EXPANSION PROPERTIES OF DOUBLE STANDARD MAPS

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Abstract. For the family of Double Standard Maps $f_{a,b} = 2x + a + \frac{b}{\pi}\sin 2\pi x$ (mod 1) we investigate the structure of the space of parameters a when $b = 1$ and when $b \in [0, 1)$. In the first case the maps have a critical point, but for a set of parameters E_1 of positive Lebesgue measure there is an invariant absolutely continuous measure for $f_{a,1}$. In the second case there is an open nonempty set E_b of parameters for which the map $f_{a,b}$ is expanding. We show that as $b \nearrow 1$, the set E_b accumulates on many points of E_1 in a regular way from the measure point of view.

1. Introduction

In one-dimensional dynamics, a lot is known about the families of smooth maps with a critical point, like quadratic maps, and about the maps that have no critical points (local diffeomorphisms of the circle). Here we start to investigate what happens at the interface of those two cases.

Consider the family of *double standard maps* of the circle onto itself, given by

$$f_{a,b} = 2x + a + \frac{b}{\pi}\sin 2\pi x \qquad (\text{mod } 1), \tag{1.1}$$

where the parameters a, b are real, $a \in [0, 1)$ and $b \in [0, 1]$. In fact, we consider a from the circle $\mathbb{R}/\mathbb{Z}$, but since we are mostly working locally (and far from $a = 0$), considering a real is simpler. These family of maps were introduced in [19].

For $b = 1$, maps of the family (1.1) have a unique cubic critical point (at $c = 1/2$) and negative Schwarzian derivative. Thus, they behave similarly to the quadratic maps. In particular, there is a set of parameters a for which there is an invariant probability measure, absolutely continuous with respect to the Lebesgue measure. For $b < 1$, there is no critical point, so the the maps are local diffeomorphisms. Complexification of the maps, obtained by conjugacy via $e^{2\pi ix}$, gives the family

$$g_{a,b}(z) = e^{2\pi ia}z^2e^{b\left(z-\frac{1}{z}\right)}.$$

Those maps are symmetric with respect to the unit circle, and factored by this symmetry, they have only one critical point and no asymptotic values in $\mathbb{C}\setminus\{0\}$. Therefore a map $f_{a,b}$ has at most one attracting or neutral periodic orbit (see [19, 8, 10]).

One can also look at the family of double standard maps as a hybrid between the family of *standard maps*, studied by V. Arnold (see [1]) and important in the creation

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of the KAM theory, and *expanding maps* of the circle (see [22]). Of course instead of maps of degree 2 one can take maps of higher degrees and the results will be practically the same (but we would introduce one more parameter and loose a nice name of the family).

Some recent work has been done for classes of families that include double standard maps. Misiurewicz and Rodrigues studied them in [19, 20]. Benedicks and Rodrigues [4] investigated symbolic dynamics for this family. Universality for critical circle covers was studied by Levin and Świątek, [15]. Levin and van Strien [14] proved complex bounds, quasisymmetric rigidity and density of hyperbolicity for a class of real analytic maps which includes the double standard maps. Fagella and Garijo [10] studied a class of complex maps containing the maps $g_{a,b}$. Dezotti [8] also considered maps $g_{a,b}$, and using complex methods obtained important results on the real case.

As for the Arnold's family, for the double standard family we call the sets for which there is an attracting periodic orbit of a given type (period plus combinatorics) *tongues*. Dezotti [8] proved that tongues are connected. The lowest tongue tip is at $b = 1/2$, for the period 1 tongue. If $0 < b < 1/2$, the map $f_{a,b}$ is expanding. At higher b-levels there may be finitely or infinitely many tongues (see [19]). In particular, at the critical level $b = 1$ all tongues are present, and it is easy to prove that they are dense at this level (see [14]). We show (in Theorem A) that at the lower levels $f_{a,b}$ can have an attracting or neutral periodic orbit, and otherwise it is expanding. Moreover, the set of expanding maps is dense in the complement of the tongues.

For simplicity, we will be using notation f_a for $f_{a,1}$. A parameter a_0 will be called an *MT parameter* if the trajectory of the critical point $c = 1/2$ is preperiodic (but not periodic).

In this case f_{a_0}has an absolutely continuous invariant measure, [17], and it is also true that the critical value $f_{a_0}(\frac{1}{2})$ satisfies the Collet-Eckmann condition, i.e. that there is $C_{\mathrm{CE}} > 0$ and $\kappa_1 > 0$ such that for $a = a_0$

$$(f_a^n)'(f_a(c)) \geq C_{\mathrm{CE}} e^{\kappa_1 n}, \qquad \forall n \geq 0, \tag{1.2}$$

which implies the existence of an absolutely continuous invariant measure, [23, 25].

Using the methods of [2] it is possible to prove

Proposition 1.1. *There is a set of positive Lebesgue measure $\tilde{E}_1$ so that for all $a \in \tilde{E}_1$ there is $n_0(a)$ so that*

$$(f_a^n)'(f_a(c)) \geq e^{n^{2/3}}, \qquad \forall n \geq n_0(a), \tag{1.3}$$

Here $\frac{2}{3}$ can be replaced by any constant $\sigma < 1$.

A parameter exclusion requiring

$$\operatorname{dist}(f_a^j(c), c) \geq e^{-\sqrt{j}}, \quad j \geq 1, \tag{1.4}$$

will be sufficient to prove (1.3) and then also Jacobson's theorem follows.

Using the methods of Large deviations of [3] it is possible to prove

Proposition 1.2. *There is a set of positive Lebesgue measure E_1 and some $\kappa > 0$ so that for all $a \in E_1$*

$$(f_a^n)'(f_a(c)) \geq Ce^{\kappa n}, \qquad \forall n \geq 0. \tag{1.5}$$

For a similar result see [26].

In the present paper we will consider the non-critical case $0 < b < 1$ and use more elementary methods based on [2], which give stretched exponential growth of the type

$$(f_{a,b}^n)'(f_{a,b}(c)) \geq e^{n^{2/3}}, \qquad n_0 \leq n \leq \hat{N}(a,b), \tag{1.6}$$

for all $a \in \tilde{E}_b$ for a set $\tilde{E}_b$, which is a finite union of intervals. To obtain this it is sufficient to do parameter exclusions of the type

$$\operatorname{dist}(f_{a,b}^j(c), c) \geq C_1 e^{-\sqrt{j}}, \quad j \geq 1, \tag{1.7}$$

and then prove exponential expansion in Section 8. The discussion of the proof is more elaborated at the end of this section.

We will outline the proof of Proposition 1.1 after the proof of Theorem A.

By the results of [5] and [6], if $f_a(c)$ satisfies the Collet-Eckmann condition, then f_a has an absolutely continuous invariant measure. This is the analogue of Jakobson's theorem [11] in this case.

It is also possible to prove (1.2) for a in a set E_1 of positive Lebesgue measure, but with the present setup this would require the method of Large deviations of [3], and this is not required when $0 < b < 1$.

Let us introduce some notations. For a fixed b, let us denote the sets of those parameters a for which $f_{a,b}$ has an attracting (resp. neutral) orbit T_b (resp. TN_b). Moreover, let E_b be the set of those parameters a for which $f_{a,b}$ is *expanding*, that is, there exist $C > 0$ and $\kappa > 0$ such that

$$(f_{a,b}^n)'(x) \geq Ce^{\kappa n}, \qquad \forall n \geq 0 \quad \forall x \in \mathbb{T}. \tag{1.8}$$

By the result of Mañé [16], if a does not belong to T_b or TN_b, then it belongs to E_b. Observe that by the definition, a small perturbation of an expanding map is also expanding, so E_b is open. In fact, the set $E = \{(a,b) : a \in E_b, 0 \leq b < 1\}$ is open in $[0,1) \times [0,1)$.

Note that our definition of E_1 or $\tilde{E}_1$ is quite different from the noncritical case, i.e. the case of E_b for $b < 1$. Nevertheless, there are some common features of the noncritical case, because if $f_{a,b}$ is expanding, then by the results of Krzyżewski and Szlenk [12], or by the Lasota-Yorke Theorem [13], there exists an absolutely continuous invariant measure.

Extending the methods of the proof of (1.3), we prove the following theorem.

Theorem A. Let a_0 be a MT parameter for the family $\{f_a\}$. Denote $\omega(\varepsilon) = (a_0 - \varepsilon, a_0 + \varepsilon)$. Then for some $\varepsilon_0 > 0$ there is a function $b_0 : (0, \varepsilon_0) \to (0,1)$ such that

$$\lim_{\varepsilon \to 0} \frac{\inf\{|E_b \cap \omega(\varepsilon)| : b \in (b_0(\varepsilon), 1]\}}{|\omega(\varepsilon)|} = 1.$$

Here $|A|$ denotes the Lebesgue measure of the set A.

This can be considered as the main result of the paper. It gives a quantitative relation between the behavior of the system for $b < 1$, where the maps are local diffeomorphisms, and for $b = 1$, the critical case.

Finally, we prove a topological result, using very different methods.

Theorem B. For each $b < 1$, the set E_b is dense in the complement of T_b. In particular, every interval of the parameters a either is contained in a closure of one tongue or intersects E_b.

The above theorem in some sense complements Theorem A. Locally it says less about the set E_b, but it applies to all $b < 1$, not only to b sufficiently close to 1 (moreover, this closeness in Theorem A depends on a_0).

The paper is organized as follows. In Section 2 we introduce notation and some definitions that we will use throughout the paper. In Section 3 we prove the *transversality condition* for maps of the family (1.1). In Section 4 we prove that we have exponential growth of the derivative for an orbit of a map $f_{a,b}$ that moves outside an open interval containing the the critical point when (a, b) is a small perturbation of an MT parameter $(a_0, 1)$. In Section 5 we describe the induction including its startup and prove that the conditions on the Induction Statement are satisfied for the first free return time. Furthermore, we define the *bound period* and prove some results concerning the derivative growth during the bound period. In Section 6 we prove the Global Distortion Lemma and in Section 7 we start the proof of Theorem A. In Section 8 we finish this proof. Finally, in Section 9 we prove Theorem B.

Let us indicate what is technically new in this paper compared to previous work. The proof of Theorem A is based on techniques in [2] and [3].

The main strategy of the proof of Theorem A is the inductive proof of

$$(f^n_{a,b})'(f_{a,b}(c)) \geq C_2 e^{n^{2/3}}, \tag{1.9}$$

up to a certain time $\hat{N}$.

The set $\tilde{E}_1$ in the critical case $b = 1$ is a Cantor set of positive measure represented as

$$E_1 = \bigcap_{n=0}^{\infty} A_n,$$

where each A_n is a disjoint union of intervals $\{I^j_n\}$. Here $A_{n+1} \subset A_n$ and the set A_{n+1} are defined by removing subintervals of each I^j_n according to two rules: First those subintervals that do not satisfy an approach rate condition (1.4) for the critical point (or inflexion point) $c = \frac{1}{2}$ are deleted. This replaces the basic assumption (BA) in [2] and [3]. In the non critical case $0 < b < 1$ this condition corresponds to (1.7).

The proof of Theorem A is different from that of Proposition 1.1 in the critical case due to the fact that the behaviour at the inflexion point $c = \frac{1}{2}$ is given by the Taylor series

$$\begin{aligned} f_{a,b}(x) &= a + (2-2b)(x-\frac{1}{2}) + \sum_{k=1}^{\infty} a_{2k+1}(x-\frac{1}{2})^{2k+1} \\ &= a + (2-2b)(x-\frac{1}{2}) + g(x). \end{aligned}$$

Here $g(x)/(x-\frac{1}{2})^3$ is bounded above and below by strictly numerical positive constants, i.e they depending only on the MT map f_{a_0}. Similarly $g'(x)/(x-\frac{1}{2})^2$ has similar bounds from above and below.

This means that when a point $y = f^n_{a,b}(x)$ is close to $c = \frac{1}{2}$, the derivative $f'_{a,b}(y)$ will either be dominated by $2-2b$ or by the quadratic term $g'(y)$ and these two cases will be treated differently.

The induction in the non-critical case $b < 1$ can actually be terminated at the time $\hat{N}$ defined by the condition that constant term in the derivative $2-2b$ can be of size comparable to $g'(x)$, which is quadratic, i.e.

$$2 - 2b \sim (e^{-\sqrt{\hat{N}}})^2, \tag{1.10}$$

where $\sim$ means that the two sides are comparable within fixed constants, which are only depending on f_{a_0}. Defining $\hat{N}$ this way we can stop the induction at time $\hat{N}$ and the remaining set $E_b = \bigcap_{n=0}^{\hat{N}} A_n(b)$ is a finite union of intervals.

Another new aspect of the present paper is that for $b < 1$ that the condition of Collet-Eckmann type (again with $c = \frac{1}{2}$)

$$(f^n_{a,b})'(f_{a,b}(c)) \geq Ce^{\kappa n} \qquad \forall n \geq 0, \tag{1.11}$$

is no longer sufficient for the existence of absolutely continuous invariant measures for $f_{a,b}$. There is however another argument given in Section 8 which uses (1.11) together with *bound period estimates*, see Section 4, which prove uniform hyperbolicity, (1.8).

2. Notation

Throughout this paper, C is a general numeric constant. For a set $A \subset \mathbb{R}$ we will denote by $|A|$ its Lebesgue measure.

Consider the family of *double standard maps* given by (1.1) with $b = 1$. Throughout this paper we denote $f_a(x) = f_{a,1}(x)$ and $\xi_j(a)$ the orbit of the critical point: $\xi_j(a) = f^j_a(c)$.

For a general $b \leq 1$, we also use the notation $\xi_j(a,b) = f^j_{a,b}(c)$. It is clear that when $b < 1$, the point $c = \frac{1}{2}$ is an inflexion point. Sometimes we also use the notations $f(x,a,b)$ for $f_{a,b}(x)$ and $f(x,a)$ for $f_a(x)$.

By $\partial_a f^j_{a,b}(x)$ we denote the partial derivative of $f^j_{a,b}(x)$ with respect to a and by $\partial_a f^j_a(x)$ the partial derivative of $f^j_a(x)$ with respect to a.

Definition 2.1. A parameter $a = a_0$ will be called an *MT parameter*, if there is an integer m and a period length ℓ such that $\xi_m(a_0)$ is a periodic point of f_{a_0} of period ℓ and the multiplier $\Lambda := (f^\ell_{a_0})'(\xi_m(a_0))$ is larger than 1.

As in [2] and [3], we define a partition $\mathcal{Q} = \{I_{r,l}\}$ of the *return interval* $I^* = (c-\delta, c+\delta)$, where $\delta = e^{-r_\delta}$. We first divide I^*

$$I^* = \bigcup_{r=r_\delta}^{\infty} I_r \cup \bigcup_{r=r_\delta}^{\infty} I_{-r},$$

where I_r is the interval $(c+e^{-r-1}, c+e^{-r})$ for $r_\delta \le r < \infty$, and I_{-r} is the interval $(c-e^{-r}, c-e^{-r-1})$.

We then subdivide I_r into r^2 intervals of equal length with disjoint interiors as follows

$$I_r = \bigcup_{\ell=0}^{r^2-1} I_{r,\ell}.$$

For convenience we also use the convention that $I_{r,r^2} = I_{r-1,0}$, $r > 0$, and the analogous convention for $r < 0$.

Note that we have $|I_{r,l}| = \frac{e^{-r}}{r^2}(1-e^{-1})$ and $|I_r| = e^{-r}(1-e^{-1})$. We will also need the extended interval

$$I^+_{r,\ell} = I_{r,\ell-1} \cup I_{r,\ell} \cup I_{r,\ell+1}.$$

For technical reasons we will also need a partition $\mathcal{Q}' = \{I_{r,l}\}$, $|r| \ge r^1_\delta$, of an interval $I^{**} = (c-\delta_1, c+\delta_1)$, where $|r| \ge r^1_\delta$, for some $r^1_\delta < r_\delta$, i.e. $\delta_1 = e^{-r^1_\delta} > \delta$.

A main tool in this paper is a sequence of partitions $\mathcal{P}_n$, $n = 0, 1, 2, \dots$ of the parameter space which is induced by the phase space partition. We define

$$\mathcal{E}_n = \bigcup_{\omega \in \mathcal{P}_n} \omega.$$

We call a time n a *free return* if there is a parameter interval ω belonging to a partition $\mathcal{P}_n$ such that

$$\xi_n(\omega) = I_{r,\ell}.$$

Similarly if we fix $b < 1$, we will have

$$\xi_n(\omega, b) = I_{r,\ell}.$$

(In some cases these two conditions for technical reasons will have to be replaced by $I_{r,\ell} \subset \xi_n(\omega) \subset I^+_{r,\ell}$ or $I_{r,\ell} \subset \xi_n(\omega, b) \subset I^+_{r,\ell}$.)

3. Transversality

In this section we prove the *transversality condition* for maps belonging to the family (1.1).

Lemma 3.1. *The following formula holds:*

$$\partial_a f^n_{a,b}(x) = \sum_{k=0}^{n-1} (f^k_{a,b})'(f^{n-k}_{a,b}(x)) = (f^{n-1}_{a,b})'(f_{a,b}(x)) \sum_{k=0}^{n-1} \frac{1}{(f^k_{a,b})'(f_{a,b}(x))}. \tag{3.1}$$

Proof. We have

$$\partial_a f^{n+1}_{a,b}(x) = 1 + f'_{a,b}(f^n_{a,b}(x)) \cdot \partial_a f^n_{a,b}(x) \tag{3.2}$$

(note that $\partial_a f^0_{a,b}(x) = 0$ and $\partial_a f^1_{a,b}(x) = 1$). Using this formula, we prove by induction

$$\partial_a f^n_{a,b}(x) = \sum_{k=0}^{n-1} (f^k_{a,b})'(f^{n-k}_{a,b}(x)). \tag{3.3}$$

If $n = 0$, then both sides of (3.3) are 0. Assume now that (3.3) holds for some n and prove it for $n+1$ instead:

$$\begin{aligned}\partial_a f^{n+1}_{a,b}(x) &= 1 + f'_{a,b}(f^n_{a,b}(x)) \cdot \sum_{k=0}^{n-1} (f^k_{a,b})'(f^{n-k}_{a,b}(x)) = 1 + \sum_{k=0}^{n-1} (f^{k+1}_{a,b})'(f^{n-k}_{a,b}(x)) \\ &= 1 + \sum_{k=1}^{n} (f^k_{a,b})'(f^{n-(k-1)}_{a,b}(x)) = \sum_{k=0}^{n} (f^k_{a,b})'(f^{(n+1)-k}_{a,b}(x)).\end{aligned}$$

Thus, by induction, (3.3) holds for every n.

Now, we have

$$(f^{n-k-1}_{a,b})'(f_{a,b}(x)) \cdot (f^k_{a,b})'(f^{n-k}_{a,b}(a)) = (f^{n-1}_{a,b})'(f_{a,b}(x)).$$

From this and (3.3) we get

$$\begin{aligned}\partial_a f^n_{a,b}(x) &= (f^{n-1}_{a,b})'(f_{a,b}(x)) \sum_{k=0}^{n-1} \frac{1}{(f^{n-k-1}_{a,b})'(f_{a,b}(x))} \\ &= (f^{n-1}_{a,b})'(f_{a,b}(x)) \sum_{k=0}^{n-1} \frac{1}{(f^{k}_{a,b})'(f_{a,b}(x))}.\end{aligned}$$

□

We get the following immediate corollary.

Corollary 3.2. *We have $\partial_a f_{a,b}(x) \equiv 1$ and if $n > 0$ then $\partial_a f^n_{a,b}(x) \geq 1$. Moreover,*

$$\frac{\partial_a \xi_n(a,b)}{(f^{n-1}_{a,b})'(f_{a,b}(c))} = \sum_{k=0}^{n-1} \frac{1}{(f^k_{a,b})'(f_{a,b}(c))}, \tag{3.4}$$

so, in particular,

$$\partial_a \xi_n(a,b) \geq (f^{n-1}_{a,b})'(f_{a,b}(c)). \tag{3.5}$$

In a special case, when there is a constant C_2 so that

$$(f^\nu_{a,b})'(f_{a,b}(c)) \geq C_2 e^{\nu^{2/3}}, \qquad \nu = 0, 1, 2, \dots, n-1 \tag{3.6}$$

then for all $\nu \leq n$ we obtain by combining the inequality with the lower bound (3.5)

$$1 \leq \frac{\partial_a \xi_n(a,b)}{(f^{n-1}_{a,b})'(f_{a,b}(c))} \leq q_*, \tag{3.7}$$

where

$$q_* = \sum_{i=0}^{\infty} C_2^{-1} e^{-i^{2/3}}.$$

Remark 3.3. We would like to emphasize the central role that Corollary 3.2 plays in this paper. We prove the estimate (3.6) successively by induction on ν. We can then conclude that (3.7) holds with n replaced by ν for a given ν. From this estimate we conclude that the x- and a-derivative are comparable at a given time ν. It is important that we prove the x-expansion first and then verify the comparison. The constant q_* will be fixed, i.e. it only depends on f_{a_0}.

We will also need the following lemma which can be viewed as a lower bound for the Radon-Nikodym derivative of $\xi_\nu(a,b) \mapsto \xi_\mu(a,b)$, $\nu < \mu$, (with respect to $a \in \omega$).

Lemma 3.4. *Suppose that ω is a parameter interval and $\nu < \mu$. Assume further that there is a constant q' such that for all $t \in \omega$*

$$(f_{t,b}^{\nu-1})'(f_{t,b}(c)) \geq \frac{1}{q'} \partial_a \xi_\nu(t,b). \tag{3.8}$$

Then

$$|\xi_\mu(\omega,b)| \geq \frac{1}{q'} \inf_{a\in\omega} (f_{a,b}^{\mu-\nu})'(f_{a,b}^{\nu}(c)) \cdot |\xi_\nu(\omega,b)|.$$

Proof. By Corollary 3.2,

$$|\xi_\mu(\omega,b)| = \int_\omega \partial_a \xi_\mu(t,b)\,dt \geq \int_\omega (f_{t,b}^{\mu-1})'(f_{t,b}(c))\,dt. \tag{3.9}$$

However, by (3.8) we have

$$(f_{t,b}^{\mu-1})'(f_{t,b}(c)) = (f_{t,b}^{\mu-\nu})'(f_{t,b}^{\nu}(c)) \cdot (f_{t,b}^{\nu-1})'(f_{t,b}(c)) \geq \inf_{a\in\omega} (f_{a,b}^{\mu-\nu})'(f_{a,b}^{\nu}(c)) \cdot \frac{1}{q'} \partial_a \xi_\nu(t,b).$$

Together with (3.9), we get

$$|\xi_\mu(\omega,b)| \geq \frac{1}{q'} \inf_{a\in\omega} (f_{a,b}^{\mu-\nu})'(f_{a,b}^{\nu}(c)) \cdot \int_\omega \partial_a \xi_\nu(t,b)\,dt = \frac{1}{q'} \inf_{a\in\omega} (f_{a,b}^{\mu-\nu})'(f_{a,b}^{\nu}(c)) \cdot |\xi_\nu(\omega,b)|.$$

□

4. The outside expansion

The aim of this section is to prove that we have exponential growth of the derivative for an orbit of a map $f_{a,b}$ that moves outside an open interval I containing c, when (a,b) is a small perturbation of an MT parameter $(a_0,1)$. We consider the parameter space $\mathbb{R}/\mathbb{Z} \times (0,1]$, and when we speak of a neighborhood of $(a_0,1)$, we mean its neighborhood in this space.

By $|x-y|$ we denote the distance between x and y on the circle. Since the points x and y will be usually close to each other, this makes perfect sense. Denote

$$\overline{d} = \min_{j\geq 1} |c - f_{a_0}^j(c)|. \tag{4.1}$$

By the definition of an MT parameter, we have $\overline{d} > 0$.

Since f_{a_0} has negative Schwarzian derivative, the following lemma follows immediately from Theorem 1.3 of [17] (in a general case one could use also the result of Mañé (see [16])).

Lemma 4.1. *Let I be an open interval containing c. Then there exists a neighborhood $\mathcal{N}$ of $(a_0,1)$, positive constants C_3, κ_2, and an integer M_1 such that if $(a,b) \in \mathcal{N}$ then*

(i) *if* $x, f_{a,b}(x), \ldots, f_{a,b}^{n-1}(x) \notin I$, *then*

$$(f_{a,b}^n)'(x) > C_3 e^{\kappa_2 n};$$

(ii) *if* $x, f_{a,b}(x) \ldots,\ f_{a,b}^{n-1}(x) \notin I$ *and* $n \geq M_1$, *then*

$$(f_{a,b}^n)'(x) > e^{\kappa_2 n}.$$

Proof. By Theorem 1.3 of [17] (or a result of Mañé [16]), there exists $L > 0$ and $\kappa_2' > 0$ such that if $x, f_{a_0}(x), \ldots, f_{a_0}^{L-1}(x) \notin I$, then $(f_{a_0}^L)'(x) > e^{\kappa_2' L}$. Therefore, if $\mathcal{N}$ is a sufficiently small neighborhood of $(a_0, 1)$, then for all $(a,b) \in \mathcal{N}$ and x such that $x, f_{a,b}(x), \ldots, f_{a,b}^{L-1}(x) \notin I$, we have $(f_{a,b}^L)'(x) > e^{\kappa_2' L}$. Since the infimum of $(f_{a,b}^i)'(x)$ over $(a,b) \in \mathcal{N}$, $x \notin I$ and $i = 0, 1, \ldots, L-1$ is positive, there exists a positive constant C_3 such that (i) holds with $\kappa_2 = \kappa_2'$. Thus it also holds with $\kappa_2 = \kappa_2'/2$, and then (ii) holds with any

$$M_1 > \frac{-\log C_3}{\kappa_2}.$$

□

Now we fix a positive constant $\beta > 0$. It will depend only on the unperturbed map f_{a_0} and can be chosen as, say $\frac{1}{100}\min(\tilde{\kappa}, \kappa_3)$. Here $\tilde{\kappa} = (1/\ell)\log \Lambda$, where Λ is the multiplier of the repelling periodic point of the MT-point, and κ_3 is the, Lyaponov exponent in Lemma 4.6.

Definition 4.2. Let $x \in I^{**} = (c-\delta_1, c+\delta_1)$. We say that x is *β-bound to the critical point c up to time p for $f_{a,b}$*, if p is the maximal integer such that

$$|f_{a,b}^j(x) - f_{a,b}^j(c)| \leq e^{-\beta j}, \qquad \forall j \leq p. \tag{4.2}$$

Observe that for every a, b (where $b \leq 1$) and every x we have

$$f_{a,b}'(x) \leq 4 \quad \text{and} \quad |f_{a,b}''(x)| \leq 4\pi < 13. \tag{4.3}$$

Let us state a version of the Bound Distortion Lemma (see [2] and [3]).

Lemma 4.3. *If δ_1 is sufficiently small, then there is a constant $C_4(\delta_1) > 1$, which converges to 1 as $\delta_1 \to 0$, such that for every $x \in I^{**} = (c-\delta_1, c+\delta_1)$ if x is β-bound to c up to time p for f_{a_0}, then*

$$\frac{1}{C_4} < \frac{(f_{a_0}^k)'(f_{a_0}(x))}{(f_{a_0}^k)'(f_{a_0}(c))} < C_4 \tag{4.4}$$

for all $k \leq p$. Moreover, there is a constant $C_5 = C_5(\delta_1) > 0$, which converges to 0 as $\delta_1 \to 0$, such that

$$|f_{a_0}^k(x) - f_{a_0}^k(c)| < C_5 \tag{4.5}$$

for all $k \leq p$.

Proof. Assume that x is bound to c up to time p. Now choose $p_1 = \frac{1}{10}\log(1/\delta_1)$.

Then by (4.3) we can estimate

$$|f_{a_0}^j(x) - f_{a_0}^j(c)| \leq \delta_1 4^j \leq \delta_1 4^{p_1}$$

if $j \leq p_1$ and

$$|f_{a_0}^j(x) - f_{a_0}^j(c)| \leq e^{-\beta j} \leq e^{-\beta p_1}$$

if $p_1 < j \le p$. Thus, if δ_1 is sufficiently small then $|f_{a_0}^j(x) - f_{a_0}^j(c)| \le \overline{d}/2$ and therefore

$$|f_{a_0}^j(x) - c| \ge \frac{\overline{d}}{2} \tag{4.6}$$

for all $j \le p$.

This also proves the last statement of the lemma.

We have

$$\begin{aligned} \frac{(f_{a_0}^k)'(f_{a_0}(x))}{(f_{a_0}^k)'(f_{a_0}(c))} &= \prod_{j=1}^{k} \left(1 + \frac{f_{a_0}'(f_{a_0}^j(x)) - f_{a_0}'(f_{a_0}^j(c))}{f_{a_0}'(f_{a_0}^j(c))} \right) \\ &\le \exp\left(\sum_{j=1}^{k} \frac{13|f_{a_0}^j(x) - f_{a_0}^j(c)|}{f_{a_0}'(f_{a_0}^j(c))} \right) \\ &\le \exp\left\{ K_1 \left(\delta_1 p_1 4^{p_1} + \sum_{j=p_1+1}^{k} e^{-\beta j} \right) \right\}. \end{aligned} \tag{4.7}$$

The last sum in the exponential may be empty.

Similarly, using (4.6), we get

$$\frac{(f_{a_0}^k)'(f_{a_0}(c))}{(f_{a_0}^k)'(f_{a_0}(x))} \le \exp\left\{ K_2 \left(\delta_1 p_1 4^{p_1} + \sum_{j=p_1+1}^{k} e^{-\beta j} \right) \right\} \tag{4.8}$$

The sums in (4.7) and (4.8) are bounded by a constant, which only depends on δ_1, so we get (4.4). Moreover, by (4.7) and (4.8), $C_4 > 1$ converges to 1 as $\delta_1 \to 0$. ☐

Set $\tilde{\kappa} = (1/\ell) \log \Lambda$. Then there is a constant $C_6 = C_6(a_0)$ so that

$$(f_{a_0}^j)'(f_{a_0}(c)) \ge C_6 e^{\tilde{\kappa} j} \tag{4.9}$$

for all $j \ge 1$.

At c, the first two derivatives of f_a vanish, but the third one does not. Therefore, there are positive constants C_7, C_8 such that for all a sufficiently close to a_0

$$C_7 |x-c|^3 < |f_a(x) - f_a(c)| < C_8 |x-c|^3 \tag{4.10}$$

whenever $x \in I^{**}$. If δ_1 is small, the constants C_7 and C_8 can be made close to each other. Similarly, for some positive constants C_9', C_{10}',

$$C_9'(x-c)^2 < f_a'(x) < C_{10}'(x-c)^2 \tag{4.11}$$

whenever $x \in I^{**}$. If instead of f_a we consider $f_{a,b}$ with b sufficiently close to 1, we similarly obtain

$$|x-c| \left(2 - 2b + C_7(x-c)^2\right) < |f_{a,b}(x) - f_{a,b}(c)| < |x-c| \left(2 - 2b + C_8(x-c)^2\right) \tag{4.12}$$

and

$$2 - 2b + C_9(x-c)^2 < f_{a,b}'(x) < 2 - 2b + C_{10}(x-c)^2, \tag{4.13}$$

and we chose C_9 and C_{10} so that these estimates are valid for all $b \le 1$. Moreover, we have

$$|f_{a,b}''(x)| \le 8\pi^2 |x-c| < 80|x-c|. \tag{4.14}$$

In the following lemma we estimate the length of the bound period.

Lemma 4.4. *If δ_1 is sufficiently small, x is β-bound to c up to time p for f_{a_0}, and $x \in I^{**} \setminus \{c\}$, then*

$$p < -\frac{4}{\tilde{\kappa}} \log |x - c|. \tag{4.15}$$

In the particular case when $x \in I_{\pm r}$ we obtain

$$p \leq \frac{4r}{\tilde{\kappa}}. \tag{4.16}$$

Proof. By Lemma 4.3, we have

$$|f_{a_0}^p(x) - f_{a_0}^p(c)| > \frac{(f_{a_0}^p)'(f_{a_0}(c))}{C_4} |f_{a_0}(x) - f_{a_0}(c)|.$$

Taking into account (4.9) and (4.10), we get

$$1 > |f_{a_0}^p(x) - f_{a_0}^p(c)| > \frac{C_6 C_7}{C_4} e^{\tilde{\kappa} p} |x - c|^3.$$

If δ_1 is small, then $C_4 < 2$, so taking logarithms gives us

$$\log \frac{C_6 C_7}{2} + \tilde{\kappa} p + 3 \log |x - c| < 0.$$

If δ_1 is small, then $-\log |x - c|$ is large, so we get $-\log(C_6 C_7/2) < -\log |x - c|$, and therefore $\tilde{\kappa} p < -4 \log |x - c|$. □

We need a derivative estimate for an orbit of f_{a_0} that moves completely outside $I^* = (c - \delta, c + \delta)$ or $I^{**} = (c - \delta_1, c + \delta_1)$ but returns to one of these intervals at time n.

In the proof of the next lemma we will use the fact that f_a has negative Schwarzian derivative. This result can be generalized to the C^2 case (see van Strien [25]).

Lemma 4.5. *Let $\overline{d}$ be as in* (4.1). *For every $\delta_1 \in (0, \overline{d}/2)$ and for every $n \geq 1$, if x is such that $f_{a_0}^j(x) \notin I^{**}$ for $j = 0, \ldots, n-1$, and $f_{a_0}^n(x) \in I^{**}$, then $(f_{a_0}^n)'(x) > \overline{d}/2$.*

Proof. On each side of x there are the two closest preimages of c of order less than n: $\eta_1 < x$ and $\eta_2 > x$. Then $f_{a_0}^n$ has positive derivative on (η_1, η_2) and has negative Schwarzian derivative on that interval. Therefore on one of the intervals $[\eta_1, x]$ and $[x, \eta_2]$ the maximum of the derivative $(f_{a_0}^n)'$ is attained at x. We may assume that this is the interval $[\eta_1, x]$. Then $f_{a_0}^n(\eta_1) = f_{a_0}^k(c)$ for some $k > 0$, so

$$|f_{a_0}^n(\eta_1) - f_{a_0}^n(x)| \geq \overline{d} - \delta_1 > \overline{d}/2.$$

By the Mean Value Theorem,

$$|f_{a_0}^n(\eta_1) - f_{a_0}^n(x)| = (f_{a_0}^n)'(t) |\eta_1 - x| \leq (f_{a_0}^n)'(t)$$

for some $t \in (\eta_1, x)$, and thus,

$$(f_{a_0}^n)'(x) \geq (f_{a_0}^n)'(t) > \overline{d}/2.$$

□

In the following lemma we consider what we call a *free period*.

Lemma 4.6. *Given δ_1 sufficiently small, there is a neighborhood $\mathcal{N}$ of $(a_0, 1)$ in the parameter space and positive constants C^* and κ_3, such that, if $(a,b) \in \mathcal{N}$ then if $x, f_{a,b}(x), \dots, f_{a,b}^{q-1}(x) \notin I^{**}$ and $f_{a,b}^q(x) \in I^{**}$ then*

$$(f_{a,b}^q)'(x) \geq C^* e^{\kappa_3 q}. \tag{4.17}$$

Here the constant C^* depends only on the unperturbed map f_{a_0}, while κ_3 depends on δ_1.

Proof. By Lemma 4.1, $(f_{a,b}^{M_1})'(x) \geq e^{\kappa_2 M_1}$. For a general q write $q = kM_1 + \ell$, $0 \leq \ell < M_1$. Choose κ_3' so that $e^{\kappa_3' M_1} \leq 2$.

Since $\ell < M_1$, then by Lemma 4.5, $(f_{a,b}^\ell)'(f_{a,b}^{kM_1}(x)) > \overline{d}/4$ (here we can extend the estimate to a neighborhood of $\mathcal{N}_2$ of $(a_0, 1)$ because we consider only finitely many iterates of the map). Then for $(a,b) \in \mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$

$$\begin{aligned}(f_{a,b}^q)'(x) &= (f_{a,b}^{kM_1})'(x)(f_{a.b}^\ell)'(f_{a,b}^{kM_1}(x)) \geq e^{\kappa_2 kM_1} \cdot \frac{\overline{d}}{4} \\ &\geq e^{\kappa_2 kM_1} \frac{\overline{d}}{8} \cdot e^{\kappa_3' M_1} \geq \frac{\overline{d}}{8} e^{\kappa_2 kM_1 + \kappa_3' \ell}\end{aligned}$$

so (4.17) follows with $C^* = \overline{d}/8$ and $\kappa_3 = \min(\kappa_2, \kappa_3')$. Note that as required, C^* only depends on the unperturbed map f_{a_0}, while κ_3 depends on δ_1.

□

We will also need an estimate of the derivative during the bound period.

Lemma 4.7. *Assume that $9\beta \leq \tilde{\kappa}$. Let C^* be the constant from Lemma 4.6. Then there is an arbitrarily small δ_1 such that if x is β-bound to c up to time p for f_{a_0} and $x \in I^{**} = (c - \delta_1, c + \delta_1)$ then*

$$(f_{a_0}^p)'(x) > \frac{1}{C^*} e^{\frac{\tilde{\kappa}}{4} p}. \tag{4.18}$$

Proof. By the Mean Value Theorem, there is an y between $f_{a_0}(x)$ and $f_{a_0}(c)$, such that

$$|f_{a_0}^p(x) - f_{a_0}^p(c)| = (f_{a_0}^{p-1})'(y)|f_{a_0}(x) - f_{a_0}(c)|.$$

By this, Lemma 4.3 and (4.10), there exists a constant K_3 such that if δ_1 is sufficiently small then

$$|f_{a_0}^p(x) - f_{a_0}^p(c)| < K_3 e^{\tilde{\kappa} p}|x - c|^3.$$

Similarly, since $(f_{a_0}^p)'(x) = f_{a_0}'(x) \cdot (f_{a_0}^{p-1})'(f_{a_0}(x))$, we get by Lemma 4.3 and (4.11) that there exists a constant K_4 such that if δ_1 is sufficiently small then

$$(f_{a_0}^p)'(x) > K_4 e^{\tilde{\kappa} p}|x - c|^2.$$

By the definition of p we have

$$|f_{a_0}^{p+1}(x) - f_{a_0}^{p+1}(c)| > e^{-\beta(p+1)},$$

and therefore for some constant K_5

$$|f_{a_0}^p(x) - f_{a_0}^p(c)| > K_5 e^{-\beta p}.$$

Thus,

$$K_3 e^{\tilde{\kappa} p}|x - c|^3 > K_5 e^{-\beta p},$$

so

$$|x-c|^2 > K_5^{2/3} K_3^{-2/3} e^{(2/3)(-\beta-\tilde{\kappa})p}.$$

Together with an earlier estimate, this gives us

$$(f_{a_0}^p)'(x) > K_4 e^{\tilde{\kappa}p} K_5^{2/3} K_3^{-2/3} e^{(2/3)(-\beta-\tilde{\kappa})p} = K_4 K_5^{2/3} K_3^{-2/3} e^{(1/3)(\tilde{\kappa}-2\beta)p}.$$

Since $9\beta \le \tilde{\kappa}$, we have

$$\frac{1}{3}(\tilde{\kappa}-2\beta) > \frac{7}{27}\tilde{\kappa},$$

and therefore (4.18) holds if

$$C^* > K_4^{-1} K_5^{-2/3} K_3^{2/3} e^{-\frac{\tilde{\kappa}}{108}p(\delta_1)}, \tag{4.19}$$

where $p(\delta_1)$ is the bound period associated with δ_1. Since $p(\delta_1) \to \infty$ as $\delta_1 \to 0$, and C^* is independent of δ_1, the above inequality holds if δ_1 is sufficiently small. □

Remark 4.8. If in (4.18) we replace (on both sides of the inequality) p by $p-1$ or $p+1$, then at the right-hand side of in (4.19) there will be one more multiplicative constant. Since it was irrelevant in the proof what constant is there, Lemma 4.7 still holds with a suitably modified inequality (4.18).

The following lemma is very similar to Lemma 4.1, but the exponent in the estimate does not depend on the size of the neighborhood of c that we consider. In this lemma we assume that δ_1 is sufficiently small (so that the lemmas that we use in the proof hold) but fixed.

Lemma 4.9. *Let I be an open symmetric interval around c, whose closure is contained in I^{**}. Fix a sufficiently small neighborhood $\mathcal{N}$ of $(a_0, 1)$ (depending on I). Then there are constants $C_{11} > 0$ and $\kappa_4 > 0$, (independent of I) and an integer M (depending on I) such that for $(a,b) \in \mathcal{N}$*

(i) *if $x, f_{a,b}(x), \dots, f_{a,b}^{n-1}(x) \notin I$ and $f_{a,b}^n(x) \in I^{**}$, then*

$$(f_{a,b}^n)'(x) \ge C_{11} e^{\kappa_4 n};$$

(ii) *if $x, f_{a,b}(x), \dots, f_{a,b}^{n-1}(x) \notin I$ and $n \ge M$, then*

$$(f_{a,b}^n)'(x) \ge e^{\kappa_4 n}.$$

Remark. Note that we state (i) with the weaker assumption $f_{a,b}^n(x) \in I^{**}$ instead of the more natural $f_{a,b}^n(x) \in I$. This slightly stronger statement will be used in the proof of (ii).

Proof. Let $0 = t_0 < t_1 < t_2 < \cdots < t_S < t_{S+1} = n$, where t_i for $i \in \{1, 2, \dots, S\}$ are the times when $f_{a,b}^{t_i}(x) \in I^{**} \setminus I$. We want to estimate

$$(f_{a,b}^n)'(x) = \prod_{j=0}^{S} (f_{a,b}^{t_{j+1}-t_j})'(f_{a,b}^{t_j}(x)).$$

The times from $[t_0, t_1)$ form a free period; let $t_1 - t_0 = q_0$. Hence, by Lemma 4.6,

$$(f_{a,b}^{q_0})'(x) \ge C^* e^{\kappa_3 q_0}.$$

Consider now times from $[t_j, t_{j+1})$, where $j > 0$. We can write it as a union of a bound period $[t_j, t_j + p_j)$ and a free period $[t_j + p_j, t_{j+1})$, and we write its length as

$t_{j+1} - t_j = p_j + q_j$. For the bound periods $[t_j, t_j + p_j)$ we can use the estimate from Lemma 4.7 if $\mathcal{N}$ is sufficiently small, because by Lemma 4.4 we work only with the finite number of iterates (for I fixed; this is why $\mathcal{N}$ depends on I). Although p may depend on the map that we are using, if $\mathcal{N}$ is sufficiently small, it only may change to $p \pm 1$, and then by Remark 4.8 we can still use Lemma 4.7.

Thus, for the bound periods $[t_j, t_j + p_j)$ we get

$$(f_{a,b}^{p_j})'(f_{a,b}^{t_j}(x)) \geq \frac{1}{C^*} e^{\frac{\tilde{\kappa}}{4} p_j} \tag{4.20}$$

and for the free period, as before, the estimate from Lemma 4.6 gives us

$$(f_{a,b}^{q_j})'(f_{a,b}^{t_j+p_j}(x)) \geq C^* e^{\kappa_3 q_j}.$$

Combining these estimates we get

$$(f_{a,b}^n)'(x) \geq C^* e^{\kappa_3 q_0} \prod_{j=1}^{S} \frac{1}{C^*} e^{\frac{\tilde{\kappa}}{4} p_j} \cdot C^* e^{\kappa_3 q_j} \geq C^* e^{\kappa_4' n},$$

with $\kappa_4' = \min(\kappa_3, \tilde{\kappa}/4)$. This completes the proof of (i). □

Under the assumptions of (ii) instead of (i) we make the same construction and estimates. The only difference is that we do not know that $f_{a,b}^n(x) \in I^{**}$, so we lose information about the last period. There are two cases.

Case 1. $f_{a,b}^n(x)$ is still in bound state to the last return to I^{**} at time t_S. At time t_S we can use the estimate of (i)

$$(f_{a,b}^{t_S})'(x) \geq C_{11} e^{\kappa_4' t_S}.$$

The derivative contribution at time t_S is

$$f_{a,b}'(f_{a,b}^{t_S}(x)) \geq C_9' \delta^2$$

Then there is a derivative contribution from the time $[t_S + 1, t_S + j]$, $j = n - t_S$. Since $1 \leq j \leq p_S$ we can use the Collet-Eckmann condition (1.2) and the distorsion estimate Lemma 4.3, combined with continuity in a for $a \in \mathcal{A}$, and the fact that p_S is bounded to conclude that, say

$$(f_{a,b}^j)'(f_{a,b}^{t_S+1}(x)) \geq \frac{1}{2} C_{\mathrm{CE}} e^{\kappa_1 j}.$$

Combining these estimates and using the the chain rule we get that

$$(f_{a,b}^n)'(x) \geq C_{11} e^{\kappa_4 t_S} \cdot C_9' \delta^2 \cdot \frac{1}{2} C_{\mathrm{CE}} e^{\frac{\tilde{\kappa}}{4} j}$$

and since $n \geq M$, where M is allowed to depend on δ this gives the estimate (ii) with a suitable $\kappa_4'' < \kappa_4'$.

Case 2. $n \geq t_S + p_S$. In this case we can use (4.20) with $j = S$ to obtain

$$(f_{a,b}^{p_S})'(f_{a,b}^{t_j}(x)) \geq \frac{1}{C^*} e^{\frac{\tilde{\kappa}}{4} p_S} \tag{4.21}$$

Then

$$(f_{a,b}^n)'(x) \geq (f_{a,b}^{t_S})'(x)(f_{a,b}^{p_S})'(f_{a,b}^{t_S}(x))((f_{a,b}^{q_S})'(f_{a,b}^{t_S+p_S}(x)),$$

where $q_S = n-(t_S+p_S)$. We get using Lemma 4.1 and the simple estimate $(f_{a,b})'(x) \geq C_9'\delta_1^2$ for $|x-c| \geq \delta_1$ that

$$(f_{a,b}^{q_S})'(f_{a,b}^{t_S+p_S}(x)) \geq \begin{cases} e^{\kappa_2 q}, & q_S \geq M_1 \\ (C_9'\delta_1^2)^{q_S}, & q_S < M_1. \end{cases}$$

Using that the constants δ_1, C_9', κ_2 and M_1 only depend on f_{a_0}, we conclude that (ii) holds with $\kappa_4 = \kappa_4'''$ for $n \geq M$, if M is sufficienty large. The final κ_4 is then chosen as $\kappa_4 = \min(\kappa_4', \kappa_4'', \kappa_4''')$.

Remark 4.10. Note that we in this setting will have an analogy of Lemma 4.4 and the estimate

$$p \leq \frac{4r}{\kappa_4} \tag{4.22}$$

holds.

Remark 4.11. We need in the future in several occasions a distorsion estimate in the situation of Lemma 4.9, i.e. for orbits located outside of I. We need the estimate for parameter dynamics, i.e we have a parameter interval ω in the space of a-parameters, and we consider $\xi_j(\omega, b)$ for j satisfying $\nu \leq j < \mu = n$, where $\xi_j(\omega, b) \cap I = \emptyset$ for $j = \nu, \ldots, n-1$ and $\xi_n(\omega, b) \cap I \neq \emptyset$. Let $\omega' \subset \omega$ be the interval that is mapped onto I. Then Lemma 4.9 (i) implies that

$$\inf_{a\in\omega'} (f_{a,b}^{n-\nu})'(f_{a,b}^{\nu}(c)) \geq C_{11} e^{\kappa_4(n-\nu)} \tag{4.23}$$

We also assume that (3.6) hold at time ν i.e. for $a \in \omega$

$$(f_{a,b}^{j})'(f_{a,b}(c)) \geq C_2 e^{j^{2/3}}, \qquad 1 \leq j \leq \nu - 1 \tag{4.24}$$

Then by Corollary 3.2

$$1 \leq \frac{\partial_a \xi_\nu(a,b)}{(f_{a,b}^{\nu-1})'(f_{a,b}(c))} \leq q_*. \tag{4.25}$$

Then we conclude from Lemma 3.4 that

$$|\xi_n(\omega', b)| \geq \frac{1}{q*} \inf_{a\in\omega'} (f_{a,b}^{n-\nu})'(f_{a,b}^{\nu}(c)) \cdot |\xi_\nu(\omega', b)|. \tag{4.26}$$

Lemma 4.12. *There exists a constant C_{12}, such that in the situation of Remark 4.11, if $a', a'' \in \omega'$ then*

$$\frac{(f_{a',b}^{n-\nu})'(f_{a',b}^{\nu}(c))}{(f_{a'',b}^{n-\nu})'(f_{a'',b}^{\nu}(c))} \leq \exp\left(C_{12} \frac{|f_{a',b}^{n}(c) - f_{a'',b}^{n}(c)|}{\delta} \right). \tag{4.27}$$

Proof. Set $x_k = f^{\nu+k}_{a',b}(c)$ and $y_k = f^{\nu+k}_{a'',b}(c)$. Note that $f'_{a,b}(x)$ is independent of a. Therefore

$$\log \frac{(f^{n-\nu}_{a',b})'(x_0)}{(f^{n-\nu}_{a'',b})'(y_0)} = \sum_{k=0}^{n-\nu-1} (\log f'_{a',b}(x_k) - \log f'_{a',b}(y_k)) \le \sum_{k=0}^{n-\nu-1} \left(\frac{|f''_{a',b}(\eta_k)|}{f'_{a',b}(\eta_k)} \cdot |x_k - y_k| \right)$$

for some η_k between x_k and y_k. Since $\eta_k \notin I^*$ for $k = 0, \ldots, n-\nu-1$, we get, by (4.13) and (4.14),

$$\frac{|f''_{a',b}(\eta_k)|}{f'_{a',b}(\eta_k)} < \frac{80}{C_9\delta}. \tag{4.28}$$

Therefore,

$$\log \frac{(f^{n-\nu}_{a',b})'(x_0)}{(f^{n-\nu}_{a'',b})'(y_0)} \le \frac{80}{C_9\delta} \sum_{k=0}^{n-\nu-1} |x_k - y_k|. \tag{4.29}$$

We have $x_k = \xi_{\nu+k}(a', b)$ and $y_k = \xi_{\nu+k}(a'', b)$. Therefore, by Remark 4.11,

$$|x_k - y_k| \le \frac{q_*}{C_{11}} e^{\kappa_4(n-\nu-k)} |x_{n-\nu} - y_{n-\nu}|.$$

Thus,

$$\sum_{k=0}^{n-\nu-1} |x_k - y_k| \le \frac{q_*}{C_{11}} \sum_{m=0}^{\infty} e^{-\kappa_4 m} |x_{n-\nu} - y_{n-\nu}| = \frac{q_*}{C_{11}(1 - e^{-\kappa_4})} |x_{n-\nu} - y_{n-\nu}|.$$

Together with (4.29), we get

$$\log \frac{(f^{n-\nu}_{a',b})'(x_0)}{(f^{n-\nu}_{a'',b})'(y_0)} \le \frac{80 q_*}{C_9 C_{11}(1 - e^{-\kappa_4})\delta} |x_{n-\nu} - y_{n-\nu}|.$$

and we have proved (4.27) with

$$C_{12} = \frac{160 q_*}{C_9 C_{11}(1 - e^{-\kappa_4})}.$$

□

Remark 4.13. We note that the distortion in Lemma 4.12 is uniformly bounded since $|f^n_{a,b}(c) - f^n_{a',b}(c)| \le 2\delta$.

We will need a distorsion estimate of the same type as Lemma 4.12 in the situation when we only assume estimates as (4.23) for all $\nu < n$ and with another Lyapunov exponent $\kappa_5 > 0$, together with (4.24). This is the case of *hyperbolic times* in the sense of Alves.

Lemma 4.14. *Assume that $\xi_j(\omega, b)$, $j = \nu, \ldots, n$, is located in $U = S^1 \setminus I^{**}$ and*

$$\inf_{a \in \omega} (f^{n-j}_{a,b})'(f^j_{a,b}(c)) \ge C_{11} e^{\kappa_5(n-j)} \quad \text{for all } j,\ \nu \le j < n. \tag{4.30}$$

Furthermore assume that (4.24) is satisfied.

Then

$$(4.31) \qquad \frac{(f_{a',b}^{n-\nu})'(f_{a',b}^{\nu}(c))}{(f_{a'',b}^{n-\nu})'(f_{a'',b}^{\nu}(c))} \leq \exp\left(C_{13}|f_{a',b}^{n}(c) - f_{a'',b}^{n}(c)|\right).$$

Here the constant C_{13} *can be chosen as* $C_{13} = C'_{13}N(f_{\mathcal{N}}, U)/(1-e^{-\kappa_5})$*, where* $N(f_{\mathcal{N}}, U)$ *is the maximal nonlinearity*

$$N(f_{\mathcal{N}}, U) = \sup_{(a,b)\in\mathcal{N}} \max_{x\in U} \frac{|f''_{a,b}(x)|}{f'_{a,b}(x)}.$$

$N(f_{\mathcal{N}}, U)$ *depends only on* f_{a_0} *and hence not on* δ *and* C'_{13} *is a constant that only depends on* f_{a_0}.

Proof. We will not give the proof since it is virtually word by word the same as that of Lemma 4.12. The only difference that the upper bound $80/(C_9\delta)$ in (4.28) is replaced by $N(\mathcal{N}, U)$.

□

5. Induction

Recall that the partition of the return interval $I^* = (c-\delta, c+\delta)$ was introduced on Section 2.

Recall also that we defined $\xi_n(a,b) = f_{a,b}^n(c)$.

The next lemma will be used for the startup of the induction.

Lemma 5.1. *Assume that* δ_1 *is sufficiently small and the neighborhood* $\mathcal{N}$ *of* $(a_0, 1)$ *is sufficiently small. Then there are constants* C_1, C_2, $\kappa_6 > 0$ *so that for every* $\varepsilon = 2^{-J_0}$ *sufficiently small, there is a function* $b_0(\varepsilon)$ *so that for every* $b_0(\varepsilon) \leq b < 1$ *one can partition* $(a_0 - \varepsilon, a_0 - \varepsilon^2)$ *into a partition* $\mathcal{Q}$ *of countable number of parameter intervals* ω *and an exceptional set* $\mathcal{E}$ *of measure* $o(\epsilon)$*, so that for all* $\omega \in \mathcal{Q}$ *there is an* $n_0 = n_0(\omega)$ *so that for some* (r, ℓ)*, with* $r \leq \sqrt{n_0}$*, (or equivalently* $e^{-r} \geq e^{-\sqrt{n_0}}$*)*

$$I_{r,\ell} \subset \xi_{n_0}(\omega, b) \subset I_{r,\ell}^+,$$

and such that for every $a \in \omega$

(a) $(f_{a,b}^j)'(f_{a,b}(c)) \geq C_2 e^{\kappa_6 j}$ *for* $0 \leq j \leq n_0 - 1$*;*
(b) $\partial_a f_{a,b}^j(c) \geq C_2 e^{\kappa_6 (j-1)}$ *for* $1 \leq j \leq n_0$*;*
(c) $|\xi_j(a,b) - c| > C_1 e^{-\sqrt{j}}$ *for* $1 \leq j < n_0$*;*
(d) $(f_{a,b}^{n_0-1})'(f_{a,b}(c)) \geq e^{2(n_0-1)^{2/3}}$*;*
(e) $|\xi_{n_0}(a,b) - c| \geq e^{-\sqrt{n_0}}$

The corresponding statement holds also for the interval $(a_0 + \varepsilon^2, a_0 + \varepsilon)$.

Proof. We partition $(a_0 - \varepsilon, a_0 - \varepsilon^2)$ into subintervals $\eta_j = (a_0 - 2^{-j}, a_0 - 2^{-j-1}) = (a'_j, a''_j)$, $j = J_0, \ldots, 2J_0 - 1$. The critical point c of unperturbed map f_{a_0} is mapped to a repelling periodic point P in m iterates. Let U_0 be a symmetric interval contained in the linearization domain of P so that

$$(f_{a,b}^{\ell})'(x) \geq \lambda_1^{\ell} = \Lambda_1 > 1 \qquad \text{for } x \in U_0.$$

Let $\tilde{\eta}_j = (a_0 - 2^{-j}, a_0)$. Then there is a constant C_{14} so that

$$(f^i_{a,b})'(\xi_m(a,b)) \geq C_{14}\lambda_1^i, \quad \text{for all } a \in \tilde{\eta}_j$$

as long as $\xi_{m+i}(\tilde{\eta}_j, b) \subset U_0$. We now state a version of Lemma 4.12 which will be used in the startup construction.

Lemma 5.2.

Suppose that for $a \in \omega$ there is a constant $\tilde{C} = \frac{1}{2}$, say, so that

$$|\xi_m(\omega', b)| \geq \frac{1}{q*} \inf_{a\in\omega'} (f^{m-\nu}_{a,b})'(f^\nu_{a,b}(c)) \cdot |\xi_\nu(\omega', b)|. \tag{5.1}$$

and

$$(f^j_{a,b})'(f_{a,b}(c)) \geq \tilde{C}e^{\frac{\hat{\kappa}}{4}}, \qquad 1 \ \leq j \leq \nu - 1. \tag{5.2}$$

Then there is a constant C_{12} so that

$$\frac{(f^{m-\nu}_{a',b})'(f^\nu_{a',b}(c))}{(f^{m-\nu}_{a'',b})'(f^\nu_{a'',b}(c))} \leq \exp\left(C_{12}\frac{|f^m_{a',b}(c) - f^n_{a'',b}(c)|}{\delta}\right). \tag{5.3}$$

We will not give the proof of this lemma since it is identical to that of Lemma 4.12. We conclude that

$$|\xi_{m+i}(\tilde{\eta}_j, b)| \geq \frac{C_{14}}{q_*}\lambda_1^i \cdot |\xi_m(\tilde{\eta}_j, b)|$$

and q_* has a uniform control by Corollary 3.2.

We also get a uniform distorsion control of $\partial_a f^\nu_{a,b}(\xi_j(a,b))$, i.e. there is a a constant $\tilde{C}$ depending only on a_0 so that for all a, a' in $\tilde{\eta}_j$

$$\frac{1}{\tilde{C}} \leq \frac{\partial_a f^\nu_{a,b}(\xi_j(a,b))}{\partial_a f^\nu_{a',b}(\xi_j(a',b))} \leq \tilde{C}, \qquad \nu = 1, 2, \ldots, \tag{5.4}$$

as long as $\xi_{m+\nu}(\tilde{\eta}_j, b) \subset U_0$.

It follows that there is a first time L so that
$\xi_{m+L+1}(\tilde{\eta}_j, b) \not\subset U_0$. We write $\tilde{\eta}_j$ as the disjoint union (except for an endpoint)

$$\tilde{\eta}_j = \eta_j \cup {\eta'}_j.$$

By (5.4) it follows that $\xi_{m+L}(\eta_j, b)$ and $\xi_{m+L}(\eta'_j, b)$ are comparable withing a fixed constant C_{15}, which only depends on f_{a_0}.

We continue to iterate $\xi_{m+L+i}(\eta_j, b)$ for $i = 1, 2, \ldots$. By Lemma 4.9, Lemma 3.4 and the control of the constant q_* it follows that at the first time J such that $\xi_{m+L+J}(\eta_j, b) \cap I^* \neq \emptyset$

$$|\xi_{m+L+J}(\tilde{\eta}_j, b)| \geq \frac{C_{11}}{q_*}e^{\kappa_4 J}|\xi_{m+L}(\tilde{\eta}_j, b)|.$$

Then $\kappa_6 = \min(\log \lambda_1, \kappa_4)$ is the required Lyapunov exponent in (a). It follows by Lemma 4.9 (ii), Lemma 3.4 and the control of q_* that the time J will be finite. At time $N_0 = m + L + J$, we partition

$$(c - \delta, c + \delta) \cap \xi_{N_0}(\eta_j, b)$$

into preimages $\{\omega\}$ under the map $a \mapsto \xi_{N_0}(a,b)$ of the partition $\mathcal{Q} = \{I_{r,l}\}$ and define $n_0 = N_0$ for these ω:s. In the special case when $\xi_{N_0}(\eta_j, b)$ only intersects partially an end interval of $\mathcal{Q} = \{I_{r,l}\}$, we just keep iterating until we cover complete intervals of $\mathcal{Q}$. In other special case when $\xi_{N_0}(\eta_j, b)$ only partially covers a $I_{r,l}$ interval we adjoin the corresponding preimage to the adjacent interval. Simultaneously we delete the part of η_j that is mapped to $(c - e^{-\sqrt{n_0}}, c + e^{-\sqrt{n_0}})$. By the uniform distorsion of both the x-derivative and a-derivative which follows from Lemma 4.9, Lemma 4.12 and Corollary 3.2 a proportion of at most $C_{16}e^{-\sqrt{n_0}}/\delta$ of the piece of η_j mapped into $(c - e^{-\sqrt{n_0}}, c + e^{-\sqrt{n_0}})$. Here C_{16} is a constant only depending on f_{a_0}. We continue to iterate $\xi_{N_0}(\eta_j, b) \setminus (c - \delta, c + \delta)$, still using Lemma 4.9, Lemma 4.12 and Corollary 3.2. For the new returning interval ω formed in this way $n_0(\omega) > N_0$ and still only a quantity proportional to $e^{-\sqrt{n_0}}/\delta$ is deleted. The conclusions (a)–(e) of Lemma 5.1 are immediately verified. $\square$

Remark 5.3. The startup argument is essentially the same as the free period argument in the main induction and the argument in Lemma 4.9 in Section 4. See the main induction below in this section for a more thorough discussion. The only difference is the initial period that is spent close to the repelling periodic point which in some sense replaces the bound period. The expansive behaviour close to the repelling periodic point allows us to avoid inessential free returns and gives the initial exclusion ratio of at most $C_{16}e^{-\sqrt{n_0}}/\delta$.

Let us now fix b, $0 < b_0(\varepsilon) \le b < 1$. Note that for every positive integer n we have a family $\mathcal{P}_n$ of subintervals of $(a_0 - \varepsilon, a_0 + \varepsilon)$ (as in Lemma 5.1) with pairwise disjoint interiors, such that each element of $\mathcal{P}_{n+1}$ is contained in some element of $\mathcal{P}_n$. In the set of pairs (n, ω) such that $\omega \in \mathcal{P}_n$ there is a natural structure of a combinatorial tree, that goes down with its branches. Pairs (n, ω) are vertices of this tree; n is the level on which the vertex lies; there is an edge from (n, ω) to $(n+1, \omega')$ if and only if $\omega' \subset \omega$.

Certain pairs with the property $\xi_n(\omega, b) \subset I^*$ will be called *free return pairs.*

The induction will be separate on every branch of the tree. Fixing the branch results in considering a descending sequence of intervals $\omega_n \in \mathcal{P}_n$. If (n, ω_n) is a free return pair, then we will call n a free return time. An important feature of the construction is that if n is not a free return time then $\omega_n = \omega_{n-1}$. The main induction step will be from a free return time to the next free return time. The constants C_2 and C_1 are as in Lemma 5.1. In the whole induction they will stay the same.

Our **Induction Statement** is the following. If n is a free return time and and $a \in \omega$, then:

(i) we have

$$(f_{a,b}^{n-1})'(f_{a,b}(c)) \ge e^{2(n-1)^{2/3}}, \tag{5.5}$$

(ii) for every $\nu \in [n_0, n)$

$$(f_{a,b}^{\nu})'(f_{a,b}(c)) \ge e^{\nu^{2/3}}, \tag{5.6}$$

(iii) for every $\nu \in [1, n)$

$$(f_{a,b}^{\nu})'(f_{a,b}(c)) \ge C_2 e^{\nu^{2/3}}, \tag{5.7}$$

(iv) if $\nu < n$ is also a free return time, then

$$(f_{a,b}^{n-\nu})'(f_{a,b}^{\nu}(c)) \geq C(\delta) >> 1, \tag{5.8}$$

(v) for every $\nu \in [n_0, n]$

$$|\xi_\nu(a,b) - c| \geq e^{-\sqrt{\nu}}, \tag{5.9}$$

(vi) for every $\nu \in [0, n]$

$$|\xi_\nu(a,b) - c| \geq C_1 e^{-\sqrt{\nu}}, \tag{5.10}$$

In [2] and [3] statements (v) and (vi) is called the basic assumption (BA).

Remember that b sufficiently close to 1 is fixed. We set $\mathcal{P}_n = \{\omega_b\}$ for $n = 1, 2, \ldots, N_0$. Thus, this is the beginning of every branch. Then we declare $n_0 = n_0(\omega)$ to be the first free return time according to the startup construction. Thus, for every branch we have to start induction by checking that that the above conditions are satisfied for $n = n_0(\omega)$.

Lemma 5.4. *The Induction Statement conditions (i)-(vi) are satisfied for $n = n_0$.*

This is a consequence of the startup construction, Lemma 5.1.

Now we make a small modification of Definition 4.2.

Definition 5.5. Let a' be the midpoint of the interval ω such that

$$\xi_n(\omega, b) \subset I_{r,l}^{+} = I_{r,l-1} \cup I_{r,l} \cup I_{r,l+1}$$

for some n, r, l. We define the *bound period* as the maximal integer p so that for all $j \leq p$, $a \in \omega$, and $x \in \xi_n(\omega, b)$

$$|f_{a,b}^{j}(x) - f_{a',b}^{j}(c)| \leq e^{-4\sqrt{j}}. \tag{5.11}$$

By (4.3) and Lemma 3.1, we get for every n, a, b, x

$$\partial_a f_{a,b}^{n}(x) \leq \sum_{k=0}^{n-1} 4^k = \frac{4^n - 1}{3} < 4^n. \tag{5.12}$$

In the several next lemmas we will be using the same set of assumptions. We formalize these in the following definition

Definition 5.6. We say that *Condition (*)* is satisfied if

- ω, n, r, l, p and a' are as in Definition 5.5,
- conditions (iii), (v) and (vi) of the Induction Statement hold.

Next we formulate another version of the Bound Distorsion Lemma

Lemma 5.7. *There is a constant C_{17} such that if Condition (*) holds, then*

$$\frac{1}{C_{17}} \leq \frac{(f_{a,b}^{k})'(f_{a,b}(y))}{(f_{a,b}^{k})'(f_{a,b}(c))} \leq C_{17} \tag{5.13}$$

and

$$\frac{1}{C_{17}} \leq \frac{(f_{a,b}^{k})'(f_{a,b}(y))}{(f_{a',b}^{k})'(f_{a',b}(c))} \leq C_{17} \tag{5.14}$$

for every $x \in I_{r,l}$, y between x and c, $a \in \omega$, and $k \leq \max(p, n)$. By making δ sufficiently small, the constant $C_{17} = C_{17}(\delta) > 1$ can be chosen arbitrarily close to 1.

Proof. The proof will proceed by induction on k. Using (4.3), we get in the same way as in the proof of Lemma 4.3

$$\frac{(f_{a,b}^k)'(f_{a,b}(y))}{(f_{a,b}^k)'(f_{a,b}(c))} \leq \exp\left(\sum_{j=1}^{k} \frac{13|f_{a,b}^j(y) - f_{a,b}^j(c)|}{f'_{a,b}(f_{a,b}^j(c))}\right). \tag{5.15}$$

Furthermore,

$$|f_{a,b}^j(y) - f_{a,b}^j(c)| \leq |f_{a,b}^j(x) - f_{a,b}^j(c)| \leq |f_{a,b}^j(x) - f_{a',b}^j(c)| + |f_{a',b}^j(c) - f_{a,b}^j(c)|, \tag{5.16}$$

and by (5.11) we have

$$|f_{a,b}^j(x) - f_{a',b}^j(c)| \leq e^{-4\sqrt{j}}. \tag{5.17}$$

Thus, we need to estimate $|f_{a',b}^j(c) - f_{a,b}^j(c)|$. Note that by the mean value theorem there is a'' between a and a' so that

$$|f_{a',b}^j(c) - f_{a,b}^j(c)| = \partial_a f_{a'',b}^j(c) \cdot |a - a'|. \tag{5.18}$$

Note that $|a - a'|$ can be interpreted as $|\xi_1(a,b) - \xi_1(a',b)|$. By Lemma 3.4

$$|\xi_n(a,b) - \xi_n(a',b)| \geq \frac{1}{q_*} \inf_{\tilde{a}\in[a,a']} (f_{\tilde{a},b}^{n-1})'(f_{\tilde{a},b}(c)) \cdot |\xi_1(a,b) - \xi_1(a',b)|.$$

By the induction statement (iii), $(f_{\tilde{a},b}^{n-1})'(f_{\tilde{a},b}(c)) \geq C_2 e^{(n-1)^{2/3}}$. We may therefore conclude that $|a - a'| \leq C_2^{-1} q_* e^{-(n-1)^{2/3}} \cdot |\xi_n(a,b) - \xi_n(a',b)|$. But by the mean value theorem

$$|f_{a,b}^j(x) - f_{a',b}^j(c)| = |f_{a,b}^{j-1}(f_{a,b}(x)) - f_{a,b}^{j-1}(f_{a,b}(c))| = (f_{a,b}^{j-1})'(f_{a,b}(y)) \cdot |f_{a,b}(c) - f_{a,b}(x)|. \tag{5.19}$$

However since $|\xi_n(a,b) - \xi_n(a',b)| \leq e^{-r}$, we have

$$|f_{a,b}(c) - f_{a,b}(x)| \geq C_7 \cdot |x - c|^3 \geq C_7 e^{2(-r-1)} \cdot e^{-1} \cdot |\xi_n(a,b) - \xi_n(a',b)|. \tag{5.20}$$

By the basic assumption $e^{-r} \geq C_1 e^{-\sqrt{n}}$. Note also that by Corollary 3.2, $\partial_a f_{a'',b}^j(c)$ is comparable within the multiplicative constant q_* to $(f_{a'',b}^{j-1})'(f_{a'',b}(c))$. But this quantity is in itself by induction comparable within a multiplicative constant C_{17} to $\inf_{\tilde{a}\in[a,a']}(f_{\tilde{a},b}^{n-1})'(f_{\tilde{a},b}(c))$. We use the statement of our result for $k = j - 1$.

We use (5.18) and note that $|a - a'|$ also can be written as $|\xi(a,b) - \xi_1(a',b)|$. By Lemma 3.4

$$|\xi_n(a,b) - \xi_n(a'b)| \geq \frac{1}{q_*} \inf_{\tilde{a}\in[a,a']} (f^{n-1})'(f_{\tilde{a},b}(c)) \tag{5.21}$$

By combining (5.19),(5.20) and (5.21), we obtain

$$\begin{aligned}|f^j_{a,b}(x) - f^j_{a,b}(c)| &= |f^{j-1}_{a,b}(f_{a,b}(x)) - f^{j-1}_{a,b}(f_{a,b}(c))| \geq (f^{j-1})'(f_{a,b}(y))|f_{a,b}(x) - f_{a,b}(c)| \\ &\geq \inf_{y\in I_r}(f^{j-1}_{a,b})'(f_{a,b}(y))\,|f_{a,b}(x) - f_{a,b}(c)| \\ &\geq C_7 e^{-3r}\inf_{y\in I_r}(f^{j-1}_{a,b})'(f_{a,b}(y)) \\ &\geq C_7 e^{-2r}\inf_{y\in I_r}(f^{j-1}_{a,b})'(f_{a,b}(y))|\xi_n(a,b) - \xi_n(a',b)| \\ &\geq C_7 e^{-2r}\inf_{y\in I_r}(f^{j-1}_{a,b})'(f_{a,b}(y))(\inf_{\tilde{a}\in[a,a']}\partial_a f^n_{a'',b}(c))|\xi_1(a,b) - \xi_1(a',b)| \\ &\geq C_7 e^{-2r}\frac{\inf_{y\in I_r}(f^{j-1}_{a,b})'(f_{a,b}(y))}{\sup_{z\in I_r}(f^{j-1}_{a,b})'(f_{a,b}(z))}(\inf_{\tilde{a}\in[a,a']}\partial_a f^n_{a'',b}(c))|)\,|\xi_j(a,b) - \xi_j(a',b)|.\end{aligned}$$

Now $\sup_{z\in I_r}(f^{j-1}_{a,b})'(f_{a,b}(z))$ and $\inf_{y\in I_r}(f^{j-1}_{a,b})'(f_{a,b}(y))$ are comparable with constant C_{17}^2, by the statement of Lemma 5.7 with $k = j-1$. This is where the inductive step is used. Moreover by Corollary 3.2 and the induction statement (i), we have $\partial_a f^n_{a,b}(c) \geq q_*^{-1}e^{2(n-1)^{2/3}}$. Furthermore $e^{-2r} \geq e^{-2\sqrt{n}}$, by the induction statement (v), (5.9). Combining these estimates we get

$$|\xi_j(a,b) - \xi_j(a',b)| \leq \frac{1}{2}|f^j_{a,b}(x) - f^j_{a,b}(c)|$$

since $n \geq n_0(a)$ and $n_0(a)$ can be choosen arbitrarily large.

When inserting this estimate in (5.16) we conclude that

$$|f^j_{a,b}(y) - f^j_{a,b}(c)| \leq 2e^{-4\sqrt{j}}. \tag{5.22}$$

To estimate $f'_{a,b}(f^j_{a,b}(c))$ from below, we use (vi) of the Induction Statement and (4.13). We get

$$f'_{a,b}(f^j_{a,b}(c)) \geq C_9 C_1^2 e^{-2\sqrt{j}}. \tag{5.23}$$

Putting together (5.15), (5.22) and (5.23), we obtain

$$\frac{(f^k_{a,b})'(f_{a,b}(y))}{(f^k_{a,b})'(f_{a,b}(c))} < \exp\left(\frac{13\cdot 2}{C_9\cdot C_1^2}\sum_{j=1}^{k}\frac{e^{-4\sqrt{j}}}{e^{-2\sqrt{j}}}\right). \tag{5.24}$$

For the lower bound, we obtain in a similar way as (5.15)

$$\frac{(f^k_{a,b})'(f_{a,b}(c))}{(f^k_{a,b})'(f_{a,b}(y))} \leq \exp\left(\sum_{j=1}^{k}\frac{13|f^j_{a,b}(y) - f^j_{a,b}(c)|}{(f_{a,b})'(f^j_{a,b}(y))}\right).$$

Note however that

$$(f_{a,b})'(f^j_{a,b}(y)) \geq C_9(f^j_{a,b}(y) - c)^2 \geq C_9\left(|c - f^j_{a,b}(c)| - |f^j_{a,b}(c) - f^j_{a,b}(y)|\right)^2,$$

and using (5.9) and (5.22) we get

$$f'_{a,b}(f^j_{a,b}(y)) > C_9(C_1 e^{-\sqrt{j}} - 2e^{-4\sqrt{j}})^2$$

whenever $C_1 e^{-\sqrt{j}} > 2e^{-4\sqrt{j}}$. Now, C_1 is fixed and thus, there is a positive integer $\tilde{N}$ such that if $j \geq \tilde{N}$ then $C_1 e^{-\sqrt{j}} > 2 \times 2e^{-4\sqrt{j}}$, and then

$$f'_{a,b}(f^j_{a,b}(y)) > \frac{C_9 C_1^2}{4} e^{-2\sqrt{j}}.$$

By making $\mathcal{N}$ and I^* sufficiently small, we can make $|f^j_{a,b}(c) - f^j_{a,b}(y)|$ smaller than $C_1 e^{-4\sqrt{j}}$ instead of $2e^{-4\sqrt{j}}$, and then we get

$$f'_{a,b}(f^j_{a,b}(x)) \geq C_9 C_1^2 K_6 e^{-2\sqrt{j}}$$

for some constant K_6 depending only of a_0. In such a way, in the same way as we obtained (5.24), we get a similar estimate for the reciprocal ratio, but with a different constant. We choose then as C_{17} the larger of those constants and we get (5.13). The proof of (5.14) is completely analogous and will be omitted. As for the statement that C_{17} can be chosen arbitrarily close to 1 (but larger than 1), we refer to the argument in Lemma 4.3. □

Lemma 5.8. *Assume that Condition (*) holds. Then the bound period p in the sense of Definition 5.5 satisfies*

$$p \leq 8r^{3/2}.$$

Proof. We claim that $p \leq 8r^{3/2}$. Note that $\leq 8n^{3/4} < n$. We argue by contradiction. Assume that there is $k > 8r^{3/2}$ so that $f^j_{a,b}(f_{a.b}(z)$ is still bound to $f^j_{a,b}(f_{a.b}(c)$ for all $x \in I_{r,l}$ and all z between x and c.. By the Mean Value Theorem, there is a point y between x and c such that

$$|f^k_{a,b}(x) - f^k_{a,b}(c)| = |f_{a,b}(x) - f_{a,b}(c)| \cdot (f^{k-1}_{a,b})'(f_{a,b}(y)).$$

By Lemma 5.7 and (iii) of the Induction Statement,

$$(f^{k-1}_{a,b})'(f_{a,b}(y)) \geq \frac{1}{C_{17}} (f^{k-1}_{a,b})'(f_{a,b}(c)) \geq \frac{C_2}{C_{17}} e^{(k-1)^{2/3}}.$$

Since $|x - c| \geq e^{-r-1}$, by (4.13) we get

$$|f_{a,b}(x) - f_{a,b}(c)| \geq \frac{C_9}{3} e^{-3r-3}.$$

Putting the last three inequalities together, we get

$$|f^k_{a,b}(x) - f^k_{a,b}(c)| \geq \frac{C_2}{C_{17}} e^{(k-1)^{2/3}} \cdot \frac{C_9}{3} e^{-3r-3}.$$

Taking into account (5.22) (which is valid also for $y = x$), we get

$$2 > 2e^{-\sqrt{k}} > \frac{C_2 C_9}{3C_{17}} e^{(k-1)^{2/3}} e^{-3r-3}.$$

Therefore,

$$(k-1)^{2/3} < 3r - \log K_7$$

where K_7 is a constant only depending on a_0.

If δ is sufficiently small, then $\frac{1}{2} r > -\log K_7$, and we get $k^{2/3} < (k-1)^{2/3} + \frac{2}{3} k^{-1/3} < 4r$.We conclude that $k \leq 8r^{3/2}$ and this gives a contradiction.

□

Let us prove an elementary lemma about our family.

Lemma 5.9. *For the family of double standard maps, if* $0 < |x - c| < 1/2$ *then*

$$f'_{a,b}(x) > \frac{|f_{a,b}(x) - f_{a,b}(c)|}{|x - c|}. \tag{5.25}$$

Proof. We have $c = 1/2$ and $f''_{a,b}(t) = -4\pi b \sin(2\pi t)$. Therefore $f_{a,b}$ is strictly convex in $(c, c + 1/2)$, and thus for $x \in (c, c + 1/2)$ we get (5.25). Similarly, in $(c - 1/2, c)$ the function $f_{a,b}$ is strictly concave, and (5.25) follows similarly. □

Lemma 5.10. *There exists a positive constant* C_{18} *such that if Condition (*) holds, then*

$$(f^{p+1}_{a,b})'(x) > C_{18} e^r \cdot e^{-4\sqrt{p}}. \tag{5.26}$$

Proof. By Definition 5.5, there exists $a \in \omega$ such that

$$|f^{p+1}_{a,b}(x) - f^{p+1}_{a',b}(c)| \geq e^{-4\sqrt{p+1}}. \tag{5.27}$$

We have

$$|f^{p+1}_{a,b}(x) - f^{p+1}_{a',b}(c)| \leq |f^{p+1}_{a,b}(x) - f^{p+1}_{a,b}(c)| + |f^{p+1}_{a,b}(c) - f^{p+1}_{a',b}(c)|. \tag{5.28}$$

By the Mean Value Theorem, there is a point y between x and c such that

$$|f^{p+1}_{a,b}(x) - f^{p+1}_{a,b}(c)| = |f_{a,b}(x) - f_{a,b}(c)| \cdot (f^p_{a,b})'(f_{a,b}(y)). \tag{5.29}$$

Now we estimate the second summand in (5.28). As in the proof of Lemma 5.7, we can prove that

$$|f^{p+1}_{a,b}(c) - f^{p+1}_{a',b}(c)| \leq \frac{1}{2}|f^{p+1}_{a',b}(c) - f^{p+1}_{a,b}(x)|.$$

Therefore

$$|f^{p+1}_{a,b}(x) - f^{p+1}_{a,b}(c)| \geq \frac{1}{2}|f^{p+1}_{a,b}(x) - f^{p+1}_{a',b}(c)| \geq \frac{1}{2}e^{-4\sqrt{p+1}}.$$

From Lemma 5.7 we get

$$(f^p_{a,b})'(f_{a,b}(x)) \geq \frac{1}{C_{17}^2}(f^p_{a,b})'(f_{a,b}(y)),$$

so

$$(f^p_{a,b})'(f_{a,b}(x)) > \frac{1}{2C_{17}^2} \cdot e^{-4\sqrt{p+1}} \cdot \frac{1}{|f_{a,b}(x) - f_{a,b}(c)|}.$$

By the Chain Rule and Lemma 5.9 we get from this inequality

$$(f^{p+1}_{a,b})'(x) > \frac{1}{2C_{17}^2} \cdot e^{-4\sqrt{p+1}} \cdot \frac{1}{|x - c|}.$$

Since $x \in I_r$, we have $|x - c| \leq e^{-r}$, and we get (5.26) with a suitable choice of C_{18}. □

Let (n, ω) be a free return pair. Consider the intervals $\xi_{n+p+1+s}(\omega, b)$, $s = 0, \dots, s_0 - 1$, where s_0 is the smallest nonnegative integer such that

$$\xi_{n+p+1+s_0}(\omega, b) \cap I^* \neq \varnothing.$$

For $0 \leq s < s_0$, we say that $\xi_{n+p+1+s}(\omega, b)$ is in free orbit and the length of this orbit is s_0. We also use the notation $n' = n + p + 1 + s_0$ and it is our new free return time.

At the first free return there are different cases that can occur.

Case 1. $\Omega_{n'} = \xi_{n'}(\omega, b)$ is completely contained in I^* but does not contain a complete interval $I_{r,\ell}$. Then either $\xi_{n+p+1+s_0}(\omega, b)$ is contained in an interval $I_{r,l}$ or it is contained in the union of two adjacent intervals $I_{r,l} \cup I_{r,l+1}$.

This is called an *inessential free return.* In this case $\omega \in \mathcal{P}_{n'}$, and we just continue to iterate. This also applies if $\Omega_{n'}$ intersects the boundary of I^* but does not contain any of the end intervals.

Case 2. $\Omega_{n'}$ contains at least one of the partition intervals $I_{r,\ell}$. This is the case of an *essential free return.* We then proceed to define a new partition on a subset of ω according to the following algorithm.

- We do not include the preimage of $(c - e^{-\sqrt{n'}}, c + e^{-\sqrt{n'}})$ under $a \mapsto \xi_{n'}(a, b)$ in $\bigcup_{\omega' \in \mathcal{P}_{n'}} \omega'$, in order that (BA) should be satisfied.
- The intervals $\omega_{r,\ell}$ and $\omega'_{r,\ell}$ are defined as the preimages of $I_{r,\ell}$ under $\omega \ni a \mapsto \xi_{n'}(a, b)$. Because of the double covering property of $f_{a,b}$, there could be 0,1 or 2 such intervals. These will be new partition intervals of $\mathcal{P}_{n'}$. At the two ends of ω we could have the property that some intervals only partially cover $I_{r,\ell}$. In that case we use the special rule that we adjoin the corresponding subintervals to the adjacent intervals of $\mathcal{P}_{n'}$.
- There may be at most three subintervals of ω, call them ω_1, ω_2 and ω_3 which are mapped outside I^* by $\omega \ni a \mapsto \xi_{n'}(a, b)$. In the beginning of the procedure there are at most two intervals mapped outside, but in later stages because of the double covering property of $f_{a,b}$, there can be three. In this case these intervals are *long*, i.e they are not contained in intervals adjacent to the end intervals in the partition of $(c - e^{-r_\delta+1}, c + e^{-r_\delta+1})$, they are considered to be still free, and the free period continues for these intervals. If one or more of the intervals ω_1, ω_2 or ω_3 are *short*, i.e not long, they are adjoined to their adjacent neighbor.

Let X_{BA} be the set that is mapped to $(c - e^{-\sqrt{n'}}, c + e^{-\sqrt{n'}})$. Then we define the partition $\mathcal{P}_{n'}|(\omega \setminus X_{\mathrm{BA}})$ as the intervals $\{\omega_{r,\ell}\}$, $\{\omega'_{r,\ell}\}$ and ω_i, $i = 1, 2, 3$. Some of these intervals may be empty.

Later we will see that deletions because of (BA) do not happen in Case 1, because the interval $\Omega_{n'}$ is too long.

In order to proceed, we need to verify, at least partially, the induction step from time n to time n'. Here n' is interpreted as the first free return to I^* after n. There may be previous returns ν, where another partition element of $\mathcal{P}_\nu$ has a free return, while the present parameter interval does not return.

Lemma 5.11. *Assume the Induction Statement (i)-(vi). Then the Induction Statement conditions (i),(ii), (iii) and (iv) hold for any free return pair (n', ω'), where n' is as above.*

Proof. Let η be the distance from $\xi_n(\omega, b)$ to c. Therefore, by Induction Statement (i) and (4.13),

$$(f^n_{a,b})'(f_{a,b}(c)) > C_9 \eta^2 e^{2(n-1)^{2/3}}.$$

However, by (v), $\eta \geq C_1 e^{-\sqrt{n}}$, so we get

$$(f_{a,b}^n)'(f_{a,b}(c)) > C_9 C_1^2 e^{2(n-1)^{2/3}} e^{-2\sqrt{n}}. \tag{5.30}$$

After time n there follows the bound period p, and by Lemma 5.7 and (iii) we get

$$(f_{a,b}^k)'(f_{a,b}^{n+1}(c)) \geq C_{17}^{-1}(f_{a,b}^k)'(f_{a,b}(c)) \geq C_{17}^{-1} C_2 e^{k^{2/3}} \tag{5.31}$$

for all $k \leq p$. Combining (5.30) and (5.31), we conclude that

$$(f_{a,b}^{n+k})'(f_{a,b}(c)) = (f_{a,b}^n)'(f_{a,b}(c)) \cdot (f_{a,b}^k)'(f_{a,b}^{n+1}(c)) > C_9 C_1^2 C_{17}^{-1} C_2 e^{2(n-1)^{2/3} - 4\sqrt{n} + k^{2/3}}. \tag{5.32}$$

For $k \leq p \leq 8r^{3/2} \leq 8n^{3/4}$, we conclude that

$$(f_{a,b}^{n+k})'(f_{a,b}(c)) \geq e^{(n+k)^{2/3}}$$

At time $n+p+1$ the bound period has expired and we have for all $a \in \omega$

$$(f_{a,b}^p)'(f_{a,b}^n(c)) e^{-3r} \geq C_{19} e^{-4\sqrt{p+1}}, \tag{5.33}$$

where C_{19} is a constant only depending on f_{a_0}.

For the total derivative we obtain

$$(f^{n+p})'(f_{a,b}^n(c)) \geq C_{19} e^{2n^{2/3}} e^{-2r} (f_{a,b}^p)'(f_{a,b}^{n-1}(c)) \tag{5.34}$$

After raising (5.33) to the power $\frac{2}{3}$ we obtain

$$(f^{n+p})'(f_{a,b}^n(c)) \geq C_{19}^{2/3} e^{2n^{2/3}} (f_{a,b}^p)'(f_{a,b}^n(c))^{1/3} e^{-\frac{8}{3}\sqrt{p+1}} \tag{5.35}$$

Looking at the exponents, we get using (5.7) the lower bound

$$2n^{2/3} + \frac{1}{3}p^{2/3} - \frac{8}{3}\sqrt{p+1} \geq 2(n+p)^{2/3} + \frac{1}{10}p^{2/3}$$

Here we have used the information from Lemma 5.8, $p \leq 8n^{3/4}$ and that if $p \leq \frac{1}{100}n$

$$2n^{2/3} + \frac{1}{3}p^{2/3} \geq 2(n+p)^{2/3} + \frac{1}{5}p^{2/3}.$$

Now, if $k = p + s$ and $0 < s \leq s_0$, then we can use Lemma 4.9 (ii). If $s \geq M$ then

$$(f_{a,b}^s)'(f_{a,b}^{n+p+1}(c)) \geq e^{\kappa_4 s}.$$

If $s < M$ we use Lemma 4.5, which allows a perturbation to $(a,b) \in \mathcal{N}$ with a worse constant $\overline{d}/4$ instead of $\overline{d}/2$ and we get

$$(f_{a,b}^s)'(f_{a,b}^{n+p+1}(c)) \geq \frac{\overline{d}}{4}.$$

Thus, independently whether $s \geq M$ or $s < M$, we have

$$(f_{a,b}^s)'(f_{a,b}^{n+p+1}(c)) \geq \frac{\overline{d}}{4} e^{(s-M)\kappa_4}.$$

Together with (5.32) (where we substitute $k = p + s$), we get

$$\begin{aligned}(f_{a,b}^{n+k})'(f_{a,b}(c)) &= (f_{a,b}^{n+p})'(f_{a,b}(c)) \cdot (f_{a,b}^s)'(f_{a,b}^{n+p+1}(c)) \\ &> C_9 C_1^2 C_{17}^{-1} C_2 \frac{\overline{d}}{4} e^{2(n-1)^{2/3} - 4\sqrt{n} + \frac{1}{10}p^{2/3} + (s-M)\kappa_4}.\end{aligned}$$

Note that since the constants C_3, C_1, C_{17}, C_2 are absolute constants and p can be made arbitrarily large by making δ sufficiently small. Doing this, we conclude that (ii) and (iii) of the induction statement holds for ν satisfying $n+p<\nu<n'$ where n' is the next free return time.

We now turn to verifying (i) at the next free return time n'. Using the previous derivative estimates and (i) of Lemma 4.9 we get after writing $n'=n+p+1+q$ that

$$(f_{a,b}^{n'-1})'(f_{a,b})(c) \geq e^{2(n-1)^{2/3}+\frac{1}{10}p^{2/3}-\frac{8}{3}\sqrt{p+1}}C_{11}e^{\kappa_4(q-1)},$$

where C_{11} is an absolute constant only depending on a_0. Arguing in different cases depending on the relative sizes of n and q, one can verify that (i) of the induction with n replaced by n' holds.

Since C_{11} and C_{18} do not depend on δ, while by making δ sufficiently small we can make p as large as we want, we conclude using Lemma 5.8 that

$$C_{11}C_{18}e^r \cdot e^{-\sqrt{p+1}} \geq C_{11}C_{18}\exp\{\frac{1}{2}p^{2/3}-4\sqrt{p+1}\} \geq 1.$$

This proves (iv) for n'. □

We now delete the parameters which are mapped to

$$(c-{C_2}^*e^{-\sqrt{n'}}, c+{C_2}^*e^{-\sqrt{n'}}),$$

where ${C_2}^*=\max(C_2,1)$. We conclude that (v) and (vi) of the induction also hold. This completes the proof of the induction step.

We note that the proof also gives the information

$$(f_{a,b}^{n'-n})'(f_{a,b}^{n}(c)) \geq e^{\frac{1}{6}(n'-n)^{2/3}}. \tag{5.36}$$

Remark 5.12. From (5.36) it immediately follows that

$$(f_{a,b}^{n'-1})'(f_{a,b}(c)) \geq (f_{a,b}^{n-1})'(f_{a,b}(c)) \cdot e^{\frac{1}{6}p^3}, \tag{5.37}$$

which will be used later. We will also later use (5.36).

Remark 5.13. Clearly, in Lemma 3.4, ω can be replaced by any subinterval $\omega' \subset \omega$. If we choose $\mu=n'$ and if $\xi_{n'}(\omega',b) \subset I^*$, and $n+p+1 \leq \nu < \mu = n'$, we can use Lemma 4.9 (i) to estimate $\inf_{a\in\omega}(f_{a,b}^{n'-\nu})'(f_{a,b}^{\nu}(c))$ from below by $C_{11}e^{\kappa_4(n'-\nu)}$. Moreover by Lemma 5.11 we know that (iii) of the induction holds for $\nu<n'$. Therefore we conclude from (3.7) that (3.8) holds with $q'=q_*$. In such a way we get

$$|\xi_{n'}(\omega',b)| \geq \frac{C_{11}}{q_*}e^{\kappa_4(n'-\nu)}|\xi_\nu(\omega',b)|.$$

6. The global distortion Lemma

Lemma 6.1. *There exists a constant C_{13}, such that if a and a' are two parameter points, so that $a,a' \in \omega \in \mathcal{P}_n$, where n is a free return time and Induction Statement for n (and all smaller free return times) holds, then*

$$\frac{(f_{a,b}^{k})'(f_{a,b}(c))}{(f_{a',b}^{k})'(f_{a',b}(c))} \leq C_{13} \tag{6.1}$$

for all $k \leq n-1$.

Proof. Let us fix $k \le n-1$. Set $t_0 = 1$ and let $\{t_j\}_{j=1}^m$ be the free return times arranged in an increasing order. Here m is defined by the condition $t_{m-1} < k \le t_m - 1$, and we can assume that $t_m = n$. Observe that for all free returns t_j there is r_j so that $\xi_{t_j}(a,b), \xi_{t_j}(a',b) \in I_{r_j}$.

Note that $f'_{a,b}(x) = f'_{a',b}(x)$. Thus, using the Mean Value Theorem, we can write the logarithm of the left hand side of (6.1) as

$$\log \frac{(f^k_{a,b})'(f_{a,b}(c))}{(f^k_{a',b})'(f_{a',b}(c))} = \log \frac{\prod_{j=1}^k f'_{a,b}(\xi_j(a,b))}{\prod_{j=1}^k f'_{a,b}(\xi_j(a',b))}$$
$$\le \sum_{i=0}^{m-1} \left(\sum_{j=t_i}^{t_i+p_i} \left| \frac{f''_{a,b}(\eta_j)}{f'_{a,b}(\eta_j)} \right| |\xi_j(a,b) - \xi_j(a',b)| + \log \frac{(f^{t_{i+1}-t_i-p_i-1}_{a,b})'(\xi_{t_i+p_i+1}(a,b))}{(f^{t_{i+1}-t_i-p_i-1}_{a',b})'(\xi_{t_i+p_i+1}(a',b))} \right)$$

for some η_j between $\xi_j(a,b)$ and $\xi_j(a',b)$, where p_i is the corresponding bound time and $p_0 = -1$. We will denote the first sum in the parenthesis above by S'_i and the second term in parenthesis by S''_i. Note that the sum S'_0 is empty.

By (4.13) and (4.14) we get

$$\left| \frac{f''_{a,b}(\eta_j)}{f'_{a,b}(\eta_j)} \right| \le \frac{80}{C_9|\eta_j - c|}. \tag{6.2}$$

Set $\sigma_i = |\xi_{t_i}(a,b) - \xi_{t_i}(a',b)|$. We claim that the sum S'_i can be estimated from above by a constant times $\sigma_i e^{r_i}$.

First we note that by (6.2), the first term of S'_i can be estimated by $80\sigma_i/(C_9 e^{-r_i})$.

For the remaining terms we introduce the reference interval $\Omega_{t_i} = I_{r_i+1}$ and intervals $\Omega_{t_i+\nu} = f^\nu_{a,b}(\Omega_{t_i})$, $\nu = 0,1,2,\dots,p_i$. We have

$$\begin{aligned}\xi_{t_i+1}(a,b) - \xi_{t_i+1}(a',b) &= (f_{a,b}(\xi_{t_i}(a,b)) - f_{a,b}(\xi_{t_i}(a',b)) \\ &\quad + (f_{a,b}(\xi_{t_i}(a',b) - f_{a',b}(\xi_{t_i}(a',b)) \\ &= f'_{a,b}(y)(\xi_{t_i}(a,b) - \xi_{t_i}(a',b)) + (a-a')\end{aligned}$$

for some y between $\xi_{t_i}(a,b)$ and $\xi_{t_i}(a',b)$. Furthermore, $|\Omega_{t_i+1}| = f'_{a,b}(y')|\Omega_{t_i}|$ for some $y' \in \Omega_{t_i}$.

We get

$$\frac{|\xi_{t_i+1}(a,b) - \xi_{t_i+1}(a',b)|}{|\Omega_{t_i+1}|} = \frac{f'_{a,b}(y)}{f'_{a,b}(y')} \cdot \frac{\sigma_i}{|\Omega_{t_i}|} \pm \frac{|a-a'|}{f'_{a,b}(y')|\Omega_{t_i}|}. \tag{6.3}$$

By the Mean Value Theorem

$$\frac{|a-a'|}{f'_{a,b}(y)\sigma_i} = \frac{1}{f'_{a,b}(y)\partial_a \xi_{t_i}(a'',b)}$$

for some a'' between a and a'. By (3.5) and the induction statement (iii),

$$\partial_a \xi_{t_i}(a'',b) \ge (f^{t_i-1}_{a'',b})'(f_{a'',b}(c)) \ge C_2 e^{\sqrt{t_i-1}}.$$

By (4.13) and the induction statement (v),

$$f'_{a,b}(y) \ge C_9 C_1^2 e^{-2\sqrt{t_i}}.$$

Therefore we get

$$\frac{|a-a'|}{f'_{a,b}(y)\sigma_i} \leq \frac{e}{C_9 C_1^2 C_2} e^{(2\sqrt{t_i}-(t_i)^{2/3})}.$$

Since t_i can be made as large as we want (because $t_i \geq n_0$), we get $|a-a'| < f'_{a,b}(y)\sigma_i/2$. Therefore from (6.3) we get

$$\frac{1}{2}\frac{f'_{a,b}(y)}{f'_{a,b}(y')} \cdot \frac{\sigma_i}{|\Omega_{t_i}|} < \frac{|\xi_{t_i+1}(a,b) - \xi_{t_i+1}(a',b)|}{|\Omega_{t_i+1}|} < 2\frac{f'_{a,b}(y)}{f'_{a,b}(y')} \cdot \frac{\sigma_i}{|\Omega_{t_i}|}.$$

Since $y, y' \in I_{r_i} \cup I_{r_i+1}$, we have $|y-c| \in [e^{-r_i-2}, e^{-r_i}]$, and the same holds for y'. Therefore, by (4.13), we get

$$\frac{f'_{a,b}(y)}{f'_{a,b}(y')} \leq \frac{2-2b+C_{10}e^{-2r}}{2-2b+C_9 e^{-2r-4}}.$$

The right-hand side above is a weighted average between $(2-2b)/(2-2b) = 1$ and $(C_{10}e^{-2r})/(C_9 e^{-2r-4}) = C_{10}e^4/C_9 > 1$, so it is smaller than $C_{10}e^4/C_9$. Since we can switch y and y', we get

$$\frac{C_9}{C_{10}e^4} < \frac{f'_{a,b}(y)}{f'_{a,b}(y')} < \frac{C_{10}e^4}{C_9}.$$

In such a way we get the following inequality with $C_{20} = 2C_{10}e^4/C_9$.

$$C_{20}^{-1}\frac{\sigma_i}{|\Omega_{t_i}|} \leq \frac{|\xi_{t_i+1}(a,b) - \xi_{t_i+1}(a',b)|}{|\Omega_{t_i+1}|} \leq C_{20}\frac{\sigma_i}{|\Omega_{t_i}|}. \tag{6.4}$$

Now we want to estimate from above

$$\frac{|\xi_{t_i+\nu}(a,b) - \xi_{t_i+\nu}(a',b)|}{|\Omega_{t_i+\nu}|}.$$

The numerator can be estimated as follows:

$$\begin{aligned} |\xi_{t_i+\nu}(a,b) - \xi_{t_i+\nu}(a',b)| &\leq |f_{a,b}^{\nu-1}(f_{a,b}^{t_i+1}(c)) - f_{a,b}^{\nu-1}(f_{a',b}^{t_i+1}(c))| \\ &\quad + |f_{a,b}^{\nu}(f_{a',b}^{t_i}(c)) - f_{a',b}^{\nu}(f_{a',b}^{t_i}(c))|. \end{aligned} \tag{6.5}$$

By a similar argument as in the proof of Lemma 5.8, in particular estimating $|a-a'|$ as in that lemma we obtain that

$$|f_{a,b}^{\nu}(f_{a',b}^{t_i}(c)) - f_{a',b}^{\nu}(f_{a',b}^{t_i}(c))| \leq \frac{1}{2}|\xi_{t_i+\nu}(a,b) - \xi_{t_i+\nu}(a',b)|.$$

Therefore we get the estimate

$$\begin{aligned} |\xi_{t_i+\nu}(a,b) - \xi_{t_i+\nu}(a',b)| &\leq 2|f_{a,b}^{\nu-1}(f_{a,b}^{t_i+1}(c)) - f_{a,b}^{\nu-1}(f_{a',b}^{t_i+1}(c))| \\ &= 2(f_{a,b}^{\nu-1})'(y)|\xi_{t_i+1}(a,b) - \xi_{t_i+1}(a',b)|, \end{aligned} \tag{6.6}$$

where y is between $\xi_{t_i+1}(a,b)$ and $\xi_{t_i+1}(a',b)$ (using the Mean Value Theorem). Again by the same theorem there is $y'' \in \Omega_{t_i+1}$ such that

$$|\Omega_{t_i+\nu}| = (f_{a,b}^{\nu-1})'(y'')|\Omega_{t_i+1}|.$$

By Lemma 5.7, $(f_{a,b}^{\nu-1})'(y)/(f_{a,b}^{\nu-1})'(y'') \leq C_{17}^2$, so we get

$$\frac{|\xi_{t_i+\nu}(a,b) - \xi_{t_i+\nu}(a',b)|}{|\Omega_{t_i+\nu}|} \leq 2C_{17}^2\frac{|\xi_{t_i+1}(a,b) - \xi_{t_i+1}(a',b)|}{|\Omega_{t_i+1}|}. \tag{6.7}$$

Let us consider the interval $\omega_i \in \mathcal{P}_{t_i}$ containing ω and denote its midpoint by $\hat{a}_i$. Then by (5.11),

$$|\xi_{t_i+\nu}(\tilde{a}, b) - \xi_\nu(\hat{a}_i, b)| \le e^{-4\sqrt{\nu}} \tag{6.8}$$

for all $\tilde{a} \in \omega_i$ and $\nu \le p_i$.

We claim that for all $\tilde{a} \in \omega_i$ and $\nu \le p_i$ we have

$$|\xi_{t_i+\nu}(\tilde{a}, b) - c| \ge \frac{1}{2}|\xi_\nu(\hat{a}_i, b) - c|. \tag{6.9}$$

There is an integer ν_0 such that

$$\frac{1}{2} \cdot C_1 e^{-\sqrt{\nu}} \ge e^{-4\sqrt{\nu}} \tag{6.10}$$

for all $\nu \ge \nu_0$. Note that ν_0 depends only on C_1, which is independent of δ. Therefore, we may assume that δ is so small that

$$2\delta \cdot 4^{\nu_0} \le \frac{1}{2} \cdot C_1 e^{-\sqrt{\nu_0}}. \tag{6.11}$$

Moreover, by Induction Statement (v),

$$|\xi_\nu(\hat{a}_i, b) - c| \ge C_1 e^{-\sqrt{\nu}}. \tag{6.12}$$

Consider $\nu \le p_i$. If $\nu \ge \nu_0$, then by (6.8), (6.10) and (6.12), we get

$$|\xi_{t_i+\nu}(\tilde{a}, b) - \xi_\nu(\hat{a}_i, b)| \le \frac{1}{2}|\xi_\nu(\hat{a}_i, b) - c|. \tag{6.13}$$

Therefore

$$|\xi_{t_i+\nu}(\tilde{a}, b) - c| \ge |\xi_\nu(\hat{a}_i, b) - c| - |\xi_{t_i+\nu}(\tilde{a}, b) - \xi_\nu(\hat{a}_i, b)| \ge \frac{1}{2}|\xi_\nu(\hat{a}_i, b) - c|$$

and (6.9) follows.

If $\nu < \nu_0$ then by (4.3) and (5.12)

$$\begin{aligned} |\xi_{t_i+\nu}(\tilde{a}, b) - \xi_\nu(\hat{a}_i, b)| &\le |\xi_{t_i+\nu}(\tilde{a}, b) - \xi_\nu(\tilde{a}, b)| + |\xi_\nu(\tilde{a}, b) - \xi_\nu(\hat{a}_i, b)| \\ &\le 4^\nu|\xi_{t_i}(\tilde{a}, b) - c| + 4^\nu|\tilde{a} - \hat{a}_i| \le 4^\nu(\delta + |\omega_i|). \end{aligned}$$

By Lemma 5.1 (b) for $j = n_0$ and by making n_0 sufficiently large, we get $\partial_a \xi_j(a, b) \ge 1$. Therefore

$$|\omega| \le |\xi_{n_0}(\omega, b)| \le \delta.$$

Thus, $|\omega_i| \le \delta$, and we get

$$|\xi_{t_i+\nu}(\tilde{a}, b) - \xi_\nu(\hat{a}_i, b)| \le 2\delta \cdot 4^\nu.$$

Together with (6.11) and (6.12) we get also in this case (6.13), and (6.9) follows.

Now for each ν we choose $\tilde{a}_\nu$ between a and a', so that $\xi_{t_i+\nu}(\tilde{a}_\nu, b) = \eta_{t_i+\nu}$. Thus, by (6.9) and (6.12),

$$|\eta_{t_i+\nu} - c| = |\xi_{t_i+\nu}(\tilde{a}_\nu) - c| \ge \frac{1}{2}|\xi_\nu(\hat{a}_i, b) - c| \ge \frac{1}{2}C_1 e^{-\sqrt{\nu}}. \tag{6.14}$$

By (6.2) we have

$$\sum_{j=t_i+1}^{t_i+p_i}\left|\frac{f''_{a,b}(\eta_j)}{f'_{a,b}(\eta_j)}\right||\xi_j(a,b)-\xi_j(a',b)| \le \sum_{\nu=1}^{p_i}\frac{80}{C_9}\cdot\frac{|\Omega_{t_i+\nu}|}{|\eta_{t_i+\nu}-c|}\cdot\frac{|\xi_{t_i+\nu}(a,b)-\xi_{t_i+\nu}(a',b)|}{|\Omega_{t_i+\nu}|} \tag{6.15}$$

By the definition of bound periods and the definition of $\Omega_{t_i+\nu}$, we have

$$|\Omega_{t_i+\nu}| \le e^{-4\sqrt{\nu}}.$$

Moreover,

$$|\Omega_{t_i}| = e^{-r_i-1}-e^{-r_i-2}.$$

Substituting those two inequalities, (6.14), (6.7) and (6.4) into the right-hand side of (6.15), we get

$$\sum_{j=t_i+1}^{t_i+p_i}\left|\frac{f''_{a,b}(\eta_j)}{f'_{a,b}(\eta_j)}\right||\xi_j(a,b)-\xi_j(a',b)| \le \sum_{\nu=1}^{p_i} C_{21}\cdot\frac{e^{-4\sqrt{\nu}}}{e^{-\sqrt{\nu}}}\cdot\frac{\sigma_i}{e^{-r_i}}$$

for some constant C_{21}. This implies that there is a constant C_{22}, such that

$$\sum_{j=t_i+1}^{t_i+p_i}\left|\frac{f''_{a,b}(\eta_j)}{f'_{a,b}(\eta_j)}\right||\xi_j(a,b)-\xi_j(a',b)| \le C_{22}\frac{\sigma_i}{e^{-r_i}}.$$

Together with the estimate on the first term of S'_i, that we obtained long ago, we get a constant C_{23} such that

$$S'_i \le C_{23}\frac{\sigma_i}{e^{-r_i}}. \tag{6.16}$$

Note that C_{23} depends on κ_2, but not on δ.

To estimate S''_i, we use Lemma 4.12, and get immediately

$$S''_i \le C_{12}\frac{\sigma_{i+1}}{\delta}.$$

However, $\delta > e^{-r_{i+1}}$, so

$$S''_i \le C_{12}\frac{\sigma_{i+1}}{e^{-r_{i+1}}}. \tag{6.17}$$

This estimate also applies to S''_0.

By Lemma 3.4 applied to a subinterval $\omega'=[a,a']$ (or $[a',a]$) of ω (we can do it by Remark 5.13) and (5.36), see Remark 5.12, we get

$$\sigma_{i+1} \ge \frac{1}{q_*}\inf_{\tilde a\in\omega'}(f_{\tilde a,b}^{t_{i+1}-t_i})'(f_{\tilde a,b}^{t_i}(c))\cdot\sigma_i \ge \frac{C_{11}C_{18}}{q_*}e^{(p_i^{2/3}-4\sqrt{p_i+1})}\cdot\sigma_i.$$

As we already noticed in the proof of Lemma 5.11, by taking δ sufficiently small we can make p_i as large as we need and we may assume that

$$\frac{C_{11}C_{18}}{q_*}\exp\{p_i^{2/3}-4\sqrt{p_i+1}\} \ge 2,$$

and therefore we get

$$\sigma_{i+1} \ge 2\sigma_i. \tag{6.18}$$

Now we are ready to estimate the logarithm of the left-hand side of (6.1), which is less then or equal to $\sum_{i=0}^{m-1}(S_i' + S_i'')$. By (6.16) and (6.17), we get

$$\sum_{i=0}^{m-1}(S_i' + S_i'') \le (C_{23} + C_{12}) \sum_{i=0}^{m} \frac{\sigma_i}{e^{-r_i}}.$$

Rearrange the sum $\sum_{i=0}^{m} \sigma_i / e^{-r_i}$ and group it according to the values of r_i. Set $W_k = \{i \in [1, m] : r_i = k\}$. Consider k such that W_k is nonempty. Then we can write $W_k = \{i_s < i_{s-1} < \cdots < i_0\}$, and by (6.18), we have $\sigma_{i_j} \le \sigma_{i_0}/2^j$. Thus,

$$\sum_{i \in W_k} \frac{\sigma_i}{e^{-r_i}} \le 2 \frac{\sigma_{\mu_k}}{e^{-k}},$$

where μ_k is the largest element of W_k. However, σ_{μ_k} is the length of an interval which is contained in the union of 3 subintervals of I_k, and the length of each of those subintervals is $|I_k|/k^2$. Moreover, $|I_k| < e^{-k}$. Thus,

$$\sum_{i \in W_k} \frac{\sigma_i}{e^{-r_i}} \le \frac{6}{k^2}. \tag{6.19}$$

If W_k is empty, then of course (6.19) also holds. In such a way we get

$$\sum_{i=0}^{m-1}(S_i' + S_i'') \le 6(C_{23} + C_{12}) \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

The right-hand side of the above inequality is finite, so we can denote its exponential by C_{13} and then (6.1) holds. □

Lemma 6.2. *There exists a constant C_{24}, such that if a and a' are two parameter points, so that $a, a' \in \omega \in \mathcal{P}_n$, where n is a free return time and Induction Statement for n (and all smaller free return times) holds, then*

$$\frac{\partial_a f_{a,b}^k(c)}{\partial_a f_{a',b}^k(c)} \le C_{24} \tag{6.20}$$

for all $k \le n$.

Proof. The lemma follows immediately from Lemma 6.1, Corollary 3.2 and Induction Statement (iii). □

7. Part I of the proof of Theorem A

In this section we prove a proposition, which is an essential part of the proof of Theorem A, and is stated as follows.

Proposition 7.1. *Let $a = a_0$ be an MT-parameter for f_a and let $\varepsilon > 0$ be given. There is a function $\eta(\varepsilon) \to 0$ and a function $b_0(\varepsilon) \to 1$ as $\varepsilon \to 0$ such that if $b_0(\varepsilon) < b < 1$, if ω_0 is a parameter interval such that*

$$\omega_0 \subset (a_0 - \varepsilon, a_0 - \varepsilon^2) \cup (a_0 + \varepsilon^2, a_0 + \varepsilon), \tag{7.1}$$

such that $I_{r,\ell} \subset \xi_{n_0}(\omega_0, b) \subset I^+_{r,\ell}$ *and such that induction assumptions (i)–(vi) are satisfied for* $n = n_0$. *Then there is a set* $\tilde{E}_b \subset \omega_0$ *so that* $|\tilde{E}_b| \geq (1 - \eta(\varepsilon))|\omega_0|$, $C = C(a_0)$ *and* $\hat{\kappa} = \hat{\kappa}(a_0) > 0$ *so that*

$$(f^n_{a,b})'(f_{a,b}(c)) \geq Ce^{\hat{\kappa}n}, \qquad \forall n \geq 0 \qquad \forall a \in \tilde{E}_b. \tag{7.2}$$

Note that the assumptions of Proposition 7.1 is satisfied by Lemma 5.1.

This, together with Proposition 8.1 in Section 8 immediately lead to the following

Corollary 7.2. *The set* E *of parameters for which the double standard map is uniformly expanding accumulates on the MT points* $(a_0, 1)$ *in the parameter space.*

However we will need more general formulation of the propositions given above in order to prove Theorem A.

The proofs will be based on the induction formulated in Section 5. In the critical case $b = 1$, which we are not treating in detail, the remaining parameter set is of positive measure, while in the non-critical case $b < 1$ the remaining parameter set is a finite union of intervals.

We first discuss the parameter deletion due to the (BA) assumption.

If n is a free essential return time for a partition element $\omega = (a, a')$ of a partition $\mathcal{P}_{n''}$, where n'' is the essential free return immediately before n.

At each time we may have to omit a fraction of the parameter interval because of (BA). Assume that the previous free return occurred in the interval $I_{r'',\ell}$. Its length is $\frac{c}{r''^2}|I_{r''}|$, $1 \leq c \leq 3$. By the (BA) assumption applied to time n'', we have

$$e^{-r''} \geq e^{-\sqrt{n''}}$$

Since $n - n''$ has a minimal length with estimate $n - n'' \geq C\log(1/\delta)$, where C is a constant only depending on f_{a_0}. Not also that $r'' \leq \sqrt{n}$.

During the bound period the interval $\mathcal{K}_{r''+1} = (c, c + e^{-r''-1)})$ of size $e^{-r''-1}$ is increased to size $e^{-\sqrt{p''+1}}$, where $p'' \leq 8(r'')^{3/2}$ by Lemma 5.8. Our present interval is of length $\frac{c'}{r''^2}|I_{r''}|$, $1 \leq c' \leq 3$.

For $a = a'$ the size of $f_{a',b}(\mathcal{K}_{r''+1})$ can be estimated by formula (4.12) as follows

$$|x - c|(2 - 2b + C_9(x - c)^2) \leq |f_{a',b}(x) - f_{a',b}(c)| \leq |x - c|(2 - 2b + C_{10}(x - c)^2).$$

By inserting $x = c + e^{-r''-1}$ we obtain an estimate for $|f_{a,b}(\mathcal{K}_{r''+1})|$ as follows:

$$e^{-r''-1}(2 - 2b + C_7e^{-2r''-2}) \leq |f_{a,b}(\mathcal{K}_{r''+1})| \leq e^{-r''-1}(2 - 2b + C_8e^{-2r''-2}). \tag{7.3}$$

For the image of ω at time $n'' + 1$ we obtain the estimate

$$|\xi_{n''+1}(a, b) - \xi_{n''+1}(a', b)| = f'_{a,b}(y) \cdot |\xi_{n''}(a, b) - \xi_{n''}(a', b)| \pm |a - a'| \tag{7.4}$$

Here $y \in I_r$ so it follows from (4.13) that $|a - a'|$ can be estimated by the first term as in the estimate of (6.3) and we obtain

$$2 - 2b + C_9e^{-2r''-2} < f'_{a,b}(y) < 2 - 2b + C_{10}e^{-2r''}.$$

By the definition of a free return we also have the estimate

$$\frac{1}{r''^2}e^{-r''} \leq |\xi_{n''}(a, b) - \xi_{n''}(a', b)| \leq \frac{3}{r''^2}e^{-r''}.$$

By Lemma 5.7 (the bound distortion lemma) and comparison with the orbit of $\mathcal{K}_{r''}$ the size of $|\xi_{n''+p''+1}(a,b) - \xi_{n''+p''+1}(a',b)|$ has the lower bound.

$$\frac{1}{C_{17}} \cdot \frac{1}{r''^2} e^{-4\sqrt{p''+1}} \geq \frac{1}{C'_{17}} \cdot \frac{1}{r''^2} e^{-(\sqrt{8(r'')^{3/2}}} \geq \frac{1}{C'_{17}} e^{-8^{3/2} \cdot n^{3/8}}.$$

We have again used that δ may be chosen arbitrarily small. Using (6.18), Lemma 4.9 and Lemma 3.4 and it follows that the relative fraction to be deleted is at most

$$C_{17}'^{-1} C_{11} \frac{1}{q_*} \frac{e^{-\sqrt{n}}}{e^{-8^{3/2} \cdot n^{3/8}}} < e^{-\frac{1}{2}\sqrt{n}}, \tag{7.5}$$

since $n \geq n_0$ which at each time n we in principle may to have to do such a deletion. The remaining fraction of the parameter interval can then be estimated from below as

$$\geq \prod_{n=N_0}^{\infty} \left(1 - e^{-\frac{1}{2}\sqrt{n}}\right) \tag{7.6}$$

Note that this is arbitrarily close to 1 as $N_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Outline of proof of Proposition 7.1. The proof of Proposition 7.1 is based on the induction. Note that the Cantor Set construction can be stopped at a finite stage $\hat{N}$, which is defined by the relation

$$2 - 2b \geq C_9 e^{-2\sqrt{\hat{N}}}.$$

After this time the term $2 - 2b$ of equation 4.12 dominates in the derivative and we conclude that for all $a \in \tilde{E}_b$ there is a constant $C > 0$ so that

$$(f_{a,b}^n)'(f_{a,b}(c)) \geq C e^{\tilde{\kappa} n} \qquad \forall n \geq 0. \tag{7.7}$$

A more general result with a more detailed proof is given in Proposition 8.1.

□

Outline of the proof of Proposition 1.1. We proceed as in the proof of Theorem A. In this case the time $\hat{N}$, after which the linear term $2-2b$ dominates in the derivative does not exist and the induction proceeds to infinite time. We now have to use the large deviation argument of [3]. The main idea is that you delete parameters for which the critical orbits spend to much fractions of the time recovering the derivative loss. from returns to $(c-\delta^2, c+\delta^2)$. However an estimate similar to (7.6) is still valid. We do not give the full details.

8. Part II of the proof of Theorem A — the uniform expansion

In this section we consider $b < 1$, and we construct a non-empty union of open intervals $\hat{E}_b \supset \tilde{E}_b$ so that for $a \in \hat{E}_b$ there is an integer N so that, $f_{a,b}^N$ is uniformly expanding. This is formulated in Proposition 8.1. The set $\hat{E}_b$ is obtained by stopping the construction of the parameter set $\tilde{E}_b$ of Proposition 7.1 at a finite stage.

Let us outline the main idea of the proof of the uniform expansion. We will heavily use that the fact that $d = 2 - 2b > 0$, i.e. that the inflexion point is non-critical.

In the case the starting point x is outside the return interval I^* we can uses (i) of Lemma 4.9 to conclude that if $x, f_{a,b}(x), \dots, f_{a,b}^{n-1}(x) \notin I^*$, and $f_{a,b}^n(x) \in I^*$ then

$$(f_{a,b}^n)'(x) \geq C_{11} e^{\kappa_4 n}.$$

Here it is important that the constant C_{11} does not depend on δ.

At the return time n we have a derivative loss but this derivative loss is compensated during the bound period by Lemma 4.7. Since $p \to \infty$ as $\delta \to 0$, we can make the factor $e^{p^{2/3}}$ compensate C_2/C^* by making δ sufficiently small. We also use that the derivative of $f_{a,b}$ is bounded below by $f_{a,b}'(\frac{1}{2}) = 2 - 2b$, and we will also denote this number by d.

We state this result as follows.

Proposition 8.1. *Let $a = a_0$ be a MT parameter. Then if $b_0 = b_0(a_0) < 1$ is sufficiently close to 1 then for all $b \in (b_0, 1)$ there is a set $\hat{E}_b$, which is a finite union of intervals $\{\omega_j\}_{j=0}^{J_0}$ so that for $a \in \omega_j$, there is an integer M_j so that for all $x \in \mathbb{T}$,*

$$(f_{a,b}^{M_j})'(x) \geq \lambda_j > 1. \tag{8.1}$$

Proof. The proof is really the same as the proof of Proposition 7.1 initially.

As before, we carry out the construction only until time $\hat{N}$. Here $\hat{N}$ is the the smallest integer $\hat{N}$ satisfying

$$e^{-\sqrt{\hat{N}}} \leq d.$$

At time $\hat{N}$ we have a partition $\mathcal{P}_{\hat{N}}$ consisting of finitely many intervals $\{\omega_j\}_{j=1}^{M_{\hat{N}}}$.

We now aim to prove that the hyperbolicity statement (8.1) is true.

We first recall the two outside expansion statements of Lemma 4.9. Suppose that $(a, b) \in \mathcal{N}$ and chose $I = I^* = (c - \delta, c + \delta)$ in that lemma. Then the following holds.

1) If $x,\ f_{a,b}x, \dots, f_{a,b}^{n-1}x \notin I^*$ and $f_{a,b}^n x \in I^*$,

$$(f_{a,b})'(x) \geq C_{11} e^{\kappa_4 n}.$$

2) There is an integer M so that if $x, f_{a,b}x, \dots, f_{a,b}^{M-1}x \notin I^*$ then

$$(f_{a,b}^M)'(x) \geq e^{\kappa_4 M}.$$

□

Let us define R_0 as the smallest integer R_0 satisfying $e^{-2R_0} \leq e^{-\sqrt{\hat{N}}}$, i.e. R_0 corresponds to the r where the square term in the expression for the derivative is of the same size as the constant term $d = 2 - 2b$. The bound period $p(x)$, $x \in (c - e^{-R_0}, c + e^{-R_0})$ is chosen to be the infimum of the bound period for $y \in I_{\pm R_0}$.

We also know by (5.6) and Lemma 5.7 that

$$(f_{a,b}^p)'(x) \geq \frac{1}{C_{17}} e^{p^{2/3}}, \qquad x \in I_{\pm r},$$

and it holds as well that

$$(f_{a,b}^p)'(x) \geq \frac{1}{C_{17}} e^{p^{2/3}}, \qquad x \in (c - e^{-R_0}, c + e^{-R_0}).$$

Introducing κ_7 as

$$\kappa_7 = \frac{1}{2} \min_{r_\delta \leq |r| \leq R_0} \min_{x \in I_r} \frac{1}{p(x)^{1/3}},$$

we can in all cases write these estimates as

$$(f_{a,b}^p)'(x) \geq e^{\kappa_7 p}. \tag{8.2}$$

The factor $\frac{1}{2}$ is here used to absorb the constant C_{17}.

Let us in the following use the notation $\hat{I}_{R_0}$ for the union of $(c-e^{-R_0-1}, c+e^{-R_0-1})$ and the previously defined I_{-R_0} and I_{R_0}. The idea is that the derivative recovery has the same estimate for these three (original) intervals since $(f_{a,b})'(x) \sim d = 2-2b$ in $\hat{I}_{R_0}$ and the bound period is defined in terms of I_{R_0}.

Divide the set $\mathbb{T} \setminus I^*$ into several pieces.

We first consider the set

$$X_M = \{x : x, f_{a,b}x, \ldots, f_{a,b}^M x \notin I^*\}.$$

For $x \in X_M$, hyperbolicity is valid by Lemma 4.9, (ii):

$$(f_{a,b}^M)'(x) \geq e^{\kappa_4 M}.$$

We also introduce the sets

$$X_k = \{x : x, \ldots, f_{a,b}^{k-1} x \notin I^* \text{ but } f_{a,b}^k x \in I^*\}, \qquad 1 \leq k \leq M-1.$$

Pick a $k \geq 1$. Now write the set

$$X_k = \bigcup_{r_\delta \leq |r| \leq R_0} X_{k,r},$$

where $X_{k,r} = \{x \in X_k : f_{a,b}^k x \in I_r\}$, $|r_\delta| \leq |r| < R_0$ and

$$X_{k,\pm R_0} = \{x \in X_k : f_{a,b}^k x \in (c - e^{-R_0}, c + e^{-R_0}\}.$$

We then know that for $x \in X_{k,r}$

$$(f_{a,b}^{k+p})'(x) \geq C_{11} e^{\kappa_4 k} e^{\kappa_7 p} \geq e^{\kappa_8 (k+p)},$$

where $\kappa_8 = \min(\kappa_4, \kappa_7/2)$.

Here we have used the fact that also for the minimal possible p the factor $e^{\frac{\kappa_7}{2} p}$ always compensates the constant C_{11} of Lemma 4.9, and this constant is independent of δ.

Hence we know that the entire set $\mathbb{T}$ can be written as a disjoint union of sets $\{Y_j\}_{j=1}^J$ so that for some κ_9 and all $x \in Y_j$

$$(f_{a,b}^{n_j})'(x) \geq e^{\kappa_9 n_j}.$$

We start with an $x \in Y_{j_0}$. After n_{j_0} steps we will end up in Y_{j_1} and after another n_{j_1} steps we will end up in Y_{j_2} etc. The total time will be $n_{j_0} + n_{j_1} + n_{j_2} + \cdots + n_{j_s}$ and

$$(f_{a,b}^{n_{j_s} + \cdots + n_{j_0}})'(x) \geq e^{\kappa_9 (n_{j_s} + \cdots + n_{j_0})},$$

where

$$n_{j_s} + \cdots + n_{j_0} = \sum_{i=1}^m k_i n_i.$$

Let $n_{\max} = \max_{1 \leq j \leq J} n_j$ and pick an integer N very large so that

$$e^{\kappa_9 N} \cdot d_1^{n_{\max}} \geq e^{\kappa_{10} N}. \tag{8.3}$$

Here $d_1 = 1/B$, where $B = 4 \geq \max_{x \in \mathbb{T}} |f'_{a,b}(x)|$.

For each point x there is an $n = n(x) = n_{j_0} + n_{j_1} + n_{j_2} + \cdots + n_{j_s}$ so that
$$N \le n \le N + n_{\max}.$$
We claim that
$$(f_{a,b}^N)'(x) \ge e^{\kappa_{10} N}.$$
This follows since
$$(f_{a,b}^N)'(x) = (f_{a,b}^n)'(x)/(f_{a,b}^{n-N})'(f_{a,b}^N(x)) \ge e^{\kappa_9 N} d_1^{n_{\max}} \ge e^{\kappa_{10} N},$$
for a suitably κ_{10}. We conclude that the statement of Proposition 8.1 holds. □

We now have all ingredients for the proof of Theorem A.

Let ω_0 be an interval as defined in Proposition 7.1 satisfying (7.1) and let $\tilde{E}_b$ be the set defined in this proposition. Let $\hat{E}_b = \hat{E}_b^{\hat{N}} \supset \tilde{E}_b$ be the set corresponding to the $\hat{N}$:th order construction of Proposition 8.1. $\hat{N}$ is here determined as the smallest integer satisfying $e^{-\hat{N}} \le d$ as in the proof of Proposition 8.1. By (8.1) it then follows that the conclusion of Theorem A holds.

9. Proof of Theorem B

In this section we are going to prove the last result of the paper. The methods of its proof will be completely different than the ones used in the rest of the paper. We will use the term "countable" in the sense "at most countable." For the definitions, see Introduction.

Proof of Theorem B. Fix $b < 1$. Each tongue is open, so the set T_b is open. Therefore it is the union of countably many components, each of them an open interval. Since the points on the boundary of a tongue belong to TN, and the sets T and TN are disjoint, each component is contained in one tongue.

We claim that the intersection of the closures of two distinct components A_1 and A_2 is empty. Suppose it is not and that a belongs to this intersection. Then $(a, b) \in TN$, so it has its type. This type must be the same as the type of each of the tongues containing A_1 and A_2, so those types are the same, that is, A_1 and A_2 are contained in the same tongue. If n is the period of the neutral periodic orbit of $f_{a,b}$, the map $f_{a,b}^n$ has an interval on which it looks like one of the Cases 1, 2 or 4 of Lemma 4.1 of [19]. By Theorem 4.1 and Lemma 2.6 of [20], this cannot be Case 4 (a neutral periodic point repelling from both sides), and by Lemma 4.2 of [19] it cannot be Case 1 or 2 (a neutral periodic point repelling from one side). This proves our claim.

If a parameter $a \in TN_b$ does not belong to a boundary of a component of T_b, then by Lemma 4.2 of [19] the neutral periodic orbit of $f_{a,b}$ is repelling from both sides (Case 4), so by Theorem 4.1 and Lemma 2.6 of [20] a is isolated in the set of elements of $T_b \cup TN_b$ which have type of the same period. This proves that there are only countably many such values of a.

By the claim, the complement of T_b is a closed set without isolated points. The set TN_b is countable. Therefore E_b (which is the complement of T_b minus TN_b) is dense in the complement of T_b.

The second part of the statement follows from the first one and the fact that each component of T_b is contained in one tongue. □

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