



Force methods for the two-relaxation-times lattice Boltzmann

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The two-relaxation-times collision benefits the steady lattice Boltzmann method by yielding viscosity-independent numerical errors. We present in an intuitive way how to incorporate popular force methods into the two-relaxation-times collision. We subsequently rewrite force methods into a generic equation to reveal commonalities and differences. We prove that force methods with a second-order velocity moment of the force break the viscosity independence. A force method with only a first-order velocity moment of the force averts this breakage. We validate our proof numerically.

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I. VISCOSITY DEPENDENCE OF THE LATTICE BOLTZMANN METHOD

Consider two steady incompressible fluid flows that are physically similar by having a similar Reynolds number. In addition, the first flow has its Reynolds number composed with kinematic viscosity ν_1 , the second flow has its Reynolds number composed with kinematic viscosity $\nu_2 \neq \nu_1$. Simulate these two flows numerically with the lattice Boltzmann method [1,2] using the common single-relaxation-time (SRT) collision [3,4]. The simulations will yield two different numerical errors, despite their flows being physically similar. Why? Because with SRT, kinematic viscosity ν scales the spatial discretization error [5,6]. That ν scales the spatial discretization error, and not merely scales the strain rate, results in unphysical behavior. Consequently, either a small or a large ν can result in large numerical errors.

The spatial discretization error can be rendered viscosity-independent with the two-relaxation-times (TRT) collision [5,6]. With TRT, user-specified constant Λ scales the spatial discretization error. TRT's viscosity independence can break if you incorporate additional functionality into TRT, e.g., many boundary conditions break TRT's viscosity independence [6–8]. Incorporating force methods into TRT has largely been neglected, in contrast to SRT [9–15]. In this paper, first we demonstrate how to incorporate popular force methods into TRT. Second, we prove and numerically validate which force methods break TRT's viscosity independence. Thereby this paper rigorously evaluates force methods for TRT.

II. NOMENCLATURE

Before we commence we introduce the nomenclature of this paper, which is a nomenclature common in the TRT

literature [5,6]. Here units are specified in lattice units. The lattice Boltzmann equation is formulated as

$$f_q(\vec{x} + \vec{c}_q, t + 1) = f_q(\vec{x}, t) + \Omega_q(\vec{x}, t),$$

with lattice quantities subscripted with a q : \vec{c}_q being the q th velocity link, f_q is the distribution function, and Ω_q is the collision operator. The SRT collision is formulated as

$$\Omega_q^{\text{SRT}} = -\frac{1}{\tau}(f_q(\vec{x}, t) - \{e_q[\rho(\vec{x}, t), \vec{u}(\vec{x}, t)] + S_q(\vec{x}, t)\}),$$

$$e_q(\rho, \vec{u}) = t_q \left(P + \hat{\rho} u_q + \frac{3}{2} \hat{\rho} u_q^2 - \frac{1}{2} \hat{\rho} \vec{u}^2 \right),$$

with e_q being the equilibrium distribution function, t_q is the weight, S_q is an external source with a force density \vec{F} , and $u_q = \vec{c}_q \cdot \vec{u}$. The relaxation time is $\tau = 3\nu + \frac{1}{2}$. The macroscopic quantities are as follows: mass density $\rho = \sum f_q$, pressure $P = \rho c_s^2$ with sound speed $c_s = \sqrt{1/3}$, velocity $\vec{u} = (\sum \vec{c}_q f_q + \vec{F}/2)/\hat{\rho}$, $\hat{\rho} = \rho$ for the conventional compressible model, and $\hat{\rho} = \rho_0$ for He and Luo's incompressible model with constant ρ_0 [16].

The TRT collision is formulated as

$$\Omega_q^{\text{TRT}} = \Omega_q^+(\vec{x}, t) + \Omega_q^-(\vec{x}, t),$$

$$\Omega_q^+ = -\frac{1}{\tau^+}(f_q^+(\vec{x}, t) - \{e_q^+[\rho(\vec{x}, t), \vec{u}(\vec{x}, t)] + S_q^+(\vec{x}, t)\}),$$

$$\Omega_q^- = -\frac{1}{\tau^-}(f_q^-(\vec{x}, t) - \{e_q^-[\rho(\vec{x}, t), \vec{u}(\vec{x}, t)] + S_q^-(\vec{x}, t)\}),$$

$$e_q^+(\rho, \vec{u}) = \frac{1}{2}(e_q + e_{\bar{q}}) = t_q \left(P + \frac{3}{2} \hat{\rho} u_q^2 - \frac{1}{2} \hat{\rho} \vec{u}^2 \right),$$

$$e_q^-(\rho, \vec{u}) = \frac{1}{2}(e_q - e_{\bar{q}}) = t_q \hat{\rho} u_q,$$

$$f_q^+ = \frac{1}{2}(f_q + f_{\bar{q}}) \quad \text{and} \quad S_q^+ = \frac{1}{2}(S_q + S_{\bar{q}}),$$

$$f_q^- = \frac{1}{2}(f_q - f_{\bar{q}}) \quad \text{and} \quad S_q^- = \frac{1}{2}(S_q - S_{\bar{q}}).$$

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TABLE I. External source term S_q of four force methods for SRT.

Buick and Greated [12]:	$S_q = (\tau - \frac{1}{2})t_q F_q$
Guo <i>et al.</i> [13]:	$S_q = \underbrace{(\tau - \frac{1}{2})t_q(F_q + 3u_q F_q - \bar{u} \cdot \bar{F})}_{\text{Buick}}$
Kupershtokh [14]:	$S_q = \underbrace{(\tau - \frac{1}{2})t_q(F_q + 3u_q F_q - \bar{u} \cdot \bar{F})}_{\text{Guo}} + \frac{t_q}{8\hat{\rho}}(3F_q^2 - \bar{F}^2)$
Shan and Chen [15]:	$S_q = \underbrace{(\tau - \frac{1}{2})t_q(F_q + 3u_q F_q - \bar{u} \cdot \bar{F})}_{\text{Kupershtokh}} + (\frac{1}{4} - \tau + \tau^2) \frac{t_q}{2\hat{\rho}}(3F_q^2 - \bar{F}^2)$

TRT decomposes an arbitrary lattice quantity ψ into a symmetric part $\psi^+ = \frac{1}{2}(\psi_q + \psi_{\bar{q}})$ and an antisymmetric part $\psi^- = \frac{1}{2}(\psi_q - \psi_{\bar{q}})$. Here \bar{q} denotes the opposite link of q , consequently $\bar{c}_{\bar{q}} = -\bar{c}_q$. Relaxation time τ^+ relaxes f_q^+ to e_q^+ , relaxation time τ^- relaxes f_q^- to e_q^- . The user-specified constant that scales the spatial discretization error is defined as $\Lambda = \Lambda^+ \Lambda^- = (\tau^+ - \frac{1}{2})(\tau^- - \frac{1}{2})$, where $\tau^+ = 3\nu + \frac{1}{2}$ and consequently $\tau^- = \frac{\Lambda}{3\nu} + \frac{1}{2}$. Finally, three convenience variables are $u_{\bar{q}} = \bar{c}_{\bar{q}} \cdot \bar{u}$, $F_q = \bar{c}_q \cdot \bar{F}$, and $F_{\bar{q}} = \bar{c}_{\bar{q}} \cdot \bar{F}$.

III. INCORPORATING FORCE METHODS

We incorporate four popular force methods [12–15] into TRT. These force methods also form a good test set, because they significantly differ among each other, either in form or in viscosity dependence, as shown later. All four methods discretize the force up to second order in space and time to avoid discrete lattice artifacts. Force methods with first-order space-time discretizations are inferior [1,9,13], and therefore disregarded.

To compare the force methods on an equal footing, we rewrite Kupershtokh's [14] and Shan and Chen's [15] SRT formulation (Table I). These two force methods use in our paper $\bar{u} = (\sum \bar{c}_q f_q + \bar{F}/2)/\hat{\rho}$ instead of the original $\bar{u} = (\sum \bar{c}_q f_q)/\hat{\rho}$. Shan and Chen's [15] force method is further rewritten to a form using an external source term S_q (Appendix A). From our SRT formulations we can intuitively map to TRT formulations (Table II). And from our TRT formulations we can easily deduce a generic TRT force equation for the remainder of our paper,

$$S_q^+ = B t_q (3u_q F_q - \bar{u} \cdot \bar{F}) + C \frac{t_q}{2\hat{\rho}} (3F_q^2 - \bar{F}^2), \quad (1)$$

$$S_q^- = \Lambda^- t_q F_q.$$

With

	B	C
Buick and Greated [12]:	0	0
Guo <i>et al.</i> [13]:	Λ^+	0
Kupershtokh [14]:	Λ^+	$\frac{1}{4}$
Shan and Chen [15]:	Λ^+	Λ^{+2} ,

where $\Lambda^+ = 3\nu$. A generic force equation for SRT would consist of $S_q = S_q^+ + S_q^-$ and $\Lambda^- = \Lambda^+$.

IV. DERIVING FORCE ERRORS

Next we derive the force errors that TRT with Eq. (1) appends to the well-known Navier-Stokes fluid flow equations. Our derivation assumes a flow is steady and incompressible, which is customary in the TRT literature. Steady, because time-discretization errors cannot be rendered viscosity-independent with TRT. Incompressible, because $\hat{\rho} = \rho_0$ reduces compressibility errors with steady flows [17,18] and simplifies the mathematics. Our derivation employs the steady-state ansatz derived by Ginzburg [19]. Ginzburg's ansatz, compared to the traditional Chapman-Enskog expansion [20], relies on less assumptions and shortens the derivation. The ansatz states

$$\Omega_q^+ = \partial_q (e_q^- + S_q^-) - \Lambda^- \partial_q^2 (e_q^+ + S_q^+) + O(\epsilon^3), \quad (2a)$$

$$\Omega_q^- = \partial_q (e_q^+ + S_q^+) - \Lambda^+ \partial_q^2 (e_q^- + S_q^-) + O(\epsilon^3), \quad (2b)$$

where $\partial_q = c_{q\alpha} \partial_\alpha$ (Greek indices denote Einstein's summation convention), $\partial_q^2 = c_{q\alpha} c_{q\beta} \partial_{\alpha\beta}$, and ϵ is a small perturbation parameter. To recover the steady incompressible Navier-Stokes mass equation, take the zeroth-order velocity moment of

TABLE II. External source terms S_q^+ and S_q^- of four force methods for TRT.

Buick and Greated [12]:	$S_q^+ = \frac{1}{2}[S_q(\tau^+) + S_{\bar{q}}(\tau^+)]$ $= \Lambda^+ (\frac{1}{2}t_q F_q + \frac{1}{2}t_{\bar{q}} F_{\bar{q}})$ $= 0$	$S_q^- = \frac{1}{2}[S_q(\tau^-) - S_{\bar{q}}(\tau^-)]$ $= \Lambda^- (\frac{1}{2}t_q F_q - \frac{1}{2}t_{\bar{q}} F_{\bar{q}})$ $= \Lambda^- t_q F_q$
Guo <i>et al.</i> [13]:	$S_q^+ = \Lambda^+ t_q (3u_q F_q - \bar{u} \cdot \bar{F})$	$S_q^- = \Lambda^- t_q F_q$
Kupershtokh [14]:	$S_q^+ = \Lambda^+ t_q (3u_q F_q - \bar{u} \cdot \bar{F}) + \frac{t_q}{8\hat{\rho}} (3F_q^2 - \bar{F}^2)$	$S_q^- = \Lambda^- t_q F_q$
Shan and Chen [15]:	$S_q^+ = \Lambda^+ t_q (3u_q F_q - \bar{u} \cdot \bar{F}) + \Lambda^{+2} \frac{t_q}{2\hat{\rho}} (3F_q^2 - \bar{F}^2)$	$S_q^- = \Lambda^- t_q F_q$

Eq. (2a),

$$\begin{aligned} \sum \Omega_q^+ &= \sum c_{q\alpha} \partial_\alpha (e_q^- + S_q^-) - \Lambda^- \sum c_{q\alpha} c_{q\beta} \partial_{\alpha\beta} (e_q^+ + S_q^+) \\ 0 &= \sum t_q c_{q\alpha} c_{q\beta} \partial_\alpha (\rho_0 u_\beta + \Lambda^- F_\beta) \\ &\quad - \Lambda^- \sum t_q c_{q\alpha} c_{q\beta} \partial_{\alpha\beta} \left(P \right. \\ &\quad \left. + c_{q\gamma} c_{q\delta} \frac{3}{2} \rho_0 u_\gamma u_\delta - \frac{1}{2} \rho_0 u_\gamma u_\gamma \right. \\ &\quad \left. + c_{q\gamma} c_{q\delta} B 3 u_\gamma F_\delta - B u_\gamma F_\gamma \right. \\ &\quad \left. + c_{q\gamma} c_{q\delta} C \frac{3}{2\rho_0} F_\gamma F_\delta - C \frac{1}{2\rho_0} F_\gamma F_\gamma \right). \end{aligned}$$

The zero on the left-hand side indicates no mass source is present. Substitute lattice symmetries $\sum t_q c_{q\alpha} c_{q\beta} = \delta_{\alpha\beta}$ and $\sum t_q c_{q\alpha} c_{q\beta} c_{q\gamma} c_{q\delta} = \frac{1}{3} (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma})$ into the equation, and work out to recover

$$\begin{aligned} \rho_0 \underbrace{\partial_\alpha u_\alpha}_{\text{solenoidal velocity}} &+ \frac{\Lambda}{3\nu} \underbrace{[\partial_\alpha F_\alpha - \partial_{\alpha\alpha} P - \rho_0 \partial_{\alpha\beta} (u_\alpha u_\beta)]}_{\text{pressure Poisson}} \\ &= B \frac{\Lambda}{3\nu} 2 \partial_{\alpha\beta} (u_\alpha F_\beta) + C \frac{\Lambda}{3\nu} \frac{1}{\rho_0} \partial_{\alpha\beta} (F_\alpha F_\beta). \end{aligned} \quad (3)$$

To recover the steady incompressible Navier-Stokes momentum equation, take the first-order velocity moment of Eq. (2b),

$$\begin{aligned} \sum c_{q\alpha} \Omega_q^- &= \sum c_{q\alpha} c_{q\beta} \partial_\beta (e_q^+ + S_q^+) \\ &\quad - \Lambda^+ \sum c_{q\alpha} c_{q\beta} c_{q\gamma} \partial_{\beta\gamma} (e_q^- + S_q^-) \\ F_\alpha &= \sum t_q c_{q\alpha} c_{q\beta} \partial_\beta \left(P \right. \\ &\quad \left. + c_{q\gamma} c_{q\delta} \frac{3}{2} \rho_0 u_\gamma u_\delta - \frac{1}{2} \rho_0 u_\gamma u_\gamma \right. \\ &\quad \left. + c_{q\gamma} c_{q\delta} B 3 u_\gamma F_\delta - B u_\gamma F_\gamma \right. \\ &\quad \left. + c_{q\gamma} c_{q\delta} C \frac{3}{2\rho_0} F_\gamma F_\delta - C \frac{1}{2\rho_0} F_\gamma F_\gamma \right. \\ &\quad \left. - c_{q\gamma} c_{q\delta} \partial_\gamma \Lambda^+ \rho_0 u_\delta - c_{q\gamma} c_{q\delta} \partial_\gamma \Lambda F_\delta \right). \end{aligned}$$

Use lattice symmetries again, to recover

$$\begin{aligned} \partial_\alpha P + \rho_0 \partial_\beta (u_\alpha u_\beta) - \rho_0 \nu \partial_\beta (\partial_\alpha u_\beta + \partial_\beta u_\alpha + \delta_{\alpha\beta} \partial_\gamma u_\gamma) - F_\alpha \\ = \frac{\Lambda}{3} \partial_\beta (\partial_\alpha F_\beta + \partial_\beta F_\alpha + \delta_{\alpha\beta} \partial_\gamma F_\gamma) \\ - B \partial_\beta (u_\alpha F_\beta + u_\beta F_\alpha) - C \frac{1}{\rho_0} \partial_\beta (F_\alpha F_\beta). \end{aligned} \quad (4)$$

Left-hand sides of Eqs. (3) and (4) contain the targeted steady incompressible Navier-Stokes equations, right-hand sides contain the force errors. The targeted equations are still expressed in nondimensional lattice units. We rewrite the targeted equations into nondimensional physical units to maintain the ‘‘similarity principle’’ of a fluid flow. To that end,

we introduce the nondimensional quantities

$$\begin{aligned} u^* &= \frac{u}{u_0}, \quad P^* = \frac{P}{\rho_0 u_0^2}, \quad F^* = F \frac{l_0}{\rho_0 u_0^2}, \\ \partial_\alpha^* &= l_0 \partial_\alpha, \quad \text{Re} = \frac{u_0 l_0}{\nu}, \end{aligned} \quad (5)$$

where l_0 is a characteristic length, or numerically viewed the spatial resolution, u_0 is a characteristic speed, and Re is the Reynolds number. Equations (3) and (4) then become

$$\begin{aligned} \partial_\alpha^* u_\alpha^* + \frac{1}{l_0^2} \text{Re} \frac{\Lambda}{3} [\partial_\alpha^* F_\alpha^* - \partial_{\alpha\alpha}^* P^* - \partial_{\alpha\beta}^* (u_\alpha^* u_\beta^*)] \\ = B \underbrace{\nu \frac{1}{l_0^4} \text{Re}^2 \frac{\Lambda}{3} 2 \partial_{\alpha\beta}^* (u_\alpha^* F_\beta^*)}_{\epsilon_B^{(m)}} + C \underbrace{\nu^2 \frac{1}{l_0^6} \text{Re}^3 \frac{\Lambda}{3} \partial_{\alpha\beta}^* (F_\alpha^* F_\beta^*)}_{\epsilon_C^{(m)}}, \end{aligned} \quad (6)$$

and

$$\begin{aligned} \partial_\alpha^* P^* + \partial_\beta^* (u_\alpha^* u_\beta^*) - \frac{1}{\text{Re}} \partial_\beta (\partial_\alpha^* u_\beta^* + \partial_\beta^* u_\alpha^* + \delta_{\alpha\beta} \partial_\gamma^* u_\gamma^*) - F_\alpha^* \\ = \frac{1}{l_0^2} \frac{\Lambda}{3} \underbrace{\partial_\beta^* (\partial_\alpha^* F_\beta^* + \partial_\beta^* F_\alpha^* + \delta_{\alpha\beta} \partial_\gamma^* F_\gamma^*)}_{\epsilon_A^{(j)}} \\ - B \underbrace{\nu \frac{1}{l_0^2} \text{Re} \partial_\beta^* (u_\alpha^* F_\beta^* + u_\beta^* F_\alpha^*)}_{\epsilon_B^{(j)}} - C \underbrace{\nu^2 \frac{1}{l_0^4} \text{Re}^2 \partial_\beta^* (F_\alpha^* F_\beta^*)}_{\epsilon_C^{(j)}}. \end{aligned} \quad (7)$$

A variant of Eq. (7), one without truncation errors, has been derived by Silva in a succeeding work [21]. His equation is however limited to $A\epsilon_A^{(j)}$ and $B\epsilon_B^{(j)}$ type force errors and steady channel flows.

Use Eqs. (6) and (7) to assess the force errors produced by each force method. Buick and Greated’s single force error $\epsilon_A^{(j)}$ stems from spatially discretizing the force. This force error decreases quadratically when spatial resolution l_0 increases and scales linearly with the constant Λ . Therefore, this force error preserves TRT’s viscosity independence. As a side remark, Eqs. (6) and (7) also imply Buick and Greated is a proper choice for steady incompressible SRT, even though $\epsilon_A^{(j)}$ then becomes viscosity-dependent, you do dispose of $B\epsilon_B$ and $C\epsilon_C$, corroborated by Ref. [10]. Guo *et al.*’s characteristic force errors $3\nu\epsilon_B^{(m)}$ and $3\nu\epsilon_C^{(j)}$ scale with $\frac{1}{l_0^4}$ and $\frac{1}{l_0^6}$. Both force errors scale with ν^2 . Kupershtokh’s characteristic force errors $\frac{1}{4}\epsilon_C^{(m)}$ and $\frac{1}{4}\epsilon_C^{(j)}$ scale with $\frac{1}{l_0^4}$ and $\frac{1}{l_0^6}$. So particularly $\frac{1}{4}\epsilon_C^{(m)}$ decreases rapidly when you increase spatial resolution. Both force errors scale with ν^2 . Lastly, Shan and Chen’s characteristic force errors $9\nu^2\epsilon_C^{(m)}$ and $9\nu^2\epsilon_C^{(j)}$ scale with ν^4 , which means these force errors increase rapidly when you increase viscosity. The force errors added by Guo *et al.*, Kupershtokh, and Shan and Chen are viscosity-dependent, thereby they break TRT’s viscosity independence. They stem from S_q^+ , i.e., from the second-order velocity moment of the force. Guo *et al.*, Kupershtokh, and Shan and Chen deliberately introduce $B\epsilon_B$, because they focus on unsteady flows, in which $B\epsilon_B$

cancel a time-dependent error. In steady flows however, this time-dependent error is not there to be canceled, causing $B\epsilon_B$ to subsist. They disregard ϵ_A and ϵ_C because generally $\vec{F} \propto \epsilon$ and $\partial_\alpha \propto \epsilon$, rendering $\epsilon_A \propto \epsilon^3$ and $\epsilon_C \propto \epsilon^3$ beyond the studied *second-order* perturbation expansion. Still, in certain cases ϵ_A and ϵ_C become relevant, as shown in Refs. [19,22] and in our numerical results.

V. NUMERICAL VALIDATION

A. Setup

To validate Eqs. (6) and (7) numerically, requires a steady incompressible flow that renders the force errors nonzero, or at least the viscosity-dependent force errors ($B\epsilon_B$ and $C\epsilon_C$) nonzero. Preferably the flow also has an analytic solution to compare with. These requirements are fulfilled by the four-roll mill and by a variant of the Poiseuille flow (Appendix B) [23]. The four-roll mill comprises a periodic square domain with four counter-rotating vortices in the corners. Its analytic solution in nondimensional physical units comprises

$$\begin{aligned} \vec{u}^*(x^*, y^*) &= \begin{bmatrix} \sin(2\pi k_x x^*) \cos(2\pi k_y y^*) \\ -\cos(2\pi k_x x^*) \sin(2\pi k_y y^*) \end{bmatrix}, \\ \vec{F}^*(x^*, y^*) &= \frac{8\pi^2}{\text{Re}} \vec{u}^*(x^*, y^*), \\ P^*(x^*, y^*) &= \frac{1}{4} [\cos(4\pi k_x x^*) + \cos(4\pi k_y y^*)] + P_{\text{avg}}^*, \end{aligned} \tag{8}$$

with $x^* = \frac{x}{l_x} \in [0, 1)$ and $y^* = \frac{y}{l_y} \in [0, 1)$. Wave numbers $k_x = k_y = 1$ yield one vortex in each corner. Average pressure P_{avg}^* is a constant specified by the user, often set to 1.

Poiseuille is a channel flow, our variant features a linear force and a linear pressure gradient. Its analytic solution in nondimensional physical units comprises $u_y^* = F_y^* = 0$,

$$\begin{aligned} u_x^*(y^*) &= 4(y^* - y^{*2}), \quad F_x^*(x^*) = \frac{8}{\text{Re}} x^*, \\ P^*(x^*) &= \frac{8}{\text{Re}} \frac{l_x}{l_y} \left(\frac{1}{2} x^{*2} - x^* + \frac{1}{3} \right) + P_{\text{avg}}^*. \end{aligned} \tag{9}$$

We employ for the inlet and outlet Kim and Pitsch’s [24] periodic pressure boundary condition and for the walls the familiar half-way bounce-back boundary condition. Both

boundary conditions preserve TRT’s viscosity independence [6,7].

We choose $\text{Re} = 1$ to stay within the laminar regime mandated by the analytic solutions. Default $\Lambda = \frac{1}{5}$, approximately halfway the prevalent values for Λ of $\{\frac{1}{12}, \frac{1}{6}, \frac{3}{16}, \frac{1}{4}, \frac{3}{8}\}$. Default $\nu = 1$, approximately halfway our test range for ν of $[0.01, 2]$. Spatial resolution is 16×16 for the four-roll mill and 23×11 for Poiseuille, deliberately low to better reveal the discrepancies among force methods. We let all force methods output a force-corrected velocity $\vec{u} = (\sum \vec{c}_q f_q + \vec{F}/2)/\rho_0$. When the simulated flows reach steady state, we measure the well-known relative ℓ^2 norm of pressure errors and velocity errors,

$$\begin{aligned} \epsilon_P &= \frac{\|\mathbf{P}_s - \mathbf{P}_a\|}{\|\mathbf{P}_a\|} = \frac{\sqrt{\sum_{\vec{x}} [P_s(\vec{x}) - P_a(\vec{x})]^2}}{\sqrt{\sum_{\vec{x}} [P_a(\vec{x})]^2}}, \\ \epsilon_{\vec{u}} &= \frac{\|\mathbf{u}_{x,s} - \mathbf{u}_{x,a}\| + \|\mathbf{u}_{y,s} - \mathbf{u}_{y,a}\|}{\|\mathbf{u}_{x,a}\| + \|\mathbf{u}_{y,a}\|}, \end{aligned}$$

where s subscripts simulated values and a analytic values.

B. Results

We start by testing how each force method depends on viscosity by plotting ϵ_P and $\epsilon_{\vec{u}}$ vs ν (Fig. 1). When increasing ν , we proportionally increase u_0 to keep the Reynolds number constant. The straight horizontal lines of Buick and Greated in all plots imply Buick and Greated does not depend on viscosity, as predicted by Eqs. (6) and (7). The steep curves of non-Buick force methods demonstrate viscosity dependence can result in large errors. Guo *et al.*’s errors are barely smaller than Kupershtokh’s errors, which implies Kupershtokh’s additional force error $\frac{1}{4}\epsilon_C$ is small compared to $3\nu\epsilon_B$. Fitting polynomial curves through the errors of Guo *et al.* and Kupershtokh reveals that a second-order polynomial fits well, which matches the $O(\nu^2)$ predicted by Eqs. (6) and (7). Exception here is $\epsilon_{\vec{u}}$ of the four-roll mill, a third-order polynomial then fits slightly better, probably because the narrow $\epsilon_{\vec{u}}$ range hinders a reliable fit. Fitting polynomial curves through Shan and Chen’s errors reveals a predicted fourth-order polynomial. Numerical experiments did show the $O(\nu^4)$ scaling only appears with very large ν , which corroborates ϵ_C is small. Finally, two side remarks, not shown by the plots. A smaller or larger Λ does not result in significant different behavior. Under rare circumstances a non-Buick curve decreases, due to $B\epsilon_B$ or $C\epsilon_C$ fortuitously canceling another error.

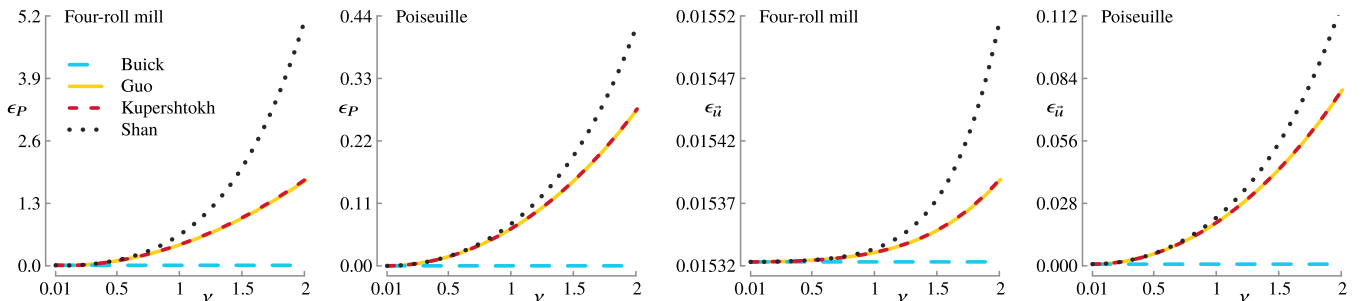


FIG. 1. Four force methods affecting the viscosity dependence of the pressure error and velocity error. $\text{Re} = 1$ and $\Lambda = \frac{1}{5}$.

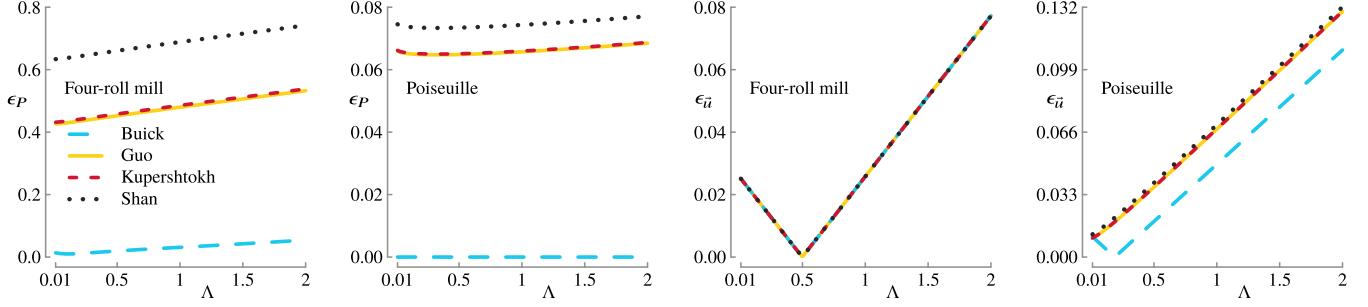


FIG. 2. Four force methods affecting the Λ dependence of the pressure error and velocity error. $Re = 1$ and $\nu = 1$.

Next, we look at how non-Buick force methods deviate from the desired behavior of Buick and Guo in plots of ϵ_P and $\epsilon_{\bar{u}}$ vs Λ (Fig. 2). The TRT literature often shows this relation, because steady incompressible ϵ_P and $\epsilon_{\bar{u}}$ solely depend on *user-specifiable* Λ and the spatial discretization error. In all four plots, non-Buick curves lie relatively close to each other. This implies ϵ_C is small, particularly if you consider $C_{Kupershtokh}$ and C_{Shan} differ by as much as factor of $(3 \times 1)^2 / (\frac{1}{4}) = 36$. In ϵ_P plots, Buick and Guo's ϵ_P is small, because it contains no first- or second-order force errors [Eq. (6)]. Non-Buick ϵ_P 's by contrast, are large, due to $B\epsilon_B^{(m)}$ being large. In $\epsilon_{\bar{u}}$ plots, all curves scale similarly, because $B\epsilon_B^{(j)}$ and $C\epsilon_C^{(j)}$ do not depend on Λ [Eq. (7)]. With the four-roll mill, all $\epsilon_{\bar{u}}$ curves even practically overlap. This implies the four-roll mill's $B\epsilon_B^{(j)}$ and $C\epsilon_C^{(j)}$ are negligible compared to other errors, corroborated by the small absolute differences in the four-roll mill's $\epsilon_{\bar{u}}$ vs ν plot (Fig. 1). As a side remark, with Buick and Guo we numerically discovered optimal $\Lambda = \frac{1}{2}$ for circular-shaped vortices. With Poiseuille, non-Buick $\epsilon_{\bar{u}}$ curves nearly overlap with each other but not with Buick and Guo's curve, implying Poiseuille's $B\epsilon_B^{(j)}$ is significant and $C\epsilon_C^{(j)}$ is not. The optimal $\Lambda = \frac{3}{16}$ of Buick and Guo equals the optimal Λ of force-free pressure-driven

Poiseuille [6]. Indeed, the single force error $\epsilon_A^{(j)}$ of Buick and Guo equals 0 for a linearly increasing force. Finally, two insights, not shown by our plots with $\nu = 1$. The optimal Λ of a non-Buick force method is not constant but depends on ν . The discrepancy between a non-Buick curve and Buick and Guo's curve scales with ν .

VI. CONCLUSIONS

From what we presented, it should be clear how to incorporate force methods into TRT and how to assess them. The equations to theoretically assess force methods agree well with the numerical results. The numerical results demonstrate force errors can become large if they are viscosity-dependent. The equations and numerical results dispel the common belief a force method always benefits from the second-order velocity moment of the force. Indeed, for steady incompressible fluids, the force method of Buick and Guo [12] is appropriate, which only contains a first-order velocity moment of the force.

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APPENDIX A: REWRITING THE FORCE EQUATIONS OF KUPERSHTOKH AND SHAN AND CHEN

We rewrite the original SRT formulations of Kupershtokh [14] and Shan and Chen [15] to the SRT formulations in Table I. Let $\bar{u} = (\sum \bar{c}_q f_q + \bar{F}/2)/\hat{\rho}$ and $e_q(\rho, \bar{u}) = t_q(\rho c_s^2 + \hat{\rho} u_q + \frac{3}{2} \hat{\rho} u_q^2 - \frac{1}{2} \hat{\rho} \bar{u}^2)$. Rewriting Kupershtokh [14] starts by moving the external source term inside the collision,

$$\begin{aligned}
 f_q(\bar{x} + \bar{c}_q, t + 1) &= f_q - \frac{1}{\tau} \left[f_q - e_q \left(\rho, \bar{u} - \frac{\bar{F}}{2\hat{\rho}} \right) \right] + \underbrace{e_q \left(\rho, \bar{u} + \frac{\bar{F}}{2\hat{\rho}} \right) - e_q \left(\rho, \bar{u} - \frac{\bar{F}}{2\hat{\rho}} \right)}_{S_q^{\text{original}}} \\
 &= f_q - \frac{1}{\tau} \left\{ f_q - \left[e_q \left(\rho, \bar{u} - \frac{\bar{F}}{2\hat{\rho}} \right) + \tau e_q \left(\rho, \bar{u} + \frac{\bar{F}}{2\hat{\rho}} \right) - \tau e_q \left(\rho, \bar{u} - \frac{\bar{F}}{2\hat{\rho}} \right) \right] \right\} \\
 e_q \left(\rho, \bar{u} - \frac{\bar{F}}{2\hat{\rho}} \right) &= t_q \left\{ c_s^2 \rho + \hat{\rho} \left(\bar{c}_q \cdot \left[\bar{u} - \frac{\bar{F}}{2\hat{\rho}} \right] \right) + \frac{3}{2} \hat{\rho} \left(\bar{c}_q \cdot \left[\bar{u} - \frac{\bar{F}}{2\hat{\rho}} \right] \right)^2 - \frac{1}{2} \hat{\rho} \left[\bar{u} - \frac{\bar{F}}{2\hat{\rho}} \right]^2 \right\},
 \end{aligned}$$

work out the terms,

$$\begin{aligned}\hat{\rho}\left(\vec{c}_q \cdot \left[\vec{u} - \frac{\vec{F}}{2\hat{\rho}}\right]\right) &= \hat{\rho}\vec{c}_q \cdot \vec{u} - \frac{1}{2}\vec{c}_q \cdot \vec{F} = \hat{\rho}u_q - \frac{1}{2}F_q, \\ \frac{3}{2}\hat{\rho}\left(\vec{c}_q \cdot \left[\vec{u} - \frac{\vec{F}}{2\hat{\rho}}\right]\right)^2 &= \frac{3}{2}\hat{\rho}u_q^2 - \frac{3}{2}u_qF_q + \frac{3}{8\hat{\rho}}F_q^2, \\ -\frac{1}{2}\hat{\rho}\left[\vec{u} - \frac{\vec{F}}{2\hat{\rho}}\right]^2 &= -\frac{1}{2}\hat{\rho}\vec{u}^2 + \frac{1}{2}\vec{u} \cdot \vec{F} - \frac{1}{8\hat{\rho}}\vec{F}^2,\end{aligned}$$

and combine them to

$$e_q\left(\rho, \vec{u} - \frac{\vec{F}}{2\hat{\rho}}\right) = e_q(\rho, \vec{u}) - \frac{t_q}{2}(F_q + 3u_qF_q - \vec{u} \cdot \vec{F}) + \frac{t_q}{8\hat{\rho}}(3F_q^2 - \vec{F}^2).$$

Analogously,

$$e_q\left(\rho, \vec{u} + \frac{\vec{F}}{2\hat{\rho}}\right) = e_q(\rho, \vec{u}) + \frac{t_q}{2}(F_q + 3u_qF_q - \vec{u} \cdot \vec{F}) + \frac{t_q}{8\hat{\rho}}(3F_q^2 - \vec{F}^2).$$

Substitute the equations of $e_q(\rho, \vec{u} - \frac{\vec{F}}{2\hat{\rho}})$ and $e_q(\rho, \vec{u} + \frac{\vec{F}}{2\hat{\rho}})$ into $f_q(\vec{x} + \vec{c}_q, t + 1)$ and work out to acquire

$$f_q(\vec{x} + \vec{c}_q, t + 1) = f_q - \frac{1}{\tau}\left(f_q - \left[e_q(\rho, \vec{u}) + \left(\tau - \frac{1}{2}\right)t_q(F_q + 3u_qF_q - \vec{u} \cdot \vec{F}) + \frac{t_q}{8\hat{\rho}}(3F_q^2 - \vec{F}^2)\right]\right).$$

Next we rewrite Shan and Chen [15], starting again from the original SRT formulation,

$$\begin{aligned}f_q(\vec{x} + \vec{c}_q, t + 1) &= f_q - \frac{1}{\tau}\left[f_q - e_q\left(\rho, \vec{u} - \frac{\vec{F}}{2\hat{\rho}} + \frac{\tau\vec{F}}{\hat{\rho}}\right)\right], \\ e_q\left(\rho, \vec{u} - \frac{\vec{F}}{2\hat{\rho}} + \frac{\tau\vec{F}}{\hat{\rho}}\right) &= t_q\left\{c_s^2\rho + \hat{\rho}\left(\vec{c}_q \cdot \left[\vec{u} - \frac{\vec{F}}{2\hat{\rho}} + \frac{\tau\vec{F}}{\hat{\rho}}\right]\right) + \frac{3}{2}\hat{\rho}\left(\vec{c}_q \cdot \left[\vec{u} - \frac{\vec{F}}{2\hat{\rho}} + \frac{\tau\vec{F}}{\hat{\rho}}\right]\right)^2 - \frac{1}{2}\hat{\rho}\left[\vec{u} - \frac{\vec{F}}{2\hat{\rho}} + \frac{\tau\vec{F}}{\hat{\rho}}\right]^2\right\},\end{aligned}$$

work out the terms,

$$\begin{aligned}\hat{\rho}\left(\vec{c}_q \cdot \left[\vec{u} - \frac{\vec{F}}{2\hat{\rho}} + \frac{\tau\vec{F}}{\hat{\rho}}\right]\right) &= \hat{\rho}u_q - \frac{1}{2}F_q + \tau F_q, \\ \frac{3}{2}\hat{\rho}\left(\vec{c}_q \cdot \left[\vec{u} - \frac{\vec{F}}{2\hat{\rho}} + \frac{\tau\vec{F}}{\hat{\rho}}\right]\right)^2 &= \frac{3}{2}\hat{\rho}u_q^2 + \left(\tau - \frac{1}{2}\right)3u_qF_q + \left(\frac{1}{8\hat{\rho}} - \frac{\tau}{2\hat{\rho}} + \frac{\tau^2}{2\hat{\rho}}\right)3F_q^2, \\ -\frac{1}{2}\hat{\rho}\left[\vec{u} - \frac{\vec{F}}{2\hat{\rho}} + \frac{\tau\vec{F}}{\hat{\rho}}\right]^2 &= -\frac{1}{2}\hat{\rho}\vec{u}^2 - \left(\tau - \frac{1}{2}\right)\vec{u} \cdot \vec{F} - \left(\frac{1}{8\hat{\rho}} - \frac{\tau}{2\hat{\rho}} + \frac{\tau^2}{2\hat{\rho}}\right)\vec{F}^2,\end{aligned}$$

and combine them to

$$e_q\left(\rho, \vec{u} - \frac{\vec{F}}{2\hat{\rho}} + \frac{\tau\vec{F}}{\hat{\rho}}\right) = e_q(\rho, \vec{u}) + \left(\tau - \frac{1}{2}\right)t_q(F_q + 3u_qF_q - \vec{u} \cdot \vec{F}) + \left(\frac{1}{4} - \tau + \tau^2\right)\frac{t_q}{2\hat{\rho}}(3F_q^2 - \vec{F}^2).$$

Substitute this equation into $f_q(\vec{x} + \vec{c}_q, t + 1)$ to finish with

$$f_q(\vec{x} + \vec{c}_q, t + 1) = f_q - \frac{1}{\tau}\left(f_q - \left[e_q(\rho, \vec{u}) + \left(\tau - \frac{1}{2}\right)t_q(F_q + 3u_qF_q - \vec{u} \cdot \vec{F}) + \left(\frac{1}{4} - \tau + \tau^2\right)\frac{t_q}{2\hat{\rho}}(3F_q^2 - \vec{F}^2)\right]\right).$$

APPENDIX B: DERIVING THE EQUATIONS OF THE BENCHMARK FLOWS

The four-roll mill is the steady version of the unsteady Taylor-Green vortex (TGV) [25] at $t = 0$. The four-roll mill's analytic velocity [Fig. 3(a)] is therefore

$$\vec{u}(x, y) = \vec{u}^{\text{TGV}}(x, y, t = 0) = \begin{bmatrix} Q \sin(qx) \cos(ry) \\ R \cos(qx) \sin(ry) \end{bmatrix} \underbrace{D(t=0)}_{=1},$$

with constants

$$Q = u_0 \sqrt{\frac{k_y l_x}{k_x l_y}}, \quad R = -u_0 \sqrt{\frac{k_x l_y}{k_y l_x}}, \quad q = k_x \frac{2\pi}{l_x}, \quad r = k_y \frac{2\pi}{l_y},$$

and with $x \in [0, l_x)$, $y \in [0, l_y)$, and $D(t) = e^{-(q^2+r^2)vt}$. Wave numbers k_x and k_y control the number of vortices in the x and y directions. The velocity magnitude is controlled by u_0 . To keep the flow steady, a force density \vec{F} counterbalances the

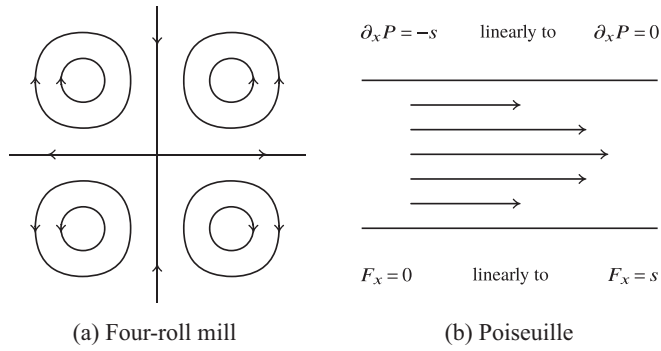


FIG. 3. Velocity streamlines of the benchmark flows.

Taylor-Green vortex's deceleration,

$$\vec{F}(x, y) = -\rho_0 \partial_t \vec{u}^{\text{TGV}}|_{t=0} = \rho_0 (q^2 + r^2) v \vec{u}(x, y).$$

To acquire an equation for the pressure P , substitute \vec{u} into the Navier-Stokes momentum equations, combine and rewrite the results to a Poisson equation for the pressure, finally integrate the Poisson equation to acquire

$$P(x, y) = \frac{\rho_0 Q^2}{4} \left(\cos(2qx) + \frac{q^2}{r^2} \cos(2ry) \right) + P_{\text{avg}},$$

with average pressure P_{avg} . Use the nondimensional quantities of Eq. (5) to nondimensionalize \vec{u} , \vec{F} , and P into \vec{u}^* , \vec{F}^* , and P^* of Eq. (8) with $l_0 = \frac{l_x}{k_x} = \frac{l_y}{k_y}$ (wavelength).

Poiseuille flow [26] has an analytic velocity of

$$u_x(y) = \frac{s}{2\rho_0\nu} y(l_y - y),$$

with constant $s = F_x - \partial_x P$. Generally, the channel flow is driven by either a constant force density F_x or a constant pressure gradient $\partial_x P$. Occasionally, it is driven by both a constant F_x and a constant $\partial_x P$. In our Poiseuille variant, F_x runs linearly from 0 to s and $\partial_x P$ runs linearly from $-s$ to 0, from $x = 0$ to $x = l_x$ [Fig. 3(b)]. Consequently, the equations for F_x and P become

$$\begin{aligned} F_x(x) &= \frac{s}{l_x} x, \\ P(x) &= \int \partial_x P dx \\ &= \int \left(\frac{s}{l_x} x - s \right) dx \\ &= \frac{1}{2} \frac{s}{l_x} x^2 - sx + P_{\text{int}}. \end{aligned}$$

Integration constant P_{int} can be determined by specifying an average pressure,

$$P_{\text{avg}} = \frac{1}{l_x} \int_0^{l_x} P(x) dx = -\frac{1}{3} s l_x + P_{\text{int}}.$$

Again use Eq. (5) to nondimensionalize \vec{u} , \vec{F} , and P into \vec{u}^* , \vec{F}^* , and P^* of Eq. (9) with $l_0 = l_y$ and $u_0 = u_x(\frac{l_y}{2})$.

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