Traffic Modelling and Some Inequalities in Banach Spaces

Miss Bouasy DOUNGSAVANH

Master Thesis in Applied Mathematics

Faculty of Natural Science
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Supervised by:

Assoc. Prof. Sackmone SIRISACK

Prof. Dr. Joaquim M.C. CORREIA

Prof. Dr. Nabil BEDJAOUI

Edited by:

Dr. Thepsavanh KITIGNAVONG

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Thesis Certification:

Supervised by:

Assoc. Prof. Sackmone SIRISACK .... S. mony.

Prof. Dr. Joaquim M.C. CORREIA ....

Prof. Dr. Nabil BEDJAOUI ....

Edited by:

Dr. Thepsavanh KITIGNAVONG ....

Thesis prepared by: Miss Bouasy DOUNGSAVANH

This thesis was defence on July 10, 2017 at National University of Laos

Committee:

Assoc. Prof. Sackmone SIRISACK .... S. mony.

Prof. Dr. Sounthone PHOMMASONE ....

Dr. Lisatteur LILIANOU ....

Dr. Latsamy VONGSENGCHANH ....

Dr. Thepsavanh KITIGNAVONG ....

Dean of Faculty of Natural Science

at National University of Laos

Date: 26/07/2017

Signature: .... ຮສ.ດຣ.ບຸນ ສີວົງໄຊ

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Abstract

Modelling traffic flow has been around since the appearance of traffic jams. Ideally, if we can correctly predict the behavior of vehicle flow given an initial set of data, then adjusting the flow in crucial areas can maximize the overall throughput of traffic along a stretch of road.

We consider a mathematical model for traffic flow on single land and without exits or entries. So, we are just observing what happens as time evolves if we fix at initial time ($t = 0$) some special distribution of cars (initial datum u_0). Because we do approximations, we need the notion of convergence and its corresponding topology. The numerical approximation of scalar conservation laws is carried out by using conservative methods such as the Lax-Friedrichs and the Lax-Wendroff schemes.

The Lax-Friedrichs scheme gives regular numerical solutions even when the exact solution is discontinuous (shock waves). We say the scheme is *diffusive* meaning that the scheme is solving in fact an evolution equation of the form $u_t+f(u)_x = \epsilon u_{xx}$, where ϵ is a small parameter depending on Δx and Δt.

The Lax-Wendroff scheme is more precise than the Lax-Friedrichs scheme, and give the right position of the discontinuities for the shock waves. But it develop oscillations. We say the scheme is *dispersive* what means the scheme is solving approximatively an evolution equation of the form $u_t + f(u)_x = \delta u_{xxx}$, where δ is a small parameter depending on Δx and Δt.

An elaboration and an implementation of Lax-Friedrichs schemes and of Lax-Wendroff schemes even extended to second order provided numerical solutions to the problem of traffic flows on the road. Since along the roads the schemes present the same features as for conservation laws, the new and original aspect is given by the treatment of the solution at junctions. Our tests show the effectiveness of the approximations, revealing that Lax-Wendroff schemes is more accurate than Lax-Friedrichs schemes.

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Chapter 1

Introduction

Interest in modelling traffic flow has been around since the appearance of traffic jams. Ideally, if we can correctly predict the behavior of vehicle flow given an initial set of data, then adjusting the flow in crucial areas can maximize the overall throughput of traffic along a stretch of road. This is of particular importance in regions of high traffic density, which may be caused by high volume peak time traffic, accidents, closure of some road lanes, etc.. Also this is an important issue as its impact in pollution, well-being or economy are huge.

More recently, following traffic flow theory, people are developing too pedestrian evacuation dynamic systems. In many ways both problems are similar. Thus it is natural that similar mathematical approaches could be applied to model the vehicle traffic flows and the pedestrian evacuation flows. E.g., in our case we are concerned with systems of partial differential equations and escape exit strategies are modelled as sink terms in those equations.

So, from the mathematical point of view we want to model such dynamical systems as partial differential equations (for short, PDEs) of evolution type (with time variable; thus): parabolic or hyperbolic PDEs like the ones in use to model fluid dynamics. But since we want to capture traffic jam phenomena, we are mainly concerned with hyperbolic PDEs as they develop shocks which provide our jam mathematical description. To be more specific, we will work with macroscopic balance models given by first order PDEs having divergence form, called hyperbolic conservation laws. It is well known that such models have solutions that develop discontinuities (shocks) at finite time, providing the jam prevision.

Meanwhile, the calculus and analysis with discontinuous functions is a hard task. From the functional analysis point of view we need work with spaces of discontinuous functions. And the theory for systems of hyperbolic conservation laws is an almost full open area of PDEs.

Actually both mathematical and numerical analysis of those problems are very active areas of research.

It is the goal of this dissertation to introduce me at this area. Then to achieve that goal, our study will be focused in a few and fundamental aspects. We will simplify as much as possible the technical and theoretical non essential difficulties, e.g., we will look for the case of just one equation, in 1-dimension of space, without source or sink terms. Of course, such mathematical assumptions are related with severe restrictions on real traffic problems: we consider just a single-lane without either cross points, exits or entries... In fact, even in such simple situation jams occur (according experiences) and the mathematical essential difficulty (discontinuities) is still there.

In chapter 2 we will briefly present a toy model for such simplified traffic flow, in chapter 3 we will be concerned with the role of a number of important, basic, inequalities in functional analysis of normed spaces, then in the last chapter 4 we will work with the Lax-Friedrich and Lax-Wendroff numerical schemes, specially drawn to deal with discontinuous solutions.

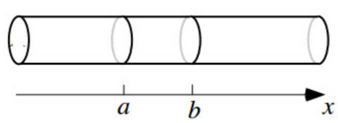
Chapter 2

Modelling traffic

2.1 Conservation Laws

Consider a flow of mass, heat, ..., in a long narrow region (a tube) of space (the x-axis) and let $u(x,t)$ and $\phi(x,t)$ denote respectively its density ([quantity] [volume]$^{-1}$) and its flux[1] ([quantity] [area]$^{-1}$[time]$^{-1}$), both functions of position x and time t assuming that density and flux keep constant in each perpendicular section of the tube[2].

The amount of that quantity (mass, heat, ...) in a tube interval, say $a \le x \le b$, at each instant of time, t, is given by the $\int_a^b u(x,t)\,dx$. So, its rate of variation is given by the $\frac{d}{dt}\int_a^b u(x,t)\,dx$. But, this is the net flux into the interval $a \le x \le b$ which is also given by $\phi(b,t) - \phi(a,t)$:



we deduce that

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \phi(b,t) - \phi(a,t) \qquad \Longleftrightarrow \tag{2.1}$$

$$\int_a^b \frac{\partial u}{\partial t}(x,t)\,dx = \int_a^b \frac{\partial \phi}{\partial x}(x,t)\,dx \qquad \Longleftrightarrow \tag{2.2}$$

$$\int_a^b u_t + \phi_x\,dx = 0 \tag{2.3}$$

[1]The product of density by velocity ([quantity] [volume]$^{-1}\times$[length][time]$^{-1}$).

[2]Each section corresponds to a fixed value of x for which u and ϕ have no variation in the (y,z) directions.

and, because $[a,b]$ is arbitrary, this implies the next conservation law holds at every (x,t)

$$u_t + \phi_x = 0\,.$$

Now to close the equation we do the assumption that $\phi = f(u)$, meaning that the flux depends solely upon the density. It is given by some known, physical, constitutive law, in general nonlinear. Our model, PDE, to rule this kind of phenomena is then given by the nonlinear transport equation

$$u_t + f(u)_x = 0\,. \tag{2.4}$$

2.2 The Model of traffic flow

Following the ideas in the last section, we will apply here the nonlinear transport equation as a toy model for traffic flow: we are looking for a long road with a single lane and without exits or entries. So, we are just observing what happens as time evolves if we fix at initial time ($t=0$) some spacial distribution of cars (initial datum u_0).

We consider then the initial value problem for the conservation law of cars number density

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & (x,t) \in \mathbb{R}\times\mathbb{R}_+ \\ u(x,0) = u_0(x), & x\in\mathbb{R} \end{cases} \tag{2.5}$$

where:

- $u = u(x,t) \in [0, u_{max}]$ represents the density of cars which cannot overpass a maximum value of cars on the road, the saturation point u_{max};

- $f(u) = u\,v$ is the flux of cars because $v = v(x,t)$ is the local speed of the cars, which to full accomplish with the assumptions in the previous section must depend upon the density: $v = v(u)$. And it is natural to assume that v is a decreasing function of the density such that $v(u_{max}) = 0$.

  If $v = v(u)$ is regular enough, then $f : [0, u_{max}] \to [0, +\infty[$ as $u, v \geq 0$ implies $f(u) = u\,v(u) \geq 0$. Then $f(0) = 0 = f(u_{max})$, $f'(u) = v(u) + u\,v'(u)$, $f''(u) = 2v'(u) + u\,v''(u)$. So $f'(0) = v(0) = v_{max} > 0$, $f'(u_{max}) = u_{max}\,v'(u_{max}) \leq 0$, $f''(0) = 2v'(0) \leq 0$.

- for simplicity take $v(u) = v_{max}\left(1 - \frac{u}{u_{max}}\right)$, then $f(u) = uv_{max}\left(1 - \frac{u}{u_{max}}\right)$ is a concave function. In fact doing a normalization we can just fix our attention in the model:

$$\begin{cases} u_t + (u(1-u))_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}. \tag{2.6}$$

Because the PDE is hyperbolic, for general initial datum its solutions develop discontinuities and are nonunique. We need a criterium to select a unique solution (the physically meaningful one):

Entropy criterium: A solution u of Problem 2.5 is said an entropy solution if for any pair (η, q) with a convex function η such that $\eta'(u)f'(u) = q'(u)$ it verifies the following entropy inequalities

$$\eta(u)_t + q(u)_x \leq 0\,. \tag{2.7}$$

This criterium can be justified in the following manner. In our traffic model if we consider the effect on drivers of the increasing density of cars in the road (given by u_x), we must wait that drivers will decrease the velocity. Then, in the right-hand side of the PDE we must add a term like ϵu_{xx}[3] with a small positive ϵ coefficient (say a coefficient of "viscosity"):

$$\begin{cases} \partial_t u + \partial_x f(u) = \epsilon u_{xx}, & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}. \tag{2.8}$$

This a parabolic PDE having regular solutions. When $\epsilon \to 0$ the equation (2.8) converges to the equation (2.5) and so it is for their solutions: the solutions of (2.8) as $\epsilon \to 0$ converge to the entropy solution of (2.5). (This is the celebrated 'vanishing viscosity method').

2.3 Riemann Problem

2.3.1 Case of the general hyperbolic equation

We will consider the **Riemann problem**

$$u_t + f(u)_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{2.9}$$

with initial condition

$$u(x,0) = u_0(x) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \tag{2.10}$$

[3] Actually we are considering instead of the term $\partial_x f(u)$, the term $\partial_x \left(f(u) - \epsilon u_x\right)$.

It is a conservation law together with piecewise constant data having a single discontinuity.

If $u(x,t)$ is a solution, then $u(\lambda t, \lambda x)$ is also a solution.

Thus, it is natural to consider solutions of the form

$$u(x,t) = U(x/t)$$

It turns out there are three cases (when f is concave):

- If $u_l = u_r$, then $u = u_0$ is the entropy solution.

- If $u_l < u_r$, $(f'(u_l) > f'(u_r))$, then

  $$u(x,t) = \begin{cases} u_l, & x < \sigma t, \\ u_r, & x > \sigma t, \end{cases}$$

  where σ satisfies the jump condition (Rankine-Hugoniot)

  $$\sigma = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

  Such a solution is called **shock wave**.

- If $u_l > u_r$ $(f'(u_l) < f'(u_r))$, then we try to find a smooth solution of the form $U(x/t)$.

  The equation becomes

  $$U'(x/t)\left[-\frac{x}{t^2} + f'(U)\frac{1}{t}\right] = 0$$

  which reduces to

  $$-\frac{x}{t} + f'(U) = 0$$

  if we assume that $U' \neq 0$ everywhere, we get $U = (f')^{-1}$ and

  $$u(x,t) = \begin{cases} u_l, & x \leq f'(u_l)t, \\ U(x/t), & f'(u_l)t < x < f'(u_r)t, \\ u_r, & x \geq f'(u_r)t. \end{cases}$$

  Such a solution is called a **rarefaction wave**.

2.3.2 Application to the traffic flow model

Here we consider the Riemann problem (2.9)-(2.10) where the flux function is given by $f(u) = u(1-u)$. This function is concave since $f''(u) = -2 < 0$. Thus, following the previous paragraph the exact entropy solution of the Riemann problem is given as follows:

- If $u_l = u_r$, then $u = u_0$ is the entropy solution.

- If $u_l < u_r$, $(f'(u_l) > f'(u_r))$, the entropy solution is a shock wave with speed σ where

$$\begin{aligned}\sigma &= \frac{f(u_r) - f(u_l)}{u_r - u_l} \\ &= \frac{u_r(1-u_r) - u_l(1-u_l)}{u_r - u_l} \\ &= 1 - (u_r + u_l).\end{aligned}$$

  Thus, u writes

$$u(x,t) = \begin{cases} u_l, & x < (1-(u_r+u_l))t, \\ u_r, & x > (1-(u_r+u_l))t, \end{cases}$$

- If $u_l > u_r$ $(f'(u_l) < f'(u_r))$, then the entropy solution is a rarefaction wave. Following the notations given for general case we have

$$\frac{x}{t} = f'(U) = 1 - 2U.$$

  Thus,

$$U(x/t) = \frac{1-\frac{x}{t}}{2} = \frac{t-x}{2t}$$

  and u is given by

$$u(x,t) = \begin{cases} u_l, & x \leq (1-2u_l)t, \\ \frac{t-x}{2t}, & (1-2u_l)t < x < (1-2u_r)t, \\ u_r, & x \geq (1-2u_r)t. \end{cases}$$

  Such a solution is called a **rarefaction wave**.

Chapter 3

Inequalities in Banach Spaces

In our study the functions we are working with are often discontinuous, still we have to operate between such functions, approximate them, etc.. So, we need concern with the algebra and the topology of such classes of functions (functional analysis). Here, we will consider appropriate vector subspaces of the real vector space $\mathcal{F}$ of all the applications $f : \mathbb{R} \to \mathbb{R}$ (by 'application' we mean a function defined in the full $\mathbb{R}$) with the usual operations of addition between functions and product of functions by real numbers.

To be explicit, we would like to work with the spaces of Lebesgue integrable functions

$$L^p(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \mid \int_{\mathbb{R}} |f(x)|^p \, dx < +\infty \right\} \qquad (\, p \in [1, +\infty] \,) \; .$$

However, to avoid the technicalities related with the Lebesgue integral and just retain the essentials behind the construction of Banach spaces, we will instead study the case of the "small" Lebesgue spaces $l^p(\mathbb{R})$ which we will define a few lines bellow in Prop. 3.1.6.

The emphasis in this chapter is on the ubiquitous role of inequalities as a basic tool to introduce an develop both algebraic and topological concepts we need to define and to work in Banach spaces.

Before to start that study we want to keep here just two basic and "frustrating" examples enrolling continuous functions. They are a simultaneous motivation of how natural it is to work with discontinuous functions and why we need the more general theory of Lebesgue integrable functions.

Example 3.0.1 *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be the sequence of continuous functions* $f_n : [0,1] \to \mathbb{R}$ *defined* *by* $f_n(x) = x^n$. *This sequence converges pointwise to the limit function* $f : [0,1] \to \mathbb{R}$ *given by*

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1 \end{cases},$$

which is a discontinuous function.

Example 3.0.2 *Consider the sequence* $\{f_n\}_{n\ge 2}$ *of continuous functions* $f_n : [0,1] \to \mathbb{R}$ *defined by*

$$f_n(x) = \begin{cases} n^2 x, & \text{if } 0 \le x \le \frac{1}{n} \\ n^2\left(\frac{2}{n} - x\right), & \text{if } \frac{1}{n} \le x \le \frac{2}{n} \\ 0, & \text{if } \frac{2}{n} \le x \le 1 \end{cases}.$$

This sequence converges pointwise to the continuous limit function $f : [0,1] \to \mathbb{R}$ *given by* $f(x) \equiv 0$. *Now, because these functions are continuous on* $[0,1]$ *they are Riemann integrable*

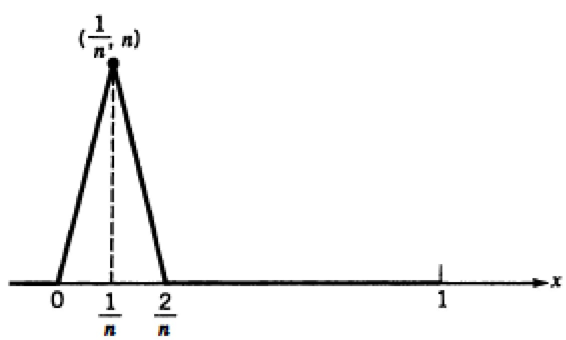


Figure 3.1: Riemann integrable functions

functions on $[0,1]$, *in fact we compute the integrals* $\int_0^1 f_n(x)\,dx \equiv 1$ *and* $\int_0^1 f(x)\,dx = 0$, *but*

$$1 = \lim_{n\to+\infty} \int_0^1 f_n(x)\,dx \ne \int_0^1 \lim_{n\to+\infty} f_n(x)\,dx = 0\,.$$

3.1 Linear spaces

Now, we want to remember from linear algebra the definitions of vector (or linear) space and subspace as to fix some notation and results which will be in use.

Definition 3.1.1 *We say* $(V, +, \cdot)$ *is a* real vector space *if* V *is a set and* $+ : V \times V \to V$, $\cdot : \mathbb{R} \times V \to V$ *are* operations[1] *subjected to the following axioms:*

Commutative law: $\forall u, v \in V \quad u + v = v + u.$

Associative law: $\forall u, v, w \in V \quad u + (v + w) = (u + v) + w.$

Zero vector: $\exists 0 \in V\colon\ \forall v \in V \quad v + 0 = v,$ *i.e., there exists in* V *an additive identity element, named 'zero' and noted* 0.

Symmetric vector: $\forall v \in V\ \exists x \in V : \quad v + x = 0,$ *such vector* x *is called the symmetric of* v *and its notation is* $-v$.

Distributive law: $\forall k \in \mathbb{R}\ \forall u, v \in V \quad k(u + v) = ku + kv.$

Distributive law: $\forall k, \lambda \in \mathbb{R}\ \forall v \in V \quad (k + \lambda)v = kv + \lambda v.$

Associative law: $\forall k, \lambda \in \mathbb{R}\ \forall v \in V \quad k(\lambda v) = (k\lambda)v.$

Unit: $\forall v \in V \quad 1v = v.$

Definition 3.1.2 *Given two real vector spaces* $(V, +, \cdot)$ *and* $(S, +, \cdot)$*, we say that* $(S, +, \cdot)$ *is a vector or linear* subspace *of* $(V, +, \cdot)$*, notation* $(S, +, \cdot) \leq (V, +, \cdot)$ *or for shortness* $S \leq V$*, when* S *is a subset of* V*,* $S \subset V$*, and both the operations on* S *are the restrictions of the operations on* V *to* S *(we say "the same" of* V*).*

Proposition 3.1.3 *Let* $(V, +, \cdot)$ *and* S *be respectively a vector space and a non-empty subset of* V*,* $\emptyset \neq S \subset V$*. Then* $(S, +, \cdot)$ *is a subspace of the vector space* $(V, +, \cdot)$ *iff*

S **is closed about the addition:** $\forall u, v \in S \quad u + v \in S.$

[1] Abstract ones but still named 'addition' and 'multiplication by scalars' to keep our intuition about. In the same line, instead of the notation $+(u, v)$ for the addition of $u, v \in V$ we will use $u + v$ and for a given scalar $k \in \mathbb{R}$ and vector $v \in V$ we will write kv instead of $\cdot(k, v)$. Remark that using the original notation the commutative law should be written as $+(u, v) = +(v, u)$ and the associative law as $+(u, +(v, w)) = +(+(u, v), w)$, somewhat hard to decipher. Finally, 'operations' are applications, meaning that the domain in the case of addition is the full $V \times V$ and of multiplication it is the full $\mathbb{R} \times V$.

S **is closed about the multiplication:** $\quad \forall k \in \mathbb{R}\ \forall v \in S \quad kv \in S.$

Proposition 3.1.4 *Consider the set of all the real applications with domain* $\mathbb{R}$*,*

$$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R} \mid \textit{the domain of } f \textit{ is } \mathbb{R}\}.$$

Given a subset $A \subset \mathbb{R}$ *and* $f \in \mathcal{F}$*, define the application with domain* A*, called the restriction of* f *to* A*,* $f|_A : A \to \mathbb{R}$ *by* $f|_A(x) = f(x)$ *for any* $x \in A$*. Finally let, for a fixed subset* A*, be*

$$\mathcal{F}|_A = \{f|_A : A \to \mathbb{R} \mid f \in \mathcal{F}\}.$$

Assuming that $(\mathcal{F}, +, \cdot)$ *is a real vector space, then* $(\mathcal{F}|_A, +, \cdot)$ *is too a real vector space for "the same" operations.*

As a consequence of this last proposition, if we fix the subset $A = \{1, 2, 3, \cdots, n\}$ for some natural number n or $A = \mathbb{N}$, then we conclude at once that both the sets of finite and infinite sequences of real numbers, with the usual operations, are real vector spaces.

Most usually we represent such finite sequences, applications $f : \{1, 2, 3, \cdots, n\} \to \mathbb{R}$ with $y = f(x)$, in extension as $(f(1), f(2), f(3), \cdots, f(n))$ or to simplify $(y_1, y_2, y_3, \cdots, y_n)$. Analogously, for infinite sequences $f : \mathbb{N} \to \mathbb{R}$ we use commonly the notation $(y_i)_{i \in \mathbb{N}}$ as synonym of the infinite sequence $(y_1, y_2, y_3, \cdots, y_i, \cdots)$. Thus the notation we will keep for the set of all the real finite sequences is, as usually, $\mathbb{R}^n$ and for the set of all the real infinite sequences it is $\mathbb{R}^\infty$. Using Prop. 3.1.4 we have the following useful[2] result

Corollary 3.1.5 *For the usual operations,* $(\mathbb{R}^n, +, \cdot)$ *and* $(\mathbb{R}^\infty, +, \cdot)$ *are real vector spaces.*

Now lets introduce the "small" Lebesgue spaces $(l^p(\mathbb{R}), +, \cdot)$ as subspaces of the real vector space $(\mathbb{R}^\infty, +, \cdot)$.

[2]More generally and usefully, we could do such construction using any field $(\mathbb{F}, +, \cdot)$ instead of the real field $(\mathbb{R}, +, \cdot)$. Then using abstract algebra we would be doing (a beginning of) abstract functional analysis.

Proposition 3.1.6 *For the usual operations and* $p \in]0, +\infty]$, $(l^p(\mathbb{R}), +, \cdot)$ *is a real vector space where*

$$l^p(\mathbb{R}) = \left\{(x_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty | \sum_{i=1}^{+\infty} |x_i|^p < +\infty\right\} \qquad (p > 0)\,;$$
$$l^\infty(\mathbb{R}) = \left\{(x_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty | \ \exists L > 0\colon \forall i \in \mathbb{N} \quad |x_i| \le L\right\} \qquad (p = +\infty)\,.$$

Proof: We use Prop. 3.1.3 together with the last Cor. 3.1.5. Consider the real vector space $(\mathbb{R}^\infty, +, \cdot)$ and his subset $l^p(\mathbb{R}) \subset \mathbb{R}^\infty$ which is a non empty set (as the null sequence shows). Let $k \in \mathbb{R}$ and $(x_i), (y_i) \in l^p(\mathbb{R})$ be arbitrary[3], then in $(\mathbb{R}^\infty, +, \cdot)$ we have $k\,(x_i) = (kx_i)$, $(x_i) + (y_i) = (x_i + y_i)$, and we have to prove that $l^p(\mathbb{R})$ is closed about those operations.

When $p = +\infty$, the sequences in $l^\infty(\mathbb{R})$ are the bounded sequences and then the sequences $k\,(x_i)$ and $(x_i) + (y_i)$ are still bounded, meaning that $l^\infty(\mathbb{R})$ is closed about the operations $+$ and $\cdot$, done for $p = +\infty$.

When $p > 0$ (finite), by hypothesis

$$\sum_{i=1}^{+\infty} |x_i|^p < +\infty\,, \quad \sum_{i=1}^{+\infty} |y_i|^p < +\infty\,,$$

thus $l^p(\mathbb{R})$ is closed about the multiplication because

$$\sum_{i=1}^{+\infty} |kx_i|^p = |k|^p \sum_{i=1}^{+\infty} |x_i|^p < +\infty \qquad (p > 0) \tag{3.1}$$

and it is closed too about the addition because for $0 < p \le 1$

$$\sum_{i=1}^{+\infty} |x_i + y_i|^p \le \sum_{i=1}^{+\infty} |x_i|^p + \sum_{i=1}^{+\infty} |y_i|^p < +\infty \qquad (0 < p \le 1), \tag{3.2}$$

where we just need remark that each term in the series verifies $|x_i + y_i|^p \le |x_i|^p + |y_i|^p$ if $0 < p \le 1$, and for $p \ge 1$[4]

$$\sum_{i=1}^{+\infty} |x_i + y_i|^p \le 2^{p-1} \left(\sum_{i=1}^{+\infty} |x_i|^p + \sum_{i=1}^{+\infty} |y_i|^p\right) < +\infty \qquad (p \ge 1), \tag{3.3}$$

where we apply the inequality $\left|\frac{1}{2}\,x_i + \frac{1}{2}\,y_i\right|^p \le \frac{1}{2}\left(|x_i|^p + |y_i|^p\right)$ to each term in the series, which is a consequence of the fact that for $p \ge 1$ the function $\varphi(x) = |x|^p$ is convex (see below the Def. 3.2.1).

[3] We abbreviate the notation using (x_i) instead of $(x_i)_{i\in\mathbb{N}}$.

[4] The constant 2^{p-1} is the best (the smaller) we can get as we see if we consider the case where $(x_i) = (y_i)$.

Actually the $l^p(\mathbb{R})$ spaces are forming an increasing chain of (sub)spaces (see the next proposition) with maximum term $l^\infty(\mathbb{R})$ but without minimum term ($q \searrow 0^+$):

$$\cdots < l^q(\mathbb{R}) < \cdots < l^1(\mathbb{R}) < \cdots < l^p(\mathbb{R}) < \cdots < l^\infty(\mathbb{R}).$$

Proposition 3.1.7 *If* $0 < p_1 \leq p_2 \leq +\infty$, *then the* $(l^{p_1}(\mathbb{R}), +, \cdot)$ *space is a subspace of the* $(l^{p_2}(\mathbb{R}), +, \cdot)$ *space,* $l^{p_1}(\mathbb{R}) \leq l^{p_2}(\mathbb{R})$. *Moreover they are proper subspaces as* $l^{p_1}(\mathbb{R}) \subsetneqq l^{p_2}(\mathbb{R})$ *for* $p_1 < p_2$:

$$l^{p_1}(\mathbb{R}) < l^{p_2}(\mathbb{R}) \qquad (p_1 < p_2). \tag{3.4}$$

Proof: First we will prove the case for $p_2 = +\infty$, each $(l^p(\mathbb{R}), +, \cdot)$ space is a subspace of the $(l^\infty(\mathbb{R}), +, \cdot)$ space, then we will prove the remaining cases for finite p_1 and p_2.

If $(x_i) \in l^p(\mathbb{R})$, then by definition $\sum |x_i|^p$ is convergent and necessarily $\lim_{i\to+\infty} |x_i|^p = 0$. Such (x_i) is a bounded sequence. So $l^p(\mathbb{R}) \subset l^\infty(\mathbb{R})$ and we proved (by Def. 3.1.2 of subspace) that $(l^p(\mathbb{R}), +, \cdot) \leq (l^\infty(\mathbb{R}), +, \cdot)$ for $p > 0$.

Now if $(x_i) \in l^{p_1}(\mathbb{R})$, let $L > 0$ be a bound for the sequence (x_i) ($\forall_{i\in\mathbb{N}}\ |x_i| \leq L$). Take $p_2 \geq p_1$, then

$$\sum_{i\in\mathbb{N}} |x_i|^{p_2} = \sum_{i\in\mathbb{N}} |x_i|^{p_1} |x_i|^{p_2-p_1} \leq L^{p_2-p_1} \sum_{i\in\mathbb{N}} |x_i|^{p_1} < +\infty.$$

Necessarily $(x_i) \in l^{p_2}(\mathbb{R})$, as we had to prove.

Finally, to prove the inclusions are proper we will use the Dirichlet criterium for series. Remember it: the series $\sum_{i\in\mathbb{N}} \frac{1}{i^\alpha}$ converge for $\alpha > 1$ and diverge for $\alpha \leq 1$. Then we consider the sequence

$$(x_i)_{i\in\mathbb{N}} = \left(\left(\frac{1}{i}\right)^{\frac{1}{p_1}}\right)_{i\in\mathbb{N}} \quad \text{for which} \quad \sum_{i\in\mathbb{N}} |x_i|^p = \sum_{i=1}^{+\infty} \frac{1}{i^\alpha} \quad \text{with} \quad \alpha = \frac{p}{p_1}.$$

So (x_i) belongs to $l^{p_2}(\mathbb{R})$ if $p_1 < p_2$ because $\alpha = \frac{p_2}{p_1} > 1$, but (x_i) do not belongs to $l^{p_1}(\mathbb{R})$ because $\alpha = \frac{p_1}{p_1} = 1$.

Proposition 3.1.8 *For any* $p \in]0, +\infty]$ *and* $n \in \mathbb{N}$ *the space* $(\mathbb{R}^n, +, \cdot)$ *is isomorphic to a subspace of* $(l^p(\mathbb{R}), +, \cdot)$.

Proof: There we have the obvious isomorphism between $(\mathbb{R}^n, +, \cdot)$ and the subspace of $(l^p(\mathbb{R}), +, \cdot)$ defined by $\varphi((x_1, x_2, x_3, \cdots, x_n)) = (x_1, x_2, x_3, \cdots, x_n, 0, \cdots, x_i, \cdots)$ such that

$x_i = 0$ for $i \geq n+1$.

Because of this isomorphism we do the usual identification of $(x_1, x_2, x_3, \cdots, x_n) \in \mathbb{R}^n$ with $(x_1, x_2, x_3, \cdots, x_n, 0, 0, 0, \cdots) \in l^p(\mathbb{R})$. Thus we got the increasing infinite chain of subspaces:

$$\mathbb{R} < \mathbb{R}^2 < \cdots < \mathbb{R}^n < \cdots < l^q(\mathbb{R}) < \cdots < l^1(\mathbb{R}) < \cdots < l^p(\mathbb{R}) < \cdots < l^\infty(\mathbb{R}) \tag{3.5}$$

and notice that the subchain before $l^1(\mathbb{R})$ is bi-infinite having the infinite discrete part ($n \in \mathbb{N}$) of all the linear spaces $\mathbb{R}^n$ preceding all the $l^q(\mathbb{R})$ of the infinite continuum part ($0 < q < 1$). The chain has a minimum, $\mathbb{R}$, and a maximum, $l^\infty(\mathbb{R})$.

As a final remark in this section we want to make explicit that all the real linear spaces $l^p(\mathbb{R})$, $0 < p \leq +\infty$, have infinite dimension. This being clear as we exhibit the infinite set of linearly independent vectors $\{(\delta_i^n)_{i\in\mathbb{N}}\}_{n\in\mathbb{N}} \subset l^p(\mathbb{R})$, where δ_i^n is the Kronecker's delta ($\delta_i^n = 1$ if $i = n$ and $\delta_i^n = 0$ if $i \neq n$): for each $n \in \mathbb{N}$ we have a vector $(x_i)_{i\in\mathbb{N}} = (\delta_i^n)_{i\in\mathbb{N}}$ with all its x_i terms equal to zero but one, in coordinate number n, which has the value one).

3.2 Convex inequalities

In the last section, inside the proof of Prop. 3.1.6 to get the inequality (3.2), we used the notion of convex function which we will now remember here.

Definition 3.2.1 *A real function φ defined in an open interval $]a, b[\subset \mathbb{R}$, $\varphi :]a, b[\to \mathbb{R}$, is called a* convex function *if the following inequality is satisfied:*

$$\forall\, x, y \in]a, b[\quad \forall \alpha \in [0, 1] \qquad \varphi((1-\alpha)\,x + \alpha\,y) \leq (1-\alpha)\,\varphi(x) + \alpha\,\varphi(y)\,. \tag{3.6}$$

If we reverse the sense of the inequality (as $\geq$ instead of $\leq$), then we call φ a concave function.

Graphical interpretation of convex (concave) function: while $t = (1-\alpha)\,x + \alpha\,y$ describes the segment $[x, y] \subset]a, b[$, as $\alpha \in [0, 1]$, the left-hand side of inequality (3.6) describes the graphic of the function φ in between the points $(x, \varphi(x))$ and $(y, \varphi(y))$ and the right-hand side describes the (straight) segment line connecting these two points. Thus, the meaning of inequality (3.6) is that each such portion of the graphic of φ should be always bellow (above) those corresponding segment lines.

Alternatively, we can interpret (3.6) in terms of slopes: still with $t = (1-\alpha)\,x + \alpha\,y$ we have $\alpha = \frac{t-x}{y-x}$ and $1-\alpha = \frac{y-t}{y-x}$, then (3.6) is equivalent to (the reverse one for concave functions)

$$\forall\, x, y \in \,]a,b[\quad \forall t \in \,]x,y[\qquad \frac{\varphi(t)-\varphi(x)}{t-x} \leq \frac{\varphi(y)-\varphi(t)}{y-t}\,. \tag{3.7}$$

Proposition 3.2.2 *For regular enough functions, $\varphi\colon]a,b[\to \mathbb{R}$, φ is a convex (concave) function iff*

differentiable φ: *when φ is differentiable in $]a,b[$, φ' is a nondecreasing (nonincreasing) function in $]a,b[$;*

twice differentiable φ: *when φ is twice differentiable in $]a,b[$, φ'' is a nonnegative (nonpositive) function in $]a,b[$.*

Proof: To prove the first we use the mean value theorem of differentiation to show that (3.7) is equivalent to the condition that φ' is a nondecreasing (nonincreasing) function in $]a,b[$:

$$\varphi'(\xi_1) = \frac{\varphi(t)-\varphi(x)}{t-x} \leq \frac{\varphi(y)-\varphi(t)}{y-t} = \varphi'(\xi_2) \qquad \text{with} \qquad x < \xi_1 < t < \xi_2 < y.$$

To prove the second it is enough to remember that for a regular function (φ'), she is a nondecreasing (nonincreasing) function in $]a,b[$ iff its derivative (φ'') is a nonnegative (nonpositive) function in $]a,b[$.

Another, direct, proof can be done if we define the auxiliary twice differentiable function $g(s) = \varphi((1-\alpha)\,s + \alpha\,y) - (1-\alpha)\,\varphi(s)$ for $s \in [x,y]$. Compute $g(x) = \varphi((1-\alpha)\,x + \alpha\,y) - (1-\alpha)\,\varphi(x)$ and $g(y) = \alpha\,\varphi(y)$. Then (3.6) translates as $g(x) - g(y) \leq 0$. But, by the mean value theorem of differentiation, $g(x)-g(y) = g'(\xi)(x-y)$ for some $\xi \in]x,y[$ and because $x - y < 0$ (3.6) is equivalent to $g'(\xi) \geq 0$ for $\xi \in]x,y[$. Now we compute, using again the mean value theorem of differentiation applied to the differentiable φ' function, $g'(\xi) = (1-\alpha)\,\varphi'((1-\alpha)\,\xi + \alpha\,y) - (1-\alpha)\,\varphi'(\xi) = (1-\alpha)\,\varphi''(\eta)\,\alpha\,(y-\xi)$ for some $\eta \in]\xi, (1-\alpha)\,\xi + \alpha\,y[$. And because $y - \xi > 0$, (3.6) becomes equivalent to $\varphi''(\eta) \geq 0$ for $\eta \in \,]a,b[$ (as x and y are arbitrary in $]a,b[$).

Corollary 3.2.3 *A differentiable function $\varphi\colon]a,b[\to\mathbb{R}$ with strictly positive (negative) derivative φ' in $]a,b[$ is an invertible convex (concave) function which inverse function is a concave (convex) function.*

Proof: This is real analysis, using the elementary derivative rule for the inverse function

$$\left(\varphi^{-1}\right)'(y) = -\frac{1}{\varphi'(x)}\,,$$

where $x=\varphi(y)$.

Proposition 3.2.4 *Any convex (concave) function $\varphi\colon]a,b[\to\mathbb{R}$ is a continuous function on the open interval $]a,b[$*[5].

Proof: For an arbitrary $x_0\in\,]a,b[$, using the definition of convexity we show that both the $\lim_{x\to x_0^\pm}\varphi(x)=\varphi(x_0)$.

Theorem 3.2.5 *Suppose $\varphi\colon]a,b[\to\mathbb{R}$ is a convex function and, for some $n\in\mathbb{N}$, there are given $\forall_{1\le i\le n}\ x_i\in\,]a,b[$ and $\alpha_i\ge 0$ such that $\sum_{i=1}^n\alpha_i=1$. Then, the following inequality holds*

$$\varphi\left(\sum_{i=1}^n\alpha_i\,x_i\right)\le\sum_{i=1}^n\alpha_i\varphi(x_i).$$

Proof: Use the definition of convexity taking three points x_i with coefficients α_i $(i=1,2,3)$ such that $\sum_{i=1}^3\alpha_i=1$. Just remark that $\alpha_1+\alpha_2=1-\alpha_3$ and

$$\varphi\left(\alpha_1x_1+\alpha_2x_2+\alpha_3x_3\right)=\varphi\left((1-\alpha_3)\left(\frac{\alpha_1}{1-\alpha_3}x_1+\frac{\alpha_2}{1-\alpha_3}x_2\right)+\alpha_3x_3\right).$$

Then do induction for any $n\in\mathbb{N}$.

Corollary 3.2.6 (Young's inequality) *Let for $n\in\mathbb{N}$ be $\forall_{1\le i\le n}\ y_i,\alpha_i\ge 0$ with $\sum_{i=1}^n\alpha_i=1$, then*

$$y_1\,y_2\,\cdots\,y_n\le\alpha_1\,y_1^{\frac{1}{\alpha_1}}+\alpha_2\,y_2^{\frac{1}{\alpha_2}}+\cdots+\alpha_n\,y_n^{\frac{1}{\alpha_n}}\,.$$

[5]Remark that we could define convex and concave functions in a closed interval $[a,b]$, but then this theorem should be false as the counterexample given by $\varphi\colon\ [0,1]\to\mathbb{R}$ such that $\varphi(x)=0$ for $x\in[0,1[$ and $\varphi(1)=1$ show us.

Proof: It is enough to take $\varphi(x) = e^x$ and to define conveniently the y_i's from the e^{x_i}'s.

Corollary 3.2.7 (Jensen's inequality) *Suppose $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a convex function and there are given $\forall_{i\in\mathbb{N}}\ x_i \in \mathbb{R}$ and $\alpha_i \geq 0$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Then*

$$\varphi\left(\sum_{i=1}^{\infty} \alpha_i\, x_i\right) \leq \sum_{i=1}^{\infty} \alpha_i \varphi(x_i).$$

Proof: We use first the continuity of φ given by Prop. 3.2.4, to reduce this case to the finite case:

$$\varphi\left(\sum_{i=1}^{\infty} \alpha_i\, x_i\right) = \varphi\left(\lim_{n\to+\infty} \sum_{i=1}^{n} \alpha_i\, x_i\right) = \lim_{n\to+\infty} \varphi\left(\sum_{i=1}^{n} \alpha_i\, x_i\right).$$

Remark that if $\sum_{i=1}^{\infty} \alpha_i = 1$, then $\sum_{i=1}^{n} \alpha_i < 1$ and[6] we cannot use immediately the Theor. 3.2.5 in the last term above. But define $A_n = \sum_{i=1}^{n} \alpha_i$, then $\forall_{1\leq i\leq n}\ \frac{\alpha_i}{A_n} \geq 0$ and $\sum_{i=1}^{n} \frac{\alpha_i}{A_n} = 1$. Now we can use the Theor. 3.2.5:

$$\varphi\left(\sum_{i=1}^{n} \alpha_i\, x_i\right) = \varphi\left(\sum_{i=1}^{n} \frac{\alpha_i}{A_n}\, A_n x_i\right) \leq \sum_{i=1}^{n} \frac{\alpha_i}{A_n}\, \varphi\left(A_n x_i\right)$$

and

$$\varphi\left(\sum_{i=1}^{\infty} \alpha_i\, x_i\right) = \lim_{n\to+\infty} \varphi\left(\sum_{i=1}^{n} \alpha_i\, x_i\right) \leq \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{\alpha_i}{A_n}\, \varphi\left(A_n x_i\right).$$

To conclude it is enough to prove that

$$\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{\alpha_i}{A_n}\, \varphi\left(A_n x_i\right) = \lim_{n\to+\infty} \sum_{i=1}^{n} \alpha_i\, \varphi\left(x_i\right) + \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{\alpha_i}{A_n} \left(\varphi\left(A_n x_i\right) - A_n\, \varphi\left(x_i\right)\right),$$

where the last limit is zero. To see this notice that

$$0 \leq \sum_{i=1}^{n} \frac{\alpha_i}{A_n} \left(\varphi\left(A_n x_i\right) - A_n\, \varphi\left(x_i\right)\right) \leq \sup_{1\leq i\leq n} \left\{\varphi\left(A_n x_i\right) - \varphi\left(x_i\right)\right\},$$

where $A_n \to 1$, so $A_n\, x_i \to x_i$ and by continuity $\varphi(A_n\, x_i) \to \varphi(x_i)$ for all $1 \leq i \leq n$. In fact because we can restrict ourselves to some closed interval, we can use the uniform continuity of φ to eliminate the dependence in n. The conclusion follows.

3.3 Normed spaces

Because we want to do approximations, we need the notion of convergence. And, fixed a linear space, each notion of convergence is introduced by its corresponding topology. A common way

[6] For the same reason we are handling the case of $\varphi\colon \mathbb{R} \to \mathbb{R}$ and not that of $\varphi\colon]a, b[\to \mathbb{R}$.

to define topologies is using semi-norms. Actually some vector spaces of infinite dimension and suitable for important applications, called *locally convex spaces*, have a topology making use of a necessarily infinite number of semi-norms. Here, we will be occupied with the case of vector spaces with a single semi-norm, in fact the *normed linear spaces* and the *Banach spaces.*

Our main examples will be constructed upon the linear spaces $(\mathbb{R}^n, +, \cdot)$ and $(l^p(\mathbb{R}), +, \cdot)$, abbreviated as $\mathbb{R}^n$ and $l^p(\mathbb{R})$. So, our next goal is to show that the $l^p(\mathbb{R})$ are normed spaces.

Before we will define and discuss the notion of semi-norm and norm. Along the exposition of this chapter we use a long list of inequalities. A major objective of this dissertation is to understand the role and meaning of those inequalities as tools of (real) functional analysis.

Definition 3.3.1 *A real application defined on a vector space V, $q : V \to \mathbb{R}$, is called a* semi-norm *on V if it satisfies the following axioms*

subadditivity: $\forall u, v \in V \quad q(u + v) \leq q(u) + q(v)$;

positive homogeneity: $\forall k \in \mathbb{R}\ \forall v \in V \quad q(kv) = |k|q(v)$.

Moreover, if the following axiom is also satisfied

separation: *for* $v \in V, \quad q(v) = 0 \implies v = 0$,

we say that q is a norm *and that $((V, +, \cdot), q)$ or abbreviated (V, q) is a* normed space.

In the case of normed spaces we will use the more common notation $\|v\|$ instead of $q(v)$. The $\|v\|$ should be interpreted as the length of the vector v and then, with the usual interpretations in vector spaces, the distance between u and v is given by $d(u, v) = \|u - v\|$.

In fact by the definition of norm and the next proposition we see that $\|v\|$ is a nonnegative number which is zero iff v is the zero vector (or $d(u, v) = 0$ iff $u = v$) such that (by homogeneity) the length of a positive multiple of a vector v or its symmetric is that positive multiple of the length of $\pm v$. The subadditivity is the *triangle inequality* saying that the length of any side of a triangle is smaller than the addition of the lengths of the remaining two sides.

Proposition 3.3.2 *A semi-norm verifies the following properties*

semidefinite: *if* $v = 0, \quad q(v) = 0$*;*

continuous: $\forall u, v \in V \quad q(u - v) \geq |q(u) - q(v)|$*;*

positive: $\forall v \in V \quad q(v) \geq 0$*.*

Proof: The first property is an immediate consequence of the homogeneity taking $k = 0$ and $v = 0$. Now use the subadditivity: $q(u) = q\left((u - v) + v\right) \leq q(u - v) + q(v)$, then $q(u-v) \geq q(u)-q(v)$. By homogeneity and the previous: $q(u-v) = |-1|q(v-u) \geq q(v)-q(u)$. So we proved the second property. Finally, the third one is a consequence of the second taking $v = 0$.

We remember the notion of *distance* or *metrics* and that of *metric space:*

Definition 3.3.3 (M, d)*, where* M *is a set and* $d : M \times M \to \mathbb{R}$*, is said to be a metric space if* d *is an application, called a metric or distance in* M*, satisfying the axioms*

positive definite: $\forall x, y \in M \quad d(x, y) \geq 0$ *and* $d(x, y) = 0$ *iff* $x = y$*;*

symmetry: $\forall x, y \in M \quad d(x, y) = d(y, x)$*;*

triangle inequality: $\forall x, y, z \in M \quad d(x, y) \leq d(x, z) + d(z, y)$*.*

Proposition 3.3.4 *A normed space with norm* $\|\cdot\|$ *is always a metric space with the metric defined by* $d(u, v) = \|u - v\|$*.*

Proof: We already proved it in the paragraph before proposition 3.3.2.

Our next goal is to prove that the linear spaces $l^p(\mathbb{R})$ are normed spaces, where the candidate applications to norms are given in the following definition.

Definition 3.3.5 *For* $p \in \,]0, +\infty]$ *we define on* $l^p(\mathbb{R})$ *the real applications* $\|\cdot\|_p : l^p(\mathbb{R}) \to \mathbb{R}$

given by the expressions

$$\|(x_i)\|_p = \sum_{i=1}^{\infty} |x_i|^p \,, \qquad \textit{for} \;\; 0 < p \le 1; \tag{3.8}$$

$$\|(x_i)\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} , \qquad \textit{for} \;\; 1 \le p < +\infty; \tag{3.9}$$

$$\|(x_i)\|_\infty = \sup_{i \in \mathbb{N}} \{|x_i|\} \;, \qquad \textit{for} \;\; p = +\infty. \tag{3.10}$$

Theorem 3.3.6 *For $0 < p \le 1$ the $(l^p(\mathbb{R}), \|\cdot\|_p)$ is a normed space.*

Proof: From the definition of the $\|\cdot\|_p$ given by (3.8), the verification of the separation and positive homogeneity axioms of Def. 3.3.1 is trivial and that of the subadditivity is a direct consequence of the inequality $|x+y|^p \le |x|^p + |y|^p$, true because $0 < p \le 1$, which we apply to each term of the series $\sum_{i=1}^{\infty} |x_i + y_i|^p$, (cf. inequality (3.2)).

To prove that $(l^p(\mathbb{R}), \|\cdot\|_p)$ are normed spaces for $p \in]1, +\infty[$ we will need an important auxiliary inequality, the Hölder inequality. To prove the $l^p(\mathbb{R})$ are linear spaces, see Prop. 3.1.6, we already used an inequality coming from convexity (cf. inequality (3.3)). The Hölder inequality is too a consequence of convexity. In fact we will use the

Lemma 3.3.7 (Young's inequality) *Let $a, b \ge 0$ and $p, p' > 0$ such that $p + p' = pp'$, then*

$$ab \le \frac{1}{p}\, a^p + \frac{1}{p'}\, b^{p'} \,, \qquad \frac{1}{p} + \frac{1}{p'} = 1 \quad \textit{and} \quad p, p' > 1 \,. \tag{3.11}$$

The p and p' are called conjugate (or dual) numbers. *Moreover, the equality occurs iff $a^p = b^{p'}$.*

Proof: The first inequality is a particular case for $n = 2$ of Young's inequality in Cor. 3.2.6 because $p + p' = pp'$ is equivalent to $\frac{1}{p} + \frac{1}{p'} = 1$ (by hypothesis $p, p' \ne 0$). From the last equation we see that forcibly $p, p' > 1$ (by hypothesis $p, p' > 0$).

A direct proof is given considering the concave function $\log \colon]0, +\infty[\to \mathbb{R}$ and $\frac{1}{p} + \frac{1}{p'} = 1$:

$$\log\left(\frac{1}{p}\, a^p + \frac{1}{p'}\, b^{p'} \right) \ge \frac{1}{p}\, \log(a^p) + \frac{1}{p'}\, \log(b^{p'}) = \log(ab).$$

Then apply to the extremes the exponential function which is an order preserving function (as its derivative is nonnegative).

From now on, because $p \searrow 1 \iff p' \nearrow +\infty$ (keeping the equality $\frac{1}{p} + \frac{1}{p'} = 1$ true), we will admit $p = 1$ and $p' = +\infty$ as conjugate (or dual) numbers too.

Theorem 3.3.8 (Hölder's inequality) *Let $p, p' \in [1, +\infty]$ be dual numbers and $(x_i) \in l^p(\mathbb{R})$, $(y_i) \in l^{p'}(\mathbb{R})$, then $(x_i\, y_i) \in l^1(\mathbb{R})$ and*

$$\|(x_i\, y_i)\|_1 \le \|(x_i)\|_p \, \|(y_i)\|_{p'} \qquad \text{or} \qquad \sum_{i=1}^{\infty} |x_i\, y_i| \le \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |y_i|^{p'} \right)^{\frac{1}{p'}} . \quad (3.12)$$

Moreover the equality holds iff the $l^1(\mathbb{R})$ vectors (x_i^p) and $(y_i^{p'})$ are parallel.

Proof: The case where $p = 1$ and $p' = +\infty$ is trivial, so let just consider the case where $p > 1$. Define $A = \|(x_i)\|_p^p$ and $B = \|(y_i)\|_{p'}^{p'}$. If $A = 0$ or $B = 0$, then the vector $(x_i) = \vec{0}$ or the vector $(y_i) = \vec{0}$ and the vector $(x_i\, y_i) = \vec{0}$. The equality (3.12) is proved in these cases. It remains to prove the inequality for $A \neq 0$ and $B \neq 0$. In this case (3.12) writes as

$$\sum_{i=1}^{\infty} |x_i\, y_i| \le A^{\frac{1}{p}}\, B^{\frac{1}{p'}} \qquad \iff \qquad \sum_{i=1}^{\infty} \frac{|x_i|}{A^{\frac{1}{p}}} \, \frac{|y_i|}{B^{\frac{1}{p'}}} \le 1 \, .$$

Define $a_i = \frac{|x_i|}{A^{\frac{1}{p}}}$ and $b_i = \frac{|y_i|}{B^{\frac{1}{p'}}}$, we have to show that $\sum_{i=1}^{\infty} a_i\, b_i \le 1$. To do so, use the Lemma 3.3.7 for each $i \in \mathbb{N}$:

$$a_i\, b_i \le \frac{1}{p}\, a_i^p + \frac{1}{p'}\, b_i^{p'} \quad \Rightarrow \quad \sum_{i=1}^{\infty} a_i\, b_i \le \frac{1}{p} \left(\sum_{i=1}^{\infty} a_i^p \right) + \frac{1}{p'} \left(\sum_{i=1}^{\infty} b_i^{p'} \right) = \frac{1}{p} + \frac{1}{p'} = 1$$

where, because of the definition of A and B,

$$\sum_{i=1}^{\infty} a_i^p = \frac{1}{A} \left(\sum_{i=1}^{\infty} |x_i|^p \right) = \frac{A}{A} \, , \qquad \sum_{i=1}^{\infty} b_i^{p'} = \frac{1}{B} \left(\sum_{i=1}^{\infty} |y_i|^{p'} \right) = \frac{B}{B} \, .$$

Theorem 3.3.9 *For $1 \le p < +\infty$, $(l^p(\mathbb{R}), \|\cdot\|_p)$ is a normed space.*

Proof: As before, the positive homogeneity and the separation axioms are easy to prove from the expression (3.9) defining $\|(x_i)\|_p$, Def. 3.3.5. To prove the subadditivity lets consider the $\|(x_i + y_i)\|_p^p$ together with

$$|x_i + y_i|^p = |x_i + y_i| |x_i + y_i|^{p-1} \le (|x_i| + |y_i|)\, |x_i + y_i|^{p-1} \le |x_i| |x_i + y_i|^{p-1} + |y_i| |x_i + y_i|^{p-1} \, ,$$

then adding each term

$$
\begin{aligned}
\|(x_i + y_i)\|_p^p &= \sum_{i=1}^{\infty} |x_i + y_i|^p \\
&\leq \sum_{i=1}^{\infty} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{\infty} |y_i||x_i + y_i|^{p-1}
\end{aligned} \tag{3.13}
$$

and using Hölder's inequality (3.12), remark that $(p-1)p' = p$ and $\frac{p}{p'} = p - 1$,

$$
\begin{aligned}
\sum_{i=1}^{\infty} |x_i||x_i + y_i|^{p-1} &\leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} \left(|x_i + y_i|^{p-1}\right)^{p'}\right)^{\frac{1}{p'}} \\
&= \|(x_i)\|_p \, \|(x_i + y_i)\|_p^{p-1} < \infty \,,
\end{aligned}
$$

so, doing analogous to both the right-hand side terms in (3.13), we have that

$$
\begin{aligned}
\|(x_i + y_i)\|_p^p &\leq \|(x_i)\|_p \, \|(x_i + y_i)\|_p^{p-1} + \|(y_i)\|_p \, \|(x_i + y_i)\|_p^{p-1} \\
&= \|(x_i + y_i)\|_p^{p-1} \left(\|(x_i)\|_p + \|(y_i)\|_p\right) \,.
\end{aligned}
$$

If $\|(x_i + y_i)\|_p \neq 0$, the last inequality writes exactly as the subadditivity

$$
\|(x_i + y_i)\|_p \leq \|(x_i)\|_p + \|(y_i)\|_p \,.
$$

If $\|(x_i + y_i)\|_p = 0$, by the positiveness of the norm, the subadditivity is immediate.

This subadditivity property has commonly the name of Minkowski inequality. We have just proved it and let put it in evidence:

Theorem 3.3.10 (Minkowski's inequality) *If $1 \leq p < +\infty$ and $(x_i), (y_i) \in l^p(\mathbb{R})$, then*

$$
\left(\sum_{i=1}^{\infty} |x_i + y_1|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{\frac{1}{p}} \,. \tag{3.14}
$$

Moreover the equality holds iff (x_i) and (y_i) are parallel vectors in the linear space $l^p(\mathbb{R})$.

Notice that, as a consequence of inequality (3.3), we already knew that

$$
\|(x_i + y_i)\|_p \leq 2^{\frac{p-1}{p}} \left(\|(x_i)\|_p + \|(y_i)\|_p\right)
$$

but here the best (smaller) constant is 1 (by Minkowski inequality) and not $2^{\frac{p-1}{p}}$ (even if in the inequality (3.3) the best constant is 2^{p-1}).

Theorem 3.3.11 *$(l^\infty(\mathbb{R}), \|\cdot\|_\infty)$ is a normed space.*

Proof: The proof is almost trivial using the properties of the *supremum.*

Proposition 3.3.12 *If $(V, \|\cdot\|)$ is a normed space and S is a linear subspace of V, then S endowed with the same application $\|\cdot\|$, or more accurately with the application $\|\cdot\|_{|S} : S \to \mathbb{R}$, is too a normed space $(S, \|\cdot\|_{|S})$, which we will abbreviate as $(S, \|\cdot\|)$.*

Because of this Proposition and Prop. 3.1.7, for $p_1 < p_2$ the space $l^{p_1}(\mathbb{R})$ (which is a subspace of $l^{p_2}(\mathbb{R})$) is endowed with its own norm $\|\cdot\|_{p_1}$ and the norm $\|\cdot\|_{p_2|_{l^{p_1}(\mathbb{R})}}$ induced from $l^{p_2}(\mathbb{R})$. But for $(x_i) \in l^{p_1}(\mathbb{R}) < l^{p_2}(\mathbb{R}) < l^\infty(\mathbb{R})$ we have that:

if $p_1 < p_2 \leq 1$, then $p_2 - p_1 > 0$ and

$$\|(x_i)\|_{p_2} = \sum_{i=1}^\infty |x_i|^{p_2} = \sum_{i=1}^\infty |x_i|^{p_1}|x_i|^{p_2-p_1} \leq \sum_{i=1}^\infty |x_i|^{p_1}\|(x_i)\|_\infty^{p_2-p_1} = \|(x_i)\|_\infty^{p_2-p_1}\|(x_i)\|_{p_1};$$

if $1 \leq p_1 < p_2$, then $\frac{p_2}{p_1} > 1$ and

$$\|(x_i)\|_{p_2} = \left(\sum_{i=1}^\infty |x_i|^{p_2}\right)^{\frac{1}{p_2}} = \left(\sum_{i=1}^\infty (|x_i|^{p_1})^{\frac{p_2}{p_1}}\right)^{\frac{1}{p_2}} \leq \left(\sum_{i=1}^\infty |x_i|^{p_1}\right)^{\frac{p_2}{p_1}\frac{1}{p_2}} = \|(x_i)\|_{p_1};$$

if $p_1 < 1 < p_2$, then using the previous two cases for $1 = p_1 < p_2$ and $p_1 < p_2 = 1$ we have

$$\|(x_i)\|_{p_2} \leq \|(x_i)\|_1 \leq \|(x_i)\|_\infty^{1-p_1}\|(x_i)\|_{p_1};$$

if $0 < p < +\infty$, then $1 < p+1 < +\infty$ and by definition of supremum for any $\varepsilon > 0$ (we do the choice of $\varepsilon = \sum_{i=1}^\infty |x_i|^{p+1}$ we have a $x_{i_*} \in (x_i)$ such that

$$\|(x_i)\|_\infty = \left(\sup_{i\in\mathbb{N}}\left\{|x_i|^{p+1}\right\}\right)^{\frac{1}{p+1}} \leq \left(|x_{i_*}|^{p+1} + \varepsilon\right)^{\frac{1}{p+1}} \leq \left(2\sum_{i=1}^\infty |x_i|^{p+1}\right)^{\frac{1}{p+1}} = 2^{\frac{1}{p+1}}\|(x_i)\|_{p+1};$$

thus, we just proved the

Theorem 3.3.13 *If $0 < p_1 < p_2 \leq +\infty$, then there exist some constant C, just depending on p_1 and p_2, such that*

$$\forall\ (x_i) \in l^{p_1}(\mathbb{R}) \qquad \|(x_i)\|_{p_2} \leq C\|(x_i)\|_{p_1}\,. \tag{3.15}$$

Definition 3.3.14 *Consider a normed space $(V, \|\cdot\|)$ and a sequence $\{v_n\}_{n\in\mathbb{N}} \subset V$, we say the sequence $\{v_n\}_{n\in\mathbb{N}}$ converges to $v \in V$ and we write $\lim_{n\to+\infty} v_n = v$, when*

$$\forall_{\epsilon>0}\ \exists_{N\in\mathbb{N}}: \quad n > N \Rightarrow \|v_n - v\| < \epsilon\,.$$

Or, equivalently, when $\lim_{n\to+\infty} \|v_n - v\| = 0$ in $\mathbb{R}$.

So by inequality (3.15), if $\{(x_i)_n\}_{n\in\mathbb{N}} \subset l^{p_1}(\mathbb{R})$ is a sequence which converges under norm $\|\cdot\|_{p_1}$ it converges too under norm $\|\cdot\|_{p_2}$ for any $p_2 > p_1$.

The convergence in the $l^p(\mathbb{R})$ subspaces of our chain (3.5) is then "compatible" (meaning that the injection of $(l^{p_1}(\mathbb{R}), \|\cdot\|_{p_1})$ into $(l^{p_2}(\mathbb{R}), \|\cdot\|_{p_2})$ is a continuous application) and we say that the normed space $(l^{p_1}(\mathbb{R}), \|\cdot\|_{p_1})$ is a normed subspace of the normed space $(l^{p_2}(\mathbb{R}), \|\cdot\|_{p_2})$.

Thus we got again, now as normed spaces, the increasing infinite chain

$$\mathbb{R} < \mathbb{R}^2 < \cdots < \mathbb{R}^n < \cdots < l^q(\mathbb{R}) < \cdots l^1(\mathbb{R}) < \cdots < l^p(\mathbb{R}) < \cdots < l^\infty(\mathbb{R}). \tag{3.16}$$

In particular, each space $\mathbb{R}^n$ is endowed with an infinite set of norms, those induced from each normed space $l^p(\mathbb{R})$ for $p \in [0, +\infty]$. Meanwhile, because of the finite dimension of $\mathbb{R}^n$ it is easy to prove that

Theorem 3.3.15 *In the space $\mathbb{R}^n$ all two norms are equivalent*[7].

Where the meaning of *equivalence of norms* is that they give the same notion of convergence, the definition, after our words above, being no surprise:

Definition 3.3.16 *Two norms, say $\|\cdot\|_1$ and $\|\cdot\|_2$, defined in a same linear space V are equivalent if there exist constants $C, c > 0$ such that*

$$\forall\ v \in V \quad c\|v\|_1 \le \|v\|_2 \le C\|v\|_1\,. \tag{3.17}$$

[7] More generally, in each real linear space of finite dimension any two norms are equivalent.

Proof of Theor. 3.3.16: Fix a basis in the vector space $\mathbb{R}^n$ and write down any vector in such basis as some linear combination, then apply the subadditivity property to it.

A final remark before to finish this section is a justification of the notation $l^\infty(\mathbb{R})$ for the space of the bounded sequences:

Proposition 3.3.17 *If for some* $p_1 > 0$ *we have* $(x_i) \in l^{p_1}(\mathbb{R})$*, then*

$$\lim_{p\to+\infty} \|(x_i)\|_p = \|(x_i)\|_\infty.$$

Proof: We saw that if $(x_i) \in l^{p_1}(\mathbb{R})$, then $(x_i) \in l^p(\mathbb{R})$ for any $p \geq p_1$ and for p big enough $\|(x_i)\|_\infty \leq 2^{\frac{1}{p+1}} \|(x_i)\|_{p+1}$. Now we remark that $\|(x_i)\|_{p+1} \leq \|(x_i)\|_\infty^{\frac{p+1-p_1}{p+1}} \|(x_i)\|_{p_1}^{\frac{p_1}{p+1}}$. So

$$\|(x_i)\|_\infty \leq \lim_{p\to+\infty} 2^{\frac{1}{p+1}} \|(x_i)\|_{p+1} \leq \lim_{p\to+\infty} 2^{\frac{1}{p+1}} \|(x_i)\|_\infty^{1-\frac{p_1}{p+1}} \|(x_i)\|_{p_1}^{\frac{p_1}{p+1}} = \|(x_i)\|_\infty .$$

3.4 Euclidean spaces

Euclidean spaces are real vector spaces with an inner product. In the "inverse sense" of Prop. 3.3.4 we have:

Proposition 3.4.1 *A Euclidean space with inner product* $\langle\cdot,\cdot\rangle$ *is always a normed space with the norm defined by* $\|v\| = \sqrt{\langle v, v\rangle}$.

Next we remember, first the notion of *inner product* and then the proof of the last proposition.

Definition 3.4.2 *An inner product on a real vector space* V *is a real application* $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ *satisfying the axioms*

definite positive $\quad \forall v \in V \quad v \neq 0 \implies \langle v, v\rangle > 0$;

symmetry: $\quad \forall u, v \in V \quad \langle u, v\rangle = \langle v, u\rangle$;

homogeneity: $\quad \forall k \in \mathbb{R}\ \forall u, v \in V \quad \langle ku, v\rangle = k\langle u, v\rangle$;

additivity: $\quad \forall u, v, w \in V \quad \langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$.

Proof (of Prop. 3.4.1): The positive homogeneity of the norm: $\|kv\| = \sqrt{\langle kv, kv\rangle} = \sqrt{k^2\langle v, v\rangle}$, using the definition of the norm and the homogeneity and symmetry of the inner product.

The separation: if $v \neq 0$ then by the definite positivity of the inner product $\|v\| = \sqrt{\langle v, v\rangle} > 0$.

The subadditivity is a consequence of the Schwarz inequality (3.18) (see the next lemma), the additivity and the symmetry of the inner product: $\|u+v\|^2 = \langle u+v, u+v\rangle = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 \leq (\|u\| + \|v\|)^2$.

Lemma 3.4.3 (Schwarz inequality) *Let V be an Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and define for $v \in V$ the application $\|v\| = \sqrt{\langle v, v\rangle}$, then*

$$\forall u, v \in V \quad |\langle u, v\rangle| \leq \|u\| \, \|v\| . \tag{3.18}$$

Proof: Consider the function in the variable $k \in \mathbb{R}$ defined by

$$\phi(k) = \langle ku + v, ku + v\rangle .$$

This is a nonnegative application on $\mathbb{R}$ because of the definite positivity and homogeneity properties of the inner product (by the homogeneity $\|0\| = 0$). Moreover, using the additivity and homogeneity properties of the inner product we see that

$$\phi(k) = k^2\langle u, u\rangle + 2k\langle u, v\rangle + \langle v, v\rangle = \|u\|^2 k^2 + 2\langle u, v\rangle k + \|v\|^2 .$$

So, ϕ is a second order polynomial in the variable k which never has negative values. It has at most one root. So, the coefficients of ϕ must verify the condition $(2\langle u, v\rangle)^2 - 4\|u\|^2\|v\|^2 \leq 0$.

The next result gives a characterization of which normed spaces are Euclidean spaces:

Theorem 3.4.4 *A normed space is an Euclidean space iff the norm satisfies the parallelogram law:*

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right) .$$

Moreover the inner product is given by:

$$\langle u, v\rangle = 4^{-1}\left(\|u+v\|^2 + \|u-v\|^2\right) .$$

Corollary 3.4.5 *The space $l^p(\mathbb{R})$ is an Euclidean space iff $p = 2$. If $(x_i), (y_i) \in l^2(\mathbb{R})$ the inner product is defined by $\langle (x_i), (y_i) \rangle = (\sum_{i=1}^{\infty} x_i\, y_i)^{\frac{1}{2}}$, in particular, the Cauchy-Schwarz inequality is the Hölder inequality for $p = 2$.*

Proof: Using the Def. 3.4.2 of inner product or the last Theor. 3.4.4 it is easy to verify that $l^2(\mathbb{R})$ is an Euclidean space.

If $p \neq 2$, then $l^p(\mathbb{R})$ cannot be an Euclidean space as the parallelogram law is false: let $u = (1, 0)$ and $v = (0, 1)$, then $\|u\|_p = \|v\|_p = 1$, $\|u + v\|_p = \|u - v\|_p = 2^{\frac{1}{p}}$. So, we have that $\|u + v\|^2 + \|u - v\|^2 \neq 2\left(\|u\|^2 + \|v\|^2\right)$.

We are using the fact that $\mathbb{R}^2$ is a subspace of any $l^p(\mathbb{R})$ space.

3.5 Banach Spaces

In metric spaces, like for real numbers, a very useful theoretical and computational criterium to show that a sequence converges is to show that the sequence is a Cauchy sequence. In general this is not an equivalence. While each convergent sequence is always a Cauchy sequence the reciprocal is false. We are then interested to work with metric spaces having that property (each Cauchy sequence is convergent), called *complete metric spaces*.

One example of the relevance of this concept comes from the *Banach fixed-point theorem* which is the usual tool we use to prove, e.g., the existence and uniqueness theorems (Picard's theorem) for ordinary differential equations, but which can be still proved in more general frameworks as that of $L^p(\mathbb{R})$ spaces.

In Prop. 3.3.4 we saw that each normed space is a metric space where the metric is defined by the norm as $d(u, v) = \|u - v\|$. Lets translate these concepts in the setting of the normed spaces and state two last results. Before we give the following definitions.

Definition 3.5.1 *Let $(V, \|\cdot\|)$ be a normed space.*

Cauchy sequence: *A sequence $\{v_n\}_{n\in\mathbb{N}} \subset V$ is a Cauchy sequence when*

$$\forall_{\epsilon>0}\, \exists_{N\in\mathbb{N}} : \; n > N, m > N \Rightarrow \|v_n - v_m\| < \epsilon\,.$$

Banach space: *$(V, \|\cdot\|)$ is a Banach space if it is complete (meaning that each Cauchy sequence is convergent).*

Contractive application: *An application $A : V \to V$ is said to be contractive when there exists a constant $0 < L < 1$:*

$$\forall_{u,v \in V} \quad \|A(u) - A(v)\| \leq L \|u - v\| .$$

Theorem 3.5.2 *Each $l^p(\mathbb{R})$ space ($p \in]0, +\infty]$) is a Banach space.*

Theorem 3.5.3 (Banach fixed-point theorem) *Let $A : V \to V$ be a contractive application defined in a Banach space V, then A has one fixed point ($u_0 \in V : A(u_0) = u_0$) which moreover is unique.*

Proof: Recalling the notation $A : V \to V$ is a contraction with contraction constant c. We want to show$A$ has a unique fixed point, which can be obtained as a limit through iteration of A from any initial value. To show A has at most one fixed point in V, let u_0 and u_0' be fixed points of A. Then

$$d(u_0, u_0') = d(A(u_0), A(u_0') \leq cd(u_0, u_0')$$

If $u_0 \neq u_0'$ then $d(u_0, u_0') > 0$ so we can divide by $d(u_0, u_0')$ to get $1 \leq c$ which is false. Thus $u_0 = u_0'$.

Next we want to show for any $u_0 \in V$, that the recursively defined iterates $u_n = A(u_{n-1})$ for $n \geq 1$ converge to the fixed point of A. How close is u_n to u_{n+1}? for any $n \geq 1$ $d(u_n, u_{n+1}) = d(A(u_{n-1}), A(u_n)) \leq cd(u_{n-1}, u_n)$. therefore

$$d(u_n, u_{n+1}) \leq cd(u_{n-1}, u_n) \leq c^2 d(u_{n-2}, u_{n-1}) \leq \cdots \leq c^n d(u_0, u_1)$$

Using the expression on the far right as an upper bound on $d(u_n, u_{n+1})$ show the u_n's are getting consecutively close at a geometric rate. This implies to the u_n's are Cauchy: for any $m > n$ using the triangle inequality several time show:

$$\begin{aligned} d(u_n, u_m) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \cdots + d(u_{m-1}, u_m) \\ &\leq c^n d(u_0, u_1) + c^{n+1} d(u_0, u_1) + \cdots + c^{m-1} d(u_0, u_1) \\ &= (c^n + c^{n+1} + c^{n+2} + \cdots + c^{m-1}) d(u_0, u_1) \\ &\leq (c^n + c^{n+1} + c^{n+2} + c^{n+2} + \cdots) d(u_0, u_1) \\ &\leq \frac{c^n}{1 - c} d(u_0, u_1) \end{aligned} \tag{3.19}$$

To prove from this bound that the u_n's are Cauchy choose $\epsilon > 0$ and then pick $N > 1$ such that $(c^N/(1-c))d(u_o, u_1) < \epsilon$ then for any $m > n \geq N$

$$d(u_n, u_m) \leq \frac{c^n}{1-c} d(u_0, u_1) \leq \frac{c^N}{1-c} d(u_0, u_1) < \epsilon$$

The prove u_n is a Cauchy sequence since V is complete the u_n's converge in V, set $u = \lim_{n\to\infty} u_n$ in V .

To show $A(u_0) = u_0$ we need to know that contraction are continuous. In fact, a contraction is uniformly continuous, this is clear when $c = 0$. Since then A is a constant function, if $c > 0$ and we are given $\epsilon > 0$, setting $\delta = \frac{\epsilon}{c}$ implies that if $d(u_0, u_0') < \delta$ then $d(A(u_0), A(u_0')) \leq cd(u_0, u_0') < c\delta = \epsilon$. That prove A is uniformly continuous. Since A is then continuous from $u_n \to u_0$ we get $A(u_n) \to A(u_0)$ since $A(u_n) = u_{n+1}$, $A(u_n) \to u_0$ as $n \to \infty$ then $A(u_0)$ and u_0 are both limit of $u_{nn\geq 0}$. From the uniqueness of limit $u_0 = A(u_0)$ this conclude the proof of the Banach fixed-point theorem.

Chapter 4

Numerical Schemes and Simulations

4.1 Characteristic Method

The characteristic method gives generally an implicit formula for the solution. It is interesting to use it when the initial condition is continuous and affine to get an explicit formula of the solution.

We take $v(u) = 1 - u$ to obtain the model

$$\begin{cases} u_t + (u(1-u))_x = 0 \\ u(x,0) = \varphi(x) \end{cases} \tag{4.1}$$

Let Γ be parametrized by r, such that $\Gamma = (r, 0)$. Now Γ will be Characteristic as long as $\gamma_1'(r) - \gamma_2'(r) \neq 0$ but $\gamma_1'(r) = 1$ and $\gamma_2'(r) = 0$, Therefore, Γ is noncharacteristic. Our characteristic equation are given by

$$\begin{cases} \dfrac{dt}{ds} = 1 \\ \dfrac{dx}{ds} = 1 - 2z \\ \dfrac{dz}{ds} = 0 \end{cases} \tag{4.2}$$

With initial condition

$$\begin{cases} t(r,0) = 0 \\ x(r,0) = r \\ z(r,0) = \varphi(r) \end{cases} \tag{4.3}$$

We solve this system as follows

$$t(r,s) = s + c_1(r)$$

$$z(r,s) = c_3(r)$$

$$x(r,s) = (1 - 2c_3(r))s + r$$

Now using the initial condition we have

$$t(r,s) = s$$

$$z(r,s) = \varphi(r)$$

$$x(r,s) = (1 - 2\varphi(r))s + r$$

Now $x(r,s) = (1 - 2\varphi(r))s + r$ and $t = s$ implies $r = 2\varphi(r)s - s + x = x + 2zt - t$ therefore letting $u(x,t) = z(r,s)$ we have $u(x,t) = \varphi(x + 2ut - t)$ and implicit formula for a solution to (4.1).

Example 4.1.1 *We consider the Cauchy problem with the continuous initial data*

$$\varphi(x) = \begin{cases} \frac{3}{4}, & x \le -a, \\ \frac{1}{2} - \frac{x}{4a}, & -a < x < a, \\ \frac{1}{4}, & x > a. \end{cases}$$

Sine u_0 is continuous we use the characteristic method:

$$u(x,t) = \varphi(x + 2ut - t) = \frac{1}{2} - \frac{x + 2ut - t}{4a} \tag{4.4}$$

provide

$$-a < x + 2ut - t < a \tag{4.5}$$

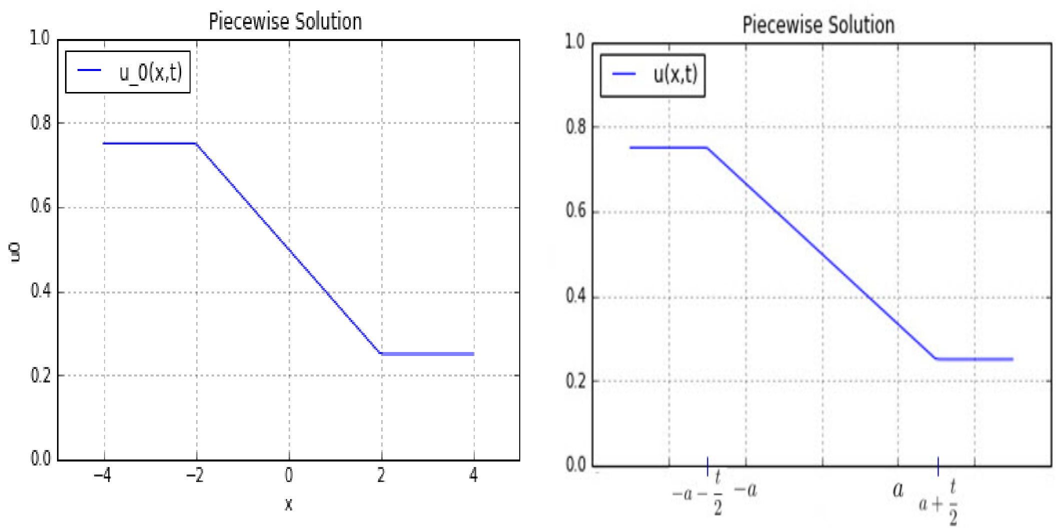
from (4.4) we have

$$u(1 + \frac{t}{2a}) = \frac{1}{2} + \frac{t}{4a} - \frac{x}{4a}$$

thus $u = \dfrac{1}{2} - \dfrac{x}{4a + 2t}$, *now the condition (4.5) writes* $-a < x - \dfrac{2xt}{4t + 2a} < a$ *that give* $-a - \dfrac{t}{2} < x < a + \dfrac{t}{2}$

Finally, the solution $u(x,t)$ is continuous and given by

$$u(x,t) = \begin{cases} \frac{3}{4}, & x \leq -a - \frac{t}{2}, \\ \frac{1}{2} - \frac{x}{4a+2t}, & -a - \frac{t}{2} < x < a + \frac{t}{2}, \\ \frac{1}{4}, & x \geq a + \frac{t}{2}. \end{cases}$$



(a) Piecewise plots u_0

(b) Piecewise plots $u(x,t)$

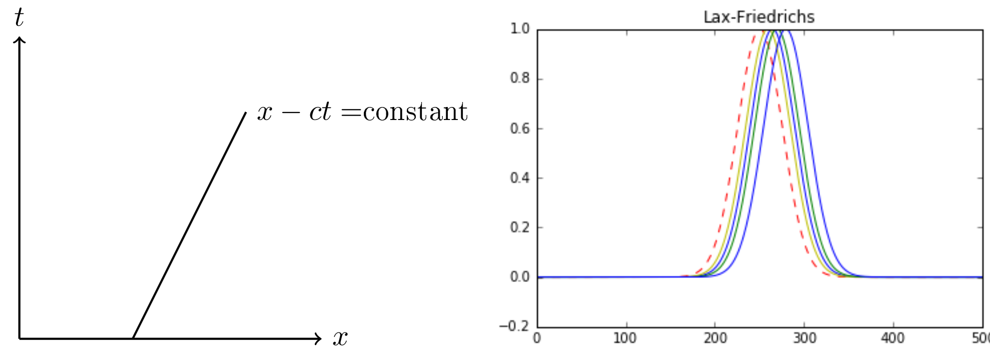
Figure 4.1: Piecewise plots Solution

Facing the traffic light the density is high, while on the other side of the light there is a small constant density.

Remark 4.1.2 *We note that if we consider the advection equation $u_t + cu_x = 0$ the characteristics are parallel and given by $x - ct = constant$. The solution is given simply by*

$$u(x,t) = \varphi(x - ct)$$

Graphically as t increases the initial function $u(x,0)$ will move with speed c to the right if $c > 0$ and to the left if $c < 0$ (see Fig4.2 for the case $c > 0$).



(a) Characteristic Line (b) Evolution

Figure 4.2: Characteristic and solution for the advection equation $u_t + cu_x = 0, c > 0$

4.2 Lax-Friedrichs scheme

4.2.1 Case of the advection equation

Let us consider the advection equation

$$u_t + cu_x = 0.$$

We introduce a time step Δt and a space step Δx and we set $t_n = n\Delta t$ and $x_i = i\Delta x$. when Δt and Δx are small we can write

$$\begin{aligned} u_t(x_i, t_n) &= \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t) \\ u_x(x_i, t_n) &= \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + O(\Delta x^2). \end{aligned}$$

An exact solution satisfies

$$u_t(x_i, t_n) + cu_x(x_i, t_n) = 0$$

so that we can write

$$u_i^{n+1} = u_i^n - c\frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + O(\Delta t^2, \Delta x^2 \Delta t)$$

Thus, allows us to introduce the numerical method FTCS (Forward-Time-Centered-space) that writes

$$u_i^{n+1} = u_i^n - c\frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

Unfortunately, this scheme is *unconditionally unstable* i.e. the numerical solution can be destroyed by numerical errors, which will be produced and grow exponential independently of the relative size of Δt and Δx.

The basis idea to get a stable scheme is based on replacing in the previous FTCS formulas the term u_i^n with the spatial average

$$u_i^n = \frac{u_{i+1}^n + u_{i-1}^n}{2}$$

to obtain for the advection equation

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n).$$

This scheme is called the Lax-Friedrichs scheme. It is a first order scheme which is stable provide the so-called CFL condition $\mid c\frac{\Delta t}{\Delta x} \mid < 1$ is satisfied.

4.2.2 Case of the general hyperbolic equation

We consider here a general hyperbolic conservation law

$$u_t + f(u)_x = 0.$$

We introduce a conservative scheme, i.e., a scheme of the form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}[F\left(u_{i-p}^n, u_{i-p+1}^n, \cdots, u_{i+q}^n\right) - F\left(u_{i-p-1}^n, u_{i-p}^n, \cdots, u_{i+q-1}^n\right)] \tag{4.6}$$

for some function F of $p+q+1$ arguments. F is called the numerical flux function. In the simplest case, $p = 0$ and $q = 1$, F is a function of only two variables and the scheme (4.6) becomes

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}[F\left(u_i^n, u_{i+1}^n\right) - F\left(u_{i-1}^n, u_i^n\right)] \tag{4.7}$$

This form is very natural if we view u_i^n as an approximation to the cell average

$$\overline{u}_i^n = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, t_n)\Delta x, \tag{4.8}$$

where $u(x, t_n)$ represents the exact weak solution at time $t_n = n\Delta t$. Also, this exact solution satisfies the following integral form of the conservation law

$$\begin{aligned}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t_{n+1})dx = &\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t_n)dx \\ &- \left[\int_{t_n}^{t_{n+1}} f(u(x_{i+\frac{1}{2}},t))dt - \int_{t_n}^{t_{n+1}} f(u(x_{i-\frac{1}{2}},t))dt\right]\end{aligned} \tag{4.9}$$

Dividing the last equality by Δx and using the cell averages defined in (4.8) this gives

$$\overline{u}_i^{n+1} = \overline{u}_i^n - \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} f(u(x_{i+\frac{1}{2}},t))dt - \int_{t_n}^{t_{n+1}} f(u(x_{i-\frac{1}{2}},t))dt\right] \tag{4.10}$$

Comparing this to (4.7) we see that the numerical flux function $F(u_i, u_{i+1})$ plays the role of an average flux through $x_{i+\frac{1}{2}}$ over the time interval $[t_n, t_{n+1}]$

$$F(u_i, u_{i+1}) \sim \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{i+\frac{1}{2}},t))dt \tag{4.11}$$

One way to derive numerical methods in conservation form is to use standard finite difference discretizations but to start with the conservative form of the PDE rather than the quasilinear form.

The generalization of the Lax-Friedrichs method given for the advection equation ($f(u) = cu$) to the general non-linear hyperbolic equation consists in taking the numerical flux F in (4.7) of the form:

$$F(u_i^n, u_{i+1}^n) = \frac{1}{2}(f(u_i^n) + f(u_{i+1}^n)) - \frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_i^n)$$

to obtain

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right)$$

This method is conservative and first order accurate, hence quite dissipative. It can, however be used as a building block for building high-order numerical schemes for solving hyperbolic partial differential equations.

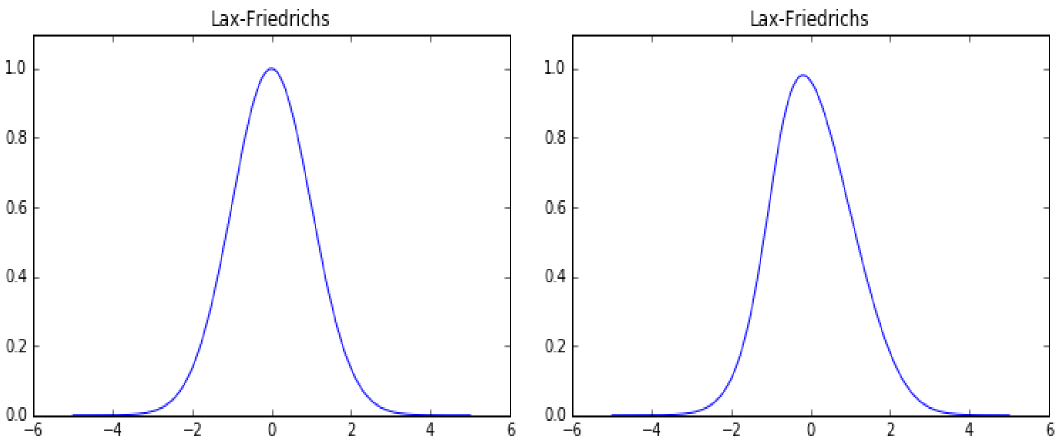
4.2.3 Application to the traffic flow model

In the case of the traffic flow model we have $f(u) = u(1-u)$.

Example 4.2.1 *First, we consider the case where $f(u) = u(1-u)$ with a Gaussian initial data:*

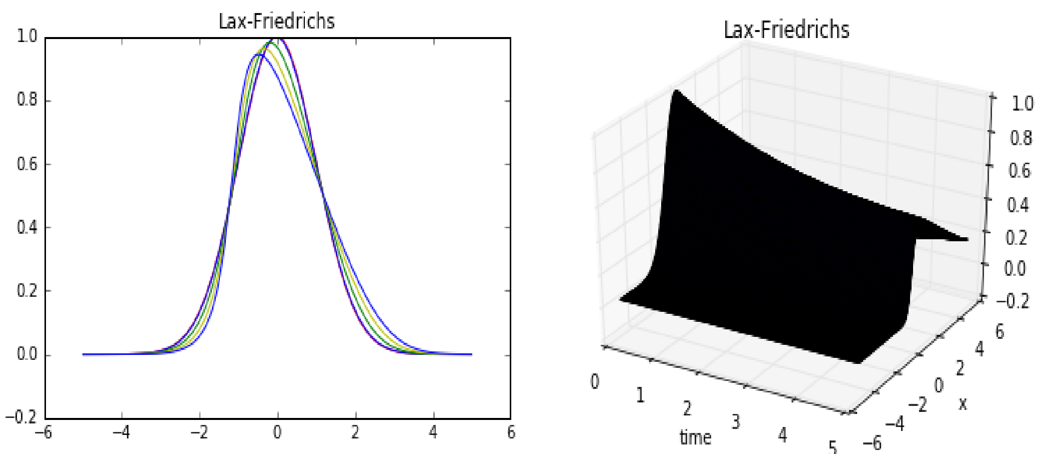
$$u_0(x) = \exp(-\frac{x^2}{2})$$

Fig4.3 shows the numerical solution with the Lax-Friedrichs scheme with $\Delta t = 0.004$ *and* $\Delta x = 0.04$ *obtained at time* $t = 1$ *when Fig4.4 shows the evolution of the solution with respect to time.*



(a) (b)

Figure 4.3: Gaussian plots of Lax-Friedrichs scheme $u_0(x) = \exp(-\frac{x^2}{2})$



(a) Some solution of Lax-Friedrichs scheme (b) Gaussian3D plots of Lax-Friedrichs scheme

Figure 4.4: Gaussian plots of Lax-Friedrichs scheme $u_0(x) = \exp(-\frac{x^2}{2})$

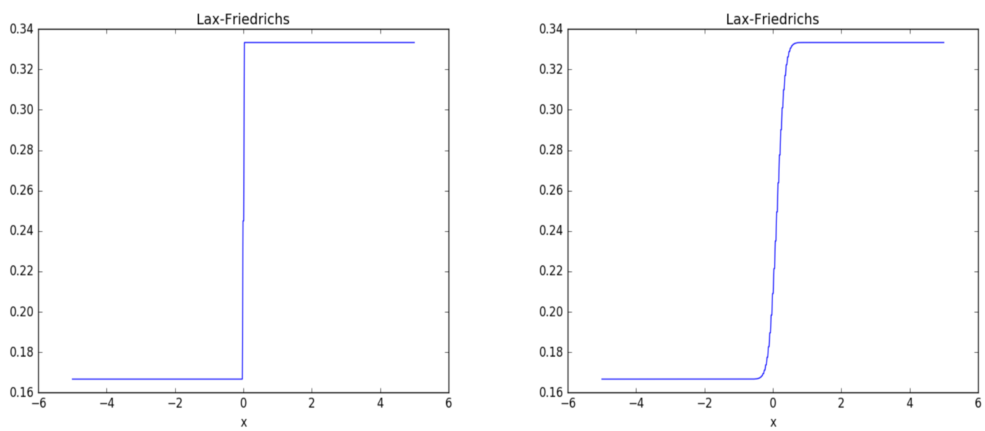
Example 4.2.2 *We consider the Riemann problem in the case* $f(u) = u(1-u)$ *with the initial data*

$$u_0(x) = \begin{cases} \frac{1}{6}, & x < 0, \\ \frac{1}{3}, & x > 0. \end{cases}$$

f is concave $u_l = 1/6 < u_r = 1/3$, The exact entropy solution is the shock wave given by

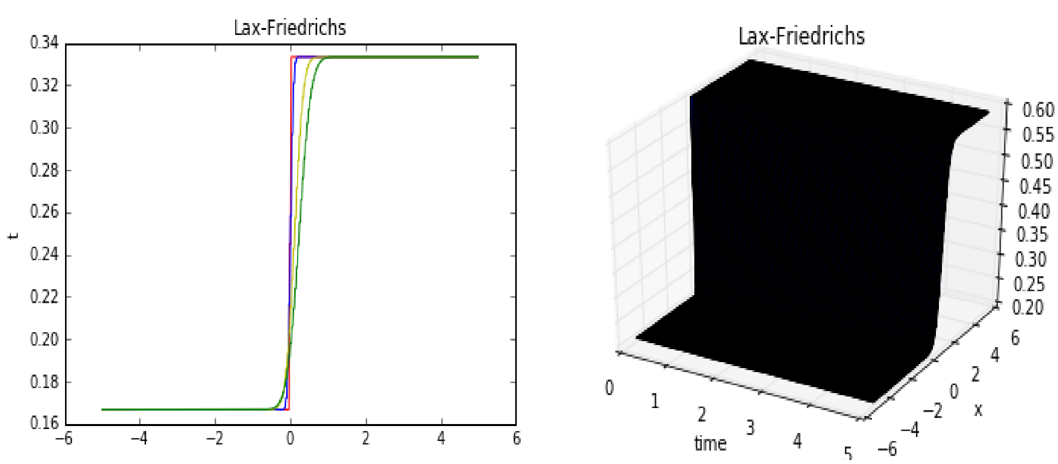
$$u(x,t) = \begin{cases} \frac{1}{6}, & x < \frac{1}{2}t, \\ \frac{1}{3}, & x > \frac{1}{2}t. \end{cases}$$

Fig4.5 shows the numerical solution with the Lax-Friedrichs scheme with $\Delta t = 0.002$ and $\Delta x = 0.02$ obtained at time $t = 4.81$ when Fig4.6 shows the evolution of the solution with respect to time.



(a) (b)

Figure 4.5: Shock plots of Lax-Friedrichs scheme $u_l = \frac{1}{6}$ and $u_r = \frac{1}{3}$



(a) Shock plot of Lax-Friedrichs scheme (b) Shock plots 3D of Lax-Friedrichs scheme

Figure 4.6: Shock plots of Lax-Friedrichs scheme $u_l = \frac{1}{6}$ and $u_r = \frac{1}{3}$

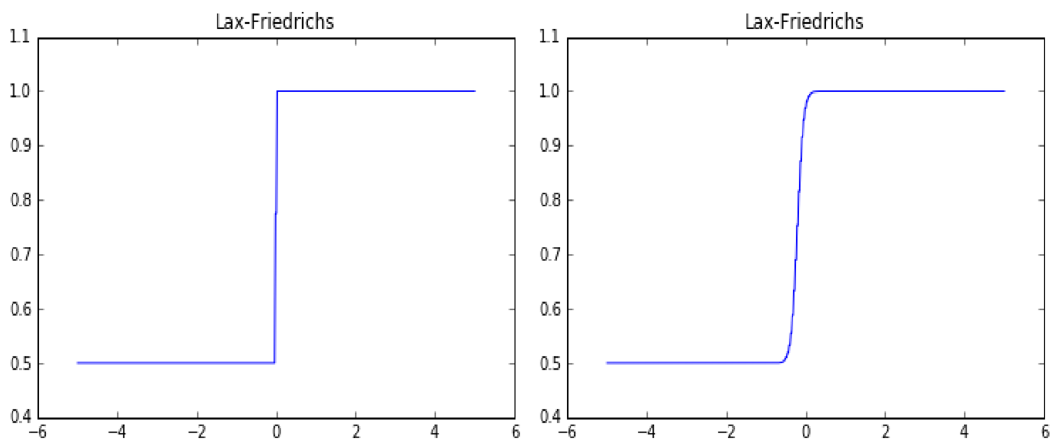
Example 4.2.3 *We consider the Riemann problem in the case $f(u) = u(1-u)$ with the initial data*

$$u_0(x) = \begin{cases} \frac{1}{2}, & x < 0, \\ 1, & x > 0. \end{cases}$$

f is concave $u_l = 1/2 < u_r = 1$, The exact entropy solution is the shock wave given by

$$u(x,t) = \begin{cases} \frac{1}{2}, & x < -\frac{1}{2}t, \\ 1, & x > -\frac{1}{2}t. \end{cases}$$

Fig4.7 shows the numerical solution with the Lax-Friedrichs scheme with $\Delta t = 0.00408$ and $\Delta x = 0.02004$ obtained at time $t = 5$ when Fig4.8 shows the evolution of the solution with respect to time.



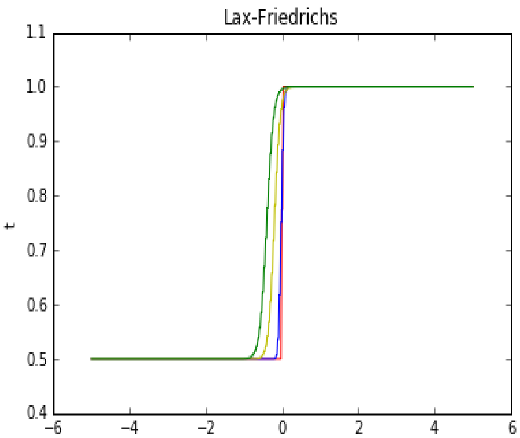
(a) (b)

Figure 4.7: Shock plots with Lax-Friedrichs scheme $u_l = \frac{1}{2}$ and $u_r = 1$

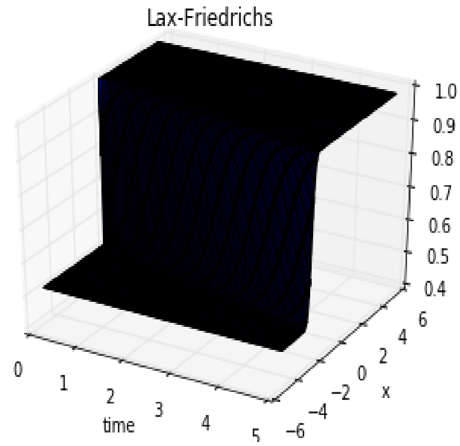
Example 4.2.4 *We consider the Riemann problem in the case $f(u) = u(1-u)$ with the initial data*

$$u_0(x) = \begin{cases} 1, & x < 0, \\ \frac{1}{2}, & x > 0. \end{cases}$$

f is concave $u_l = 1 > u_r = 1/2$, the exact entropy solution now is the rarefaction wave given



(a) Shock plots with Lax-Friedrichs scheme



(b) Shock plots 3D with Lax-Friedrichs scheme

Figure 4.8: Shock plots with Lax-Friedrichs scheme $u_l = \frac{1}{2}$ and $u_r = 1$

by

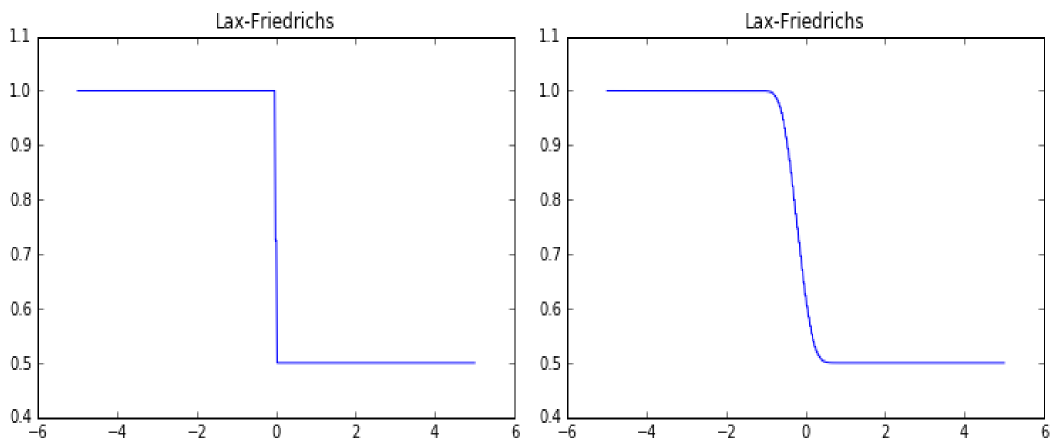
$$u(x,t) = \begin{cases} u_l, & x < f'(1)t, \\ U(\frac{x}{t}), & f'(1)t < x < f'(1/2)t, \\ u_r, & x > f'(1/2)t. \end{cases}$$

$$u(x,t) = \begin{cases} 1, & x \leq -t, \\ \dfrac{t-x}{2t}, & -t \leq x \leq 0, \\ \frac{1}{2}, & x \geq 0. \end{cases}$$

Fig4.9and Fig 4.10 shows the numerical solution with the Lax-Friedrichs scheme with $\Delta t = 0.004008$ *and* $\Delta x = 0.02004$ *obtained at different times.*

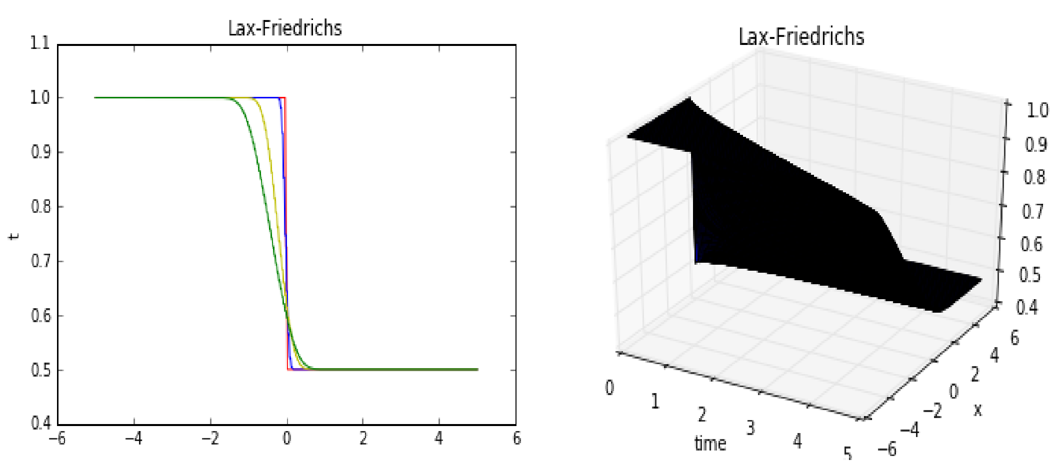
Example 4.2.5 *We consider the Cauchy problem with the continuous initial data*

$$u_0(x) = \begin{cases} \dfrac{3}{4}, & x \leq -a, \\ \frac{1}{2} - \dfrac{x}{4a}, & -a < x < a, \\ \dfrac{1}{4}, & x > a. \end{cases}$$



(a) (b)

Figure 4.9: Rarefaction plots with Lax-Friedrichs scheme $u_l = 1$ and $u_r = \frac{1}{2}$

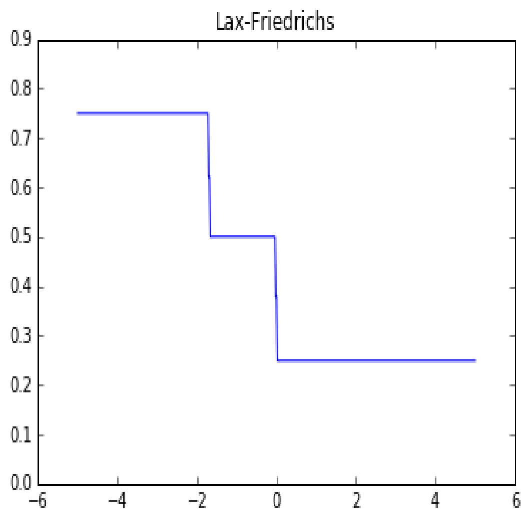


(a) Shock plots with Lax-Friedrichs scheme (b) Shock left plots 3D with Lax-Friedrichs scheme

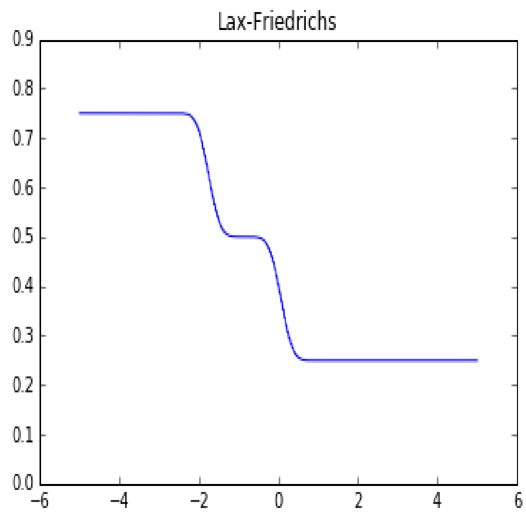
Figure 4.10: Rarefaction plots with Lax-Friedrichs scheme $u_l = 1$ and $u_r = \frac{1}{2}$

The exact solution $u(x,t)$ we find that

$$u(x,t) = \begin{cases} \dfrac{3}{4}, & x \leq -a - \dfrac{t}{2}, \\ \frac{1}{2} - \dfrac{x}{4a+2t}, & -a - \dfrac{t}{2} < x < a + \dfrac{t}{2}, \\ \dfrac{1}{4}, & x \geq a + \dfrac{t}{2}. \end{cases}$$

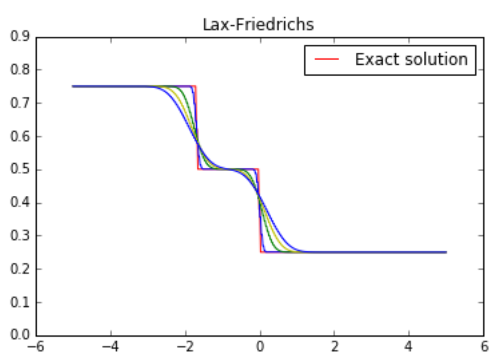


(a) Shock plots with Lax-Friedrichs scheme

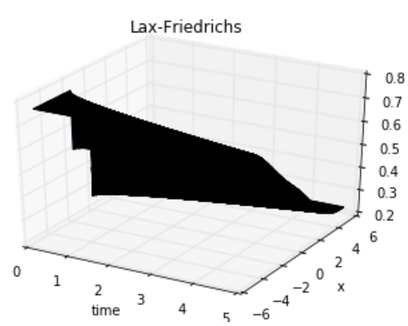


(b) Shock left plots 3D with Lax-Friedrichs scheme

Figure 4.11: Piecewise plots with Lax-Friedrichs scheme



(a) Shock plots with Lax-Friedrichs scheme



(b) Shock left plots 3D with Lax-Friedrichs scheme

Figure 4.12: Piecewise plots with Lax-Friedrichs scheme

Facing the traffic light the density is high, while on the other side of the light there is a small constant density.

Following the numerical results we can see that the Lax-Friedrichs scheme gives regular numerical solutions even when the exact solution is discontinuous (shock waves). We say that the scheme is *diffusive* that means that the scheme is solving approximatively an evolution equation of the form

$$u_t + f(u)_x = \epsilon u_{xx},$$

where ϵ is a small parameter depending on Δx and Δt.

4.3 Lax-Wendroff scheme

4.3.1 Case of the advection equation

The Lax-Wendroff scheme is a correction of the Lax-Friedrichs scheme to get a second order accuracy. For the advection equation $u_t + cu_x = 0$ it writes

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \tag{4.12}$$

4.3.2 Case of the general hyperbolic equation

The generalization of the Lax-Friedrichs method given for the advection equation ($f(u) = cu$) to the general non-linear hyperbolic equation consists in taking the numerical flux F in (4.7) of the form:

$$F(u_i^n, u_{i+1}^n) = \frac{1}{2}(f(u_i^n) + f(u_{i+1}^n) + \frac{\Delta t}{\Delta x} f'(u_{i+\frac{1}{2}})(f(u_{i+1}^n) - f(u_i^n))$$

to obtain

$$\begin{aligned} u_i^{n+1} &= u_i^n - \frac{\Delta t}{2\Delta x}\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right) \\ &+ \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[f'(u_{i+\frac{1}{2}})(f(u_{i+1}^n) - f(u_i^n)) - f'(u_{i-\frac{1}{2}})(f(u_i^n) - f(u_{i-1}^n))\right] \end{aligned} \tag{4.13}$$

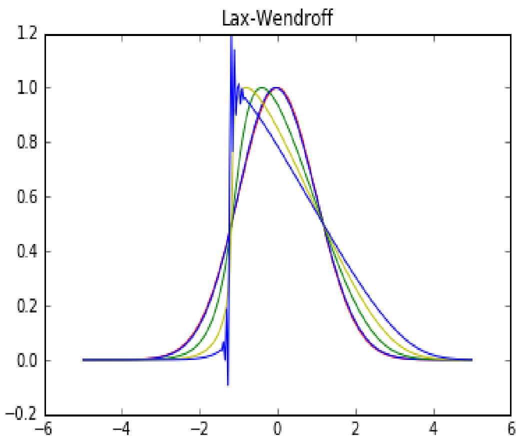
4.3.3 Application to the traffic flow model

In the case of the traffic flow model we have $f(u) = u(1-u)$ and $f'(u) = 1 - 2u$

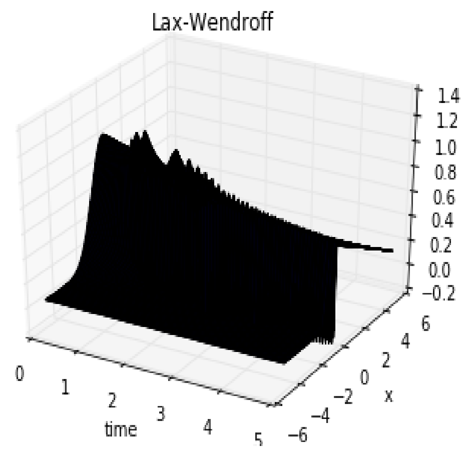
Example 4.3.1 *First, as for the Lax-Friedrichs scheme, we consider a Gaussian initial data:*

$$u_0(x) = \exp(-\frac{x^2}{2})$$

Fig4.13 shows the numerical solution with the Lax-Wendroff scheme with $\Delta t = 0.004$ *and* $\Delta x = 0.04$ *obtained at different times.*



(a)



(b)

Figure 4.13: Gaussian plot for the Lax-Wendroff scheme $u_0(x) = \exp(-\frac{x^2}{2})$

Example 4.3.2 *We consider the Riemann problem with the initial data*

$$u_0(x) = \begin{cases} \frac{1}{6}, & x < 0, \\ \frac{1}{3}, & x > 0. \end{cases}$$

f *is concave* $u_l = 1/6 < u_r = 1/3$, *The exact entropy solution is the shock wave given by*

$$u(x,t) = \begin{cases} \frac{1}{6}, & x < \frac{1}{2}t, \\ \frac{1}{3}, & x > \frac{1}{2}t. \end{cases}$$

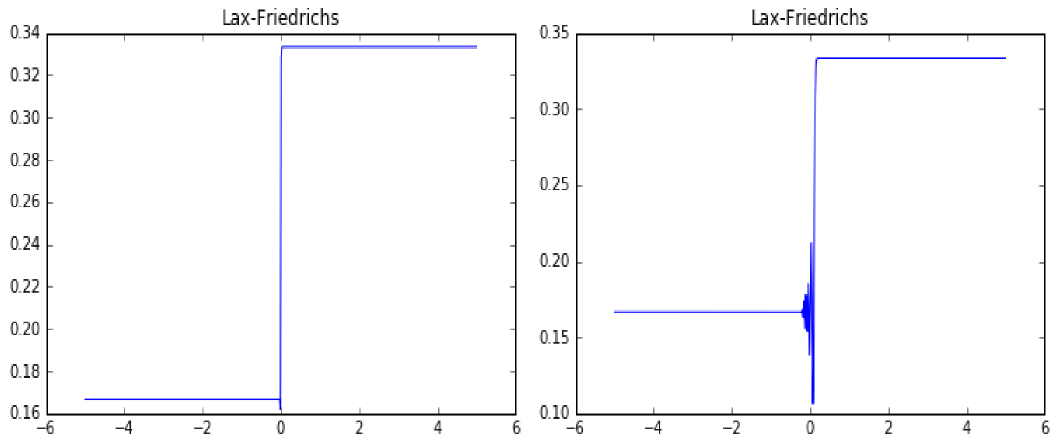
Fig4.14 shows the numerical solution with the Lax-Wendroff scheme with $\Delta t = 0.0024$ *and* $\Delta x = 0.02$ *obtained at time* $t = 1$ *when Fig4.15 shows the evolution of the solution with respect to time.*

Example 4.3.3 *We consider the Riemann problem with the initial data*

$$u_0(x) = \begin{cases} \frac{1}{2}, & x < 0, \\ 1, & x > 0. \end{cases}$$

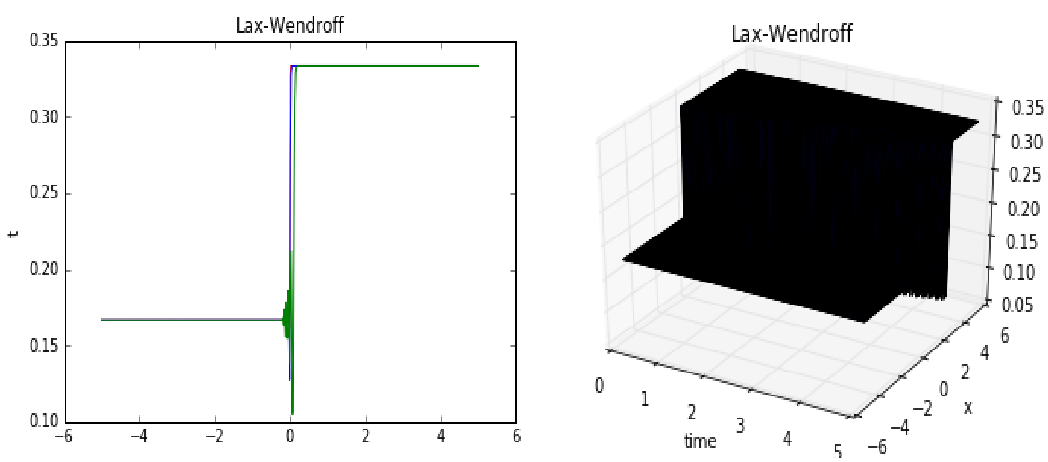
f *is concave* $u_l = 1/2 < u_r = 1$, *The exact entropy solution is the shock wave given by*

$$u(x,t) = \begin{cases} \frac{1}{2}, & x < -\frac{1}{2}t, \\ 1, & x > -\frac{1}{2}t. \end{cases}$$



(a) (b)

Figure 4.14: Sock plot for the Lax-Wendroff scheme $u_l = \frac{1}{6}$ and $u_r = \frac{1}{3}$



(a) (b)

Figure 4.15: Sock plot for the Lax-Wendroff scheme $u_l = \frac{1}{6}$ and $u_r = \frac{1}{3}$

Fig4.16 shows the numerical solution with the Lax-Wendroff scheme with $\Delta t = 0.002004$ and $\Delta x = 0.02004$ obtained at time $t = 1$ when Fig4.17 shows the evolution of the solution with respect to time.

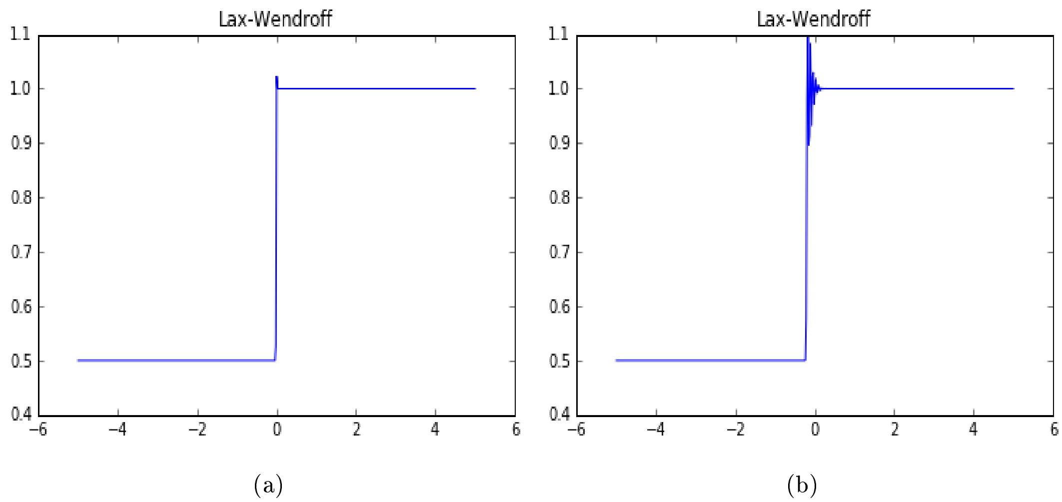


Figure 4.16: Sock plot for the Lax-Wendroff scheme $u_l = \frac{1}{2}$ and $u_r = 1$

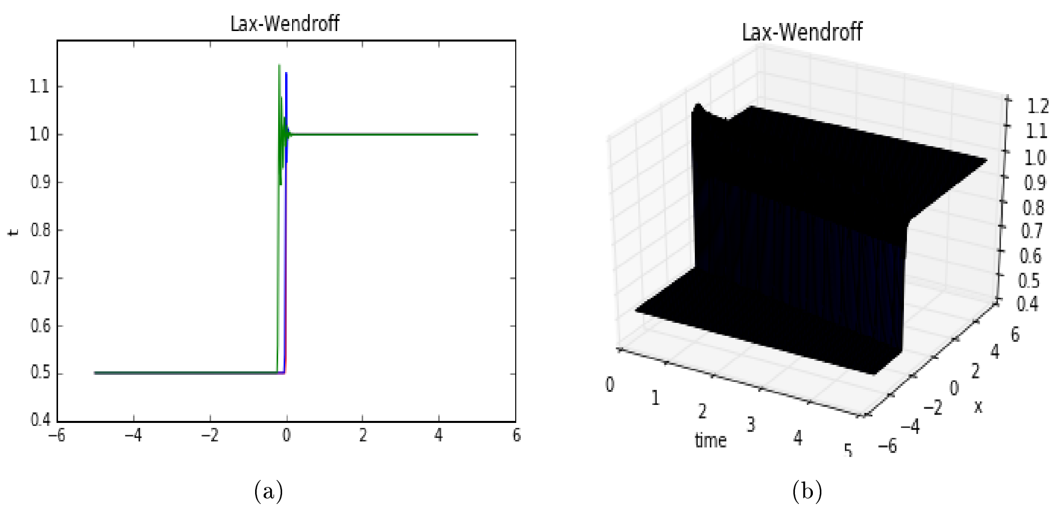


Figure 4.17: Sock plot for the Lax-Wendroff scheme $u_l = \frac{1}{2}$ and $u_r = 1$

Example 4.3.4 *We consider the Riemann problem with the initial data*

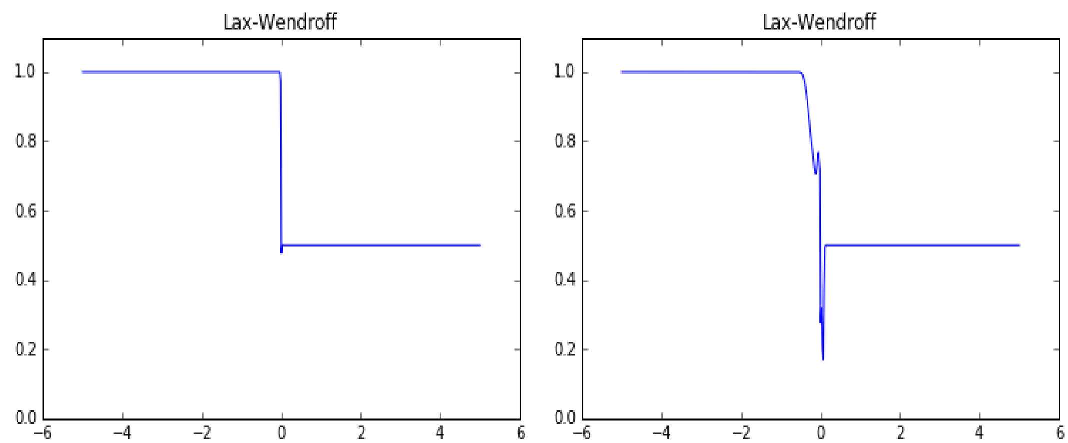
$$u_0(x) = \begin{cases} 1, & x < 0, \\ \frac{1}{2}, & x > 0. \end{cases}$$

f is concave $u_l = 1 > u_r = 1/2$, the exact entropy solution now is the rarefaction wave given by

$$u(x,t) = \begin{cases} u_l, & x < f'(1)t, \\ U(\frac{x}{t}), & f'(1)t < x < f'(1/2)t, \\ u_r, & x > f'(1/2)t. \end{cases}$$

$$u(x,t) = \begin{cases} 1, & x \leq -t, \\ \dfrac{t-x}{2t}, & -t \leq x \leq 0, \\ \frac{1}{2}, & x \geq 0. \end{cases}$$

Fig4.18 shows the numerical solution with the Lax-Wendroff scheme with $\Delta t = 0.002004$ and $\Delta x = 0.02004$ obtained at time $t = 1$ when Fig4.19 shows the evolution of the solution with respect to time.

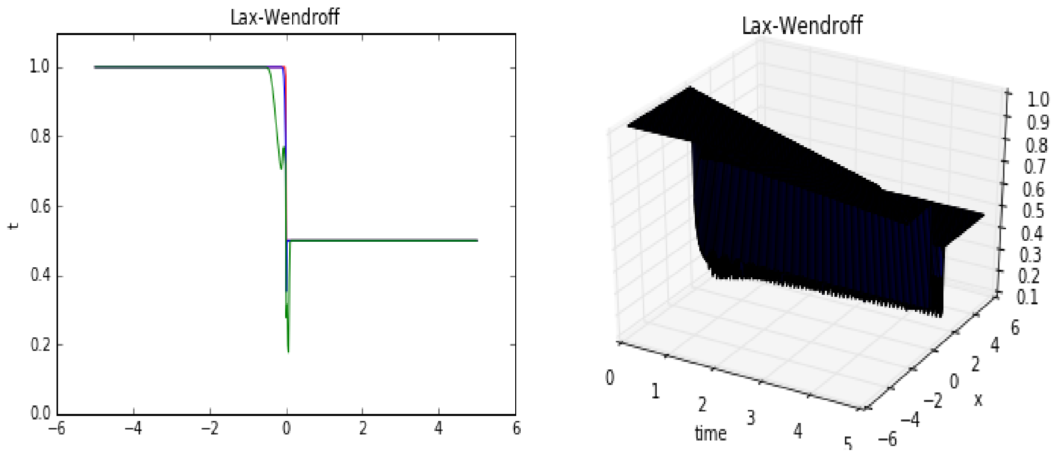


(a) (b)

Figure 4.18: Rarefaction plot for the Lax-Wendroff scheme $u_l = 1$ and $u_r = \frac{1}{2}$

Example 4.3.5 *We consider the Riemann problem with the initial data*

$$u_0(x) = \begin{cases} \dfrac{3}{4}, & x \leq -a, \\ \dfrac{1}{2} - \dfrac{x}{4a}, & -a < x < a, \\ \dfrac{1}{4}, & x > a. \end{cases}$$



(a) (b)

Figure 4.19: Rarefaction plot for the Lax-Wendroff scheme $u_l = 1$ and $u_r = \frac{1}{2}$

The solution $u(x,t)$ we find that

$$u(x,t) = \begin{cases} \dfrac{3}{4}, & x \leq -a - \dfrac{t}{2}, \\ \dfrac{1}{2} - \dfrac{x}{4a}, & -a - \dfrac{t}{2} < x < a + \dfrac{t}{2}, \\ \dfrac{1}{4}, & x \geq a + \dfrac{t}{2}. \end{cases}$$

This solution models a situation where the traffic density initially is small positive x and high for negative x, if we let a tend to zero the solution reads

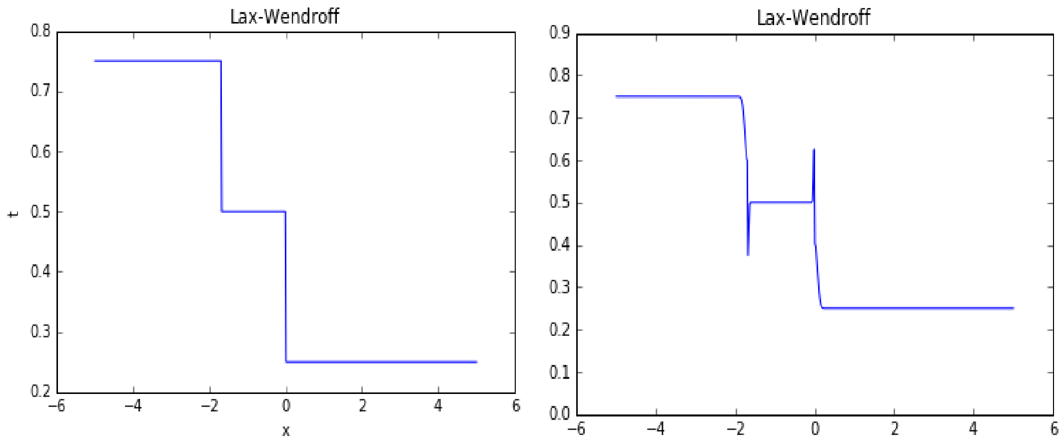
$$u(x,t) = \begin{cases} \dfrac{3}{4}, & x \leq -\dfrac{t}{2}, \\ \dfrac{1}{2} - \dfrac{x}{4a}, & -\dfrac{t}{2} < x < \dfrac{t}{2}, \\ \dfrac{1}{4}, & x \geq \dfrac{t}{2}. \end{cases}$$

As the reader may check directly, this is also a classical solution everywhere except at $x = \dfrac{t}{2}$ it takes discontinuous initial value

$$u_0(x) = \begin{cases} \dfrac{3}{4}, & x < 0 \\ \dfrac{1}{4}, & \textit{otherwise.} \end{cases}$$

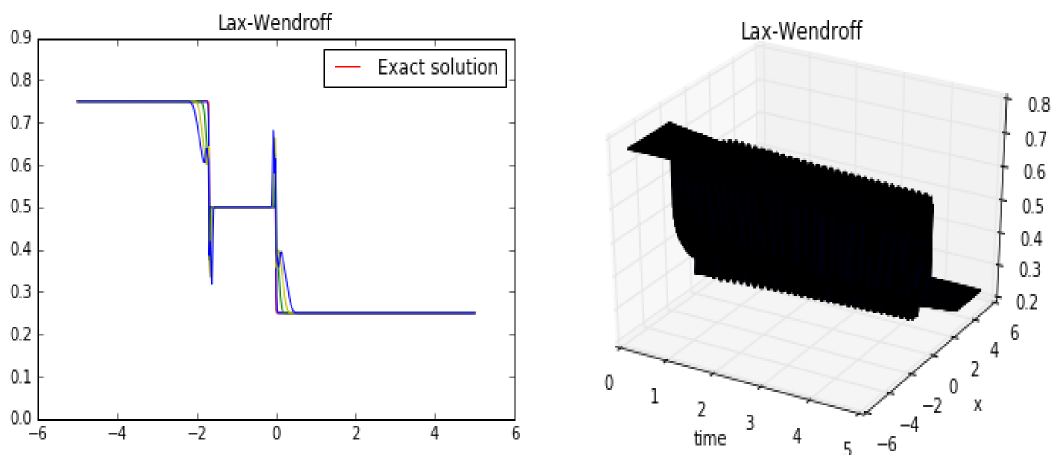
This initial function may model the situation when at traffic light turn green at $t = 0$ see the Fig 4.20 and Fig 4.21 .

Facing the traffic light the density is high, while on the other side of the light there is a small constant density.



(a) Shock plots with Lax-Wendroff scheme (b) Shock left plots 3D with Lax-Wendroff scheme

Figure 4.20: Piecewise plots with Lax-Wendroff scheme



(a) Shock plots with Lax-Wendroff scheme (b) Shock left plots 3D with Lax-Wendroff scheme

Figure 4.21: Piecewise plots with Lax-Wendroff scheme

Following the numerical results we can see that the Lax-Wendroff scheme is more precise that the Lax-Friedrichs scheme, and give the right position of the discontinuities for the shock waves. But it gives oscillations. We say that the scheme is *dispersive* that means that the scheme is solving approximatively an evolution equation of the form

$$u_t + f(u)_x = \delta u_{xxx},$$

where δ is a small parameter depending on Δx and Δt.

4.4 Conclusion

An elaboration and an implementation of Lax-Friedrichs schemes and of Lax-Wendroff schemes even extended to second order provided numerical solutions to the problem of traffic flows on the road. Since along the roads the schemes present the same features as for conservation laws, the new and original aspect is given by the treatment of the solution at junctions. Our tests show the effectiveness of the approximations, revealing that Lax-Wendroff schemes is more accurate than Lax-Friedrichs schemes.

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CURRICULUM VITAE

- Name and Surname: Miss. Bouasy DOUNGSAVANH
- Date of Birth: March 01, 1990
- Staff: October 01, 2014
- Youth Union: April 15, 2003
- Laos Trade Union: August 20, 2010
- Women Union: April 26, 2003
- Status: Single
- Place of Birth: Nakho Village, Phaxai District, Xiengkhouang Province
- EXPERIENCE since 1998 - Recent

  1998-2000 study at Horkhang Primary school, Horkhang Village, Phaxai District, Xiengkhouang Province

  2001-2003 study at Nama Secondary school, Yon Village,Phek District, Xiengkhouang Province

  2001-2003 study at Nama Secondary school, Yon Village,Phek District, Xiengkhouang Province

  2004-2006 study at Phonesavanh hight school,Phonmeexai Village,Phek District, Xiengkhouang Province

  2007-2011 study at Department of Mathematics, Faculty of Natural Science National University of Laos

  2012-2014 Teacher Volunteer at Department of Mathematics, Faculty of Natural Science National University of Laos

  2015-2017 Master student at Department of Mathematics, Faculty of Natural Science National University of Laos
- Tel: +856 20 99396894
- Email: bouasyd@hotmail.com