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Higher-Order Functional Discontinuous Boundary Value Problems on the Half-Line

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Abstract: In this paper, we consider a discontinuous, fully nonlinear, higher-order equation on the half-line, together with functional boundary conditions, given by general continuous functions with dependence on the several derivatives and asymptotic information on the $(n-1)th$ derivative of the unknown function. These functional conditions generalize the usual boundary data and allow other types of global assumptions on the unknown function and its derivatives, such as nonlocal, integro-differential, infinite multipoint, with maximum or minimum arguments, among others. Considering the half-line as the domain carries on a lack of compactness, which is overcome with the definition of a space of weighted functions and norms, and the equiconvergence at ∞. In the last section, an example illustrates the applicability of our main result.

Keywords: functional higher-order problems; unbounded solutions; half-line; fixed-point theory

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1. Introduction

This work considers a fully nonlinear, higher-order discontinuous equation on the half-line

$$u^{(n)}(t) = f(t, u(t), u'(t), \dots, u^{(n-1)}(t)), \quad t \in [0, +\infty[, \tag{1}$$

where $f : [0, +\infty[\times\mathbb{R}^n \to \mathbb{R}$ is a L^1- Carathéodory function, with the functional boundary conditions,

$$\begin{cases} L_i\Big(u^{(i)}, u(0), u'(0), \dots, u^{(n-2)}(0)\Big) = 0, \ i = 0, \dots, n-2, \\ L_{n-1}\Big(u, u^{(n-1)}(+\infty)\Big) = 0, \end{cases} \tag{2}$$

with $u^{(n-1)}(+\infty) := \lim\limits_{t\to+\infty} u^{(n-1)}(t)$, $L_i : C([0, +\infty[) \times \mathbb{R}^{n-1} \to \mathbb{R}$, $i = 0, 1, \dots, n-2$, and $L_{n-1} : C([0, +\infty[) \times \mathbb{R} \to \mathbb{R}$ continuous functions.

These types of higher-order boundary value problems have been considered by many authors, not only with a general higher-order derivative n, but also for particular cases of n. Most of all, they are studied for continuous nonlinearities, and in bounded intervals, with classical boundary conditions, such as [1,2], for linear problems [3,4], for two-point separated and Sturm-Liouville boundary conditions [5–7], for multipoint problems [8], and for periodic solutions, among others.

The functional boundary conditions in higher-order problems can include global data on the unknown variable and its derivatives, and, in this way, they generalize the usual boundary assumptions, considering local, nonlocal, or integro-differential conditions, with deviating arguments, delays or advances, maxima or minima of some variables. For work dealing with these features, see [9–18] and the references therein.

On unbounded intervals, there is a lack of compacity on the operator, which can be overcome by applying some adequate techniques to guarantee the solvability. As examples, we mention the extension by continuity of some adequate bounded intervals by a diagonalization method, the definition of suitable Banach spaces and norms to obtain sufficient conditions for the existence of fixed points, and the lower and upper solutions technique. Interested readers can see these methods in, for example, [19–25] and in their references.

In more detail, we refer to [26], where the authors study the the nth-order differential equation on the half-line

$$-u^{(n)}(t) = q(t)f(t,\, u(t), \dots, u^{(n-1)}(t)), \;\; t \in (0, +\infty),$$

where $q : (0, +\infty) \to (0, +\infty)$, $f : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ are continuous, together with the boundary conditions

$$\begin{cases} u^{(i)}(0) = A_i, \;\; i = 0, 1, \dots, n-3, \\ u^{(n-2)}(0) - a u^{(n-1)}(0) = B, \\ u^{(n-1)}(+\infty) = C, \end{cases}$$

with $a > 0$, $A_i, B, C \in \mathbb{R}$, $i = 0, 1, \dots, n-3$. Applying the lower and upper solutions method and the Schäuder fixed-point theorem, the authors prove the existence of a solution, and from the topological degree theory, of triple solutions.

In [27], it is considered the problem with the ϕ–Laplacian type differential equation

$$-\left(\phi(u^{(n-1)}(t))\right)' = f\left(t, u(t), u'(t), \dots, u^{(n-1)}(t)\right),$$

defined on the bounded interval $(0, 1)$, where $n \geq 2$, ϕ is an increasing homeomorphism and $f : (0, 1) \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, and the functional boundary conditions

$$\begin{cases} g_i\left(u, u', \dots, u^{(n-1)}, u^{(i)}(0)\right) = 0, \; i = 0, \dots, n-2, \\ g_{n-1}\left(u, u', \dots, u^{(n-1)}, u^{(n-2)}(1)\right) = 0, \end{cases}$$

with $g_i : (C[0, 1])^n \times \mathbb{R} \to \mathbb{R}$, $i = 0, \dots, n-1$, continuous functions. Applying the lower and upper solutions method, together with a Nagumo-type condition, it is proved that, for $n \geq 3$, the order between the upper and lower solutions and the correspondent derivatives is not relevant. The type of order depends on whether n is even or odd, and on the existent relationship between the $(n-2) - nd$ derivatives of the upper and lower functions. Moreover, the monotonic behavior of the nonlinearities is related to the parity of n.

In our problem, we combine, for the first time, as far as we know, all these features, taking advantage of all of them and allowing their application to a wider range of real-life problems and phenomena. In short, the method is based on the definition of an auxiliary problem, composed by a truncated and perturbed equation, with initial values and the asymptotic behavior of the higher derivative given by truncated functions, which include the functional data. An adequate operator is defined in a weighted Banach space, and the lack of compactness is overcome by considering weighted norms. Sufficient conditions are given to have fixed points, via Schauder's fixed point theorem. The lower and upper solutions method is used to prove that these fixed points, solutions of the auxiliary problem, are solutions to the initial problem, too. Moreover, despite the localization part, we stress that these solutions may be unbounded.

The fact that the non-linearity of (1) and the boundary conditions (2) are very general, allows the problem to cover a wide number of applications. As an example, for $n = 2$, we mention an industrial micro-engineering problem to study a membrane MEMS device via an elliptic semilinear 1D model, referred to in [28]. Another possible application for higher-order problems defined on unbounded intervals is, for $n = 4$, the study of the bending of infinite beams with different types of foundations, as can be seen, for example, in [29–32]. We point out that the functional boundary conditions, as (2), allow us to consider new types of models, where, for example, global data on the beam could be considered, which is new in the literature.

The paper's structure is the following: Section 2 contains the definitions of the weighted Banach space and norms, some *a priori* bounds, and other auxiliary results. In Section 3, the main result is presented: an existence and localization theorem for the functional problem. The last section is concerned with a numerical example subject to global boundary conditions.

2. Definitions and Auxiliary Results

In this work, we apply the so-called Bielecki's method and the correspondent weighted Bielecki's norms. As far as we know, this technique was introduced in [33], and, originally, it was used so that the exponential function was the weighted function. Some authors still used it, as in [34], but we may use weaker weighted functions: polynomial functions, as in [35] or [26].

To the best of our knowledge, it is the first time that this method is applied to functional problems of order n on unbounded intervals.

Consider the space

$$X = \left\{ x \in C^{n-1}[0,+\infty[: \lim_{t\to+\infty} \frac{x^{(i)}(t)}{1+t^{n-1-i}} \text{ exists in } \mathbb{R},\ i = 0,1,\ldots,n-1 \right\}$$

with the norm $\|x\|_X := \max\left\{ \left\| x^{(i)} \right\|_i \right\}$, $i = 0,1,\ldots,n-1$, where

$$\left\| \omega^{(i)} \right\|_i = \sup_{0\le t<+\infty} \left| \frac{\omega^{(i)}(t)}{1+t^{n-1-i}} \right|,\ i = 0,1,\ldots,n-1.$$

In this way, it is clear that $(X, \|.\|_X)$ is a Banach space.

The next definition gives the regularity of the nonlinear part:

Definition 1. *A function $f : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}$ is called a L^1- Carathéodory function if it verifies:*

(i) *For each $(y_0,\ldots,y_{n-1}) \in \mathbb{R}^n$, $t \mapsto f(t,y_0,\ldots,y_{n-1})$ is measurable on $[0,+\infty[$;*
(ii) *For almost every $t \in [0,+\infty[$, $(y_0,\ldots,y_{n-1}) \mapsto f(t,y_0,\ldots,y_{n-1})$ is continuous in $\mathbb{R}^n$;*
(iii) *For each $\rho > 0$, there exists a positive function $\varphi_\rho \in L^1[0,+\infty[$, $j = 0,1,\ldots,n-1$, such that whenever $(t,y_0,\ldots,y_{n-1}) \in [0,+\infty[\times\mathbb{R}^n$ satisfies $|y_i| < \rho\left(1+t^{n-1-i}\right)$, $i = 0,1,\ldots,n-1$, one has*

$$|f(t,y_0,\ldots,y_{n-1})| \le \varphi_\rho(t),\ a.e.\ t \in [0,+\infty[.$$

Solutions of the linear problem associated to (1)–(2) are defined with kernels given by Green's function, which can be obtained by standard calculus, as in [26], Lemma 2.1:

Lemma 1. *For $h \in L^1[0,+\infty[$, the linear boundary value problem*

$$\left\{ \begin{array}{l} u^{(n)}(t) = h(t),\ a.e.\ t \in [0,+\infty[, \\ u^{(i)}(0) = A_i,\ i = 0,1,\ldots,n-2, \\ u^{(n-1)}(+\infty) = B, \end{array} \right. \tag{3}$$

with $A_i, B \in \mathbb{R}$, $i = 0,1,\ldots,n-2$, has a unique solution, given by

$$u(t) = \sum_{i=0}^{n-2} \frac{A_i}{i!} t^i + \frac{B}{(n-1)!} t^{n-1} + \int_0^{+\infty} G(t,s)h(s)ds, \tag{4}$$

where

$$G(t,s) = \begin{cases} \sum_{j=0}^{n-2} \left(\frac{(-1)^j}{(n-2-j)!(j+1)!} s^{j+1} t^{n-2-j} \right), & 0 \le s \le t < +\infty \\ \frac{1}{(n-1)!} t^{n-1}, & 0 \le t \le s < +\infty. \end{cases} \tag{5}$$

Remark 1. *The Green function given by (5) satisfies*

$$\lim_{t \to +\infty} \frac{G_i(t,s)}{1 + t^{n-1-i}} \in \mathbb{R}, \text{for } i = 0, 1, \dots, n-1,$$

with

$$G_i(t,s) := \frac{\partial^i G}{\partial t^i}(t,s) = \sum_{j=0}^{n-2-i} \left(\frac{(-1)^{j+i}}{(n-2-j-i)!(j+1)!} s^{j+1} t^{n-2-j-i} \right). \tag{6}$$

To apply the fixed-point theory, we need an a priori estimation for $u^{(n-1)}(t)$, given by a Nagumo-type condition:

Let $\gamma, \Gamma \in X$, and define the set

$$E = \left\{ \begin{array}{l} (t, y_0, \dots, y_{n-1}) \in [0, +\infty[\times \mathbb{R}^n : \gamma^{(i)}(t) \le y_i \le \Gamma^{(i)}(t), \\ i = 0, 1, \dots, n-2, \ \gamma^{(n-1)}(+\infty) \le y_{n-1} \le \Gamma^{(n-1)}(+\infty) \end{array} \right\}. \tag{7}$$

Definition 2. *A L^1- Carathéodory function $f : E \to \mathbb{R}$ satisfies a Nagumo-type growth condition in E if it verifies*

$$|f(t, y_0, \dots, y_{n-1})| \le \psi(t) \Phi(|y_{n-1}|), \forall (t, y_0, \dots, y_{n-1}) \in E, \tag{8}$$

for some positive continuous functions ψ, Φ, and some $\nu > 1$, such that

$$\sup_{0 \le t < +\infty} \psi(t)(1+t)^\nu < +\infty, \int_0^{+\infty} \frac{s}{\Phi(s)} ds = +\infty. \tag{9}$$

The next lemma provides an a priori bound:

Lemma 2. *Let $f : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}$ be a L^1- Carathéodory function such that (8) and (9) hold on E. Then, for every $r > 0$, there is $R > 0$ (which does not depend on u) such that every u solution of (1), with*

$$\begin{gathered} \gamma^{(i)}(t) \le u^{(i)}(t) \le \Gamma^{(i)}(t), i = 0, 1, \dots, n-2, \\ \gamma^{(n-1)}(+\infty) \le u^{(n-1)}(+\infty) \le \Gamma^{(n-1)}(+\infty), \end{gathered} \tag{10}$$

for $t \in [0, +\infty[$, verifies

$$\left\| u^{(n-1)} \right\|_{n-1} < R. \tag{11}$$

Proof. Let u be a solution of (1) such that (10) holds. Consider $r > 0$ such that

$$r > \max\left\{ \left| \gamma^{(n-1)}(+\infty) \right|, \left| \Gamma^{(n-1)}(+\infty) \right| \right\}. \tag{12}$$

By the previous inequality and (7), $\left| u^{(n-1)}(t) \right| > r, \forall t \in [0, +\infty[$, cannot happen.

If $\left| u^{(n-1)}(t) \right| \le r, \forall t \in [0, +\infty)$, taking $R > r/2$, the proof is complete, as

$$\left\| u^{(n-1)} \right\|_{n-1} = \sup_{0 \le t < +\infty} \left| \frac{u^{(n-1)}(t)}{2} \right| \le \frac{r}{2} < R.$$

If there is $t_0 \in [0, +\infty)$ such that $\left|u^{(n-1)}(t_0)\right| > r$, then, in the case that $u^{(n-1)}(t_0) > r$, by (9), we take $R > r$ such that

$$\int_r^R \frac{s}{\Phi(s)} ds > M \max \left\{ \begin{array}{c} M_1 + \sup_{0 \leq t < +\infty} \frac{\Gamma^{(n-2)}(t)}{1+t} \frac{\nu}{\nu-1}, \\ \\ M_1 - \inf_{0 \leq t < +\infty} \frac{\gamma^{(n-2)}(t)}{1+t} \frac{\nu}{\nu-1} \end{array} \right\}$$

with

$$M := \sup_{0 \leq t < +\infty} \psi(t)(1+t)^\nu \text{ and } M_1 := \sup_{0 \leq t < +\infty} \frac{\Gamma^{(n-2)}(t)}{(1+t)^\nu} - \inf_{0 \leq t < +\infty} \frac{\gamma^{(n-2)}(t)}{(1+t)^\nu}.$$

By (12), choose $t_1 \in (0, +\infty)$ such that $t_1 > t_0$ and

$$u^{(n-1)}(t_1) = r,\ u^{(n-1)}(t) > r, \forall t \in [t_0, t_1[.$$

Then,

$$\begin{aligned}
\int_{u^{(n-1)}(t_1)}^{u^{(n-1)}(t_0)} \frac{s}{\Phi(s)} ds &= \int_{t_1}^{t_0} \frac{u^{(n-1)}(s)}{\Phi|u^{(n-1)}(s)|} u^{(n)}(s) ds \\
&\leq \int_{t_0}^{t_1} \frac{\left|f(s, u(s), ..., u^{(n-1)}(s))\right|}{\Phi(u^{(n-1)}(s))} u^{(n-1)}(s) ds \\
&\leq \int_{t_0}^{t_1} \psi(s)\, u^{(n-1)}(s)\, ds \leq M \int_{t_0}^{t_1} \frac{u^{(n-1)}(s)}{(1+s)^\nu} ds \\
&= M \int_{t_0}^{t_1} \left(\frac{u^{(n-2)}(s)}{(1+s)^\nu} \right)' + \frac{\nu u^{(n-2)}(s)}{(1+s)^{1+\nu}} ds \\
&= M \left(\frac{u^{(n-2)}(t_1)}{(1+t_+)^\nu} - \frac{u^{(n-2)}(t_0)}{(1+t_*)^\nu} + \int_{t_0}^{t_1} \frac{\nu u^{(n-2)}(s)}{(1+s)^{1+\nu}} ds \right) \\
&\leq M \left(M_1 + \sup_{0 \leq t < +\infty} \frac{\Gamma^{(n-2)}(t)}{1+t} \int_0^{+\infty} \frac{\nu}{(1+s)^\nu} ds \right) \\
&< \int_r^R \frac{s}{\Phi(s)} ds.
\end{aligned}$$

So, $u^{(n-1)}(t_0) < R$, and as t_1, t_0 are arbitrary in $[0, +\infty)$, for the values where $u^{(n-1)}(t) > r$, we have that $u^{(n-1)}(t) < R, \forall t \in [0, +\infty[$.

By the same technique, considering t_- and t_* such that $u^{(n-1)}(t_-) < -r$, $u^{(n-1)}(t_*) = -r$, $u^{(n-1)}(t) < -r, \forall t \in [t_-, t_*]$, it can be proved that $u^{(n-1)}(t) > -R, \forall t \in [0, +\infty[$, and, therefore, $\left\|u^{(n-1)}\right\|_{n-1} < \frac{R}{2} < R, \forall t \in [0, +\infty)$. □

The next result will play a key role to apply a fixed-point theorem.

Lemma 3 ([19])**.** *A set $M \subset X$ is relatively compact if the following conditions hold:*

1. *All functions from M are uniformly bounded;*
2. *All functions from M are equicontinuous on any compact interval of $[0, +\infty[$;*
3. *All functions from M are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a $t_\epsilon > 0$ such that, for $i = 0, 1, \dots, n-1$,*

$$\left| \frac{u^{(i)}(t)}{1+t^{n-1-i}} - \lim_{t \to +\infty} \frac{u^{(i)}(t)}{1+t^{n-1-i}} \right| < \epsilon, \text{ for all } t > t_\epsilon,\ x \in M.$$

Upper and lower solutions are defined as follows:

Definition 3. *A function $\alpha \in C^n[0,+\infty[\cap X$ is a lower solution of problem (1) and (2) if*

$$\begin{cases} \alpha^{(n)}(t) \geq f(t,\alpha(t),\alpha'(t),\ldots,\alpha^{(n-1)}(t)), \quad t \in [0,+\infty[, \\ L_i(\alpha^{(i)},\alpha(0),\ldots,\alpha^{(n-2)}(0) \geq 0, \; i=0,1,\ldots,n-2, \\ L_{n-1}\Big(\alpha,\alpha^{(n-1)}(+\infty)\Big) > 0 \end{cases}$$

A function β is an upper solution of problem (1) and (2) if it verifies the reversed inequalities.

Forward, the boundary functions L_j, for $j=0,1,\ldots,n-1$, must verify the following assumptions:

(H_1) For $i=0,1,\ldots,n-2$, $L_i(w,y_0,y_1,\ldots,y_{n-2})$ is nondecreasing in all the arguments except in the $(i+2)-nd$ variable;

(H_2) $\lim\limits_{t\to\infty} L_{n-1}(w,z) \in \mathbb{R}$ for $\alpha \leq w \leq \beta$ and $\alpha^{(n-1)}(+\infty) \leq z \leq \beta^{(n-1)}(+\infty)$;

(H_3) $L_{n-1}(w,z)$ is nondecreasing on w for fixed z.

3. Main Result

This section contains an existence and localization result—that is, not only is the existence of at least a solution for problem (1) and (2) proved, but it also provides some localization data for this solution and its derivatives.

Theorem 1. *Let $f:[0,+\infty[\times\mathbb{R}^n \to \mathbb{R}$ be a L^1- Carathéodory function, and α,β lower and upper solutions of (1) and (2), respectively, such that*

$$\alpha^{(n-2)}(t) \leq \beta^{(n-2)}(t), \forall t \in [0,+\infty[. \tag{13}$$

$$\alpha^{(i)}(0) \leq \beta^{(i)}(0), \; i=0,1,\ldots,n-3, \tag{14}$$

and

$$\alpha^{(n-1)}(+\infty) \leq \beta^{(n-1)}(+\infty). \tag{15}$$

Assume that f verifies the Nagumo conditions (8) and (9) in the set

$$E_* = \left\{ \begin{array}{c} (t,y_0,\ldots,y_{n-1}) \in [0,+\infty[\times\mathbb{R}^n, \alpha^{(i)}(t) \leq y_i \leq \beta^{(i)}(t), i=0,1,\ldots,n-2, \\ \alpha^{(n-1)}(+\infty) \leq y_{n-1} \leq \beta^{(n-1)}(+\infty) \end{array} \right\},$$

and

$$\begin{aligned} f\Big(t,\alpha(t),\alpha'(t),\ldots,\alpha^{(n-3)}(t),y_{n-2},y_{n-1}\Big) &\geq f(t,y_0,\ldots,y_{n-1}) \\ &\geq f\Big(t,\beta(t),\beta'(t),\ldots,\beta^{(n-3)}(t),y_{n-2},y_{n-1}\Big) \end{aligned} \tag{16}$$

and the assumptions (H_1)–(H_3) hold.

If there is $\rho > 0$ such that,

$$\max\left\{ \begin{array}{c} \|\alpha\|_X, \|\beta\|_X, R, \\ \max\limits_{i=0,1,\ldots,n-2} \left\{ \sum\limits_{j=i}^{n-2} \frac{M_j}{(j-i)!} + \frac{M_\infty}{(n-1-i)!} + \int\limits_0^{+\infty} M_i(s)\Big(\varphi_\rho(s) + \frac{1}{1+s^{2n}}\Big)ds \right\}, \\ \frac{M_\infty}{2} + \frac{1}{2}\int\limits_0^{+\infty}\Big(\varphi_\rho(s) + \frac{1}{1+s^{2n}}\Big)ds \end{array} \right\} < \rho \tag{17}$$

where

$$M_j := \max\Big\{\Big|\alpha^{(j)}(0)\Big|, \Big|\beta^{(j)}(0)\Big|\Big\},$$

$$M_\infty := \max\left\{ \left|\alpha^{(n-1)}(+\infty)\right|, \left|\beta^{(n-1)}(+\infty)\right| \right\},$$

$$M_i(s) := \sup_{0\le t<+\infty} \frac{|G_i(t,s)|}{1+t^{n-1-i}},\ i=0,1,\ldots,n-1, \tag{18}$$

and $G_i(t,s)$ are given by (6), then, for R given by (11), there is $u \in X$, a solution of problem (1) and (2), such that

$$\begin{aligned} \alpha^{(i)}(t) &\le u^{(i)}(t) \le \beta^{(i)}(t), i=0,1,\ldots,n-2, \\ -R &< u^{(n-1)}(t) < R, \textit{ for } t \in [0,+\infty[, \\ \alpha^{(n-1)}(+\infty) &\le u^{(n-1)}(+\infty) \le \beta^{(n-1)}(+\infty). \end{aligned}$$

Proof. By integration of (13) and (14), $\alpha^{(j)}(t) \le \beta^{(j)}(t)$, $j=0,1,\ldots,n-3$, for $t\in[0,+\infty[$. So, let us consider the truncated and perturbed equation

$$\begin{aligned} u^{(n)}(t) &= f\Big(t,\delta_0(t,u(t)),\delta_1(t,u'(t)),\ldots,\delta_{n-2}(t,u^{(n-2)}(t)),u^{(n-1)}(t))\Big) \\ &+\frac{1}{1+t^{2n}}\,\frac{u^{(n-2)}(t)-\delta_{n-2}(t,u^{(n-2)}(t))}{1+\left|u^{(n-2)}(t)-\delta_{n-2}(t,u^{(n-2)}(t))\right|}, t\in[0,+\infty[, \end{aligned} \tag{19}$$

where the functions $\delta_j : [0,+\infty[\times\mathbb{R}\to\mathbb{R}, i=0,1,\ldots,n-2,$ are given by

$$\delta_i(t,y_i) = \begin{cases} \beta^{(i)}(t) & , y_i > \beta^{(i)}(t) \\ y_i & , \alpha^{(i)}(t) \le y_i \le \beta^{(i)}(t) \\ \alpha^{(i)}(t) & , y_i < \alpha^{(i)}(t) \end{cases},\ i=0,1,2,\ldots,n-3, \tag{20}$$

together with the truncated boundary conditions and

$$\begin{cases} u^{(i)}(0) = \delta_i\Big(0,u^{(i)}(0)+L_i\Big(u^{(i)},u(0),u'(0),\ldots,u^{(n-1)}(0)\Big)\Big), \\ \qquad \text{for } i=0,1,\ldots,n-2, \\ u^{(n-1)}(+\infty) = \delta_\infty\Big(u^{(n-1)}(+\infty)+L_{n-1}\Big(u,\delta_\infty\Big(u^{(n-1)}(+\infty)\Big)\Big)\Big) \end{cases} \tag{21}$$

where

$$\delta_\infty(t,y) = \begin{cases} \beta^{(n-1)}(+\infty) & , & y>\beta^{(n-1)}(+\infty) \\ y & , & \alpha^{(n-1)}(+\infty)\le y\le\beta^{(n-1)}(+\infty) \\ \alpha^{(n-1)}(+\infty) & , & y<\alpha^{(n-1)}(+\infty). \end{cases} \tag{22}$$

Let us define the operator $T : X \to X$

$$Tu(t) = \sum_{j=0}^{n-2}\frac{A_j}{j!}t^j + \frac{B}{(n-1)!}t^{n-1} + \int_0^{+\infty} G(t,s)F_u(s)ds$$

with

$$A_j := \delta_j\Big(0,u^{(j)}(0)+L_j\Big(u^{(j)},u(0),u'(0),\ldots,u^{(n-2)}(0)\Big)\Big),$$

$$B := \delta_\infty\Big(u^{(n-1)}(+\infty)+L_{n-1}(u,\delta_\infty\Big(u^{(n-1)}(+\infty)\Big))\Big),$$

$$\begin{aligned} F_u(s) \quad : \quad &= f\Big(s,\delta_0(s,u(s))),\ldots,\delta_{n-2}\Big(s,u^{(n-2)}(s)\Big),u^{(n-1)}(s)\Big) + \\ &\frac{1}{1+s^{2n}}\frac{u^{(n-2)}(s)-\delta_{n-2}(s,u^{(n-2)}(s))}{1+\left|u^{(n-2)}(s)-\delta_{n-2}(s,u^{(n-2)}(s))\right|} \end{aligned}$$

and $G(t,s)$ given by (5).

The proof is divided into several steps:

STEP 1: *T is compact*

(i) $T : X \to X$ *is well-defined.*

Let $u \in X$. As f is a L^1-Carathéodory function by, $Tu \in C^{n-1}([0,+\infty[)$ and by Definition 1, for $u \in X$ such that $\|u\|_X < \rho_0$, with

$$\rho_0 > \max\{\|\alpha\|_X, \|\beta\|_X, R\}, \tag{23}$$

there is a positive function $\varphi_{\rho_0} \in L^1[0,+\infty[$, such that

$$\int_0^{+\infty} |F_u(s)|ds \leq \int_0^{+\infty} \left(\varphi_{\rho_0}(s) + \frac{1}{1+s^{2n}}\right)ds < +\infty.$$

Therefore, F_u is also a L^1- Carathéodory function.

Moreover, for $i = 0,1,\ldots,n-1$, and $G_i(t,s)$ given by (6), we have

$$\lim_{t\to+\infty} \frac{(Tu)^i(t)}{1+t^{n-1-i}} =$$

$$\lim_{t\to+\infty} \frac{1}{1+t^{n-1-i}} \left[\sum_{j=i}^{n-2} A_j \frac{t^{j-i}}{(j-i)!} + B\frac{t^{n-1-i}}{(n-1-i)!} + \int_0^{+\infty} G_i(t,s)F_u(s)ds\right] < +\infty,$$

that is, $Tu \in X$.

(ii) *T is continuous.*

For every convergent sequence $u_n \to u$ in X, there is $\rho > 0$ such that $\sup_n \|u_n\|_X < \rho$, and

$$\begin{aligned}
\|Tu_n - Tu\|_X &= \max\left\{\left\|(Tu_n)^{(i)} - (Tu)^{(i)}\right\|_i, i = 0,1,\ldots,n-1\right\} \\
&= \max_{i=0,1,\ldots,n-1}\left\{\sup_{0\leq t<+\infty} \frac{\left|(Tu_n)^{(i)}(t) - (Tu_n)^{(i)}(t)\right|}{1+t^{n-1-i}}\right\} \\
&\leq \max_{i=0,1,\ldots,n-1} \int_0^{+\infty} \frac{|G_i(t,s)|}{1+t^{n-1-i}} |F_{u_n}(s) - F_u(s)|ds \\
&\leq \max_{i=0,1,\ldots,n-1} \int_0^{+\infty} M_i(s)|F_{u_n}(s) - F_u(s)|ds \longrightarrow 0\,, \quad n \to +\infty,
\end{aligned}$$

with $M_i(s)$ given by (18).

(iii) *T is compact*.

Let $B \subset X$ be a bounded subset. So, there exists $r > 0$ such that $\|u\|_X < r, \forall u \in B$.

By (i) and (ii), it is clear that

$$\begin{aligned}
\|Tu\|_X &= \max\left\{\left\|(Tu)^{(i)}\right\|_i, i = 0,1,\ldots,n-1\right\} \\
&= \max_{i=0,1,\ldots,n-1}\left\{\sup_{0\leq t<+\infty} \frac{\left|(Tu)^{(i)}\right|}{1+t^{n-1-i}}\right\} < +\infty
\end{aligned}$$

and so, TB is uniformly bounded.

In order to prove that TB is equicontinuous, consider $L > 0$ and $t_1, t_2 \in [0, L]$. Suppose, without loss of generality, that $t_1 < t_2$. Then,

$$\left| \frac{(Tu)^{(i)}(t_1)}{1+t_1^{n-1-i}} - \frac{(Tu)^{(i)}(t_2)}{1+t_2^{n-1-i}} \right| \leq$$

$$\left| \begin{array}{c} \frac{1}{1+t_1^{n-1-i}} \left[\sum\limits_{j=i}^{n-2} A_j \frac{t_1^{j-i}}{(j-i)!} + B \frac{t_1^{n-1-i}}{(n-1-i)!} \right] \\ -\frac{1}{1+t_2^{n-1-i}} \left[\sum\limits_{j=i}^{n-2} A_j \frac{t_2^{j-i}}{(j-i)!} + B \frac{t_2^{n-1-i}}{(n-1-i)!} \right] \end{array} \right|$$

$$+ \int_0^{+\infty} \left| \frac{G_i(t,s)}{1+t_1^{n-1-i}} - \frac{G_i(t,s)}{1+t_2^{n-1-i}} \right| |F_u(s)| ds \to 0, \text{ as } t_1 \to t_2.$$

For $i = n-1$, the function G_{n-1} is not continuous for $s = t$, and

$$\left| \frac{(Tu)^{(n-1)}(t_1)}{2} - \frac{(Tu)^{(n-1)}(t_2)}{2} \right| \leq \int_0^{+\infty} \left| \frac{G_{n-1}(t_1,s) - G_{n-1}(t_2,s)}{2} \right| |F_u(s)| ds$$

$$\leq \frac{1}{2} \int_{t_1}^{t_2} |F_u(s)| ds \to 0, \text{ as } t_1 \to t_2.$$

Moreover, TB is equiconvergent at infinity because, by Lebesgue's Dominated Convergence Theorem, we obtain, for $i = 0, 1, \ldots, n-2$,

$$\left| \frac{(Tu)^{(i)}(t)}{1+t^{n-1-i}} - \lim_{t \to +\infty} \frac{(Tu)^{(i)}(t)}{1+t^{n-1-i}} \right| =$$

$$\left| \begin{array}{c} \frac{1}{1+t^{n-1-i}} \left(\sum\limits_{j=i}^{n-2} A_j \frac{t^{j-i}}{(j-i)!} + B \frac{t^{n-1-i}}{(n-1-i)!} \right) \\ + \int_0^{+\infty} G_i(t,s) F_u(s) ds - \frac{B}{(n-1-i)!} \end{array} \right| \to 0,$$

as $t \to +\infty$, and, for $i = n-1$,

$$\left| \frac{(Tu)^{(n-1)}(t)}{2} - \lim_{t \to +\infty} \frac{(Tu)^{(n-1)}(t)}{2} \right| = \frac{1}{2} \int_t^{+\infty} |F_u(s)| ds \to 0$$

as $t \to +\infty$.

Therefore, by Lemma 3, TB is relatively compact, and so, T is compact.

STEP 2. *The problems (19) and (21) at least have a solution.*

By Lemma 1, the fixed points of the operator T are solutions of (19) and (21). Therefore, it will be enough to show that T has a fixed point.

To apply Schauder's fixed-point theorem, we consider the non-empty, closed, bounded, and convex set $D \subset X$, defined by

$$D := \{u \in X : \|u\|_X \leq \rho_1\}$$

with $\rho_1 > 0$ given by

$$\max_{i=0,1,\dots,n-2} \left\{ \begin{array}{c} \sum\limits_{j=i}^{n-2} \frac{|A_j|}{(j-i)!} + \frac{|B|}{(n-1-i)!} + \int\limits_0^{+\infty} M_i(s)\left(\varphi_{\rho_1} + \frac{1}{1+s^{2n}}\right)ds, \\ \frac{|B|}{2} + \frac{1}{2}\int\limits_0^{+\infty}\left(\varphi_{\rho_1} + \frac{1}{1+s^{2n}}\right)ds \end{array} \right\} < \rho_1.$$

Let us prove that $TD \subset D$.
For $i = 0, 1, \dots, n-2$, and $u \in D$,

$$\begin{aligned}
\left\|(Tu)^{(i)}\right\|_X &= \max_i \left\|(Tu)^{(i)}\right\|_i = \max_i \left\{ \sup_{0 \le t < +\infty} \left| \frac{(Tu)^{(i)}(t)}{1+t^{n-1-i}} \right| \right\} \\
&\le \max_i \left[\sup_{0 \le t < +\infty} \left(\begin{array}{c} \sum\limits_{j=i}^{n-2} |A_j| \frac{t^{j-i}}{(1+t^{n-1-i})(j-i)!} + |B| \frac{t^{n-1-i}}{(1+t^{n-1-i})(n-1-i)!} \\ + \int\limits_0^{+\infty} |G_i(t,s)||F_u(s)|ds \end{array} \right) \right] \\
&\le \max_i \left\{ \sum_{j=i}^{n-2} \frac{|A_j|}{(j-i)!} + \frac{|B|}{(n-1-i)!} + \int_0^{+\infty} M_i(s)|F_u(s)|ds \right\} \\
&\le \max_i \left\{ \sum_{j=i}^{n-2} \frac{|A_j|}{(j-i)!} + \frac{|B|}{(n-1-i)!} + \int_0^{+\infty} M_i(s)\left(\varphi_{\rho_1} + \frac{1}{1+s^{2n}}\right)ds \right\} \\
&< \rho_1.
\end{aligned}$$

For $i = n-1$,

$$\begin{aligned}
\left\|(Tu)^{(n-1)}\right\|_{n-1} &= \sup_{0 \le t < +\infty} \left| \frac{(Tu)^{(n-1)}(t)}{2} \right| = \frac{1}{2}\left| B + \int_0^{+\infty} F_u(s)ds \right| \\
&\le \frac{|B|}{2} + \frac{1}{2}\int_0^{+\infty} \left(\varphi_{\rho_1} + \frac{1}{1+s^{2n}}\right)ds < \rho_1.
\end{aligned}$$

So, $TD \subset D$, and, by Schauder's fixed-point theorem, the operator T has a fixed point u_*, solution of problem (19) and (21).

STEP 3. *Every solution of problem (19) and (21) verifies*

$$\alpha^{(i)}(t) \le u^{(i)}(t) \le \beta^{(i)}(t), i = 0, 1, \dots, n-2, \tag{24}$$

$$-R < u^{(n-1)}(t) < R, \text{ for } t \in [0, +\infty[, \tag{25}$$

$$\alpha^{(n-1)}(+\infty) \le u^{(n-1)}(+\infty) \le \beta^{(n-1)}(+\infty). \tag{26}$$

Let u be a solution of (19) and (21) and assume that, by contradiction, there is $t \in [0, +\infty)$, such that $\alpha^{(n-2)}(t) > u^{(n-2)}(t)$. Therefore,

$$\inf_{0 \le t < +\infty} (u^{(n-2)}(t) - \alpha^{(n-2)}(t)) < 0.$$

By (14), this infimum cannot be attained on 0 as, by (20) and (21),

$$\begin{aligned}
(u^{(n-2)}(0) - \alpha^{(n-2)}(0)) &= \delta_{n-2}\left(0, u^{(n-2)}(0) + L_{n-2}\left(u^{(n-2)}, u(0), \dots, u^{(n-2)}(0)\right)\right) \\
&\quad - \alpha^{(n-2)}(0) \ge 0,
\end{aligned}$$

nor at $+\infty$, as, by (22),

$$\begin{gathered}(u^{(n-1)}(+\infty)-\alpha^{(n-1)}(+\infty))= \\ \delta_\infty\left(u^{(n-1)}(\infty)+L_{n-1}\left(u,\delta_\infty u^{(n-1)}(+\infty)\right)\right)-\alpha^{(n-1)}(+\infty)\geq 0.\end{gathered}$$

Then there is an interior point $t_* \in (0,+\infty)$, such that

$$\min_{0\leq t<+\infty}(u^{(n-2)}(t)-\alpha^{(n-2)}(t)):=u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)<0,$$

with $u^{(n-1)}(t_*)=\alpha^{(n-1)}(t_*)$ and $u^{(n)}(t_*)-\alpha^{(n)}(t_*)\geq 0$. Therefore, by (16) and Definition 3, we get the contradiction

$$\begin{aligned}0 &\leq u^{(n)}(t_*)-\alpha^{(n)}(t_*) \\ &= f(t_*,\delta_0(t_*,u(t_*)),\ldots,\delta_{n-2}(t_*,u^{(n-2)}(t_*)),u^{n-1}(t_*)) \\ &\quad +\frac{1}{1+t_*^{2n}}\frac{u^{(n-2)}(t_*)-\delta_{n-2}(t_*,u^{(n-2)}(t_*))}{1+|u^{n-2}(t_*)-\delta_{n-2}(t_*,u^{n-2}(t_*))|}-\alpha^{(n)}(t_*) \\ &= f(t_*,\delta_0(t_*,u(t_*)),\ldots,\delta_{n-3}(t_*,u^{(n-3)}(t_*)),\alpha^{(n-2)}(t_*),\alpha^{(n-1)}(t_*)) \\ &\quad +\frac{1}{1+t_*^{2n}}\frac{u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)}{1+\left|u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)\right|}-\alpha^{(n)}(t_*) \\ &\leq f\left(t_*,\alpha(t_*),\alpha'(t_*),\ldots,\alpha^{(n-1)}(t_*)\right) \\ &\quad +\frac{1}{1+t_*^{2n}}\frac{u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)}{1+\left|u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)\right|}-\alpha^{(n)}(t_*) \\ &\leq \frac{1}{1+t_*^{2n}}\frac{u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)}{1+\left|u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)\right|}<0.\end{aligned}$$

So, $u^{(n-2)}(t)\geq\alpha^{(n-2)}(t), \forall t\in[0,+\infty[$.
Analogously, it can be shown that $u^{(n-2)}(t)\leq\beta^{(n-2)}(t), \forall t\in[0,+\infty[$.
By (14), integrating on $[0,+\infty[$, we have

$$\begin{aligned}\alpha^{(n-3)}(t)\leq u^{(n-3)}(t)-\delta_{n-3}\left(0,u^{(n-3)}(0)+L_{n-3}\left(u^{(n-3)},u(0),\ldots,u^{(n-2)}(0)\right)\right) \\ +\alpha^{(n-3)}(0)\leq u^{(n-3)}(t),\end{aligned}$$

and by similar arguments,

$$\alpha^{(i)}(t)\leq u^{(i)}(t), \text{ for } t\in[0.+\infty[, \text{ and } i=0,1,\ldots,n-3.$$

With the same technique, it can be proved that

$$u^{(i)}(t)\leq\beta^{(i)}(t), \text{ for } t\in[0+\infty[, \text{ and } i=0,1,\ldots,n-2,$$

and, therefore,

$$\alpha^{(i)}(t)\leq u^{(i)}(t)\leq\beta^{(i)}(t), \forall t\in[0,+\infty[,\ i=0,1,\ldots,n-2.$$

So, this solution of problems (19) and (21) belongs to E_* and condition (25) is a direct consequence of Lemma 2.

Moreover, (26) is trivially verified, by (22).

STEP 4. *Let u_* be a solution of problems (19) and (21). Then, u_* is a solution of problems (1) and (2).*

According to Step 3, to prove this claim, it is enough to show that

$$\alpha^{(i)}(0) \leq u_*^{(i)}(0) + L_i\left(u_*^{(i)}, u_*(0), \ldots, u_*^{(n-2)}(0)\right) \leq \beta^{(i)}(0), \tag{27}$$

for $i = 0, 1, \ldots, n-2$, and

$$\alpha^{(n-1)}(+\infty) \leq u_*^{(n-1)}(+\infty) + L_{n-1}\left(\delta_\infty\left(u_*^{(n-1)}(+\infty)\right)\right) \leq \beta^{(n-1)}(+\infty).$$

Suppose that the first inequality of (27) does not hold for $i = n-2$. That is,

$$\alpha^{(n-2)}(0) > u_*^{(n-2)}(0) + L_{n-2}\left(u_*^{(n-2)}, u_*(0), \ldots, u_*^{(n-2)}(0)\right). \tag{28}$$

Therefore, by (20) and (21), we have

$$u_*^{(n-2)}(0) = \alpha^{(n-2)}(0).$$

By Definition 3 and (H_1), the following contradiction with (28) holds:

$$\begin{aligned} & u_*^{(n-2)}(0) + L_{n-2}\left(u_*^{(n-2)}, u_*(0), \ldots, u_*^{(n-3)}(0), u_*^{(n-2)}(0)\right) \\ = \; & \alpha^{(n-2)}(0) + L_{n-2}\left(u_*^{(n-2)}, u_*(0), \ldots, u_*^{(n-3)}(0), \alpha^{(n-2)}(0)\right) \\ \geq \; & \alpha^{(n-2)}(0) + L_{n-2}\left(\alpha^{(n-2)}, \alpha(0), \ldots, \alpha^{(n-3)}(0), \alpha^{(n-2)}(0)\right) \\ \geq \; & \alpha^{(n-2)}(0) \end{aligned}$$

.

A similar contradiction can be obtained in the remaining inequalities.
So, (27) holds.
Assume now that

$$u_*^{(n-1)}(+\infty) + L_{n-1}\left(u_*, \delta_\infty\left(u_*^{(n-1)}(+\infty)\right) < \alpha^{(n-1)}(+\infty)\right). \tag{29}$$

Therefore, by (21),

$$u_*^{(n-1)}(+\infty) = \alpha^{(n-1)}(+\infty)$$

which, by (H_3) and Definition 3, leads to a contradiction with (29):

$$\begin{aligned} & u_*^{(n-1)}(+\infty) + L_{n-1}\left(u_*, \delta_\infty\left(u_*^{(n-1)}(+\infty)\right)\right) \\ = \; & \alpha^{(n-1)}(+\infty) + L_{n-1}\left(u_*, \alpha^{(n-1)}(+\infty)\right) \\ \geq \; & \alpha^{(n-1)}(+\infty) + L_{n-1}\left(\alpha, \alpha^{(n-1)}(+\infty)\right) \\ > \; & \alpha^{(n-1)}(+\infty) \end{aligned}$$

.

Therefore,

$$u_*^{(n-1)}(+\infty) + L_{n-1}\left(u_*, \delta_\infty\left(u_*^{(n-1)}(+\infty)\right) \geq \alpha^{(n-1)}(+\infty)\right)$$

.

Applying the same technique, it can be proved that

$$u_*^{(n-1)}(+\infty) + L_{n-1}\left(u_*, \delta_\infty\left(u_*^{(n-1)}(+\infty)\right) \leq \beta^{(n-1)}(+\infty)\right).$$

Therefore, u_* is a solution of problems (1) and (2). □

4. Example

Let us consider a problem composed by the nonlinear, fourth-order differential equation

$$u^{(4)}(t) = \frac{1}{10}\frac{1}{1+t^4}\left[\frac{|u''(t)|}{1+t}e^{-u'(t)} - k\arctan(u(t)) + 2u'''(t)\right], \tag{30}$$

for $t \in [0, +\infty[$, and $k > 0$.

Assume that $u \in X$, the sum $\sum_{i=1}^{+\infty} \frac{u''(i^2)}{i^4}$, and the integral $\int_0^{+\infty} \frac{|u(t)|}{1+t^6} dt$ are finite. Then, define the functional boundary conditions

$$\begin{cases} u(0) = \min\limits_{t\in[0,+\infty[} u(t) \\ u'(0) = \frac{1}{4}\|u'\|_1, \\ u''(0) = \frac{1}{7}\sum\limits_{i=1}^{+\infty} \frac{u''(i^2)}{i^4} \\ u'''(+\infty) = \int_0^{+\infty} \frac{|u(t)|}{1+t^6} dt. \end{cases} \tag{31}$$

Indeed, the problems (30) and (31) are particular cases of the initial problems (1) and (2) with $n = 4$,

$$\begin{aligned} f(t, y_0, y_1, y_2, y_3) &= \frac{1}{10}\frac{1}{1+t^4}\left[\frac{|y_2|}{1+t}e^{-y_1} - k\arctan y_0 + 2y_3\right], \\ L_0(w, w_0, w_1, w_2) &= \min_{t\in[0,+\infty[} w(t) - w_0, \\ L_1(w', w_0, w_1, w_2) &= \frac{1}{4}\|w'\|_1 - w_1, \\ L_2(w'', w_0, w_1, w_2) &= \frac{1}{7}\sum_{i=1}^{+\infty}\frac{w''(i^2)}{i^4} - w_2, \\ L_3(w, w_3) &= \int_0^{+\infty}\frac{|w(t)|}{1+t^6}dt - w_3. \end{aligned}$$

The functions $\alpha, \beta \in X$, defined by $\alpha = \frac{1}{2}$ and $\beta = t^3 + t^2 + t + 1$ are, respectively, lower and upper solutions of (30) and (31), for $0 < k \leq 7$, verifying (13)–(15).

The assumptions (H_1), (H_2), and (H_3) are fulfilled, and the Nagumo condition is verified in the set

$$E^* = \left\{(t, y_0, y_1, y_2, y_3) \in [0, +\infty[\times\mathbb{R}^4, \alpha^{(i)}(t) \leq y_i \leq \beta^{(i)}(t), i = 0, 1, 2\right\},$$

with

$$\psi(t) = \frac{1}{10}\frac{1}{1+t^4}, \Phi(|y_3|) = 6 + \frac{7\pi}{2} + 2|y_3|, \text{ and } \nu = 4.$$

To define Green's function for the homogeneous problem

$$\begin{cases} u^{(4)}(t) = 0 \\ u(0) = 0 \\ u'(0) = 0 \\ u''(0) = 0 \\ u'''(+\infty) = 0, \end{cases}$$

we need to find functions g_i and $h_i, i = 1, 2, 3, 4$, such that

$$G(t,s) = \begin{cases} g_1(s) + g_2(s)t + g_3(s)t^2 + g_4(s)t^3, 0 \leq s \leq t \\ h_1(s) + h_2(s)t + h_3(s)t^2 + h_4(s)t^3, t \leq s < \infty. \end{cases}$$

By the properties of Green's function (see [36]), from the boundary conditions, we get $h_j(s) = 0$, for $j = 1, 2, 3$.

From

$$\lim_{s \to t^+} \frac{\partial^3 G}{\partial t^3}(t,s) - \lim_{s \to t^-} \frac{\partial^3 G}{\partial t^3}(t,s) = 1,$$

one has, for $g_4(s) = 0$, $h_4(s) = \frac{1}{6}$.

Finally, from

$$\lim_{s \to t^+} \frac{\partial^k G}{\partial t^k}(t,s) = \lim_{s \to t^-} \frac{\partial^k G}{\partial t^k}(t,s), \text{ for } k = 0, 1, 2,$$

we have

$$g_3(s) = \frac{s}{2}, g_2(s) = -\frac{s^2}{2} \text{ and } g_1(s) = \frac{s^3}{6}.$$

Therefore,

$$G(t,s) = \begin{cases} \frac{s^3}{6} - \frac{ts^2}{2} + \frac{t^2 s}{2}, 0 \le s \le t \\ \\ \frac{t^3}{6}, 0 \le t \le s < \infty, \end{cases}$$

and the constants, in (17), are

$$\begin{aligned} \|\alpha\|_X &= \frac{1}{2}, \|\beta\|_X = 6, R = 3, \\ M_0 &= 1, M_1 = 1, M_2 = 2, M_3 = 6, M_\infty = 6, \\ M_0(s) &= \frac{1}{6}, M_1(s) = \frac{1}{2}, M_2(s) = 1, M_3(s) = \frac{1}{2}, \end{aligned}$$

and

$$\varphi_\rho(t) = \frac{1}{10} \frac{1}{1+t^4} \left(\frac{\rho}{1+t} + \frac{7\pi}{2} + 2\rho \right).$$

It can be easily seen, from the corresponding calculus, that condition (17) holds for $\rho > 14,52$.

Therefore, by Theorem 1, there is a solution u of problem (30) and (31), for $0 < k \le 7$, such that, for $t \in [0, +\infty[$,

$$\begin{aligned} \frac{1}{2} &\le u(t) \le t^3 + t^2 + t + 1, \\ 0 &\le u'(t) \le 3t^2 + 2t + 1, \\ 0 &\le u''(t) \le 6t + 2, \\ -6 &< u'''(t) < 6, \end{aligned}$$

and

$$0 \le u'''(+\infty) \le 6.$$

5. Conclusions

This paper provided a technique to deal with discontinuous, fully nonlinear, higher-order boundary value problems defined on the half-line with functional boundary conditions. Weighted spaces and their weighted norms, together with the equiconvergence at infinity, are essential tools to recover the compacity of the correspondent operator on unbounded intervals. Moreover, the lower and upper solutions method allows for the definition of a modified and perturbed auxiliary problem with very general boundary conditions on the unknown function and its derivatives, which may include nonlocal, integro-differential, infinite-multipoint, and maximum and/or minimum arguments, among others.

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