| \\ & + \left| \int_a^{x_i} \phi^{-1}\left(\phi(B) + \int_s^b F(\xi, \delta_1(\xi, u(\xi)), u'(\xi), \delta_2(\xi, v(\xi)), v'(\xi)) d\xi\right) ds \right. \\ & \left. - \lim_{x\to x_i^+} \int_a^{x} \phi^{-1}\left(\phi(B) + \int_s^b F(\xi, \delta_1(\xi, u(\xi)), u'(\xi), \delta_2(\xi, v(\xi)), v'(\xi)) d\xi\right) ds \right| \\ \leq \; & \left| \int_x^{x_i} \phi^{-1}\left(|\phi(B)| + \int_s^b \rho_{1L}(\xi) d\xi\right) ds \right| \longrightarrow 0, \end{aligned}$$

uniformly on $(u, v) \in \mathcal{B}$, as $x \longrightarrow x_i^+$, for $i = 1, 2, ..., m$ and

$$\begin{aligned} & \left| \left(T_1\left(u,v\right)\right)'\left(x_i\right) - \lim_{x\to x_i^+} \left(T_1\left(u,v\right)\right)'\left(x\right) \right| = \\ & \left| \phi^{-1}\left(\phi(B) + \int_{x_i}^b F(\xi, \delta_1(\xi, u(\xi)), u'(\xi), \delta_2(\xi, v(\xi)), v'(\xi)) d\xi\right) \right. \\ & \left. - \lim_{x\to x_i^+} \phi^{-1}\left(\phi(B) + \int_x^b F(\xi, \delta_1(\xi, u(\xi)), u'(\xi), \delta_2(\xi, v(\xi)), v'(\xi)) d\xi\right) \right| \longrightarrow 0, \end{aligned}$$

uniformly on $(u, v) \in \mathcal{B}$, as $x \longrightarrow x_i^+$, for $i = 1, 2, ..., m$.

Therefore, $T_1\mathcal{B}$ is equiconvergent at each point $x = x_i^+$ , for $i = 1, 2, ..., m$. Analogously, it can be proved that $T_2\mathcal{B}$ is equiconvergent at each point $\tau = \tau_j^+$, for $j = 1, 2, ..., n$.

So, $T\mathcal{B}$ is equiconvergent at each impulsive point.

**Claim 4**: $T : X^2 \to X^2$ *has a fixed point* .

Consider

$$\Omega := \{(u, v) \in X^2 : \|(u, v)\|_{X^2} \leq K\},$$

with $K > 0$ such that

$$K := \max \left\{ \begin{array}{l} L, |A| + M + \int_a^b \phi^{-1}\left(|\phi(B)| + \int_a^b \rho_{1L}(\xi)d\xi\right) ds, \\ \phi^{-1}\left(|\phi(B)| + \int_a^b \rho_{1L}(\xi)d\xi\right), \\ |C| + M + \int_a^b \psi^{-1}\left(|\psi(D)| + \int_a^b \rho_{2L}(\xi)d\xi\right) ds, \\ \psi^{-1}\left(|\psi(D)| + \int_a^b \rho_{2L}(\xi)d\xi\right) \end{array} \right\},$$

with $L > 0$ and $M > 0$ given by (4.18) and (4.19). According to Claim 1, we have

$$\begin{array}{rcl} \|T(u,v)\|_{X^2} & = & \|(T_1(u,v), T_2(u,v))\|_{X^2} \\ & = & \max\{\|T_1(u,v)\|_{X_1}, \|T_2(u,v)\|_{X_2}\} \\ & = & \max\{\|T_1(u,v)\|_\infty, \|T_1'(u,v)\|_\infty, \|T_2(u,v)\|_\infty, \|T_2'(u,v)\|_\infty\} \\ & \leq & K. \end{array}$$

So, $T\Omega \subset \Omega$, an by Theorem 4.1.8, the operator $T(u,v) = (T_1(u,v), T_2(u,v))$, has a fixed point $(u^*, v^*)$, which is a solution of the problem (4.15), (4.2), (4.16) .

**Step 3**: *This pair of functions $(u^*(x), v^*(x))$ is a solution of (4.1), (4.2), (4.3).*

To prove this claim it is enough to show that

$$\alpha_1(x) \leq u^*(x) \leq \beta_1(x), \quad \alpha_2(x) \leq v^*(x) \leq \beta_2(x), \quad \forall x \in [a,b].$$

For the second inequality, assume, by contradiction, that there is $x \in [a,b]$ such that $u^*(x) > \beta_1(x)$. Therefore,

$$\sup_{a \leq x \leq b} (u^*(x) - \beta_1(x)) := u^*(\bar{x}) - \beta_1(\bar{x}) > 0. \tag{4.22}$$

As, by boundary conditions (4.2) and Definition 4.1.6, $u^*(a) - \beta_1(a) \leq 0$, then $\bar{x} \neq a$.

Likewise, $u^{*\prime}(b^-) - \beta_1'(b^-) \leq 0$, therefore the $\sup_{x\in[a,b]}(u^* - \beta_1)(x)$ cannot be reached at $x = b$.

Then $\bar{x} \neq b$, and, therefore $\bar{x} \in ]a,b[$.

Two possibilities remain to be studied:

- Assume that there is $p \in \{0,1,2,...,n\}$ such that $\bar{x} \in (x_p, x_{p+1})$. Therefore

$$\max_{x\in(x_p,x_{p+1})} (u^*(x) - \beta_1(x)) := u^*(\bar{x}) - \beta_1(\bar{x}) > 0,$$

and

$$u'(\bar{x}) - \beta_1'(\bar{x}) = 0. \tag{4.23}$$

Choose $\epsilon > 0$, sufficiently small, such that

$$u^*(x) - \beta_1(x) > 0 \text{ and } u^{*\prime}(x) - \beta_1'(x) \leq 0, \forall x \in (\bar{x}, \bar{x}+\epsilon). \tag{4.24}$$

By Definition 4.1.6, for all $x \in (\bar{x}, \bar{x} + \epsilon)$,

$$\begin{aligned}
(\phi(u^{*\prime}(x)))' - (\phi(\beta_1'(x)))' &\geq -f(x, \delta_1(x, u^*(x)), u^{*\prime}(x), \delta_2(x, v^*(x)), v^{*\prime}(x)) \\
&\quad - \frac{\delta_1(x, u^*(x)) - u^*(x)}{1 + |\delta_1(x, u^*(x)) - u^*(x)|} - (\phi(\beta_1'(x)))' \\
&\geq -f(x, \beta_1(x), \beta_1'(x), \delta_2(x, v^*(x)), v^{*\prime}(x)) \\
&\quad - \frac{\beta_1(x) - u^*(x)}{1 + |\beta_1(x) - u^*(x)|} - (\phi(\beta_1'(x)))' \\
&\geq \frac{u^*(x) - \beta_1(x)}{1 + |u^*(x) - \beta_1(x)|} > 0.
\end{aligned}$$

So $(\phi(u^{*\prime}(x)) - \phi(\beta_1'(x)))$ is increasing for $\forall x \in (\bar{x}, \bar{x} + \epsilon)$, and, by (4.24), we obtain the contradiction in $(\bar{x}, \bar{x} + \epsilon)$, by (4.23):

$$0 = \phi(u^{*\prime}(\bar{x})) - \phi(\beta_1'(\bar{x})) < \phi(u^{*\prime}(x)) - \phi(\beta_1'(x) \leq 0.$$

- Suppose, now, that there is $p \in \{1, 2, ..., n\}$ such that, $\bar{x} = x_p$. That is,

$$\sup_{x \in [a,b]} (u^*(x) - \beta_1(x)) := u^*(x_p) - \beta_1(x_p) > 0. \tag{4.25}$$

By (4.16), (H3) and Definition 4.1.6, we obtain the contradiction:

$$\begin{aligned}
\Delta\left(u^* - \beta_1\right)(x_p) &\geq I_p(x_p, \delta_1(x_p, u^*(x_p)), \frac{\partial \delta_1}{\partial x}(x_p, u^*(x_p)), \delta_2(x_p, v(x_p))) - \Delta\beta_1(x_p) \\
&= I_p(x_p, \beta_1(x_p), \beta_1'(x_p), \delta_2(x_p, v(x_p))) - \Delta\beta_1(x_p) \\
&\geq I_p(x_p, \beta_1(x_p), \beta_1'(x_p), \beta_2(x_p)) - \Delta\beta_1(x_p) > 0,
\end{aligned}$$

but, by the initial condition and the previous item, we have

$$u^*(x_p) = \lim_{x \to x_p^-} u^*(x) \leq \lim_{x \to x_p^-} \beta_1(x) = \beta_1(x_p).$$

If $\bar{x} = x_p^+$, then

$$\sup_{x \in [a,b]} (u^*(x) - \beta_1(x)) := u^*(x_p^+) - \beta_1(x_p^+) > 0,$$

and, for $\epsilon > 0$, small enough,

$$u^{*\prime}(x) - \beta_1'(x) \leq 0, \forall x \in (x_p^+, x_p^+ + \epsilon),$$

which leads to a similar contradiction with the interior case (4.24).

Therefore, $u^*(x) \leq \beta(x)$, for $x \in [a, b]$.

The other inequalities follow similar arguments. □

## 4.3 Example

Consider the following system of coupled differential equations

$$\begin{cases} (|u'(x)|^{p-2}\,u'(x))' - (u(x))^3 + u'(x) + \dfrac{v(x)}{1+(v'(x))^2} = 0,\ x \in [0,1] \setminus \left\{\frac{1}{2}\right\}, \\ \\ (\arctan(v'(x)))' + \sqrt[3]{u(x)} + \cos(u'(x))v(x) + v'(x) = 0,\ x \in [0,1] \setminus \left\{\frac{1}{2}, \frac{3}{4}\right\} \end{cases} \tag{4.26}$$

where $p > 1$ and $p \in \mathbb{Z}$, with the boundary conditions

$$\begin{cases} u(0) = 0,\ u'(1) = 0, \\ v(0) = 0,\ v'(1) = 0, \end{cases} \tag{4.27}$$

and impulsive conditions given by

$$\begin{cases} \Delta u\left(\dfrac{1}{2}\right) &= 2u\left(\dfrac{1}{2}\right) - u'\left(\dfrac{1}{2}\right) - v\left(\dfrac{1}{2}\right), \\ \\ \Delta v\left(\dfrac{1}{2}\right) &= -u\left(\dfrac{1}{2}\right) + 3v\left(\dfrac{1}{2}\right) - v'\left(\dfrac{1}{2}\right), \\ \\ \Delta v\left(\dfrac{3}{4}\right) &= -u\left(\dfrac{3}{4}\right) + v\left(\dfrac{3}{4}\right) + 4v'\left(\dfrac{3}{4}\right). \end{cases} \tag{4.28}$$

The system (4.26)-(4.28) is a particular case of the problem (4.1)-(4.3), with $a = 0$, $b = 1$, $A = B = C = D = 0$, $m = 1$, $n = 2$,

$$\begin{aligned} f(x, y_0, y_1, z_0, z_1) &= -y_0^3 + y_1 + \frac{z_0}{1+z_1^2}, \\ g(x, y_0, y_1, z_0, z_1) &= \sqrt[3]{y_0} + z_0\cos(y_1) + z_1, \\ \phi(y_1) &= |y_1|^{p-2}y_1, \\ \psi(z_1) &= \arctan(z_1), \\ I_1(x_1, y_0, y_1, z_0) &= 2y_0 - y_1 - z_0, \quad x_1 = \frac{1}{2}, \\ J_1(\tau_1, y_0, z_0, z_1) &= -y_0 + 3z_0 - z_1, \quad \tau_1 = \frac{1}{2}, \\ J_2(\tau_2, y_0, z_0, z_1) &= -y_0 + z_0 + 4z_1, \quad \tau_2 = \frac{3}{4}. \end{aligned}$$

It is clear that the homeomorphisms $\phi, \psi : \mathbb{R} \to \mathbb{R}$ satisfy (H1), and $f$, $g$ are $L^1$-Carathéodory functions, as, for $L > 0$ such that

$$\max\{|y_0|, |y_1|, |z_0|, |z_1|\} < L,$$

we have

$$\begin{aligned} |f(x, y_0, y_1, z_0, z_1)| &\leq L^3 + 2L := \rho_{1L}(x), \\ |g(x, y_0, y_1, z_0, z_1)| &\leq \sqrt[3]{L} + 2L := \rho_{2L}(x). \end{aligned}$$

Moreover, $f$ and $g$ satisfy a Nagumo condition relative to the sets

$$S_1 = \{(x, y_0, y_1, z_0, z_1) \in [0,1] \times \mathbb{R}^4 : \alpha_1(x) \le y_0 \le \beta_1(x),\ \alpha_2(x) \le z_0 \le \beta_2(x)\}.$$

In fact, for some constant $\mathcal{K} > 0$,

$$\mathcal{K} := \max_{x\in[0,1]} \{|\alpha_k(x)|, |\beta_k(x)|, k = 1, 2\},$$

and $\mu_i$ defined as in (4.7), then

$$\begin{aligned}|f(x, y_0, y_1, z_0, z_1)| &= \left|-y_0^3 + y_1 + \frac{z_0}{1 + z_0^2}\right| \\ &\le |y_1| + \mathcal{K} + \mathcal{K}^3 := \varphi_1(|y_1|),\end{aligned}$$

$$\begin{aligned}|g(x, y_0, y_1, z_0, z_1)| &= |\sqrt[3]{y_0} + z_0 \cos(y_1) + z_1| \\ &\le |z_1| + \sqrt[3]{\mathcal{K}} + \mathcal{K} := \varphi_2(|z_1|),\end{aligned}$$

and, by (4.8),

$$\int_{\phi(\mu_1)}^{\phi(+\infty)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds = \int_{\phi(\mu_1)}^{\phi(+\infty)} \frac{|\phi^{-1}(s)|}{|\phi^{-1}(s)| + \mathcal{K} + \mathcal{K}^3} ds = +\infty,$$

and

$$\int_{\psi(\mu_2)}^{\psi(+\infty)} \frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)} ds = \int_{\psi(\mu_2)}^{\psi(+\infty)} \frac{|\psi^{-1}(s)|}{|\psi^{-1}(s)| + \sqrt[3]{\mathcal{K}} + \mathcal{K}} ds = +\infty.$$

The functions $\alpha_k, \beta_k : [0,1] \to \mathbb{R}$, $k = 1, 2$, given by

$$\alpha_1(x) = \begin{cases} \frac{1}{2}x & \text{if } 0 \le x \le \frac{1}{2} \\ -x + 1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}, \quad \alpha_2(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2} \\ -\frac{1}{2}x^2 + 2 & \text{if } \frac{1}{2} < x \le \frac{3}{4} \\ -\frac{1}{2}x + 2 & \text{if } \frac{3}{4} < x \le 1 \end{cases},$$

and

$$\beta_1(x) = \begin{cases} \pi x + 2 & \text{if } 0 \le x \le \frac{1}{2} \\ \pi x + 1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}, \quad \beta_2(x) = \begin{cases} \frac{1}{2}x + 2 & \text{if } 0 \le x \le \frac{1}{2} \\ -x^2 + 2x + 2 & \text{if } \frac{1}{2} < x \le \frac{3}{4} \\ -e^{-x} + e & \text{if } \frac{3}{4} < x \le 1 \end{cases},$$

are, respectively, lower and upper solutions of de problem (4.26)-(4.28), according to Definition 4.1.6.

The four differential inequalities are verified in the interval $[0, 1]$, as shown in Figure 4.1, the boundary conditions verify

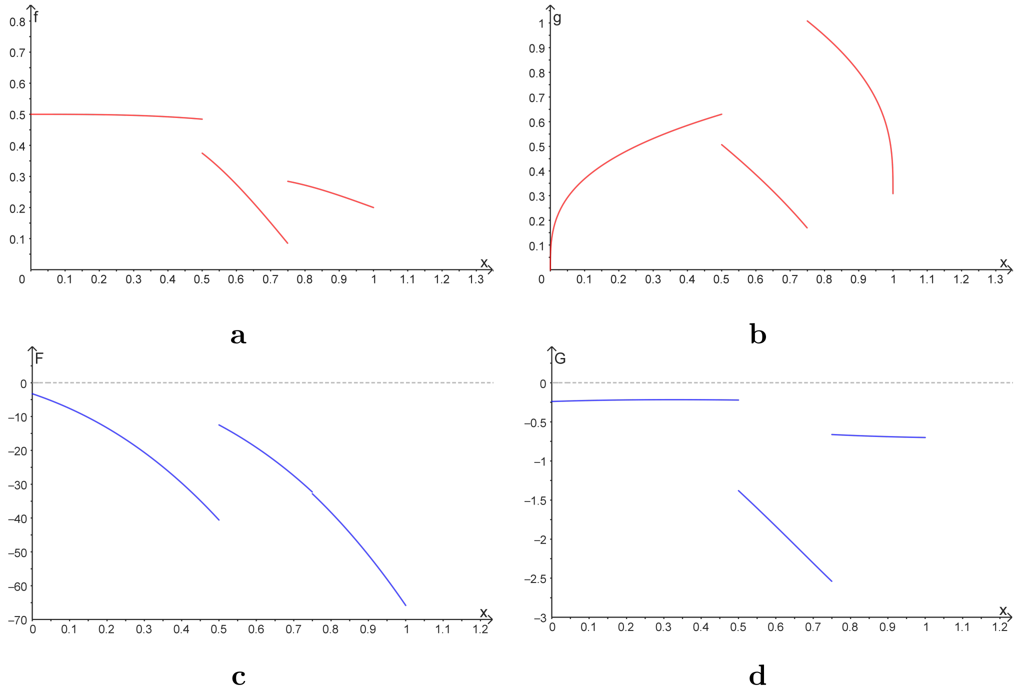



**Fig. 4.1:** Relationship between nonlinearities depending on the lower and upper solutions, given by the inequalities of the Definition 4.1.6: a) substituting lower solution $(\alpha_1, \alpha_2)$ into the first equation; b) substituting lower solution $(\alpha_1, \alpha_2)$ in the second equation; c) substituting upper solution $(\beta_1, \beta_2)$ into the first equation ; d) substituting upper solution $(\beta_1, \beta_2)$ in the second equation.

$$\alpha_1(0) = 0, \quad \alpha_1'(1) = -1 < 0, \quad \alpha_2(0) = 0, \quad \alpha_2'(1) = -\frac{1}{2} < 0,$$
$$\beta_1(0) = 2 > 0, \quad \beta_1'(1) = \pi > 0, \quad \beta_2(0) = 2 > 0, \quad \beta_2'(1) = \frac{1}{e} > 0,$$

and, for the impulsive conditions

$$\Delta\alpha_1\left(\frac{1}{2}\right) = \frac{1}{4} > I_1\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, 0\right) = 0,$$
$$\Delta\alpha_2\left(\frac{1}{2}\right) = \frac{15}{8} > J_1\left(\frac{1}{2}, \frac{1}{4}, 0, 0\right) = -\frac{1}{4},$$
$$\Delta\alpha_2\left(\frac{3}{4}\right) = -\frac{3}{32} > J_2\left(\frac{3}{4}, \frac{55}{32}, \frac{1}{4}, -\frac{3}{4}\right) = -\frac{49}{32},$$
$$\Delta\beta_1\left(\frac{1}{2}\right) = -1 < I_1\left(\frac{1}{2}, \frac{\pi+4}{2}, \pi, \frac{9}{4}\right) = \frac{7}{4},$$
$$\Delta\beta_2\left(\frac{1}{2}\right) = -\frac{1}{2} < J_1\left(\frac{1}{2}, \frac{\pi+4}{2}, \frac{9}{4}, \frac{1}{2}\right) = \frac{17-2\pi}{4},$$
$$\Delta\beta_2\left(\frac{3}{4}\right) = e - e^{-\frac{3}{4}} - \frac{47}{16} < J_2\left(\frac{3}{4}, \frac{3\pi+4}{4}, \frac{47}{16}, \frac{1}{2}\right) = \frac{63-12\pi}{16}.$$

So, by Theorem 4.2.1, there is at least one pair of functions $(u(x), v(x)) \in X^2$, solution of the problem (4.26)-(4.28).

Moreover, this solution can be localized in the strips

$$\alpha_1(x) \le u(x) \le \beta_1(x), \quad \alpha_2(x) \le v(x) \le \beta_2(x), \quad \forall x \in [0, 1],$$

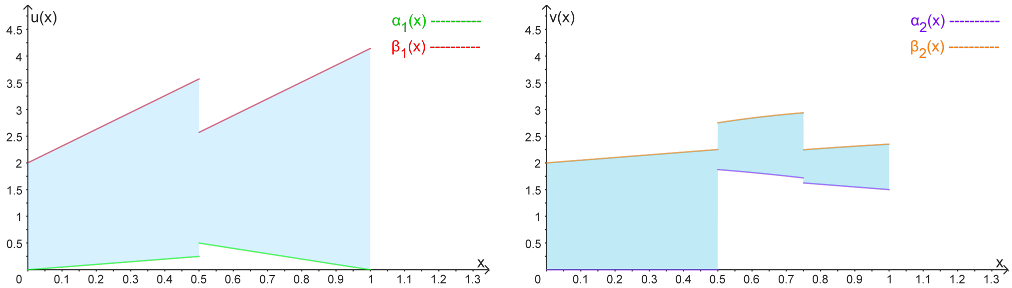



**Fig. 4.2:** At least one solution $(u(x), v(x))$ of problem (5.19)-(5.21) is located in the cracked region, when $x \in [0, 1]$.

illustrated in Figure 4.2, and

$$\|u'\|_\infty \leq N_1 \quad \text{and} \quad \|v'\|_\infty \leq N_2,$$

with $N_1$ and $N_2$ given by Lemma 4.1.4.

Chapter 5

# Impulsive coupled systems with regular and singular $\phi$-Laplacians

In this chapter, based on the article [70], we consider the third-order impulsive coupled system

$$\begin{cases}(\phi(u''(x)))' + f(x,u(x),u'(x),u''(x),v(x),v'(x),v''(x)) = 0, & x \in M,\\ (\psi(v''(x)))' + g(x,u(x),u'(x),u''(x),v(x),v'(x),v''(x)) = 0, & x \in N,\end{cases} \tag{5.1}$$

where $M = [a,b]\setminus\{x_1,...,x_m\}$ and $N = [a,b]\setminus\{\tau_1,...,\tau_n\}$, $\phi,\psi : \mathbb{R}\to\mathbb{R}$ are increasing homeomorphisms such that $\phi(0)=\psi(0)=0$ and $\phi(\mathbb{R})=\psi(\mathbb{R})=\mathbb{R}$, $f,g:[a,b]\times\mathbb{R}^6 \mapsto \mathbb{R}$, are $L^1$-Carathéodory functions, together with the boundary conditions

$$\begin{cases}u(a)=A_0,\ u'(a)=A_1,\ u''(b)=A_2,\\ v(a)=B_0,\ v'(a)=B_1,\ v''(b)=B_2,\end{cases} \tag{5.2}$$

with $A_k, B_k \in \mathbb{R}$, $k=0,1,2$.

The impulsive conditions are given by

$$\begin{aligned}\Delta u(x_i) &= I_{0i}(x_i,u(x_i),u'(x_i),v(x_i),v'(x_i)),\\ \Delta u'(x_i) &= I_{1i}(x_i,u(x_i),u'(x_i),u''(x_i),v(x_i),v'(x_i)),\\ \Delta\phi(u''(x_i)) &= I_{2i}(x_i,u(x_i),u'(x_i),u''(x_i),v(x_i),v'(x_i),v''(x_i)),\\ \Delta v(\tau_j) &= J_{0j}(\tau_j,u(\tau_j),u'(\tau_j),v(\tau_j),v'(\tau_j)),\\ \Delta v'(\tau_j) &= J_{1j}(\tau_j,u(\tau_j),u'(\tau_j),v(\tau_j),v'(\tau_j),v''(\tau_j)),\\ \Delta\psi(v''(\tau_j)) &= J_{2j}(\tau_j,u(\tau_j),u'(\tau_j),u''(\tau_j),v(\tau_j),v'(\tau_j),v''(\tau_j)),\end{aligned} \tag{5.3}$$

being $\Delta u(x_i) = u(x_i^+) - u(x_i^-)$, $i=1,2,...,m$, $\Delta v(\tau_j) = v(\tau_j^+) - v(\tau_j^-)$, $j=1,2,...,n$, $I_{0i}, J_{0j} \in C([a,b]\times\mathbb{R}^4,\mathbb{R})$, $I_{1i}, J_{1j} \in C([a,b]\times\mathbb{R}^5,\mathbb{R})$, $I_{2i}, J_{2j} \in C([a,b]\times\mathbb{R}^6,\mathbb{R})$, and $x_k$ fixed points such that $a = x_0 < x_1 < x_2 < ... < x_m < x_{m+1} = b$, $a = \tau_0 < \tau_1 < \tau_2 < ... < \tau_n < \tau_{n+1} = b$.

For a particular case, without jumps on the $\phi,\psi-$Laplacians, that is, for

$$\begin{aligned}\Delta u(x_i) &= I_{0i}(x_i,u(x_i),u'(x_i),v(x_i),v'(x_i)),\\ \Delta u'(x_i) &= I_{1i}(x_i,u(x_i),u'(x_i),u''(x_i),v(x_i),v'(x_i)),\\ \Delta v(\tau_j) &= J_{0j}(\tau_j,u(\tau_j),u'(\tau_j),v(\tau_j),v'(\tau_j)),\\ \Delta v'(\tau_j) &= J_{1j}(\tau_j,u(\tau_j),u'(\tau_j),v(\tau_j),v'(\tau_j),v''(\tau_j)),\end{aligned} \tag{5.4}$$

it is proved an existence and localization theorem, where we present the sufficient assumptions to localize a solution in a strip bound by lower and upper solutions.

Usually $\phi$, and $\psi$ are known as $\phi,\psi-$Laplacian as they generalize the one-dimensional Laplacian and the $p$-Laplacian, and they were used by many authors in a broad range of problems. Some examples: [102], to obtain a positive periodic solution for a $\varphi$-Laplacian Liénard equation with a singularity; [56], proving the multiplicity of solutions of $p$-Laplacian Dirichlet boundary value problem with discontinuous nonlinearities; [103], giving sufficient conditions for the existence of at least three positive solutions of one-dimensional $p$-Laplacian boundary value problem; [28, 95], to obtain positive solutions for some $p$-Laplacian problem in superlinear cases; [93], based on nonnegative nonlinearities under a version of the Krasnosel'skii expansion and compression cone theory.

Beyond the classic regular laplacians, the singular cases, that is, homeomorphisms $\phi : (-a, a) \to \mathbb{R}$ with $0 < a < +\infty$, have been recently studied by several authors, such as, for example: [46, 47], for $p$-Laplacian; [10, 11, 84], with existence and multiplicity results; [14], obtaining heteroclinic solutions; [38], for equations on the half-line with functional boundary conditions.

Nonlinear coupled systems, where the unknown functions and their derivatives can interact, have been considered in several works in recent years, such as, among others: [94], via Schauder's fixed point theorem; [35], for fractional differential equations at resonance applying coincidence degree theory; [77], including the study of different types of differential and integral equations; [105], via lower and upper solutions technique;[50], applied to reaction-diffusion Robin problems.

Impulsive differential equations model many real phenomena in which the nonlinearities have sudden discontinuous jumps in their values. These types of events can occur in population dynamics, control, and optimization theory, ecology, biology and biotechnology, economics, pharmacokinetics, and other physics and mechanics problems. As some examples of the approach to impulsive differential equations, we refer: [58], for a general theory; [52], via fixed point index; [61, 78], applied to functional impulsive problems; [108], with a monotone iterative technique for approximating the solution. The study of $\phi$-Laplacian impulsive problems can be seen, for instance in: [49], in periodic problems applying a continuation theorem; [72, 73], for bounded and unbounded intervals; [101], for fractional equations with $p$-Laplacian.

Combining all these areas and results, we consider, for the first time, the methods and techniques suggested in, for example, [15, 45], to an impulsive coupled system with fully differential equations including different regular and singular Laplacians and generalized impulsive conditions, which jumps depend of both variables and some of its derivatives.

The chapter is organized as follows: Section 5.1 contains the functional framework and some preliminary results, namely the explicit solution for the impulsive linear problem associated, Nagumo-type growth conditions, and *a priori* bounds for the second-derivatives. In Section 5.2, we present an existence theorem for the general case. Section 5.3 contains an existence and localization result applied to a particular case of the initial impulsive conditions, and a concrete example to show the applicability of the localization tool. Section 5.4 applies our method to the singular case and to special relativity theory.

## 5.1 Definitions and preliminary results

This section will introduce some preliminary results and the functional framework,

Define

$$y(x_\kappa^\pm) := \lim_{x \to x_\kappa^\pm} y(x),$$

and consider the sets of piece-wise continuous functions:

$$PC_1([a,b]) := \left\{ \begin{array}{c} u : u \in C([a,b],\mathbb{R}) \text{ continuous for } x \neq x_i, u(x_i) = u(x_i^-), \\ u(x_i^+) \text{ is finite for } i = 1,2,3,...,m \end{array} \right\},$$

$$PC_2([a,b]) := \left\{ \begin{array}{c} v : v \in C([a,b],\mathbb{R}) \text{ continuous for } x \neq \tau_j, v(\tau_j) = v(\tau_j^-), \\ v(\tau_j^+) \text{ is finite for } j = 1,2,3,...,n \end{array} \right\},$$

and

$$PC_k^l[a,b] = \left\{ y : y^{(l)} \in PC_k[a,b],\ l,k = 1,2 \right\}.$$

Let $X_k := PC_k^2[a,b]$, $k = 1,2$, be the usual Banach space equipped with the norm $\|\cdot\|_\infty$, defined by

$$\|y\|_{X_k} := \max\{\|y\|_\infty, \|y'\|_\infty, \|y''\|_\infty\},$$

where

$$\|y\|_\infty := \sup_{a \le x \le b} |y(x)|$$

and $X^2 := X_1 \times X_2$ with the norm

$$\|(u,v)\|_{X^2} = \max\{\|u\|_{X_1}, \|v\|_{X_2}\}.$$

**Definition 5.1.1.** *A function $h : [a,b] \times \mathbb{R}^6 \to \mathbb{R}$ is $L^1$-Carathéodory if*

*i. for each $(y_0, y_1, y_2, z_0, z_1, z_2) \in \mathbb{R}^6$, $x \mapsto h(x, y_0, y_1, y_2, z_0, z_1, z_2)$ is measurable on $[a,b]$;*

*ii. for almost every $x \in [a,b]$, $(y_0, y_1, y_2, z_0, z_1, z_2) \mapsto h(x, y_0, y_1, y_2, z_0, z_1, z_2)$ is continuous on $\mathbb{R}^6$;*

*iii. for each $L > 0$, there is a positive function $\rho_L \in L^1[a,b]$ such that, for a.e. $x \in [a,b]$, and $(y_0, y_1, y_2, z_0, z_1, z_2) \in \mathbb{R}^6$ with*

$$\max\{|y_0|, |y_1|, |y_2|, |z_0|, |z_1|, |z_2|\} < L,$$

*we have*

$$|h(x, y_0, y_1, y_2, z_0, z_1, z_2)| \le \rho_L(x).$$

For $(u,v)$ solution of problem (5.1)-(5.3), one must consider $(u(x), v(x)) \in X^2$, satisfying (5.1), the boundary conditions (5.2) and the impulsive effects (5.3).

The next lemma gives the unique solution for the homogeneous problem related to (5.1)-(5.3).

**Lemma 5.1.2.** *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be increasing homeomorphisms and $p, q \in L^1[a,b]$. The problem composed by the differential system*

$$\begin{cases} (\phi(u''(x)))' + p(x) = 0 \\ (\psi(v''(x)))' + q(x) = 0 \end{cases} \tag{5.5}$$

*and conditions (5.2), and (5.3), has a unique solution given by*

$$\begin{aligned} u(x) &= A_0 + A_1(x-a) + \sum_{i\,:\,x_i<x} I_{0i}(x_i, u(x_i), u'(x_i), v(x_i), v'(x_i)) \\ &+ (x-a) \sum_{i\,:\,x_i<x} I_{1i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i)) \\ &+ \int_a^x \int_a^r \phi^{-1} \left( \phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)) \right. \\ &+ \left. \int_s^b p(\xi)d\xi \right) dsdr \end{aligned}$$

*and*

$$\begin{aligned} v(x) &= B_0 + B_1(x-a) + \sum_{j\,:\,\tau_j<x} J_{0j}(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j)) \\ &+ (x-a) \sum_{j\,:\,\tau_j<x} (\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)) \\ &+ \int_a^x \int_a^r \psi^{-1} \left( \psi(B_2) - \sum_{j\,:\,\tau_j>s} J_{2j}(\tau_j, u(\tau_j), u'(\tau_j), u''(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)) \right. \\ &+ \left. \int_s^b q(\xi)d\xi \right) dsdr. \end{aligned}$$

*Proof.* Integrating the first equation of (5.5), for $x \in (x_n, b]$, we have, by (5.2),

$$\phi(u''(x)) = \phi(A_2) + \int_x^b p(\xi)d\xi, \tag{5.6}$$

For $x \in (x_{n-1}, x_n]$, integrating (5.5), by (5.3) and (5.6),

$$\begin{aligned} &\phi(u''(x)) = \phi(u''(x_n^-)) + \int_x^{x_n} p(\xi)d\xi \\ =&\phi(u''(x_n^+)) - I_{2n}(x_n, u(x_n), u'(x_n), u''(x_n), v(x_n), v'(x_n), v''(x_n)) + \int_x^{x_n} p(\xi)d\xi \\ =&\phi(A_2) + \int_{x_n}^b p(\xi)d\xi - I_{2n}(x_n, u(x_n), u'(x_n), u''(x_n), v(x_n), v'(x_n), v''(x_n)) \\ &+ \int_x^{x_n} p(\xi)d\xi \\ =&\phi(A_2) - I_{2n}(x_n, u(x_n), u'(x_n), u''(x_n), v(x_n), v'(x_n), v''(x_n)) + \int_x^b p(\xi)d\xi. \end{aligned}$$

So, by mathematical induction, for $x \in [a,b]$,

$$\begin{aligned} &\phi(u''(x)) = \phi(A_2) \\ &- \sum_{i\,:\,x_i>x} I_{2i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)) + \int_x^b p(\xi)d\xi, \end{aligned}$$

and, therefore

$$u''(x) = \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>x} I_{2i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)) + \int_s^b p(\xi)d\xi\right). \tag{5.7}$$

By a new integration of (5.7) from $a$ to $x$, when $x \in [a, x_1]$,

$$\begin{aligned} u'(x) &= A_1 \\ &+ \int_a^x \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)) \right. \\ &\left. + \int_s^b p(\xi)d\xi\right) ds. \end{aligned} \tag{5.8}$$

According to (5.3), when $x \to x_1^+$, we have

$$u'(x_1^+) = u'(x_1^-) + I_{11}(x_1, u(x_1), u'(x_i), u''(x_1), v(x_1), v'(x_1)),$$

by (5.8),

$$\begin{aligned} u'(x_1^+) &= A_1 \\ &+ \int_a^{x_1} \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)) \right. \\ &\left. + \int_s^b p(\xi)d\xi\right) ds \\ &+ I_{11}(x_1, u(x_1), u'(x_i), u''(x_1), v(x_1), v'(x_1)), \end{aligned}$$

and, for $x \in [a, b]$,

$$\begin{aligned} u'(x) &= A_1 + \sum_{i\,:\,x_i<x} I_{1i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i)) \\ &+ \int_a^x \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)) \right. \\ &\left. + \int_s^b p(\xi)d\xi\right) ds. \end{aligned}$$

Similarly,

$$\begin{aligned} u(x_1^+) &= A_0 + A_1(x_1 - a) + I_{01}(x_1, u(x_1), u'(x_1), v(x_1), v'(x_1)) \\ &+ (x_1 - a) \sum_{i\,:\,x_i<x} I_{1i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i)) \\ &+ \int_a^{x_1}\int_a^r \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)) \right. \\ &\left. + \int_s^b p(\xi)d\xi\right) dsdr \end{aligned}$$

and, by induction, it can be proved that the solution of the first equation of the problem (5.5), (5.2), (5.3), for $x \in [a,b]$, is given by

$$\begin{aligned}
u(x) &= A_0 + A_1(x-a) + \sum_{i\,:\,x_i<x} I_{0i}(x_i, u(x_i), u'(x_i), v(x_i), v'(x_i)) \\
&+ (x-a) \sum_{i\,:\,x_i<x} I_{1i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i)) \\
&+ \int_a^x \int_a^r \phi^{-1} \left( \phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)) \right. \\
&\left. + \int_s^b p(\xi) d\xi \right) ds dr
\end{aligned}$$

Likewise, for the second equation, we have

$$\begin{aligned}
v(x) &= B_0 + B_1(x-a) + \sum_{j\,:\,\tau_j<x} J_{0j}(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j)) \\
&+ (x-a) \sum_{j\,:\,\tau_j<x} J_{0j}(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)) \\
&+ \int_a^x \int_a^r \psi^{-1} \left( \psi(B_2) - \sum_{j\,:\,\tau_j>s} J_{2j}(\tau_j, u(\tau_j), u'(\tau_j), u''(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)) \right. \\
&\left. + \int_s^b q(\xi) d\xi \right) ds dr.
\end{aligned}$$

□

The Nagumo condition, introduced in [80], is an important tool for controlling the second derivatives. We consider here a Nagumo-type condition given by the next definition:

**Definition 5.1.3.** *Let $\gamma_k^{(l)}(x),\ \Gamma_k^{(l)}(x),\ k=1,2,\ l=0,1,$ be piecewise continuous functions such that*

$$\gamma_1^{(l)}(x) \le \Gamma_1^{(l)}(x),\ \gamma_2^{(l)}(x) \le \Gamma_2^{(l)}(x),\ a.e.\ x \in [a,b],$$

*and consider the set*

$$S = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, z_0, z_1, z_2) \in [a,b] \times \mathbb{R}^6 : \\ \gamma_1^{(l)}(x) \le y_l \le \Gamma_1^{(l)}(x),\ \gamma_2^{(l)}(x) \le z_l \le \Gamma_2^{(l)}(x),\ l=0,1 \end{array} \right\}. \tag{S}$$

*The $L^1-$Carathéodory functions $f, g : [a,b] \times \mathbb{R}^6 \to \mathbb{R}$ satisfy a Nagumo-type condition, if, there are $\mu_k > 0,\ k=1,2,$ with*

$$\begin{aligned}
\mu_1 : &= \max_{i=0,1,2,\dots,m} \left\{ \frac{|\Gamma_1'(x_{i+1}) - \gamma_1'(x_i)|}{x_{i+1} - x_i}, \frac{|\gamma_1'(x_{i+1}) - \Gamma_1'(x_i)|}{x_{i+1} - x_i} \right\}, \\
\mu_2 : &= \max_{j=0,1,2,\dots,n} \left\{ \frac{|\Gamma_2'(\tau_{j+1}) - \gamma_2'(\tau_j)|}{\tau_{j+1} - \tau_j}, \frac{|\gamma_2'(\tau_{j+1}) - \Gamma_2'(\tau_j)|}{\tau_{j+1} - \tau_j} \right\},
\end{aligned} \tag{5.9}$$

*and continuous positive functions* $\varphi_k : [0,+\infty) \to (0,+\infty),\ k = 1,2,$ *verifying*

$$\begin{aligned} |f(x,y_0,y_1,y_2,z_0,z_1,z_2)| &\leq \varphi_1(|y_2|),\ \forall(x,y_0,y_1,y_2,z_0,z_1,z_2) \in S, \\ |g(x,y_0,y_1,y_2,z_0,z_1,z_2)| &\leq \varphi_2(|z_2|),\ \forall(x,y_0,y_1,y_2,z_0,z_1,z_2) \in S, \end{aligned} \tag{5.10}$$

*with*

$$\int_{\phi(\mu_1)}^{\phi(+\infty)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds = +\infty,\ \int_{\psi(\mu_2)}^{\psi(+\infty)} \frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)} ds = +\infty. \tag{5.11}$$

This growth condition allows *a priori* estimations on the second derivatives.

**Lemma 5.1.4.** *Consider* $\gamma_k', \Gamma_k' \in PC^1[a,b],\ k = 1,2,$ *such that*

$$\gamma_k'(x) \leq \Gamma_k'(x),\ a.e.\ x \in [a,b],$$

*and let* $f, g : [a,b] \times \mathbb{R}^6 \to \mathbb{R}$ *be* $L^1-$*Carathéodory functions satisfying a Nagumo-type condition, according to Definition 5.1.3. Then, there exist* $N_k > 0,\ k = 1,2,$ *such that for every solution* $(u,v)$ *of (5.1) on the set* $S$ *satisfies*

$$\|u''\|_\infty \leq N_1 \text{ and } \|v''\|_\infty \leq N_2.$$

**Remark 5.1.5.** *Note that* $N_1$ *depends only on* $\gamma_1', \Gamma_1'$ *and* $\varphi_1$*, and* $N_2$ *on* $\gamma_2', \Gamma_2'$ *and* $\varphi_2$.

*Proof.* Let $(u(x), v(x))$ be a solution of (5.1) on the set S.

By the Mean Value Theorem, there are $\bar{x} \in (x_i, x_{i+1})$ and $\tilde{x} \in (\tau_j, \tau_{j+1})$ such that

$$u''(\bar{x}) = \frac{u'(x_{i+1}) - u'(x_i^+)}{x_{i+1} - x_i} \quad \text{and} \quad v''(\tilde{x}) = \frac{v'(\tau_{j+1}) - v'(\tau_j^+)}{\tau_{j+1} - \tau_j}. \tag{5.12}$$

If $|u''(x)| \leq \mu_1,\ \forall x \in [a,b]$, then it is enough to define $N_1 := \mu_1$ and the proof is complete.

The case $|u''(t)| > \mu_1,\ \forall x \in [a,b]$, with $\mu_1$ defined in (5.9), is not possible.

In fact, if $u''(x) > \mu_1,\ \forall x \in (x_i, x_{i+1})$, we obtain, by (5.12), (S) and (5.9), the contradiction

$$u''(\bar{x}) = \frac{u'(x_{i+1}) - u'(x_i^+)}{x_{i+1} - x_i} \leq \frac{\Gamma_1'(x_{i+1}) - \gamma_1'(x_i)}{x_{i+1} - x_i} \leq \mu_1. \tag{5.13}$$

If $u''(x) < -\mu_1,\ \forall x \in [a,b]$, the contradiction is similar.

Assume, now, that there are $\breve{x}, x^* \in (x_i, x_{i+1})$ with $\breve{x} < x^*$, such that

$$u''(\breve{x}) \leq \mu_1 \quad \text{and} \quad u''(x^*) > \mu_1.$$

By continuity of $u''(x)$, there exists $\hat{x} \in [\breve{x}, x^*]$ such that $u''(\hat{x}) = \mu_1$, and $u''(x) > 0, \forall x \in [\breve{x}, x^*]$.

Consider $N_k > \mu_k$, $k = 1, 2$, such that

$$\int_{\phi(\mu_1)}^{\phi(N_1)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds > \mu_1(b-a) \quad \text{and} \quad \int_{\psi(\mu_2)}^{\psi(N_2)} \frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)} ds > \mu_2(b-a). \qquad (5.14)$$

Making a convenient change of variable and using (5.10) and (5.13),

$$\begin{aligned}
\int_{\phi(u''(\hat{x}))}^{\phi(u''(x^*))} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds &= \int_{\hat{x}}^{x^*} \frac{|\phi^{-1}(\phi(u''(x)))|}{\varphi_1(|\phi^{-1}(\phi(u''(x)))|)} (\phi(u''(x)))' dx \\
&\leq \int_{\hat{x}}^{x^*} \frac{|u''(x)|}{\varphi_1(|u''(x)|)} (\phi(u''(x)))' dx \\
&\leq \int_{\hat{x}}^{x^*} \frac{|u''(x)| \cdot |-f(x, u(x), u'(x), u''(x), v(x), v'(x), v''(x))|}{\varphi_1(|u''(x)|)} dx, \\
&\leq \int_{\hat{x}}^{x^*} \frac{|u''(x)| \cdot |\varphi_1(|u''(x)|)|}{\varphi(|u''(t)|)} dx, \\
&\leq \int_{\hat{x}}^{x^*} u''(x) dx = u'(x^*) - u'(\hat{x}) \leq \mu_1(b-a),
\end{aligned}$$

and by (5.14)

$$\int_{\phi(u''(\hat{x}))}^{\phi(u''(x^*))} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds \leq \mu_1(b-a) < \int_{\phi(\mu_1)}^{\phi(N_1)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds.$$

Therefore $u''(x^*) < N_1$, and as $x^*$ is taken arbitrarily, then $u''(x) < N_1$, for the values of $x$ whenever $u''(x) > \mu_1$.

The case for $\breve{x} > x^*$ follows similar arguments.

The other possible case where

$$u''(\breve{x}) > -\mu_1 \quad \text{and} \quad u''(x^*) < -\mu_1.$$

can be proved by the previous techniques. Therefore $\|u''\|_\infty \leq N_1$.

By a similar method, it can be shown that $\|v''\|_\infty \leq N_2$. □

The arguments forward will require the following lemma, of [99]:

**Lemma 5.1.6.** *For $v, w \in C(I)$ such that $v(x) \leq w(x)$, for every $x \in I$, define*

$$q(x, u) = max\{v, min\{u, w\}\}.$$

*Then, for each $u \in C^1(I)$ the next two properties hold:*

*(a) $\frac{d}{dx} q(x, u(x))$ exists for a.e. $x \in I$.*

*(b) If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$ then*

$$\frac{d}{dx} q(x, u_m(x)) \to \frac{d}{dx} q(x, u(x)) \text{ for a.e. } x \in I.$$

Schauder's fixed point theorem will be the key existence tool:

**Theorem 5.1.7.** *([106]) Let $Y$ be a nonempty, closed, bounded, and convex subset of a Banach space $X$, and suppose that $P : Y \to Y$ is a compact operator. Then $P$ has at least one fixed point in $Y$.*

## 5.2 Existence result

The next theorem will guarantee the existence of a solution of (5.1)-(5.3), through the existence of fixed points of a convenient operator.

**(H1)** $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are increasing homeomorphisms such that $\phi(0) = \psi(0) = 0$ and $\phi(\mathbb{R}) = \psi(\mathbb{R}) = \mathbb{R}$, and

$$\left|\phi^{-1}(w)\right| \leq \phi^{-1}(|w|), \text{ and } \left|\psi^{-1}(w)\right| \leq \psi^{-1}(|w|).$$

**Theorem 5.2.1.** *Consider $A_k, B_k \in \mathbb{R}$, $k = 0, 1, 2$, and the homeomorphisms $\phi$ and $\psi$ verifying (H1). Let $f, g : [a, b] \times \mathbb{R}^6 \to \mathbb{R}$ be $L^1$-Carathéodory functions, satisfying a Nagumo-type conditions as in Definition 5.1.3, and $I_{ki}, J_{kj}$, $k = 0, 1, 2$, $i = 1, ..., m$, $j = 1, ..., n$, continuous functions. Then there is at least one pair of functions $(u, v) \in X^2$ solution to the problem (5.1)-(5.3).*

*Proof.* Define the operators $T_1 : X^2 \to X_1$, $T_2 : X^2 \to X_2$, and $T : X^2 \to X^2$ given by

$$T(u, v) = (T_1(u, v), T_2(u, v)), \tag{5.15}$$

with

$$\begin{aligned}
(T_1(u, v))(x) &= A_0 + A_1(x-a) + \sum_{i\,:\,x_i<x} I_{0i}(x_i, u(x_i), u'(x_i), v(x_i), v'(x_i)) \\
&+ (x-a) \sum_{i\,:\,x_i<x} I_{1i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i)) \\
&+ \int_a^x \int_a^r \phi^{-1} \Bigg( \phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)) \\
&+ \int_s^b f(\xi, u(\xi), u'(\xi), u''(\xi), v(\xi), v'(\xi), v''(\xi)) d\xi \Bigg) dsdr
\end{aligned}$$

and

$$\begin{aligned}
(T_2(u, v))(x) &= B_0 + B_1(x-a) + \sum_{j\,:\,\tau_j<x} J_{0j}(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j)) \\
&+ (x-a) \sum_{j\,:\,\tau_j<x} J_{1j}(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)) \\
&+ \int_a^x \int_a^r \psi^{-1} \Bigg( \psi(B_2) - \sum_{j\,:\,\tau_j>s} J_{2j}(\tau_j, u(\tau_j), u'(\tau_j), u''(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)) \\
&+ \int_s^b g(\xi, u(\xi), u'(\xi), u''(\xi), v(\xi), v'(\xi), v''(\xi)) d\xi \Bigg) dsdr.
\end{aligned}$$

Define $L > 0$ and $M > 0$, such that

$$L > \|(u,v)\|_{X^2} \tag{5.16}$$

and

$$M > \max_{k=0,1,2} \left\{ \sum_{i=1}^{m} |I_{ki}|, \sum_{j=1}^{n} |J_{kj}| \right\}. \tag{5.17}$$

Since $f$ and $g$ are $L^1-$Carathéodory functions and non-negative function $\rho_{\kappa L}(x) \in L^1([a,b])$, $\kappa = 1, 2$, such that

$$\begin{aligned} &|f(x,u(x),u'(x),u''(x),v(x),v'(x),v''(x))| \leq \rho_{1L}(x), \\ &|g(x,u(x),u'(x),u''(x),v(x),v'(x),v''(x))| \leq \rho_{2L}(x), \quad a.e.\ x \in [a,b]. \end{aligned} \tag{5.18}$$

The proof will follow several steps that, for clarity, are detailed for the $T_1(u,v)$ operator. The technique for the $T_2(u,v)$ operator is similar.

**Step 1**: *$T$ is well defined, continuous and uniformly bounded.*

By the Lebesgue dominated convergence Theorem, (5.18), (5.3), (H1), and (5.17), then

$$\begin{aligned} |(T_1(u,v))(x)| \leq\ & |A_0| + |A_1(x-a)| + \sum_{i\,:\,x_i<x} |I_{0i}| + |(x-a)| \sum_{i\,:\,x_i<x} |I_{1i}| \\ & + \left| \int_a^x \int_a^r \phi^{-1} \left( \phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i} \right.\right. \\ & \left.\left. + \int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi \right) ds dr \right| \\ \leq\ & |A_0| + |A_1(x-a)| + M + M\,|(x-a)| \\ & + \int_a^x \int_a^r \phi^{-1} \left( |\phi(A_2)| + M + \int_s^b |f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))|\, d\xi \right) ds dr \\ \leq\ & |A_0| + |A_1(b-a)| + M + M\,|(b-a)| \\ & + \int_a^b \int_a^r \phi^{-1} \left( |\phi(A_2)| + M + \int_s^b \rho_{1L}(\xi) d\xi \right) ds dr < +\infty, \end{aligned}$$

$$\begin{aligned} \left|(T_1(u,v))'(x)\right| \leq\ & |A_1| + \sum_{i\,:\,x_i<x} |I_{1i}| \\ & + \left| \int_a^x \phi^{-1} \left( \phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i} \right.\right. \\ & \left.\left. + \int_s^b |f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))|\, d\xi \right) ds \right| \\ \leq\ & |A_1| + M + \int_a^b \phi^{-1} \left( |\phi(A_2)| + M + \int_s^b \rho_{1L}(\xi) d\xi \right) ds < +\infty \end{aligned}$$

and

$$\left|(T_1(u,v))''(x)\right| \le \left|\phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i} + \int_s^b \left|f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))\right| d\xi\right)\right| \le \phi^{-1}\left(|\phi(A_2)| + M + \int_a^b \rho_{1L}(\xi)d\xi\right) < +\infty$$

Therefore $(T_1(u,v))(x) \in X_1$. The proof that $(T_2(u,v))(x) \in X_2$ is similar, and, so, therefore $T$ is well defined in $X^2$.

Moreover, defining $\mathcal{B} \subseteq X^2$ as

$$\mathcal{B}.=\{(u,v) \in X^2 : \|(u,v)\|_{X^2} \le L\},$$

from the above, it is clear that $T\mathcal{B}$ is uniformly bounded.

**Step 2**: *$T$ is equicontinuous, that is, $T_1\mathcal{B}$ is equicontinuous on each interval $]x_i,x_{i+1}]$, for $i=0,1,...,m$, with $x_0=a$ and $x_{m+1}=b$, and $T_2\mathcal{B}$ is equicontinuous on each interval $]\tau_j,\tau_{j+1}]$, for $j=0,1,...,n$, with $\tau_0=a$ and $\tau_{n+1}=b$.*

Consider $\mathcal{I} \subseteq ]x_i,x_{i+1}]$ and $\tilde{x},x^* \in \mathcal{I}$ such that, without loss of generality, $\tilde{x} \le x^*$. For $(u,v) \in \mathcal{B}$, we have

$$\begin{aligned}
&\lim_{\tilde{x}\to x^*} |(T_1(u,v))(x^*) - (T_1(u,v))(\tilde{x})| \le \lim_{\tilde{x}\to x^*} |A_1(x^*-a) - A_1(\tilde{x}-a)| \\
&+ \lim_{\tilde{x}\to x^*}\left|\sum_{a<x_i<x^*} I_{0i} - \sum_{a<x_i<\tilde{x}} I_{0i} + (x^*-a)\sum_{a<x_i<x^*} I_{1i} - (\tilde{x}-a)\sum_{a<x_i<\tilde{x}} I_{1i}\right| \\
&+ \lim_{\tilde{x}\to x^*}\left|\int_{x_i}^{x^*}\int_a^r \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i} + \int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right) dsdr\right. \\
&\left. - \int_{x_i}^{\tilde{x}}\int_a^r \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i} + \int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right) dsdr\right| \\
&\le \lim_{\tilde{x}\to x^*}\left|\int_{\tilde{x}}^{x^*}\int_a^r \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i} + \int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right) dsdr\right| = 0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{\tilde{x}\to x^*}\left|(T_1(u,v))'(x^*)-(T_1(u,v))'(\tilde{x})\right| \leq \lim_{\tilde{x}\to x^*}\left|\sum_{a<x_i<x^*} I_{1i}-\sum_{a<x_i<\tilde{x}} I_{1i}\right| \\
&+\lim_{\tilde{x}\to x^*}\left|\int_{x_i}^{x^*}\phi^{-1}\left(\phi(A_2)-\sum_{i\,:\,x_i>s} I_{2i}\right.\right. \\
&\left.+\int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right)ds \\
&\left.-\int_{x_i}^{\tilde{x}}\phi^{-1}\left(\phi(A_2)-\sum_{i\,:\,x_i>s} I_{2i}+\int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right)ds\right| \\
&\leq \lim_{\tilde{x}\to x^*}\left|\int_{\tilde{x}}^{x^*}\phi^{-1}\left(\phi(A_2)-\sum_{i\,:\,x_i>s} I_{2i}\right.\right. \\
&\left.\left.+\int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right)ds\right|=0,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\tilde{x}\to x^*}\left|(T_1(u,v))''(x^*)-(T_1(u,v))''(\tilde{x})\right| \leq \\
&\lim_{\tilde{x}\to x^*}\left|\phi^{-1}\left(\phi(A_2)-\sum_{i:\,x_i>x^*} I_{2i}+\int_{x^*}^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right)\right. \\
&\left.-\phi^{-1}\left(\phi(A_2)-\sum_{i:\,x_i>\tilde{x}} I_{2i}+\int_{\tilde{x}}^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right)\right|=0.
\end{aligned}$$

Therefore, $T_1\mathcal{B}$ is equicontinuous on $X_1$. Similarly, we can show that $T_2\mathcal{B}$ is equicontinuous on $X_2$, too. Thus, $T\mathcal{B}$ is equicontinuous on $X^2$.

**Step 3**: $T\mathcal{B}: X^2 \to X^2$ *is equiconvergent at* $x = x_i$ and $x = \tau_j$.

First, let us prove the equiconvergence at $x = x_i^+$, for $i = 1, 2, ...m$. The proof of equiconvergence at $x = \tau_j^+$, for $j = 1, 2, ...n$, is analogous.

So, it follows, for $i = 1, 2, ..., m$,

$$\left|\left(T_1\left(u,v\right)\right)(x_i) - \lim_{x\to x_i^+}\left(T_1\left(u,v\right)\right)(x)\right| \leq \left|A_1(x_i-a) - \lim_{x\to x_i^+} A_1(x-a)\right|$$
$$+\left|\sum_{a<x_\lambda<x_i} I_{0\lambda} - \lim_{x\to x_i^+}\sum_{a<x_\lambda<x} I_{0\lambda} + (x_i-a)\sum_{a<x_\lambda<x_i} I_{1\lambda} - \lim_{x\to x_i^+}(x-a)\sum_{a<x_\lambda<x} I_{1\lambda}\right|$$
$$+\left|\int_a^{x_i}\int_a^r \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}\right.\right.$$
$$\left.+\int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right) dsdr$$
$$-\lim_{x\to x_i^+}\int_a^x\int_a^r \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}\right.$$
$$\left.\left.+\int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right) dsdr\right|$$
$$\leq \lim_{x\to x_i^+}\left|\int_x^{x_i}\int_a^r \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}\right.\right.$$
$$\left.\left.+\int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right) dsdr\right| = 0,$$

$$\left|\left(T_1\left(u,v\right)\right)'(x_i) - \lim_{x\to x_i^+}\left(T_1\left(u,v\right)\right)'(x)\right| \leq \left|\sum_{a<x_\lambda<x_i} I_{1\lambda} - \lim_{x\to x_i^+}\sum_{a<x_\lambda<x} I_{1\lambda}\right|$$
$$+\left|\int_a^{x_i}\phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i} + \int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right) ds\right.$$
$$-\lim_{x\to x_i^+}\int_a^x \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}\right.$$
$$\left.\left.+\int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right) ds\right|$$
$$\leq \lim_{x\to x_i^+}\left|\int_{x_i}^x \phi^{-1}\left(\phi(A_2) - \sum_{i\,:\,x_i>s} I_{2i}\right.\right.$$
$$\left.\left.+\int_s^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right) ds\right| = 0,$$

and

$$\left|\left(T_1(u,v)\right)''(x_i) - \lim_{x\to x_i^+}\left(T_1(u,v)\right)''(x)\right| \leq$$

$$+\left|\phi^{-1}\left(\phi(A_2) - \sum_{a<x_\lambda<x_i} I_{2\lambda} + \int_{x_i}^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right)\right.$$

$$- \lim_{x\to x_i^+}\phi^{-1}\left(\phi(A_2) - \sum_{a<x_\lambda<x} I_{2\lambda}\right.$$

$$\left.\left.+\int_x^b f(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi))d\xi\right)\right| = 0.$$

Therefore, $T_1\mathcal{B}$ is equiconvergent at each point $x = x_i^+$ , for $i = 1,2,...,m$.

Analogously, it can be proved that $T_2\mathcal{B}$ is equiconvergent at each point $x = \tau_j^+$, for $j = 1,2,...,n$.

So, $T\mathcal{B}$ is equiconvergent at each impulsive point.

**Step 4**: $T : X^2 \to X^2$ *has a fixed point* .

Consider

$$\Omega := \{(u,v) \in X^2 : \|(u,v)\|_{X^2} \leq K\},$$

with $K > 0$ such that

$$K := \max\left\{\begin{array}{l} L, |A_0| + |A_1(b-a)| + M + M\,|(b-a)| \\ +\int_a^b\int_a^r \phi^{-1}\left(|\phi(A_2)| + M + \int_s^b \rho_{1L}(\xi)d\xi\right)dsdr, \\ |A_1| + M + \int_a^b \phi^{-1}\left(|\phi(A_2)| + M + \int_s^b \rho_{1L}(\xi)d\xi\right)ds, \\ \phi^{-1}\left(|\phi(A_2)| + M + \int_a^b \rho_{1L}(\xi)d\xi\right), \\ |B_0| + |B_1(b-a)| + M + M\,|(b-a)| \\ +\int_a^b\int_a^r \psi^{-1}\left(|\psi(B_2)| + M + \int_s^b \rho_{2L}(\xi)d\xi\right)dsdr, \\ |B_1| + M + \int_a^b \psi^{-1}\left(|\psi(B_2)| + M + \int_s^b \rho_{2L}(\xi)d\xi\right)ds, \\ \psi^{-1}\left(|\psi(B_2)| + M + \int_a^b \rho_{2L}(\xi)d\xi\right) \end{array}\right\},$$

with $L > 0$ and $M > 0$ given by (5.16) and (5.17).

According to Step 1, we have

$$\begin{aligned} \|T(u,v)\|_{X^2} &= \|(T_1(u,v), T_2(u,v))\|_{X^2} \\ &= \max\{\|T_1(u,v)\|_{X_1}, \|T_2(u,v)\|_{X_2}\} \\ &= \max\left\{\begin{array}{c} \|T_1(u,v)\|_\infty, \|T_1'(u,v)\|_\infty, \|T_1''(u,v)\|_\infty, \\ \|T_2(u,v)\|_\infty, \|T_2'(u,v)\|_\infty, \|T_2''(u,v)\|_\infty \end{array}\right\} \\ &\leq K. \end{aligned}$$

So, $T\Omega \subset \Omega$, and by Theorem 5.1.7, the operator $T(u,v) = (T_1(u,v), T_2(u,v))$, has a fixed point $(u^*, v^*)$.

By standard techniques, and Lemma 5.1.2, it can be shown that this fixed point is a solution of problems (5.1)-(5.3). □

**Example 5.2.2.** *Consider the following system of coupled differential equations*

$$\begin{cases} \dfrac{u'''(x)}{\sqrt{1+(u''(x))^2}} - (u(x))^3\, v(x) + u'(x)\arctan(v'(x)) + (u''(x))^2 - \sqrt[3]{v''(x)} = 0, \\ x \in [0,1] \setminus \{(x_i)\}, \\ \\ v'''(x) - u(x)v'(x) + \cos(u''(x))v(x) + (u'(x))^2 - v''(x) = 0, \\ x \in [0,1] \setminus \{(\tau_j)\} \end{cases} \tag{5.19}$$

*with the boundary conditions*

$$\begin{cases} u(0) = 0,\ u'(0) = 1,\ u''(1) = 0 \\ v(0) = 0,\ v'(0) = -1,\ v''(1) = 0, \end{cases} \tag{5.20}$$

*and the impulsive effects given by*

$$\begin{cases} \Delta u(x_i) &=& 2u(x_i) + u'(x_i) + v(x_i), \\ \Delta u'(x_i) &=& (u'(x_i))^2 - u''(x_i) + v(x_i), \\ \Delta\phi(u''(x_i)) &=& \dfrac{\sin(x_i)}{x_i}, \\ \Delta v(\tau_i) &=& u(\tau_i) + 3v(\tau_i) + v'(\tau_i), \\ \Delta v'(\tau_i) &=& u(\tau_i) + v(\tau_i) + 4\sin(v'(\tau_i)) - (v''(\tau_i))^2, \\ \Delta\psi(v''(\tau_j)) &=& e^{-\tau_j} \end{cases} \tag{5.21}$$

*with* $x_i = \dfrac{i}{5}$, *for* $i = 1,2,3,4$ *and* $\tau_j = \dfrac{j^2}{10}$, *for* $j = 1,2,3.$

This problem is a particular case of (5.1)-(5.3), with $[a,b] = [0,1]$,

$$\phi(y_2) = \operatorname{arcsinh} y_2,\ \psi(z_2) = z_2, \tag{5.22}$$

$$\begin{array}{rcl} f(x, y_0, y_1, y_2, z_0, z_1, z_2) &=& -y_0^3\ z_0 + y_1\ \arctan z_1 + y_2^2 - \sqrt[3]{z_2}, \\ g(x, y_0, y_1, y_2, z_0, z_1, z_2) &=& -y_0\ z_1 + \cos(y_2)\ z_0 + y_1^2 - z_2, \end{array}$$

$$A_0 = A_2 = B_0 = B_2 = 0,\ A_1 = 1, B_1 = -1,$$

$$\begin{array}{rcl} I_0(\cdot\ , y_0, y_1, z_0, z_1) &=& 2y_0 + y_1 + z_0, \\ I_1(\cdot\ , y_0, y_1, y_2, z_0, z_1) &=& y_1^2 - y_2 + z_0, \\ I_2(\cdot\ , y_0, y_1, y_2, z_0, z_1) &=& \dfrac{\sin(x_i)}{x_i}, \\ J_0(\cdot\ , y_0, y_1, z_0, z_1) &=& y_0 + 3z_0 + z_1, \\ J_1(\cdot\ , y_0, y_1, z_0, z_1, z_2) &=& y_0 + z_0 + 4\sin z_1 - z_2^2, \\ J_1(\cdot\ , y_0, y_1, z_0, z_1, z_2) &=& e^{-\tau_j} \end{array}$$

and $m = 4$, $n = 3$.

It is clear that the functions in (5.22) verify assumption (H1) and $f$ and $g$ satisfy a Nagumo-type condition in sets such as, for some piecewise continuous functions $\gamma_k^{(l)}(x)$, $\Gamma_k^{(l)}(x),\ k=1,2,\ l=0,1,$

$$\left\{\begin{array}{c}(x,y_0,y_1,y_2,z_0,z_1,z_2)\in[0,1]\times\mathbb{R}^6:\gamma_1(x)\le y_0\le\Gamma_1(x),\\ \gamma_1'(x)\le y_1\le\Gamma_1'(x),\gamma_2(x)\le z_0\le\Gamma_2(x),\gamma_2'(x)\le z_1\le\Gamma_2'(x)\end{array}\right\},$$

with

$$\varphi_1(|y_2|):=K_0+y_2^2,\ \text{and}\ \varphi_2(|z_2|):=K_1+|z_2|\,,$$

being $K_0,K_1$ some real positive numbers.

Therefore by Theorem 5.2.1, the problem (5.19)-(5.21) has at least a solution.

## 5.3 Existence and localization results

In addition to the existence of a solution, it is possible to obtain an existence and localization theorem, that is, not only it guarantees the existence of at least a solution, but provides also a strip where this solution is localized.

However, the localization part is obtained for a particular case of the impulsive conditions (5.4), applying lower and upper functions, defined as follows:

**Definition 5.3.1.** *The pair of functions* $(\alpha_1(x)\alpha_2(x))\in X^2$ *such that* $(\phi(\alpha_1''(x)),\psi(\alpha_2''(x)))\in(AC[a,b])^2$*, is a lower solution of problem (5.1), (5.2), (5.4), if*

$$\begin{cases}(\phi(\alpha_1''(x)))'+f(x,\alpha_1(x),\alpha_1'(x),\alpha_1''(x),\alpha_2(x),\alpha_2'(x),z)\ge 0,\ \textit{for } z\in\mathbb{R}\\ (\psi(\alpha_2''(x)))'+g(x,\alpha_1(x),\alpha_1'(x),y,\alpha_2(x),\alpha_2'(x),\alpha_2''(x))\ge 0,\ \textit{for } y\in\mathbb{R},\\ \alpha_1^{(l)}(a)\le A_l,\ \alpha_2^{(l)}(a)\le B_l,\ l=0,1,\\ \alpha_1''(b)\le A_2,\ \alpha_2''(b)\le B_2,\\ \Delta\alpha_1(x_i)\le I_{0i}(x_i,\alpha_1(x_i),\alpha_1'(x_i),\alpha_2(x_i),\alpha_2'(x_i)),\\ \Delta\alpha_1'(x_i)\le I_{1i}(x_i,\alpha_1(x_i),\alpha_1'(x_i),\alpha_1''(x_i),\alpha_2(x_i),\alpha_2'(x_i)),\\ \Delta\alpha_2(\tau_j)\le J_{0j}(\tau_j,\alpha_1(\tau_j),\alpha_1'(\tau_j),\alpha_2(\tau_j),\alpha_2'(\tau_j)),\\ \Delta\alpha_2'(\tau_j)\le J_{1j}(\tau_j,\alpha_1(\tau_j),\alpha_1'(\tau_j),\alpha_2(\tau_j),\alpha_2'(\tau_j),\alpha_2''(\tau_j)),\end{cases}\tag{5.23}$$

*A pair of functions* $(\beta_1(x),\beta_2(x))\in X^2$ *such that* $(\phi(\beta_1''(x)),\psi(\beta_2''(x)))\in(AC[a,b])^2$ *is an upper solution of problem (5.1), (5.2), (5.4) if the opposite inequalities hold.*

To obtain this goal, we consider local monotone assumptions:

**(H2)** $f,g:[a,b]\times\mathbb{R}^4\mapsto\mathbb{R}$, are $L^1$ -Carathéodory such that

$$\begin{aligned}f(x,\alpha_1(x),y_1,\alpha_1''(x),\alpha_2(x),\alpha_2'(x),z_2)&\le f(x,y_0,y_1,y_2,z_0,z_1,z_2)\\ &\le f(x,\beta_1(x),y_1,\beta_1''(x),\beta_2(x),\beta_2'(x),z_2),\end{aligned}$$

for $\alpha_1(x) \leq y_0 \leq \beta_1(x)$, $\alpha_1''(x) \leq y_2 \leq \beta_1''(x)$, $\alpha_2(x) \leq z_0 \leq \beta_2(x)$, $\alpha_2'(x) \leq z_1 \leq \beta_2'(x)$, and $(x, y_1, z_2) \in [a,b] \times \mathbb{R}^2$, and

$$\begin{aligned} g(x, \alpha_1(x), \alpha_1'(x), y_2, \alpha_2(x), z_1, \alpha_2''(x)) &\leq g(x, y_0, y_1, y_2, z_0, z_1, z_2) \\ &\leq g(x, \beta_1(x), \beta_1'(x), y_2, \beta_2(x), z_1, \beta_2''(x)), \end{aligned}$$

for $\alpha_1(x) \leq y_0 \leq \beta_1(x)$, $\alpha_1'(x) \leq y_1 \leq \beta_1'(x)$, $\alpha_2(x) \leq z_0 \leq \beta_2(x)$, $\alpha_2''(x) \leq z_2 \leq \beta_2''(x)$, and $(x, y_2, z_1) \in [a,b] \times \mathbb{R}^2$.

**(H3)** $I_{0i}, J_{0j} \in C([a,b] \times \mathbb{R}^4, \mathbb{R})$, verify

$$\begin{aligned} I_{0i}(x_i, \alpha_1(x_i), \alpha_1'(x_i), \alpha_2(x_i), \alpha_2'(x_i)) &\leq I_{0i}(x_i, y_0, y_1, z_0, z_1) \\ &\leq I_{0i}(x_i, \beta_1(x_i), \beta_1'(x_i), \beta_2(x_i), \beta_2'(x_i)), \end{aligned}$$

for $i = 1, 2, ..., m$, $l = 0, 1$, $\alpha_1^{(l)}(x) \leq y_l \leq \beta_1^{(l)}(x)$, $\alpha_2^{(l)}(x) \leq z_l \leq \beta_2^{(l)}(x)$, and

$$\begin{aligned} J_{0j}(\tau_j, \alpha_1(\tau_j), \alpha_1'(\tau_j), \alpha_2(\tau_j), \alpha_2'(\tau_j)) &\leq J_{0j}(\tau_j, y_0, y_1, z_0, z_1) \\ &\leq J_{0j}(\tau_j, \beta_1(\tau_j), \beta_1'(\tau_j), \beta_2(\tau_j), \beta_2'(\tau_j)). \end{aligned}$$

for $j = 1, 2, ..., n$, $l = 0, 1$, $\alpha_1^{(l)}(x) \leq y_l \leq \beta_1^{(l)}(x)$, $\alpha_2^{(l)}(x) \leq z_l \leq \beta_2^{(l)}(x)$,
and $I_{1i}, J_{1j} \in C([a,b] \times \mathbb{R}^5, \mathbb{R})$, satisfy

$$\begin{aligned} I_{1i}(x_i, \alpha_1(x_i), y_1, y_2, \alpha_2(x_i), \alpha_2'(x_i)) &\leq I_{1i}(x_i, y_0, y_1, y_2, z_0, z_1) \\ &\leq I_i(x_i, \beta_1(x_i), y_1, y_2, \beta_2(x_i), \beta_2'(x_i)), \end{aligned}$$

for $i = 1, 2, ..., m$, $l = 0, 1$, $\alpha_1(x) \leq y_0 \leq \beta_1(x)$, $\alpha_2^{(l)}(x) \leq z_l \leq \beta_2^{(l)}(x)$, $\forall (y_1, y_2) \in \mathbb{R}^2$ and

$$\begin{aligned} J_{1j}(\tau_j, \alpha_1(\tau_j), \alpha_1'(\tau_j), \alpha_2(\tau_j), z_1, z_2) &\leq J_{1j}(\tau_j, y_0, y_1, z_0, z_1, z_2) \\ &\leq J_{1j}(\tau_j, \beta_1(\tau_j), \beta_1'(\tau_j), \beta_2(\tau_j), z_1, z_2). \end{aligned}$$

for $j = 1, 2, ..., n$, $l = 0, 1$, $\alpha_1^{(l)}(x) \leq y_l \leq \beta_1^{(l)}(x)$, $\alpha_2(x) \leq z_0 \leq \beta_2(x)$, $\forall (z_1, z_2) \in \mathbb{R}^2$.

The existence and localization theorem is given as follows:

**Theorem 5.3.2.** *Let $A_k, B_k \in \mathbb{R}$, $k = 0, 1, 2$, and the homeomorphisms $\phi$ and $\psi$ verify (H1). Assume that there are lower and upper solutions of (5.1), (5.2), (5.4), $(\alpha_1^{(l)}, \alpha_2^{(l)})$ and $(\beta_1^{(l)}, \beta_2^{(l)})$, respectively, such that*

$$\alpha_\kappa^{(l)}(x) \leq \beta_\kappa^{(l)}(x), \quad \kappa = 1, 2, \quad l = 0, 1, \ \forall x \in [a,b],$$

*the $L^1-$Carathéodory functions $f, g : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy Nagumo conditions as in Definition 5.1.3, in the set*

$$S^* = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, z_0, z_1, z_2) \in [a,b] \times \mathbb{R}^6 : \\ \alpha_1^{(l)}(x) \leq y_l \leq \beta_1^{(l)}(x), \alpha_2^{(l)}(x) \leq z_l \leq \beta_2^{(l)}(x), \ l = 0, 1 \end{array} \right\}.$$

*If assumptions (H2), and (H3) hold, then there is at least a pair $(u(x), v(x)) \in X^2$ solution of (5.1), (5.2), (5.4) and, moreover,*

$$\alpha_1^{(l)}(x) \leq u^{(l)}(x) \leq \beta_1^{(l)}(x), \quad \alpha_2^{(l)}(x) \leq v^{(l)}(x) \leq \beta_2^{(l)}(x), \quad l = 0, 1, \quad \forall x \in [a,b],$$

*and*

$$\|u''\| \leq N_1 \quad \text{and} \quad \|v''\| \leq N_2,$$

*with $N_1$ and $N_2$ given by Lemma 5.1.4.*

*Proof.* Define the truncature functions $\delta_{im} : [a,b] \times \mathbb{R} \to \mathbb{R}$, for $\kappa = 1,2$ and $l = 0,1$, given by

$$\delta_{\kappa l}(x, w_l) = \begin{cases} \beta_\kappa^{(l)}(x) & \text{if } \ w_l > \beta_\kappa^{(l)}(x) \\ w_l & \text{if } \ \alpha_\kappa^{(l)}(x) \le w_l \le \beta_\kappa^{(l)}(x) \\ \alpha_\kappa^{(l)}(x) & \text{if } \ w_l < \alpha_\kappa^{(l)}(x). \end{cases} \tag{5.24}$$

Consider the following modified coupled system composed by the truncated and perturbed differential equations

$$\begin{cases} (\phi(u''(x)))' + f\begin{pmatrix} x, \delta_{10}(x,u(x)), \delta_{11}(x,u'(x)), u''(x), \\ \delta_{20}(x,v(x)), \delta_{21}(x,v'(x)), v''(x) \end{pmatrix} \\ +\dfrac{\delta_{11}(x,u'(x)) - u'(x)}{1 + |\delta_{11}(x,u'(x)) - u'(x)|} = 0, \\ \\ (\psi(v''(x)))' + g\begin{pmatrix} x, \delta_{10}(x,u(x)), \delta_{11}(x,u'(x)), u''(x), \\ \delta_{20}(x,v(x)), \delta_{21}(x,v'(x)), v''(x) \end{pmatrix} \\ +\dfrac{\delta_{21}(x,v'(x)) - v'(x)}{1 + |\delta_{21}(x,v'(x)) - v'(x)|} = 0, \end{cases} \tag{5.25}$$

with the truncated impulsive conditions

$$\begin{aligned} \Delta u(x_i) &= I_{0i}(x_i, \delta_{10}(x_i,u(x_i)), \delta_{11}(x_i,u'(x_i)), \delta_{20}(x_i,v(x_i)), \delta_{21}(x_i,v'(x_i))), \\ \Delta u'(x_i) &= I_{1i}\begin{pmatrix} x_i, \delta_{10}(x_i,u(x_i)), \delta_{11}(x_i,u'(x_i)), \dfrac{d}{dx}\delta_{11}(x_i,u'(x_i)), \\ \delta_{20}(x_i,v(x_i)), \delta_{21}(x_i,v'(x_i)) \end{pmatrix} \\ \Delta v(\tau_j) &= J_{0j}(\tau_j, \delta_{10}(\tau_j,u(\tau_j)), \delta_{11}(\tau_j,u'(\tau_j)), \delta_{20}(\tau_j,v(\tau_j)), \delta_{21}(\tau_j,v'(\tau_j))), \\ \Delta v'(\tau_j) &= J_{1j}\begin{pmatrix} \tau_j, \delta_{10}(\tau_j,u(\tau_j)), \delta_{11}(\tau_j,u'(\tau_j)), \delta_{20}(\tau_j,v(\tau_j)), \\ \delta_{21}(\tau_j,v'(\tau_j)), \dfrac{d}{dx}\delta_{21}(x_i,v'(\tau_j)) \end{pmatrix}, \end{aligned} \tag{5.26}$$

for $i = 1,2,...,m$, $j = 1,2,...,n$, and the boundary conditions (5.2).

It is clear that the functions $F$ and $G$, given by

$$\begin{aligned} F(x) := {} & f(x, \delta_{10}(x,u(x)), \delta_{11}(x,u'(x)), u''(x), \delta_{20}(x,v(x)), \delta_{21}(x,v'(x)), v''(x)) \\ & +\frac{\delta_{11}(x,u'(x)) - u'(x)}{1 + |\delta_{11}(x,u'(x)) - u'(x)|} \end{aligned}$$

and

$$\begin{aligned} G(x) := {} & g(x, \delta_{10}(x,u(x)), \delta_{11}(x,u'(x)), u''(x), \delta_{20}(x,v(x)), \delta_{21}(x,v'(x)), v''(x)) \\ & +\frac{\delta_{21}(x,v'(x)) - v'(x)}{1 + |\delta_{21}(x,v'(x)) - v'(x)|} \end{aligned}$$

satisfy the Nagumo type conditions, as in Definition 5.1.3, relative to the set $S^*$, with

$$|F(x, y_0, y_1, y_2, z_0, z_1, z_2)| \le \varphi_1(|y_2|) + 1$$

and

$$|G(x, y_0, y_1, y_2, z_0, z_1, z_2)| \le \varphi_2(|y_2|) + 1.$$

Therefore, applying the same arguments as in the Theorem 5.2.1, it can be proved that problem (5.25), (5.2), (5.26) has, at least a solution $(u(x), v(x))$, To prove that this solution is also a solution to the initial problem (5.1), (5.2), (5.4), it will be enough to show that

$$\alpha_1^{(l)}(x) \le u^{(l)}(x) \le \beta_1^{(l)}(x), \quad \alpha_2^{(l)}(x) \le v^{(l)}(x) \le \beta_2^{(l)}(x), \quad l = 0, 1, \quad \forall x \in [a, b].$$

For the second inequality, assume, by contradiction, that there is $x \in [a, b]$ such that $u'(x) > \beta_1'(x)$, and define

$$\sup_{a \le x \le b} (u'(x) - \beta_1'(x)) := u'(\bar{x}) - \beta_1'(\bar{x}) > 0. \tag{5.27}$$

As, by boundary conditions (5.2) and Definition 5.3.1, $u'(a) - \beta_1'(a) \le 0$, then $\bar{x} \ne a$. In the same way, $u''(b^-) - \beta_1''(b^-) \le 0$, therefore $\bar{x} \ne b$.

Then $\bar{x} \in (a, b)$, two possibilities remain to be studied:

(i) Assume that there is $p \in \{0, 1, 2, ..., n\}$ such that $\bar{x} \in (x_p, x_{p+1})$. Therefore

$$\max_{x \in (x_p, x_{p+1})} (u'(x) - \beta_1'(x)) := u'(\bar{x}) - \beta_1'(\bar{x}) > 0,$$

and

$$u''(\bar{x}) - \beta_1''(\bar{x}) = 0. \tag{5.28}$$

Choose $\epsilon > 0$, sufficiently small, such that

$$u'(x) - \beta_1'(x) > 0 \text{ and } u''(x) - \beta_1''(x) \le 0, \forall x \in (\bar{x}, \bar{x} + \epsilon). \tag{5.29}$$

By (H2), for all $x \in (\bar{x}, \bar{x} + \epsilon)$,

$$\begin{aligned}
&(\phi(u''(x)))' - (\phi(\beta_1''(x)))' \\
&\ge -f(x, \delta_{10}(x, u(x)), \delta_{11}(x, u'(x)), u''(x), \delta_{20}(x, v(x)), \delta_{21}(x, v'(x)), v''(x)) \\
&- \frac{\delta_{11}(x, u'(x)) - u'(x)}{1 + |\delta_{11}(x, u'(x)) - u'(x)|} + f(x, \beta_1(x), \beta_1'(x), \beta_1''(x), \beta_2(x), \beta_2'(x), \beta_2''(x)) \\
&\ge -f(x, \delta_{10}(x, u(x)), \beta_1'(x), \beta_1''(x), \delta_{20}(x, v(x)), \delta_{21}(x, v'(x)), v''(x)) \\
&- \frac{\beta_1'(x) - u'(x)}{1 + |\beta_1'(x) - u'(x)|} + f(x, \beta_1(x), \beta_1'(x), \beta_1''(x), \beta_2(x), \beta_2'(x), \beta_2''(x)) \\
&\ge \frac{u'(x) - \beta_1'(x)}{1 + |u'(x) - \beta_1'(x)|} > 0.
\end{aligned}$$

So $(\phi(u''(x)) - \phi(\beta_1''(x)))$ is increasing for $\forall x \in (\bar{x}, \bar{x} + \epsilon)$, and, by (5.29 ), we obtain the contradiction in $(\bar{x}, \bar{x} + \epsilon)$, by (5.28) and (5.29):

$$0 = \phi(u''(\bar{x})) - \phi(\beta_1''(\bar{x})) < \phi(u''(x)) - \phi(\beta_1''(x) \le 0.$$

Therefore, for $x \in (x_p, x_{p+1})$, $p = 0, 1, 2, ..., n$,

$$u'(x) \le \beta_1'(x).$$

(ii) Suppose, now, that there is $p_* \in \{1, 2, ..., n\}$ such that, $\bar{x} = x_{p_*}$. That is,

$$\sup_{x\in[a,b]} (u'(x) - \beta_1'(x)) := u'(x_{p_*}) - \beta_1'(x_{p_*}) > 0. \tag{5.30}$$

As $u, \beta_1 \in X$, by (i), we obtain the contradiction:

$$u'(x_{p_*}) = \lim_{x\to x_{p_*}^-} u'(x) \le \lim_{x\to x_{p_*}^-} \beta_1'(x) = \beta_1'(x_{p_*}).$$

If $\bar{x} = x_{p_*}^+$, suppose

$$\sup_{x\in[a,b]} (u'(x) - \beta_1'(x)) := u'(x_{p_*}^+) - \beta_1'(x_{p_*}^+) > 0,$$

By (5.26), (H3) and Definition 5.3.1, we obtain the contradiction:

$$\begin{array}{rcl} 0 & < & u'(x_{p_*}^+) - \beta_1'(x_{p_*}^+) = u'(x_{p_*}) \\ & + & I_{1p_*}\left( \begin{array}{c} x_{p_*}, \delta_{10}(x_{p_*}, u(x_{p_*})), \delta_{11}(x_{p_*}, u'(x_{p_*})), \dfrac{d}{dx}\delta_{11}(x_{p_*}, u'(x_{p_*})), \\ \delta_{20}(x_{p_*}, v(x_{p_*})), \delta_{21}(x_{p_*}, v'(x_{p_*})) \end{array} \right) \\ & - & \beta_1'(x_{p_*}) - I_{p_*}(x_{p_*}, \beta_1(x_{p_*}), \beta_1'(x_{p_*}), \beta_1''(x_{p_*}), \beta_2(x_{p_*}), \beta_2'(x_{p_*})) \\ & \le & I_{1p_*}(x_{p_*}, \beta_1(x_{p_*}), \beta_1'(x_{p_*}), \beta_1''(x_{p_*}), \delta_{20}(x_{p_*}, v(x_{p_*})), \delta_{21}(x_{p_*}, v'(x_{p_*}))) \\ & - & I_{1p_*}(x_{p_*}, \beta_1(x_{p_*}), \beta_1'(x_{p_*}), \beta_1''(x_{p_*}), \beta_2(x_{p_*}), \beta_2'(x_{p_*})) \le 0. \end{array}$$

Therefore, $u'(x) \le \beta'(x)$, for $x \in [a, b]$.

By similar arguments, it can be proved the remaining inequality and, therefore,

$$\alpha_1'(x) \le u(x) \le \beta_1'(x), \text{ for all } x \in [a, b]. \tag{5.31}$$

The other inequalities follow similar steps.

By integration of (5.31) for $x \in [a, x_1]$,

$$\alpha_1(x) \le u(x) - u(a) + \alpha_1(a) \le u(x),$$

and for $x \in (x_1, x_2]$, we have, by (H3),

$$\begin{aligned} \alpha_1(x) &\le u(x) - u(x_1^+) + \alpha_1(x_1^+) \\ &\le u(x) - u(x_1) \\ &- I_{01}(x_1, \delta_{10}(x_1, u(x_1)), \delta_{11}(x_1, u'(x_1)), \delta_{20}(x_1, v(x_1)), \delta_{21}(x_1, v'(x_1))) \\ &+ \alpha_1(x_1) + I_{01}(x_i, \alpha_1(x_1), \alpha_1'(x_i), \alpha_2(x_i), \alpha_2'(x_i)) \\ &\le u(x). \end{aligned}$$

By recurrence, it can be shown, analogously, that

$$\alpha_1(x) \le u(x), \quad \forall x \in (x_i, x_{i+1}], \text{ for } i = 1, ..., m.$$

Therefore, $\alpha_1(x) \le u(x)$, $\forall x \in [a, b]$.

Analogously, it can be proved the remaining inequality and, therefore,

$$\alpha_1(x) \leq u(x) \leq \beta_1(x), \text{ for all } x \in [a,b]. \tag{5.32}$$

Analogously, it can be proved that

$$\alpha_2^{(l)}(x) \leq v^{(l)}(x) \leq \beta_2^{(l)}(x), \quad l = 0,1, \quad \forall x \in [a,b].$$

□

To illustrate the importance of the location arguments, we consider the following example:

**Example 5.3.3.** *Let be the problem composed system composed by the strongly nonlinear $\phi-$Laplacian and $p-$Laplacian differential equations*

$$\begin{cases} \dfrac{u'''(x)}{1+(u''(x))^2} + u(x) - 4u'(x) + v(x) + \arctan(v''(x)) = 0, \\ x \in [0,1] \setminus \{(x_i)\}, \\ \\ \left(|v''(x)|^{p-2}\, v''(x)\right)' + u(x) - 2v'(x)\sin^2(u''(x)) + v(x) - 2\sqrt[3]{v'(x)} = 0, \\ x \in [0,1] \setminus \{(\tau_j)\} \end{cases} \tag{5.33}$$

*with $p > 1$, the boundary conditions*

$$\begin{cases} u(0) = 0, \ u'(0) = \dfrac{1}{2}, \ u''(1) = 1 \\ v(0) = 0, \ v'(0) = 0, \ v''(1) = 1, \end{cases} \tag{5.34}$$

*and impulsive conditions are given by*

$$\begin{cases} \Delta u(x_i) &= \dfrac{1}{30}u(x_i) + \dfrac{1}{40}u'(x_i) + \dfrac{1}{50}v(x_i), \\ \\ \Delta u'(x_i) &= \dfrac{1}{2\pi}\sin(\pi u'(x_i)) + \dfrac{1}{10}u''(x_i) + \dfrac{1}{10}v(x_i), \\ \\ \Delta v(\tau_j) &= \dfrac{1}{50}u(\tau_j) + \dfrac{1}{200e^2}v(\tau_j) + \dfrac{1}{1000}\sqrt[3]{v'(\tau_j)}, \\ \\ \Delta v'(\tau_j) &= \dfrac{1}{20}u(\tau_j) + \dfrac{1}{20}v(\tau_j) + \dfrac{1}{\pi}\sin\left(\dfrac{2\pi}{5}v'(\tau_j)\right), \end{cases} \tag{5.35}$$

*with $x_i = \dfrac{i}{5}$, for $i = 1,2,3,4$ and $\tau_j = \dfrac{j^2}{10}$, for $j = 1,2,3$.*

The system (5.33)-(5.35) is a particular case of the problem (5.1), (5.2), (5.4), with $[a,b] = [0,1]$,

$$\phi(y_2) = \arctan(y_2), \quad \psi(z_2) = |z_2|^{p-2}z_2, \tag{5.36}$$

$$f(x,y_0,y_1,y_2,z_0,z_1,z_2) = y_0 - 4y_1 + z_0 + \arctan z_2,$$
$$g(x,y_0,y_1,y_2,z_0,z_1,z_2) = y_0 - 2z_1\sin^2(y_2) + z_0 - 2\sqrt[3]{z_1},$$

$$A_0 = B_0 = B_1 = 0,\ A_1 = \frac{1}{2},\ A_2 = B_2 = 1,$$

$$\begin{aligned}
I_0(\cdot\ ,y_0,y_1,z_0,z_1) &= \frac{1}{30}y_0 + \frac{1}{40}y_1 + \frac{1}{50}z_0,\\
I_1(\cdot\ ,y_0,y_1,y_2,z_0,z_1) &= \frac{1}{2\pi}\sin(\pi y_1) + \frac{1}{10}y_2 + \frac{1}{10}z_0,\\
J_0(\cdot\ ,y_0,y_1,z_0,z_1) &= \frac{1}{50}y_0 + \frac{1}{200e^2}z_0 + \frac{1}{1000}\sqrt[3]{z_1},\\
J_1(\cdot\ ,y_0,y_1,z_0,z_1,z_2) &= \frac{1}{20}y_0 + \frac{1}{20}z_0 + \frac{1}{\pi}\sin\left(\frac{2\pi}{5}z_1\right)
\end{aligned}$$

and $m = 4$, $n = 3$.

It is easy to see that the functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$, given in (5.36), verify assumption (H1), and are increasing homeomorphisms such that $\phi(0) = \psi(0) = 0$, $\phi(\mathbb{R}) = \psi(\mathbb{R}) = \mathbb{R}$.

The functions $\alpha_\kappa : [0,1] \to \mathbb{R}$, $\kappa = 1,2$, given by

$$\alpha_1(x) = \begin{cases} -e^{x-1} - \frac{1}{5}x & \text{if } 0 \le x \le \frac{1}{5} \\ -e^{x-1} - \frac{2}{5}x & \text{if } \frac{1}{5} < x \le \frac{2}{5} \\ -e^{x-1} - \frac{3}{5}x & \text{if } \frac{2}{5} < x \le \frac{3}{5} \\ -e^{x-1} - \frac{4}{5}x & \text{if } \frac{3}{5} < x \le \frac{4}{5} \\ -e^{x-1} - x & \text{if } \frac{4}{5} < x \le 1 \end{cases}, \quad \alpha_2(x) = \begin{cases} -x^2 - \frac{1}{10}x & \text{if } 0 \le x \le \frac{1}{10} \\ -x^2 - \frac{2}{5}x & \text{if } \frac{1}{10} < x \le \frac{2}{5} \\ -x^2 - \frac{9}{10}x & \text{if } \frac{2}{5} < x \le \frac{9}{10} \\ -x^2 - x & \text{if } \frac{9}{10} < x \le 1 \end{cases},$$

and $\beta_\kappa : [0,1] \to \mathbb{R}$, $\kappa = 1,2$, given by

$$\beta_1(x) = \begin{cases} e^{x-1} + \frac{1}{5}x & \text{if } 0 \le x \le \frac{1}{5} \\ e^{x-1} + \frac{2}{5}x & \text{if } \frac{1}{5} < x \le \frac{2}{5} \\ e^{x-1} + \frac{3}{5}x & \text{if } \frac{2}{5} < x \le \frac{3}{5} \\ e^{x-1} + \frac{4}{5}x & \text{if } \frac{3}{5} < x \le \frac{4}{5} \\ e^{x-1} + x & \text{if } \frac{4}{5} < x \le 1 \end{cases}, \quad \beta_2(x) = \begin{cases} x^2 + \frac{1}{10}x & \text{if } 0 \le x \le \frac{1}{10} \\ x^2 + \frac{2}{5}x & \text{if } \frac{1}{10} < x \le \frac{2}{5} \\ x^2 + \frac{9}{10}x & \text{if } \frac{2}{5} < x \le \frac{9}{10} \\ x^2 + x & \text{if } \frac{9}{10} < x \le 1 \end{cases},$$

when $x_5 = \tau_4 = 1$, are, respectively, lower and upper solutions of the problem (5.33)-(5.35), according to Definition 5.3.1. The differential inequalities are verified in the interval $[0,1]$, as shown in Figure 5.1.

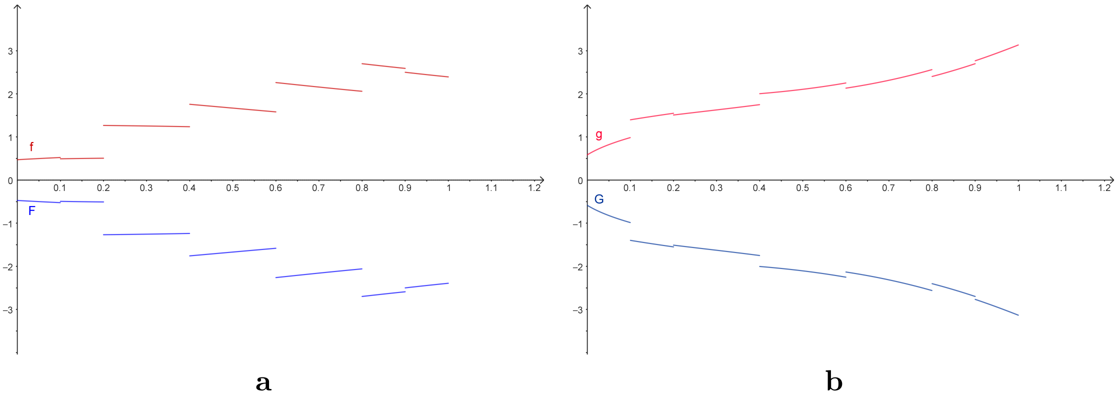



**Fig. 5.1:** Relationship between nonlinearities depending on the lower and upper solutions, given by the inequalities: a) ; b) .

The boundary conditions

$$\alpha_1(0) = -\frac{1}{e} < 0, \quad \alpha_1'(0) = -\frac{1}{e} - \frac{1}{5} < \frac{1}{2}, \quad \alpha_1''(1) = -1 < 1,$$
$$\alpha_2(0) = 0, \quad \alpha_2'(0) = -\frac{1}{10} < 0, \quad \alpha_2''(1) = -2 < 1,$$
$$\beta_1(0) = \frac{1}{e} > 0, \quad \beta_1'(0) = \frac{1}{e} + \frac{1}{5} > \frac{1}{2}, \quad \beta_1''(1) = 1,$$
$$\beta_2(0) = 0, \quad \beta_2'(0) = \frac{1}{e} > 0, \quad \beta_2''(1) = 2 > 1,$$

and impulsive conditions verify the inequalities of Definition 5.3.1, as shown in Table 5.1 and Table 5.2.

**Table 5.1:** Impulse conditions for functions $\alpha_1$ and $\beta_1$.

| $i$ | $x_i$ | $\Delta\alpha_1(x_i)$ | $I_{0i}(\alpha_1)$ | $I_{0i}(\beta_1)$ | $\Delta\beta_1(x_i)$ | $\Delta\alpha_1'(x_i)$ | $I_{1i}(\alpha_1)$ | $I_{1i}(\beta_1)$ | $\Delta\beta_1'(x_i)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.2 | -0.0400 | -0.0349 | 0.0349 | 0.0400 | -0.2000 | -0.1989 | 0.1989 | 0.2000 |
| 2 | 0.4 | -0.0800 | -0.0537 | 0.0537 | 0.0800 | -0.2000 | -0.1124 | 0.1124 | 0.2000 |
| 3 | 0.6 | -0.1200 | -0.0841 | 0.0841 | 0.1200 | -0.2000 | -0.0375 | 0.0375 | 0.2000 |
| 4 | 0.8 | -0.1600 | -0.1163 | 0.1163 | 0.1600 | -0.2000 | -0.0697 | 0.0697 | 0.2000 |

**Table 5.2:** Impulse conditions for functions $\alpha_2$ and $\beta_2$.

| $j$ | $\tau_j$ | $\Delta\alpha_2(\tau_j)$ | $J_{0j}(\alpha_2)$ | $J_{0j}(\beta_2)$ | $\Delta\beta_2(\tau_j)$ | $\Delta\alpha_2'(\tau_j)$ | $J_{1j}(\alpha_2)$ | $J_{1j}(\beta_2)$ | $\Delta\beta_2'(\tau_j)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.1 | -0.0300 | -0.0092 | 0.0092 | 0.0300 | -0.3000 | -0.1395 | 0.1395 | 0.2000 |
| 2 | 0.4 | -0.2000 | -0.0155 | 0.0155 | 0.2000 | -0.5000 | -0.3691 | 0.3691 | 0.5000 |
| 3 | 0.9 | -0.0900 | -0.0386 | 0.0386 | 0.0900 | -0.1000 | -0.0921 | 0.0921 | 0.1000 |

Let

$$L > \max\{\|\alpha_1(x)\|_{X_1}, \|\beta_1(x)\|_{X_1}, \|\alpha_2(x)\|_{X_2}, \|\beta_2(x)\|_{X_2}\},$$

then $f$ and $g$ are $L^1$-Carathéodory functions, with

$$\begin{aligned} |f(x, y_0, y_1, y_2, z_0, z_1, z_2)| &\leq 6L + 1 := \rho_{1L}(x), \\ |g(x, y_0, y_1, y_2, z_0, z_1, z_2)| &\leq 4L + 2\sqrt[3]{L} := \rho_{2L}(x), \end{aligned}$$

and the sum of the jumps is bounded.

The functions $f$ and $g$ satisfy Nagumo condition relative to the sets

$$S_1 = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, z_0, z_1, z_2) \in [0,1] \times \mathbb{R}^6 : \\ \alpha_1^{(l)}(x) \leq y_l \leq \beta_1^{(l)}(x),\ \alpha_2^{(l)}(x) \leq z_l \leq \beta_2^{(l)}(x), l = 0, 1 \end{array} \right\}.$$

Consider a constant $\mathcal{K}_k > 0$, $k = 1, 2$, and $\mu_k$ as defined in (5.9), then, in $S_1$,

$$\begin{aligned} |f(x, y_0, y_1, y_2, z_0, z_1, z_2)| &= |y_0 - 4y_1 + z_0 + \arctan z_2| \\ &\leq \mathcal{K}_1 := \varphi_1(|y_2|) \end{aligned}$$

and

$$\begin{aligned} |g(x, y_0, y_1, y_2, z_0, z_1, z_2)| &= \left| y_0 - 2z_1 \sin^2(y_2) + z_0 - 2\sqrt[3]{z_1} \right| \\ &\leq \mathcal{K}_2 := \varphi_2(|z_2|), \end{aligned}$$

it is trivial that

$$\int_{\phi(\mu_1)}^{\phi(+\infty)} \frac{|\phi^{-1}(s)|}{|\phi^{-1}(s)| + \mathcal{K}_1} ds = +\infty \text{ and } \int_{\psi(\mu_2)}^{\psi(+\infty)} \frac{|\psi^{-1}(s)|}{|\psi^{-1}(s)| + \mathcal{K}_2} ds = +\infty.$$

So, by Theorem 5.3.2, there is at least one pair of functions $(u(x), v(x)) \in X^2$, solution of the problem (5.33)-(5.35) and, moreover,

$$\begin{array}{ll} \alpha_1(x) \leq u(x) \leq \beta_1(x), & \alpha_2(x) \leq v(x) \leq \beta_2(x), \\ \alpha_1'(x) \leq u'(x) \leq \beta_1'(x), & \alpha_2'(x) \leq v'(x) \leq \beta_2'(x), \quad \forall x \in [0, 1], \end{array}$$

as shown in Figure 5.2 and Figure 5.3, and

$$\|u''\| \leq N_1 \quad \text{and} \quad \|v''\| \leq N_2,$$

with $N_1$ and $N_2$ given by Lemma 5.1.4.

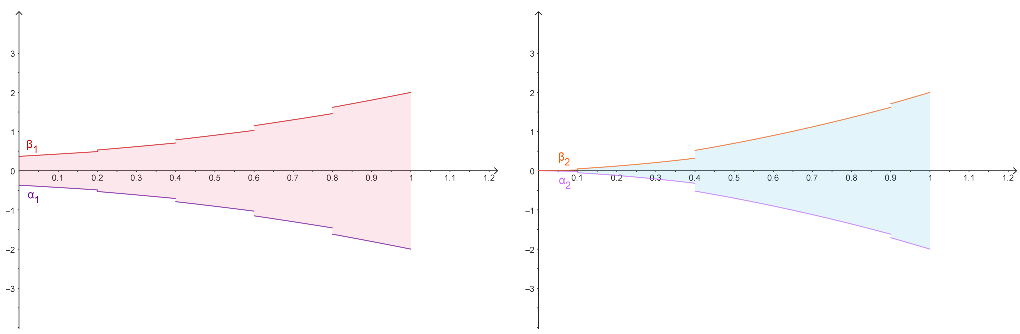



**Fig. 5.2:** At least one solution $(u(x), v(x))$ of problem (5.33)-(5.35) is located in the colored region, when $x \in [0, 1]$.

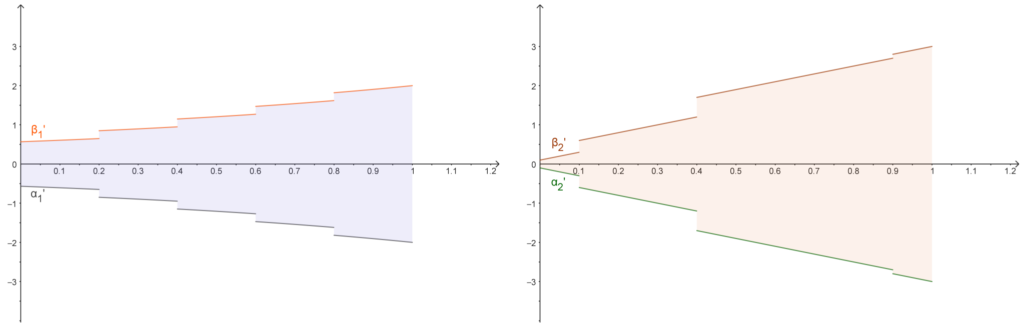



**Fig. 5.3:** Localization for the first derivative of the solution of the problem (5.33)-(5.35), when $x \in [0,1]$.

## 5.4 Singular $\phi-$Laplacian equations in special relativity

Relativity implies that physical laws do not depend on the chosen reference frame. In special relativity, the speed of light $c$ is recognized as the maximum speed with which information can travel in free space from one frame of reference to another, [12]. Let us consider two frames of reference $\mathcal{P}_0$ and $\mathcal{P}$ in uniform relative motion to each other, that is, moving with relative speed $v$. Taking into account the upper limit $c$ of the speed of information propagation, the space-time coordinates of the frames $\mathcal{P}_0$ and $\mathcal{P}$ must be related by *Lorentz transformations*, [41]. The Lorentz factor depends non-linearly on the relative velocity $v$ and is defined by

$$\Gamma \equiv \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}.$$

The theory of special relativity is fundamental in the development of the modern theory of classical electrodynamics. The fact that an electric charge $q$ generates an electric field $\mathbf{E}$ and in motion generates a magnetic field $\mathbf{B}$ is intuitively compatible with the statement that the electric and magnetic fields are covariant under a Lorentz transformation from one inertial system to another, [54].

The study developed in this article can be adapted and applied to a system of singular $\phi$-Laplacian equations, that is, to the system of equations (5.1), with two restrictions:

- In Lemma 5.1.4, the constants $N_1$ and $N_2$ must be chosen such that

$$0 < N_1 < \eta \text{ and } 0 < N_2 < \gamma \; ;$$

- Assumption (H1) must be replaced by

**(Hs)** $\phi : (-\eta,\eta) \to \mathbb{R}$ and $\psi : (-\gamma,\gamma) \to \mathbb{R}$, for some $0<\eta<+\infty$ and $0<\gamma<+\infty$, are increasing homeomorphisms with $\phi(0)=\psi(0)=0$, $\phi(-\eta,\eta)=\mathbb{R}$ and $\psi(-\gamma,\gamma)=\mathbb{R}$, such that

$$\left|\phi^{-1}(w)\right| \le \phi^{-1}(|w|), \text{ and } \left|\psi^{-1}(w)\right| \le \psi^{-1}(|w|).$$

In this case, a solution to the problem (5.1)-(5.3) is a pair of functions $(u(x),v(x)) \in X^2$ such that $(u''(x),v''(x)) \in (-\eta,\eta)\times(-\gamma,\gamma)$ for all $x\in[a,b]$, satisfying (5.1)-(5.3).

**Example 5.4.1.** *Consider the problem*

$$\begin{cases} \left(\dfrac{u''(x)}{\sqrt{1-(u''(x))^2}}\right)' + \dfrac{1}{12}u(x) - \dfrac{5}{2}(u'(x))^2 + \dfrac{1}{6}v'(x)|v'(x)| = 0, \\ x \in [-1,1] \setminus \{(x_i)\}, \\ \left(\dfrac{v''(x)}{\sqrt{1-\dfrac{(v''(x))^2}{9}}}\right)' + \dfrac{1}{12}u(x) - \dfrac{5}{2}(v'(x))^2 + \dfrac{1}{6}u'(x)|u'(x)| = 0, \\ x \in [-1,1] \setminus \{(\tau_j)\}, \end{cases} \tag{5.37}$$

*with the boundary conditions*

$$\begin{cases} u(-1) = 0, \ u'(-1) = \dfrac{1}{3}, \ u''(1) = \dfrac{1}{2} \\ v(-1) = 0, \ v'(-1) = -\dfrac{1}{2}, \ v''(1) = 0, \end{cases} \tag{5.38}$$

*and impulsive conditions are given by*

$$\begin{cases} \Delta u(x_i) &= \dfrac{1}{4\pi}\arctan(u(x_i)+1), \\ \Delta u'(x_i) &= \dfrac{1}{10}u(x_i) - \dfrac{1}{10}u'(x_i), \\ \Delta v(\tau_j) &= \dfrac{1}{100}v(\tau_j) + \dfrac{1}{6}, \\ \Delta v'(\tau_j) &= \dfrac{1}{3\pi}\sin^2(2\pi v'(\tau_j)), \end{cases} \tag{5.39}$$

*with* $x_1 = 0$ *and* $\tau_1 = -\frac{1}{2}$, $\tau_2 = 0$, $\tau_3 = \frac{1}{2}$.

The system (5.37)-(5.39) is a particular case of the problem (5.1), (5.2), (5.4), with $[a,b] = [-1,1]$,

$$\phi(y_2) = \frac{y_2}{\sqrt{1-y_2^2}}, \quad \psi(z_2) = \frac{z_2}{\sqrt{1-\frac{z_2^2}{9}}}, \tag{5.40}$$

$$\begin{aligned} f(x,y_0,y_1,y_2,z_0,z_1,z_2) &= \frac{1}{12}y_0 - \frac{5}{2}y_1^2 + \frac{1}{6}z_1|z_1|, \\ g(x,y_0,y_1,y_2,z_0,z_1,z_2) &= \frac{1}{12}y_0 - \frac{5}{2}z_1^2 + \frac{1}{6}y_1|y_1|, \end{aligned}$$

$$A_0 = B_0 = B_2 = 0, \ A_1 = \frac{1}{3}, \ A_2 = \frac{1}{2}, B_1 = -\frac{1}{2},$$

$$\begin{aligned}
I_0(\cdot\,,y_0,y_1,z_0,z_1) &= \frac{1}{4\pi}\arctan(y_0+1),\\
I_1(\cdot\,,y_0,y_1,y_2,z_0,z_1) &= \frac{1}{10}y_0-\frac{1}{10}y_1,\\
J_0(\cdot\,,y_0,y_1,z_0,z_1) &= \frac{1}{100}z_0+\frac{1}{6},\\
J_1(\cdot\,,y_0,y_1,z_0,z_1,z_2) &= \frac{1}{3\pi}\sin^2(2\pi z_1),
\end{aligned}$$

and $m=1$, $n=3$.

It is easy to see that the functions $\phi:(-1,1)\to\mathbb{R}$ and $\psi:(-3,3)\to\mathbb{R}$, given in (5.40), verify assumption (Hs), are increasing homeomorphisms such that $\phi(0)=\psi(0)=0$, $\phi(-1,1)=\mathbb{R}$ and $\psi(-3,3)=\mathbb{R}$.

The functions $\alpha_\kappa:[-1,1]\to\mathbb{R}$, $\kappa=1,2$, given by

$$\alpha_1(x)=\begin{cases}\frac{1}{60}x^5+\frac{1}{18}x^3+\frac{1}{20} & \text{if } -1\le x\le 0\\[2mm] \frac{1}{60}x^5+\frac{1}{36}x^3+\frac{1}{10} & \text{if } 0<x\le 1\end{cases},$$

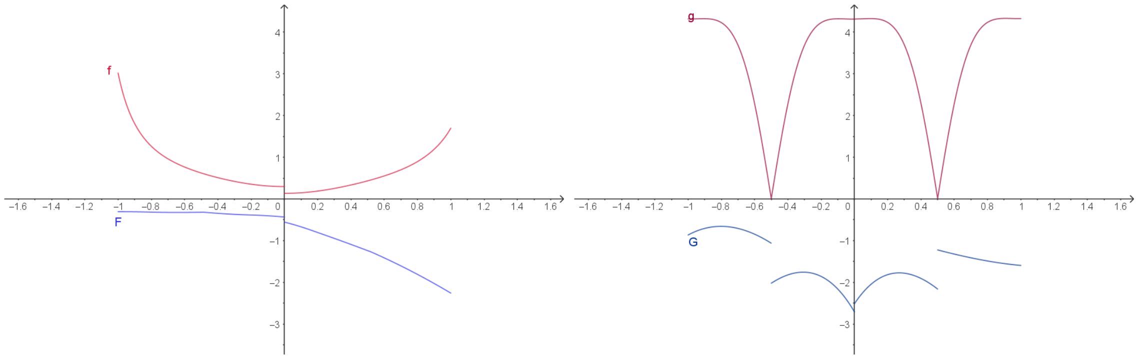
$$\alpha_2(x)=\begin{cases}\frac{1}{2\pi}\sin(\pi x) & \text{if } -1\le x\le -\frac{1}{2}\\[2mm] -\frac{1}{2\pi}\left(\sin(\pi x)+1\right) & \text{if } -\frac{1}{2}<x\le 0\\[2mm] -\frac{1}{2\pi}\sin(\pi x) & \text{if } 0<x\le\frac{1}{2}\\[2mm] \frac{1}{2\pi}\left(\sin(\pi x)-1\right) & \text{if } \frac{1}{2}<x\le 1\end{cases},$$

and $\beta_\kappa:[-1,1]\to\mathbb{R}$, $\kappa=1,2$, given by

$$\beta_1(x)=\begin{cases}\frac{1}{6}\sqrt{(x+3)^3} & \text{if } -1\le x\le 0\\[2mm] \frac{1}{4}x^2+\frac{1}{2}x+1 & \text{if } 0<x\le 1\end{cases},$$

$$\beta_2(x)=\begin{cases}-\frac{1}{12}x^3+\frac{1}{2}x^2+x+\frac{1}{2} & \text{if } -1\le x\le -\frac{1}{2}\\[2mm] -\frac{1}{6}x^3-x^2-\frac{2}{5}x+\frac{9}{25} & \text{if } -\frac{1}{2}<x\le 0\\[2mm] -\frac{1}{6}x^3+x^2-\frac{3}{10}x+\frac{3}{5} & \text{if } 0<x\le\frac{1}{2}\\[2mm] -\frac{1}{12}x^3+\frac{1}{3}x^2+\frac{1}{3}x+\frac{2}{3} & \text{if } \frac{1}{2}<x\le 1\end{cases},$$

are, respectively, lower and upper solutions of the problem (5.37)-(5.39), according to Definition 5.3.1. In fact, the differential inequalities are verified in the interval $[-1,1]$, as shown in Figure 5.4.




**a** **b**

**Fig. 5.4:** Relationship between nonlinearities depending on the lower and upper solutions, given by the inequalities: a) substituting lower e upper solution into the first equation; b) substituting lower e upper solution in the second equation.

The boundary conditions

$$\alpha_1(-1) = -\frac{1}{45} < 0, \quad \alpha_1'(-1) = \frac{1}{4} < \frac{1}{3}, \quad \alpha_1''(1) = \frac{1}{2},$$
$$\alpha_2(-1) = 0, \quad \alpha_2'(-1) = -\frac{1}{2}, \quad \alpha_2''(1) = 0,$$
$$\beta_1(-1) = \frac{1}{3}\sqrt{2} > 0, \quad \beta_1'(-1) = \frac{\sqrt{2}}{4} > \frac{1}{3}, \quad \beta_1''(1) = \frac{1}{2},$$
$$\beta_2(-1) = \frac{1}{12} > 0, \quad \beta_2'(-1) = -\frac{1}{4} > -\frac{1}{2}, \quad \beta_2''(1) = \frac{1}{6} > 0,$$

and impulsive conditions verify the inequalities of Definition 5.3.1, as shown in Table 5.3 and Table 5.4.

Let
$$L > \max\{\|\alpha_1(x)\|_{X_1}, \|\beta_1(x)\|_{X_1}, \|\alpha_2(x)\|_{X_2}, \|\beta_2(x)\|_{X_2}\},$$
then $f$ and $g$ are $L^1$-Carathéodory functions, with

$$|f(x, y_0, y_1, y_2, z_0, z_1, z_2)| \leq \frac{1}{12}L + \frac{8}{3}L^2 := \rho_{1L}(x),$$

$$|g(x, y_0, y_1, y_2, z_0, z_1, z_2)| \leq \frac{1}{12}L + \frac{8}{3}L^2 := \rho_{2L}(x),$$

**Table 5.3:** Impulse conditions for functions $\alpha_1$ and $\beta_1$.

| $i$ | $x_i$ | $\Delta\alpha_1(x_i)$ | $I_{0i}(\alpha_1)$ | $I_{0i}(\beta_1)$ | $\Delta\beta_1(x_i)$ | $\Delta\alpha_1'(x_i)$ | $I_{1i}(\alpha_1)$ | $I_{1i}(\beta_1)$ | $\Delta\beta_1'(x_i)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.0 | 0.0500 | 0.0644 | 0.0859 | 0.1340 | 0.0000 | 0.0050 | 0.0433 | 0.0670 |

**Table 5.4:** Impulse conditions for functions $\alpha_2$ and $\beta_2$.

| $j$ | $\tau_j$ | $\Delta\alpha_2(\tau_j)$ | $J_{0j}(\alpha_2)$ | $J_{0j}(\beta_2)$ | $\Delta\beta_2(\tau_j)$ | $\Delta\alpha_2'(\tau_j)$ | $J_{1j}(\alpha_2)$ | $J_{1j}(\beta_2)$ | $\Delta\beta_2'(\tau_j)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | -0.5 | 0.1592 | 0.1651 | 0.1680 | 0.1954 | 0.0000 | 0.0000 | 0.0155 | 0.0375 |
| 2 | 0.0 | 0.1592 | 0.1651 | 0.1703 | 0.2400 | 0.0000 | 0.0000 | 0.0367 | 0.1000 |
| 3 | 0.5 | 0.1592 | 0.1651 | 0.1735 | 0.2271 | 0.0000 | 0.0000 | 0.0219 | 0.0292 |

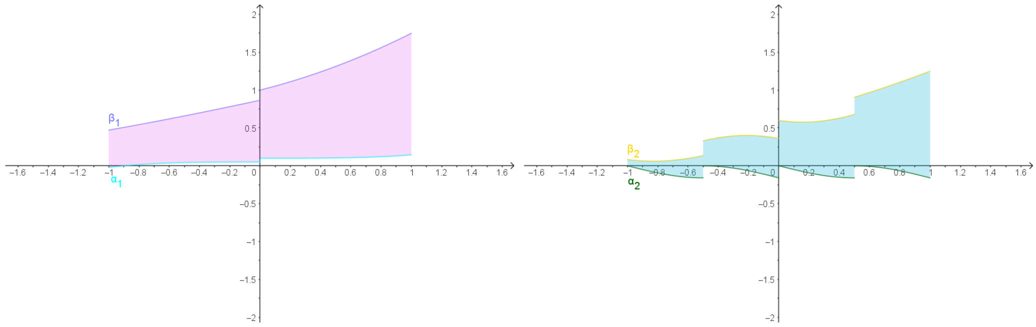


**Fig. 5.5:** At least one solution $(u(x), v(x))$ of problem (5.37)-(5.39) is located in the colored region, when $x \in [-1,1]$.

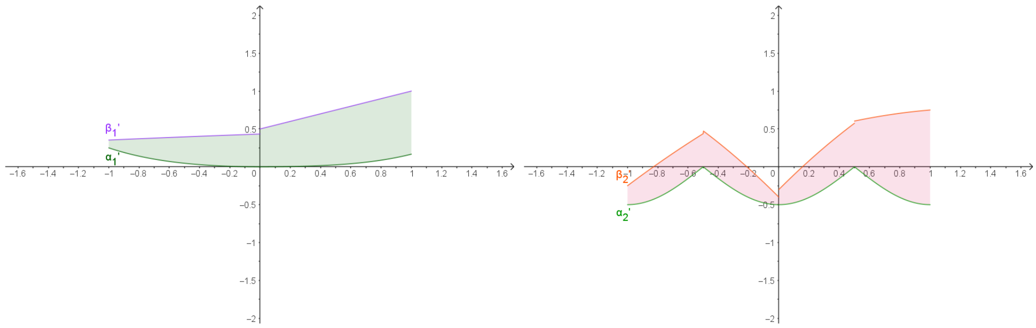


**Fig. 5.6:** Localization for the first derivative of the solution of the problem (5.37)-(5.39), when $x \in [-1,1]$.

and the sum of the jumps is bounded.

The functions $f$ and $g$ satisfy Nagumo condition relative to the sets

$$S_2 = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, z_0, z_1, z_2) \in [-1,1] \times \mathbb{R}^6 : \\ \\ \alpha_1^{(l)}(x) \le y_l \le \beta_1^{(l)}(x),\ \alpha_2^{(l)}(x) \le z_l \le \beta_2^{(l)}(x), l = 0,1 \end{array} \right\}.$$

Consider a constant $\mathcal{K}_k > 0$, $k = 1,2$, and $\mu_k$ as defined in (5.9), then, in $S_2$,

$$\begin{aligned} |f(x, y_0, y_1, y_2, z_0, z_1, z_2)| &= \left| \frac{1}{12} y_0 - \frac{5}{2} y_1^2 + \frac{1}{6} z_1 |z_1| \right| \\ &\le \mathcal{K}_1 := \varphi_1(|y_2|) \end{aligned}$$

and

$$\begin{aligned} |g(x, y_0, y_1, y_2, z_0, z_1, z_2)| &= \left| \frac{1}{12} y_0 - \frac{5}{2} z_1^2 + \frac{1}{6} y_1 |y_1| \right| \\ &\le \mathcal{K}_2 := \varphi_2(|z_2|), \end{aligned}$$

it is trivial that

$$\int_{\phi(\mu_1)}^{+\infty} \frac{|\phi^{-1}(s)|}{|\phi^{-1}(s)| + \mathcal{K}_1} ds = +\infty$$

and

$$\int_{\psi(\mu_2)}^{+\infty} \frac{|\psi^{-1}(s)|}{|\psi^{-1}(s)| + \mathcal{K}_2} ds = +\infty.$$

So, by Theorem 5.3.2, there is at least one pair of functions $(u(x), v(x)) \in X^2$, solution of the problem (5.37)-(5.39) and, moreover,

$$\begin{array}{ll} \alpha_1(x) \leq u(x) \leq \beta_1(x), & \alpha_2(x) \leq v(x) \leq \beta_2(x), \\ \alpha_1'(x) \leq u'(x) \leq \beta_1'(x), & \alpha_2'(x) \leq v'(x) \leq \beta_2'(x), \quad \forall x \in [-1,1], \end{array}$$

as shown in Figure 5.2 and Figure 5.3, and

$$\|u''\| \leq N_1 < 1 \quad \text{and} \quad \|v''\| \leq N_2 < 3,$$

with $N_1$ and $N_2$ given by Lemma 5.1.4.

Chapter **6**

# Higher-order impulsive coupled systems with $\phi$-Laplacian

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In this chapter we consider the higher order impulsive coupled system with $\phi$-Laplacian

$$\begin{cases} (\phi(u^{(\eta-1)}(t)))' + f(t, u(t), ..., u^{(\eta-1)}(t), v(t), ..., v^{(\eta-1)}(t)) = 0, & t \in M, \\ (\psi(v^{(\xi-1)}(t)))' + g(t, u(t), ..., u^{(\xi-1)}(t), v(t), ..., v^{(\xi-1)}(t)) = 0, & t \in N, \end{cases} \tag{6.1}$$

where $\eta, \xi \geq 2$, $M = [a,b]\setminus\{t_1, ..., t_m\}$ and $N = [a,b]\setminus\{\tau_1, ..., \tau_n\}$, $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are increasing homeomorphisms such that $\phi(0) = \psi(0) = 0$ and $\phi(\mathbb{R}) = \psi(\mathbb{R}) = \mathbb{R}$, $f : [a,b] \times \mathbb{R}^{2\eta} \mapsto \mathbb{R}$, $g : [a,b] \times \mathbb{R}^{2\xi} \mapsto \mathbb{R}$,are $L^1$-Carathéodory functions and $m, n \in \mathbb{N}$, together with the boundary conditions

$$\begin{cases} u^{(l)}(a) = A_l, \; u^{(\eta-1)}(b) = A_{\eta-1}, \\ v^{(k)}(a) = B_k, \; v^{(\xi-1)}(b) = B_{\xi-1}, \end{cases} \tag{6.2}$$

with $A_l, A_{\eta-1}, B_k, B_{\xi-1} \in \mathbb{R}$, $l = 0, 1, ..., \eta - 2$, $k = 0, 1, ..., \xi - 2$.

The impulsive conditions are given by

$$\begin{aligned} \Delta u^{(l)}(t_i) &= I_{l,i}(t_i, u(t_i), ..., u^{(\eta-1)}(t_i), v(t_i), ..., v^{(\xi-1)}(t_i)), \\ \Delta\phi(u^{(\eta-1)}(t_i)) &= I_{\eta-1,i}(t_i, u(t_i), ..., u^{(\eta-1)}(t_i), v(t_i), ..., v^{(\xi-1)}(t_i)), \\ \Delta v^{(k)}(\tau_j) &= J_{k,j}(\tau_j, u(\tau_j), ..., u^{(\eta-1)}(\tau_j), v(\tau_j), ..., v^{(\xi-1)}(\tau_j)), \\ \Delta\psi(v^{(\xi-1)}(\tau_j)) &= J_{\xi-1,j}(\tau_j, u(\tau_j), ..., u^{(\eta-1)}(\tau_j), v(\tau_j), ..., v^{(\xi-1)}(\tau_j)), \end{aligned} \tag{6.3}$$

being $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$, $i = 1, 2, ..., m$, $\Delta v(\tau_j) = v(\tau_j^+) - v(\tau_j^-)$, $j = 1, 2, ..., n$, $I_{l,i}, J_{kj} \in C([a,b] \times \mathbb{R}^{\eta+\xi}, \mathbb{R})$, $l = 0, 1, ..., \eta - 1$, $k = 0, 1, ..., \xi - 1$, and $t_i$,$\tau_j$ are fixed points such that $a = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = b$, $a = \tau_0 < \tau_1 < \tau_2 < ... < \tau_n < \tau_{n+1} = b$.

In addition to the existence of a solution, it is possible to obtain an existence and location theorem, that is, it not only guarantees the existence of at least one solution, but also provides an interval where this solution is located.

The location of the solution is obtained for a particular case of the impulsive conditions (6.3), that is, for $I_{l,i}, J_{kj} \in C([a,b] \times \mathbb{R}^{\eta+\xi-2}, \mathbb{R})$, $l = 0, 1, ..., \eta - 3$, $k = 0, 1, ..., \xi - 3$, $I_{\eta-2,i}, J_{\xi-2,j} \in C([a,b] \times \mathbb{R}^{\eta+\xi-1}, \mathbb{R})$, and

$$\begin{aligned} \Delta u^{(l)}(t_i) &= I_{l,i}(t_i, u(t_i), ..., u^{(\eta-2)}(t_i), v(t_i), ..., v^{(\xi-2)}(t_i)), \; l = 0, 1, ..., \eta - 3, \\ \Delta u^{(\eta-2)}(t_i) &= I_{\eta-2,i}(t_i, u(t_i), ..., u^{(\eta-1)}(t_i), v(t_i), ..., v^{(\xi-2)}(t_i)) \\ \Delta v^{(k)}(\tau_j) &= J_{k,j}(\tau_j, u(\tau_j), ..., u^{(\eta-2)}(\tau_j), v(\tau_j), ..., v^{(\xi-2)}(\tau_j)), k = 0, 1, ..., \xi - 3, \\ \Delta v^{(\xi-2)}(\tau_j) &= J_{\xi-2,j}(\tau_j, u(\tau_j), ..., u^{(\eta-2)}(\tau_j), v(\tau_j), ..., v^{(\xi-1)}(\tau_j)), \end{aligned} \tag{6.4}$$

for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.

Nonlinear coupled systems, meaning that the unknown functions and some of their derivatives interact, have been considered recently, such as, among others: [94], applying Schauder's fixed point theorem; [35], in fractional order at resonance via coincidence degree theory; [77], for different types of differential and integral equations; [105], using lower and upper solutions method; [50], related to reaction-diffusion Robin problems.

For the first time, this work contains, a general theory that allows coupled systems, with regular and singular semi-linear differential equations of a different order. For example, it can be applied to nonlinear coupled systems combining a fourth-order differential equation with a second-order one, as it is commonly used in elasticity theory.

Impulsive differential equations can model many real phenomena, with sudden discontinuous jumps. These events can occur in many fields, such as population dynamics, control, and optimization theory, chemistry, biology and biotechnology, economics, pharmacokinetics, and other physics and mechanics problems. A classical work in impulsive differential equations, is the book [58], but, for example, we can mention: [52], applying fixed point index; [61, 78], dealing with functional impulsive problems; [108], for approximation of solution via a monotone iterative technique. Impulsive problems containing $\phi$-Laplacians can be found, for instance in: [49], for periodic problems and a continuation theorem; [72, 73], applied to bounded and unbounded intervals; [101], for fractional $p$-Laplacian.

To the best of our knowledge, this is the first work combining the method suggested in, for example, [15, 45], to an impulsive coupled system, with the possibility of equations of different order, with fully differential equations including different regular and singular Laplacians and generalized impulsive effects, depending on both variables and some derivatives.

The Chapter is organized as follows: In Section 6.1 we have definitions, the functional framework, the explicit solution for the impulsive linear problem associated, Nagumo-type growth conditions, some *a priori* bounds for the highest order in the nonlinearities, and classical results. Section 6.2, contains an existence result for the more general case. Section 6.3 presents an existence and localization result applied to a particular case of the initial impulsive conditions, applying the lower and upper solutions method. In Section 6.4 we present an application of our theory to the flexural vibration of a single-span suspension bridge. Section 6.5 shows how to adapt this technique to singular $\phi-$Laplacians and illustrates it with an example.

## 6.1 Definitions and preliminary results

This section will introduce some preliminary results and the functional framework,

Define

$$x(t_\kappa^\pm) := \lim_{t \to t_\kappa^\pm} x(t),$$

and consider the sets of piecewise continuous functions:

$$PC_1^{\eta-1}([a,b]) := \left\{ \begin{array}{c} u : u \in C^{\eta-1}([a,b],\mathbb{R}) \text{ continuous for } t \neq t_i, u^{(l)}(t_i) = u^{(l)}(t_i^-), \\ u^{(l)}(t_i^+) \text{ is finite for } i = 1,2,3,...,m \text{ and } l = 0,1,...,\eta-1 \end{array} \right\}$$

and

$$PC_2^{\xi-1}([a,b]) := \left\{ \begin{array}{c} v : v \in C^{\xi-1}([a,b],\mathbb{R}) \text{ continuous for } \tau \neq \tau_j, v^{(k)}(\tau_j) = v^{(k)}(\tau_j^-), \\ v^{(k)}(\tau_j^+) \text{ is finite for } j = 1,2,3,...,n \text{ and } k = 0,1,...,V-1 \end{array} \right\}.$$

Let $X_1 := PC_1^{\eta-1}[a,b]$, and $X_2 := PC_1^{\xi-1}[a,b]$ be the usual Banach space equipped with the norms defined by

$$\begin{aligned} \|x\|_{X_1} \quad &: \quad = \max\{\|x^{(l)}\|_\infty, l = 0,1,...,\eta-1\}, \\ \|x\|_{X_2} \quad &: \quad = \max\{\|x^{(k)}\|_\infty, k = 0,1,...,\xi-1\}, \end{aligned}$$

where

$$\|x\|_\infty := \sup_{a \leq t \leq b} |x(t)|$$

and $X^2 := X_1 \times X_2$ with the norm

$$\|(u,v)\|_{X^2} = \max\{\|u\|_{X_1}, \|v\|_{X_2}\}.$$

The ordered pair $(u,v)$ is a solution to problem (6.1)-(6.3) when $(u(t),v(t)) \in X^2$.

**Definition 6.1.1.** *A function $h : [a,b] \times \mathbb{R}^{p+q} \to \mathbb{R}$ is $L^1$-Carathéodory if*

*i. for each $(x_0,...,x_{p-1},y_0,...,y_{q-1}) \in \mathbb{R}^{p+q}$, $t \mapsto h(t,x_0,...,x_{p-1},y_0,...,y_{q-1})$ is measurable on $[a,b]$;*

*ii. for almost every $t \in [a,b]$, $(x_0,...,x_{p-1},y_0,...,y_{q-1}) \mapsto h(t,x_0,...,x_{p-1},y_0,...,y_{q-1})$ is continuous on $\mathbb{R}^{p+q}$;*

*iii. for each $K > 0$, there is a positive function $\rho_K \in L^1[a,b]$ such that, for a.e. $t \in [a,b]$, and $(x_0,...,x_{p-1},y_0,...,y_{q-1}) \in \mathbb{R}^{p+q}$ with*

$$\max\{|x_l|, |y_k|, l = 0,1,...,p-1, k = 0,1,...,q-1\} < K,$$

*we have*

$$|h(t,x_0,...,x_{p-1},y_0,...,y_{q-1})| \leq \rho_K(t).$$

Forward, where there are no doubts, we write the nonlinearities in a short way

$$\begin{aligned} f(U(\cdot),V(\cdot)) \quad &: \quad = f(t,u(\cdot),...,u^{(\eta-1)}(\cdot),v(\cdot),...,v^{(\eta-1)}(\cdot)), \\ g(U(\cdot),V(\cdot)) \quad &: \quad = g(t,u(\cdot),...,u^{(\xi-1)}(\cdot),v(\cdot),...,v^{(\xi-1)}(\cdot)), \end{aligned}$$

and the impulsive conditions as

$$I_{l,i}(\cdot) := I_{l,i}(t_i,u(t_i),...,u^{(\eta-1)}(t_i),v(t_i),...,v^{(\xi-1)}(t_i)),$$

for $l=0,1,...,\eta-1$, $i=1,2,...,m$, and

$$J_{k,j}(\cdot):=J_{k,j}(\tau_j,u(\tau_j),...,u^{(\eta-1)}(\tau_j),v(\tau_j),...,v^{(\xi-1)}(\tau_j)),$$

for $k=0,1,...,\xi-1$, and $j=1,2,...,n$.

Next lemma provides a uniqueness result for a linear problem related to (6.1)-(6.3).

**Lemma 6.1.2.** *Let $\phi,\psi:\ \mathbb{R}\to\mathbb{R}$ be increasing homeomorphisms and $p,q\in L^1[a,b]$. The problem composed by the differential system*

$$\begin{cases}(\phi(u^{(\eta-1)}(t)))'+p(t)=0\\(\psi(v^{(\xi-1)}(t)))'+q(t)=0\end{cases}\tag{6.5}$$

*and conditions (6.2), and (6.3), has a unique solution given by*

$$\begin{aligned}u(t)=&\sum_{l=0}^{\eta-2}\left(\left(A_l+\sum_{i\,:\,t_i<t}I_{l,i}(\cdot)\right)\frac{(t-a)^l}{l!}\right)\\&+\int_a^t\frac{(t-s)^{\eta-2}}{(\eta-2)!}\phi^{-1}\left(\phi(A_{\eta-1})-\sum_{i\,:\,t_i>s}I_{\eta-1,i}(\cdot)+\int_s^b p(\sigma)d\sigma\right)ds\end{aligned}$$

*and*

$$\begin{aligned}v(t)=&\sum_{k=0}^{\xi-2}\left(\left(B_k+\sum_{j\,:\,\tau_j<t}J_{k,j}(\cdot)\right)\frac{(t-a)^k}{k!}\right)\\&+\int_a^t\frac{(t-s)^{\xi-2}}{(\xi-2)!}\psi^{-1}\left(\psi(B_{\xi-1})-\sum_{j\,:\,\tau_j>s}J_{\xi-1,j}(\cdot)+\int_s^b q(\sigma)d\sigma\right)ds.\end{aligned}$$

*Proof.* Integrating the first equation of (6.5), for $t\in(t_m,b]$, we have, by (6.2),

$$\phi(u^{(\eta-1)}(t))=\phi(A_{\eta-1})+\int_t^b p(s_\eta)ds_\eta,\tag{6.6}$$

For $t\in(t_{m-1},t_m]$, integrating (6.5), by (6.3) and (6.6),

$$\begin{aligned}\phi(u^{(\eta-1)}(t))=&\phi(u^{(\eta-1)}(t_m^-))+\int_t^{t_m}p(s_\eta)ds_\eta\\=&\phi(u^{(\eta-1)}(t_m^+))-I_{\eta-1,m}(\cdot)+\int_t^{t_m}p(s_\eta)ds_\eta,\\=&\phi(A_{\eta-1})+\int_{t_n}^b p(s_\eta)ds_\eta-I_{\eta-1,m}(\cdot)+\int_t^{t_m}p(s_\eta)ds_\eta,\\=&\phi(A_{\eta-1})-I_{\eta-1,m}(\cdot)+\int_t^b p(s_\eta)ds_\eta.\end{aligned}$$

So, by mathematical induction, for $t\in[a,b]$,

$$\phi(u^{(\eta-1)}(t))=\phi(A_{\eta-1})-\sum_{i\,:\,t_i>t}I_{\eta-1,i}(\cdot)+\int_t^b p(s_\eta)ds_\eta,$$

and, therefore

$$u^{(\eta-1)}(t) = \phi^{-1}\left(\phi(A_{\eta-1}) - \sum_{i\,:\,t_i>t} I_{\eta-1,i}(\cdot) + \int_t^b p(s_\eta)ds_\eta\right). \tag{6.7}$$

By a new integration of (6.7) from $a$ to $t$, by (6.3) and by mathematical induction, when $t \in [a,b]$,

$$\begin{aligned} u^{(\eta-2)}(t) =& A_{\eta-2} + \sum_{i\,:\,t_i<t} I_{\eta-2,i}(\cdot) \\ &+ \int_a^t \phi^{-1}\left(\phi(A_{\eta-1}) - \sum_{i\,:\,t_i>s_{\eta-1}} I_{\eta-1,i}(\cdot) + \int_{s_{\eta-1}}^b p(s_\eta)ds_\eta\right) ds_{\eta-1}. \end{aligned}$$

By iterative integrations, of (6.6), we obtain for $t \in [a,b]$,

$$\begin{aligned} u(t) =& \sum_{l=0}^{\eta-2}\left(\left(A_l + \sum_{i\,:\,t_i<t} I_{l,i}(\cdot)\right)\frac{(t-a)^l}{l!}\right) \\ &+ \int_a^t \frac{(t-s)^{\eta-2}}{(\eta-2)!}\phi^{-1}\left(\phi(A_{\eta-1}) - \sum_{i\,:\,t_i>s} I_{\eta-1,i}(\cdot) + \int_s^b p(\sigma)d\sigma\right) ds. \end{aligned}$$

Likewise, for the second equation, we have

$$\begin{aligned} v(t) =& \sum_{k=0}^{\xi-2}\left(\left(B_k + \sum_{j\,:\,\tau_j<t} J_{k,,j}(\cdot)\right)\frac{(t-a)^k}{k!}\right) \\ &+ \int_a^t \frac{(t-s)^{\xi-2}}{(\xi-2)!}\psi^{-1}\left(\psi(B_{\xi-1}) - \sum_{i\,:\,\tau_j>s} J_{\xi-1,j}(\cdot) + \int_s^b q(\sigma)d\sigma\right) ds. \end{aligned}$$

□

The Nagumo condition, introduced in [80], is an important tool for controlling the derivatives $u^{(\eta-1)}(t)$ and $v^{(\xi-1)}(t)$:

**Definition 6.1.3.** *Let* $\gamma_1^{(l)}(t),\ \Gamma_1^{(l)}(t) \in X_1,\ l = 0,1,...,\eta-2,\ \gamma_2^{(k)}(t),\ \Gamma_2^{(k)}(t) \in X_2,$ $k = 0,1,...,\xi-2,$ *be* $L^1$*-Carathéodory functions such that*

$$\gamma_1^{(l)}(t) \le \Gamma_1^{(l)}(t),\ \gamma_2^{(k)}(t) \le \Gamma_2^{(k)}(t),\ a.e.\ t \in [a,b],$$

*and consider the sets*

$$S_1 = \left\{ \begin{array}{c} (t, x_0, x_1, ..., x_{\eta-1}, y_0, y_1, ..., y_{\eta-1}) \in [a,b]\times\mathbb{R}^{2\eta} : \\ \gamma_1^{(l)}(t) \le x_l \le \Gamma_1^{(l)}(t),\ \gamma_2^{(l)}(t) \le y_l \le \Gamma_2^{(l)}(t),\ l = 0,1,...,\eta-2 \end{array} \right\}, \tag{$S_1$}$$

*and*

$$S_2 = \left\{ \begin{array}{c} (t, x_0, x_1, ..., x_{\xi-1}, y_0, y_1, ..., y_{\xi-1}) \in [a,b]\times\mathbb{R}^{2\xi} : \\ \gamma_1^{(k)}(t) \le x_k \le \Gamma_1^{(k)}(t),\ \gamma_2^{(k)}(t) \le y_k \le \Gamma_2^{(k)}(t),\ k = 0,1,...,\xi-2 \end{array} \right\}, \tag{$S_2$}$$

*The $L^1-$Carathéodory functions $f:[a,b]\times\mathbb{R}^{2\eta}\to\mathbb{R}$ satisfies a Nagumo-type condition in $S_1$, and $g:[a,b]\times\mathbb{R}^{2\xi}\to\mathbb{R}$ in $S_2$, if, there are $\mu_1,\mu_2>0$, with*

$$\begin{aligned}\mu_1 &:= \max_{i=0,1,2,...,m}\left\{\left|\frac{\Gamma_1^{(\eta-2)}(x_{i+1})-\gamma_1^{(\eta-2)}(x_i)}{x_{i+1}-x_i}\right|,\left|\frac{\gamma_1^{(\eta-2)}(x_{i+1})-\Gamma_1^{(\eta-2)}(x_i)}{x_{i+1}-x_i}\right|\right\},\\ \mu_2 &:= \max_{j=0,1,2,...,n}\left\{\left|\frac{\Gamma_2^{(\xi-2)}(\tau_{j+1})-\gamma_2^{(\xi-2)}(\tau_j)}{\tau_{j+1}-\tau_j}\right|,\left|\frac{\gamma_2^{(\xi-2)}(\tau_{j+1})-\Gamma_2^{(\xi-2)}(\tau_j)}{\tau_{j+1}-\tau_j}\right|\right\},\end{aligned} \tag{6.8}$$

*and continuous positive functions $\varphi_k:[0,+\infty)\to(0,+\infty),\ k=1,2,$ verifying*

$$\begin{aligned}|f(t,x_0,...,x_{\eta-1},y_0,...,y_{\eta-1})| &\le \varphi_1(|x_{\eta-1}|),\ \forall(t,x_0,...,x_{\eta-1},y_0,...,y_{\eta-1})\in S_1,\\ |g(t,x_0,...,x_{\xi-1},y_0,...,y_{\xi-1})| &\le \varphi_2(|y_{\xi-1}|),\ \forall(t,x_0,...,x_{\xi-1},y_0,...,y_{\xi-1})\in S_2,\end{aligned} \tag{6.9}$$

*with*

$$\int_{\phi(\mu_1)}^{\phi(+\infty)}\frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)}ds=+\infty,\ \int_{\psi(\mu_2)}^{\psi(+\infty)}\frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)}ds=+\infty. \tag{6.10}$$

This growth condition allows *a priori* estimations on the derivatives $u^{(\eta-1)}(t)$ and $v^{(\xi-1)}(t)$.

**Lemma 6.1.4.** *Consider $\gamma_1,\Gamma_1\in PC_1^{\eta-1}[a,b]$, and $\gamma_2,\Gamma_2\in PC_2^{\xi-1}[a,b]$, such that*

$$\begin{aligned}\gamma_1^{(l)}(t) &\le \Gamma_1^{(l)}(t),\ \text{for } l=0,1,...,\eta-2,\\ \gamma_2^{(k)}(t) &\le \Gamma_2^{(k)}(t),\ \text{for } k=0,1,...,\xi-2,\ \text{and a.e. } t\in[a,b].\end{aligned}$$

*Let $f:[a,b]\times\mathbb{R}^{2\eta}\to\mathbb{R}$, and $g:[a,b]\times\mathbb{R}^{2\xi}\to\mathbb{R}$ be $L^1-$Carathéodory functions satisfying a Nagumo-type condition, according to Definition 6.1.3. Then, there exist $N_1\ge\mu_1$, and $N_2\ge\mu_2$, such that for every solution $(u,v)$ of (6.1) on set $S_1$ and $S_2$ satisfies*

$$\|u^{(\eta-1)}\|_\infty<N_1 \text{ and } \|v^{(\xi-1)}\|_\infty<N_2.$$

**Remark 6.1.5.** *Note that $N_1$ depends only on $\gamma_1^{(\eta-2)},\Gamma_1^{(\eta-2)}$ and $\varphi_1$, and $N_2$ on $\gamma_2^{(\xi-2)},\Gamma_2^{(\xi-2)}$ and $\varphi_2$.*

*Proof.* Let $(u(t),v(t))\in S_1\times S_2$ be a solution of (6.1).

By the Mean Value Theorem, there are $\lambda_0\in(t_i,t_{i+1})$ and $\lambda_1\in(\tau_j,\tau_{j+1})$ such that

$$u^{(\eta-1)}(\lambda_0)=\frac{u^{(\eta-2)}(t_{i+1})-u^{(\eta-2)}(t_i^+)}{t_{i+1}-t_i} \tag{6.11}$$

and

$$v^{(\xi-1)}(\lambda_1)=\frac{v^{(\xi-2)}(\tau_{j+1})-v^{(\xi-2)}(\tau_j^+)}{\tau_{j+1}-\tau_j}. \tag{6.12}$$

If $|u^{(\eta-1)}(t)|\le\mu_1,\ \forall t\in[a,b]$, then it is enough to define $N_1:=\mu_1$ and the proof is complete.

The case $|u^{(\eta-1)}(t)| > \mu_1, \ \forall t \in [a,b]$ , with $\mu_1$ defined in (6.8), is not possible.

In fact, if $u^{(\eta-1)}(t) > \mu_1, \ \forall t \in (t_i, t_{i+1})$, we obtain, by (6.11), $(S_1)$ and (6.8), the contradiction

$$u^{(\eta-1)}(\lambda_0) = \frac{u^{(\eta-2)}(t_{i+1}) - u^{(\eta-2)}(t_i^+)}{t_{i+1}-t_i} \leq \frac{\Gamma_1^{(\eta-2)}(t_{i+1}) - \gamma_1^{(\eta-2)}(t_i)}{t_{i+1}-t_i} \leq \mu_1. \tag{6.13}$$

If $u^{(\eta-1)}(t) < -\mu_1, \ \forall t \in [a,b]$, the contradiction is similar.

Assume, now, that there are $\lambda_2, \lambda_3 \in (t_i, t_{i+1})$ with, without loss of generality, $\lambda_2 < \lambda_3$, such that

$$u^{(\eta-1)}(\lambda_2) \leq \mu_1 \quad \text{and} \quad u^{(\eta-1)}(\lambda_3) > \mu_1.$$

By continuity of $u^{(\eta-1)}(t)$, there exists $\lambda_4 \in [\lambda_2, \lambda_3]$ such that $u^{(\eta-1)}(\lambda_4) = \mu_1$, and $u^{(\eta-1)}(t) > 0, \ \forall t \in [\lambda_4, \lambda_3]$.

Consider $N_k > \mu_k$, $k = 1, 2$, such that

$$\int_{\phi(\mu_1)}^{\phi(N_1)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds > \mu_1(b-a) \quad \text{and} \quad \int_{\psi(\mu_2)}^{\psi(N_2)} \frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)} ds > \mu_2(b-a). \tag{6.14}$$

Making a convenient change of variable and using (6.9) and (6.13),

$$\begin{aligned}
\int_{\phi(u^{(\eta-1)}(\lambda_4))}^{\phi(u^{(\eta-1)}(\lambda_3))} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds &= \int_{\lambda_4}^{\lambda_3} \frac{|\phi^{-1}(\phi(u^{(\eta-1)}(t)))|}{\varphi_1(|\phi^{-1}(\phi(u^{(\eta-1)}(t)))|)} (\phi(u^{(\eta-1)}(t)))' dt \\
&\leq \int_{\lambda_4}^{\lambda_3} \frac{|u^{(\eta-1)}(t)|}{\varphi_1(|u^{(\eta-1)}(t)|)} (\phi(u^{(\eta-1)}(t)))' dt \\
&\leq \int_{\lambda_4}^{\lambda_3} \frac{|u^{(\eta-1)}(t)| \ |f(t, x_0, ..., x_{\eta-1}, y_0, ..., y_{\eta-1})|}{\varphi_1(|u^{(\eta-1)}(t)|)} dt, \\
&\leq \int_{\lambda_4}^{\lambda_3} \frac{|u^{(\eta-1)}(t)| \ |\varphi_1(|u^{(\eta-1)}(t)|)|}{\varphi(|u^{(\eta-1)}(t)|)} dt, \\
&\leq \int_{\lambda_4}^{\lambda_3} u^{(\eta-1)}(t) dt = u^{(\eta-2)}(\lambda_3) - u^{(\eta-2)}(\lambda_4) \leq \mu_1(b-a),
\end{aligned}$$

and by (6.14)

$$\int_{\phi(u^{(\eta-1)}(\lambda_4))}^{\phi(u^{(\eta-1)}(\lambda_3))} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds \leq \mu_1(b-a) < \int_{\phi(\mu_1)}^{\phi(N_1)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)} ds.$$

Therefore $u^{(\eta-1)}(\lambda_3) < N_1$, and as $\lambda_3$ is taken arbitrarily, then $u^{(\eta-1)}(t) < N_1$, for the values of $t$ whenever $u^{(\eta-1)}(t) > \mu_1$.

The case for $\lambda_2 > \lambda_3$ follows similar arguments.

The other possible case where

$$u^{(\eta-1)}(\lambda_2) \geq -\mu_1 \quad \text{and} \quad u^{(\eta-1)}(\lambda_3) < -\mu_1.$$

can be proved by the previous techniques. Therefore $\|u^{(\eta-1)}\|_\infty \leq N_1$.

By a similar method, it can be shown that $\|v^{(\xi-1)}\|_\infty \leq N_2$. □

The arguments forward will require the following lemma, of [99]:

**Lemma 6.1.6.** *For $v, w \in C(I)$ such that $v(x) \leq w(x)$, for every $x \in I$, define*

$$q(x,u) = \max\{v, \min\{u, w\}\}.$$

*Then, for each $u \in C^1(I)$ the next two properties hold:*

*(a) $\frac{d}{dx}q(x,u(x))$ exists for a.e. $x \in I$.*

*(b) If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$ then*

$$\frac{d}{dx}q(x,u_m(x)) \to \frac{d}{dx}q(x,u(x)) \text{ for a.e. } x \in I.$$

The Schauder's fixed point theorem, will be the key existence tool:

**Theorem 6.1.7.** *([106]) Let $Y$ be a nonempty, closed, bounded and convex subset of a Banach space $X$, and suppose that $P: Y \to Y$ is a compact operator. Then $P$ has at least one fixed point in $Y$.*

## 6.2 Existence result

The next theorem will guarantee the existence of a solution of (6.1)-(6.3), through the existence of fixed points of a convenient operator.

**(H1)** $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are increasing homeomorphisms such that $\phi(0) = \psi(0) = 0$ and $\phi(\mathbb{R}) = \psi(\mathbb{R}) = \mathbb{R}$, and

$$\left|\phi^{-1}(w)\right| \leq \phi^{-1}(|w|), \text{ and } \left|\psi^{-1}(w)\right| \leq \psi^{-1}(|w|).$$

**Theorem 6.2.1.** *Consider $A_l, B_k \in \mathbb{R}$, $l = 0,1,..,\eta-1$, $k = 0,1,...,\xi-1$, and the homeomorphisms $\phi$ and $\psi$ verifying (H1). Let $f : [a,b]\times\mathbb{R}^{2\eta} \to \mathbb{R}$ , and $g : [a,b]\times\mathbb{R}^{2\xi} \to \mathbb{R}$ be $L^1$-Carathéodory functions, satisfying a Nagumo-type conditions as in Definition 6.1.3, and $I_{l,i}, J_{k,j} \in C([a,b]\times\mathbb{R}^{\eta+\xi},\mathbb{R})$, $l = 0,1,...,\eta-1$, $k = 0,1,...,\xi-1$, $i = 1,...,m$, $j = 1,...,n$.*
*Then there is at least one pair of functions $(u,v) \in X^2$ solution to the problem (6.1)-(6.3).*

*Proof.* Define the operators $T_1 : X^2 \to X_1$, $T_2 : X^2 \to X_2$, and $T : X^2 \to X^2$ given by

$$T(u,v) = (T_1(u,v), T_2(u,v)), \tag{6.15}$$

with

$$
\begin{aligned}
(T_1(u,v))(t) = & \sum_{l=0}^{\eta-2}\left(\left(A_l + \sum_{i\,:\,t_i<t} I_{l,i}(\cdot)\right)\frac{(t-a)^l}{l!}\right) \\
& + \int_a^t \frac{(t-s)^{\eta-2}}{(\eta-2)!}\phi^{-1}\left(\phi(A_{\eta-1}) - \sum_{i\,:\,t_i>s} I_{\eta-1,i}(\cdot) + \int_s^b f(U(\sigma),V(\sigma))d\sigma\right) ds
\end{aligned}
$$

and

$$
\begin{aligned}
(T_2(u,v))(t) = & \sum_{k=0}^{\xi-2}\left(\left(B_k + \sum_{j\,:\,\tau_j<t} J_{k,,j}(\cdot)\right)\frac{(t-a)^k}{k!}\right) \\
& + \int_a^t \frac{(t-s)^{\xi-2}}{(\xi-2)!}\psi^{-1}\left(\psi(B_{\xi-1}) - \sum_{j\,:\,\tau_j>s} J_{\xi-1,j}(\cdot) + \int_s^b g(U(\sigma),V(\sigma))d\sigma\right) ds.
\end{aligned}
$$

Define $K > 0$ and $M > 0$, such that

$$K > \|(u,v)\|_{X^2} \tag{6.16}$$

and

$$M > \max_{\substack{l=0,1,\dots,\eta-1,\\ k=0,1,\dots,\xi-1}} \left\{\sum_{i=1}^{m} |I_{l,i}(\cdot)|, \sum_{j=1}^{n} |J_{k,j}(\cdot)|\right\}. \tag{6.17}$$

Since $f$ and $g$ are $L^1-$Carathéodory functions and non-negative functions $\rho_{1K}(t), \rho_{2K}(t) \in L^1([a,b])$, such that

$$
\begin{aligned}
&\left|f(t,u(t),...,u^{(\eta-1)}(t),v(t),...,v^{(\eta-1)}(t))\right| \leq \rho_{1K}(t), \\
&\left|g(t,u(t),...,u^{(\xi-1)}(t),v(t),...,v^{(\xi-1)}(t))\right| \leq \rho_{2K}(t), \quad a.e. t \in [a,b].
\end{aligned} \tag{6.18}
$$

The proof will follow several steps that, for clarity, which are detailed for the $T_1(u,v)$ operator. For the operator $T_2(u,v)$ the technique is similar.

**Step 1**: *$T$ is well defined, continuous and uniformly bounded.*

By the Lebesgue dominated convergence Theorem, (6.3), (H1), (6.17) and (6.18), then, for $p = 0,1,...,\eta-2$,

$$
\begin{aligned}
\left|(T_1(u,v))^{(p)}(t)\right| \leq & \left|\sum_{l=p}^{\eta-2}\left(\left(A_l+\sum_{i\,:\,t_i<t} I_{l,i}(\cdot)\right)\frac{(t-a)^{l-p}}{(l-p)!}\right)\right| \\
& + \left|\int_a^t \frac{(t-s)^{\eta-2-p}}{(\eta-2-p)!}\phi^{-1}\left(\phi(A_{\eta-1})-\sum_{i\,:\,t_i>s} I_{\eta-1,i}(\cdot)+\int_s^b f(U(\sigma),V(\sigma))d\sigma\right)ds\right| \\
\leq & \sum_{l=p}^{\eta-2}\left((|A_l|+M)\frac{(t-a)^{l-p}}{(l-p)!}\right) \\
& + \int_a^t \frac{(t-s)^{\eta-2-p}}{(\eta-2-p)!}\phi^{-1}\left(|\phi(A_{\eta-1})|+M+\int_s^b |f(U(\sigma),V(\sigma)|\,d\sigma\right)ds \\
\leq & \sum_{l=p}^{\eta-2}\left((|A_l|+M)\frac{(b-a)^{l-p}}{(l-p)!}\right) \\
& + \int_a^b \frac{(b-s)^{\eta-2-p}}{(\eta-2-p)!}\phi^{-1}\left(|\phi(A_{\eta-1})|+M+\int_s^b \rho_{1K}(\sigma)d\sigma\right)ds < +\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|(T_1(u,v))^{(\eta-1)}(t)\right| = & \left|\phi^{-1}\left(\phi(A_{\eta-1})-\sum_{i\,:\,t_i>s} I_{\eta-1,i}(\cdot)+\int_s^b f(U(\sigma),V(\sigma))d\sigma\right)\right| \\
\leq & \phi^{-1}\left(|\phi(A_{\eta-1})|+M+\int_a^b \rho_{1K}(\sigma)d\sigma\right) < +\infty
\end{aligned}
$$

Therefore, $(T_1(u,v))(t) \in X_1$.

The proof that $(T_2(u,v))(t) \in X_2$ is similar, and, so, $T$ is well defined in $X^2$.

Define $\mathcal{B} \subseteq X^2$ as

$$
\mathcal{B}=\{(u,v)\in X^2 : \|(u,v)\|_{X^2} \leq \mathcal{K}\},
$$

with $\mathcal{K}>0$ such that

$$
\mathcal{K} := \max\left\{\begin{array}{l}
K, \sum_{l=p}^{\eta-2}\left((|A_l|+M)\dfrac{(b-a)^{l-p}}{(l-p)!}\right) \\
+\int_a^b \dfrac{(b-s)^{\eta-2-p}}{(\eta-2-p)!}\phi^{-1}\left(|\phi(A_{\eta-1})|+M+\int_s^b \rho_{1K}(\sigma)d\sigma\right)ds, \\
\text{ for } p=0,1,...,\eta-2, \\
\phi^{-1}\left(|\phi(A_{\eta-1})|+M+\int_a^b \rho_{1K}(\sigma)d\sigma\right), \\
\sum_{k=q}^{\xi-2}\left((|B_k|+M)\dfrac{(b-a)^{l-q}}{(l-q)!}\right) \\
+\int_a^b \dfrac{(b-s)^{\xi-2-q}}{(\xi-2-q)!}\psi^{-1}\left(|\psi(B_{\xi-1})|+M+\int_s^b \rho_{2K}(\sigma)d\sigma\right)ds, \\
\text{for } q=0,1,...,\xi-2, \\
\psi^{-1}\left(|\psi(B_{\xi-1})|+M+\int_a^b \rho_{2K}(\sigma)d\sigma\right)
\end{array}\right\}, \tag{6.19}
$$

with $K > 0$ and $M > 0$ given by (6.16) and (6.17).

From the above, it is clear that $T\mathcal{B}$ is uniformly bounded.

**Step 2**: *$T$ is equicontinuous, that is, $T_1\mathcal{B}$ is equicontinuous on each interval $(t_i, t_{i+1}]$, for $i = 0, 1, ..., m$, with $t_0 = a$ and $t_{m+1} = b$, and $T_2\mathcal{B}$ is equicontinuous on each interval $(\tau_j, \tau_{j+1}]$, for $j = 0, 1, ..., n$, with $\tau_0 = a$ and $\tau_{n+1} = b$.*

Consider $\mathcal{I} \subseteq (t_i, t_{i+1}]$ and $\lambda_0, \lambda_1 \in \mathcal{I}$ such that, without loss of generality, $\lambda_0 \leq \lambda_1$. For $(u, v) \in \mathcal{B}$, we have, for $p = 0, 1, ..., \eta - 2$,

$$\lim_{\lambda_0 \to \lambda_1} \left| (T_1(u,v))^{(p)}(\lambda_1) - (T_1(u,v))^{(p)}(\lambda_0) \right| \leq \lim_{\lambda_0 \to \lambda_1} \left| \sum_{l=p}^{\eta-2} \left( \left( A_l + \sum_{i\,:\,t_i<\lambda_1} I_{l,i}(\cdot) \right) \frac{(\lambda_1 - a)^{l-p}}{(l-p)!} \right) \right.$$
$$\left. - \sum_{l=p}^{\eta-2} \left( \left( A_l + \sum_{i\,:\,t_i<\lambda_0} I_{l,i}(\cdot) \right) \frac{(\lambda_0 - a)^{l-p}}{(l-p)!} \right) \right|$$
$$+ \lim_{\lambda_0 \to \lambda_1} \left| \int_a^{\lambda_1} \frac{(\lambda_1 - s)^{\eta-2-p}}{(\eta-2-p)!} \phi^{-1} \left( \phi(A_{\eta-1}) - \sum_{i\,:\,t_i>s} I_{\eta-1,i}(\cdot) + \int_s^b f(U(\sigma), V(\sigma)) d\sigma \right) ds \right.$$
$$\left. - \int_a^{\lambda_0} \frac{(\lambda_0 - s)^{\eta-2-p}}{(\eta-2-p)!} \phi^{-1} \left( \phi(A_{\eta-1}) - \sum_{i\,:\,t_i>s} I_{\eta-1,i}(\cdot) + \int_s^b f(U(\sigma), V(\sigma)) d\sigma \right) ds \right| = 0,$$

and

$$\lim_{\lambda_0 \to \lambda_1} \left| (T_1(u,v))^{(\eta-1)}(\lambda_1) - (T_1(u,v))^{(\eta-1)}(\lambda_0) \right| =$$
$$\lim_{\lambda_0 \to \lambda_1} \left| \phi^{-1} \left( \phi(A_{\eta-1}) - \sum_{i\,:\,t_i>\lambda_1} I_{\eta-1,i}(\cdot) + \int_{\lambda_1}^b f(U(\sigma), V(\sigma)) d\sigma \right) \right.$$
$$\left. - \phi^{-1} \left( \phi(A_{\eta-1}) - \sum_{i\,:\,t_i>\lambda_0} I_{\eta-1,i}(\cdot) + \int_{\lambda_0}^b f(U(\sigma), V(\sigma)) d\sigma \right) \right| = 0.$$

Therefore, $T_1\mathcal{B}$ is equicontinuous on $X_1$.

Similarly, it can be shown that $T_2\mathcal{B}$ is equicontinuous on $X_2$, too. Thus, $T\mathcal{B}$ is equicontinuous on $X^2$.

**Step 3**: *$T\mathcal{B} : X^2 \to X^2$ is equiconvergent at $t = t_i$* and *$t = \tau_j$.*

First, let us prove the equiconvergence at $t = t_i^+$, for $i = 1, 2, ...m$. The proof of equiconvergence at $t = \tau_j^+$, for $j = 1, 2, ...n$, is analogous.

So, for $t_i < t < t_{i+1}$, and $p = 0, 1, ...\eta - 2$,

$$\left|(T_1(u,v))^{(p)}(t_i) - \lim_{t\to t_i^+}(T_1(u,v))^{(p)}(t)\right| \le \left|\sum_{l=p}^{\eta-2}\left(\left(A_l + \sum_{k\,:\,t_k<t} I_{l,k}(\cdot)\right)\frac{(t_i-a)^{l-p}}{(l-p)!}\right)\right.$$
$$\left. - \lim_{t\to t_i^+}\sum_{l=p}^{\eta-2}\left(\left(A_l + \sum_{k\,:\,t_k<t_i^+} I_{l,k}(\cdot)\right)\frac{(t-a)^{l-p}}{(l-p)!}\right)\right|$$
$$+\left|\int_a^{t_i}\frac{(t_i-s)^{\eta-2-p}}{(\eta-2-p)!}\phi^{-1}\left(\phi(A_{\eta-1}) - \sum_{i\,:\,t_i>s} I_{\eta-1,i}(\cdot) + \int_s^b f(U(\sigma),V(\sigma))d\sigma\right)ds\right.$$
$$\left. - \lim_{t\to t_i^+}\int_a^t\frac{(\lambda_0-s)^{\eta-2-p}}{(\eta-2-p)!}\phi^{-1}\left(\phi(A_{\eta-1}) - \sum_{i\,:\,t_i>s} I_{\eta-1,i}(\cdot) + \int_s^b f(U(\sigma),V(\sigma))d\sigma\right)ds\right| \longrightarrow 0$$

uniformly, when $t \longrightarrow t_i^+$, and

$$\left|(T_1(u,v))^{(\eta-1)}(t_i) - \lim_{t\to t_i^+}(T_1(u,v))^{(\eta-1)}(t)\right| =$$
$$\left|\phi^{-1}\left(\phi(A_{\eta-1}) - \sum_{k\,:\,t_k>t_i} I_{\eta-1,k}(\cdot) + \int_{t_i}^b f(U(\sigma),V(\sigma))d\sigma\right)\right.$$
$$\left. -\phi^{-1}\left(\phi(A_{\eta-1}) - \lim_{t\to t_i^+}\sum_{i\,:\,t_k>t} I_{\eta-1,k}(\cdot) + \int_t^b f(U(\sigma),V(\sigma))d\sigma\right)\right| \longrightarrow 0$$

uniformly when $t \longrightarrow t_i^+$, is valid for all $\eta \ge 2$. Therefore, $T_1\mathcal{B}$ is equiconvergent at each point $t = t_i^+$, for $i = 1, 2, ..., m$.

Analogously, it can be proved that $T_2\mathcal{B}$ is equiconvergent at each point $t = \tau_j^+$, for $j = 1, 2, ..., n$.

So, $T\mathcal{B}$ is equiconvergent at each impulsive point.

**Step 4**: $T : X^2 \to X^2$ *has a fixed point* .

Consider

$$\Omega := \{(u,v) \in X^2 : \|(u,v)\|_{X^2} \le \mathcal{K}\},$$

with $\mathcal{K} > 0$ such that with $K > 0$ and $M > 0$ given by (6.16) and (6.17). According to Step 1, we have

$$\begin{aligned}
\|T(u,v)\|_{X^2} &= \|(T_1(u,v), T_2(u,v))\|_{X^2} \\
&= \max\{\|T_1(u,v)\|_{X_1}, \|T_2(u,v)\|_{X_2}\} \\
&= \max\{\|T_1(u,v)\|_\infty, ..., \|T_1^{(\eta-1)}(u,v)\|_\infty, \|T_2(u,v)\|_\infty, ..., \|T_2^{(\eta-1)}(u,v)\|_\infty\} \\
&\le \mathcal{K}.
\end{aligned}$$

So, $T\Omega \subset \Omega$, and by Theorem 6.1.7, the operator $T(u,v) = (T_1(u,v), T_2(u,v))$, has a fixed point $(u^*, v^*)$.

By Lemma 6.1.2, this fixed point is a solution of problem (6.1)-(6.3). □

## 6.3 Existence and localization results

The localization of the solution is obtained for a particular case of the impulsive conditions (6.3), that is, for (6.4).

The lower and upper solutions will play a key role in determining the location of the solution and are defined, in this particular case, as follows:

**Definition 6.3.1.** *The pair of functions* $(\alpha_1(t),\alpha_2(t))\in X^2$ *with* $(\phi(\alpha_1^{(\eta-1)}(t)),\psi(\alpha_2^{(\xi-1)}(t)))\in \left(PC^1[a,b]\right)^2$*, is a lower solution of problem (6.1), (6.2), (6.4), if*

$$\begin{cases}(\phi(\alpha_1^{(\eta-1)}(t)))'+f(t,\alpha_1(t),...,\alpha_1^{(\eta-1)}(t),\alpha_2(t),...,\alpha_2^{(\eta-2)}(t),z)\geq 0,\\ \text{ for } z\in\mathbb{R},\\ (\psi(\alpha_2^{(\xi-1)}(t)))'+g(t,\alpha_1(t),...,\alpha_1^{(\xi-2)}(t),y,\alpha_2(t),...,\alpha_2^{(\xi-1)}(t))\geq 0,\\ \text{ for } y\in\mathbb{R},\\ \alpha_1^{(l)}(a)\leq A_l,\ l=0,1,...,\eta-2,\ \alpha_1^{(\eta-1)}(b)\leq A_{\eta-1},\\ \alpha_2^{(k)}(a)\leq B_k,\ k=0,1,...,\xi-2,\ \alpha_2^{(\xi-1)}(b)\leq B_{\xi-1},\\ \Delta\alpha_1^{(l)}(t_i)\leq I_{l,i}(t_i,\alpha_1(t_i),...,\alpha_1^{(\eta-2)}(t_i),\alpha_2(t_i),...,\alpha_2^{(\xi-2)}(t_i)),\\ l=0,1,...,\eta-3,\\ \Delta\alpha_1^{(\eta-2)}(t_i)\leq I_{\eta-2,i}(t_i,\alpha_1(t_i),...,\alpha_1^{(\eta-1)}(t_i),\alpha_2(t_i),...,\alpha_2^{(\xi-2)}(t_i))\\ \Delta\alpha_2^{(k)}(\tau_j)\leq J_{k,j}(\tau_j,\alpha_1(\tau_j),...,\alpha_1^{(\eta-2)}(\tau_j),\alpha_2(\tau_j),...,\alpha_2^{(\xi-2)}(\tau_j)),\\ k=0,1,...,\xi-3,\\ \Delta\alpha_2^{(\xi-2)}(\tau_j)\leq J_{\xi-2,j}(\tau_j,\alpha_1(\tau_j),...,\alpha_1^{(\eta-2)}(\tau_j),\alpha_2(\tau_j),...,\alpha_2^{(\xi-1)}(\tau_j)),\end{cases}\tag{6.20}$$

*for* $i=1,2,...,m,\ j=1,2,...,n.$

*A pair of functions* $(\beta_1(t),\beta_2(t))\in X^2$ *such that* $(\phi(\beta_1^{(\eta-1)}(t)),\psi(\beta_2^{(\xi-1)}(t)))\in\left(PC^1[a,b]\right)^2$ *is an upper solution of problem (6.1)-(6.3) if the opposite inequalities hold.*

The nonlinearities and the impulsive functions must satisfy local monotone assumptions:

**(H2)** $f:[a,b]\times\mathbb{R}^{2\eta}\mapsto\mathbb{R}$, $g:[a,b]\times\mathbb{R}^{2\xi}\mapsto\mathbb{R}$ are $L^1$-Carathéodory such that

$$\begin{aligned}f(t,\alpha_1(t),...,\alpha_1^{(\eta-3)}(t),x_{\eta-2},x_{\eta-1},\alpha_2(t),...,\alpha_2^{(\eta-2)}(t),y_{\eta-1})&\leq f(t,x_0,...,x_{\eta-1},y_0,...,y_{\eta-1})\\ &\leq f(t,\beta_1(t),...,\beta_1^{(\eta-3)}(t),x_{\eta-2},x_{\eta-1},\beta_2(t),...,\beta_2^{(\eta-2)}(t),y_{\eta-1}),\end{aligned}$$

for $\alpha_1^{(l)}(t)\leq x_l\leq\beta_1^{(l)}(t)$, when $l=0,1,...,\eta-3$, $\alpha_2^{(q)}(t)\leq y_q\leq\beta_2^{(q)}(t)$, $q=0,1,...,\eta-2$, for fixed $(t,x_{\eta-2},x_{\eta-1},y_{\eta-1})\in[a,b]\times\mathbb{R}^3$;

$$\begin{aligned}g(t,\alpha_1(t),...,\alpha_1^{(\xi-2)}(t),x_{\xi-1},\alpha_2(t),...,\alpha_2^{(\xi-3)}(t),y_{\xi-2},y_{\xi-1})&\leq g(t,x_0,...,x_{\xi-1},y_0,...,y_{\xi-1})\\ &\leq g(t,\beta_1(t),...,\beta_1^{(\xi-2)}(t),x_{\xi-1},\beta_2(t),...,\beta_2^{(\xi-3)}(t),y_{\xi-2},y_{\xi-1}),\end{aligned}$$

for $\alpha_1^{(k)}(t)\leq x_k\leq\beta_1^{(k)}(t)$, when $k=0,1,...,\xi-2$, $\alpha_2^{(q)}(t)\leq y_q\leq\beta_2^{(q)}(t)$, when $q=0,1,...,\xi-3$, fixed $(t,x_{\xi-1},y_{\xi-2},y_{\xi-1})\in[a,b]\times\mathbb{R}^3$.

**(H3)** $I_{l,i}, J_{kj} \in C([a,b] \times \mathbb{R}^{\eta+\xi-2}, \mathbb{R})$, $l = 0,1,...,\eta-3$, $k = 0,1,...,\xi-3$, $I_{\eta-2,i}, J_{\xi-2,j} \in C([a,b] \times \mathbb{R}^{\eta+\xi-1}, \mathbb{R})$, verify

$$\begin{aligned} I_{l,i}(t_i, \alpha_1(t_i), ..., \alpha_1^{(\eta-2)}(t_i), \alpha_2(t_i), ..., \alpha_2^{(\xi-2)}(t_i)) &\leq I_{l,i}(t_i, x_0, ..., x_{\eta-2}, y_0, ..., y_{\eta-2}) \\ &\leq I_{l,i}(t_i, \beta_1(t_i), ..., \beta_1^{(\eta-2)}(t_i), \beta_2(t_i), ..., \beta_2^{(\xi-2)}(t_i)), \end{aligned}$$

for $i = 1,2,...,m$, and

$$\begin{aligned} J_{k,j}(\tau_j, \alpha_1(\tau_j), ..., \alpha_1^{(\eta-2)}(\tau_j), \alpha_2(\tau_j), ..., \alpha_2^{(\xi-2)}(\tau_j)) &\leq J_{l,j}(\tau_j, x_0, ..., x_{\eta-2}, y_0, ..., y_{\xi-2}) \\ &\leq J_{k,j}(\tau_j, \beta_1(\tau_j), ..., \beta_1^{(\eta-2)}(\tau_j), \beta_2(\tau_j), ..., \beta_2^{(\xi-2)}(\tau_j)), \end{aligned}$$

for $j = 1,2,...,n$;

$$\begin{aligned} &I_{\eta-2,i}(t_i, \alpha_1(t_i), ..., \alpha_1^{(\eta-3)}(t_i), x_{\eta-2}, x_{\eta-1}, \alpha_2(t_i), ..., \alpha_2^{(\xi-2)}(t_i)) \\ &\quad \leq I_{\eta-2,i}(t_i, x_0, ..., x_{\eta-1}, y_0, ..., y_{\xi-2}) \\ &\quad \leq I_{\eta-2,i}(t_i, \beta_1(t_i), ..., \beta_1^{(\eta-3)}(t_i), x_{\eta-2}, x_{\eta-1}, \beta_2(t_i), ..., \beta_2^{(\xi-2)}(t_i)), \end{aligned}$$

for $i = 1,2,...,m$, $\alpha_1^{(l)}(t_i) \leq x_l \leq \beta_1^{(l)}(t_i)$, $l = 0,1,...,\eta-3$, $(x_{\eta-2}, x_{\eta-1}) \in \mathbb{R}^2$, and

$$\begin{aligned} &J_{\xi-2,j}(\tau_j, \alpha_1(\tau_j), ..., \alpha_1^{(\eta-2)}(\tau_j), \alpha_2(\tau_j), ..., \alpha_2^{(\xi-3)}(\tau_j), y_{\xi-2}, y_{\xi-1}) \\ &\quad \leq J_{\eta-2,j}(\tau_j, x_0, ..., x_{\eta-2}, y_0, ..., y_{\xi-1}) \\ &\quad \leq J_{\eta-2,j}(\tau_j, \beta_1(\tau_j), ..., \beta_1^{(\eta-2)}(\tau_j), \beta_2(\tau_j), ..., \beta_2^{(\eta-3)}(\tau_j), y_{\eta-2}, y_{\eta-1}). \end{aligned}$$

for $j = 1,2,...,n$, $\alpha_2^{(k)}(\tau_j) \leq y_k \leq \beta_2^{(k)}(\tau_j)$, $k = 0,1,...,\eta-3$, $(y_{\eta-2}, y_{\eta-1}) \in \mathbb{R}^2$.

The existence and localization theorem is given as follows:

**Theorem 6.3.2.** *Let $A_l, B_k \in \mathbb{R}$, $l = 0,1,...,\eta-1$, $k = 0,1,...,\xi-1$, and the homeomorphisms $\phi$ and $\psi$ verify (H1).*
*Assume that there are lower and upper solutions of (6.1), (6.2), (6.4), $(\alpha_1^{(l)}, \alpha_2^{(k)})$ and $(\beta_1^{(l)}, \beta_2^{(k)})$, respectively, such that*

$$\begin{aligned} \alpha_1^{(l)}(t) &\leq \beta_1^{(l)}(t), \quad l = 0,1,...,\eta-2, \\ \alpha_2^{(k)}(t) &\leq \beta_2^{(k)}(t), \quad k = 0,1,...,\xi-2 \quad \forall t \in [a,b], \end{aligned}$$

*$L^1-$Carathéodory functions $f : [a,b] \times \mathbb{R}^{2\eta} \to \mathbb{R}$ , and $g : [a,b] \times \mathbb{R}^{2\xi} \to \mathbb{R}$, satisfying a Nagumo-type conditions, as in Definition 6.1.3, in the sets*

$$S_1^* = \left\{ \begin{array}{c} (t, x_0, ..., x_{\eta-1}, y_0, ..., y_{\eta-1}) \in [a,b] \times \mathbb{R}^{2\eta} : \\ \alpha_1^{(l)}(t) \leq x_l \leq \beta_1^{(l)}(t),\ \alpha_2^{(l)}(t) \leq y_l \leq \beta_2^{(l)}(t),\ l = 0,1,...,\eta-2 \end{array} \right\},$$

$$S_2^* = \left\{ \begin{array}{c} (t, x_0, ..., x_{\xi-1}, y_0, ..., y_{\xi-1}) \in [a,b] \times \mathbb{R}^{2\xi} : \\ \alpha_1^{(k)}(t) \leq x_k \leq \beta_1^{(k)}(t),\ \alpha_2^{(k)}(t) \leq y_k \leq \beta_2^{(k)}(t),\ k = 0,1,...,\xi-2 \end{array} \right\},$$

*and (H2).*
*If the impulsive functions $I_{l,i}, J_{kj} \in C([a,b] \times \mathbb{R}^{\eta+\xi-2}, \mathbb{R})$, $l = 0,1,...,\eta-3$, $k = 0,1,...,\xi-3$,*

$I_{\eta-2,i}, J_{\xi-2,j} \in C([a,b]\times\mathbb{R}^{\eta+\xi-1},\mathbb{R})$, *verify (H3), then there is at least a pair* $(u(t),v(t)) \in X^2$ *solution of (6.1), (6.2), (6.4), and, moreover,*

$$\begin{aligned} \alpha_1^{(l)}(t) &\leq u^{(l)}(t) \leq \beta_1^{(l)}(t), \quad l=0,1,...,\eta-2, \\ \alpha_2^{(k)}(t) &\leq v^{(k)}(t) \leq \beta_2^{(k)}(t), \quad k=0,1,...,\xi-2, \quad \forall t\in[a,b], \end{aligned}$$

$$\|u^{(\eta-1)}\| \leq N_1 \quad \textit{and} \quad \|v^{(\xi-1)}\| \leq N_2,$$

*with* $N_1$ *and* $N_2$ *given by Lemma 6.1.4.*

*Proof.* Define the truncature functions $\delta_{1,l}, \delta_{2,k} : [a,b]\times\mathbb{R} \to \mathbb{R}$, for $l=0,1,...,\eta-2$, and $k=0,1,...,\xi-2$, given by

$$\delta_{1,l}(t,x_l) = \begin{cases} \beta_1^{(l)}(t) & \text{if } \ x_l > \beta_1^{(l)}(t) \\ x_l & \text{if } \ \alpha_1^{(l)}(t) \leq x_l \leq \beta_1^{(l)}(t) \\ \alpha_1^{(l)}(t) & \text{if } \ x_l < \alpha_1^{(l)}(t), \end{cases} \tag{6.21}$$

$$\delta_{2,k}(t,y_k) = \begin{cases} \beta_2^{(k)}(t) & \text{if } \ y_k > \beta_2^{(k)}(t) \\ y_k & \text{if } \ \alpha_2^{(k)}(t) \leq y_k \leq \beta_2^{(k)}(t) \\ \alpha_2^{(k)}(t) & \text{if } \ y_k < \alpha_2^{(k)}(t), \end{cases} \tag{6.22}$$

and the truncated and perturbed functions

$$F(t) := f\begin{pmatrix} t, \delta_{1,0}(t,u(t)),...,\delta_{1,\eta-2}(t,u^{(\eta-2)}(t)), \\ \dfrac{d}{dt}\left(\delta_{1,\eta-2}(t,u^{(\eta-2)}(t))\right), \\ \delta_{2,0}(t,v(t)),...,\delta_{2,\eta-2}(t,v^{(\eta-2)}(t)), \\ \dfrac{d}{dt}\left(\delta_{2,\eta-2}(t,v^{(\eta-2)}(t))\right) \end{pmatrix} + \frac{\delta_{1,\eta-2}(t,u^{(\eta-2)}(t)) - u^{(\eta-2)}(t)}{1+|\delta_{1,\eta-2}(t,u^{(\eta-2)}(t)) - u^{(\eta-2)}(t)|},$$

and

$$G(t) := g\begin{pmatrix} t, \delta_{1,0}(t,u(t)),...,\delta_{1,\xi-2}(t,u^{(\xi-2)}(t)), \\ \dfrac{d}{dt}\left(\delta_{1,\xi-2}(t,u^{(\xi-2)}(t))\right), \\ \delta_{2,0}(t,v(t)),...,\delta_{2,\xi-2}(t,v^{(\xi-2)}(t)), \\ \dfrac{d}{dt}\left(\delta_{2,\xi-2}(t,v^{(\xi-2)}(t))\right) \end{pmatrix} + \frac{\delta_{2,\xi-2}(t,v^{(\xi-2)}(t)) - v^{(\xi-2)}(t)}{1+|\delta_{2,\xi-2}(t,v^{(\xi-2)}(t)) - v^{(\xi-2)}(t)|}.$$

Consider the following modified coupled system composed by differential equations

$$\begin{cases} (\phi(u^{(\eta-1)}(t)))' + F(t) = 0, \ t\in[a,b]\setminus\{t_1,...,t_m\}, \\ \\ (\psi(v^{(\xi-1)}(t)))' + G(t) = 0, \ t\in[a,b]\setminus\{\tau_1,...,\tau_n\}, \end{cases} \tag{6.23}$$

with the truncated impulsive conditions for $i = 1, 2, ..., m$, $l = 1, 2, ..., \eta - 3$,

$$\Delta u^{(l)}(t_i) = I_{l,i}\begin{pmatrix} t_i, \delta_{1,0}(t_i, u(t_i)), ..., \delta_{1,\eta-2}(t_i, u^{(\eta-2)}(t_i)), \\ \delta_{2,0}(t_i, v(t_i)), ..., \delta_{2,\xi-2}(t_i, v^{(\xi-2)}(t_i)) \end{pmatrix},$$

$$\Delta u^{(\eta-2)}(t_i) = I_{\eta-2,i}\begin{pmatrix} t_i, \delta_{1,0}(t_i, u(t_i)), ..., \delta_{1,\eta-2}(t_i, u^{(\eta-2)}(t_i)), \\ \frac{d}{dt}\left(\delta_{1,\eta-2}(t_i, u^{(\eta-2)}(t_i))\right), \\ \delta_{2,0}(t_i, v(t_i)), ..., \delta_{2,\xi-2}(t_i, v^{(\xi-2)}(t_i)) \end{pmatrix}, \tag{6.24}$$

and, for $j = 1, 2, ..., n$, $k = 1, 2, ..., \xi - 3$,

$$\Delta v^{(k)}(\tau_j) = J_{k,j}\begin{pmatrix} \tau_j, \delta_{1,0}(\tau_j, u(\tau_j)), ..., \delta_{1,\eta-2}(\tau_j, u^{(\eta-2)}(\tau_j)), \\ \delta_{2,0}(\tau_j, v(\tau_j)), ..., \delta_{2,\xi-2}(\tau_j, v^{(\xi-2)}(\tau_j)) \end{pmatrix},$$

$$\Delta v^{(\xi-2)}(\tau_j) = J_{\xi-2,j}\begin{pmatrix} \tau_j, \delta_{2,0}(\tau_j, u(\tau_j)), ..., \delta_{2,\eta-2}(\tau_j, u^{(\eta-2)}(\tau_j)), \\ \delta_{2,0}(\tau_j, v(\tau_j)), ..., \delta_{2,\xi-2}(\tau_j, v^{(\xi-2)}(\tau_j)), \\ \frac{d}{dt}\left(\delta_{2,\xi-2}(\tau_j, v^{(\xi-2)}(\tau_j))\right) \end{pmatrix}, \tag{6.25}$$

and the boundary conditions (6.2).

It is clear that the functions $F$ and $G$ satisfy the Nagumo type conditions, as in Definition 6.1.3, relative to the sets $S_1^*$ and $S_2^*$, with

$$|F(t, x_0, ..., x_{\eta-1}, y_0, ..., y_{\eta-1})| \leq \varphi_1(|x_{\eta-1}|) + 1,$$

and

$$|G(t, x_0, ..., x_{\xi-1}, y_0, ..., y_{\xi-1})| \leq \varphi_2(|y_{\xi-1}|) + 1,$$

and are $L^1$-Carathéodory, when

$$\max\left\{|\alpha_k^{(l)}|, |\beta_k^{(l)}|, N_k,\ k = 1, 2,\ l = 0, 1, ..., p-2\right\} < K,$$

we have

$$|F(t, x_0, ..., x_{\eta-1}, y_0, ..., y_{\eta-1})| \leq \rho_{1K}(t) + 1$$

and

$$|G(t, x_0, ..., x_{\xi-1}, y_0, ..., y_{\xi-1})| \leq \rho_{2K}(t) + 1.$$

Therefore, applying the same arguments as in the Theorem 6.2.1, it can be proved that problem (6.23), (6.2), (6.24), (6.25) has, at least a solution $(u^*(t), v^*(t))$.

To prove that the solution of (6.23), (6.2), (6.24), (6.25), are solutions of the initial problem (6.1), (6.2), (6.4), it will be enough to show that

$$\alpha_1^{(l)}(t) \leq u^{(l)}(t) \leq \beta_1^{(l)}(t), \quad \alpha_2^{(k)}(t) \leq v^{(k)}(t) \leq \beta_2^{(k)}(t),$$

for $l=0,1,...,\eta-2$, $k=0,1,...,\xi-2$, and $t\in[a,b]$.

For $l=\eta-2$, suppose, by contradiction, that there exists $t\in[a,b]$ such that $u^{(\eta-2)}(t)<\alpha_1^{(\eta-2)}(t)$, and define

$$\inf_{a\le t\le b}(u^{(\eta-2)}(t)-\alpha_1^{(\eta-2)}(t)):=u^{(\eta-2)}(\lambda)-\alpha_1^{(\eta-2)}(\lambda)<0. \tag{6.26}$$

As, by boundary conditions (6.2) and Definition 6.3.1, $u^{(\eta-2)}(a)-\alpha_1^{(\eta-2)}(a)\ge 0$, then $\lambda\neq a$. In the same way, $u^{(\eta-1)}(b^-)-\alpha_1^{(\eta-1)}(b^-)>0$, therefore $\lambda\neq b$.

Then $\lambda\in(a,b)$, two possibilities remain to be studied:

(i) Assume that there is $p\in\{0,1,2,...,m\}$ such that $\lambda\in(t_p,t_{p+1})$. Therefore

$$\min_{t\in(t_p,t_{p+1})}(u^{(\eta-2)}(t)-\alpha_1^{(\eta-2)}(x)):=u^{(\eta-2)}(\lambda)-\alpha_1^{(\eta-2)}(\lambda)<0.$$

and

$$u^{(\eta-1)}(\lambda)-\alpha_1^{(\eta-1)}(\lambda)=0. \tag{6.27}$$

Choose $\epsilon>0$, sufficiently small, such that

$$u^{(\eta-2)}(t)-\alpha_2^{(\eta-2)}(t)<0 \text{ and } u^{(\eta-1)}(t)-\alpha_1^{(\eta-1)}(t)\ge 0, \forall t\in(\lambda,\lambda+\epsilon). \tag{6.28}$$

By (H2) and (6.28), for all $t\in(\lambda,\lambda+\epsilon)$,

$$\begin{aligned}
&(\phi(u^{(\eta-1)}(t)))'-(\phi(\alpha_1^{(\eta-1)}(t)))'\\
&\le -f\left(\begin{array}{c} t,\delta_{1,0}(t,u(t)),...,\delta_{1,\eta-2}(t,u^{(\eta-2)}(t)),\dfrac{d}{dt}\left(\delta_{1,\eta-2}(t,u^{(\eta-2)}(t))\right),\\ \delta_{2,0}(t,v(t)),...,\delta_{2,\eta-2}(t,v^{(\eta-2)}(t)),\dfrac{d}{dt}\left(\delta_{2,\eta-2}(t,v^{(\eta-2)}(t))\right)\end{array}\right)\\
&-\frac{\delta_{1,\eta-2}(t,u^{(\eta-2)}(t))-u^{(\eta-2)}(t)}{1+|\delta_{1,\eta-2}(t,u^{(\eta-2)}(t))-u^{(\eta-2)}(t)|}+f(t,\alpha_1(t),...,\alpha_1^{(\eta-1)}(t),\alpha_2(t),...,\alpha_2^{(\eta-1)}(t))\\
&=-f\left(\begin{array}{c} t,\delta_{1,0}(t,u(t)),...,\delta_{1,\eta-3}(t,u^{(\eta-3)}(t)),\alpha_1^{(\eta-2)}(t),\alpha_1^{(\eta-1)}(t),\\ \delta_{2,0}(t,v(t)),...,\delta_{2,\eta-2}(t,v^{(\eta-2)}(t)),\dfrac{d}{dt}\left(\delta_{2,\eta-2}(t,v^{(\eta-2)}(t))\right)\end{array}\right)\\
&-\frac{\alpha_1^{(\eta-2)}(t)-u^{(\eta-2)}(t)}{1+|\alpha_1^{(\eta-2)}(t)-u^{(\eta-2)}(t)|}+f(t,\alpha_1(t),...,\alpha_1^{(\eta-1)}(t),\alpha_2(t),...,\alpha_2^{(\eta-1)}(t))\\
&\le\frac{u^{(\eta-2)}(t)-\alpha_1^{(\eta-2)}(t)}{1+|u^{(\eta-2)}(t)-\alpha_1^{(\eta-2)}(t)|}<0.
\end{aligned}$$

So $(\phi(u^{(\eta-1)}(t))-\phi(\alpha_1^{(\eta-1)}(t)))$ is strictly decreasing for $\forall t\in(\lambda,\lambda+\epsilon)$, and, by (6.27) and (6.28), in $[\lambda,\lambda+\epsilon)$, we obtain the contradiction:

$$0=\phi(u^{(\eta-1)}(\lambda))-\phi(\alpha_1^{(\eta-1)}(\lambda))>\phi(u^{(\eta-1)}(t))-\phi(\alpha_1^{(\eta-1)}(t))\ge 0.$$

Therefore, for $t\in(t_p,t_{p+1})$, $p=0,1,2,...,m$,

$$u^{(\eta-2)}(t)\ge\alpha_1^{(\eta-2)}(t). \tag{6.29}$$

(ii) Suppose, now, that there is $p^* \in \{1,2,...,n\}$ such that, $\lambda = t_{p^*}$. That is,

$$\inf_{t\in[a,b]}(u^{(\eta-2)}(t) - \alpha_1^{(\eta-2)}(t)) := u^{(\eta-2)}(t_{p^*}) - \alpha_1^{(\eta-2)}(t_{p^*}) < 0. \tag{6.30}$$

As $u, \alpha_1 \in X_1$, by (6.29), we obtain the contradiction with (6.30):

$$u^{(\eta-2)}(t_{p^*}) = \lim_{x\to t_{p^*}^-} u^{(\eta-2)}(t) \geq \lim_{x\to t_{p^*}^-} \alpha_1^{(\eta-2)}(t) = \alpha_1^{(\eta-2)}(t_{p^*}). \tag{6.31}$$

If $\lambda = t_{p^*}^+$, suppose

$$\inf_{t\in[a,b]}(u^{(\eta-2)}(t) - \alpha_1^{(\eta-2)}(t)) := u^{(\eta-2)}(t_{p^*}^+) - \alpha_1^{(\eta-2)}(t_{p^*}^+) < 0,$$

By (6.24), (6.31), (H3) and Definition 6.3.1, we obtain the contradiction:

$$\begin{aligned}
0 &> u^{(\eta-2)}(t_{p^*}^+) - \alpha_1^{(\eta-2)}(t_{p^*}^+) \\
&= u^{(\eta-2)}(t_{p^*}) + I_{\eta-2,p^*}\begin{pmatrix} t_{p^*}, \delta_{1,0}(t_{p^*}, u(t_{p^*})), ..., \delta_{1,\eta-2}(t_{p^*}, u^{(\eta-2)}(t_{p^*})), \\ \frac{d}{dt}\left(\delta_{1,\eta-2}(t_{p^*}, u^{(\eta-2)}(t_{p^*}))\right), \\ \delta_{2,0}(t_{p^*}, v(t_{p^*})), ..., \delta_{2,\xi-2}(t_{p^*}, v^{(\xi-2)}(t_{p^*})) \end{pmatrix} \\
&- \alpha_1^{(\eta-2)}(t_{p^*}) - I_{\eta-2,p^*}(t_{p^*}, \alpha_1(t_{p^*}), ..., \alpha_1^{(\eta-1)}(t_{p^*}), \alpha_2(t_{p^*}), ..., \alpha_2^{(\xi-2)}(t_{p^*})) \\
&\geq I_{\eta-2,p^*}\begin{pmatrix} t_{p^*}, \delta_{1,0}(t_{p^*}, u(t_{p^*})), ..., \delta_{1,\eta-2}(t_{p^*}, u^{(\eta-3)}(t_{p^*})), \\ \alpha_1^{(\eta-2)}(t_{p^*}), \alpha_1^{(\eta-1)}(t_{p^*}), \\ \delta_{2,0}(t_{p^*}, v(t_{p^*})), ..., \delta_{2,\eta-2}(t_{p^*}, v^{(\xi-2)}(t_{p^*})) \end{pmatrix} \\
&- I_{\eta-2,p^*}(t_{p^*}, \alpha_1(t_{p^*}), ..., \alpha_1^{(\eta-1)}(t_{p^*}), \alpha_2(t_{p^*}), ..., \alpha_2^{(\xi-2)}(t_{p^*})) \geq 0.
\end{aligned}$$

Therefore, $u^{(\eta-2)}(t) \geq \alpha^{(\eta-2)}(t)$, for $t \in [a,b]$.

By similar arguments, it can be proved the remaining inequality and, therefore,

$$\alpha_1^{(\eta-2)}(t) \leq u^{(\eta-2)}(t) \leq \beta_1^{(\eta-2)}(t), \text{ for all } t \in [a,b]. \tag{6.32}$$

By integration of (6.32) for $t \in [a, t_1]$,

$$\alpha^{(\eta-3)}(t) \leq u^{(\eta-3)}(t) - u^{(\eta-3)}(a) + \alpha_1^{(\eta-3)}(a) \leq u^{(\eta-3)}(t),$$

and for $t \in (t_1, t_2]$, we have, by (H3),

$$\begin{aligned}
\alpha^{(\eta-3)}(t) &\leq u^{(\eta-3)}(t) - u^{(\eta-3)}(t_1^+) + \alpha_1^{(\eta-3)}(t_1^+) \\
&\leq u^{(\eta-3)}(t) - u^{(\eta-3)}(t_1) - I_{\eta-3,1}\begin{pmatrix} t_1, \delta_{1,0}(t_1, u(t_1)), ..., \delta_{1,\eta-2}(t_1, u^{(\eta-2)}(t_1)), \\ \delta_{2,0}(t_1, v(t_1)), ..., \delta_{2,\xi-2}(t_1, v^{(\xi-2)}(t_1)) \end{pmatrix} \\
&+ \alpha_1^{(\eta-3)}(t_1) + I_{\eta-3,1}(t_1, \alpha_1(t_1), ..., \alpha_1^{(\eta-2)}(t_1), \alpha_2(t_1), ..., \alpha_2^{(\xi-2)}(t_1) \\
&\leq u^{(\eta-3)}(t).
\end{aligned}$$

Applying this method for each interval $(t_i, t_{i+1}]$, $i = 2, ..., m$, we obtain

$$\alpha_1^{(\eta-3)}(t) \leq u^{(\eta-3)}(t), \text{ for all } t \in [a,b],$$

and, by the same technique,

$$u^{(\eta-3)}(t) \leq \beta_1^{(\eta-3)}(t), \text{ for all } t \in [a,b].$$

By iteration of these arguments, we conclude

$$\alpha_1^{(l)}(t) \leq u^{(l)}(t) \leq \beta_1^{(l)}(t), \quad l = 0,1,...,\eta-2, \quad \forall t \in [a,b].$$

Analogously, it can be proved that

$$\alpha_2^{(k)}(t) \leq v^{(k)}(t) \leq \beta_2^{(k)}(t), \quad k = 0,1,...,\xi-2, \quad \forall t \in [a,b].$$

The estimates $\|u^{(\eta-1)}(t)\| \leq N_1$ and $\|v^{(\xi-1)}(t)\| \leq N_2$ are trivial consequences do Lemma (6.1.4). □

## 6.4 Flexural vibration of a single-span suspension bridge

Problems related to suspension bridge structures, which are typically designed as flexible structures with long spans, are part of a vast field of investigation due to their vulnerability to external loads and large deformations, see, for example, [55, 62, 92, 107].

In this application, we consider a model, based on the works [55, 107], to describe the deflection curves $u(x)$ and $v(x)$ of the girder and cable, respectively, of a suspension bridge of single span with length $L > 1$, given by the system of fourth order equations

$$\begin{cases} E_c I_c u^{(4)}(x) - H(u''(x) + u'''(x)) + E_c A_c (v(x))^2 u''(x) = 0, \\ x \in \left[0, \dfrac{L}{2}\right] \setminus \{(x_i)\}, \\ \\ E_g I_g v^{(4)}(x) - E_g A_g (v''(x))^2 v'''(x) + \gamma(u(x) + v(x)) = 0, \\ x \in \left[0, \dfrac{L}{2}\right] \setminus \{(\tau_j)\}, \end{cases} \tag{6.33}$$

where $E_c, E_g$ is the Young's modulus, $I_c, I_g$ the moment of inertia of the mass, $A_c, A_g$ the cross-sectional area, with $c$ for the cable and $g$ for the girder; $\gamma > 0$ is the tension of a spring force applied vertically to the assembly and $H > 0$ is the horizontal tension of the main cable under permanent loads.

At the ends, the behavior of the cable and beam assembly is given by the following boundary conditions

$$\begin{cases} u(0) = 0, \ u'(0) = 0, \ u''(0) = 0, \ u'''\left(\dfrac{L}{2}\right) = 1, \\ \\ v(0) = 0, \ v'(0) = 0, \ v''(0) = 0, \ v'''\left(\dfrac{L}{2}\right) = 1. \end{cases} \tag{6.34}$$

For simplicity, we only consider the impulse moment that occurs at $i = 1,2$ and $j = 1,2,3$. The impulsive effects are given by generalized functions with dependence on the unknown function itself and its derivatives,

$$\begin{cases} \Delta u(x_i) &= \dfrac{1}{20}u(x_i)+\dfrac{1}{25}v(x_i), \quad \Delta u'(x_i)=\dfrac{1}{30}u'(x_i)+\dfrac{1}{40}v'(x_i), \\ \Delta u''(x_i) &= -\dfrac{1}{6}u'''(x_i)+\dfrac{1}{5}v''(x_i), \quad \Delta v\left(\tau_j\right)=\dfrac{1}{10}u(\tau_j)+\dfrac{1}{5}v(\tau_j), \\ \Delta v'\left(\tau_j\right) &= \dfrac{1}{5}u'(\tau_j)+\dfrac{1}{10}v'(\tau_j), \quad \Delta v''\left(\tau_j\right)=-x_i+u''(\tau_j)-v'''(\tau_j), \end{cases} \tag{6.35}$$

with $x_1=\dfrac{L}{8}$, $x_2=\dfrac{L}{4}$, $x_3=\dfrac{3L}{8}$, $\tau_1=\dfrac{L}{6}$ and $\tau_3=\dfrac{2L}{6}$.

The system (6.33)-(6.35) is a particular case of the problem (6.1), (6.2), (6.4), with $[a,b]=\left[0,\frac{L}{2}\right]$,

$$\phi(w_3)=E_cI_cw_3, \quad \psi(z_3)=E_gI_gz_3, \tag{6.36}$$

$$\begin{aligned} f(x,w_0,w_1,w_2,w_3,z_0,z_1,z_2,z_3) &= -H(w_2+w_3)+E_cA_cz_0^2w_2, \\ g(x,w_0,w_1,w_2,w_3,z_0,z_1,z_2,z_3) &= -E_gA_gz_2^2z_3+\gamma(w_0+z_0), \end{aligned}$$

$$A_0=A_1=A_2=B_0=B_1=B_2=0, \ A_3=B_3=1,$$

$$\begin{aligned} I_{0,i}(x_i,w_0,w_1,w_2,z_0,z_1,z_2) &= \frac{1}{20}w_0+\frac{1}{25}z_0, \\ I_{1,i}(x_i,w_0,w_1,w_2,z_0,z_1,z_2) &= \frac{1}{30}w_1+\frac{1}{40}z_1, \\ I_{2,i}(x_i,w_0,w_1,w_2,w_3,z_0,z_1,z_2) &= -\frac{1}{6}w_3+\frac{1}{5}z_2, \\ J_{0,j}(\tau_j,w_0,w_1,w_2,z_0,z_1,z_2) &= \frac{1}{10}w_0+\frac{1}{5}z_0, \\ J_{1,j}(\tau_j,w_0,w_1,w_2,z_0,z_1,z_2) &= \frac{1}{5}w_1+\frac{1}{10}z_1, \\ J_{2,j}(\tau_j,w_0,w_1,w_2,z_0,z_1,z_2,z_3) &= -x_i+z_2-z_3, \end{aligned}$$

and $m=3$, $n=2$.

It is easy to see that the functions $\phi:\mathbb{R}\to\mathbb{R}$ and $\psi:\mathbb{R}\to\mathbb{R}$, given in (6.36), verify assumption (H1), are increasing homeomorphisms such that $\phi(0)=\psi(0)=0$, $\phi(\mathbb{R})=\mathbb{R}$ and $\psi(\mathbb{R})=\mathbb{R}$.

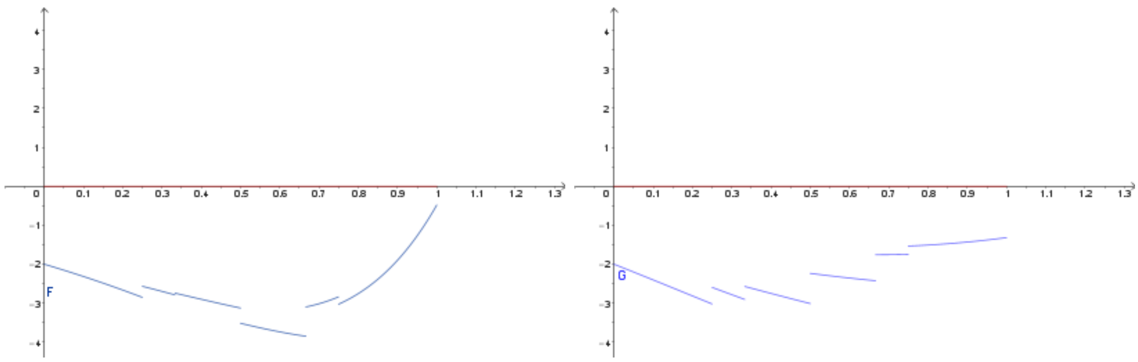
As a numerical example we can consider $E_cI_c=1$, $E_gI_g=1$, $E_cA_c=1$, $E_gA_g=2$, $H=2$, $L=2$ and $\gamma=2$. The functions $\alpha_\kappa:[0,1]\to\mathbb{R}$, $\kappa=1,2$, given by $\alpha_1(x)\equiv 0$,

$\alpha_1(x) \equiv 0$ and $\beta_\kappa : [0,1] \to \mathbb{R}$, $\kappa = 1, 2$, given by

$$\beta_1(x) = \begin{cases} e^x - \frac{1}{4}x^2 - x - 1 & \text{if} \quad 0 \leq x \leq \frac{1}{4} \\ \frac{2}{3}e^x - \frac{1}{2}x - \frac{1}{2} & \text{if} \quad \frac{1}{4} < x \leq \frac{1}{2} \\ \frac{3}{4}e^x - \frac{1}{2} & \text{if} \quad \frac{3}{4} < x \leq 1 \\ \frac{4}{5}e^x - \frac{1}{2} & \text{if} \quad \frac{1}{2} < x \leq \frac{3}{4} \end{cases},$$

$$\beta_2(x) = \begin{cases} \frac{1}{6}x^3 + \frac{1}{2}x^2 & \text{if} \quad \frac{\pi}{2} < t \leq \pi \\ \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{2}x & \text{if} \quad \frac{\pi}{2} < t \leq \pi \\ \frac{1}{6}x^3 + \frac{1}{2}x^2 + x & \text{if} \quad \frac{\pi}{2} < t \leq \pi \end{cases},$$

are, respectively, lower and upper solutions of the problem (6.33)-(6.35), according to Definition 6.3.1. In fact, the differential inequalities are verified in the interval $[0,1]$, as shown in Figure 6.1.




**a** **b**

**Fig. 6.1:** Relationship between nonlinearities as a function of the lower and upper solutions, given by the inequalities of the Definition 5.3.1: a) first equation of the system; b) second equation of the system.

The boundary conditions

$$\begin{aligned} &\alpha_1(0) = 0, \quad \alpha_1'(0) = 0, \quad \alpha_1''(0) = 0, \quad \alpha_1'''(1) = 0 < 1, \\ &\alpha_2(0) = 0, \quad \alpha_2'(0) = 0, \quad \alpha_2''(0) = 0, \quad \alpha_2'''(1) = 0 < 1, \\ &\beta_1(0) = 0, \quad \beta_1'(0) = 0, \quad \beta_1''(0) = \frac{1}{2} > 0, \quad \beta_1'''(1) = \frac{4e}{5} > 1, \\ &\beta_2(0) = 0, \quad \beta_2'(0) = 0, \quad \beta_2''(0) = 1 > 0, \quad \beta_2'''(1) = 1, \end{aligned}$$

and impulsive conditions verify the inequalities of Definition 6.3.1, as shown in Table 6.1 and Table 6.2.

Let

$$\max\left\{|\alpha_1^{(l)}|, |\alpha_2^{(k)}|, |\beta_1^{(l)}|, |\beta_2^{(k)}|, N_1, N_2, \ l = 0,1,2, \ k = 0,1,2\right\} < K,$$

**Table 6.1:** Impulse conditions for functions $\alpha_1$ and $\beta_1$.

| $l$ | $i$ | $t_i$ | $\Delta\alpha_1^{(l)}(t_i)$ | $\leq I_{l,i}(\alpha_1)$ | $\leq I_{l,i}(\beta_1)$ | $\leq \Delta\beta_1^{(l)}(t_i)$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0.2500 | 0.0000 | 0.0000 | 0.0023 | 0.2126 |
| 0 | 2 | 0.5000 | 0.0000 | 0.0000 | 0.0333 | 0.3874 |
| 0 | 3 | 0.7500 | 0.0000 | 0.0000 | 0.0985 | 0.1059 |
| 1 | 1 | 0.2500 | 0.0000 | 0.0000 | 0.0123 | 0.1970 |
| 1 | 2 | 0.5000 | 0.0000 | 0.0000 | 0.0481 | 0.6374 |
| 1 | 3 | 0.7500 | 0.0000 | 0.0000 | 0.1037 | 0.1059 |
| 2 | 1 | 0.2500 | 0.0000 | 0.0000 | 0.0360 | 0.0720 |
| 2 | 2 | 0.5000 | 0.0000 | 0.0000 | 0.1168 | 0.1374 |
| 2 | 3 | 0.7500 | 0.0000 | 0.0000 | 0.0854 | 0.1059 |

**Table 6.2:** Impulse conditions for functions $\alpha_2$ and $\beta_2$.

| $k$ | $j$ | $\tau_j$ | $\Delta\alpha_2^{(k)}(\tau_j)$ | $\leq J_{k,j}(\alpha_2)$ | $\leq J_{k,j}(\beta_2)$ | $\leq \Delta\beta_2^{(k)}(\tau_j)$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0.3333 | 0.0000 | 0.0000 | 0.0387 | 0.1667 |
| 0 | 2 | 0.6667 | 0.0000 | 0.0000 | 0.2171 | 0.3333 |
| 1 | 1 | 0.3333 | 0.0000 | 0.0000 | 0.1250 | 0.5000 |
| 1 | 2 | 0.6667 | 0.0000 | 0.0000 | 0.4310 | 0.5000 |
| 2 | 1 | 0.3333 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2 | 2 | 0.6667 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

then $f$ and $g$ are $L^1$-Carathéodory functions, with

$$\begin{aligned} |f(x,w_0,w_1,w_2,w_3,z_0,z_1,z_2,z_3)| &\leq 4K+K^3 := \rho_{1K}(x), \\ |g(x,w_0,w_1,w_2,w_3,z_0,z_1,z_2,z_3)| &\leq 4K+2K^3 := \rho_{2K}(x), \end{aligned}$$

and the sum of the jumps is bounded.

The functions $f$ and $g$ satisfy Nagumo's condition relating to sets

$$S_1^0 = \left\{ \begin{array}{c} (x,w_0,w_1,w_2,w_3,z_0,z_1,z_2,z_3) \in [0,1]\times\mathbb{R}^8 : \\ \alpha_1^{(l)}(t) \leq w_l \leq \beta_1^{(l)}(t),\ \alpha_2^{(l)}(t) \leq z_l \leq \beta_2^{(l)}(t),\ l=0,1,2 \end{array} \right\}$$

and

$$S_2^0 = \left\{ \begin{array}{c} (x,w_0,w_1,w_2,w_3,z_0,z_1,z_2,z_3) \in [0,1]\times\mathbb{R}^8 : \\ \alpha_1^{(k)}(t) \leq x_k \leq \beta_1^{(k)}(t),\ \alpha_2^{(k)}(t) \leq y_k \leq \beta_2^{(k)}(t),\ k=0,1,2 \end{array} \right\}.$$

Consider a constant $\mathcal{K}_k > 0$, $k=1,2$, and $\mu_k$ as defined in (6.8), then, in $S_1^0$ and $S_2^0$,

$$\begin{aligned} |f(x,w_0,w_1,w_2,w_3,z_0,z_1,z_2,z_3)| &= \left|-2(w_2+w_3)+z_0^2 w_2\right| \\ &\leq \mathcal{K}_1 + 2w_3 := \varphi_1(|w_3|) \end{aligned}$$

and

$$\begin{aligned} |g(x,w_0,w_1,w_2,w_3,z_0,z_1,z_2,z_3)| &= \left|-2z_2^2 z_3 + 2(w_0+z_0)\right| \\ &\leq \mathcal{K}_2 z_3 + \mathcal{K}_2 := \varphi_2(|z_3|), \end{aligned}$$

it is trivial that

$$\int_{\phi(\mu_1)}^{\phi(+\infty)} \frac{|\phi^{-1}(s)|}{2|\phi^{-1}(s)|+\mathcal{K}_1}ds = +\infty \text{ and } \int_{\psi(\mu_2)}^{\psi(+\infty)} \frac{|\psi^{-1}(s)|}{\mathcal{K}_2|\psi^{-1}(s)|+\mathcal{K}_2}ds = +\infty.$$

So, by Theorem 6.3.2, there is at least one pair of functions $(u(x), v(x)) \in X^2$, solution of the problem (6.33)–(6.35) and, moreover,

$$\begin{aligned} \alpha_1^{(l)}(x) &\le u^{(l)}(x) \le \beta_1^{(l)}(x), \quad l = 0,1,2, \\ \alpha_2^{(k)}(x) &\le v^{(k)}(x) \le \beta_2^{(k)}(x), \quad k = 0,1,2, \quad \forall x \in [0,1], \end{aligned}$$

and

$$\|u'''\| \le N_1 \quad \text{and} \quad \|v'''\| \le N_2,$$

with $N_1$ and $N_2$ given by Lemma 6.1.4.

The colored regions in the Figure 6.2 show the locations of the solution $(u(x), v(x))$ of problem (6.33)–(6.35).

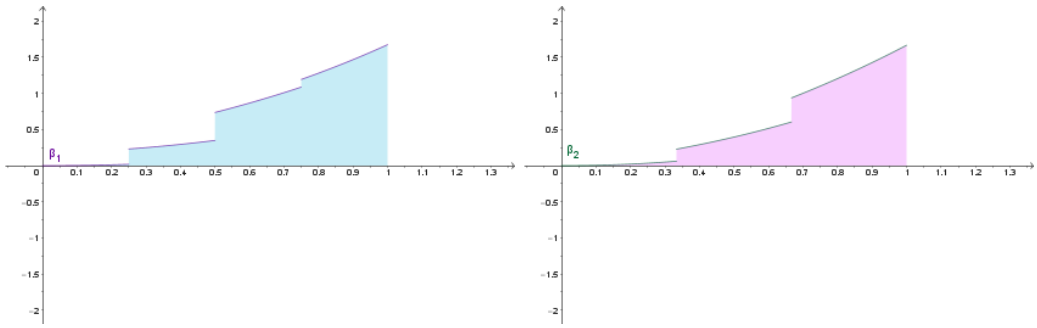



**Fig. 6.2:** At least one solution $(u(x), v(x))$ of problem (6.33)-(6.35) is located in the colored region, when $x \in [0, 1]$.

## 6.5 Singular $\phi$-Laplacian equations

The study developed in this chapter can be adapted and applied to a system of singular $\phi$-Laplacian equations, that is, to the system of equations (6.1). To do this, two assumptions are necessary:

**(Hs)** $\phi : (-c, c) \to \mathbb{R}$ and $\psi : (-d, d) \to \mathbb{R}$, for some $0 < c < +\infty$ and $0 < d < +\infty$, are increasing homeomorphisms with $\phi(0) = \psi(0) = 0$, $\phi(-c, c) = \mathbb{R}$ and $\psi(-d, d) = \mathbb{R}$, such that

$$\left|\phi^{-1}(w)\right| \le \phi^{-1}(|w|), \text{ and } \left|\psi^{-1}(w)\right| \le \psi^{-1}(|w|).$$

**(H4)** In the Nagumo-type condition of the Definition 6.1.3, let us consider the modification:

$$\int_{\phi(\mu_1)}^{\phi(<c)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)}ds = +\infty \text{ and } \int_{\psi(\mu_2)}^{\psi(<d)} \frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)}ds = +\infty,$$

therefore $0 < \mu_1 \le N_1 < c$ and $0 < \mu_2 \le N_2 < d$.

In this case, a solution to the problem (6.1)-(6.3) is a pair of functions $(u(t), v(t)) \in X^2$ such that $(u^{(\eta-1)}(t), v^{(\xi-1)}(t)) \in (-c, c) \times (-d, d)$ for all $t \in [a, b]$, satisfying (6.1)-(6.3). The theory for singular $\phi$-Laplacian equations is analogous to Theorems 6.2.1 and Theorems 6.3.2, replacing the assumption (H1) by (Hs) and adding (H4).

**Example 6.5.1.** *Consider the coupled nonlinear system*

$$\begin{cases} (\operatorname{arctanh}(u''(t)))' - 5u'(t) + v(t)(u'(t))^2 = 0, \ t \in [0, \pi] \setminus \{(t_i)\}, \\ \\ (\tan(v'''(t)))' + u(t) - 3v''(t) - \sin(v''(t)) = 0, \ t \in [0, \pi] \setminus \{(\tau_j)\}, \end{cases} \tag{6.37}$$

*with the boundary conditions*

$$\begin{cases} u(0) = -\dfrac{1}{\pi}, \ u'(0) = 0, \ u''(\pi) = 0 \\ \\ v(0) = 0, \ v'(0) = 0, \ v''(0) = 0, \ v'''(\pi) = 1, \end{cases} \tag{6.38}$$

*and impulsive conditions are given by*

$$\begin{cases} \Delta u(t_i) = \sin(\pi t_i), \quad \Delta u'(t_i) = \dfrac{1}{2}\cos(\pi u'(t_i)), \\ \\ \Delta v(\tau_j) = \dfrac{1}{4} v''(\tau_j), \quad \Delta v'(\tau_j) = \cos(\tau_j), \\ \\ \Delta v''(\tau_j) = \dfrac{1}{3\pi} \sin^2(2\pi v'''(\tau_j)), \end{cases} \tag{6.39}$$

*with* $t_1 = 1$, $t_2 = 2$ *and* $\tau_1 = -\dfrac{\pi}{2}$.

The system (6.37)-(6.39) is a particular case of the problem (6.1), (6.2), (6.4), with $[a, b] = [0, \pi]$,

$$\phi(x_2) = \operatorname{arctanh}(x_2), \quad \psi(y_3) = \tan(y_3), \tag{6.40}$$

$$\begin{aligned} f(t, x_0, x_1, x_2, y_0, y_1, y_2) &= -5x_1 + y_0(x_1)^2, \\ g(t, x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3) &= x_0 - 3y_2 - \sin(y_2), \end{aligned}$$

$$A_0 = -\frac{1}{\pi}, \ A_1 = A_2 = B_0 = B_1 = B_2 = 0, \ B_3 = 1,$$

$$\begin{aligned} I_{0,i}(t_1, x_0, x_1, y_0, y_1, y_2) &= \sin(\pi t_i), \\ I_{1,i}(t_i, x_0, x_1, x_2, y_0, y_1, y_2) &= \frac{1}{2}\cos(\pi x_1), \\ J_{0,j}(\tau_j, x_0, x_1, y_0, y_1, y_2) &= \frac{1}{4} y_2, \\ J_{1,j}(\tau_j, x_0, x_1, y_0, y_1, y_2) &= \cos(\tau_j), \\ J_{2,j}(\tau_j, x_0, x_1, y_0, y_1, y_2, y_3) &= \frac{1}{3\pi}\sin^2(2\pi y_3), \end{aligned}$$

and $m = 2$, $n = 1$.

It is easy to see that the functions $\phi : (-1,1) \to \mathbb{R}$ and $\psi : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, given in (6.40), verify assumption (Hs), are increasing homeomorphisms such that $\phi(0) = \psi(0) = 0$, $\phi(-1,1) = \mathbb{R}$ and $\psi(-\frac{\pi}{2}, \frac{\pi}{2}) = \mathbb{R}$.

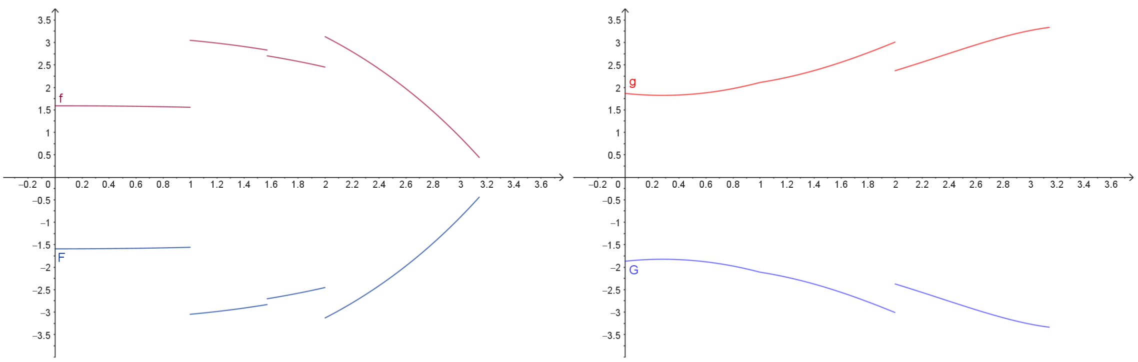
The functions $\alpha_\kappa : [0,\pi] \to \mathbb{R}$, $\kappa = 1,2$, given by

$$\alpha_1(t) = \begin{cases} -\dfrac{1}{\pi}(t+1) & \text{if} \quad 0 \le t \le 1 \\[2ex] -\dfrac{2}{\pi}t & \text{if} \quad 1 < t \le 2 \\[2ex] -\dfrac{3}{\pi}t & \text{if} \quad 2 < t \le \pi \end{cases}, \qquad \alpha_2(t) = \begin{cases} -\dfrac{1}{\pi}\left(\dfrac{t^4}{24} + t^2\right) & \text{if} \quad 0 \le t \le \dfrac{\pi}{2} \\[2ex] -\dfrac{1}{\pi}\left(\dfrac{t^4}{24} + t^2 + 1\right) & \text{if} \quad \dfrac{\pi}{2} < t \le \pi \end{cases},$$

and $\beta_\kappa : [0,\pi] \to \mathbb{R}$, $\kappa = 1,2$, given by

$$\beta_1(x) = \begin{cases} \dfrac{1}{\pi}(t+1) & \text{if} \quad 0 \le t \le 1 \\[2ex] \dfrac{2}{\pi}t & \text{if} \quad 1 < t \le 2 \\[2ex] \dfrac{3}{\pi}t & \text{if} \quad 2 < t \le \pi \end{cases}, \qquad \beta_2(x) = \begin{cases} \dfrac{1}{\pi}\left(\dfrac{t^4}{24} + t^2\right) & \text{if} \quad 0 \le t \le \dfrac{\pi}{2} \\[2ex] \dfrac{1}{\pi}\left(\dfrac{t^4}{24} + t^2 + 1\right) & \text{if} \quad \dfrac{\pi}{2} < t \le \pi \end{cases},$$

are, respectively, lower and upper solutions of the problem (6.37)-(6.39), according to Definition 6.3.1. In fact, the differential inequalities are verified in the interval $[0,\pi]$, as shown in Figure 6.3.




**a** **b**

**Fig. 6.3:** Relationship between nonlinearities as a function of the lower and upper solutions, given by the inequalities of the Definition 6.3.1: a) first equation of the system; b) second equation of the system.

The boundary conditions

$$\begin{aligned}
&\alpha_1(0) = -\frac{1}{\pi}, \quad \alpha_1'(0) = -\frac{1}{\pi} < 0, \quad \alpha_1''(\pi) = 0, \\
&\alpha_2(0) = 0, \quad \alpha_2'(0) = 0, \quad \alpha_2''(0) = -\frac{2}{\pi} < 0, \quad \alpha_2'''(\pi) = -1 < 1, \\
&\beta_1(0) = \frac{1}{\pi} > -\frac{1}{\pi}, \quad \beta_1'(0) = \frac{1}{\pi} > 0, \quad \beta_1''(1) = 0, \\
&\beta_2(0) = 0, \quad \beta_2'(0) = 0, \quad \beta_2''(0) = \frac{1}{\pi} > 0, \quad \beta_2'''(\pi) = 1,
\end{aligned}$$

and impulsive conditions verify the inequalities of Definition 6.3.1, as shown in Table 6.3 and Table 6.4.

**Table 6.3:** Impulse conditions for functions $\alpha_1$ and $\beta_1$.

| $l$ | $i$ | $t_i$ | $\Delta\alpha_1^{(l)}(t_i)$ | $\leq I_{l,i}(\alpha_1)$ | $\leq I_{l,i}(\beta_1)$ | $\leq \Delta\beta_1^{(l)}(t_i)$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 1.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 | 2 | 2.0000 | -0.6366 | 0.0000 | 0.0000 | 0.6366 |
| 1 | 1 | 1.0000 | -0.3183 | 0.2702 | 0.2702 | 0.3183 |
| 1 | 2 | 2.0000 | -0.3183 | -0.2081 | -0.2081 | 0.3183 |

**Table 6.4:** Impulse conditions for functions $\alpha_2$ and $\beta_2$.

| $k$ | $j$ | $\tau_j$ | $\Delta\alpha_2^{(k)}(\tau_j)$ | $\leq J_{k,j}(\alpha_2)$ | $\leq J_{k,j}(\beta_2)$ | $\leq \Delta\beta_2^{(k)}(\tau_j)$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 1.5708 | -0.3183 | -0.2573 | 0.2573 | 0.3183 |
| 1 | 1 | 1.5708 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2 | 1 | 1.5708 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

Let
$$\max\left\{|\alpha_1^{(l)}|, |\alpha_2^{(k)}|, |\beta_1^{(l)}|, |\beta_2^{(k)}|, N_1, N_2,\ l=0,1,\ k=0,1,2\right\} < K,$$
then $f$ and $g$ are $L^1$-Carathéodory functions, with
$$\begin{aligned} |f(t,x_0,x_1,x_2,y_0,y_1,y_2)| &\leq 5K+K^3 := \rho_{1K}(t), \\ |g(t,x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3)| &\leq 4K+1 := \rho_{2K}(t), \end{aligned}$$
and the sum of the jumps is bounded.

The functions $f$ and $g$ satisfy Nagumo's condition relating to sets
$$S_1^1 = \left\{ \begin{array}{c} (t,x_0,x_1,x_2,y_0,y_1,y_2) \in [0,\pi]\times\mathbb{R}^6 : \\ \alpha_1^{(l)}(t) \leq x_l \leq \beta_1^{(l)}(t),\ \alpha_2^{(l)}(t) \leq y_l \leq \beta_2^{(l)}(t),\ l=0,1 \end{array} \right\}$$
and
$$S_2^1 = \left\{ \begin{array}{c} (t,x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3) \in [0,\pi]\times\mathbb{R}^8 : \\ \alpha_1^{(k)}(t) \leq x_k \leq \beta_1^{(k)}(t),\ \alpha_2^{(k)}(t) \leq y_k \leq \beta_2^{(k)}(t),\ k=0,1,2 \end{array} \right\}.$$
Consider a constant $\mathcal{K}_k > 0$, $k=1,2$, and $\mu_k$ as defined in (6.8), then, in $S_1^1$ and $S_2^1$,
$$\begin{aligned} |f(t,x_0,x_1,x_2,y_0,y_1,y_2)| &= \left|-5x_1 + y_0(x_1)^2\right| \\ &\leq \mathcal{K}_1 := \varphi_1(|x_2|) \end{aligned}$$
and
$$\begin{aligned} |g(t,x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3)| &= |x_0 - 3y_2 - \sin(y_2)| \\ &\leq \mathcal{K}_2 := \varphi_2(|y_3|), \end{aligned}$$
it is trivial that
$$\int_{\phi(\mu_1)}^{+\infty} \frac{|\phi^{-1}(s)|}{|\phi^{-1}(s)|+\mathcal{K}_1} ds = +\infty \text{ and } \int_{\psi(\mu_2)}^{+\infty} \frac{|\psi^{-1}(s)|}{|\psi^{-1}(s)|+\mathcal{K}_2} ds = +\infty.$$

So, by Theorem 6.3.2, there is at least one pair of functions $(u(t), v(t)) \in X^2$, solution of the problem (6.37)–(6.39) and, moreover,

$$\begin{aligned} \alpha_1^{(l)}(t) &\leq u^{(l)}(t) \leq \beta_1^{(l)}(t), \quad l = 0, 1, \\ \alpha_2^{(k)}(t) &\leq v^{(k)}(t) \leq \beta_2^{(k)}(t), \quad k = 0, 1, 2, \quad \forall t \in [a, b], \end{aligned}$$

and

$$\|u'''\| \leq N_1 < 1 \quad \text{and} \quad \|v'''\| \leq N_2 < \frac{\pi}{2},$$

with $N_1$ and $N_2$ given by Lemma 6.1.4.

The colored regions in the Figure 6.4 show the locations of the solution $(u(t), v(t))$ of problem (6.37)–(6.39).

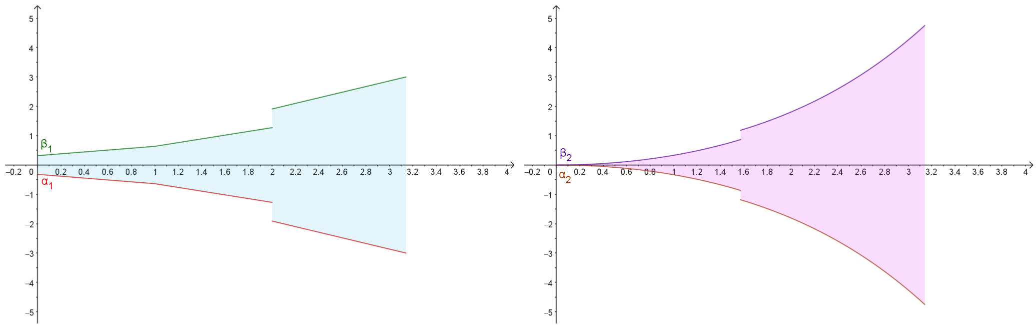



**Fig. 6.4:** At least one solution $(u(t), v(t))$ of problem (6.37)-(6.39) is located in the colored region, when $x \in [0, \pi]$.

CHAPTER 7

# Three-dimensional velocity field related to a generalized third-grade fluid model

Based on the work of Lan et al. (see [59]) the equations relating to the $\phi$-Laplacians problems of one or higher-dimensional are a generalization of the $p$-Laplacians problems. It is known that $p$-Laplacian equations with one or more dimensions arise in the study of Newtonian fluids and non-Newtonian fluids, particularly in shear-thickening (or dilatant) fluids and shear-thinning (or pseudoplastic) fluids. Therefore, this chapter follows the previous chapters and aims to present numerical results to the unsteady volume flow rate, to the unsteady three-dimensional velocity field and including an analysis on perturbed flows, for a new three-dimensional non-Newtonian fluid model, where the viscosity and the viscoelasticity terms dependent on the shear rate.

The following results obtained for the new three-dimensional proposed model for a non-Newtonian fluid with shear-dependent viscoelasticity have been published in a specialist journal in the field of mathematics related with the fluid dynamics, see [21].

Considering the work of Truesdell and Noll [83], we consider the constitutive equation for viscoelastic fluids of differential type (Rivlin-Ericksen fluids) with complexity 3, i.e., fluids of third-grade, given by

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 + \beta_1\boldsymbol{A}_3 + \beta_2(\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1) + \beta_3(tr(\boldsymbol{A}_1^2))\boldsymbol{A}_1 \tag{7.1}$$

where $p$ is the pressure, $-p\boldsymbol{I}$ is the spherical part of the stress due to incompressibility, $\mu$ is the constant viscosity of the fluid, "$tr$" denotes the trace operator and $\alpha_1, \alpha_2$, $\beta_1, \beta_2, \beta_3$ are the normal stress coefficients - also called viscoelastic parameters. The first three Rivlin-Ericksen kinematic tensors $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ are defined through (see Rivlin and Ericksen [87])

$$\boldsymbol{A}_1 = \nabla\boldsymbol{\vartheta} + (\nabla\boldsymbol{\vartheta})^T \tag{7.2}$$

$$\boldsymbol{A}_2 = \frac{d}{dt}(\boldsymbol{A}_1) + \boldsymbol{A}_1\nabla\boldsymbol{\vartheta} + (\nabla\boldsymbol{\vartheta})^T\boldsymbol{A}_1 \tag{7.3}$$

and

$$\boldsymbol{A}_3 = \frac{d}{dt}(\boldsymbol{A}_2) + \boldsymbol{A}_2\nabla\boldsymbol{\vartheta} + (\nabla\boldsymbol{\vartheta})^T\boldsymbol{A}_2 \tag{7.4}$$

where[1] $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}(\boldsymbol{x}, t)$ is the three-dimensional velocity field of the fluid, $\nabla\boldsymbol{\vartheta}$ is the spacial velocity gradient, $(\nabla\boldsymbol{\vartheta})^T$ is the transpose of $\nabla\boldsymbol{\vartheta}$ and $\frac{d}{dt}(\cdot)$ denotes the material time derivative, determined by:

$$\frac{d}{dt}(\,\cdot\,) = \frac{\partial}{\partial t}(\,\cdot\,) + \boldsymbol{\vartheta}\cdot\nabla(\cdot). \tag{7.5}$$

[1]Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be the rectangular space Cartesian coordinates (for convenience we set $x_3 = z$) and $t$ is the time variable.

The thermodynamics and stability of the fluids related with the constitutive equation (7.1) have been studied in detail by Fosdick and Rajagopal [40], who showed that if the fluid is to be compatible with thermodynamics in the sense that all motions of the fluid meet the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid is a minimum in equilibrium, then

$$\mu \geqslant 0,\ \alpha_1 \geqslant 0,\ \mid \alpha_1 + \alpha_2 \mid \leqslant \sqrt{24\mu\beta_3},\ \beta_1 = 0,\ \beta_2 = 0,\ \beta_3 \geqslant 0. \tag{7.6}$$

Now, taking into account the thermodynamically condition (7.6), the constitutive equation (7.1) can be rewritten as

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 + \beta_3(tr(\boldsymbol{A}_1^2))\boldsymbol{A}_1. \tag{7.7}$$

Considering the work of Truesdell and Noll [83] the material parameters $\mu$, $\alpha_1$, $\alpha_2$ and $\beta_3$ may depend on several factors. Then, with this assumption, we will consider in this work that the viscosity ($\mu$) and the viscoelastic terms ($\alpha_1$, $\alpha_2$, $\beta_3$) dependent on the shear rate. Therefore, using this assumption, we propose the following new constitutive equation:

$$\boldsymbol{T} = -p\boldsymbol{I} + \Upsilon(|\dot{\gamma}|)\Big(\mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 + \beta_3(tr(\boldsymbol{A}_1^2))\boldsymbol{A}_1\Big) \tag{7.8}$$

where

$$\Upsilon(|\dot{\gamma}|) : \mathbb{R}^+ \to \mathbb{R}^+ \tag{7.9}$$

is the shear-dependent viscoelastic function, $\dot{\gamma}$ is a scalar measure of the rate of shear defined by $|\dot{\gamma}| = \sqrt{2\boldsymbol{D} : \boldsymbol{D}}$ with

$$\boldsymbol{D} := \frac{1}{2}(\nabla\boldsymbol{\vartheta} + (\nabla\boldsymbol{\vartheta})^T)$$

being the rate of deformation tensor.

Experimental studies with polymers (see Beracea et al. [13]), suspensions (see Mall-Gleissle et al. [64]) and liquid crystals (see Tao et al. [96]) indicate that for several fluids, one observes a significant variation in the viscoelastic effects with the shear rate of the power-law type. Taking into account the above experimental studies, we consider the shear-dependent viscoelastic power-law type function (7.9) at equation (7.8), given by

$$\Upsilon(|\dot{\gamma}|) = |\dot{\gamma}|^{n-1} \tag{7.10}$$

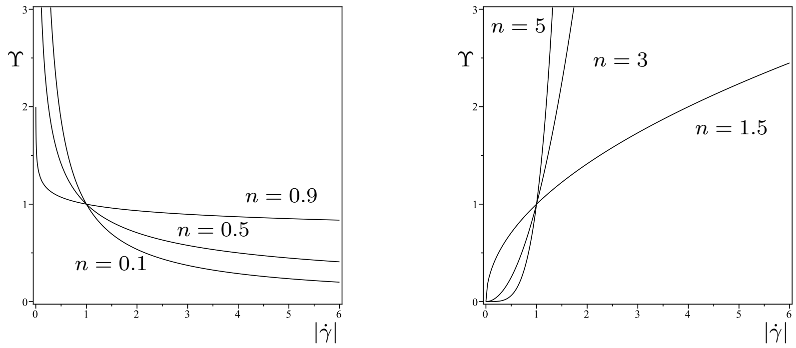
where the positive parameter $n$ is the flow index. If, in condition (7.10), $n < 1$, then

$$\lim_{|\dot{\gamma}|\to+\infty} \Upsilon(|\dot{\gamma}|) = 0, \quad \lim_{|\dot{\gamma}|\to 0} \Upsilon(|\dot{\gamma}|) = +\infty,$$

and we have a shear-thinning (or pseudoplastic) viscoelastic fluid, i.e., viscoelasticity decreases with increasing shear rate, see Figure 7.1(a). For $n > 1$, the result is

$$\lim_{|\dot{\gamma}|\to+\infty} \Upsilon(|\dot{\gamma}|) = +\infty, \quad \lim_{|\dot{\gamma}|\to 0} \Upsilon(|\dot{\gamma}|) = 0,$$

and the viscoelastic fluid is shear-thickening (or dilatant), i.e., viscoelasticity increases with shear rate, see Figure 7.1(b).




(a) Shear-thinning viscoelastic fluid

(b) Shear-thickening viscoelastic fluid

**Fig. 7.1:** Power-Law model: (a) shear-thinning viscoelastic fluid and (b) shear-thickening viscoelastic fluid, for different values of flow index.

Finally, considering $n = 1$ in condition (7.10) applied to equation (7.8), we recover the constitutive equation (7.7) for a third-grade non-Newtonian fluid with stability condition (7.6). Furthermore, considering $\beta_3 = 0$ the equation (7.7) becames the standard constitutive equation for a second-grade non-Newtonian fluid, see Coleman and Noll [29]. Also if $\alpha_1 = \alpha_2 = 0$ and $\beta_3 = 0$ the equation (7.7) becames the constitutive equation for Newtonian fluid.

The condition (7.10) has limited applications to real fluids due to the unboundedness of the viscoelastic asymptotic limits, but is widely used and can be accurate for some specific flow regimes. The present investigation can be relevant in several physical, biological and engineering applications. In particular, the constitutive equation (7.8), with condition (7.10) for $n < 1$ (shear-thinning viscoelastic fluid), may be pertinent in the study of blood flow in small vessels where viscoelastic effects are taken into account because phenomena like aggregation and deformability of red blood cells (see Chien et al. [27]). The case of $n = 1$ at the viscoelastic power-law function (7.10) on equation (7.8) was studied by Carapau and Correia [17] for a straight tube with constant circular cross-section and the case of a straight tube with variable circular cross-section was studied, also by Carapau and Correia [18].

The three-dimensional numerical study of the flow associated with an incompressible fluid that follows from the constitutive equation (7.8) in a circular cross-section tube with variable radius is in fact a challenging and complex study in terms of computational effort and infeasible in many relevant issues. In this sense, we will only study the simplest case, that is, the flow of an incompressible fluid that follows the equation (7.8) in a tube with a circular cross-section and constant radius. To get around the difficulty related to the dimensions of space $(x_1, x_2, x_3)$ and time $t$, we will use the director approach (also called Cosserat Theory) developed by Caulk and Naghdi [26]. This theory (see [42, 26, 43, 44]) includes an additional structure of directors (deformable vectors) assigned to each point on a space curve (Cosserat curve), where a 3D system of equations is replaced by a 1D system depending on time and on a single spatial variable, which is the axis of the symmetrical flow. Our one-dimensional approach is obtained by integrating the linear momentum equation over the cross-section of the tube, taking the velocity field approximation provided by the nine-directors theory. This velocity field approximation satisfies exactly both the incompressibility condition and the kinematic boundary condition. Finally, based on the work of Caulk and Naghdi [26], we consider the three-dimensional velocity field[2]

$$\boldsymbol{\vartheta}(\boldsymbol{x},t) = \vartheta_i(\boldsymbol{x},t)\boldsymbol{e}_i$$

approximated by:

$$\boldsymbol{\vartheta}(\boldsymbol{x},t) = \boldsymbol{v} + \sum_{N=1}^{k} x_{\theta_1}\dots x_{\theta_N}\boldsymbol{W}_{\theta_1\dots\theta_N}, \tag{7.11}$$

with

$$\boldsymbol{v} = v_i(z,t)\,\boldsymbol{e}_i, \quad \boldsymbol{W}_{\theta_1\dots\theta_N} = W^i_{\theta_1\dots\theta_N}(z,t)\,\boldsymbol{e}_i. \tag{7.12}$$

In condition (7.11), $\boldsymbol{v}$ denotes the velocity along the axis of symmetry $z$ at time $t$, $x_{\theta_1}\dots x_{\theta_N}$ are the polynomial weighting functions with order $k$, the vectors $\boldsymbol{W}_{\theta_1\dots\theta_N}$ are the director velocities which are symmetric with respect to their indices and $\boldsymbol{e}_i$ are the associated unit basis vectors. We remark that the number $k$ identifies the order in the hierarchical theory and is related to the number of directors. In applications these director velocities are associated to physical characteristics of the fluid. Considering the velocity field approximation (7.11) with nine-directors (see [26]), i.e., $k = 3$ in (7.11), and using the propose constitutive equation (7.8) with conditions (7.10) we obtain the unsteady equation for mean pressure gradient and wall shear stress both depending on the volume flow rate, Womersley number, viscosity and viscoelastic coefficients over a finite section of a straight, rigid and impermeable tube with constant circular cross-section. Some numerical simulations are provided by using a Runge-Kutta method for unsteady flow regimes, including an analysis on perturbed flows.

**Remark 7.0.1.** *This approach one-dimensional Cosserat theory, related to fluid dynamics, has already been applied for second-grade fluids, i.e., considering $\beta_3 = 0$ into equation* (7.7)*, in different concepts and perspectives, see e.g. [23, 24, 25, 29].*

## 7.1 Equations of motion

Let us consider a homogeneous fluid moving within a circular straight and impermeable tube, the domain $\Omega$ (see Figure 7.2) contained in $\mathbb{R}^3$. Also, let us consider the general scalar function $\phi(z,t)$ (meridian radius), that is related to the circular cross-section of the tube by

$$\phi^2(z,t) = x_1^2 + x_2^2. \tag{7.13}$$

The boundary $\partial\Omega$ consists of the proximal cross-section $\Gamma_1$, the distal cross-section $\Gamma_2$, and the lateral wall of the tube denoted by $\Gamma_w$. Now, let us consider the motion of an incompressible fluid inside a straight tube with circular cross-section, without external body forces. Therefore, the equations of motion stating the conservation of linear momentum

[2] In the sequel, latin indices take the values $1, 2, 3$, greek indices $1, 2$ and we use the convention of summing over repeated indices.

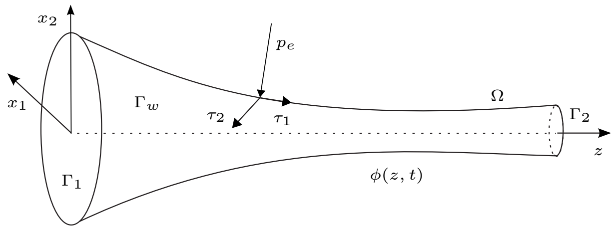



**Fig. 7.2:** General fluid domain $\Omega \subset \mathbb{R}^3$ with normal and tangential components of the surface traction vector $p_e$, $\tau_1$, $\tau_2$, where $\phi(z,t)$ denotes the radius of the domain surface along the axis of symmetry $z$ at time $t$. In particular $\tau_1$ is the wall shear stress.

and mass are given, in $\Omega \times (0,T)$, by

$$\begin{cases} \rho\Big(\dfrac{\partial \boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta}\cdot\nabla\boldsymbol{\vartheta}\Big) = \nabla\cdot\boldsymbol{T}, \\ \nabla\cdot\boldsymbol{\vartheta} = 0, \\ \boldsymbol{T} = -p\boldsymbol{I} + \Upsilon(|\dot\gamma|)\Big(\mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 + \beta_3(tr(\boldsymbol{A}_1^2))\boldsymbol{A}_1\Big), \ \ \boldsymbol{t}_w = \boldsymbol{T}\cdot\boldsymbol{\eta}, \end{cases} \tag{7.14}$$

where $\boldsymbol{\vartheta} = \vartheta_i\boldsymbol{e}_i$ is the three-dimensional velocity field and $\rho$ is the constant fluid density. Equation $(7.14)_1$ represents the balance of linear momentum and $(7.14)_2$ is the incompressibility condition. In equation $(7.14)_3$, $\boldsymbol{T}$ is the constitutive equation proposed in (7.8) with condition (7.10), and $\boldsymbol{t}_w$ denotes the stress vector on the surface whose outward unit normal vector is $\boldsymbol{\eta} = \eta_i\boldsymbol{e}_i$. The components of the outward unit normal vector to the meridian radius $\phi(z,t)$ are given by

$$\eta_1 = \frac{x_1}{\phi\sqrt{1+\phi_z^2}}, \quad \eta_2 = \frac{x_2}{\phi\sqrt{1+\phi_z^2}}, \quad \eta_3 = -\frac{\phi_z}{\sqrt{1+\phi_z^2}}, \tag{7.15}$$

where the subscripted variable denotes partial differentiation. Since equation (7.13) defines a material surface, the three-dimensional velocity field $\boldsymbol{\vartheta}$ must satisfy the kinematic condition (see condition (7.5))

$$\frac{d}{dt}(\phi^2(z,t) - x_1^2 - x_2^2) = 0,$$

i.e.,

$$\phi\phi_t + \phi\phi_z\vartheta_3 - x_1\vartheta_1 - x_2\vartheta_2 = 0, \tag{7.16}$$

on the boundary (7.13).

Aiming to obtain a one-dimensional model by approach Cosserat theory, we present averaged quantities such as volume flow rate and average pressure. Consider $S(z,t)$ as a generic axial cross-section of the tube geometry at time $t$ defined by the spatial variable $z$ and bounded by the circle defined in (7.13) and let $A(z,t)$ be the area of this section $S(z,t)$. Then, the volume flow rate $Q$ is defined by

$$Q(z,t) = \int_{S(z,t)} \vartheta_3(x_1,x_2,z,t)da, \tag{7.17}$$

and the average pressure $\bar p$, by

$$\bar p(z,t) = \frac{1}{A(z,t)}\int_{S(z,t)} p(x_1,x_2,z,t)da. \tag{7.18}$$

Using the hierarchical theory approach (7.11) with $k = 3$, it follows (see Caulk and Naghdi [26]) that the approximation of the three-dimensional velocity field $\boldsymbol{\vartheta}(\boldsymbol{x}, t) = \vartheta_i \boldsymbol{e}_i$, with nine directors, is given by

$$\begin{aligned} \boldsymbol{\vartheta}(\boldsymbol{x}, t) &= \Big[x_1(\xi + \sigma(x_1^2 + x_2^2)) - x_2(\omega + \psi(x_1^2 + x_2^2))\Big]\boldsymbol{e}_1 \\ &+ \Big[x_1(\omega + \psi(x_1^2 + x_2^2)) + x_2(\xi + \sigma(x_1^2 + x_2^2))\Big]\boldsymbol{e}_2 \\ &+ \Big[v_3 + \gamma(x_1^2 + x_2^2)\Big]\boldsymbol{e}_3 \end{aligned} \tag{7.19}$$

where $\xi(z,t), \omega(z,t), \gamma(z,t), \sigma(z,t)$ and $\psi(z,t)$ are scalar functions related to the director velocities (see [26]). The physical significance of these scalar functions in (7.19) is the following: $\gamma$ is related to transverse shearing motion, $\omega$ and $\psi$ are related to rotational motion (also called swirling motion) about $\boldsymbol{e}_3$, while $\xi$ and $\sigma$ are related to transverse elongation.

Considering the velocity equation (7.19), the kinematic condition (7.16) on the lateral boundary reduces to

$$\phi_t + (v_3 + \phi^2\gamma)\phi_z - (\xi + \phi^2\sigma)\phi = 0 \tag{7.20}$$

and the incompressibility condition $(7.14)_2$, is specialized as

$$(v_3)_z + 2\xi + (x_1^2 + x_2^2)(\gamma_z + 4\sigma) = 0. \tag{7.21}$$

For equation (7.21) to hold at every point in the fluid, the velocity coefficients must satisfy both conditions

$$(v_3)_z + 2\xi = 0 \quad \text{and} \quad \gamma_z + 4\sigma = 0. \tag{7.22}$$

Therefore, the kinematic condition (7.16) and the incompressibility condition $(7.14)_2$ are exactly satisfied by the velocity field (7.19), provided conditions (7.20) and (7.22).

Now, let us consider flow in a rigid walled tube, i.e.,

$$\phi = \phi(z). \tag{7.23}$$

On the boundary of a rigid straight tube, we impose a no-slip condition and require that the assumed velocity (7.19) vanishes identically on the surface (7.13). It follows that

$$\xi + \phi^2\sigma = 0, \quad \omega + \phi^2\psi = 0, \quad v_3 + \phi^2\gamma = 0. \tag{7.24}$$

Moreover, taking into account (7.23), equation (7.20) is satisfied identically and the two independent incompressibility conditions (7.22) reduce to

$$(v_3)_z + 2\xi = 0, \quad (\phi^2 v_3)_z = 0. \tag{7.25}$$

Conditions (7.17), (7.19), $(7.24)_3$ and $(7.25)_2$ imply that the volume flow rate $Q$ is function of time $t$ only, given by

$$Q(t) = \frac{\pi}{2}\,\phi^2(z)v_3(z,t). \tag{7.26}$$

As an alternative to the pointwise satisfying of the momentum equation $(7.14)_1$, we impose the following integral conditions

$$\int_{S(z,t)} \Big[\nabla\cdot\boldsymbol{T} - \rho\Big(\frac{\partial\boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta}\cdot\nabla\boldsymbol{\vartheta}\Big)\Big]da = 0, \tag{7.27}$$

$$\int_{S(z,t)} \Big[\nabla \cdot \boldsymbol{T} - \rho\Big(\frac{\partial \boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta} \cdot \nabla \boldsymbol{\vartheta}\Big)\Big] x_{\theta_1} \dots x_{\theta_N} da = 0, \tag{7.28}$$

where $N = 1, 2, 3$. Using the divergence theorem and integration by parts, equations (7.27) – (7.28) for nine directors, can be reduced to four vector equations:

$$\frac{\partial \boldsymbol{n}}{\partial z} + \boldsymbol{f} = \boldsymbol{a}, \tag{7.29}$$

$$\frac{\partial \boldsymbol{m}^{\theta_1 \dots \theta_N}}{\partial z} + \boldsymbol{l}^{\theta_1 \dots \theta_N} = \boldsymbol{h}^{\theta_1 \dots \theta_N} + \boldsymbol{b}^{\theta_1 \dots \theta_N}, \tag{7.30}$$

where $\boldsymbol{n}$, $\boldsymbol{h}^{\theta_1 \dots \theta_N}$, $\boldsymbol{m}^{\theta_1 \dots \theta_N}$ are resultant forces defined by

$$\boldsymbol{n} = \int_S \boldsymbol{T}_3 da, \quad \boldsymbol{h}^\alpha = \int_S \boldsymbol{T}_\alpha da, \tag{7.31}$$

$$\boldsymbol{h}^{\alpha\beta} = \int_S \Big(\boldsymbol{T}_\alpha x_\beta + \boldsymbol{T}_\beta x_\alpha\Big) da, \tag{7.32}$$

$$\boldsymbol{h}^{\alpha\beta\gamma} = \int_S \Big(\boldsymbol{T}_\alpha x_\beta x_\gamma + \boldsymbol{T}_\beta x_\alpha x_\gamma + \boldsymbol{T}_\gamma x_\alpha x_\beta\Big) da, \tag{7.33}$$

and

$$\boldsymbol{m}^{\theta_1 \dots \theta_N} = \int_S \boldsymbol{T}_3 x_{\theta_1} \dots x_{\theta_N} da. \tag{7.34}$$

The quantities $\boldsymbol{a}$ and $\boldsymbol{b}^{\theta_1 \dots \theta_N}$ are inertia terms defined by

$$\boldsymbol{a} = \int_S \rho\Big(\frac{\partial \boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta} \cdot \nabla \boldsymbol{\vartheta}\Big) da, \tag{7.35}$$

$$\boldsymbol{b}^{\theta_1 \dots \theta_N} = \int_S \rho\Big(\frac{\partial \boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta} \cdot \nabla \boldsymbol{\vartheta}\Big) x_{\theta_1} \dots x_{\theta_N} da, \tag{7.36}$$

and $\boldsymbol{f}$, $\boldsymbol{l}^{\theta_1 \dots \theta_N}$, which arise due to surface traction on the lateral boundary, are defined by

$$\boldsymbol{f} = \int_{\partial S} \sqrt{1 + \phi_z^2}\, \boldsymbol{t}_w ds, \tag{7.37}$$

$$\boldsymbol{l}^{\theta_1 \dots \theta_N} = \int_{\partial S} \sqrt{1 + \phi_z^2}\, \boldsymbol{t}_w x_{\theta_1} \dots x_{\theta_N} ds. \tag{7.38}$$

Also, from Caulk and Naghdi [26] the stress vector $\boldsymbol{t}_w$ on the lateral surface $\Gamma_w$ (see $(7.14)_3$) in terms of its outward unit normal vector $\boldsymbol{\eta}$ and normal and tangential components $p_e, \tau_1, \tau_2$ (see Figure 7.1), is given by

$$\begin{aligned}
\boldsymbol{t}_w &= \Big[\frac{1}{\phi(1+\phi_z^2)^{1/2}}\Big(\tau_1 x_1 \phi_z - p_e x_1 - \tau_2 x_2 (1+\phi_z^2)^{1/2}\Big)\Big]\boldsymbol{e}_1 \\
&+ \Big[\frac{1}{\phi(1+\phi_z^2)^{1/2}}\Big(\tau_1 x_2 \phi_z - p_e x_2 + \tau_2 x_1 (1+\phi_z^2)^{1/2}\Big)\Big]\boldsymbol{e}_2 \\
&+ \Big[\frac{1}{(1+\phi_z^2)^{1/2}}\Big(\tau_1 + p_e \phi_z\Big)\Big]\boldsymbol{e}_3,
\end{aligned} \tag{7.39}$$

where $\tau_1$ is the scalar function for the wall shear stress. In equations (7.37) – (7.38), we will apply the stress vector $\boldsymbol{t}_w$ given by (7.39). Considering a flow in a rigid tube (7.23) without rotation (i.e., $\psi = \omega = 0$) with volume flow rate (7.26), conditions $(7.24)_{1,3}$ and $(7.25)_1$, then the three-dimensional velocity field (7.19) becomes

$$\begin{aligned}\boldsymbol{\vartheta}(\boldsymbol{x},t) &= \Big[x_1\Big(1-\frac{x_1^2+x_2^2}{\phi^2}\Big)\frac{2\phi_z Q(t)}{\pi\phi^3}\Big]\boldsymbol{e}_1+\Big[x_2\Big(1-\frac{x_1^2+x_2^2}{\phi^2}\Big)\frac{2\phi_z Q(t)}{\pi\phi^3}\Big]\boldsymbol{e}_2\\ &+ \Big[\frac{2Q(t)}{\pi\phi^2}\Big(1-\frac{x_1^2+x_2^2}{\phi^2}\Big)\Big]\boldsymbol{e}_3. \end{aligned} \tag{7.40}$$

**Remark 7.1.1.** *This approach one-dimensional Cosserat theory has been validated in the works [26, 88, 22, 16, 23] and in this sense it is relevant to the study of physical problems involving the flow of Newtonian and non-Newtonian fluids under different geometries and perspectives, being a valid alternative to the classics one-dimensional models.*

## 7.2 One-dimensional results

Due to the complexity of the model under study, we will only present results for the simplest case. Therefore, we consider the specific case of a straight rigid tube with uniform circular cross-section, i.e., $\phi$ is constant. Consequently, the three-dimensional velocity field (7.40) can be written as

$$\boldsymbol{\vartheta}(\boldsymbol{x},t) = \Big[\frac{2Q(t)}{\pi\phi^2}\Big(1-\frac{x_1^2+x_2^2}{\phi^2}\Big)\Big]\boldsymbol{e}_3 \tag{7.41}$$

and the stress vector (7.39) by

$$\boldsymbol{t}_w = \frac{1}{\phi}\Big(-p_e x_1-\tau_2 x_2\Big)\boldsymbol{e}_1+\frac{1}{\phi}\Big(-p_e x_2+\tau_2 x_1\Big)\boldsymbol{e}_2+\tau_1\boldsymbol{e}_3. \tag{7.42}$$

Now, under the conditions of the system (7.14) and replacing the velocity field (7.41) and the stress vector (7.42) in equations (7.31) – (7.38), we can calculate explicitly the forces $\boldsymbol{n}$, $\boldsymbol{k}^{\alpha_1\dots\alpha_n}$, $\boldsymbol{m}^{\alpha_1\dots\alpha_n}$, the inertia terms $\boldsymbol{a}$, $\boldsymbol{b}^{\alpha_1\dots\alpha_N}$ and the surface traction $\boldsymbol{f}$, $\boldsymbol{l}^{\alpha_1\dots\alpha_N}$ which arise from nine-directors theory. Hence, plugging the solutions of equations (7.31) – (7.38) into equations (7.29) – (7.30) and using (7.18), we get the equation for the average pressure

$$\begin{aligned}\bar{p}_z(z,t) &= -\frac{8\mu(32)^{n/2-1/2}}{\pi^n\phi^{3n+1}}Q^n(t)-\frac{512(32)^{n/2-1/2}\beta_3}{3\pi^{n+2}\phi^{3n+7}}Q^{n+2}(t)\\ &- \frac{4\rho}{3\phi^2\pi}\Big(1+\frac{6\alpha_1(32)^{n/2-1/2}}{\rho\pi^{n-1}\phi^{3n-1}}Q^{n-1}(t)\Big)Q_t(t)\end{aligned} \tag{7.43}$$

and, the equation to the wall shear stress

$$\begin{aligned}\tau_1(z,t) &= \frac{4\mu(32)^{n/2-1/2}}{\pi^n\phi^{3n}}Q^n(t)+\frac{256(32)^{n/2-1/2}\beta_3}{\pi^{n+2}\phi^{3n+6}}Q^{n+2}(t)\\ &+ \frac{\rho}{6\pi\phi}\Big(1+\frac{24\alpha_1(32)^{n/2-1/2}}{\rho\pi^{n-1}\phi^{3n-1}}Q^{n-1}(t)\Big)Q_t(t).\end{aligned} \tag{7.44}$$

Integrating the condition (7.43) over a finite section of the tube with $z_1 < z_2$, we obtain the mean pressure gradient, given by

$$\begin{aligned} G(t) &= \frac{\bar{p}(z_1,t)-\bar{p}(z_2,t)}{z_2-z_1} \\ &= \frac{8\mu(32)^{n/2-1/2}}{\pi^n\phi^{3n+1}}Q^n(t)+\frac{512(32)^{n/2-1/2}\beta_3}{3\pi^{n+2}\phi^{3n+7}}Q^{n+2}(t) \\ &+ \frac{4\rho}{3\phi^2\pi}\Big(1+\frac{6\alpha_1(32)^{n/2-1/2}}{\rho\pi^{n-1}\phi^{3n-1}}Q^{n-1}(t)\Big)Q_t(t). \end{aligned} \tag{7.45}$$

Considering the following dimensionless variables (here $\omega_0$ is the characteristic frequency for unsteady flow)

$$\hat{t}=\omega_0 t,\ \ \hat{Q}=\frac{2\rho}{\pi\phi\mu}Q,\ \ \hat{\beta}_3=\frac{\mu}{\phi^4\rho^2}\beta_3,\ \ \hat{\alpha}_1=\frac{\mu^{n-1}(32)^{n/2-1/2}}{\phi^{2n}2^{n-1}\rho^n}\alpha_1$$

$$\hat{G}=\frac{\rho^n\phi^{2n+1}}{(32)^{n/2-1/2}\mu^{n+1}}G,\ \ \hat{\tau}_1=\frac{\rho^n\phi^{2n}}{(32)^{n/2-1/2}\mu^{n+1}}\tau_1$$

and, substituting this new variables into equations (7.45) and (7.44), we obtain the nondimensional mean pressure gradient

$$\hat{G}(\hat{t})=\frac{8}{2^n}\hat{Q}^n(\hat{t})+\frac{512}{3}\frac{1}{2^{n+2}}\hat{\beta}_3\hat{Q}^{n+2}(\hat{t})+\frac{2}{3}\mathcal{W}_0^2\Big(1+6\hat{\alpha}_1\hat{Q}^{n-1}(\hat{t})\Big)\hat{Q}_{\hat{t}}(\hat{t}) \tag{7.46}$$

and, the nondimensional wall shear stress

$$\hat{\tau}_1(\hat{z},\hat{t})=\frac{4}{2^n}\hat{Q}^n(\hat{t})+\frac{256}{2^{n+2}}\frac{\hat{\beta}_3}{3}\hat{Q}^{n+2}(\hat{t})+\frac{1}{12}\mathcal{W}_0^2\Big(1+24\hat{\alpha}_1\hat{Q}^{n-1}(\hat{t})\Big)\hat{Q}_{\hat{t}}(\hat{t}) \tag{7.47}$$

where $\hat{\alpha}_1$, $\hat{\beta}_3$ are viscoelastic coefficients and $\mathcal{W}_0$ is the Womersley number, given by

$$\mathcal{W}_0=\phi^n\sqrt{\frac{\omega_o\rho^n}{(32)^{n/2-1/2}\mu^n}},$$

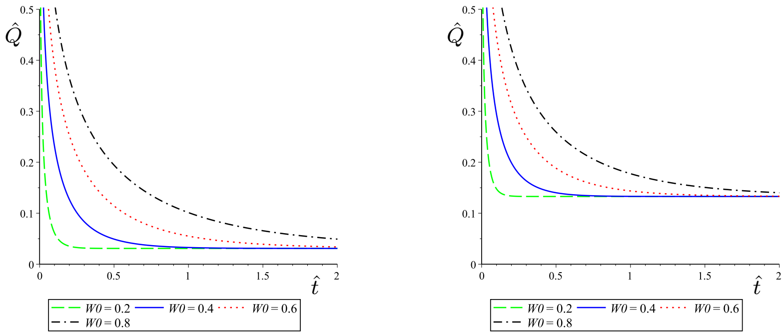
this nondimensional number is related with the pulsatility of the flow. Finally, using the dimensionless variables

$$\hat{x}_1=\frac{x_1}{\phi},\ \hat{x}_2=\frac{x_2}{\phi},\ \hat{\boldsymbol{\vartheta}}=\frac{\phi\rho}{\mu}\boldsymbol{\vartheta}$$

at the equation (7.41), we obtain the nondimensional three-dimensional velocity field

$$\hat{\boldsymbol{\vartheta}}(\hat{\boldsymbol{x}},\hat{t})=\hat{Q}(\hat{t})\Big(1-(\hat{x}_1^2+\hat{x}_2^2)\Big)\boldsymbol{e}_3. \tag{7.48}$$

Considering the flow index $n=1$ at equations (7.46) and (7.47) we recovered the results obtained by Carapau and Correia [17]. Next, we provide numerical simulations by using a Runge-Kutta method for specific unsteady flow regimes relating with shear-thinning viscoelastic and shear-thickening viscoelastic fluids.




(a) Flow index $n = 0.5$ (b) Flow index $n = 0.8$

**Fig. 7.3:** Unsteady volume flow rate (7.46) with constant men pressure gradient $\hat{G}(\hat{t}) = 1$, where $\hat{Q}(0) = 1$, $\hat{\alpha}_1 = 1$, $\hat{\beta}_3 = 1$ and $\mathcal{W}_0 = (0.2, 0.4, 0.6, 0.8)$ for shear-thinning viscoelastic fluid.

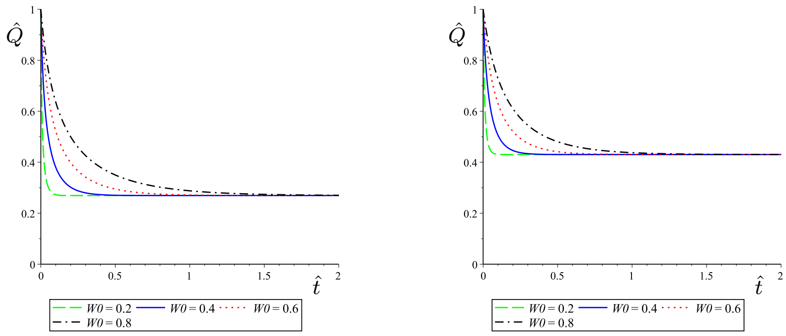
## 7.3 Numerical simulations

For specific flow regimes, we could obtain numerical results for both equation (7.46) and (7.47), but as we intend to present results for the three-dimensional velocity (7.48), we will only focus on equation (7.46). Therefore, considering the equation (7.46) with constant and non-constant mean pressure gradient $\hat{G}$ for specific flow regimes then, we will obtain the solution for the unsteady volume flow rate $\hat{Q}$ by using a Runge-Kutta method and with this volume flow rate data, we can present the three-dimensional velocity (7.48) in the circular cross-section of the tube in the cases of shear-thinning and shear-thickening viscoelastic fluids.

### 7.3.1 Flow under constant mean pressure gradient

Considering $\hat{G} = 1$ at equation (7.46), with $\hat{Q}(0) = 1$, $\hat{\alpha}_1 = 1$ and $\hat{\beta}_3 = 1$, we obtain using a Runge-Kutta method the behavior of the unsteady solution of the volume flow rate $\hat{Q}$ for various values of the Womersley number and power index $n$. The Figure 7.3 is related with shear-thinning viscoelastic fluid and, the Figure 7.4 is related with shear-thickening viscoelastic fluid. The Figure 7.3 and Figure 7.4 show as the behavior of the unsteady volume flow rate given by equation (7.46) considering shear-thinning and shear-thickening viscoelastic fluids, respectively. In this case, after the transient phase the system converges toward a steady state solution. This steady state volume flow rate is obtained solving the time dependent problem but, if we are not interested in the behavior during the initial transient phase, the steady (asymptotic) value of the volume flow rate can be obtained directly from (7.46) setting $\hat{Q}_{\hat{t}}(\hat{t}) = 0$, since at constant mean pressure gradient $\hat{Q}_{\hat{t}}(\hat{t})$ converges to zero as $t \to +\infty$.

Finally, using the unsteady volume flow rate solution obtained by (7.46) with constant mean pressure gradient $\hat{G}(\hat{t}) = 1$, we illustrate in Figure 7.5 and Figure 7.6 the behavior of the three-dimensional velocity field (7.48) on the tube cross-section in time for specific parameters, in both situations of shear-thinning and shear-thickening viscoelastic fluids.




(a) Flow index $n = 1.2$ (b) Flow index $n = 1.8$

**Fig. 7.4:** Unsteady volume flow rate (7.46) with constant men pressure gradient $\hat{G}(\hat{t}) = 1$, where $\hat{Q}(0) = 1$, $\hat{\alpha}_1 = 1$, $\hat{\beta}_3 = 1$ and $\mathcal{W}_0 = (0.2, 0.4, 0.6, 0.8)$ for shear-thickening viscoelastic fluid.

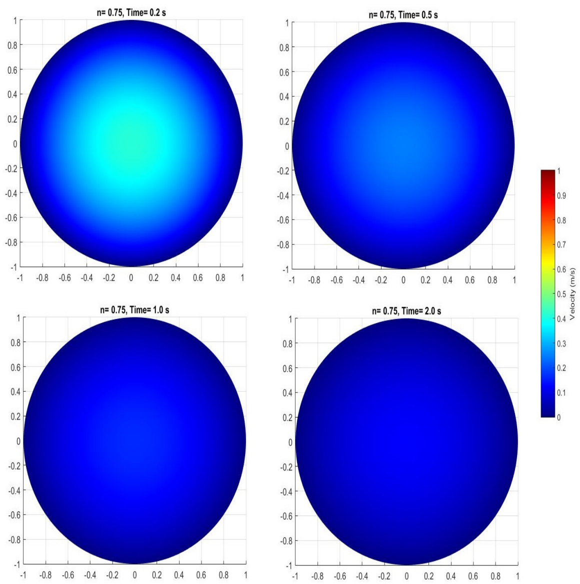



**Fig. 7.5:** Three-dimensional velocity field (7.48) at the circular constant cross-section of the tube, where the volume flow rate is obtained by (7.46) with constant mean pressure gradient $\hat{G} = 1$ and, $\hat{Q}(0) = 1$, $\mathcal{W}_0 = 0.8$, $\hat{\alpha}_1 = 1$, $\hat{\beta}_3 = 1$ to shear-thinning fluid with $n = 0.75$. Time parameters: $\hat{t} = 0.2$, $\hat{t} = 0.5$, $\hat{t} = 1$, $\hat{t} = 2$.

### 7.3.2 Flow under non-constant mean pressure gradient

Now, let us considering a non-constant mean pressure gradient, given by

$$\hat{G}(\hat{t}) = 1 + \frac{\sin^2(\hat{t}^2)}{e^{\hat{t}}}, \tag{7.49}$$

which shows an interesting behavior after the initial transient phase. Specifically, it shows a strong variation in the initial stage followed by smaller fluctuations, which tend to decrease with time, see Figure 7.7.

In Figure 7.8 and Figure 7.9, we can observe the behavior of the unsteady volume

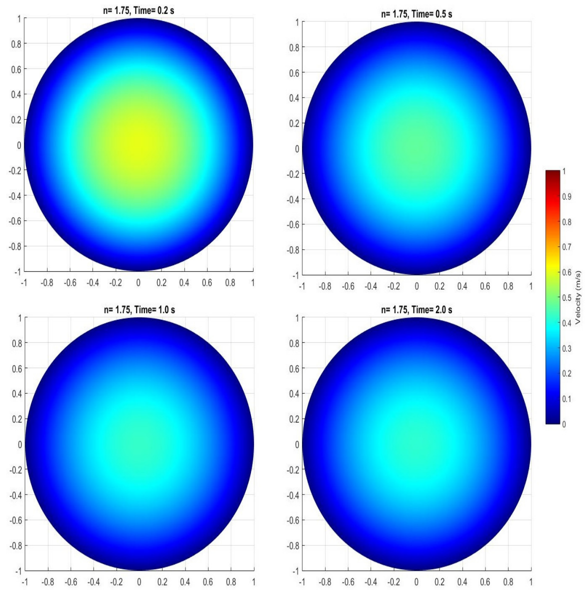



**Fig. 7.6:** Three-dimensional velocity field (7.48) at the circular constant cross-section of the tube, where the volume flow rate is obtained by (7.46) with constant mean pressure gradient $\hat{G} = 1$ and, $\hat{Q}(0) = 1$, $\mathcal{W}_0 = 0.8$, $\hat{\alpha}_1 = 1$, $\hat{\beta}_3 = 1$ to shear-thickening fluid with $n = 1.75$. Time parameters: $\hat{t} = 0.2$, $\hat{t} = 0.5$, $\hat{t} = 1$, $\hat{t} = 2$.

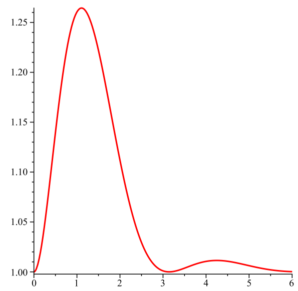
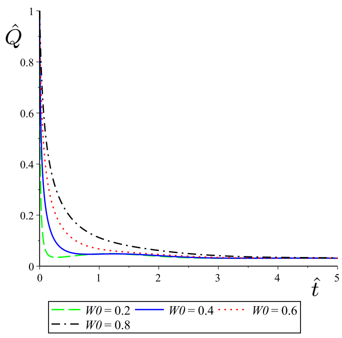
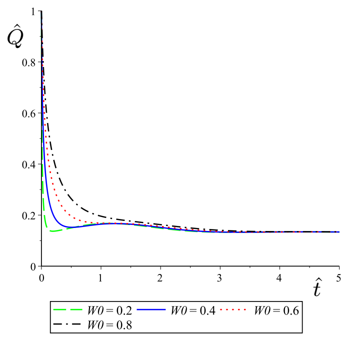


**Fig. 7.7:** Mean pressure gradient dependent on time given by (7.49).



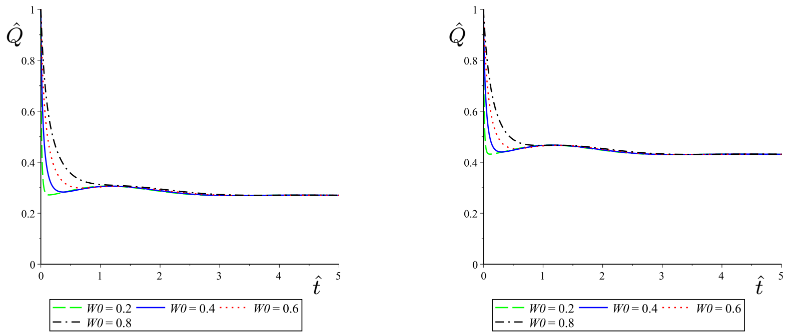

(a) Flow index $n = 0.5$




(b) Flow index $n = 0.8$

**Fig. 7.8:** Unsteady volume flow rate (7.46) with non-constant men pressure gradient (7.49), where $\hat{Q}(0) = 1$, $\hat{\alpha}_1 = 1$, $\hat{\beta}_3 = 1$ and $\mathcal{W}_0 = (0.2, 0.4, 0.6, 0.8)$ for shear-thinning viscoelastic fluid.

flow rate $\hat{Q}$ with non-constant mean pressure gradient (7.49). Again we use a Runge-Kutta method for solving the ordinary differential equation (7.46) for specific flow regimes,



(a) Flow index $n = 1.2$ (b) Flow index $n = 1.8$

**Fig. 7.9:** Unsteady volume flow rate (7.46) with non-constant men pressure gradient (7.49), where $\hat{Q}(0) = 1$, $\hat{\alpha}_1 = 1$, $\hat{\beta}_3 = 1$ and $\mathcal{W}_0 = (0.2, 0.4, 0.6, 0.8)$ for shear-thickening viscoelastic fluid.

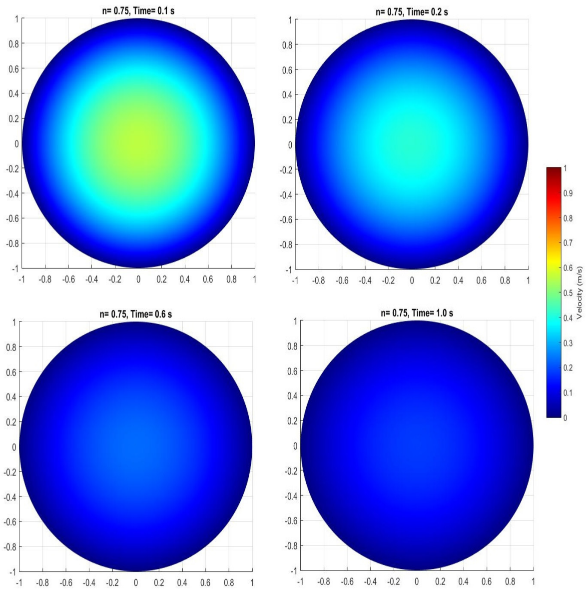


**Fig. 7.10:** Three-dimensional velocity field (7.48) at the circular constant cross-section of the tube, where the volume flow rate is obtained by (7.46) with non-constant mean pressure gradient (7.49) and, $\hat{Q}(0) = 1$, $\mathcal{W}_0 = 0.8$, $\hat{\alpha}_1 = 1$, $\hat{\beta}_3 = 1$ to shear-thinning fluid with $n = 0.75$. Time parameters: $\hat{t} = 0.1$, $\hat{t} = 0.2$, $\hat{t} = 0.6$, $\hat{t} = 1$.

considering shear-thinning and shear-thickening viscoelastic fluids, respectively. Like for constant mean pressure gradient, after the transient phase the system converges toward a steady state solution.

Finally, considering the data solution for the unsteady volume flow rate $\hat{Q}$ on equation (7.48), we illustrate the behavior of the three-dimensional velocity field (7.48) at the constant circular cross-section of the tube, in both situations of shear-thinning and shear-thickening viscoelastic fluids, see Figure 7.10 and Figure 7.11, respectively.

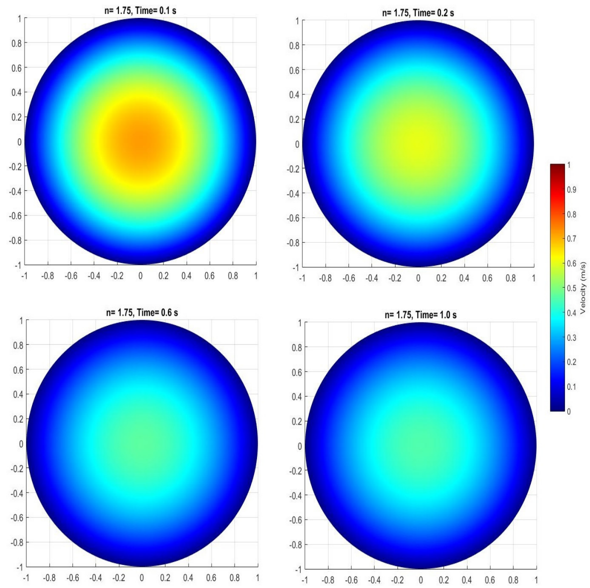



**Fig. 7.11:** Three-dimensional velocity field (7.48) at the circular constant cross-section of the tube, where the volume flow rate is obtained by (7.46) with non-constant mean pressure gradient (7.49) and, $\hat{Q}(0) = 1$, $\mathcal{W}_0 = 0.8$, $\hat{\alpha}_1 = 1$, $\hat{\beta}_3 = 1$ to shear-thickening fluid with $n = 1.75$. Time parameters: $\hat{t} = 0.1$, $\hat{t} = 0.2$, $\hat{t} = 0.6$, $\hat{t} = 1$.

## 7.4 Perburbed flows

When we consider the new constitutive equation (7.8), we lose the guarantee of stability for the solution mentioned in condition (7.6). In this sense, here we intend to take a first approach to the study of the stability of the solution for unsteady volume flow rate $\hat{Q}$, obtained by the proposed model (7.8) with condition (7.10). In this sense, we only study, for specific cases, the stability of the solution for the volume flow rate with $\hat{Q}(0) = 1$ here parameters $\mathcal{W}_0 = 0.8$, $\hat{\alpha}_1 = 1$ and $\hat{\beta}_3 = 1$ are fixed, in both situations of shear-thinning and shear-thickening viscoelastic fluids. Therefore, let us consider the perturbation of magnitude $\epsilon > 0$, given by

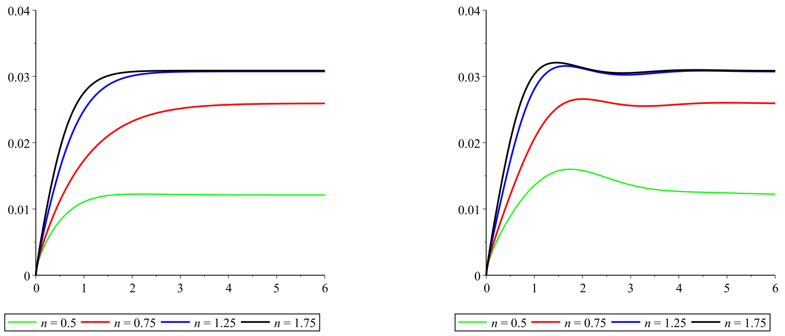
$$\hat{G}_\epsilon^\pm = (1 \pm \epsilon)\hat{G} \tag{7.50}$$

and, denote by $\hat{Q}_\epsilon^\pm$ the perturbed volume flow rate related with the perturbation $\hat{G}_\epsilon^\pm$. Considering constant (or non-constant) mean pressure gradient, for sufficiently large $\hat{t}$ (after the transient period), the unsteady volume flow rate converge to the steady solution (see Section 7.3). Therefore, assuming $\hat{Q}_{\hat{t}}(\hat{t}) = 0$ in equation (7.46) is not possible calculate explicitly the exact expression to the perturbed volume flow rate. However, we can compute the time evolution of the perturbation by the amplitude

$$|\hat{Q}_\epsilon^+ - \hat{Q}_\epsilon^-| \tag{7.51}$$

for specific magnitude $\epsilon > 0$.

Consequently, Figure 7.12 show us the perturbation of the unsteady volume flow rate given by (7.51) with magnitude $\epsilon = 0.1$ for some power index values. Therefore, for the data associated with the solutions, we provide a first step towards stability analysis of the model and, we can conclude that after the transition phase, the unsteady volume flow rate




(a) Contant mean pressure gradient $\hat{G}(\hat{t}) = 1$

(b) Non-contant mean pressure gradient (7.49)

**Fig. 7.12:** Time evolution of perturbation (7.51) with magnitude $\epsilon = 0.1$ in both situations of shear-thinning and shear-thickening viscoelastic fluids with constant and non-constant mean pressure gradient.

behavior is stable in both situations of shear-thinning and shear-thickening viscoelastic fluids, for constant and non-constant mean pressure gradient (7.49).

Chapter 8

# Conclusion

The Ambrosetti-Prodi differential equations are considered, for the first time, in coupled systems with different parameters. The main result allows evaluating some parameter values for which there are lower and upper solutions and, therefore, shows that, for these values, there is also at least one solution, and this solution is located in a range delimited by lower and upper solutions. The existence and non-existence of solutions are obtained for coupled systems, i.e., for cases in which there are strong relationships between the two unknown functions. However, the existence of multiple solutions was predicted only for independent systems, i.e., without the interaction of both variables.

We emphasize that the assumptions are based only on local monotonous assumptions about the nonlinearities and the existence of a kind of bifurcation values for the parameters.

We highlight two aspects in this work:

- A new type of definition of lower and upper solutions for coupled systems (see Definition 2.1.1), which overcomes the non-linear dependence of the two unknown functions and will be crucial for future work;
- In the presence of such lower and upper solutions, the parameters always belong to bounded sets, as illustrated in Example 2.2.2 and in the application.

The method of lower and upper solutions appears to be a powerful tool for dealing with these Ambrosetti-Prodi type problems, as it allows not only some estimates on both parameters for which there are solutions of the coupled systems but also because it provides ranges for the location of such solutions. Indeed, this localization tool, in our opinion, has been undervalued, as it is perhaps very useful to know some qualitative properties of unknown functions.

Several questions still remain open, justifying future work. For example:

- In coupled systems, what are the assumptions to allow nonlinearities to depend on the first derivatives of both variables?
- How to obtain the multiplicity result for the case of the coupled system?

In the chapters dealing with fully-coupled systems of impulsive differential equations, fully discontinuous nonlinearities, and generalized impulse effects are considered simultaneously. This work mainly shows that local monotonies in nonlinearities and impulsive functions are sufficient conditions for the solubility of an impulsive coupled system, with two differential equations involving different Laplacians, totally discontinuous nonlinearities, and two-point boundary conditions.

Following the natural flow, we generalize the results obtained in the previous chapters to a coupled impulsive system of the highest order, with the possibility of equations of different order, with fully differential equations including different regular and singular Laplacians and generalized impulsive effects, dependent on variables and of some derivatives.

The location information provided by the lower and upper solutions was underutilized to obtain qualitative data about the solutions, such as growth type, sign, and estimation of the unknown function and its derivatives, as illustrated in the examples in this work.

The arguments and techniques to be used to obtain the localization part for coupled systems with jumps in the Laplacians remain open.

Relating to the fluids chapter, we present a new three-dimensional model for an incompressible fluid where the parameters related to viscosity and viscoelasticity depend on the shear rate, by a power-law function. This model is relevant for studying the behavior of various non-Newtonian fluids, in particular for blood flow in small vessels. Using an alternative approximation theory, i.e., the Cosserat theory, it was possible for certain specific flow regimes data to present the behavior of the unsteady volume flow rate and, consequently, the behavior of the unsteady three-dimensional velocity field. Also, we present a preliminary study regarding the perturbation of the unsteady volume flow rate solution, this study should be considered as a preliminary study for the solution stability. Finally, for future work, we intend to use this Cosserat theory related to fluid dynamics, to investigate open problems for non-Newtonian fluids, namely, situations associated with curved tubes and fluid-structure interaction.

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