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ORIGINAL ARTICLE

Non-regular Frameworks and the Mean-of-Order *p* Extreme Value Index Estimation

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Abstract

Most of the estimators of parameters of rare and large events, among which we distinguish the *extreme value index* (EVI) for maxima, one of the primary parameters in statistical *extreme value theory*, are averages of statistics, based on the k upper observations. They can thus be regarded as the logarithm of the geometric mean, i.e. the logarithm of the power mean of order $p = 0$ of a certain set of statistics. Only for heavy tails, i.e. a positive EVI, quite common in many areas of application, and trying to improve the performance of the classical Hill EVI-estimators, instead of the aforementioned geometric mean, we can more generally consider the power *mean of order-p* (MO_p) and build associated MO_p EVI-estimators. The normal asymptotic behaviour of MO_p EVI-estimators has already been obtained for $p < 1/(2\xi)$, with consistency achieved for $p < 1/\xi$, where ξ denotes the EVI. We shall now consider the non-regular case, $p \geq 1/(2\xi)$, a situation in which either normal or non-normal sum-stable laws can be obtained, together with the possibility of an '*almost degenerate*' EVI-estimation.

Keywords Extreme value theory · Heavy right tails · Generalized means · Semi-parametric estimation · Sum-stable laws

1 Introduction and Preliminaries

1.1 A Brief Motivation for the Need of Extreme Value Theory (EVT)

EVT helps us to control potentially disastrous events, of high relevance to society and with a high social impact. Domains of application of EVT are quite diverse. We mention the fields of *biostatistics*, *finance*, *insurance*, *seismology* and *structural engineering*,

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among many others. Statistics of univariate extremes, as well as multivariate and spatial extremes, have recently faced a huge development, partially due to the fact that rare events can have catastrophic consequences for human activities, through their impact on both natural and constructed environments. In the eighties, there has been a shift from the area of parametric statistics of extremes, based on probabilistic asymptotic results in EVT, towards semi-parametric or even nonparametric approaches. Despite of the fact that parametric modeling is becoming again quite popular, particularly in spatial applications of EVT, we shall here consider the semi-parametric framework, placing ourselves in the situation briefly described in Sect. 1.2.

1.2 The EV Semi-Parametric Framework

In an univariate set-up, let us assume that, after a possible adequate transformation, $\underline{\mathbf{X}}_n = (X_1, \ldots, X_n)$ can be regarded as a random sample of n independent, identically distributed or more generally stationary weakly dependent *random variables* (RVs) from a *cumulative distribution function* (CDF) F. Let us denote by $X_{1:n} \leq \cdots \leq X_{n:n}$ the associated ascending order statistics. Further assume that there exist real constants $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$, the so-called max-attraction coefficients, such that the maximum, linearly normalized, i.e. $(X_{n:n} - b_n)/a_n$, converges in distribution to a non-degenerate RV. Then, the attraction coefficients can be chosen so that the limiting CDF is the *general extreme value* (GEV) CDF, with the functional form,

$$\mathrm{GEV}_\xi(x) = \begin{cases} \mathrm{e}^{-(1+\xi x)^{-1/\xi}}, \ 1+\xi x > 0, & \text{if } \xi \neq 0, \\ \mathrm{e}^{-\mathrm{e}^{-x}}, \ x \in \mathbb{R}, & \text{if } \xi = 0, \end{cases} \tag{1.1}$$

where ξ is the so-called *extreme value index* (EVI). The EVI, ξ, is the primary parameter of extreme and large events, and measures the heaviness of the right-tail function, $\overline{F}(x) := 1 - F(x)$. The underlying CDF F is then said to belong to the max-domain of attraction of GEV_ξ, in (1.1), and we write $F \in \mathcal{D}_\mathcal{M}\left(\mathrm{GEV}_\xi\right)$. Despite of the fact that similar results are valid for a general tail, we shall consider *heavy-tailed* models, i.e. *Pareto-type* underlying CDFs, with a positive EVI, working thus in $\mathcal{D}_\mathcal{M}^+ := \mathcal{D}_\mathcal{M}\left(\mathrm{GEV}_{\xi>0}\right)$. Note that, in an univariate framework, $F \in \mathcal{D}_\mathcal{M}^+ \iff \overline{F} = 1 - F \in \mathcal{R}_{-1/\xi}$, where $\mathcal{R}_a$ denotes the set of regularly varying functions with an exponent a, i.e. positive functions $r(t)$ such that $r(tx)/r(t) \to x^a$, for all $x > 0$ and as $t \to \infty$ (see [1], for details on regular variation theory). We can further consider the *reciprocal tail quantile function*, $U(t) := F^{\leftarrow}(1 - 1/t),\ t \geq 1$, with $F^{\leftarrow}(x) := \inf\{y : F(y) \geq x\}$. Then, the first-order condition can be written as:

$$F \in \mathcal{D}_\mathcal{M}^+ \iff \lim_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-1/\xi} \iff \lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\xi}, \tag{1.2}$$

for all $x > 0$. The first necessary and sufficient condition in (1.2) was due to Gnedenko ([2]) and the second one to de Haan ([3]).

The second-order parameter ρ (≤ 0) measures the rate of convergence in the aforementioned first-order condition, in (1.2), and it is the non-positive parameter in the so-called second-order condition,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \varphi_\rho(x) := \begin{cases} \frac{x^\rho - 1}{\rho}, & \text{if } \rho < 0, \\ \ln x, & \text{if } \rho = 0, \end{cases} \tag{1.3}$$

$x > 0$, where $|A|$ must be of regular variation with an index ρ ([4]). For technical simplicity, we further assume that $\rho < 0$, writing

$$A(t) = \xi\beta t^\rho, \quad \xi > 0, \ \rho < 0, \ \beta \neq 0, \tag{1.4}$$

and working thus in Hall–Welsh class of models ([5]), where $U(t) = Ct^\xi \,(1 + \xi\beta t^\rho/\rho + o(t^\rho))$, as $t \to \infty$, with $C > 0$.

1.3 The Class of Mean-of-Order-*p* EVI-Estimators

For heavy-tailed models, the classical EVI-estimators are the Hill estimators ([6]), which are the average of the log-excesses V_{ik}, $1 \le i \le k < n$, i.e.

$$\mathrm{H}(k) \equiv \mathrm{H}(k; \underline{\mathbf{X}}_n) := \frac{1}{k}\sum_{i=1}^{k} \{\ln X_{n-i+1:n} - \ln X_{n-k:n}\} =: \frac{1}{k}\sum_{i=1}^{k} V_{ik}, \qquad 1 \le k < n. \tag{1.5}$$

Note now that we can write

$$\mathrm{H}(k) = \sum_{i=1}^{k} \ln\left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k} = \ln\left(\prod_{i=1}^{k} \frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k} =: \ln\left(\prod_{i=1}^{k} U_{ik}\right)^{1/k}. \tag{1.6}$$

The Hill estimator in (1.5) is thus the logarithm of the *geometric mean* (or *mean-of-order*-0) of $U_{ik} = X_{n-i+1:n}/X_{n-k:n}$, $1 \le i \le k < n$, already defined in (1.6).

For the EVI-estimation, the authors in [7], and almost simultaneously the authors in [8], and in [9], considered then as basic statistics, the power (or Hölder) *mean-of-order-p* (MO_p) of U_{ik}, $1 \le i \le k$, for $p \ge 0$. More generally, the authors in [10], and also in [11], considered those same statistics for any $p \in \mathbb{R}$. Let us now think also on

$$\mathrm{A}_p(k) \equiv \mathrm{A}_p(k; \underline{\mathbf{X}}_n) := \begin{cases} \left(\frac{1}{k}\sum_{i=1}^{k} U_{ik}^p\right)^{1/p}, & \text{if } p \neq 0, \\ \left(\prod_{i=1}^{k} U_{ik}\right)^{1/k}, & \text{if } p = 0, \end{cases} \tag{1.7}$$

and the associated class of $\mathrm{MO}_p \equiv \mathrm{H}_p$ statistics,

$$\mathrm{MO}_p(k) \equiv \mathrm{H}_p(k) \equiv \mathrm{H}_p(k; \underline{\mathbf{X}}_n) := \begin{cases} \left(1 - \mathrm{A}_p^{-p}(k)\right)/p, & \text{if } p \neq 0, \\ \ln \mathrm{A}_0(k) = \mathrm{H}(k), & \text{if } p = 0, \end{cases} \tag{1.8}$$

with $\mathrm{H}(k)$ and $\mathrm{A}_p(k)$, respectively, given in (1.5) and in (1.7).

1.4 Scope of the Article

In semi-parametric statistical EVT, working thus in the whole $\mathcal{D}_{\mathcal{M}}(\mathrm{GEV}_\xi)$, with $\mathrm{GEV}_\xi(\cdot)$ given in (1.1), we often look for consistency of statistics, like $\mathrm{H}_p(k)$, in (1.8), as well as for the asymptotic normality of those same statistics in the whole max-domain of attraction or in sub-classes of such a max-domain of attraction. Despite of the fact that it is well-known that EVT is needed because the world is not always normal, a simple model with quadratic exponential tails, we believe that even researchers in EVT are quite 'linked' to an asymptotic normal behaviour of estimators of any parameter of extreme events. And possibly this needs to change, due to reasons like the one discussed in this paper. In Sect. 2, and on the basis of Monte Carlo simulation experiments, the interesting behaviour of the H_p class of statistics when $p = 1/\xi$ is enhanced, motivating the need for the study of the asymptotic behaviour of those statistics. First, in Sect. 3, a few indications on the asymptotic behaviour of sums of Pareto RVs are given. Section 4 is devoted to the asymptotic behaviour of $\mathrm{H}_p(k)$, in (1.8), for non-regular frameworks, i.e. for $p > 1/(2\xi)$. Finally, a few overall conclusions are drawn in Sect. 5.

2 The MO$_p$ Statistics: A Small-Scale Simulation to Motivate the Consideration of a Non-regular Framework

In order to have consistency of the Hill estimators in (1.5), in the whole $\mathcal{D}^+_{\mathcal{M}}$, we need to work with intermediate values of k, i.e. a sequence of integers $k = k_n$, $1 \le k < n$, such that

$$k = k_n \to \infty \quad \text{and} \quad k_n = o(n), \ \text{as } n \to \infty. \tag{2.1}$$

Under the aforementioned second-order framework, in (1.3), the asymptotic distributional representation

$$\mathrm{H}(k) \overset{d}{=} \xi + \frac{\xi}{\sqrt{k}} Z_k + \frac{1}{1-\rho} A(n/k)(1 + o_{\mathbb{P}}(1)) \tag{2.2}$$

holds ([12]), where Z_k is an asymptotically standard normal RV. More generally, we can state the following theorem:

Theorem 2.1 *([7]). Under the validity of the first-order condition, in (1.2), and for intermediate sequences $k = k_n$, i.e. if (2.1) holds, the class of statistics $\mathrm{H}_p(k)$, in (1.8), is consistent for the estimation of ξ provided that $p < 1/\xi$. If we moreover assume the validity of the aforementioned second-order framework, in (1.3), the asymptotic* *distributional representation*

$$\mathrm{H}_p(k) \stackrel{d}{=} \xi + \frac{\xi(1-p\xi)Z_k^{(p)}}{\sqrt{k}\sqrt{1-2p\xi}} + \frac{1-p\xi}{1-p\xi-\rho}\, A(n/k)(1+o_{\mathbb{P}}(1)) \tag{2.3}$$

holds for $p < 1/(2\xi)$, *with* $Z_k^{(p)}$ *asymptotically standard normal.*

Remark 2.1 As stated before, note that $\mathrm{H}_0(k) \equiv \mathrm{H}(k)$, and consequently, the replacement in (2.3) of p by 0, and the notation $Z_k^{(0)} \equiv Z_k$ leads to (2.2).

Then, as noticed in [13], and with RMSE denoting *root mean square error*, there is an optimal value for p, given by

$$p^* := \underset{(p,k)}{\arg\min}\, \mathrm{RMSE}(\mathrm{H}_p(k)) = \varphi_\rho/\xi,$$
$$\text{with} \quad \varphi_\rho := 1 - \rho/2 - \sqrt{\rho^2 - 4\rho + 2}\Big/2 \in \left(0, 1 - \sqrt{2}/2\right], \tag{2.4}$$

which maximizes the asymptotic efficiency of the class of estimators in (1.8) with respect to the Hill estimator, leading to an optimal MO_p class of random variables, $\mathrm{H}_{p_\mathrm{M}}(k)$, dependent on the unknown parameters (ξ, ρ). Just as done in [11], we can also estimate the optimal k-value for the H EVI-estimation, again in the sense of minimal RMSE, but now in k, i.e. $k_{0|\mathrm{H}_0} := \arg\min_k \mathrm{RMSE}(\mathrm{H}_0(k))$, as given in [14], computing $\hat{k}_{0|\mathrm{H}_0} = \left((1-\hat{\rho})n^{-\hat{\rho}}/\left(\hat{\beta}\sqrt{-2\hat{\rho}}\right)\right)^{2/(1-2\hat{\rho})}$ and next computing $\mathrm{H}_\mathrm{A} := \mathrm{H}(\hat{k}_{0|\mathrm{H}_0})$. With φ_ρ given in (2.4) and $p^* \equiv \hat{p}_\mathrm{M} = \varphi_{\hat{\rho}}/\,\mathrm{H}_\mathrm{A}$, we can further consider the optimal MO_p EVI-estimators, $\mathrm{H}^*(k) := \mathrm{H}_{p^*}(k)$.

Remark 2.2 We have so far advised the use of the ρ-estimators in [15] and the β-estimators in [16], but more recent classes of (β, ρ)-estimators have been developed, with the aim of bias reduction (see [11], for some references). Moreover, overviews on reduced-bias estimation can be found in [17], Chapter 6.6 of [18], in [19] and in [20].

2.1 Additional EVI-Estimators Related to the MO_p Estimators

At optimal levels, in the sense of minimal RMSE as a function of p whenever (2.3) holds, the optimal MO_p class of EVI-estimators, above denoted by $\mathrm{H}^*(k)$, being a generalization of $\mathrm{H}(k)$, obviously outperforms the optimal H EVI-estimator, again in the sense of minimal RMSE but as a function of k, in the whole (ξ, ρ)-plane. And as shown in [21], while the promising class of *Pareto probability weighted moments* (PPWM≡P), studied in [22] and [23], beats $\mathrm{H}(k)$, at optimal levels, in a wide region of the (ξ, ρ)–plane, it is beaten by $\mathrm{H}^*(k)$, also at optimal levels, in an even much wider class of models.

Moreover, whenever working in Hall–Welsh class of models, i.e. under the validity of (1.4), the aforementioned representation in (2.3) immediately suggested the consideration of associated *reduced-bias* MO_p (RBMO_p) EVI-estimators, denoted by

Table 1 Simulated mean values, at optimal levels, of the aforementioned statistics, for a few values of $p = \ell/(10\xi)$, $\ell = 0, 2, 4, 5, 10, 11, 12$, and for a $\mathrm{GEV}_{0.25}$ underlying parent

n	100	200	500	1000	2000	5000
$\mathrm{H} \equiv \mathrm{H}_0$	0.427	0.391	0.365	0.348	0.335	0.321
$\mathrm{H}_{0.8}$	0.384	0.350	0.338	0.330	0.321	0.312
H^*	0.340	0.335	0.330	0.323	0.317	0.308
$\mathrm{H}_{1.6}$	0.336	0.309	0.303	0.301	0.300	0.294
$\mathrm{P} \equiv \mathrm{PPWM}$	0.344	0.331	0.323	0.318	0.313	0.305
$\overline{\mathrm{H}}$	0.382	0.372	0.353	0.342	0.330	0.317
$\overline{\mathrm{H}}^*$	0.354	0.345	0.335	0.327	0.319	0.310
$\mathrm{H}_2(p = 1/(2\xi))$	0.312	0.295	0.293	0.289	0.288	0.284
$\mathrm{H}_4(p = 1/\xi)$	**0.250**	**0.250**	**0.250**	**0.250**	**0.250**	**0.250**
$\mathrm{H}_{4.4}$	0.227	0.227	0.227	0.227	0.227	0.227
$\mathrm{H}_{4.8}$	0.208	0.208	0.208	0.208	0.208	0.208

$\overline{\mathrm{H}}_p(k)$, and associated *optimal* RBMO_p (ORBMO_{p^*}), now denoted by $\overline{\mathrm{H}}^*(k)$ (see [24], [25] and [11] for details on these estimators). Note that $\overline{\mathrm{H}}_0(k)$ is the simplest class of *minimum variance reduced-bias* (MVRB) EVI-estimators in [26]. These estimators will not be discussed here, but will be taken into account in the simulation study presented in the following. For the most common heavy tailed models, among which the GEV_ξ model, in (1.1), we have run multi-sample Monte Carlo simulation experiments (of size 5000×20, i.e. 20 independent replicates with 5000 runs each for $n = \{100, 200, 500, 1000, 2000, 5000\}$, the most common size in the aforementioned articles), including values of $p \geq 1/(2\xi)$. For details on multi-sample Monte Carlo simulation, we refer [27] and more recently, [28], among others. Note that in this paper, and since we only present simulated mean values, it would be equivalent to perform a single-sample Monte Carlo simulation with a size equal to 100000 ($= 5000 \times 20$). We have been led to a quite astonishing but interesting result, which had indeed already been mentioned in [7], where there appears the conjecture that the choice $p = 1/\xi$ could be an adequate one. The simulated mean values at optimal levels, of the aforementioned EVI-estimators, for a few values of p, and for a $\mathrm{GEV}_{0.25}$ underlying parent, are presented in Table 1.

As can be seen, $p = 1/\xi = 4$ provides the best and very interesting results, written in bold in Table 1, for all sample sizes n. But if we choose $p > 1/\xi$, we are led to a simulated mean value quite close to $1/p < \xi$. For instance, for $p = 4.4$, $1/p = 0.2(27)$ and for $p = 4.8$, $1/p = 0.208(3)$. An answer to the question, 'What's happening when $1/(2\xi) \leq p \leq 1/\xi$?', and even to 'What's happening when $p > 1/\xi$?', are thus needed and provided later on, in Sect. 4. Further note that this happens also for other simulated models and for different values of ξ, as can be seen in Tables 2 and 3, respectively, associated with a Student-t_2 ($\xi = 1/2 = 0.5$) and a generalized Pareto GP_1 model ($\xi = 1$), where we present the simulated mean values at optimal levels of H_p only, for several values of p. For each sample size, the smallest absolute bias is written in bold in Tables 1, 2, 3

Table 2 Simulated mean values, at optimal levels, of the H_p statistics, again for $p = \ell/(10\xi)$, $\ell = 0, 2, 4, 5, 10, 11, 12$, and for a Student t_ν underlying parent, with $\nu = 2$ ($\xi = 1/2$)

n	100	200	500	1000	2000	5000
$\mathrm{H} \equiv \mathrm{H}_0$	0.602	0.577	0.556	0.544	0.536	0.526
$\mathrm{H}_{0.4}$	0.580	0.566	0.550	0.540	0.532	0.523
$\mathrm{H}_{0.8}$	0.555	0.549	0.539	0.533	0.526	0.518
H_1 $(p = 1/(2\xi))$	0.543	0.539	0.533	0.528	0.522	0.515
H_2 $(p = 1/\xi)$	**0.474**	**0.488**	**0.496**	**0.498**	**0.499**	**0.500**
$\mathrm{H}_{2.2}$	0.438	0.448	0.453	0.454	0.454	0.454
$\mathrm{H}_{2.4}$	0.406	0.413	0.416	0.416	0.417	0.417

Table 3 Simulated mean values, at optimal levels, of the H_p statistics, also for $p = \ell/(10\xi)$, $\ell = 0, 2, 4, 5, 10, 11, 12$, and for a GP_1 underlying parent ($\xi = 1$)

n	100	200	500	1000	2000	5000
$\mathrm{H} \equiv \mathrm{H}_0$	1.138	1.110	1.079	1.064	1.050	1.037
$\mathrm{H}_{0.2}$	1.118	1.097	1.073	1.060	1.047	1.035
$\mathrm{H}_{0.4}$	1.092	1.080	1.065	1.054	1.046	1.035
$\mathrm{H}_{0.5}(p = 1((2\xi))$	1.075	1.065	1.057	1.048	1.042	1.034
$\mathrm{H}_1(p = 1/\xi)$	**0.998**	**0.999**	**1.000**	**1.000**	**1.000**	**1.000**
$\mathrm{H}_{1.1}$	0.908	0.909	0.909	0.909	0.909	0.909
$\mathrm{H}_{1.2}$	0.833	0.833	0.833	0.833	0.833	0.833

More extensive simulation results can be found in [33, 45]. Using the computer as a laboratory in Statistics, we are then led to conjecture that even fo $p = 1/\xi$, $\mathrm{H}_p(k)$ can provide a consistent estimation of ξ, and that we have convergence towards $1/p \neq \xi$, when $p > 1/\xi$, and for adequate values of k.

2.2 Competitive PORT EVI-Estimators and the MO$_p$ Statistics

All the aforementioned EVI-estimators, generally denoted by $\mathrm{T}(k)$, are NOT location-invariant, as often desired, contrarily to what we call the PORT-T EVI-estimators, similar in spirit to the PORT-H EVI-estimators, introduced in [29] and further studied in [30], with PORT standing for *peaks over random threshold*. PORT-P EVI-estimation has been considered in [31]. PORT-MO$_p$ EVI-estimators were introduced and studied in [21] and have already exhibited a very competitive behaviour both asymptotically and for finite samples. The class of PORT-RBMO$_p$ EVI-estimators was studied in [32], for $p = 0$ (PORT-$\overline{\mathrm{H}}$), and later for a general p ([44]; [34]). All classes of PORT EVI-estimators are based on a sample of excesses over a random threshold $X_{n_q:n}$, $n_q := \lfloor nq \rfloor + 1$, $0 \le q < 1$, where $\lfloor x \rfloor$ denotes the integer part of x, depending thus on an extra tuning parameter q. We can have $q = 0$, only if the underlying CDF has a finite left endpoint $x_F := \inf\{x : F(x) > 0\}$ (the random threshold can then be the *minimum*), and $0 < q < 1$, for CDFs with finite or infinite left endpoint x_F (the

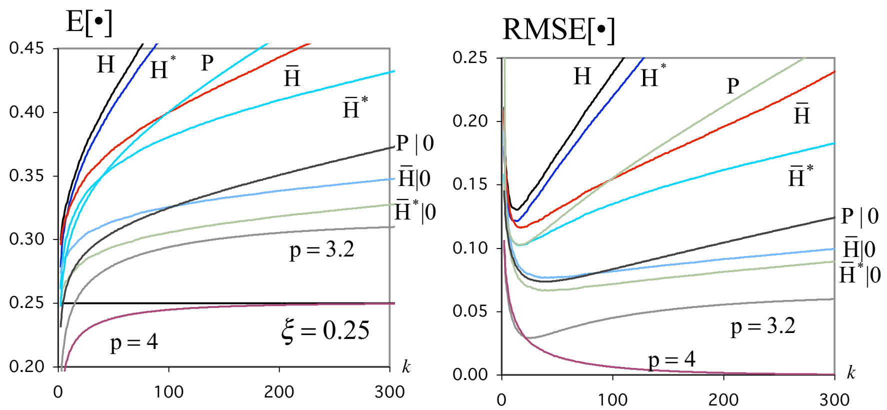

Fig. 1 Mean values (*left*) and RMSEs (*right*) of the EVI-estimators under consideration for a $\mathrm{GEV}_{0.25}$ underlying parent and a sample size $n = 1000$

random threshold is any *empirical quantile*). Notice that a tuning parameter $q = 0$, i.e. the consideration of excesses over the minimum of the sample, is appealing but should be used carefully. Indeed, if the underlying parent has not a finite left endpoint, we are led to non-consistent estimators, with sample paths that may be erroneously flat around a value quite far away from the real target. Further details can be found in [29] and [30], among others.

For $0 \le q < 1$ and $k < n - n_q$, the PORT-T classes of EVI-estimators have the same functional form of the T EVI-estimators, but with $\underline{\mathbf{X}}_n = (X_1, \ldots, X_n)$ replaced by the sample of excesses,

$$\underline{\mathbf{X}}_n^{(q)} := \big(X_{n:n} - X_{n_q:n}, \ldots, X_{n_q+1:n} - X_{n_q:n}\big),$$

with a size $n^{(q)} = n - n_q$. For any of the aforementioned EVI-estimators, generally denoted by $\mathrm{T}(k) \equiv \mathrm{T}(k; \underline{\mathbf{X}}_n)$, possibly dependent on p, we have the PORT-T EVI-estimator,

$$\mathrm{T}^{(q)}(k) \equiv \mathrm{T}\big(k; \underline{\mathbf{X}}_n^{(q)}\big).$$

These estimators are now invariant for both changes of scale and location in the data and depend on this *tuning parameter* q, that again provides a highly flexible class of EVI-estimators.

Associated now with the first bulk of 5000 runs, and as an illustration of the simulation experiment, we present in Fig. 1 the simulated mean values (E) and RMSEs of some of the aforementioned EVI-estimators, as a function of k, for a sample size $n = 1000$, and for a GEV underlying parent, with an EVI $\xi = 0.25$. We put altogether in the simulation experiment H ($p = 0$, in (1.8)), H^*, P, $\overline{\mathrm{H}}$, $\overline{\mathrm{H}}^*$, the PORT-P (P|0), the PORT-$\overline{\mathrm{H}}$ ($\overline{\mathrm{H}}$|0) and the PORT- $\overline{\mathrm{H}}^*$ ($\overline{\mathrm{H}}^*$|0), associated with $q = 0$, and H_p for $p = \ell/(10\xi)$, $\ell = 8$ ($p = 3.2$) and $\ell = 10$ ($p = 4$).

Generally, for all k and not only for this model, but for all other simulated models, there is a clear reduction in RMSE, as well as in bias, for the ORBMO$_{p^*}$ $\left(\overline{\mathrm{H}}^*\right)$ EVI-estimators comparatively to the MVRB EVI-estimators $\left(\overline{\mathrm{H}} \equiv \overline{\mathrm{H}}_0\right)$, with the obtention of estimates closer to the target value ξ, and more reliable. The difference between the PORT-P $\left(\mathrm{P}|0\right)$, the PORT-MVRB $\left(\overline{\mathrm{H}}|0\right)$ and the PORT-ORBMO$_{p^*}$ $\left(\overline{\mathrm{H}}^*|0\right)$ is not so clear, but still exists, being obviously in favour of the PORT-ORBMO$_{p^*}$. The improvement is often even stronger than in the illustration above, it is valid for the estimation of other parameters of extreme events, and the patterns of the estimates are always of the same type, in the sense that, for all k, the MVRB clearly beat the H, the ORBMO$_{p^*}$ clearly beat the MVRB, the PORT-MVRB strongly beat the MVRB when the sample has a negative left endpoint, like happens here, and the PORT-ORBMO$_{p^*}$ moderately beat the PORT-MVRB EVI-estimators. We were thus totally in favor of a PORT-ORBMO$_{p^*}$–estimation, prior to the consideration of H_p for values of p close to $1/\xi$, and not above $1/\xi$. In the following section, we shall thus look at the asymptotic behaviour of sums of the adequate powers of Pareto RVs and sum-stable laws.

3 Additive Stable Laws

Given the sequence $\{X_n\}_{n\geq 1}$, if for each $k \in \mathbb{N}$, up to an affine linear transformation, the limit is standard additive stable, there exist normalizing constants $A_k > 0$, $B_k \in \mathbb{R}$ and a non-degenerate RV, denoted by $S_{\alpha,\beta}$, such that

$$\frac{\sum_{i=1}^{k} X_i - B_k}{A_k} \xrightarrow{d} S_{\alpha,\beta}, \tag{3.1}$$

where $S_{\alpha,\beta}$ is necessarily a standard additive stable (or sum stable) law. Among other books, see [35], [36], [37] and [38], for details on sum-stable laws. The parameter $\alpha \in (0, 2]$ is the so-called characteristic exponent, related to the tail weight of the CDF, and strongly linked to the EVI ξ, in (1.1) and the parameter $\beta \in [-1, 1]$ characterizes the degree of skewness, i.e. if $\beta > 0$, the distribution is right skewed and if $\beta < 0$, the distribution is left skewed. The unique sum-stable models with known and explicit CDFs are the Lévy model ($\alpha = 1/2,\ \beta = 1$), the Cauchy model ($\alpha = 1,\ \beta = 0$) and the Normal model ($\alpha = 2, \beta = 0$). The stable distributions with $\alpha < 1$ and $\beta = 1$ have support on the positive x-axis, whereas the stable distributions with $\alpha \geq 1$ and $\beta = 1$ have support over the whole x-axis.

Regarding the common CDF F, the generalized central limit theorem ([37]; [38]) states that, as $k \to \infty$, (3.1) holds if and only if

$$1 - F(x) + F(-x) \in \mathcal{R}_{-\alpha} \quad \text{and} \quad \frac{F(-x)}{1 - F(x) + F(-x)} \underset{x\to+\infty}{\longrightarrow} \frac{1-\beta}{2}. \tag{3.2}$$

In the lines of [39], we can write the normalization and shift coefficients in (3.1), where $\mu(x) = \int_{-x}^{x} y^2 dF(y)$, as

α	A_k	B_k
$0<\alpha<1$	$\inf\left\{x : \frac{k\mu(x)}{x^2} \le 1\right\}$	$-$
$\alpha=1$	$\inf\left\{x : \frac{k\mu(x)}{x^2} \le 1\right\}$	$k^2\mathbb{E}[\sin(X/A_k)]$
$1<\alpha<2$	$\inf\left\{x : \frac{k\mu(x)}{x^2} \le 1\right\}$	$k\mathbb{E}[X]$

Remark 3.1 If $S \equiv S_{\alpha,\beta}$ is a standard additive stable law, then with $\sigma > 0$ being a scale parameter and $\mu \in \mathbb{R}$ a location (shift) parameter, $\sigma S + \mu$ is a $S_{\alpha,\beta,\sigma,\mu}$ stable law.

3.1 Sum-Stable Behaviour of Pareto CDFs

Let us now consider a unit Pareto RV, Y, with CDF $F_1(y) = 1 - 1/y,\ y \ge 1$, and for $\xi > 0$, let us consider Y^ξ, an RV with CDF $F_\xi(y) = 1 - y^{-1/\xi},\ y \ge 1$, and a right-tail function

$$\overline{F}_\xi(y) := 1 - F_\xi(y) = y^{-1/\xi},\ y \ge 1.$$

Recall that, $\mathbb{E}(Y^\xi) = 1/(1-\xi)$, for $\xi < 1$, and since $\mathbb{V}\mathrm{aR}(Y^\xi) = \xi^2/((1-\xi)^2(1-2\xi)) < \infty$ only if $\xi < 1/2$, the CLT applies under such a condition, and

$$\frac{(1-\xi)\sqrt{k(1-2\xi)}}{\xi}\left(\frac{1}{k}\sum_{i=1}^{k} Y_i^\xi - \frac{1}{1-\xi}\right) \xrightarrow{d} \mathcal{N}(0,1).$$

The case $\xi = 1/2$, where $\mathbb{E}(Y^\xi) = 1/(1-\xi) = 2$, is also of particular interest due to the fact that the variance is infinite but

$$\mathbb{E}_\tau(Y^{2\xi}) := \mathbb{E}(Y^{2\xi}|Y^{2\xi} < \tau) = \ln\tau \in \mathcal{R}_0,$$

and we are in the non-standard additive domain of attraction of the normal CDF, i.e. $\alpha = 1/\xi = 2$, and

$$\frac{1}{\sqrt{\ln k/k}}\left(\frac{1}{k}\sum_{i=1}^{k} Y_i^\xi - \frac{1}{1-\xi}\right) \xrightarrow{d} S_{2,0} \equiv \mathcal{N}(0,1). \tag{3.3}$$

For values of $\xi > 1/2$, we can apply the generalized central limit theorem, and from (3.2), we have $\beta = 1$ and $\overline{F}_\xi \in \mathcal{R}_{-1/\xi}$. Consequently, Y^ξ belongs to the additive domain of attraction of a stable law with $\beta = 1$ and a characteristic exponent given by $\alpha(\xi) = \min\{2, 1/\xi\}$, i.e. the properly normalized sum $\sum_{i=1}^{k} Y_i^\xi$ of a large number k of independent, identically distributed Pareto$(1/\xi)$ RVs can be approximated by a stable RV $S_{\min\{2,1/\xi\}}$, and we use the notation $S_\alpha = S_{\alpha,1}$,

$$\frac{\sum_{i=1}^{k} Y_i^\xi - B_k}{A_k} \underset{k\to\infty}{\xrightarrow{d}} S_\alpha, \quad \alpha = \min(2, 1/\xi).$$

Following [40] and [39], the normalization coefficients $A_k := k^{\xi} C_{\xi}$ are such that

$$C_{\xi} = \begin{cases} \left(\Gamma(1-1/\xi)\cos(\pi/(2\xi))\right)^{\xi}, & \text{if } \xi > 1, \\ \pi/2, & \text{if } \xi = 1, \\ \left(\xi\ \Gamma(2-1/\xi)\left|\cos(\pi/(2\xi))\right|/(1-\xi)\right)^{\xi}, & \text{if } 1/2 < \xi < 1, \end{cases}$$

and the shift coefficients

$$B_k = \begin{cases} 0, & \text{if } \xi > 1, \\ \frac{\pi k^2}{2}\int_1^{\infty} \sin\left(\frac{2x}{\pi k}\right) dF(x), & \text{if } \xi = 1, \\ k/(1-\xi), & \text{if } 1/2 < \xi < 1, \end{cases}$$

where $\Gamma(\cdot)$ is the gamma function. The integral for $\xi = 1$ can be expanded as ([41], Eqs. 3.761.3, 8.230.2)

$$B_k = k\ln k + k\left(\frac{\pi k}{2}\sin\left(\frac{2}{\pi k}\right) - \gamma - \ln\frac{2}{\pi} - \int_0^{2/(\pi k)} \frac{\cos t - 1}{t} dt\right), \quad (3.4)$$

where, with $\Gamma'(\cdot)$ the derivative of the gamma function, $\gamma = -\Gamma'(1) \approx 0.5772\ldots$ is usually known as the Euler's constant. We can further write the asymptotic equivalence,

$$B_k \simeq k\ln k + k(1-\gamma-\ln(2/\pi)) = k(\ln k + 1 - \gamma - \ln(2/\pi)) =: B'_k, \quad (3.5)$$

in the sense that $(B_k - B'_k)/A_k \to 0$, as $k \to \infty$, i.e. (A_k, B_k) and (A_k, B'_k) are equivalent attraction coefficients for $\sum_{i=1}^k Y_i^{\xi}$.

Remark 3.2 Note that when $1/2 < \xi < 1$, $\cos(\pi/(2\xi)) < 0$ and thus, the need to use $|\cos(\pi/(2\xi))|$. The coefficient C_{ξ} for $1/2 < \xi < 1$, $(\xi\Gamma(2-1/\xi)\,|\cos(\pi/(2\xi))| /(1-\xi))^{\xi}$ can be rewritten as $(\Gamma(1-1/\xi)\,|\cos(\pi/(2\xi))|)^{\xi}$.

Consequently, and asymptotically,

$$\frac{1}{k}\sum_{i=1}^{k} Y_i^{\xi} = \begin{cases} \frac{1}{1-\xi} + \frac{\xi}{1-\xi}\sqrt{\frac{1}{k(1-2\xi)}}\, S_{2,0}(1+o_{\mathbb{P}}(1)), & \text{if } \xi < 1/2, \\ \frac{1}{1-\xi} + \sqrt{\frac{\ln k}{k}}\, S_{2,0}(1+o_{\mathbb{P}}(1)), & \text{if } \xi = 1/2, \\ \frac{1}{1-\xi} + k^{\xi-1}\{\xi\Gamma(2-1/\xi)|\cos(\pi/(2\xi))|/(1-\xi)\}^{\xi}\, S_{1/\xi}(1+o_{\mathbb{P}}(1)), & \text{if } 1/2 < \xi < 1, \\ \ln k + 1 - \gamma - \ln(2/\pi) + \frac{\pi}{2} S_1(1+o_{\mathbb{P}}(1)), & \text{if } \xi = 1, \\ k^{\xi-1}\{\Gamma(1-1/\xi)\cos(\pi/(2\xi))\}^{\xi}\, S_{1/\xi}(1+o_{\mathbb{P}}(1)), & \text{if } \xi > 1. \end{cases} \quad (3.6)$$

4 Asymptotic Behaviour of the Class of MO$_p$ EVI-Estimators Under Non-regular Frameworks

We now state a generalization of Theorem 2.1 to non-regular cases.

Theorem 4.1 *Under the validity of the first-order condition, and for intermediate sequences $k = k_n$, the class of functionals $\mathrm{H}_p(k)$ is consistent for the estimation of ξ, provided that $p \le 1/\xi$ and converges to $1/p$ $(< \xi)$ if $p > 1/\xi$. In the region of values of p that guarantee consistency and out of the scope of Theorem 2.1, i.e. $1/(2\xi) \le p \le 1/\xi$, if we assume the validity of the second-order framework, and with S_α an asymptotically sum-stable standard law with a characteristic exponent, α, the following asymptotic distributional representations hold.*

(i) *If $1/(2\xi) < p < 1/\xi$, with $\alpha = 1/(p\xi)$, and $\sigma_p^*(\xi) = (1-p\xi)^2 C_p(\xi)/p$, with $C_p(\xi) = [p\xi\Gamma(2-1/(p\xi))|\cos(\pi/(2p\xi))|/(1-p\xi)]^{p\xi}$, we get*

$$\mathrm{H}_p(k) \stackrel{d}{=} \xi + \frac{\sigma_p^*(\xi)\ S_\alpha}{k^{1-p\xi}} + \frac{1-p\xi}{1-p\xi-\rho} A(n/k) + o_{\mathbb{P}}(A(n/k)), \qquad (4.1)$$

with $\mathrm{H}_p(k)$ in (4.1) having the same bias term as in (2.3), but a rate of convergence of the order of $k^{1-p\xi}$, $0 < 1-p\xi < 1/2$, towards a sum-stable RV, *with $1 < \alpha = 1/(p\xi) < 2$.*

(ii) *If $p = 1/(2\xi)$,*

$$\mathrm{H}_p(k) \stackrel{d}{=} \xi + \frac{S_{2,0}}{4p\sqrt{k/\ln k}} + \frac{A(n/k)}{1-2\rho} + o_{\mathbb{P}}(A(n/k)), \qquad (4.2)$$

with $\mathrm{H}_p(k)$ in (4.2) having again the same bias term as in (2.3), but a rate of convergence of the order of $\sqrt{k/\ln k}$, and towards a sum-stable RV, *with $\alpha = 1/(p\xi) = 2$.*

(iii) *If $p = 1/\xi$, with $w = 1-\gamma-\ln(2/\pi)$, and on the basis of (3.5),*

$$\mathrm{H}_p(k) \stackrel{d}{=} \xi\left(1 - \frac{1}{\ln k} + \left(\frac{w+\pi/2\ S_1}{\ln^2 k} - \frac{pA(n/k)}{\rho\ln k}\right)(1+o_{\mathbb{P}}(1))\right). \quad (4.3)$$

Proof Since $X_i \stackrel{d}{=} Y_i$, with $Y_1, \ldots, Y_n$ independent, identically distributed unit Pareto RVs (i.e. with CDF $G(y) = 1 - 1/y$, $y > 1$), the associated order statistics are such that $Y_{n-i+1:n}/Y_{n-k:n} \stackrel{d}{=} Y_{k-i+1:k}$, $1 \le i \le k$. Moreover, $kY_{n-k:n}/n \underset{n\to\infty}{\overset{p}{\longrightarrow}} 1$, i.e. $Y_{n-k:n} \stackrel{p}{\sim} n/k$. With $k \to \infty$ and $k/n \to 0$, as $n \to \infty$, we have from equation (1.2),

$$U_{ik} := \frac{X_{n-i+1:n}}{X_{n-k:n}} \stackrel{d}{=} \frac{U(Y_{n-i+1:n})}{U(Y_{n-k:n})} = \frac{U(Y_{n-k:n}Y_{k-i+1:k})}{U(Y_{n-k:n})} \stackrel{d}{=} Y_{k-i+1:k}(1+o_{\mathbb{P}}(1)).$$

Just as mentioned in [11], the term $o_{\mathbb{P}}(1)$ above is uniform in i, $1 \le i \le k$. This comes from the results in [42] (see Theorem B.2.18 in [43]), jointly with the fact that for uniform order statistics $U_{i:n}$, $1 \le i \le n$, we have that $1/U_{i:n}$ can be uniformly bounded in probability by $C[i/(n+1)]^{-1}$ (for some constant C). With $\mathrm{A}_p(k)$ defined in (1.7), we can write

$$p \ln \mathrm{A}_p(k) = p \ln \left(\frac{1}{k} \sum_{i=1}^{k} \left(\frac{X_{n-i+1:n}}{X_{n-k:n}} \right)^p \right)^{1/p} \stackrel{d}{=} \ln \left(\frac{1}{k} \sum_{i=1}^{k} \left(\frac{U(Y_{n-i+1:n})}{U(Y_{n-k:n})} \right)^p \right).$$

Consequently,

$$p \ln \mathrm{A}_p(k) \stackrel{d}{=} \ln \left(\frac{1}{k} \sum_{i=1}^{k} Y_i^{p\xi} (1 + o_{\mathbb{P}}(1)) \right).$$

Let's go back to the functional in (1.8):

$$\mathrm{H}_p(k) = \frac{1 - \exp\left(-p \ln \mathrm{A}_p(k)\right)}{p} \stackrel{d}{=} \frac{1}{p} \left(1 - \left(\frac{1}{k} \sum_{i=1}^{k} Y_i^{\xi p} (1 + o_{\mathbb{P}}(1)) \right)^{-1} \right). \tag{4.4}$$

If $p < 1/\xi$, and on the basis of (3.6) and (4.4), $\mathrm{H}_p(k)$, in (1.8), converges weakly to ξ, as $n \to \infty$. For $p = 1/\xi$, we can write (4.4) as

$$\mathrm{H}_p(k) \stackrel{d}{=} \xi \left(1 - \left(\frac{1}{k} \sum_{i=1}^{k} Y_i (1 + o_{\mathbb{P}}(1)) \right)^{-1} \right).$$

Then, on the basis of (3.6), $\mathrm{H}_p(k) = \xi(1 + O_{\mathbb{P}}(1/\ln k))$ is consistent for the estimation of ξ even if $p = 1/\xi$.

But if $p > 1/\xi$, we see from (3.6) that $\sum_{i=1}^{k} Y_i^{p\xi}/k = O_{\mathbb{P}}(k^{p\xi - 1}) \to \infty$, as $k \to \infty$. Then, from (4.4), $\mathrm{H}_p(k)$ converges to $1/p < \xi$, i.e. $\mathrm{H}_p(k)$ is no longer consistent for the estimation of ξ.

Working under the second-order framework in (1.3), we can write, for all $x > 0$, with $t \to \infty$, $U(tx)/U(t) = x^{\xi}(1 + A(t)(x^{\rho} - 1)/\rho)(1 + o(1))$, and (4.4) can be written as

$$\mathrm{H}_p(k) = \frac{1}{p} \left(1 - \Big(\frac{1}{k} \sum_{i=1}^{k} Y_i^{p\xi} + pA(n/k)\, \frac{1}{k} \sum_{i=1}^{k} Y_i^{p\xi} (Y_i^{\rho} - 1)/\rho + o_{\mathbb{P}}(A(n/k)) \Big)^{-1} \right).$$

For $p < 1/\xi$, and for any $a < 0$,

$$\mathbb{E}\left(Y^{p\xi}(Y^a - 1)/a\right) = \frac{1}{a} \left(\frac{1}{1 - p\xi - a} - \frac{1}{1 - p\xi} \right) = \frac{1}{(1 - p\xi)(1 - p\xi - a)}.$$

Let us now consider separately the three regions in the statement of the theorem:

1. If $1/2 < p\xi < 1$, from (3.6), with $C_p(\xi) = [p\xi\Gamma(2-1/(p\xi))|\cos(\pi/(2p\xi))|/(1-p\xi)]^{p\xi}$ and $\alpha = 1/(p\xi)$, we can write

$$\begin{aligned} &p\mathrm{H}_p(k)\\ &= 1-(1-p\xi)/\left(1+\frac{(1-p\xi)C_p(\xi)S_\alpha}{k^{1-p\xi}}+\frac{pA(n/k)}{1-p\xi-\rho}+o_{\mathbb{P}}(A(n/k))\right)\\ &= p\xi+\frac{(1-p\xi)^2C_p(\xi)S_\alpha}{k^{1-p\xi}}+\frac{p(1-p\xi)A(n/k)}{1-p\xi-\rho}+o_{\mathbb{P}}(A(n/k)), \end{aligned}$$

and (4.1) follows.
2. If $p\xi = 1/2$, from (3.3) and (3.6), we can write

$$p\mathrm{H}_p(k) = p\xi+\frac{(1-p\xi)^2S_2}{\sqrt{k/\ln k}}+\frac{p(1-p\xi)A(n/k)}{1-p\xi-\rho}+o_{\mathbb{P}}(A(n/k)),$$

and (4.2) follows.
3. If $p\xi = 1$, $\mathbb{E}(Y^{1+\rho}/\rho) = -1/\rho^2,\ \rho < 0$, and

$$\begin{aligned} \mathrm{H}_p(k) = \xi\Bigg(1-1/\Bigg(&\frac{1}{k}\sum_{i=1}^{k}Y_i - pA(n/k)\,\frac{1}{k}\sum_{i=1}^{k}\frac{Y_i}{\rho}\\ &+pA(n/k)\,\frac{1}{k}\sum_{i=1}^{k}\frac{Y_i^{\rho+1}}{\rho}+o_{\mathbb{P}}(A(n/k))\Bigg)\Bigg). \end{aligned}$$

From (3.6), with $w = 1-\gamma-\ln(2/\pi)$, we have

$$\begin{aligned} &\mathrm{H}_p(k)\\ &\stackrel{d}{=}\xi\Big(1-1/\Big(\big(1-pA(n/k)/\rho\big)(\ln k+w+\pi/2\ S_1)\\ &\quad -pA(n/k)/\rho^2+o_{\mathbb{P}}(A(n/k))\Big)\Big)\\ &\stackrel{d}{=}\xi\left(1-\frac{1}{\ln k}\left(\frac{1}{\left(1-\frac{pA(n/k)}{\rho}\right)\left(1+\frac{w+\pi/2\ S_1}{\ln k}\right)-\frac{pA(n/k)}{\rho^2\ln k}+o_{\mathbb{P}}\left(\frac{A(n/k)}{\ln k}\right)}\right)\right), \end{aligned}$$

and (4.3) follows.

□

Remark 4.1 Note that, contrarily to the random behaviour, either normal or non-normal, of $\mathrm{H}_p(k)-\xi$, as $k\to\infty$, when $p < 1/\xi$ (see items (i) and (ii) of Theorem 4.1), for $p = 1/\xi$, item (iii) of the same theorem enables us to conclude that, despite of at the slow rate of $1/\ln k$, $\mathrm{H}_p(k)-\xi$ converges in a degenerate way to zero, as $k\to\infty$, a point that is obviously in favour of what has been detected by simulation in Sect. 2.

5 Overall Comments

It has been clear for a long time that the H EVI-estimators lead often to a high over-estimation of the EVI, even at optimal levels, in the sense of minimal MSE. The use of the extra tuning parameter $p \in \mathbb{R}$ and the MO_p methodology can thus provide a much more adequate EVI-estimation, with the PORT MO_p, and *a fortiori* the PORT-RBMO$_p$ EVI-estimation, providing even better results. But for a consistent EVI-estimation we can now go up to $p = 1/\xi$, getting then a sum-stable behaviour, with an index of sum-stability $\alpha = 1/(p\xi)$. Under non-regular frameworks, an improvement similar to the one obtained for $\mathrm{GEV}_{0.25}$, Student t_4 and GP_1 underlying models happens for the great majority of simulated models, with no need to go further to a PORT-RBMO$_p$ EVI-estimation being the MO_p EVI-estimation highly reliable, provided that we adequately choose the tuning parameter p. Several algorithms have been conceived for an 'almost degenerate' EVI-estimation. The obvious challenge is not to go beyond $p = 1/\xi$, and the devised algorithms are often able to perform such a goal, being their comparative performance still under study, and out of the scope of this article. Notice that a double-bootstrap algorithm, of the type of the one in [7], can be used, with some minor modifications. However, such an algorithm relies too much of the finite variance of the normal asymptotic RV associated with the asymptotic behaviour of $\mathrm{H}_p(k)$, needs to be slightly modified, being still under study. But very simple heuristic algorithms, like the one associated with $\arg\min_{(p,k)}(H_p(k) - 1/p)^2$, seem to work adequately in a great variety of situations. For an adaptive choice of the tuning parameters p and k, essentially on the basis of sample path stability, see [33].

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