The Halpern-Mann iteration in CAT(0) spaces

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(joint work with Pedro Pinto)

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This talk in a nutshell

In the nonlinear setting of $CAT(0)$ *spaces*, we studied an iteration alternating between the Halpern and the Krasnoselskii-Mann iterative schemas:

$$\text{(HM)} \quad x_0 \in C, \quad \begin{cases} x_{2n+1} := (1-\alpha_n)T(x_{2n}) \oplus \alpha_n u \\ x_{2n+2} := (1-\beta_n)U(x_{2n+1}) \oplus \beta_n x_{2n+1} \end{cases}$$

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Our argument uses proof mining ideas and a technique that allows to bypass sequential weak compactness (when in Hilbert spaces)

Some remarks

- Our results extend recent work of Boț, Csetnek and Meier, and of Leuștean and Cheval.
- If we take $U = Id_C$ and $\beta_n \equiv \frac{1}{2}$, then we recover in $z_n = x_{2n}$ the Halpern iteration

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- So in particular, we have also established the strong convergence of the Halpern iteration in $CAT(0)$ spaces, recovering Saejung's result (2010), and obtained the relevant quantitative information.
- Saejung's proof was previously analysed in the context of proof mining by Kohlenbach and Leuștean (2012) relying on a technique to eliminate the use of Banach limits needed in the original proof.

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- It was also possible to extend our results to a relaxed iteration which allows for error terms (δ_n):

$$(\mathrm{HM}_e) \quad \begin{cases} d(x_{2n+1}, (1-\alpha_n)T(x_{2n}) \oplus \alpha_n u) & \leq \delta_{2n} \\ d(x_{2n+2}, (1-\beta_n)U(x_{2n+1}) \oplus \beta_n x_{2n+1}) & \leq \delta_{2n+1} \end{cases}$$

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- Motivated by the strong convergence of **(HM)**, we defined strongly convergent versions of the Forward-Backwards and Douglas-Rachford splitting methods for finding zeros for the sum of two operators, in the setting of Hilbert spaces.

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- In fact, there exist explicit examples ("Specker sequences") of sequences of computable reals with no computable limit and thus with no computable rate of convergence.

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Metastability

$$\forall \varepsilon > 0 \, \forall f : \mathbb{N} \to \mathbb{N} \, \exists N \, \forall i, j \in [N, N + f(N)] (d(x_i, x_j) \leq \varepsilon)$$

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$$\forall k \in \mathbb{N} \forall f : \mathbb{N} \to \mathbb{N} \exists N \forall i, j \in [N, N + f(N)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right)$$

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which is a Herbrandization of the Cauchy property of a sequence.

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Proof mining program $\rightarrow$ analyses of mathematical proofs with the help of proof theoretic techniques, including functional interpretations, in search of concrete new information: effective bounds, algorithms, weakening of premisses, ...

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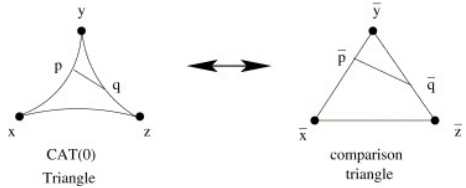
- Allow to avoid non-essential principles
- Allow to obtain explicit bounds

CAT(0) spaces

The metric space (X, d) is said to be $CAT(0)$ if every two points of X can be joined by a geodesic and every geodesic triangle $\Delta(x, y, z)$ of X verifies the hypothesis

$$\forall p, q \in \Delta(x, y, z)\,(d(p, q) \leq d_E(\bar{p}, \bar{q}))$$



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$\Downarrow$ Proof mining

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(x_n) is metastable — $\lim x_n = P_F(u)$

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$CAT(0)$ spaces:

—— (x_n) is metastable —?— $\lim x_n = P_F(u)$

Main Theorem

We consider the following conditions:

$$\text{(i)} \lim \alpha_n = 0, \quad \text{(ii)} \sum \alpha_n = \infty, \quad \text{(iii)} \sum |\alpha_{n+1} - \alpha_n| < \infty,$$
$$\text{(iv)} \sum |\beta_{n+1} - \beta_n| < \infty, \quad \text{(v)}\ 0 < \liminf \beta_n \le \limsup \beta_n < 1.$$

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Theorem (D., Pinto (2021))

Let X be a complete $CAT(0)$ space and C a nonempty closed convex subset. Consider nonexpansive maps $T, U : C \to C$ such that $F := Fix(T) \cap Fix(U) \neq \emptyset$ and $u, x_0 \in C$. Assume that $(\alpha_n) \subset [0, 1], (\beta_n) \subset (0, 1)$ are sequences of real numbers satisfying (i)-(v). Then (x_n) generated by **(HM)** *converges strongly to $P_F(u)$.*

Meanwhile in Hilbert spaces

Let us briefly look at the proof in the particular setting of Hilbert spaces. First we recall three useful results.

A: Projection characterization

For $S \subset X$ a nonempty closed convex subset and $u \in X$, we have

$$\forall y \in S \ (\langle u - P_S(u), y - P_S(u) \rangle \leq 0) \, .$$

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B: Demiclosedness

For C closed convex subset and $T : C \to C$ a nonexpansive map,

$$(x_n \rightharpoonup y \wedge T(x_n) - x_n \to 0) \Rightarrow y \in Fix(T).$$

Meanwhile in Hilbert spaces

C: Xu's Lemma

For $(a_n) \subset [0, 1]$, $(r_n) \subset \mathrm{R}$ and $(s_n) \subset \mathrm{R}_0^+$, we have

$$\left(s_{n+1} \leq (1 - a_n)s_n + a_n r_n \wedge \begin{cases} \sum a_n = \infty \\ \limsup r_n \leq 0 \end{cases} \right) \Rightarrow s_n \to 0$$

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- Combinatorial part: With $s_n = d^2(x_{2n}, \tilde{x}) = \|x_{2n} - \tilde{x}\|^2$, deduce

$$\boxed{s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n R_n},$$

where $R_n = S_n + K\langle u - \tilde{x}, x_{2n} - \tilde{x}\rangle$, with $S_n \to 0$ and $K > 0$.

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where $R_n = S_n + K\langle u - \tilde{x}, x_{2n} - \tilde{x}\rangle$, with $S_n \to 0$ and $K > 0$.
- Sequential weak compactness: Take a subsequence (x_{2n_j}) of (x_{2n}) such that $x_{2n_j} \rightharpoonup y$ and
$$\boxed{\limsup\langle u - \tilde{x}, x_{2n} - \tilde{x}\rangle = \lim_j \langle u - \tilde{x}, x_{2n_j} - \tilde{x}\rangle}.$$
By B (twice) and A, we conclude that $\limsup R_n \leq 0$, and applying C we derive $\boxed{x_n \to P_F(u)}$.

If we know that $x_n \to z$ for some $z \in F$, since

$$\langle u - \tilde{x}, x_{2n} - \tilde{x} \rangle = \langle u - \tilde{x}, x_{2n} - z \rangle + \langle u - \tilde{x}, z - \tilde{x} \rangle \leq \langle u - \tilde{x}, x_{2n} - z \rangle \to 0$$

we again conclude that $x_n \to P_F(u)$ [without weak compactness].

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Theorem

Let X *be a* $CAT(0)$ *space and* C *a nonempty convex subset. Consider n.e. maps* $T, U : C \to C$ *with* $Fix(T) \cap Fix(U) \neq \emptyset$*. Let* $(\alpha_n), (\beta_n) \subset [0, 1]$ *and* $x_0, u \in C$*. Assume that the conditions* $(Q1)$*-*$(Q5)$ *hold, and let* $N \in \mathbb{N} \setminus \{0\}$ *be such that* $N \geq \max\{d(x_0, p), 2d(u, p)\}$ *for some* $p \in F$*.*
Then (x_n) *generated by* **(HM)** *has the metastability property with rate of metastability* $\mu[\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \gamma, N](\varepsilon, f) = \cdots$ *.*

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Thank you!