Siberian Advances in Mathematics
2006, v. 16, N3, 1–13

MINIMIZERS OF A FUNCTIONAL OF THE GRADIENT WHICH ARE STABLE WITH RESPECT TO AFFINE BOUNDARY DATA

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Abstract

We study the family of minimizers of an integral functional of the gradient over all Sobolev functions $u(\cdot) \in \langle v, \cdot \rangle + \mathbf{W}_0^{1,p}(\Omega)$ and give some results (including a category theorem) on continuous dependence of such minimizers on the vector $v \in \mathbb{R}^n$ with respect to the uniform topology.

Key words and phrases: scalar variational problem, nonconvex Lagrangian, Baire category theorem, continuous selection, Lipschitz selection, density.

1. Introduction

Assume given an open bounded set $\Omega \subset \mathbb{R}^n$ and a lower semicontinuous function $g\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. In the nineties many researchers have paid attention to the following scalar minimization problem

$$\min\left\{ \int_\Omega g\big(\nabla u(x)\big)\,dx : u(\cdot) \in \langle v, \cdot \rangle + \mathbf{W}_0^{1,p}(\Omega) \right\} \qquad (\mathcal{P}_v)$$

where $1 \le p \le +\infty$, $v \in \mathbb{R}^n$, and $\langle \cdot\,, \cdot \rangle$ means the inner product in the space $\mathbb{R}^n$. In applications to elasticity theory e.g., the integral functional in $(\mathcal{P}_v)$ can be the free energy of a homogeneous body undergoing antiplane shear deformations. It was shown in [5, 6, 12, 17] that the problem $(\mathcal{P}_v)$ may have or fail to have solutions, depending on the position of the vector v in the geometric

Received July, 2005.
Dedicated to Yu. G.Reshetnyak on the occasion of his 75th birthday.
Financially supported by FCT (Fundação para a Ciência e Tecnologia, Portugal), research projects N 787/2003-2005 and POCTI/MAT/55524/2004.
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structure of the Lagrangian. In the general case the necessary and sufficient condition of existence of a minimizer can be written as (see [17]):

$$\begin{aligned}&\text{either}\ \ \partial g(v) \neq \varnothing,\\ &\text{or there exist vectors } v_1, \dots, v_k \text{ such that}\\ &v \in \operatorname{int}\ \operatorname{co}\{v_1, \dots, v_k\} \ \text{ and }\ \bigcap_{i=1}^{k} \partial g(v_i) \neq \varnothing\,.\end{aligned} \tag{$\mathbb{C}$}$$

Here $\partial g(\cdot)$ is the subdifferential of $g(\cdot)$ in the sense of convex analysis, "int" and "co" stand for the interior and the convex hull, respectively.

We always assume further that g satisfies the *superlinear growth* (*coercivity*) *hypothesis*:

$$\lim_{|v| \to +\infty} \frac{g(v)}{|v|} = +\infty. \tag{$\mathbb{H}$}$$

Together with $(\mathcal{P}_v)$, consider the *relaxed variational problem*

$$\min \left\{ \int_\Omega g^{**}\big(\nabla u(x)\big)\, dx : u(\cdot) \in \big\langle v, \cdot \big\rangle + \mathbf{W}_0^{1,p}(\Omega) \right\} \tag{$\mathcal{RP}_v$}$$

where $g^{**}\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the *bipolar function*, i.e., the greatest lower semicontinuous convex function below $g(\cdot)$. The hypothesis $(\mathbb{H})$, in fact, is the easiest way to guarantee boundedness of all proper faces of the epigraph epi (g^{**}). Using the well-known geometrical fact, to each vector v with $g^{**}(v) < +\infty$ we can associate a unique (nonvertical) closed face $F(v)$ of epi (g^{**}) to whose relative interior the point $\big(v, g^{**}(v)\big)$ belongs. Denoting by $\widehat{F}(v)$ its projection to $\mathbb{R}^n$, we reduce the condition $(\mathbb{C})$ to the form:

$$\text{either}\ \ g(v) = g^{**}(v) \ \text{ or }\ \operatorname{int}\ \widehat{F}(v) \neq \varnothing. \tag{$\mathbb{C}'$}$$

On the other hand, the condition $\operatorname{int} \widehat{F}(v) = \varnothing$ is necessary and sufficient for uniqueness of a solution to the relaxed problem (see [5, 6]). The smoothness of the boundary $\partial\Omega$ $\big($as assumed in [5]$\big)$ can be dropped here. Observe that the affine function $\langle v, \cdot\rangle$ is itself the trivial solution of $(\mathcal{RP}_v)$. Furthermore, as was shown in [5, 6, 12], $u(\cdot) \in \langle v, \cdot\rangle + W_0^{1,p}(\Omega)$ is a solution to the problem $(\mathcal{RP}_v)$ if and only if the inclusion

$$\nabla u(x) \in \widehat{F}(v) \tag{1}$$

holds for almost every (a. e.) $x \in \Omega$, this property depends neither on the smoothness of $\partial\Omega$ nor on the growth of the Lagrangian (see [12, Lemma 3.3]). Due to the hypothesis $(\mathbb{H})$ this implies the Lipschitz continuity of all minimizers. Therefore, without loss of generality, we can always put $p = +\infty$.

Notice that, by the relaxation result of [17, Lemma 2.3], every minimizer in $(\mathcal{P}_v)$ solves $(\mathcal{RP}_v)$ as well whenever $v \in \operatorname{int} \operatorname{dom}(g^{**})$, and the minimum is $\mu(\Omega)g^{**}(v)$. Here $\operatorname{dom}(g^{**}) := \{v \in \mathbb{R}^n : g^{**}(v) < +\infty\}$, and μ is the n-dimensional Lebesgue measure. Moreover, the minimizers of the original (nonconvex) functional can be searched as solutions of the gradient inclusion

$$\nabla u(x) \in \operatorname{ext} \widehat{F}(v) \tag{2}$$

with the extreme boundary on the right-hand side. In other words, each Lipschitz continuous function $u(\cdot)$, $u\big|_{\partial\Omega} = \langle v, \cdot\rangle$, satisfying (2) is, necessarily, a minimizer in the problem $(\mathcal{P}_v)$.

The next natural question is stability in some sense of the solutions of the problem $(\mathcal{P}_v)$ with respect to the boundary slope v. Various approaches to this destination were developed in [7, 11, 13]. In particular, in the works [11, 13] we have searched a solution of $(\mathcal{P}_v)$ continuous in $v \in \operatorname{dom}(g^{**})$ with respect to the uniform topology on the space $\mathbf{C}(\overline{\Omega})$ of continuous functions $u\colon \overline{\Omega} \to \mathbb{R}$. More precisely, with each $v \in \operatorname{dom}(g^{**})$ we associate the set $\mathbb{S}(v)$ (respectively, $\mathbb{S}^{**}(v)$) of all minimizers in the variational problem $(\mathcal{P}_v)$ (respectively, $(\mathcal{RP}_v)$). Notice that always $\langle v, \cdot\rangle \in \mathbb{S}^{**}(v)$ while the multifunction $\mathbb{S}(v)$ may admit empty values. So, the problem is to find a selection $s(v) \in \mathbb{S}^{**}(v)$ continuous as a mapping $\operatorname{dom}(g^{**}) \to \mathbf{C}(\overline{\Omega})$ and $s(v) \in \mathbb{S}(v)$ whenever v satisfies the existence criterion $(\mathbb{C})$ (or $(\mathbb{C}')$). In [11] such continuous (even Lipschitz continuous) selection was constructed by a series of local perturbations of the affine function $\langle v, \cdot\rangle$ due to a constructive version of the Vitali covering theorem. Making these perturbations arbitrarily small, we can approximate the trivial solution of the relaxed problem by a sequence of minimizers in the original problem uniformly in v and keeping the continuous dependence on v. However, this approach is not appropriate in describing the family of all solutions and approximating the relaxed minimizers other than the affine minimizer.

In this paper, following [14], we develop another, in some sense complementary, approach to studying solutions of the problems $(\mathcal{P}_v)$ and $(\mathcal{RP}_v)$ and their stability with respect to v. Namely, in Section 2 we define two (Lipschitz) continuous functions $v \mapsto s^+(v)(x)$ and $v \mapsto s^-(v)(x)$ that enclose all solutions of the relaxed problem and show existence of a continuous selection of $v \mapsto \mathbb{S}^{**}(v)$ passing through an arbitrary point of its graph. In particular, this implies lower semicontinuity of the multifunction $\mathbb{S}^{**}(\cdot)$.

Then, in Section 3, the simple compactness argument permits us to prove the density result: arbitrarily near to each continuous selection $\widetilde{s}(v) \in \mathbb{S}^{**}(v)$ there exists another selection $\widehat{s}(v) \in \mathbb{S}^{**}(v)$ such that $\widehat{s}(v) \in \mathbb{S}(v)$ whenever $\mathbb{S}(v) \neq \varnothing$. Moreover, we can choose a continuous "selection" of the multivalued map $\mathbb{S}(\cdot)$ passing through each given point (v^0, u^0), where $v^0 \in \operatorname{dom}(g^{**})$ (not necessarily $\operatorname{int} \widehat{F}(v^0) \neq \varnothing$) and $u^0(\cdot)$ is a solution of $(\mathcal{RP}_{v^0})$. As a convenient tool to treat such kind of problems we use the Baire category theorem and a Choquet function characterizing extreme points of a compact convex set.

In the last section we announce a result on well-posedness of the problem $(\mathcal{P}_v)$ with the best Lipschitz constant and give some useful observations.

2. Properties of relaxed solutions

In what follows, we assume Ω to be an open bounded set in $\mathbb{R}^n$ with the boundary $\partial\Omega$, and $g\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ to be a lower semicontinuous proper function satisfying the hypothesis $(\mathbb{H})$.

Given a convex compact set $A\subset\mathbb{R}^n$, we denote by $\sigma_A(\cdot)$ the *support function*, i.e., $\sigma_A(v) := \sup\big\{\langle v,x\rangle : x\in A\big\}$, $v\in\mathbb{R}^n$, and by $A^* := \big\{v\in\mathbb{R}^n : \sigma_A(v)\le 1\big\}$, the *polar set.*

The following functions

$$s^+(v)(x) := \inf_{x'\in\partial\Omega}\big\{\langle v,x'\rangle + \sigma_{\widehat{F}(v)}(x-x')\big\}, \tag{3}$$

$$s^-(v)(x) := \sup_{x'\in\partial\Omega}\big\{\langle v,x'\rangle - \sigma_{\widehat{F}(v)}(x'-x)\big\} \tag{4}$$

are crucial in our considerations and can be interpreted in various ways. First, they are connected with the expansion factors defined by

$$\rho^+(v)(x) := \sup\Big\{\rho>0 : x-\rho\big(\widehat{F}(v)-v\big)^*\subset\Omega\Big\},$$
$$\rho^-(v)(x) := -\sup\Big\{\rho>0 : x+\rho\big(\widehat{F}(v)-v\big)^*\subset\Omega\Big\}.$$

Namely, for all $v\in\operatorname{dom}(g^{**})$ and $x\in\overline{\Omega}$

$$s^\pm(v)(x) = \langle v,x\rangle + \rho^\pm(v)(x)\,.$$

Notice that in the case either $x\in\partial\Omega$ or $\operatorname{int}\widehat{F}(v)=\varnothing$ $\big($equivalently, $v\notin\operatorname{int}\widehat{F}(v)\big)$ there is no $\rho>0$ with $x\pm\rho\big(\widehat{F}(v)-v\big)^*\subset\Omega$, and we may naturally put $\rho^+(v)(x)=\rho^-(v)(x)=0$. In fact, given $v\in\operatorname{int}\widehat{F}(v)$, we have that $\rho^+(v)(\cdot)$ is the *minimum time function* in the optimal control problem with the constant dynamics $-\big(\widehat{F}(v)-v\big)^*$ and the target set $\partial\Omega$ to be reached from inside the domain Ω. Similarly, we can achieve a point $x\in\Omega$ from the boundary for the maximum (negative) time $\rho^-(v)(x)$ (with the same dynamics). Furthermore, the function $s^\pm(v)(\cdot)$ is the *viscosity solution* to the Hamilton–Jacobi equation

$$\pm\Big(\sigma_{(\widehat{F}(v)-v)^*}\big(\nabla u(x)\big)-1\Big)=0,\quad u\big|_{\partial\Omega}=\langle v,\cdot\rangle,$$

as defined in [9] (also see [3, 16]). Finally, we have the following result.

Theorem 1. *The functions $s^+(v)(\cdot)$ and $s^-(v)(\cdot)$ are minimizers in the relaxed problem $(\mathcal{RP}_v)$ whenever $v \in \operatorname{dom}(g^{**})$, and for each minimizer $u(\cdot) \in \mathbb{S}^{**}(v)$ the inequalities*

$$s^-(v)(x) \leq u(x) \leq s^+(v)(x), \quad x \in \overline{\Omega}, \tag{5}$$

hold. Moreover, the mappings $v \mapsto s^+(v)(\cdot)$ and $v \mapsto s^-(v)(\cdot)$ are Lipschitz continuous:

$$\left\| s^\pm(v_1) - s^\pm(v_2) \right\|_\infty \leq \|\Omega\| |v_1 - v_2|$$

whenever $v_1, v_2 \in \operatorname{dom}(g^{**})$. Here $\|\Omega\| := \max_{x\in\overline{\Omega}} |x|$, and $\|\cdot\|_\infty$ is the norm in $\mathbf{C}(\overline{\Omega})$.

We call the functions (3) and (4) *upper* and *lower solutions*, respectively, taking it into account that (as we will see later) they do not, in general, minimize the original functional but they can be approximated by solutions of $(\mathcal{P}_v)$. A complete proof of Theorem 1 can be found in [14] (see Proposition 2.1 and Theorem 2.1 there).

Now, using a minimum time function, we find a solution of $(\mathcal{RP}_v)$ continuous (even Lipschitz continuous) with respect to v such that with a fixed $v^0 \in \operatorname{dom}(g^{**})$ it associates some given minimizer $u^0(\cdot) \in \mathbb{S}^{**}(v^0)$.

This is obvious in the case $\operatorname{int} \widehat{F}(v^0) = \varnothing$. Otherwise, putting by $V := \widehat{F}(v^0)$, consider the function

$$T(v) := \inf_{v' \in \overline{V^c}} \sigma_{(V-v^0)^*}(v' - v), \quad v \in \operatorname{dom}(g^{**}), \tag{6}$$

where V^c is the complement of V and $\overline{V^c}$ is the closure of V^c. It is immediate from (6) that the function $T(\cdot)$ is Lipschitz continuous with the Lipschitz constant $\left\|(V - v^0)^*\right\| := \max_{x\in(V-v^0)^*} |x|$ and can be represented as

$$T(v) = \sup\left\{t > 0 : v + t(V - v^0) \subset V\right\}, \quad v \in \operatorname{dom}(g^{**}). \tag{7}$$

Clearly, $T(v^0) = 1$ and $T(v) = 0$ if $v \notin \operatorname{int} \widehat{F}(v^0)$ while for other points $0 < T(v) < 1$. Define

$$s(v)(x) := T(v)\big(u^0(x) - \langle v^0, x\rangle\big) + \langle v, x\rangle, \tag{8}$$

$x \in \overline{\Omega}$, $v \in \operatorname{dom}(g^{**})$, and, by the above properties, $s(v^0)(\cdot) \equiv u^0(\cdot)$ and $s(v)(\cdot) \equiv \langle v, \cdot\rangle$ whenever $v \in \operatorname{dom}(g^{**}) \setminus \operatorname{int} \widehat{F}(v^0)$. If $v \in \operatorname{int} \widehat{F}(v^0)$ then, by (8) and (7),

$$\nabla s(v)(x) = T(v)\big(\nabla u^0(x) - v^0\big) + v \in v + T(v)\big(\widehat{F}(v^0) - v^0\big) \subset \widehat{F}(v^0) = \widehat{F}(v)$$

a. e. in Ω. Since $s(v)(x) = \langle v, x\rangle$ for $x \in \partial\Omega$, it follows from (1) that $s(v) \in \mathbb{S}^{**}(v)$ for each $v \in \operatorname{dom}(g^{**})$. Taking into account the Lipschitz continuity of $T(\cdot)$ and the inequalities (5), we can choose a Lipschitz constant L of the mapping $v \mapsto s(v)(\cdot)$ that does not depend on $u^0(\cdot)$. Namely, put

$$L := \|\Omega\| + \big\|\widehat{F}(v^0) - v^0\big\| \cdot \big\|\big(\widehat{F}(v^0) - v^0\big)^*\big\| \cdot \max_{x \in \overline{\Omega}} \operatorname{dist}_{\partial\Omega}(x) \tag{9}$$

where $\operatorname{dist}_A(\cdot)$ means the distance from a point to A. Thus, the following theorem takes place.

Theorem 2. *Given arbitrary* $v^0 \in \operatorname{dom}(g^{**})$ *and* $u^0(\cdot) \in \mathbb{S}^{**}(v^0)$ *there exists a Lipschitz continuous selection* $s(v) \in \mathbb{S}^{**}(v)$, $v \in \operatorname{dom}(g^{**})$, *with the Lipschitz constant* L *defined by* (9), *such that* $s(v^0) = u^0$. *Hence, in particular, the multivalued mapping* $v \mapsto \mathbb{S}^{**}(v)$ *is lower semicontinuous on* $\operatorname{dom}(g^{**})$ *and is locally Lipschitz on each projection* $\widehat{F}(v^0)$.

3. Minimizers of the original (nonconvex) functional. A density theorem

As we have already said, the set of minimizers $\mathbb{S}(v)$, $v \in \operatorname{int}\operatorname{dom}(g^{**})$, contains the set $\mathbb{S}^{\mathrm{ext}}(v)$ of solutions to the extremal gradient inclusion (2) $\big($it may be larger if $v \in \operatorname{int}\widehat{F}(v)$ and $g(v) = g^{**}(v)$ simultaneously$\big)$. Denoting by $\mathfrak{S}$ (respectively, $\mathfrak{S}^{**}$ or $\mathfrak{S}^{\mathrm{ext}}$) the set of all continuous selections $s\colon \operatorname{dom}(g^{**}) \to \mathbf{C}(\overline{\Omega})$ of the multivalued mapping $v \mapsto \mathbb{S}(v)$ $\big($respectively, $v \mapsto \mathbb{S}^{**}(v)$ or $v \mapsto \mathbb{S}^{\mathrm{ext}}(v)\big)$, we obtain, in particular, that $\mathfrak{S}^{\mathrm{ext}}$ is a dense $\mathcal{G}_\delta$-subset of $\mathfrak{S}^{**}$ (a set of the *second category*) with respect to the topology of uniform convergence on compact subsets in $\operatorname{dom}(g^{**})$. Here and in the sequel, we naturally agree that $\mathbb{S}(v) = \mathbb{S}^{\mathrm{ext}}(v) = \big\{\langle v, \cdot\rangle\big\}$ whenever $\operatorname{int}\widehat{F}(v) = \varnothing$. In the same way, we extend our mappings outside $\operatorname{dom}(g^{**})$. The above-announced fact is a multidimensional continuous version of A. Cellina's result (see [4, Theorem 1]). It implies the density of the selection family $\mathfrak{S}$ in $\mathfrak{S}^{**}$, which means that every relaxed solution stable with respect to $v \in \operatorname{dom}(g^{**})$ can be approximated by a sequence of stable minimizers in the original problem $(\mathcal{P}_v)$ uniformly in v. For proving this result, it is more convenient to consider $\mathfrak{S}$, $\mathfrak{S}^{**}$, and $\mathfrak{S}^{\mathrm{ext}}$ translated by the affine selection $v \mapsto \langle v, \cdot\rangle$. Denote the so-obtained sets by $\mathcal{H}$, $\mathcal{H}^{**}$, and $\mathcal{H}^{\mathrm{ext}}$.

Observe that there exists at most countable family of the different (disjoint) nonempty sets $\operatorname{int}\widehat{F}(v)$, say V_m, $m = 1, 2, \ldots$. Then the set $\mathcal{H}^{**}$ consists clearly of all $\rho(\cdot) \in \mathcal{C}\big(\mathbb{R}^n, \mathbf{C}_0(\overline{\Omega})\big)$ such that $\rho(v)(\cdot) \in \mathbf{W}_0^{1,\infty}(\Omega)$ with $v + \nabla\rho(v)(x) \in \overline{V_m}$ for a. e. $x \in \Omega$ whenever $v \in V_m$, $m = 1, 2, \ldots$, and $\rho(v) \equiv 0$ outside of $\bigcup_{m=1}^{\infty} V_m$. Here $\mathcal{C}\big(\mathbb{R}^n, \mathbf{C}_0(\overline{\Omega})\big)$ is the space of all continuous mappings $\rho\colon \mathbb{R}^n \to \mathbf{C}(\overline{\Omega})$, $\rho(v)\big|_{\partial\Omega} = 0$, with the topology of uniform convergence on compact sets. We can easily show that $\mathcal{H}^{**}$ is closed in $\mathcal{C}\big(\mathbb{R}^n, \mathbf{C}_0(\overline{\Omega})\big)$; hence, it is a complete metric space in the induced topology. Similarly,

$$\mathcal{H}^{\text{ext}} = \Big\{\rho(\cdot) \in \mathcal{H}^{**} : v + \nabla\rho(v)(x) \in \operatorname{ext} V_m \ \text{ for a.e. } \ x \in \Omega \\ \text{whenever } v \in V_m,\ m = 1, 2, \ldots\Big\}.$$

Thus, the main result (see [14, Theorem 3.1]) can be formulated as

Theorem 3. *The set $\mathcal{H}^{\text{ext}}$ is a dense $\mathcal{G}_\delta$-subset of $\mathcal{H}^{**}$.*

We give a more precise version of this theorem. Namely, fix a vector $v^0 \in \operatorname{dom}(g^{**})$ and a function $\theta^0(\cdot) \in \mathbb{S}^{**}(v^0) - \langle v^0, \cdot\rangle$. Then the set $\mathcal{H}^{**}_{(v^0,\theta^0)} := \big\{\rho(\cdot) \in \mathcal{H}^{**} : \rho(v^0) = \theta^0\big\}$ is closed in $\mathcal{H}^{**}$ (it is nonempty by Theorem 2). Consider the family

$$\mathcal{H}^{\text{ext}}_{(v^0,\theta^0)} := \big\{\rho(\cdot) \in \mathcal{H}^{**}_{(v^0,\theta^0)} : v + \nabla\rho(v)(x) \in \operatorname{ext} V_m \ \text{ for a.e. } \ x \in \Omega \\ \text{and for all } v \in V_m,\ v \neq v^0,\ m = 1, 2, \ldots\big\}.$$

Notice that a continuous mapping $v \mapsto \langle v, \cdot\rangle + \rho(v)(\cdot)$, where $\rho(\cdot) \in \mathcal{H}^{\text{ext}}_{(v^0,\theta^0)}$, associates with each $v \in V_m$, $m = 1, 2, \ldots,\ \ v \neq v^0$, a solution of (2), with the affine boundary datum $\langle v, \cdot\rangle$ (hence, a minimizer in the variational problem $(\mathcal{P}_v)$), passing through the point $\big(v^0, \langle v^0, \cdot\rangle + \theta^0(\cdot)\big) \in \operatorname{graph} \mathbb{S}^{**}$. The set of such mappings is not empty as follows from the theorem below.

Theorem 4. *The set $\mathcal{H}^{\text{ext}}_{(v^0,\theta^0)}$ is a dense $\mathcal{G}_\delta$-subset of $\mathcal{H}^{**}_{(v^0,\theta^0)}$ whenever $v^0 \in \operatorname{dom}(g^{**})$ and $\theta^0(\cdot) \in \mathbb{S}^{**}(v^0) - \langle v^0, \cdot\rangle$.*

Proof. Without loss of generality we may assume that $\operatorname{int} \widehat{F}(v^0) \neq \varnothing$; and let $m^0 = 1, 2, \ldots$ be such that $v^0 \in V_{m^0}$.

In the next we follow the same lines as in the proof of Theorem 3.1 [14] with slight modifications in order to satisfy the condition $\rho(v^0) = \theta^0$.

With each compact convex set $K \subset \mathbb{R}^n$ we associate a *Choquet function* $l(\cdot, K)\colon K \to \mathbb{R}^+$ that is concave, upper semicontinuous, bounded $\big(0 \leq l(x, K) \leq \operatorname{diam} K\big)$, and such that

$$l(x, K) = 0 \ \text{ if and only if } \ x \in \operatorname{ext} K \tag{10}$$

(see [2] for an example of such a function). Here diam K is the diameter of K.

Given $\eta > 0$ small enough, we define

$$V^\eta_{m^0} := \big\{v \in V_{m^0} : \operatorname{dist}_{\partial V_{m^0}}(v) \geq \eta,\ |v - v^0| \geq \eta\big\}$$

and, for $m = 1, 2, \ldots$, $m \neq m^0$,

$$V^\eta_m := \big\{v \in V_m : \operatorname{dist}_{\partial V_m}(v) \geq \eta\big\}.$$

Using the compactness of V_m^η, $m = 1, 2, \dots$, and the convergence result of [10, p. 49], we can easily prove (see [14, Lemma 3.2]) upper semicontinuity of the functional $\mathfrak{L}_m^\eta\colon \mathcal{H}^{**} \to \mathbb{R}^+$,

$$\mathfrak{L}_m^\eta(\rho) := \sup_{v \in V_m^\eta} \int_\Omega l\big(v + \nabla\rho(v)(x), \overline{V_m}\big)\, dx.$$

Hence, the sublevel set $\big($localized at the point (v^0, θ^0) if necessary$\big)$

$$\mathcal{H}_{(v^0,\theta^0)}^{\eta,m} := \big\{\rho(\cdot) \in \mathcal{H}_{(v^0,\theta^0)}^{**} : \mathfrak{L}_m^\eta(\rho) < \eta\big\}$$

is open in $\mathcal{H}_{(v^0,\theta^0)}^{**}$ (in the induced topology); and, by (10), we have

$$\mathcal{H}_{(v^0,\theta^0)}^{\text{ext}} = \bigcap_{m,\,k=1}^{\infty} \mathcal{H}_{(v^0,\theta^0)}^{1/k,m}.$$

Therefore $\mathcal{H}_{(v^0,\theta^0)}^{\text{ext}}$ is a $\mathcal{G}_\delta$-subset of $\mathcal{H}_{(v^0,\theta^0)}^{**}$. In order to prove density, it suffices (by the Baire category theorem) to prove density of each $\mathcal{H}_{(v^0,\theta^0)}^{\eta,m}$, $m = 1, 2, \dots$, $\eta > 0$.

Let $m = m^0$, and fix $\eta > 0$, $\widetilde{\rho}(\cdot) \in \mathcal{H}_{(v^0,\theta^0)}^{**}$, $\varepsilon > 0$. First, applying some type of polynomial approximations as in [14, Section 4], we find a function $\bar{\rho}(\cdot) \in \mathcal{H}^{**}$ such that for every $v \in V_{m^0}$

$$\bar{\rho}(v)(x) \neq 0 \quad \text{for a. e. } x \in \Omega, \tag{11}$$

$\bar{\rho}(v) = \widetilde{\rho}(v)$ whenever $v \notin V_{m_0}$, and $\big\|\bar{\rho}(v) - \widetilde{\rho}(v)\big\|_\infty < \varepsilon/2$, $v \in \mathbb{R}^n$. From lower semicontinuity of the mapping $v \mapsto \mathbb{S}^{**}(v)$ (see Theorem 2) it follows [1, p. 44] that the multifunction $\mathfrak{F}\colon \mathbb{R}^n \to \mathbf{C}_0(\overline{\Omega})$,

$$\mathfrak{F}(v) := \big\{\theta(\cdot) \in \mathbb{S}^{**}(v) - \langle v, \cdot\rangle : \|\theta - \widetilde{\rho}(v)\|_\infty < \varepsilon/2\big\},$$

is also lower semicontinuous. Then, by Michael's classical theorem (e.g. see [1, p. 82]), taking into account the convexity of $\mathbb{S}^{**}(v)$, we can choose a continuous selection of the mapping $v \mapsto \overline{\mathfrak{F}(v)}$ $\big($the closure in the space $\mathbf{C}(\overline{\Omega})\big)$ which admits the value θ^0 at the point v^0 and equals $\bar{\rho}(v)$ for all $v \in V_{m^0}^\eta$. Therefore, this selection $\big($denote it by $\bar{\rho}(v)$ as well$\big)$ satisfies the inequality $\big\|\bar{\rho}(v) - \widetilde{\rho}(v)\big\|_\infty \le \varepsilon/2$, $v \in \mathbb{R}^n$, and belongs to the set $\mathcal{H}_{(v^0,\theta^0)}^{**}$, and the property (11) holds for all $v \in V_{m^0}^\eta$. Combining (11) with compactness of the set $V_{m^0}^\eta$, we can choose $0 < \delta < \varepsilon/2$ such that

$$\mu\Big\{x \in \Omega : \big|\rho(v)(x)\big| < \delta\Big\} < \eta/D_{m^0} \quad \text{for all} \quad v \in V_{m^0}^\eta, \tag{12}$$

where D_{m^0} is the diameter of V_{m^0}. For details we refer to Lemma 5.1 of [14].

Now, consider the compact subsets $\Gamma^{\pm} \subset V_{m^0} \times \Omega$,

$$\Gamma^{+} := \big\{(v,x) \in V^{\eta}_{m^0} \times \Omega : \bar{\rho}(v)(x) \geq \delta\big\},$$
$$\Gamma^{-} := \big\{(v,x) \in V^{\eta}_{m^0} \times \Omega : \bar{\rho}(v)(x) \leq -\delta\big\},$$

which can be covered by some finite families of the open sets

$$U^{+}(x) := \Big\{(v,y) \in V_{m^0} \times \Omega : \bar{\rho}(v)(x) > \delta/2,\ |x-y| < \varepsilon/\big(8D_{m^0}\big),$$
$$y \in x - \big(\bar{\rho}(v)(x) - \delta/2\big)\,\mathrm{int}\,\big(V_{m^0} - v\big)^{*} \subset \Omega\Big\}$$

and

$$U^{-}(x) := \Big\{(v,y) \in V_{m^0} \times \Omega : \bar{\rho}(v)(x) < -\delta/2,\ |x-y| < \varepsilon/\big(8D_{m^0}\big),$$
$$y \in x - \big(\bar{\rho}(v)(x) + \delta/2\big)\,\mathrm{int}\,(V_{m^0} - v)^{*} \subset \Omega\Big\},$$

respectively, $x \in \Omega$. Namely, there exist points $x_1^{+}, \dots, x_{N^+}^{+}$ and $x_1^{-}, \dots, x_{N^-}^{-}$ in Ω such that

$$\Gamma^{\pm} \subset \bigcup_{i=1}^{N^{\pm}} U^{\pm}(x_i^{\pm}). \tag{13}$$

The continuous functions $f^{\pm}\colon \overline{V_{m^0}} \to \mathbf{C}(\overline{\Omega})$ defined by

$$f^{+}(v)(x) := \max_{1\leq i\leq N^+} \big\{\bar{\rho}(v)(x_i^{+}) - \sigma_{V_{m^0}-v}(x_i^{+} - x), \delta/2\big\}; \tag{14}$$

$$f^{-}(v)(x) := \min_{1\leq i\leq N^-} \big\{\bar{\rho}(v)(x_i^{-}) + \sigma_{V_{m^0}-v}(x - x_i^{-}), -\delta/2\big\}, \tag{15}$$

$v \in \overline{V_{m^0}}$, $x \in \overline{\Omega}$, are suitable approximations of the positive and negative parts of $\bar{\rho}(v)(x)$ in the sense that $v + \nabla f^{\pm}(v)(x) \in \mathrm{ext}\, V_{m^0}$ for all $v \in V^{\eta}_{m^0}$ and a. e. $x \in \Omega$ with $\bar{\rho}(v)(x) \geq \delta$ $\big(\bar{\rho}(v)(x) \leq -\delta$, respectively$\big)$ while $v + \nabla f^{\pm}(v)(x) \in \overline{V_{m^0}}$ for the others $(v,x) \in \overline{V_{m^0}} \times \Omega$ provided existence of the gradients (see [15, p. 50]); moreover, $f^{\pm}(v)(\cdot)$ remains in a $\|\cdot\|_{\infty}$-neighborhood of the function $\bar{\rho}^{\pm}(v)(\cdot)$ whenever $v \in V^{\eta}_{m^0}$. Here $\bar{\rho}^{+}(v)(x) := \max\big\{\bar{\rho}(v)(x), 0\big\}$ and $\bar{\rho}^{-}(v)(x) := \min\big\{\bar{\rho}(v)(x), 0\big\}$. See [14, Section 5] for a detailed discussion.

It follows from (14), (15), and (13) that the function

$$f(v)(x) := f^{+}(v)(x) + f^{-}(v)(x), \quad v \in \overline{V_{m^0}},\ x \in \overline{\Omega}, \tag{16}$$

is well defined in the sense that it is never equal to the sum of two nonconstant terms. Hence, always $\nabla f(v)(x) \in \overline{V_{m^0}} - v$ and $v + \nabla f(v)(x) \in \mathrm{ext}\, V_{m^0}$ for all $v \in V^{\eta}_{m^0}$ and a. e. $x \in \Omega$ such that either $\bar\rho(v)(x) \ge \delta$ or $\bar\rho(v)(x) \le -\delta$. By the choice of $\delta > 0$ (see (12)), this implies $\mathfrak{L}^{\eta}_{m^0}(f) < \eta$. Furthermore, as is easy to see,

$$\big\|f(v) - \bar\rho(v)\big\|_{\infty} < \varepsilon/2 \tag{17}$$

for all $v \in V^{\eta}_{m^0}$. Choosing an open set $W \subset \mathbb{R}^n$ such that $V^{\eta}_{m^0} \subset W \subset V_{m^0}$, $v^0 \notin W$, and (17) holds for all $v \in W$, we can find a continuous function $\psi\colon \mathbb{R}^n \to [0,1]$ such that $\psi(v) = 1$ for $v \in V^{\eta}_{m^0}$ and $\psi(v) = 0$ for $v \notin W$. Then the function $\widehat\rho(\cdot)$,

$$\widehat\rho(v)(x) := \psi(v) f(v)(x) + \big(1 - \psi(v)\big)\bar\rho(v)(x), \tag{18}$$

$v \in \mathbb{R}^n$, $x \in \overline{\Omega}$, belongs obviously to $\mathcal{H}^{**}$; $\widehat\rho(v^0) = \bar\rho(v^0) = \theta^0$; and the inequality $\mathfrak{L}^{\eta}_{m^0}(\widehat\rho) < \eta$ holds. Thus, we have $\widehat\rho(\cdot) \in \mathcal{H}^{\eta, m^0}_{(v^0,\theta^0)}$ and $\big\|\widehat\rho(v) - \tilde\rho(v)\big\|_{\infty} \le \varepsilon$ for all $v \in \mathbb{R}^n$, which proves the density of $\mathcal{H}^{\eta, m^0}_{(v^0,\theta^0)}$ in $\mathcal{H}^{**}_{(v^0,\theta^0)}$.

In the case $m \ne m^0$, the proof is exactly the same except for the condition $\widehat\rho(v^0) = \theta^0$. □

4. Final remarks

Theorem 1 shows that not only the affine function $\langle v, \cdot\rangle$ but also the upper and lower solutions $s^{\pm}(v)(\cdot)$ of the relaxed problem $(\mathcal{RP}_v)$ are Lipschitz continuous with respect to v with the constant $\|\Omega\|$. Furthermore, the function $\langle v, \cdot\rangle + f(v)(\cdot)$ in the proof of Theorem 4 (see (16)) is also Lipschitz continuous with respect to v with the same constant and is “almost” a minimizer of the original nonconvex functional. However, we loose the Lipschitz property after extension of this function outside $V^{\eta}_{m^0}$ (see (18)). But there is no necessity in such extension whenever the relaxed minimizer $\tilde\rho(v)(\cdot)$ admits always either nonnegative or nonpositive values. Taking into account these observations, we might wonder whether the original problem $(\mathcal{P}_v)$ should also admit Lipschitz continuous (with respect to v) solutions approximating some of the relaxed minimizers with the same property.

In order to formulate a density result, denote by $\mathcal{H}^{**}_{\|\Omega\|}$ the set of all mappings $\rho(\cdot) \in \mathcal{H}^{**}$ such that $v \mapsto \langle v, \cdot\rangle + \rho(v)(\cdot)$ is Lipschitz continuous with Lipschitz constant $\|\Omega\|$ (further, called simply $\|\Omega\|$-Lipschitz continuous). Let also $\mathcal{C}^+$ and $\mathcal{C}^-$ be the positive and negative cones in $\mathcal{C}\big(\mathbb{R}^n, \mathbf{C}_0(\overline{\Omega})\big)$, i.e.,

$$\mathcal{C}^+ := \Big\{\rho(\cdot) \in \mathcal{C}\big(\mathbb{R}^n, \mathbf{C}_0(\overline{\Omega})\big) : \rho(v)(x) \ge 0, \quad v \in \mathbb{R}^n,\, x \in \overline{\Omega}\Big\},$$
$$\mathcal{C}^- := \Big\{\rho(\cdot) \in \mathcal{C}\big(\mathbb{R}^n, \mathbf{C}_0(\overline{\Omega})\big) : \rho(v)(x) \le 0, \quad v \in \mathbb{R}^n,\, x \in \overline{\Omega}\Big\}.$$

Theorem 5. *The set $\mathcal{H}^{\rm ext} \cap \mathcal{H}^{**}_{\|\Omega\|} \cap \mathcal{C}^{\pm}$ is a dense $\mathcal{G}_\delta$-subset of $\mathcal{H}^{**}_{\|\Omega\|} \cap \mathcal{C}^{\pm}$. In particular, the variational problem $(\mathcal{P}_v)$ admits a minimizer $\widehat{s}(v)(\cdot)$, defined on the set of those v where the minimum exists, which is $\|\Omega\|$-Lipschitz continuous with respect to v; and this minimizer can be chosen arbitrarily near to every $\|\Omega\|$-Lipschitz continuous solution $\widetilde{s}(v)(\cdot)$ of the relaxed problem $(\mathcal{RP}_v)$ which always remains above (respectively, below) the affine function.*

It follows from this theorem that the affine function itself can be uniformly approximated by a sequence of $\|\Omega\|$-Lipschitz continuous (with respect to v) solutions of the problem $(\mathcal{P}_v)$. This improves the result of [11] where the Lipschitz property was established with a larger Lipschitz constant. It is clear that $\|\Omega\|$ is the best possible Lipschitz constant.

In such a form Theorem 5 is proved in [14, Section 6], but, using Theorem 2, we can also find a Lipschitz continuous selection of $v \mapsto \mathbb{S}(v)$ passing through an arbitrary point $(v^0, u^0) \in \operatorname{graph} \mathbb{S}^{**}$. However, the Lipschitz constant here may be different $\big($it depends on the choice of v^0, see (9)$\big)$.

In conclusion, we observe that the upper and lower solutions $s^{\pm}(v)(\cdot)$ defined by (3) and (4) always satisfy the inclusions $\nabla s^{\pm}(v)(x) \in \partial \widehat{F}(v)$ $\big($the boundary of $\widehat{F}(v)\big)$ for a. e. $x \in \Omega$ (see [8, Theorem 3.1]) but, in general, they are not solutions of the original problem $(\mathcal{P}_v)$ $\big($equivalently, of the inclusion (2)$\big)$. We illustrate this fact by the following simple example.

E x a m p l e 1. Let $n = 2$, $\Omega := \Big\{x = (x_1, x_2) : \max\big(|x_1|, |x_2|\big) < 1\Big\}$, and

$$g(v) = \begin{cases} 0 & \text{if } \ |v_1| = 1 \wedge |v_2| = 1; \\ +\infty & \text{otherwise.} \end{cases}$$

Then the unique proper face of $\operatorname{epi}(g^{**})$ with nonempty interior (in $\mathbb{R}^2$) is $\widehat{F} = \overline{\Omega}$, and the variational problem $(\mathcal{RP}_v)$ is equivalent to the differential inclusion $\nabla u(x) \in \widehat{F}$, $u\big|_{\partial\Omega} = \langle v, \cdot\rangle$, for $v \in \operatorname{int} \widehat{F}$ while, for $v \in \partial \widehat{F}$, the unique solution is the affine function $x \mapsto \langle v, x\rangle$. By (3) and (4), we find

$$s^{+}(v)(x) = 1 + \min\big\{v_1 x_1 - |x_2 - v_2|,\ v_2 x_2 - |x_1 - v_1|\big\},$$
$$s^{-}(v)(x) = -1 + \max\big\{v_1 x_1 + |x_2 + v_2|,\ v_2 x_2 + |x_1 + v_1|\big\},$$

where $x = (x_1, x_2) \in \overline{\Omega}$ and $v = (v_1, v_2) \in \widehat{F}$; hence, $\nabla s^{+}(v)(x), \nabla s^{-}(v)(x) \in \big\{(v_1, \pm 1), (\pm 1, v_2)\big\}$ for a. e. $x \in \Omega$, i.e., the "extremal" solutions $s^{+}(v)(\cdot)$ and $s^{-}(v)(\cdot)$ do not satisfy the extremal inclusion.

Acknowledgement. The author is indebted to the referee for his attention to the article and kind suggestions.

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